Magnetic rings

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We study functional and spectral properties of perturbations of the operator $-(\partial_s+i\,a)^2$ in $\mathrm{L}^2(\mathbb{S}^1)$. This operator appears when considering the restriction to the unit circle of a two dimensional Schrödinger operator with the Bohm-Aharonov vector potential. We prove a Hardy-type inequality on $\mathbb{R}^2$ and, on $\mathbb{S}^1$, a sharp interpolation inequality and a sharp Keller-Lieb-Thirring inequality.

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I. INTRODUCTION

On the two-dimensional Euclidean space $\mathbb{R}^2$, let us introduce the polar coordinates $(r,\vartheta)\in[0,+\infty)\times\mathbb{S}^1$ of $\mathbf{x}\in\mathbb{R}^2$ and consider a magnetic potential $\mathbf{a}$ in a transversal gauge, or Poincaré gauge[?] , so that $(\mathbf{a},\mathbf{e}_r)=0$ and $(\mathbf{a},\mathbf{e}_\vartheta)=a_\vartheta(r,\vartheta)$, where $(\mathbf{e}_r,\mathbf{e}_\vartheta)$ is the oriented orthogonal basis associated with the polar coordinates such that, for any $\mathbf{x}\in\mathbb{R}^2\setminus\{\mathbf{0}\}$, $\mathbf{e}_r=\mathbf{x}/r$, $r=|\mathbf{x}|$. With this notation, the energy $\int_{\mathbb{R}^2}|(i\,\nabla+\mathbf{a})\,\Psi|^2\,d\mathbf{x}$ corresponding to the magnetic Schrödinger operator $-\Delta_{\mathbf{a}}$ can be rewritten as

$$\int_0^{+\infty}\int_{-\pi}^{\pi}\left(|\partial_r\Psi|^2+\frac{1}{r^2}\,|\,\partial_\vartheta\Psi+i\,r\,a_\vartheta\,\Psi|^2\right)r\,\mathrm{d}\vartheta\,\mathrm{d}r\,.$$

One of the main motivations is the study of *Bohm-Aharonov magnetic fields*[? ?] with $a_\vartheta(r,\vartheta)=a/r$ for some constant $a\in\mathbb{R}$. We recall that Stokes' formula applied to the magnetic field $b=\operatorname{curl}\mathbf{a}$ shows that the *magnetic flux* is given by

$$\int_{|\mathbf{x}|<r}b\,d\mathbf{x}=\frac{1}{2\pi}\int_{-\pi}^{\pi}a_\vartheta(r,\vartheta)\,r\,\mathrm{d}\vartheta=a\,.$$

The main result concerning Bohm-Aharonov magnetic fields is, for an arbitrary non-negative function φ in $L^q(\mathbb{S}^1)$, $q\in(1,+\infty)$, the Hardy-type inequality

$$\int_{\mathbb{R}^2}|(i\,\nabla+\mathbf{a})\,\Psi|^2\,d\mathbf{x}\ge\tau\int_{\mathbb{R}^2}\frac{\varphi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2}\,|\Psi|^2\,\mathrm{d}\mathbf{x}\qquad(1)$$

which holds for some constant τ depending on $\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}$. A precise statement will be given in Corollary II.3.

The proof relies on a method[?] developed recently and uses a *Keller-Lieb-Thirring inequality* for the first eigenvalue of a magnetic Schrödinger operator on a *magnetic ring* (see Corollary II.2). This spectral estimate is equivalent to *sharp interpolation inequalities* for a magnetic Laplacian on the circle and has been inspired by a series of previous papers[? ? ?] on interpolation inequalities and their spectral counterparts. Let us mention that some semiclassical properties of the spectrum of magnetic rings were recently studied including an electric potential that admits a double symmetric well[?] . Our results are not limited to the semi-classical regime.

II. MAIN RESULTS

On $(-\pi,\pi]\approx\mathbb{S}^1$, let us consider the uniform probability measure $\mathrm{d}\sigma=\mathrm{d}s/(2\pi)$ and denote by $\|\psi\|_{\mathrm{L}^p(\mathbb{S}^1)}$ the corresponding L^p norm, for any $p\ge1$. Assume that $a:\mathbb{R}\to\mathbb{R}$ is a 2π-periodic function such that its restriction to $(-\pi,\pi]\approx\mathbb{S}^1$ is in $\mathrm{L}^1(\mathbb{S}^1)$ and define the subspace

$$X_a:=\left\{\psi\in C_{\mathrm{per}}(\mathbb{R})\,:\,\psi'+i\,a\,\psi\in\mathrm{L}^2(\mathbb{S}^1)\right\}$$

of the space $C_{\mathrm{per}}(\mathbb{R})$ of the continuous 2π-periodic functions on $\mathbb{R}$. The change of function

$$\psi(s)\mapsto e^{i\int_{-\pi}^{s}(a(s)-\bar{a})\mathrm{d}\sigma}\,\psi(s)\,,$$

where $\bar{a}:=\int_{-\pi}^{\pi}a(s)\,\mathrm{d}\sigma$ is the *magnetic flux*, reduces the problem to the case of a constant: in the sequel of this paper we shall always assume that

a is a constant function.

Replacing ψ by $s\mapsto e^{iks}\,\psi(s)$ for any $k\in\mathbb{Z}$ shows that $\mu_{a,p}(\alpha)=\mu_{k+a,p}(\alpha)$ so that we can restrict the problem to $a\in[0,1]$. By considering $\chi(s)=e^{-is}\,\overline{\psi(s)}$, we find

$$|\psi'+i\,a\,\psi|^2=|\chi'+i\,(1-a)\,\chi|^2=\left|\overline{\psi}'-i\,a\,\overline{\psi}\right|^2,$$

and thus $\mu_{a,p}(\alpha)=\mu_{1-a,p}(\alpha)$: it is thus enough to consider the case $a\in[0,1/2]$.

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Using a Fourier series $\psi(s)=\sum_{k\in\mathbb{Z}}\psi_k\,e^{iks}$, we obtain that

$$\|\psi'+i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}=\sum_{k\in\mathbb{Z}}(a+k)^2\,|\psi_k|^2\ge a^2\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\,,$$

so that $\psi\mapsto\|\psi'+i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}+\alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$ is coercive for any $\alpha>-\,a^2$. Moreover, the optimal constant $\mu_{a,p}(\alpha)$ in the interpolation inequality

$$\|\psi'+i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}+\alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\ge\mu_{a,p}(\alpha)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}\quad(2)$$

written for any $\psi\in X_a$, is an increasing concave function of $\alpha>-\,a^2$ characterized by

$$\mu_{a,p}(\alpha):=\inf_{\psi\in X_a\setminus\{0\}}\frac{\int_{-\pi}^{\pi}\left(|\psi'+i\,a\,\psi|^2+\alpha\,|\psi|^2\right)\mathrm{d}\sigma}{\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}\quad(3)$$

and? $\lim_{\alpha\to-\,a^2}\mu_{a,p}(\alpha)=0$. We know that equality in (2) is realized if either $p=+\infty$,? or $p=2$,? . Our first result is the extension of this *interpolation* result to the case $p\in(2,+\infty)$.

Theorem II.1 *For any $p>2$, $a\in\mathbb{R}$, and $\alpha>-\,a^2$, the infimum in (3) is achieved and*

(i) if $a\in[0,1/2]$ and $a^2\,(p+2)+\alpha\,(p-2)\le1$, then $\mu_{a,p}(\alpha)=a^2+\alpha$ and equality in (2) is achieved only by the constant functions,

(ii) if $a\in[0,1/2]$ and $a^2\,(p+2)+\alpha\,(p-2)>1$, then $\mu_{a,p}(\alpha)<a^2+\alpha$ and equality in (2) is not achieved by the constant functions.

Moreover, for any $\alpha>-\,a^2$, $a\mapsto\mu_{a,p}(\alpha)$ is monotone increasing on $(0,1/2)$.

More can be said on $\mu_{a,p}(\alpha)$: see Theorem III.7. The region $a^2\,(p+2)+\alpha\,(p-2)\le1$ is exactly the set where the constant functions are linearly stable critical points. See Figs. 1 and 2.

With the results of Theorem II.1 in hand, we study some spectral properties of the magnetic Schrödinger operator $H_a-\varphi$ on the unit circle $\mathbb{S}^1\approx(-\pi,\pi]\ni s$ where φ is a potential and H_a is the magnetic Laplacian given by

$$H_a\psi(s)=-\left(\frac{d}{ds}+i\,a\right)^2\psi(s)\,.$$

The presence of a non-trivial magnetic field a in H_a "lifts" the spectrum up and the final result substantially depends on its value. Note that Lieb-Thirring inequalities with magnetic field? , in particular, imply an inequality for the first eigenvalue. However, it is not known if the constant is sharp. A somewhat similar result where the lifting of the spectrum is provided by a constant magnetic field was proved with different methods? .

The first spectral consequence of Theorem II.1 is a *Keller-Lieb-Thirring inequality* for the first eigenvalue $\lambda_1(H_a-\varphi)$ of the Schrödinger operator $H_a-\varphi$. The function $\alpha\mapsto\mu_{a,p}(\alpha)$ is monotone increasing, concave, and therefore has an inverse, denoted by $\alpha_{a,p}:\mathbb{R}^+\to\mathbb{R}^+$, which is monotone increasing, and convex.

Corollary II.2 *Let $p>2$, $a\in[0,1/2]$, $q=p/(p-2)$ and assume that φ is a non-negative function in $\mathrm{L}^q(\mathbb{S}^1)$. Then*

$$\lambda_1(H_a-\varphi)\ge-\,\alpha_{a,p}\left(\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\right).\quad(4)$$

If $4\,a^2+\mu\,(p-2)\le1$, then $\alpha_{a,p}(\mu)=\mu-a^2$; if $4\,a^2+\mu\,(p-2)>1$, then $\alpha_{a,p}(\mu)>\mu-a^2$.

These estimates are optimal in the sense that there exists a non-negative function φ such that $\lambda_1(H_a-\varphi)=-\,\alpha_{a,p}\left(\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\right)$. If $4\,a^2+\mu\,(p-2)\le1$, then the equality in (4) is achieved by constant potentials.

The second application of Theorem II.1 is related to a *Hardy inequality* in $\mathbb{R}^2$. Let us consider the *Bohm-Aharonov vector potential*

$$\mathbf{a}(\mathbf{x})=a\left(\frac{x_2}{|\mathbf{x}|^2},\frac{-x_1}{|\mathbf{x}|^2}\right),\quad\mathbf{x}=(x_1,x_2)\in\mathbb{R}^2\,,\quad a\in\mathbb{R}\,.$$

and recall the inequality?

$$\int_{\mathbb{R}^2}|(i\,\nabla+\mathbf{a})\,\Psi|^2\,\mathrm{d}\mathbf{x}\ge\min_{k\in\mathbb{Z}}(a-k)^2\int_{\mathbb{R}^2}\frac{|\Psi|}{|\mathbf{x}|^2}\,\mathrm{d}\mathbf{x}\,.\quad(5)$$

Using interpolation inequalities? , the following version? of Hardy's inequality in the case $d\ge3$ was proved:

$$\int_{\mathbb{R}^d}|\nabla\Psi|^2\,\mathrm{d}\mathbf{x}\ge\tau\int_{\mathbb{R}^d}\frac{\varphi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2}\,|\Psi|^2\,\mathrm{d}\mathbf{x}\,,$$

where the constant τ depends on the value of $\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^{d-1})}$. Using similar arguments we are now able to prove the following result.

Corollary II.3 *Let $p>2$, $a\in[0,1/2]$, $q=p/(p-2)$ and assume that φ is a non-negative function in $\mathrm{L}^q(\mathbb{S}^1)$. Then Inequality (1) holds with $\tau>0$ being the unique solution of the equation*

$$\alpha_{a,p}(\tau\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)})=0\,.$$

Moreover, $\tau=a^2/\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}$ if $4\,a^2+\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\,(p-2)\le1$.

Notice that for any $a\in(0,1/2)$, by taking φ constant, small enough in order that $4\,a^2+\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\,(p-2)\le1$, we recover the inequality

$$\int_{\mathbb{R}^2}|(i\,\nabla+\mathbf{a})\,\Psi|^2\,\mathrm{d}\mathbf{x}\ge a^2\int_{\mathbb{R}^2}\frac{|\Psi|^2}{|\mathbf{x}|^2}\,\mathrm{d}\mathbf{x}\,,$$

which is a equivalent to (5). The case $a=1/2$ is obtained by a limiting procedure and for arbitrary values of $a\in\mathbb{R}$, we refer to the observations of Section III.

III. PROOF OF THEOREM II.1 AND FURTHER RESULTS

Lemma III.1 *For all $a \in \mathbb{R}$, $p \in (2,\infty)$ and $\alpha \ge -a^2$, equality in (2) is achieved by at least one function in X_a.*

Indeed, by the diamagnetic inequality

$$\big|\,|\psi|'\big| \le |\psi' + i\,a\,\psi| \quad \text{a.e.}\,,$$

which holds for any $\psi \in X_a$, we infer that any minimizing sequence $\{\psi_n\}$ for (3) can be taken bounded in $\mathrm{H}^1(\mathbb{S}^1)$. By the compact Sobolev embeddings, this sequence is relatively compact in $\mathrm{L}^p(\mathbb{S}^1)$ and in $C(\mathbb{S}^1)$. The maps $\psi \mapsto \int_{-\pi}^{\pi} |\psi|^2\,d\sigma$ and $\psi \mapsto \int_{-\pi}^{\pi} |\psi' + i\,a\,\psi|^2\,d\sigma$ are lower semicontinuous by Fatou's lemma, which proves the claim. □

The minimization problem (3) has several reformulations.

1) Any solution $\psi \in X_a$ of the minimization problem (3) satisfies the Euler-Lagrange equation

$$(H_a + \alpha)\,\psi = |\psi|^{p-2}\,\psi$$

up to a multiplication by a constant. We observe that $v(s) = \psi(s)\,e^{ias}$ satisfies the condition

$$v(s+2\pi) = e^{2i\pi a}\,v(s) \quad \forall\, s \in \mathbb{R}\,, \tag{6}$$

and we can reformulate (3) as

$$\mu_{a,p}(\alpha) = \min_{v \in Y_a \setminus \{0\}} \mathsf{Q}_{p,\alpha}[v]$$

where $Y_a := \{v \in C(\mathbb{R}) \,:\, v' \in \mathrm{L}^2(\mathbb{S}^1)\,,\ (6) \text{ holds}\}$ and

$$\mathsf{Q}_{p,\alpha}[v] := \frac{\|v'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|v\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}{\|v\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}\,.$$

2) With $v = u\,e^{i\phi}$ written in polar form, the boundary condition becomes

$$u(\pi) = u(-\pi)\,, \quad \phi(\pi) = 2\pi\,(a+k) + \phi(-\pi) \tag{7}$$

for some $k \in \mathbb{Z}$, and $\|v'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} = \|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \|u\,\phi'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$. We can reformulate (3) as

$$\mu_{a,p}(\alpha) = \min_{(u,\phi)\in Z_a\setminus\{0\}} \frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \|u\,\phi'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}$$

where
$Z_a := \{(u,\phi) \in C(\mathbb{R})^2 \,:\, u',\, u\,\phi' \in \mathrm{L}^2(\mathbb{S}^1)\,,\ (7) \text{ holds}\}$.

3) The third reformulation of (3) relies on the Euler-Lagrange equations

$$-\,u'' + |\phi'|^2\,u + \alpha\,u = |u|^{p-2}\,u \quad \text{and} \quad (\phi'\,u^2)' = 0\,.$$

Integrating the second equation, and *assuming that u never vanishes*, we find a constant L such that $\phi' = L/u^2$. Taking (7) into account, we deduce from

$$L\int_{-\pi}^{\pi} \frac{ds}{u^2} = \int_{-\pi}^{\pi} \phi'\,ds = 2\pi\,(a+k)$$

that

$$\|u\,\phi'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} = L^2 \int_{-\pi}^{\pi} \frac{d\sigma}{u^2} = \frac{(a+k)^2}{\|u^{-1}\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}\,.$$

Hence

$$\phi(s) - \phi(0) = \frac{a+k}{\|u^{-1}\|^2_{\mathrm{L}^2(\mathbb{S}^1)}} \int_{-\pi}^{s} \frac{ds}{u^2}\,.$$

Let us define

$$\mathcal{Q}_{a,p,\alpha}[u] := \frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}\,.$$

In what follows, we denote by $\mathrm{H}^1(\mathbb{S}^1)$ the subspace of the continuous functions u on $(-\pi,\pi]$ such that $u(\pi) = u(-\pi)$ and $u' \in \mathrm{L}^2(\mathbb{S}^1)$. Notice that if $u \in \mathrm{H}^1(\mathbb{S}^1)$ is such that $u(s_0) = 0$ for some $s_0 \in (-\pi,\pi]$, then

$$|u(s)|^2 = \left(\int_{s_0}^{s} u'\,ds\right)^2 \le \sqrt{2\pi}\,\|u'\|_{\mathrm{L}^2(\mathbb{S}^1)}\,\sqrt{|s-s_0|}$$

and u^{-2} is not integrable. In this case we adopt the convention that $\mathcal{Q}_{a,p,\alpha}[u] = \mathsf{Q}_{p,\alpha}[u]$.

Lemma III.2 *For any $a \in (0,1/2)$, $p > 2$, $\alpha > -\,a^2$,*

$$\mu_{a,p}(\alpha) = \min_{u \in \mathrm{H}^1(\mathbb{S}^1)\setminus\{0\}} \mathcal{Q}_{a,p,\alpha}[u]$$

is achieved by a function $u > 0$.

To prove this result, it is enough to check that the infimum (3) is achieved by a function $\psi \in X_a$ such that $\psi(s) \neq 0$ for any $s \in (-\pi,\pi]$. Without loss of generality, we can assume that ψ is an optimal function for (2) with $\|\psi\|_{\mathrm{L}^p(\mathbb{S}^1)} = 1$. Let us decompose $v(s) = \psi(s)\,e^{ias}$ as a real and an imaginary part, $v = v_1 + i\,v_2$, which both solve the same Euler-Lagrange equation

$$-\,v_j'' + \alpha\,v_j = (v_1^2 + v_2^2)^{\frac{p}{2}-1}\,v_j\,, \quad j = 1\,,\ 2\,.$$

The Wronskian $w = (v_1\,v_2' - v_1'\,v_2)$ is constant.

Neither v_1 nor v_2 vanishes identically on $\mathbb{S}^1$ because of (6). If both v_1 and v_2 vanish at the same point, then w vanishes identically, which means that v_1 and v_2 are proportional. Again, this cannot be true because of the twisted boundary condition (6). □

If $a = 0$, $\mathcal{Q}_{a=0,p,\alpha}[u] = \mathsf{Q}_{p,\alpha}[u]$ for any $u \in \mathrm{H}^1(\mathbb{S}^1)\setminus\{0\}$.

Lemma III.3 *For any $p > 2$, if $0 < \alpha \le 1/(p-2)$, then $\mu_{0,p}(\alpha) = \alpha$ is achieved only by constant functions. Inequality (2) also holds with $p = -2$ and $\alpha = 1/(p-$*

2) $= -1/4 = \mu_{0,p}(-1/4)$, *with equality achieved only by constant functions.*

Both results (case $p>2$,? and case $p=-2$,?) were already known. As a consequence we have the inequalities

$$\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}+\beta\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\ge\beta\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^1)\quad(8)$$

for any $p>2$ and $\beta\in(0,1/(p-2)]$, and

$$\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}+\frac14\,\|u^{-1}\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\ge\frac14\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^1)\,.\quad(9)$$

Inequality (9) actually enters in the family of inequalities (8), with the parameter $\beta=-1/4=1/(p-2)$ corresponding to the critical exponent $p=2\,d/(d-2)=-2$ since here $d=1$. This exponent is critical from the point of view of scalings because, at least for a function u with compact support in $(-\pi,\pi)$, $\|u\|_{\mathrm{L}^p(\mathbb{S}^1)}$ scales like $\|u'\|_{\mathrm{L}^2(\mathbb{S}^1)}$. This is why a unified proof of both cases can be done with the Bakry-Emery method: see Appendix A.

We are now ready to study the key issues of Theorem II.1.

Lemma III.4 *Let $p>2$, $a\in[0,1/2]$, and $\alpha>-a^2$.*

(i) if $a^2\,(p+2)+\alpha\,(p-2)\le1$, then $\mu_{a,p}(\alpha)=a^2+\alpha$ and equality in (2) is achieved only by the constants,

(ii) if $a^2\,(p+2)+\alpha\,(p-2)>1$, then $\mu_{a,p}(\alpha)<a^2+\alpha$ and equality in (2) is not achieved by the constants.

In case (i), we can write

$$\begin{aligned}&\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}+a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)}+\alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\\&=(1-4\,a^2)\,\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}+\alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\\&\quad+4\,a^2\left(\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}+\frac14\,\|u^{-1}\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\right)\end{aligned}$$

and conclude using (9) and then (8) with

$$\beta=\frac{a^2+\alpha}{1-4\,a^2}\le\frac{1}{p-2}\,.$$

In case (ii), let us consider the test function $u_\varepsilon:=1+\varepsilon\,w_1$, where w_1 is the eigenfunction corresponding to the first non zero eigenvalue of $-d^2/ds^2$ on $\mathrm{H}^1(\mathbb{S}^1)$, with Neumann boundary conditions, namely, $\lambda_1=1$ and $w_1(s)=1+\cos s$. A Taylor expansion shows that

$$\mathcal{Q}_{a,p,\alpha}[u_\varepsilon]=a^2+\alpha+\big(1-a^2\,(p+2)-\alpha\,(p-2)\big)\,\varepsilon^2+o(\varepsilon^2)\,,$$

which proves the result. □

The proof of Lemma III.4, (i) relies on (8) and (9). It is remarkable that it does not use rigidity results based on the *carré du champ* method, at least directly.

It follows from the definition of $\mathcal{Q}_{a,p,\alpha}[u]$ that $a\mapsto\mu_{a,p}(\alpha)$ is nondecreasing on $[0,1/2)$. The strict monotonicity follows from the existence of an optimal function, which is known by Lemma III.1. This concludes the proof of Theorem II.1. The remainder of this section is devoted to complementary results, which specify the range of $\mu_{a,p}(\alpha)$ when a varies in $[0,1/2)$.

Let us consider

$$\nu_p(\alpha):=\inf_{v\in\mathrm{H}^1_0(\mathbb{S}^1)\setminus\{0\}}\mathsf{Q}_{p,\alpha}[v]\,.$$

Here $\mathrm{H}^1_0(\mathbb{S}^1)$ denotes the subspace of the functions $v\in\mathrm{H}^1(\mathbb{S}^1)$ such that $v(\pm\pi)=0$. Since (6) is satisfied by any function in $\mathrm{H}^1_0(\mathbb{S}^1)$, we have the following estimate.

Lemma III.5 *If $p>2$, $\alpha>-a^2$ and $a\in\mathbb{R}$, then*

$$\mu_{a,p}(\alpha)\le\nu_p(\alpha)\,.$$

Moreover, this inequality is strict if $a\in[0,1/2)$.

If $\{u_n\}_{n\in\mathbb{N}}$ is a minimizing sequence such that $\|u_n\|_{\mathrm{L}^p(\mathbb{S}^1)}=1$ for any $n\in\mathbb{N}$, then it is clearly bounded in $\mathrm{H}^1(\mathbb{S}^1)$, and so, by the compact Sobolev embeddings, it is relatively compact in $\mathrm{L}^2(\mathbb{S}^1)$, $\mathrm{L}^p(\mathbb{S}^1)$ and $C(\mathbb{S}^1)$. Up to subsequences, $\{u_n\}_{n\in\mathbb{N}}$ converges to some function u weekly in H^1 and strongly in $\mathrm{L}^2(\mathbb{S}^1)$, $\mathrm{L}^p(\mathbb{S}^1)$ and $C(\mathbb{S}^1)$. After noticing that $\mathsf{Q}_{p,\alpha}[\,|u|\,]=\mathsf{Q}_{p,\alpha}[u]$, we obtain the following result.

Lemma III.6 *If $p>2$, $\alpha>-a^2$, then $\nu_p(\alpha)$ admits a non-negative minimizer.*

The strict monotonicity of $a\mapsto\mu_{a,p}(\alpha)$ is a consequence of Lemma III.6 and, as a consequence, we know that

$$\mu_{a,p}(\alpha)<\mu_{1/2,p}(\alpha)\le\nu_p(\alpha)$$

for any $a\in[0,1/2)$. It turns out that the last inequality is an equality.

Theorem III.7 *For any $p>2$ and $\alpha>-a^2$, we have*

$$\mu_{1/2,p}(\alpha)=\nu_p(\alpha)\,.$$

This result was already known for the limit cases $p=2$,? and $p=+\infty$,? . To prove it, we set $v(s)=e^{is/2}\,\psi(s)$ and note that $v(s+2\pi)=-\,v(s)$ for all s, which follows from the periodicity condition (6) with $a=1/2$. Moreover, the derivative v' satisfies $v'(s+2\pi)=-\,v'(s)$. Note that these boundary conditions also hold for the real part and the imaginary part of v separately. We call them v_1 and v_2. Our problem is to minimize $\mathsf{Q}_{p,\alpha}[v]$ subject to these conditions. Both v_1 and v_2 must vanish at some point but *a priori* these points need not be the same. We set $\eta_j=|v_j|$, $j=1$, 2, and note that

$$\mathsf{Q}_{p,\alpha}[v]=\frac{\int_{-\pi}^{\pi}\left[\eta_1'^2+\eta_2'^2\right]\,\mathrm{d}\sigma+\alpha\int_{-\pi}^{\pi}\left[\eta_1^2+\eta_2^2\right]\,\mathrm{d}\sigma}{\|\eta\|_p^2}\,.$$

The functions η_j are now periodic. They are not necessarily smooth but are at least continuous. Now we replace both η_1 and η_2 by their symmetric decreasing rearrangements around the point 0. The numerator decreases for the usual reasons and the denominator increases (see Lemma B.1, in Appendix B). Thus, the symmetrically decreasing rearranged functions η_1^* and η_2^* have a maximum at 0 and vanish at $\pm\pi$, so that $\eta_1^* + i\,\eta_2^* \in \mathrm{H}^1_0(\mathbb{S}^1)$. If v is a minimizer of $\mathsf{Q}_{p,\alpha}$ under Condition (6) with $a = 1/2$, then

$$\nu_p(\alpha) \le \mathsf{Q}_{p,\alpha}[\eta_1^* + i\,\eta_2^*] \le \mathsf{Q}_{p,\alpha}[v] = \mu_{1/2,p}(\alpha)\,.$$

□

With the convention that $\mathcal{Q}_{a,p,\alpha}[u] = \mathsf{Q}_{p,\alpha}[u]$ if $u \in \mathrm{H}^1_0(\mathbb{S}^1)$, we can claim that the infimum of $\mathcal{Q}_{a,p,\alpha}$ is attained by some $u \in \mathrm{H}^1(\mathbb{S}^1) \setminus \{0\}$ for any $a \in [0,1/2]$, including in the case $a = 1/2$ for which the minimizer can be taken in $\mathrm{H}^1_0(\mathbb{S}^1) \setminus \{0\}$.

IV. PROOF OF COROLLARIES II.2 AND II.3

Let us start with the proof of Corollary II.2. Consider the quadratic form associated with $H_a - \varphi$. Using Hölder's inequality, we obtain

$$\|\psi' + i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} - \int_{-\pi}^{\pi} \varphi\,|\psi|^2\,\mathrm{d}\sigma \ge \|\psi' + i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} - \mu\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}$$

where $\mu = \|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}$ and $\frac{1}{q} + \frac{2}{p} = 1$. Let us choose α such that $\mu_{a,p}(\alpha) = \mu$. It follows from (2) that

$$\|\psi' + i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} - \mu\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)} \ge -\,\alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$$

and from Theorem II.1 that $\mu_{a,p}(\alpha) = a^2 + \alpha$ if $a^2\,(p+2) + \alpha\,(p-2) \le 1$. This implies that

$$\lambda_1(H_a - \varphi) \ge a^2 - \|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}$$

if $4\,a^2 + \|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\,(p-2) \le 1$. In that case the equality is achieved by $\varphi \equiv const$. The proof is complete. □

Now let us prove Corollary II.3. Let $\mathbf{x} = (r,\vartheta) \in \mathbb{R}^2$ be polar coordinates in $\mathbb{R}^2$. Then we find

$$\int_{\mathbb{R}^2} |(i\,\nabla + \mathbf{a})\,\Psi|^2\,\mathrm{d}\mathbf{x} = \int_0^\infty \int_{\mathbb{S}^1} \left(r\,|\partial_r \Psi|^2 + \frac{1}{r}\,|\partial_\vartheta \Psi + i\,a\,\Psi|^2 \right) \mathrm{d}\vartheta\,\mathrm{d}r\,.$$

Let $\tau > 0$. Then

$$\begin{aligned}\int_0^\infty \int_{\mathbb{S}^1} \frac{1}{r}\left(|\partial_\vartheta \Psi + i\,a\,\Psi|^2 - \tau\,\varphi\,|\Psi|^2 \right) \mathrm{d}\vartheta\,\mathrm{d}r &\ge \lambda_1\,(H_a - \tau\,\varphi) \int_0^\infty \int_{\mathbb{S}^1} \frac{1}{r}\,|\Psi|^2\,\mathrm{d}\vartheta\,\mathrm{d}r \\ &\ge -\,\alpha_{a,p}(\tau\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}) \int_0^\infty \int_{\mathbb{S}^1} \frac{1}{r}\,|\Psi|^2\,\mathrm{d}\vartheta\,.\end{aligned}$$

Note that if $\tau = 0$, then

$$\alpha_{a,p}(\tau\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}) = \alpha_{a,p}(0) = -\,a^2\,,$$

and for a sufficiently large τ the value of $\alpha_{a,p}(\tau\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)})$ is positive. Therefore we can find $\tau > 0$ such that $\alpha_{a,p}(\tau\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}) = 0$. This value is unique since $\alpha_{a,p}(\mu)$ is strictly monotone with respect to μ. The conclusion easily follows. □

Appendix A: A proof of Lemma III.3 by the carré du champ method

Let $\mathcal{F}_\beta[u] := \|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \beta\left(\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} - \|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}\right)$. If $p > 2$, it is enough to prove $\mu_{0,p}(\beta) = \beta$ for $\beta = \alpha_\star$, $\alpha_\star := 1/(p-2)$, because

$$\mathcal{F}_\beta[u] = \big(1 - \beta\,(p-2)\big)\,\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \beta\,(p-2)\,\mathcal{F}_{\alpha_\star}[u]$$

if $0 < \beta \le \alpha_\star$. Let us consider a positive solution of the parabolic equation

$$\frac{\partial u}{\partial t} = u'' + (p-1)\,\frac{|u'|^2}{u}$$

and compute

$$-\,\frac{d}{dt}\mathcal{F}_{\alpha_\star}[u(t,\cdot)] = \int_{-\pi}^{\pi} \left(|u''|^2 - |u'|^2\right) \mathrm{d}\sigma + \frac{p-1}{3} \int_{-\pi}^{\pi} \frac{|u'|^4}{u^2}\,\mathrm{d}\sigma$$

using several integrations by parts. The first term in the r.h.s. is non-negative by the Poincaré inequality, as well as the second one. Notice that $\rho = |u|^p$ is a solution of the heat equation, so that positivity is preserved by the flow and $\mathcal{F}_{\alpha_\star}[u(t=0,\cdot)] \ge \lim_{t\to+\infty} \mathcal{F}_{\alpha_\star}[u(t,\cdot)] = 0$, which is exactly (8) written with $u = u(t=0,\cdot)$. The strict positivity condition is easily removed by an approximation procedure. Exactly the same computations give the result in the case $p = -\,2$ and establish (9).

For $p > 2$, the method is well known[?,?]. The result for $p = -\,2$ was established earlier[?] but, as far as we know, this proof is new.

Appendix B: A symmetrization result

Here f^* denotes the symmetric decreasing rearrangement of f.

Lemma B.1 *Let $p \geq 2$. For any non-negative functions f, $g \in \mathrm{L}^p(\mathbb{S}^1)$ we have that*

$$\int_{-\pi}^{\pi} \left(f^2 + g^2\right)^{p/2} \mathrm{d}\sigma \leq \int_{-\pi}^{\pi} \left(f^{*2} + g^{*2}\right)^{p/2} \mathrm{d}\sigma\,.$$

The case $p = 2$ is obvious, in fact there is equality. Hence we assume that $p > 2$. Write

$$\left(\int_{-\pi}^{\pi} \left(f^2 + g^2\right)^{p/2} \mathrm{d}\sigma\right)^{2/p} = \sup_{\|v\|_{\mathrm{L}^q(\mathbb{S}^1)}=1} \int_{-\pi}^{\pi} \left(f^2 + g^2\right) v \, \mathrm{d}\sigma$$

where $1/q + 2/p = 1$. Clearly, we may choose v to be positive. By standard rearrangement inequalities,

$$\int_{-\pi}^{\pi} \left(f^2 + g^2\right) v \, \mathrm{d}\sigma \leq \int_{-\pi}^{\pi} \left(f^{*2} + g^{*2}\right) v^* \, \mathrm{d}\sigma$$

and $\|v^*\|_{\mathrm{L}^q(\mathbb{S}^1)} = \|v\|_{\mathrm{L}^q(\mathbb{S}^1)}$: the proof is completed with

$$\sup_{\|v^*\|_{\mathrm{L}^q(\mathbb{S}^1)}=1} \int_{-\pi}^{\pi} \left(f^{*2} + g^{*2}\right) v^* \, \mathrm{d}\sigma = \left(\int_{-\pi}^{\pi} \left(f^{*2} + g^{*2}\right)^{p/2} \mathrm{d}\sigma\right)^{2/p}.$$

Appendix C: Some numerical results

To compute the curve $\alpha \mapsto \mu_{a,p}(\alpha)$, we systematically solve the Euler-Lagrange equation associated with the variation of $\mathcal{Q}_{a,p,\alpha}$, *i.e.*,

$$-\,u'' + \frac{a^2}{u^3 \left(\int_{-\pi}^{\pi} \frac{1}{u^2} \, \mathrm{d}\sigma\right)^2} + \alpha\, u = u^{p-1} \tag{C1}$$

where the solution $u > 0$ is normalized by

$$\mu_{a,p}(\alpha) \left(\int_{-\pi}^{\pi} u^p \, \mathrm{d}\sigma\right)^{\frac{2}{p}-1} = 1\,.$$

This condition *a posteriori* provides the numerical value of $\mu_{a,p}(\alpha)$. To impose the boundary conditions $u'(0) = u'(\pi) = 0$, we use a shooting method and solve the ODE on $\mathbb{R}$ with the conditions $u'(0) = 0$ and $u(0) = \lambda > 0$. To emphasize the dependence in λ, let us denote it by u_λ. For any $\lambda > 0$, $\lambda \neq (a^2 + \alpha)^{1/(p-2)}$, the solution is non-constant and periodic so that

$$\rho(\lambda) = \min\{s > 0 \, : \, u_\lambda'(s) = 0\}$$

is well defined. The shooting parameter λ is then determined by the condition that $\rho(\lambda) = \pi$. Since (C1) involves a nonlocal term, an additional fixed-point procedure is needed to adjust the coefficient of u^{-3} in the equation. Some plots are shown in Figs. 1 and 2.

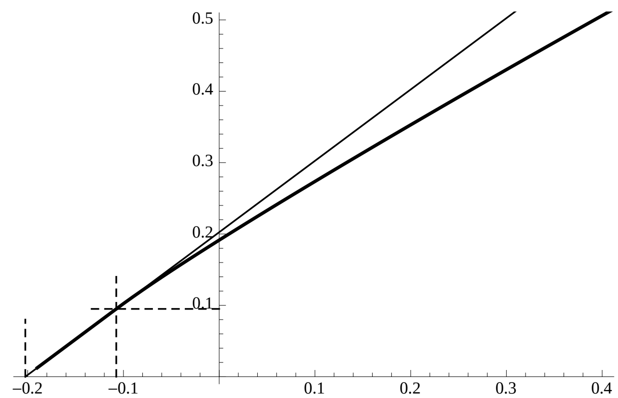

FIG. 1. The curve $\alpha \mapsto \mu_{a,p}(\alpha)$ with $p = 4$ and $a = 0.45$. The only solutions to (C1) are the constant functions for any α such that $-\,a^2 = -\,0.2025 \leq \alpha \leq -\,0.1075$ and, in this range, $\mu_{a,p}(\alpha) = a^2 + \alpha$. A branch of non-constant optimizers of (2) bifurcates at $\alpha = -\,0.1075$.

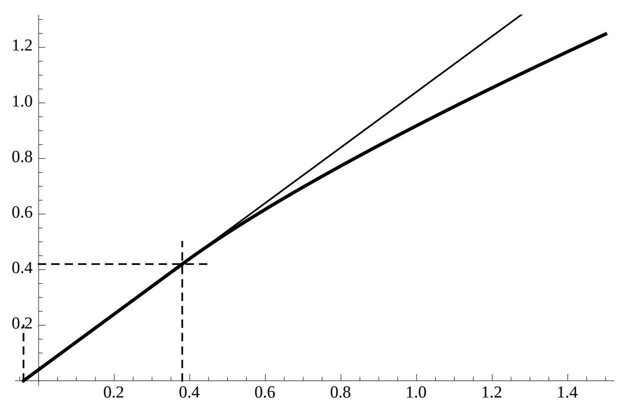

FIG. 2. The curve $\alpha \mapsto \mu_{a,p}(\alpha)$ with $p = 4$ and $a = 0.2$. Here the branch of non-constant optimizers of (2) bifurcates at $\alpha = 0.38$ which corresponds to $a^2\,(p+2) + \alpha\,(p-2) = 1$.

Equality in (2) is achieved only by constant functions according to Lemma III.4 if $a^2\,(p+2) + \alpha\,(p-2) \leq 1$: in this case, $\lambda = (a^2 + \alpha)^{1/(p-2)} \equiv u_\lambda$. For any $a \in (0, 1/2)$ such that $a^2\,(p+2) + \alpha\,(p-2) > 1$, our method provides us with a non-constant solution u of (C1) which realizes the equality in (2). As $a \to 1/2$, the integral $\int_{-\pi}^{\pi} u^{-2} \, \mathrm{d}\sigma$ diverges, so that the limit curve is described by the solution of

$$-\,u'' + \alpha\, u = u^{p-1} \tag{C2}$$

with boundary conditions $u'(0) = 0$ and $u(\pi) = 0$. See Fig. 3.

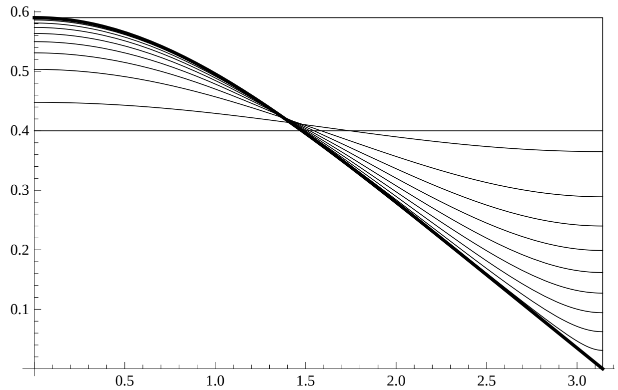

FIG. 3. Here $p = 4$ and $\alpha = 0$. Plot of the solution of (C1) for $a = 0.40, 0.41, \ldots 0.49$. The thick curve solves $u'' + u^{p-1} = 0$ and it is explicit. Similar patterns are found when $\alpha \neq 0$, with a non-explicit curve solving (C2) in the limit as $a \to 1/2$.

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REFERENCES