ELEMENTS OF POTENTIAL THEORY ON CARNOT GROUPS

MICHAEL RUZHANSKY AND DURVUDKHAN SURAGAN

Abstract. We propose and study elements of potential theory for the sub-Laplacian on homogeneous Carnot groups. In particular, we show continuity of the single layer potential and establish the Plemelj type jump relations for the double layer potential. As a consequence, we derive the formula for the trace to smooth surfaces of the Newton potential for the sub-Laplacian. Using this we construct a sub-Laplacian version of Kac's boundary value problem.

1. Introduction

In last decades the ideas combining the group theory with the analysis of partial differential equations have been a subject of active research by many authors. For example, nilpotent Lie groups play an important role in deriving sharp subelliptic estimates for differential operators on manifolds, starting from the seminal paper by Rothschild and Stein [15]. In particular, homogeneous Carnot groups appear in approximations for general Hörmander's sums of squares of vector fields on manifolds in view of the Rothschild-Stein lifting theorem [15] (see also [7]). They are also natural models for sub-Riemannian geometry ([2]). For a pseudo-differential approach to such problems we can refer to [5] on nilpotent, and to [18] on compact Lie groups, respectively.

In this paper we discuss elements of the potential theory and the theory of boundary layer operators on homogeneous Carnot groups. The applications briefly described here include the solution to Kac's boundary value problem and a refined version of local Hardy's inequality including boundary terms. The present paper is mainly based on our preprint [16] and detailed proofs can be found there.

There are several equivalent definitions of homogeneous Carnot groups. Following the definition in [1], a Lie group $\mathbb{G}=(\mathbb{R}^N,\circ)$ is called homogeneous Carnot group (or homogeneous stratified group) if it satisfies the following conditions (a) and (b):

(a) For some natural numbers $N_1+...+N_r=N$ the decomposition $\mathbb{R}^N=\mathbb{R}^{N_1}\times...\times\mathbb{R}^{N_r}$ is valid, and for every $\lambda>0$ the dilation $\delta_\lambda:\mathbb{R}^N\to\mathbb{R}^N$ given by

$$\delta_\lambda(x)\equiv\delta_\lambda(x^{(1)},...,x^{(r)}):=(\lambda x^{(1)},...,\lambda^r x^{(r)})$$

is an automorphism of the group $\mathbb{G}$. Here $x^{(k)}\in\mathbb{R}^{N_k}$ for $k=1,...,r$.

(b) Let N_1 be as in (a) and let $X_1,...,X_{N_1}$ be the left invariant vector fields on $\mathbb{G}$ such that $X_k(0)=\frac{\partial}{\partial x_k}|_0$ for $k=1,...,N_1$. Then

$$\operatorname{rank}(\operatorname{Lie}\{X_1,...,X_{N_1}\})=N,$$

for every $x\in\mathbb{R}^N$, i.e. the iterated commutators of $X_1,...,X_{N_1}$ span the Lie algebra of $\mathbb{G}$.

The number

$$Q=\sum_{k=1}^{r}kN_k.$$

is called the homogeneous dimension of G. Throughout this paper we assume $Q\geq 3$. This is not very restrictive since it effectively rules out only the spaces $\mathbb{R}$ and $\mathbb{R}^2$ where the fundamental solution assumes a different form and where most things are already known. The second order differential operator

$$\mathcal{L}=\sum_{k=1}^{N_1}X_k^2 \tag{1}$$

Date: May 3, 2018.

2010 *Mathematics Subject Classification.* Primary 35R03; Secondary 35S15.

Key words and phrases. sub-Laplacian, integral boundary condition, homogeneous Carnot group, Newton potential, layer potentials.

The authors were supported in parts by the EPSRC grant EP/K039407/1 and by the Leverhulme Grant RPG-2014-02, as well as by the MESRK grant 5127/GF4.

is called the (canonical) sub-Laplacian on $\mathbb{G}$. It is known that the sub-Laplacian has a unique fundamental solution ε on $\mathbb{G}$ (see Folland [6]),

$$\mathcal{L}\varepsilon = \delta,$$

and $\varepsilon(x, y) = \varepsilon(y^{-1}x)$ is given in terms of a homogeneous quasi-norm d ($\mathcal{L}$-gauge) in the form

$$\varepsilon(x, y) = \beta_d[d(x, y)]^{2-Q}, \tag{2}$$

where β_d is a positive constant, so the function ε is homogeneous of degree $2 - Q$. We refer to [5] for further information on stratified groups.

From now on, let $Q \geq 3$ be the homogeneous dimension of $\mathbb{G}$, $\partial\Omega$ the piecewise smooth boundary of a bounded domain Ω in $\mathbb{G}$, and $d\nu$ the volume element on $\mathbb{G}$. The standard Lebesque measure on $\mathbb{R}^N \simeq \mathbb{G}$ is the Haar measure for $\mathbb{G}$. Let $\langle X_k, d\nu\rangle$ be the natural pairing between vector fields and differential forms given by

$$\langle X_k, d\nu(x)\rangle = \bigwedge_{j=1, j\neq k}^{N_1} dx_j^{(1)} \bigwedge_{l=2}^{r} \bigwedge_{m=1}^{N_l} \theta_{l,m}, \tag{3}$$

with

$$\theta_{l,m} = -\sum_{k=1}^{N_1} a_{k,m}^{(l)}(x^{(1)}, \ldots, x^{(l-1)})dx_k^{(1)} + dx_m^{(l)}, \tag{4}$$

where $l = 2, \ldots, r$, $m = 1, \ldots, N_l$, and $a_{k,m}^{(l)}$ is a δ_λ-homogeneous polynomial of degree $l - 1$ such that

$$X_k = \frac{\partial}{\partial x_k^{(1)}} + \sum_{l=2}^{r} \sum_{m=1}^{N_l} a_{k,m}^{(l)}(x^{(1)}, ..., x^{(l-1)}) \frac{\partial}{\partial x_m^{(l)}}. \tag{5}$$

Throughout this paper we assume that the piecewise boundary $\partial\Omega$ of a bounded domain Ω is simple, that is, it has no self-intersections.

2. Single and double layer potentials

On $\mathbb{R}^n$, the basic fundamental building blocks for dealing with boundary value problems in domains are the layer potentials constructed from the fundamental solution of the Laplacian. In the setting of Carnot groups it is natural to use the sub-Laplacian. In [9], Jerison used the single layer potential defined by

$$\mathcal{S}_0 u(x) = \int_{\partial\Omega} u(y)\varepsilon(y, x)dS(y),$$

which, however, is not integrable over characteristic points. We refer to [14] for examples. On the contrary, the functionals

$$\mathcal{S}_j u(x) = \int_{\partial\Omega} u(y)\varepsilon(y, x)\langle X_j, d\nu(y)\rangle, \quad j = 1, ..., N_1, \tag{6}$$

where $\langle X_j, d\nu\rangle$ is the canonical pairing between vector fields and differential forms, are integrable over the whole boundary $\partial\Omega \in C^\infty$, see Theorem 2.1 with $u \equiv 1$. Parallel to $\mathcal{S}_j$, it will be natural to use the operator

$$\mathcal{D}u(x) = \int_{\partial\Omega} u(y)\langle\widetilde{\nabla}\varepsilon(y, x), d\nu(y)\rangle, \tag{7}$$

as a double layer potential, where $\widetilde{\nabla}\varepsilon = \sum_{k=1}^{N_1}(X_k\varepsilon)X_k$, with X_k acting on the y-variable. These potentials possess the continuity/jump properties that we now describe.

Theorem 2.1. *Let $u \in L^\infty(\partial\Omega)$. Then the single layer potential $\mathcal{S}_j u$ is absolutely convergent and continuous on $\mathbb{G}$, for all $j = 1, \ldots, N_1$.*

We now state the Plemelj type jump relations for the double layer potential $\mathcal{D}$ defined in (7).

Theorem 2.2. *Let $u \in C^1(\Omega)\bigcap C(\overline{\Omega})$, $\Omega \subset \mathbb{G}$ and $\partial\Omega \in C^\infty$. Define*

$$\mathcal{D}^0 u(x_0) := \int_{\partial\Omega} u(y)\langle \widetilde{\nabla}\varepsilon(y,x_0), d\nu(y)\rangle, \quad \mathcal{D}^+ u(x_0) := \lim_{x\to x_0,\, x\in\Omega} \int_{\partial\Omega} u(y)\langle \widetilde{\nabla}\varepsilon(y,x), d\nu(y)\rangle$$

and

$$\mathcal{D}^- u(x_0) := \lim_{x\to x_0,\, x\notin\overline{\Omega}} \int_{\partial\Omega} u(y)\langle \widetilde{\nabla}\varepsilon(y,x), d\nu(y)\rangle,$$

for $x_0 \in \partial\Omega$. Then $\mathcal{D}^+ u(x_0), \mathcal{D}^- u(x_0)$ and $\mathcal{D}^0 u(x_0)$ exist and verify the following jump relations:

$$\mathcal{D}^+ u(x_0) - \mathcal{D}^- u(x_0) = u(x_0), \quad \mathcal{D}^0 u(x_0) - \mathcal{D}^- u(x_0) = \mathcal{J}(x_0)u(x_0), \quad \mathcal{D}^+ u(x_0) - \mathcal{D}^0 u(x_0) = (1-\mathcal{J}(x_0))u(x_0),$$

where the jump value $\mathcal{J}(x_0)$ is given by the formula

$$\mathcal{J}(x_0) = \int_{\partial\Omega} \langle \widetilde{\nabla}\varepsilon(y,x_0), d\nu(y)\rangle, \quad x_0 \in \partial\Omega, \tag{8}$$

in the sense of the (Cauchy) principal value and $\widetilde{\nabla} = \widetilde{\nabla}_y$ is defined by $\widetilde{\nabla}\varepsilon = \sum_{k=1}^{N_1} (X_k\varepsilon) X_k$.

3. Kac's boundary value problem

For $0 < \alpha < 1$, Folland and Stein (see [8] and see also [6]) defined the anisotropic Hölder spaces $\Gamma_\alpha(\Omega)$, $\Omega \subset \mathbb{G}$, by

$$\Gamma_\alpha(\Omega) = \{f : \Omega \to \mathbb{C} : \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x)-f(y)|}{[d(x,y)]^\alpha} < \infty\}.$$

Let $\Omega \subset \mathbb{G}$ be a bounded domain with a smooth boundary $\partial\Omega \in C^\infty$. Consider the following analogy of the Newton potential

$$u(x) = \int_\Omega f(y)\varepsilon(y,x)d\nu(y), \quad x \in \Omega, \quad f \in \Gamma_\alpha(\Omega), \tag{9}$$

where $\varepsilon(y,x) = \varepsilon(x,y) = \varepsilon(y^{-1}x, 0) = \varepsilon(y^{-1}x)$ is the fundamental solution (2) of the sub-Laplacian $\mathcal{L}$. The function u is a solution of $\mathcal{L}u = f$ in Ω.

The aim of this section is to describe a version of Kac's boundary value problem on the homogeneous Carnot group $\mathbb{G}$, namely, to find a boundary condition for u such that with this boundary condition the equation $\mathcal{L}u = f$ has a unique solution in $C^2(\Omega)$, say, and this solution is the Newton potential (9). This amounts to finding the trace of the integral operator in (9) on $\partial\Omega$.

A starting point for this analysis is the observation that if $f \in \Gamma_\alpha(\Omega)$ for $\alpha > 0$ then u defined by (9) is twice differentiable and satisfies the equation $\mathcal{L}u = f$. We refer to Folland [6] for this property. These results extend those known for the Laplacian, in suitably redefined anisotropic Hölder spaces.

Theorem 3.1. *Let $\varepsilon(y,x) = \varepsilon(y^{-1}x)$ be the fundamental solution to $\mathcal{L}$, so that*

$$\mathcal{L}\varepsilon = \delta \quad \text{on } \mathbb{G}. \tag{10}$$

Let $\Omega \subset \mathbb{G}$ be a bounded domain with a smooth boundary $\partial\Omega \in C^\infty$. For any $f \in \Gamma_\alpha(\Omega)$, $0 < \alpha < 1$, the Newton potential (9) *is the unique solution in $C^2(\Omega)\cap C^1(\overline{\Omega})$ of the equation*

$$\mathcal{L}u = f \quad \text{in } \Omega, \tag{11}$$

with the boundary condition

$$(1-\mathcal{J}(x))u(x) + \int_{\partial\Omega} u(y)\langle \widetilde{\nabla}\varepsilon(y,x), d\nu(y)\rangle - \int_{\partial\Omega} \varepsilon(y,x)\langle \widetilde{\nabla}u(y), d\nu(y)\rangle = 0, \qquad \text{for} \quad x \in \partial\Omega, \tag{12}$$

where the jump value is given by the formula (8) *with $\widetilde{\nabla}g = \sum_{k=1}^{N_1}(X_k g)X_k$.*

It follows from Theorem 3.1 that the kernel $\varepsilon(y,x) = \varepsilon(y^{-1}x)$, which is a fundamental solution of the sub-Laplacian, is the Green function of the boundary value problem (11), (12) in Ω. Therefore, the boundary value problem (11), (12) can serve as an example of an explicitly solvable boundary value problem for the sub-Laplacian in any domain Ω (with smooth boundary) on the homogeneous Carnot group.

In the case of $\mathbb{G} = (\mathbb{R}^N, +)$ and the sub-Laplacian $\mathcal{L}$ being the Laplacian, the boundary condition (12) appeared in M. Kac's work [10] where he called it and the subsequent spectral analysis "the principle of not feeling the boundary". This was further expanded in Kac's book [11] with several further applications to the spectral theory and the asymptotics of the Weyl's eigenvalue counting function. In this direction, there are also systematic study of Kalmenov and the second author (see, e.g. [12] and [13]). The analogues of the problem (11), (12) for the Kohn Laplacian and its powers on the Heisenberg group have been recently investigated by the authors in [17], see also [3, 5] for the more general pseudo-differential analysis in the setting of the stratified and graded Lie groups, as well as [4] for the specific setting of the Heisenberg group.

References

[1] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni. Stratified Lie groups and potential theory for their sub-Laplacians. Springer- Verlag, Berlin, Heidelberg, 2007.

[2] D. Danielli, N. Garofalo and D. Nhieu. Sub-Riemannian calculus on hypersurfaces in Carnot groups. *Adv. Math.* 215:292–378, 2007.

[3] V. Fischer and M. Ruzhansky. A pseudo-differential calculus on graded nilpotent Lie groups, in Fourier Analysis, pp. 107–132, Trends in Mathematics, Birkhäuser, 2014.

[4] V. Fischer and M. Ruzhansky. A pseudo-differential calculus on the Heisenberg group. *C. R. Acad. Sci. Paris*, Ser I, 352, 197–204, 2014.

[5] V. Fischer and M. Ruzhansky. Quantization on nilpotent Lie groups, Progress in Mathematics, Vol. 314, Birkhäuser, 2016.

[6] G. B. Folland, Subelliptic estimates and function spaces on nilpotent Lie groups. *Ark. Mat.*, 13:161–207, 1975.

[7] G. B. Folland. On the Rothschild-Stein lifting theorem. *Comm. Partial Differential Equations*, 2(2):165–191, 1977.

[8] G. B. Folland and E. M. Stein. Estimates for the $\overline{\partial_b}$ complex and analysis on the Heisenberg group. *Comm. Pure Appl. Math.*, 27:429–522, 1974.

[9] D. S. Jerison. The Dirichlet problem for the Kohn Laplacian on the Heisenberg group. I and II. *J. Funct. Anal.*, 43:97–142, 224–257, 1981.

[10] M. Kac. On some connections between probability theory and differential and integral equations. In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950*, pages 189–215. University of California Press, Berkeley and Los Angeles, 1951.

[11] M. Kac. *Integration in function spaces and some of its applications.* Accademia Nazionale dei Lincei, Pisa, 1980. Lezioni Fermiane. [Fermi Lectures].

[12] T.Sh. Kalmenov, D. Suragan. To spectral problems for the volume potential. *Doklady Mathematics*, 80(2):646–649, 2009.

[13] T.Sh. Kalmenov, D. Suragan. Boundary conditions for the volume potential for the polyharmonic equation. *Differential Equations*, 48 (4):604–608, 2012.

[14] C. Romero. Potential theory for the Kohn Laplacian on the Heisenberg group. PhD thesis, University of Minnesota, 1991.

[15] L. P. Rothschild and E. M. Stein. Hypoelliptic differential operators and nilpotent groups. *Acta Math.*, 137:247–320, 1976.

[16] M. Ruzhansky and D. Suragan. Layer potentials, Kac's problem, and refined Hardy inequality on homogeneous Carnot groups. *Adv. Math.*, 308:483–528, 2017.

[17] M. Ruzhansky and D. Suragan. On Kac's principle of not feeling the boundary for the Kohn Laplacian on the Heisenberg group. *Proc. Amer. Math. Soc.*, 144(2):709–721, 2016.

[18] M. Ruzhansky and V. Turunen. Pseudo-differential operators and symmetries. Background analysis and advanced topics, volume 2 of Pseudo-Differential Operators. Theory and Applications. Birkhäuser Verlag, Basel, 2010.

Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, United Kingdom
E-mail address: m.ruzhansky@imperial.ac.uk

Institute of Mathematics and Mathematical Modelling, 125 Pushkin str., Almaty 050010, Kazakhstan, and Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, United Kingdom
E-mail address: d.suragan@imperial.ac.uk