The linear stability of the Schwarzschild solution to gravitational perturbations in the generalised wave gauge

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I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

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Abstract

We prove in this thesis that the Schwarzschild family of black holes are linearly stable as solutions to the Einstein vacuum equations as expressed in a generalised wave gauge: all sufficiently regular solutions to the system of equations that result from linearising the Einstein vacuum equations, as expressed in a generalised wave gauge, about a fixed Schwarzschild solution remain uniformly bounded on the Schwarzschild exterior region and in fact decay to a member of the linearised Kerr family. The dispersion is at an inverse polynomial rate and therefore in principle sufficient for future nonlinear applications. The result thus fits into the wider goal of establishing the full nonlinear stability of the exterior Kerr family as solutions to the Einstein vacuum equations by employing a generalised wave gauge.

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List of figures

Contents

1
Introduction

The celebrated Kerr family [8] of spacetimes, discovered in 1963, comprise a 2-parameter family of solutions to the Einstein vacuum equations of general relativity,

$$\mathrm{Ric}[g] = 0. \tag{1.1}$$

They putatively describe isolated gravitating systems that contain a rotating *black hole* – a region of spacetime which cannot communicate with distant observers. The two parameters thus determine the mass and angular momentum of the black hole.

The physical reality of such objects, as opposed to being mere mathematical fiction, requires at the very least a positive resolution to the conjectured stability of their exterior[1] regions as solutions to the Einstein vacuum equations:

Conjecture. The sub-extremal[2] Kerr exterior family is stable as a family of solutions to (1.1).

A precise mathematical formulation of this conjecture can be found in [2] where it is posed in the context of general relativity as a *hyperbolic Cauchy initial value problem*, a correspondence first identified in the pioneering [11] of Choquet-Bruhat. Observe in particular that it is the exterior Kerr family itself which is posited to be stable, as opposed to a single member of this family, for it is expected that a general perturbation contributes both mass and angular momentum to the final state of the black hole.

[1]The region of spacetime that lies outside the black hole. For the interior region, the situation is drastically different – see the recent [10].

[2]Kerr solutions for which the absolute value of the angular momentum parameter is smaller than the mass parameter. Kerr solutions for which the two parameters are equal are known as extremal Kerr black holes and stability is not expected in this case – see [9].

This conjecture remains open. Indeed, to even attempt to give a positive resolution one must first address the issue of *gauge*. For in the theory of general relativity one can only distinguish between spacetimes up to an equivalence class that is determined by diffeomorphisms. In fact, the Einstein vacuum equations as formulated in (1.1) impose an underdetermined system on the spacetime metric g, with this degenerancy arising from the diffeomorphism invariance of the theory. It follows that any attempt to resolve the conjecture must necesarily *specify a gauge*.

In regards to questions pertaining to the issue of stability, particular success has been achieved via the specification of a *wave gauge* in which g and its first order derivatives are constrained in such a way as to reduce the equation (1.1) to a a quasilinear wave equation on g, thereby making the hyperbolicity of the Einstein equations manifest. Indeed, such a gauge was originally employed by Choquet-Bruhat in[11] to demonstrate *local* well-posedness of the Einstein equations. That this gauge could in fact be utilised to understand the *global* dynamics of solutions to the Einstein equations was exhibited by Lindblad and Rodnianski in their pioneering[3], in which they established the nonlinear stability of the Minkowski space as the trivial (Riemann-flat) solution to the equation that results from expressing (1.1) in a wave gauge. This gave an alternative proof of the fact that Minkowski space is stable as a solution to (1.1), a result which was originally established in the monumental work[4] of Christodoulou and Klainerman by imposing a so-called maximal-null gauge. The novelty of this approach due to Lindblad and Rodnianski however, is not only that the proof is dramatically simpler (although one has the caveat of obtaining less detailed asymptotics on the spacetimes constructed) but moreover that it succeeds despite the fact that the Einstein equations as expressed in a wave gauge do not verify the so-called[13] null condition[3]. We note that various authors have since extended this result of Lindblad and Rodnianski by either allowing for matter models in (1.1) (see[3],[17],[18],[16],[19]) or different asymptotics (see[65]). See also[20] and[40].

Motivated by this success, it is the intention of this thesis to lend credence to the notion that the specification of a *generalised wave gauge* ([64],[12]) will be sufficient to resolve the above conjecture in the affirmative. The advantage of imposing a generalised wave gauge is that it allows for cancellations in the lower order terms that arise in the equation that results from expressing (1.1) in such a gauge as compared to that which results from expressing (1.1) in a wave gauge. A more precise definition of the generalised wave gauge can be found in section 2.1.1 of the overview of this thesis.

Indeed, in this thesis we shall establish the *linear stability* of the Schwarzschild exterior subfamily of the Kerr exterior family with vanishing angular momentum parameter as solutions to (1.1) under the imposition of a judicious generalised wave gauge:

[3]This is a sufficient ([14] and[15]) condition on the non-linearities present in a system of quasilinear wave equations under which one has global existence of solutions for small data. It was long believed until the work of Lindblad and Rodnianski that this was in fact also necessary.

Theorem. *All sufficiently regular solutions to the equations of linearised gravity around Schwarzschild i.e. the equations that result from linearising the Einstein vacuum equations* (1.1)*, as expressed in a (particular and explicit) generalised wave gauge, about a fixed member of the Schwarzschild exterior family*

i) *are uniformly bounded and asymptotically flat on the Schwarzschild exterior*

ii) *decay to a member of the linearised Kerr family.*

Note that the dynamic convergence to a member of the linearised Kerr family is to be understood within the wider context of the stability of the Kerr exterior family. In terms of resolving the conjectured stability of the Kerr family, the generalised wave gauge thus passes the first test put to it by the (less elaborate) Schwarzschild exterior subfamily.

A more comprehensive version of the above theorem is to be found in section 2.4 of the overview. However, already at this stage it is proper to discuss the issue surrounding *residual gauge freedom*. This freedom arises from the fact that imposing the generalised wave gauge on a spacetime does not fully specify the gauge. This is directly manifested in the linear theory by the existence of a *residual* class of infinitesimal diffeomorphisms on the Schwarzschild exterior spacetime which preserve the generalised wave gauge, thus generating an explicit class of solutions to the equations of linearised gravity known as *pure gauge solutions*. The existence of such solutions, along with the presence of the linearised Kerr family, implies that one can only prove a decay statement for solutions to the equations of linearised gravity *up to* the addition of some pure gauge solution and some member of the linearised Kerr family. Indeed, even part *i*) of our theorem requires a quantitative gauge-normalisation of initial data. However, this 'initial-data-normalisation' is in fact sufficient to obtain part *ii*) of our theorem. In addition, we emphasise that part *i*) should be considered as a boundedness statement at the level of certain natural energy fluxes that does not lose derivatives.

The linear stability of the Schwarzschild exterior family was originally proven by Dafermos, Holzegel and Rodnianski in [1] via the imposition of a *double null gauge*. More specifically, their analysis focused on the null decomposed linearised Bianchi equations for the Weyl curvature, coupled to the linearised null structure equations. This approach is therefore in keeping with that of Christodoulou and Klainerman in [4]. In particular, the body of work presented here complements that of Dafermos, Holzegel and Rodnianski in a similar vein as to how the result [3] of Lindblad and Rodnianski complemented that of Christodoulou and Klainerman. However, we highlight the fact that in [1] decay is obtained *only after the addition of a dynamically determined residual pure gauge solution*. We also note our recent [5] where a version of the above theorem was obtained under the imposition of a different generalised wave gauge than that considered in this thesis and for which the asymptotic flatness criterion of part *i*) was absent.

The proof of the theorem stated above relies crucially on the fact that one can extract two fully decoupled scalar wave equations from the equations of linearised gravity. That this is possible is well-known in the literature and corresponds to the remarkable discovery by Regge–Wheeler [23] and Zerilli [24] that certain *gauge-invariant* quantities decouple from the full system of linearised Einstein equations into the celebrated Regge–Wheeler and Zerilli equations. Indeed, by combining this decoupling with a sagacious choice of generalised wave gauge one can in fact 'gauge-normalise' initial data in such a way as to cause all linearised metric quantities in this gauge to be fully determined by those two quantities that satisfy the Regge–Wheeler and Zerilli equations respectively. This has the effect of essentially reducing the above theorem to a boundedness and decay statement for solutions to said equations. Key to this of course is identifying the correct generalised wave gauge and this identification is inspired by certain classical insights ([23]) into the linearised Einstein equations about Schwarzschild. We emphasize that making this identification, and then developing the corresponding well-posedness theory for the linearised Einstein equations around Schwarzschild in such a gauge, perhaps comprises the key content of this thesis in regards to potential future nonlinear applications.

A decay statement for solutions to the Regge–Wheeler equation was established by Holzegel in [27], with earlier results of [29] due to Blue and Soffer, whereas a decay statement for solutions to the Zerilli was obtained independently by the author [6] and Hung–Keller–Wang in [30]. We also note [32]. We shall in fact reprove these results in this thesis for reasons of completeness. We note that in doing so we rely heavily upon the fundamental techniques developed by Dafermos–Rodnianski in [34] and [35] by which one establishes a quantitative rate of dispersion for solutions to the scalar wave equation on the Schwarzschild exterior.

We now discuss other results that are related to the work contained within this thesis. We first note that the recent remarkable result [39] of Hintz and Vasy whereby the global nonlinear stability of the Kerr–De Sitter family[4] ([8],[41]) of black holes was established, for small rotation parameter, proceeded by employing a generalised wave gauge. This was later extended by Hintz in [42] to the Kerr–Newman–De Sitter family ([8],[41],[43]). A particular feature of this problem however is the presence of a positive cosmological constant which has the effect of making it fundamentally different in nature than one which concerns the nonlinear stability of the Kerr family[5].

Moreover, for a result towards establishing the linear stability of the exterior Schwarzschild family which employs a so-called Chandrasekhar gauge, see [30]. Conversely, for a partial result regarding the linear stability of the exterior Schwarzschild family in a wave gauge, see [31] (see also our upcoming [7]). Finally, for further references pertaining to the linearised Einstein equations about the Schwarzschild exterior family, see [28], [26] and [44]–[58].

[4] These are a family of solutions to $\mathrm{Ric}[g] = \Lambda g$ with $\Lambda > 0$.

[5] In particular, one expects the rate of dispersion to be exponentially fast.

We end the introduction with a brief discussion towards nonlinear applications and future work. Indeed, in view of the fact that one must in effect linearise about the solution one expects to approach, providing a positive resolution to the conjectured stability of the Kerr exterior family by utilising a generalised wave gauge, even for the Schwarzschild exterior subfamily, would require upgrading the linear theory established here to the full subextremal Kerr exterior family. Nevertheless, in [1] Dafermos, Holzegel and Rodnianski formulated a restricted nonlinear stability conjecture regarding the Schwarzschild exterior family for which the improved rate of dispersion embodied in part ii) of our Theorem is in principle sufficient, when coupled with the understanding gained by Lindblad–Rodnianski in [3], to treat the nonlinearities present in the system of equations that results from expressing (1.1) in a generalised wave gauge, thus paving the way for a resolution of this conjecture by means of a generalised wave gauge. A precise formulation of the conjecture can be found in section B of the Appendix. Remarkably, a proof of said conjecture in the symmetry class of axially symmetric and polarised perturations has very recently been announced by Klainerman and Szeftel over a series of three papers, the first of which can be found in [59].

2

The linear stability of the Schwarzschild solution to gravitational perturbations in the generalised wave gauge: An overview

We shall now give a complete overview of the thesis. The overview (and indeed the thesis) is to be divided into five parts.

In the first part, section 2.1, we describe the process behind which one arrives at the equations of linearised gravity around Schwarzschild. In addition, two special classes of solutions to the equations of linearised gravity are discussed, namely the pure gauge and linearised Kerr solutions.

In the second part, section 2.2, we discuss how, motivated by the existence of the special solutions discussed in section 2.1, one extracts the two scalar wave equations described by the Regge–Wheeler and Zerilli equations from the equations of linearised gravity.

In the third part, section 2.3, we discuss both the Cauchy problem for the equations of linearised gravity and solutions to the equations of linearised gravity the Cauchy data of which has been 'gauge-normalised'.

In the fourth part, section 2.4, we give rough statements of the main two theorems of this thesis, the first of which concerns a decay statement for solutions to the Regge–Wheeler and Zerilli equations and the second of which concerns a decay statement for the ‘gauge-normalised’ solutions to the equations of linearised gravity discussed in second 2.3. We note this latter statement comprises the quantitative statement of linear stability of the Schwarzschild exterior family in the generalised wave gauge discussed in the introduction.

In the fifth part, section 2.5, we make an aside to discuss the problem of the scalar wave equation on the Schwarzschild exterior spacetime with the insights gained from this problem motivating the proofs of the theorems of section 2.4.

Finally in the sixth part, section 2.5.5, we give an outline of the proofs of said theorems.

2.1 The equations of linearised gravity around Schwarzschild

We commence the overview with a discussion regarding the derivation of the equations of interest in this thesis, namely the equations of linearised gravity around Schwarzschild. Special classes of solutions to the latter are also discussed.

This section of the overview corresponds to Chapter 3 in the main body of the thesis.

2.1.1 The Einstein equations in a generalised wave gauge

We begin by describing the notion of a generalised wave gauge for an abstract Lorentzian manifold, giving then a description of the vacuum Einstein equations when expressed in this gauge.

This section of the overview corresponds to section 3.1 in the main body of the thesis.

Let $\left(\boldsymbol{\mathcal{M}},\boldsymbol{g}\right)$ and $\left(\boldsymbol{\mathcal{M}},\overline{\boldsymbol{g}}\right)$ be $3+1$ globally hyperbolic Lorentzian manifolds with[1] $\boldsymbol{f} : T^2(\boldsymbol{\mathcal{M}}) \to T^1(\boldsymbol{\mathcal{M}})$ a map.

Then following[12], $\boldsymbol{g}$ is said to be in a generalised $\boldsymbol{f}$-wave gauge with respect to $\overline{\boldsymbol{g}}$ iff the identity map

$$\mathrm{Id} : \left(\boldsymbol{\mathcal{M}},\boldsymbol{g}\right) \to \left(\boldsymbol{\mathcal{M}},\overline{\boldsymbol{g}}\right)$$

is an $\boldsymbol{f}(\boldsymbol{g})$-wave map. Denoting by $\boldsymbol{C}_{g,\overline{g}}$ the connection tensor[2] between $\boldsymbol{g}$ and $\overline{\boldsymbol{g}}$, this amounts to

$$\boldsymbol{g}^{-1} \cdot \boldsymbol{C}_{g,\overline{g}} = \boldsymbol{f}(\boldsymbol{g})^{\sharp} \tag{2.1}$$

[1] Here we recall the notation $T^k(\mathcal{M})$ for the space of k-covariant tensor fields on $\mathcal{M}$.

[2] $\left(\boldsymbol{C}_{g,\overline{g}}\right)^{\boldsymbol{\alpha}}_{\boldsymbol{\beta\gamma}} := \frac{1}{2}\left(\boldsymbol{g}^{-1}\right)^{\boldsymbol{\alpha\delta}}\left(2\overline{\boldsymbol{\nabla}}_{(\boldsymbol{\beta}}\boldsymbol{g}_{\boldsymbol{\gamma})\boldsymbol{\delta}} - \overline{\boldsymbol{\nabla}}_{\boldsymbol{\delta}}\boldsymbol{g}_{\boldsymbol{\beta\gamma}}\right)$.

where $\overline{\sharp}$ is the (raising) musical isomorphism associated to $\overline{\boldsymbol{g}}$.

We note that if $\boldsymbol{f}(\boldsymbol{g})$ is sufficiently regular[3] then the condition (2.1) is equivalent to solving a system of semilinear wave equations on $\boldsymbol{\mathcal{M}}$ and thus under sufficient regularity one can always find an open set $\boldsymbol{U} \subset \boldsymbol{\mathcal{M}}$ such that

$$\mathrm{Id} : \left(\boldsymbol{U}, \boldsymbol{g}\big|_{\boldsymbol{U}}\right) \to \left(\boldsymbol{U}, \overline{\boldsymbol{g}}\big|_{\boldsymbol{U}}\right)$$

is an $\boldsymbol{f}$-wave map. The generalised wave gauge is thus *locally well-posed* if the map $\boldsymbol{f}$ is defined appropriately. Moreover, if one sets

$$\boldsymbol{f} = 0, \qquad \boldsymbol{\mathcal{M}} = \mathbb{R}^4 \qquad \text{and} \qquad \overline{\boldsymbol{g}} = \eta,$$

with η the Minkowski metric, then choosing a globally inertial system of coordinates on $\mathbb{R}^4$ one recovers the wave gauge employed so successfully by Lindblad and Rodnianski in[3].

If one assumes that $\boldsymbol{g}$ is a generalised $\boldsymbol{f}$-wave gauge with respect to $\overline{\boldsymbol{g}}$ then the Einstein vacuum equations for the metric $\boldsymbol{g}$,

$$\mathbf{Ric}[\boldsymbol{g}] = 0,$$

reduce to a *quasilinear tensorial wave equation* on $\boldsymbol{g}$. A schematic description[4] is as follows:

$$\left(\boldsymbol{g}^{-1} \cdot \overline{\boldsymbol{\nabla}}^2\right)\boldsymbol{g} + \boldsymbol{C}_{\boldsymbol{g},\overline{\boldsymbol{g}}} \cdot \boldsymbol{C}_{\boldsymbol{g},\overline{\boldsymbol{g}}} + \overline{\mathbf{Riem}} \cdot \boldsymbol{g} = \boldsymbol{\mathcal{L}}_{\boldsymbol{f}(\boldsymbol{g})^{\overline{\sharp}}}\boldsymbol{g}, \tag{2.2}$$

$$\boldsymbol{g}^{-1} \cdot \boldsymbol{C}_{\boldsymbol{g},\overline{\boldsymbol{g}}} = \boldsymbol{f}(\boldsymbol{g}). \tag{2.3}$$

Here, $\overline{\mathbf{Riem}}$ and $\overline{\boldsymbol{\nabla}}$ are the Riemann tensor and Levi-Civita connection of $\overline{\boldsymbol{g}}$ respectively.

In particular, if the expression $\boldsymbol{f}(\boldsymbol{g})$ is sufficiently regular[5], then under sufficient regularity the system of equations given by (2.2) coupled with (2.3), which correspond to the *Einstein vacuum equations as expressed in a generalised $\boldsymbol{f}$-wave gauge with respect to* $\overline{\boldsymbol{g}}$, are always locally well-posed as a hyperbolic initial value problem with constrained[6] initial data. The generalised wave gauge thus captures the essential hyperbolicity of the Einstein equations. See the book[12] of Choquet-Bruhat for details.

In the introduction, we discussed the conjecture relating to the stability of the Kerr exterior family and how one might seek to resolve it in the affirmative by utilising a

[3]For instance, if $\boldsymbol{f}(\boldsymbol{g})$ is at the same level of regularity as $\boldsymbol{g}$.
[4]The wave-like nature of this system is a consequence of the Lorentzian character of $\boldsymbol{g}$.
[5]For instance, if $\boldsymbol{f}(\boldsymbol{g})$ is at the same level of regularity as $\boldsymbol{g}$.
[6]The constraints arise as a consequence of the Gauss–Codazzi equations.

generalised wave gauge. One can now make this aim slightly more precise with the statement that one wishes to establish an appropriate notion of stability for solutions to (2.2) and (2.3) **for which $\left(\boldsymbol{\mathcal{M}}, \overline{\boldsymbol{g}}\right)$ is set to be any fixed member of the subextremal exterior Kerr family.** This strategy was employed successfully by Hint–Vasy and Hintz in[39] and[42] respectively for the case of the Kerr–De Sitter and Kerr–Newman–De Sitter family of black holes with small angular momentum parameter (see also[40]).

In this thesis, we shall concern ourselves with the instance for which this member resides within the Schwarzschild exterior subfamily. Of course, that still leaves the freedom in specifying the map $\boldsymbol{f}$!

For a non-schematic description of the Einstein vacuum equations as expressed in a generalised wave gauge, see section 3.1.2.

2.1.2 The exterior Schwarzschild spacetime

The Schwarzschild family $\left(\mathcal{S}, g_M\right)$, with $M \in \mathbb{R}^+$, constitute the unique family of spherically symmetric solutions to the Einstein vacuum equations. A particularly relevant local description of this family is motivated by the notion that the Einstein equations, as expressed in a generalised wave gauge, are most naturally formulated in terms of a *Cauchy* problem[7]. This suggests describing g_M in a system of coordinates on $\mathcal{S}$ which adequately captures the fact that one can foliate $\left(\mathcal{S}, g_M\right)$ by Cauchy hypersurfaces. A natural candidate which also describes the event horizon in a regular fashion are the so-called Schwarzschild-star coordinates.

In this coordinate system the metric takes the form

$$g_M = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t^{*2} + \frac{4M}{r} \mathrm{d}t^* \,\mathrm{d}r + \left(1 + \frac{2M}{r}\right) \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \,\mathrm{d}\varphi^2\right) \tag{2.4}$$

where the coordinates take values

$$(t^*, r) \in (-\infty, \infty) \times (0, \infty), \qquad (\theta, \varphi) \in S^2.$$

Restricting now the coordinate r to the range $r \in [2M, \infty)$, with the null hypersurface

$$\mathcal{H}^+ := \left\{(t^*, 2M, \theta, \varphi) | (t^*, \theta, \varphi) \in \mathbb{R} \times S^2\right\}$$

[7] In particular, note that the generalised wave gauge condition (2.1) is a condition on the first order derivatives of the metric.

describing the so-called future event horizon, one thus has the Schwarzschild-star coordinate system on the Schwarzschild exterior spacetime viewed now as a submanifold with boundary $\left(\mathcal{M}, g_M\right)$ of $\left(\mathcal{S}, g_M\right)$. Since moreover t^* is now a globally regular function on $\mathcal{M}$ whose gradient is everywhere time-like, it follows that the hypersurfaces of constant t^*, which we denote by Σ_{t^*}, describe a foliation of $\left(\mathcal{M}, g_M\right)$ by Cauchy hypersurfaces. In addition, observe that the causal vector field

$$T = \partial_{t^*}$$

is manifestly Killing. The Schwarzschild exterior spacetime is thus *static*. Moreover, this same vector field determines a global time orientation on $\left(\mathcal{M}, g_M\right)$.

A Penrose diagram of the spacetime $\left(\mathcal{M}, g_M\right)$ is given in Figure 1.

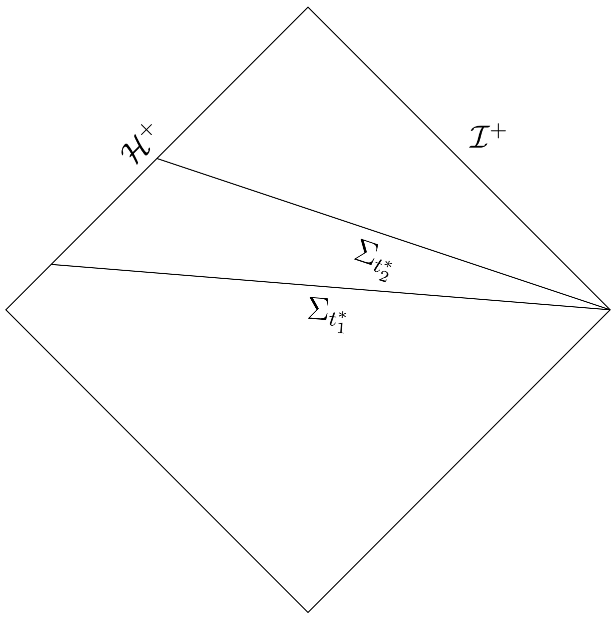

Figure 2.1: A Penrose diagram of $\left(\mathcal{M}, g_M\right)$ depicting the Cauchy hypersurfaces $\Sigma_{t_1^*}$ and $\Sigma_{t_2^*}$.

In the main body of the thesis, we will actually use the above Schwarzschild-star coordinates to *define* the Schwarzschild exterior spacetime, modulo the standard degeneration of the coordinates on S^2, without reference to the ambient spacetime $\left(\mathcal{S}, g_M\right)$. See section 3.2.1 for details.

Moreover, the limit along future-directed outgoing null cones as constructed in the spacetime $\left(\mathcal{M}, g_M\right)$ will informally be referred to as *future null infinity*, depicted in Figure 1 as $\mathcal{I}^+$. See section 3.2.4.1 for details.

2.1.3 The equations of linearised gravity

The equations of interest in this thesis are those that result from linearising the Einstein vacuum equations, as expressed in a generalised f-wave gauge defined with respect to a fixed Schwarzschild exterior solution and a fixed map f, about the very same Schwarzschild exterior solution. To that end, we now describe the linearisation process and present, following[1], the so-called equations of linearised gravity.

This section of the overview corresponds to section 3.3 of the main body of the thesis.

One first identifies the abstract Lorentzian manifold $\left(\boldsymbol{\mathcal{M}}, \overline{\boldsymbol{g}}\right)$ of section 2.1.1 with that of a *fixed* member of the Schwarzschild exterior family $\left(\mathcal{M}, g_M\right)$. Next we define the linear map[8] $f : \mathscr{T}^2_{\text{sym}}(\mathcal{M}) \to \mathscr{T}^1(\mathcal{M})$ defined as in section 3.3.1 in the bulk of the thesis, noting that $f(g_M) = 0$[9].

The schematic description of the Einstein vaccum equations as expressed in the generalised f-wave gauge with respect to g_M, the system (2.2) and (2.3), then translates to to the following:

$$\left(\boldsymbol{g}^{-1} \cdot \nabla^2_M\right)\boldsymbol{g} + \boldsymbol{C}_{\boldsymbol{g},g_M} \cdot \boldsymbol{C}_{\boldsymbol{g},g_M} + \text{Riem}_M \cdot \boldsymbol{g} = \mathcal{L}_{f(\boldsymbol{g})^\sharp}\boldsymbol{g}, \tag{2.5}$$

$$\boldsymbol{g}^{-1} \cdot \boldsymbol{C}_{\boldsymbol{g},g_M} = f(\boldsymbol{g})^\sharp. \tag{2.6}$$

In particular, ∇_M is now the Levi-Civita connection of g_M with $\boldsymbol{g}$ assumed to be a Lorentzian metric.

In order to formally linearise, we consider a smooth 1-parameter family of solutions $\boldsymbol{g}(\epsilon)$ to the Einstein vacuum equations, defined on $\mathcal{M}$, with $\boldsymbol{g}(0) = g_M$. We will also demand that each solution $\boldsymbol{g}(\epsilon)$ is in a generalised f-wave gauge with respect to g_M.

Observing thus that since $f(g_M) = 0$ then g_M is indeed a solution to the Einstein vacuum equations as expressed in a generalised f-wave gauge with respect to g_M, to linearise we consider a formal power series expansion of $\boldsymbol{g}(\epsilon)$ in terms of ϵ:

$$\boldsymbol{g}(\epsilon) = g_M + \epsilon \cdot \overset{\scriptscriptstyle(1)}{g} + o\left(\epsilon^2\right).$$

Here, $\overset{\scriptscriptstyle(1)}{g}$ is a symmetric 2-covariant tensor field on $\mathcal{M}$ denoting the linearised metric. Thus, in keeping with[1], linearised quantities are denoted by a superscript (1). We also

[8] Here we denote by $\mathscr{T}^k(\mathcal{M})$ the space of k-covariant tensors on $\mathcal{M}$ with $\mathscr{T}^2_{\text{sym}}(\mathcal{M})$ the space of smooth, symmetric 2-covariant tensors. Note that restricting the domain of the map f (when compared to section 2.1.1) will be natural in its application to the equations of linearised gravity in the sequel.

[9] It would not be appropriate (and indeed unnecessary) to define the map f precisely at this point in the overview.

write

$$f(\boldsymbol{g}(\epsilon)) = \epsilon \cdot Df\Big|_{g_M}(\overset{\scriptscriptstyle(1)}{g}) + o\Big(\epsilon^2\Big)$$

where $Df\Big|_{g_M} : \mathscr{T}^2_{\text{sym}}(\mathcal{M}) \to \mathscr{T}^1(\mathcal{M})$ is a smooth linear map denoting the linearisation of the map $f(\cdot)$ at g_M. In particular, as the map f defined in section 3.3.1 is already linear, we have $Df\Big|_{g_M} = f$.

One then arrives at the linearised equations by inserting this formal power series expansion into the system of equations defined by Einstein vacuum equations, as expressed in the generalised f-wave gauge with respect to g_M, and discarding those terms that appear to higher than linear order in ϵ. Proceeding in this manner leads to the following[10] system of equations:

$$\Box\overset{\scriptscriptstyle(1)}{g} - 2\text{Riem} \cdot \overset{\scriptscriptstyle(1)}{g} = \mathcal{L}_{\overset{\scriptscriptstyle(1)}{f}} g_M, \tag{2.7}$$

$$\text{div}\overset{\scriptscriptstyle(1)}{g} - \frac{1}{2}\,\text{dtr}\overset{\scriptscriptstyle(1)}{g} = \overset{\scriptscriptstyle(1)}{f}. \tag{2.8}$$

Here, div and $\Box$ are the spacetime divergence and Laplacian of g_M respectively, with d the exterior derivative on $\mathcal{M}$. Moreover, we have defined the 1-form $\overset{\scriptscriptstyle(1)}{f}$ according to

$$\overset{\scriptscriptstyle(1)}{f} := f(\overset{\scriptscriptstyle(1)}{g}).$$

The linearisation of the Einstein vacuum equations, as expressed in a generalised wave gauge, around Schwarzschild thus comprise of the *tensorial system of linear wave equations* (2.7) *coupled with the divergence relation* (2.8) on the Schwarzschild exterior spacetime. We remark that these are nothing but the linearised Einstein equations in a *generalised* Lorentz gauge. See the book of Wald[22].

Now, although the system (2.7)-(2.8) appears fairly innocuous, the fact that it corresponds to a tensorial system defined with respect to the Lorentzian metric g_M means the 'norm' one would most readily use to measure the solution $\overset{\scriptscriptstyle(1)}{g}$ (2.7)-(2.8) is non positive-definite. The tensorial structure of (2.7)-(2.8) thus renders them a highly complicated system of equations.

However, the structure of the system of equations (2.7)-(2.8) is made more transparent by employing the so called $2+2$ formalism. In utilising this $2+2$ formalism, a formalism which was first adopted in[49], one exploits the fact that the topology of $\mathcal{M}$ has the product structure $\mathcal{Q} \times S^2$ where $\mathcal{Q}$ is a 2-dimensional manifold with boundary. Informally, this allows a decomposition of objects on $\mathcal{M}$ into their parts 'tangent' to $\mathcal{Q}$ and S^2. In particular, one can decompose tensor fields on $\mathcal{M}$ into what we term as $\mathcal{Q}$-tensor fields

[10] Note we have dropped the subscript M notation.

and S-tensor fields respectively (see section 3.2.2.1 in the bulk of the paper for a precise definition).

Indeed, applying this decomposition to the linearised metric $\overset{(1)}{g}$ yields

$$\overset{(1)}{g} \to \overset{(1)}{\tilde{g}}, \overset{(1)}{\rlap{-}g}, \overset{(1)}{\not{g}}$$

where $\overset{(1)}{\tilde{g}}$ is a symmetric 2-covariant $\mathcal{Q}$-tensor, $\overset{(1)}{\not{g}}$ is a symmetric 2-covariant S-tensor and $\overset{(1)}{\rlap{-}g}$ is a $\mathcal{Q} \otimes S$ 1-form. Conversely, the decomposition applied to the 1-form $\overset{(1)}{f}$ on $\mathcal{M}$ returns

$$\overset{(1)}{f} \to \overset{(1)}{\tilde{f}}, \overset{(1)}{\not{f}}$$

where $\overset{(1)}{\tilde{f}}$ is $\mathcal{Q}$ 1-form and $\overset{(1)}{\not{f}}$ is an S 1-form.

Decomposed quantities in hand, the system of equations (2.7)-(2.8) for

$$\overset{(1)}{g}$$

will now reduce to a system of coupled wave equations for the tuple

$$\left(\overset{(1)}{\tilde{g}}, \overset{(1)}{\rlap{-}g}, \overset{(1)}{\not{g}}\right).$$

Before we present a sample of the reduced equations however, it will behoove us to perform a further decomposition upon $\overset{(1)}{\tilde{g}}$ and $\overset{(1)}{\not{g}}$ into a symmetric, traceless 2-covariant $\mathcal{Q}$-tensor field and a symmetric, traceless 2-covariant S-tensor field respectively:

$$\overset{(1)}{\tilde{g}} = \overset{(1)}{\hat{\tilde{g}}} + \frac{1}{2}\tilde{g}_M \cdot \mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}}, \qquad \overset{(1)}{\not{g}} = \overset{(1)}{\hat{\not{g}}} + \frac{1}{2}\not{g}_M \cdot \not{\mathrm{tr}}\overset{(1)}{\not{g}}$$

with $\tilde{g}_M$ and $\not{g}_M$ the Lorentzian $\mathcal{Q}$-metric and the Riemannian S-metric resulting from applying the $2+2$ decomposition to g_M accordingly. In particular, the scalar functions $\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}}$ and $\not{\mathrm{tr}}\overset{(1)}{\not{g}}$ on $\mathcal{M}$ correspond to the respective traces of $\overset{(1)}{\tilde{g}}$ and $\overset{(1)}{\not{g}}$ with $\tilde{g}_M$ and $\not{g}_M$.

One thus reduces the system of equations (2.7)-(2.8) to the system of coupled tensorial wave equations satisfied by the collection

$$\left(\overset{(1)}{\hat{\tilde{g}}}, \mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}}, \overset{(1)}{\rlap{-}g}, \overset{(1)}{\hat{\not{g}}}, \not{\mathrm{tr}}\overset{(1)}{\not{g}}\right).$$

Motivated by[1], we use the collective notation

$$\mathscr{S} = \left(\overset{(1)}{\hat{\tilde{g}}}, \mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}}, \overset{(1)}{\rlap{-}g}, \overset{(1)}{\hat{\not{g}}}, \not{\mathrm{tr}}\overset{(1)}{\not{g}}\right)$$

to denote solutions to the equations of linearised gravity decomposed under this $2+2$

formalism. A sample of this *system of gravitational perturbations* is presented below – see section 3.3.3 for the full presentation.

For the $\mathcal{Q}\otimes S$ 1-form $\overset{(1)}{\cancel{g}}$,

$$\widetilde{\square}\overset{(1)}{\cancel{g}}+\not{\Delta}\overset{(1)}{\cancel{g}}-\frac{1}{r^2}(1-\mu)\overset{(1)}{\cancel{g}}=\frac{2}{r}\,\mathrm{d}r\otimes_{\mathrm{s}}\left(\frac{2}{r}\overset{(1)}{\cancel{g}}_P+\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\hat{\not{g}}}+\frac{1}{2}\not{\nabla}\not{\mathrm{t}}\mathrm{r}\overset{(1)}{\not{g}}-\overset{(1)}{\not{f}}\right)-\frac{2}{r}\not{\nabla}\otimes\overset{(1)}{\hat{\tilde{g}}}_P-\frac{1}{r}\,\mathrm{d}r\,\not{\nabla}\otimes\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}}$$
$$+\widetilde{\nabla}\otimes\overset{(1)}{\not{f}}+\not{\nabla}\otimes\overset{(1)}{\tilde{f}}. \tag{2.9}$$

For the symmetric, traceless 2-covariant S-tensor $\overset{(1)}{\hat{\not{g}}}$,

$$\widetilde{\square}\overset{(1)}{\hat{\not{g}}}+\not{\Delta}\overset{(1)}{\hat{\not{g}}}-\frac{2}{r}\widetilde{\nabla}_P\overset{(1)}{\hat{\not{g}}}-\frac{4}{r}\frac{\mu}{r}\overset{(1)}{\hat{\not{g}}}=-\frac{2}{r}\not{\nabla}\hat{\otimes}\overset{(1)}{\cancel{g}}_P+\not{\nabla}\hat{\otimes}\overset{(1)}{\not{f}}. \tag{2.10}$$

Finally, the S-component of the wave gauge condition

$$-\tilde{\delta}\overset{(1)}{\cancel{g}}-\frac{1}{2}\not{\nabla}\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}}+\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\hat{\not{g}}}+\frac{2}{r}\overset{(1)}{\cancel{g}}_P=\overset{(1)}{\not{f}}. \tag{2.11}$$

Here, all geometric objects with a tilde above them relate to the $\mathcal{Q}$-metric $\tilde{g}_M$ whereas all geometric objects with a slashed through them relate to the S-metric $\not{g}_M$. In particular, $-\tilde{\delta}$ is the divergence operator associated to $\tilde{g}_M$, $\not{\nabla}\hat{\otimes}\xi$ is the trace-free part of the Lie derivative of $\not{g}_M$ with respect to the S-vector field ξ and we emphasize that $\widetilde{\square}$ and $\not{\Delta}$ are the wave operator and Laplacian associated to $\tilde{g}_M$ and $\not{g}_M$ respectively. In addition, $\mu=\frac{2M}{r}$ is the mass aspect function, P is the vector field associated to the 1-form $\mathrm{d}r$ under the musical isomorphism on $\big(\mathcal{M},g_M\big)$, $\otimes_{\mathrm{s}}$ denotes the symmetrised tensor product and we use the notation

$$\omega_X:=\omega(X)$$

for ω and X a 1-form and vector field on $\mathcal{M}$ respectively.

We can in addition use this $2+2$ formalism to reveal a bit better how the map f of section 3.3.1 in the bulk of the thesis is defined. Indeed, the map f is defined as the sum of three constituent maps $\mathring{f}$, $\overset{\mathscr{K}}{f}$ and $\overset{\diamond}{f}$ the latter two of which we shall not explicitly state in this overview (see sections 3.3.1.2 and 3.3.1.3 in the bulk of the thesis) but the former of which is defined according to[11]

$$\mathring{f}(X):=\frac{2}{r}\tilde{X}_P+\frac{2}{r}\cancel{X}_P-\frac{1}{r}\,\mathrm{d}r\,\not{\mathrm{t}}\mathrm{r}\not{X}. \tag{2.12}$$

Here, $\tilde{X}$, $\cancel{X}$ and $\not{X}$ are the projections of the smooth 2-covariant tensor X onto a

[11] Note this map first appeared in our recent [5].

smooth 2-covariant $\mathcal{Q}$-tensor, a smooth $\mathcal{Q}\otimes S$ 1-form and a smooth 2-covariant S-tensor respectively.

In particular, writing out the equations (2.9)-(2.11) more explicitly we have

$$\widetilde{\Box}\overset{(1)}{g\!\!\!-}+\not\Delta\overset{(1)}{g\!\!\!-}-\frac{2}{r}\widetilde{\nabla}\otimes\overset{(1)}{g\!\!\!-}_P-\frac{1}{r^2}(1-\mu)\overset{(1)}{g\!\!\!-}+\frac{2}{r^2}\,\mathrm{d}r\otimes_{\mathrm{s}}\overset{(1)}{g\!\!\!-}_P=\frac{2}{r}\tilde{\mathrm{d}}r\otimes_{\mathrm{s}}\mathrm{d}\not{\mathrm{i}}\mathrm{v}\overset{(1)}{\hat{\not g}}+\widetilde{\nabla}\otimes\underset{\circ}{\overset{(1)}{\not f}}+\not\nabla\otimes\underset{\circ}{\overset{(1)}{\tilde{f}}}-\frac{2}{r}\,\mathrm{d}r\otimes_{\mathrm{s}}\underset{\circ}{\overset{(1)}{\not f}},\tag{2.13}$$

$$\widetilde{\Box}\overset{(1)}{\hat{\not g}}+\not\Delta\overset{(1)}{\hat{\not g}}-\frac{2}{r}\widetilde{\nabla}_P\overset{(1)}{\hat{\not g}}-\frac{4}{r}\frac{\mu}{r}\overset{(1)}{\hat{\not g}}=\not\nabla\hat{\otimes}\underset{\circ}{\overset{(1)}{\not f}}\tag{2.14}$$

and

$$-\tilde{\delta}\overset{(1)}{g\!\!\!-}-\frac{1}{2}\not\nabla\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}}+\mathrm{d}\not{\mathrm{i}}\mathrm{v}\overset{(1)}{\hat{\not g}}=\underset{\circ}{\overset{(1)}{\not f}}\tag{2.15}$$

where $\underset{\circ}{\overset{(1)}{\tilde{f}}}$ and $\underset{\circ}{\overset{(1)}{\not f}}$ are the $\mathcal{Q}$ 1-form and S 1-form associated to the 1-form $(\overset{\mathscr{K}}{f}+\overset{\diamond}{f})(\overset{(1)}{g})$.

We thus see that the constituent part $\mathring{f}$ of the map f that defines the generalised wave gauge we imposed as part of our linearisation procedure in the above has the effect of causing certain cancellations in the first order terms of the resulting equations of linearised gravity when compared, for instance, to the linearised equations (2.9)-(2.11) with the map f taken to be trivial. This will prove important later, as will the additional terms arising from the remaining constituent parts $\overset{\mathscr{K}}{f}$ and $\overset{\diamond}{f}$ of f.

We further emphasize that a considerable advantage of employing a generalised wave gauge with respect to g_M, i.e. with respect to the metric one is to linearise about, is the absence of any explict first order derivatives of $\overset{(1)}{g}$ in the linearisation of the Einstein vacuum equations as expressed in this gauge, the system of equations given by (2.7), and the absence of any explicit zero'th order terms of $\overset{(1)}{g}$ in the linearisation of the generalised wave gauge condition itself, the system of equations given by (2.8). For instance, returning to the schematic description of the Einstein vacuum equations (2.5), one sees that under linearisation[12] the terms quadratic in the connection tensor vanish. This would be patently false if one instead linearises the Einstein vacuum equations, as expressed in *wave coordinates*, about the solution[13] g_M unless $M=0$, that is, one is linearising about the trivial solution Minkowski space. Indeed, in this latter case the linearised equations actually completely decouple – see the book of Wald[22].

[12]Since $\boldsymbol{C}_{g_M,g_M}=0$.

[13]Since, in a system of wave coordinates on $(\mathcal{M},g_M)$, $\Gamma_M\neq 0$.

As to the significance of the terms quadratic in the Christoffel symbols vanishing, see the non-schematic description of the Einstein vacuum equations as expressed in a generalised wave gauge presented in section 3.3.2.2.

Finally, see section 3.2.2.2 in the bulk of the paper for a precise definition of the various operations and differential operations on $\left(\mathcal{M}, g_M\right)$ that follow naturally from this $2+2$ formalism and which shall appear frequently in the remainder of the overview.

2.1.4 Special solutions to the equations of linearised gravity: Pure gauge and linearised Kerr

We end this first part of the overview by discussing two special classes of solutions to the equations of linearised gravity, namely the pure gauge and linearised Kerr solutions.

This section of the overview corresponds to section 3.4 in the main body of the thesis.

The first such class of solutions arises from the fact that, on an abstract Lorentzian manifold $\boldsymbol{\mathcal{M}}$, there exists a residual class of diffeomorphisms on $\boldsymbol{\mathcal{M}}$ under which the generalised wave gauge is preserved.

Indeed, suppose that the smooth, globally hyperbolic Lorentzian manifolds $\left(\boldsymbol{\mathcal{M}}, \boldsymbol{g}\right)$ and $\left(\boldsymbol{\mathcal{M}}, \overline{\boldsymbol{g}}\right)$ are solutions to the Einstein vacuum equations for which $\boldsymbol{g}$ is in a generalised $\boldsymbol{f}$-wave gauge with respect to $\overline{\boldsymbol{g}}$ where $\boldsymbol{f} : T^2(\boldsymbol{\mathcal{M}}) \to T^1(\boldsymbol{\mathcal{M}})$ is a (sufficiently regular) map. Then there exists a non-trivial class of diffeomorphisms $\boldsymbol{\phi}$ on $\boldsymbol{\mathcal{M}}$ for which $\boldsymbol{\phi}^*\boldsymbol{g}$ is in a generalised $\boldsymbol{f}$-wave gauge with respect to $\overline{\boldsymbol{g}}$. In particular, $\boldsymbol{\phi}^*\boldsymbol{g}$ and $\boldsymbol{g}$ thus reside within the same equivalence class of solutions to the Einstein vacuum equations.

This phenomenon is manifested in the linear theory by the existence of a 1-parameter family of diffeomorphisms on $\mathcal{M}$ for which each of the 1-parameter family of Lorentzian metrics given as the pullback of g_M under this 1-parameter family of diffeomorphisms are in a generalised f-wave gauge with respect to g_M to first order. This yields, upon linearisation, a family of solutions to the equations of linearised gravity corresponding to *residual gauge freedom* in the non-linear theory. Following[1], we shall denote this special class of *pure gauge solutions* by $\mathscr{G}$.

Indeed, let v be a smooth 1-form $\mathcal{M}$ satisfying

$$\Box v = f(\mathcal{L}_v g_M)$$

Then the following

$$\overset{\scriptscriptstyle(1)}{g} = \mathcal{L}_{v^\sharp} g_M \tag{2.16}$$

is a smooth solution to the equations of linearised gravity arising from the linearisation of the 1-parameter family of Lorentzian metrics given as the pullback of g_M under the 1-parameter family of diffeomorphisms generated by v. That the above solve the equations of linearised gravity can indeed by verified from equations (2.7) and (2.8).

The second such class of solutions arises from suitably expressing members of the 2-parameter Kerr exterior family in the generalised wave gauge identified in the previous section and then linearising about the fixed Schwarzschild exterior solution g_M. This yields an explicit *4-parameter*[14] *family* of solutions to the equations of linearised gravity corresponding to the fact that g_M actually sits within a family of solutions to the Einstein vacuum equations.

Indeed, consider the following 1-parameter family of Schwarzschild exterior metrics with mass $M+\epsilon\mathfrak{m}$ as expressed in Schwarzschild-star coordinates on $\mathcal{M}$:

$$g_{M+\epsilon\mathfrak{m}} := -\left(1-\frac{2(M+\epsilon\mathfrak{m})}{r}\right)\mathrm{d}t^{*2}+\frac{4(M+\epsilon\mathfrak{m})}{r}\,\mathrm{d}t^*\,\mathrm{d}r+\left(1+\frac{2(M+\epsilon\mathfrak{m})}{r}\right)\mathrm{d}r^2+r^2\mathring{g} \tag{2.17}$$

where $\mathring{g}$ is the metric of the unit round sphere. Then in section 3.4.2 in the bulk of the thesis we compute explicitly that each $g_{M+\epsilon\mathfrak{m}}$ is in a generalised f-wave gauge with respect to g_M to first order in ϵ. Linearising the thus yields the following 1-parameter family of solutions to the equations of linearised gravity:

$$\overset{\scriptscriptstyle(1)}{g}_{\mathfrak{m}} := \frac{2\mathfrak{m}}{r}\,\mathrm{d}t^{*2}+\frac{4\mathfrak{m}}{r}\,\mathrm{d}t^*\,\mathrm{d}r+\frac{2\mathfrak{m}}{r}\,\mathrm{d}r^2.$$

For the full linearised Kerr family, see section 3.4.2 in the bulk of the thesis. Following[1], we shall denote this special class of *linearised Kerr solutions* by $\mathscr{K}_{\mathfrak{m},\mathfrak{a}}$ where $\mathfrak{m}\in\mathbb{R}$ and $\mathfrak{a}$ is a smooth function on S^2 supported only on the classical $l=1$ spherical harmonics[15]. Consequently, we note two key properties of this family – stationarity[16] and being supported entirely on the $l=0,1$ spherical harmonics (see section 3.2.5 for a precise definition). In addition, we stress that the reason for introducing the map $\overset{\mathscr{K}}{f}$ as a constituent component of the map f of section 2.1.3 is to realise the Kerr family, as expressed in the Schwarzschild-star coordinate system, as being in a generalised f-wave gauge with respect to g_M, at least to first order (cf. Remark 13 in the bulk of the thesis).

A consequence of the existence of these solutions is the expectation that, at best, a general solution to the equations of linearised gravity decays to a member of the linearised Kerr

[14]One indeed has a 4-parameter family of solutions since the Schwarzschild exterior background around which ones linearises has no preferred axis of rotation – see Remark 12.

[15]The coefficients of the Y^1_i modes thus yield 3 parameters.

[16]That is, $\mathcal{L}_T\overset{\scriptscriptstyle(1)}{g}=\mathcal{L}_T\overset{\scriptscriptstyle(1)}{f}=0$ where T is the causal Killing field introduced in section 2.1.2.

family, after the addition of some pure gauge solution.

2.2 Pure gauge and linearised Kerr invariant quantities: The Regge–Wheeler and Zerilli equations

In this second part of the overview we discuss how one extracts from the equations of linearised gravity the two fully decoupled scalar wave equations described by the Regge–Wheeler and Zerilli equations respectively.

This decoupling will prove key to unlocking the tensorial structure of the equations of linearised gravity.

This section of the overview corresponds to Chapter 4 of the main body of the thesis.

Owing to the potential complications arising from the existence of the linearised Kerr and pure gauge solutions of section 2.1.4, it is natural to consider those quantities which vanish for all such solutions.

Two such quantities are the smooth scalars $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ constructed from a smooth solution $\mathscr{S}$ to the equations of linearised gravity as in Proposition 4.1 in the bulk of the thesis. Remarkably, the linearised Einstein equations i.e. the system of equations that result from linearising the Einstein vacuum equations (1.1) about $\left(\mathcal{M}, g_M\right)$ without imposing any gauge, and therefore in particular the equations of linearised gravity, actually force these two scalars to decouple from the full system of gravitational perturbations into the Regge–Wheeler and Zerilli equations respectively:

$$\widetilde{\Box}\overset{(1)}{\Phi} + \not{\Delta}\overset{(1)}{\Phi} = -\frac{6}{r^2}\frac{M}{r}\overset{(1)}{\Phi}, \tag{2.18}$$

$$\widetilde{\Box}\overset{(1)}{\Psi} + \not{\Delta}\overset{(1)}{\Psi} = -\frac{6}{r^2}\frac{M}{r}\overset{(1)}{\Psi} + \frac{24}{r^3}\frac{M}{r}(r-3M)\not{c}^{[1]}\overset{(1)}{\Psi} + \frac{72}{r^3}\frac{M}{r}\frac{M}{r}(r-2M)\not{c}^{[2]}\overset{(1)}{\Psi}. \tag{2.19}$$

Here, $\not{c}^{[p]}$ is the inverse of the elliptic operator $r^2\not{\Delta} + 2 - \frac{6M}{r}$ applied p-times – this is well defined over the space of smooth functions on $\mathcal{M}$ supported on the $l \geq 2$ spherical harmonics (see section 3.2.5 in the bulk of the thesis for a precise definition of tensor fields on $\mathcal{M}$ having support on the $l \geq 2$ spherical harmonics). Subsequently, that $\overset{(1)}{\Psi}$ (and indeed $\overset{(1)}{\Phi}$) are supported on the $l \geq 2$ spherical harmonics is a consequence of the fact they were constructed so as to vanish for all linearised Kerr solutions. To see that $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ do indeed decouple in such a manner, see Theorem 4.3 in the bulk of the thesis.

This remarkable decoupling was originally discovered by Regge–Wheeler [23] and Zerilli [24] in the context of a full mode and spherical harmonic decomposition of the linearised

Einstein equations although it took the later work of Moncrief in [25] to clarify the gauge-invariance of these quantities in that they vanish for all solutions to (1.1) of the form $\mathcal{L}_v g_M$ where v is a smooth vector field on $\mathcal{M}$. On the other hand the non-mode decomposed version of these equations presented above is ultimately[17] due to Chaverra, Ortiz and Sarbach in [45].

Whilst this decoupling is indeed remarkable given the algebraic complexity of the system of gravitational perturbations in question (and indeed the full system of linearised Einstein equations), the key point behind this decoupling at the level of an analysis of the equations of linearised gravity is that the two equations (2.18) and (2.19) can be understood using the techniques developed for studying the scalar wave equation $\Box_{g_M}\psi = 0$ on $\mathcal{M}$, techniques that shall be discussed in section 2.5 of the overview. Indeed, it is by establishing a decay statement for solutions to the Regge–Wheeler and Zerilli equations that we shall ultimately establish a decay statement for solutions to the equations of linearised gravity.

Before we end this section of the overview we note the interesting result of Corollary 5.9 in the bulk of the thesis which states that a sufficiently regular solution to the equations of linearised gravity for which both $\overset{\scriptscriptstyle(1)}{\Phi}$ and $\overset{\scriptscriptstyle(1)}{\Psi}$ vanish is in fact the sum of a linearised Kerr and pure gauge solution. We further remark that, as discovered by Chandrasekhar, there exists a remarkable transformation theory mapping solutions of the Zerilli equation to solutions of the Regge–Wheeler equation, although we will not make use of this in this thesis. See [54] for details.

Lastly, we further note that the Regge–Wheeler equation appears and plays a major role in the work of Dafermos, Holzegel and Rodnianski in [1]. It is remarkable that the same equation appears in these two different contexts. Note that the linearised Robinson–Trautman [60] solutions that appear in their work as solutions with *vanishing* ‘Regge–Wheeler quantities’ are related to the aforementioned transformation mapping solutions of the Zerilli equation to the Regge–Wheeler equation, which has non-trivial kernel consisting of the so-called algebraically special modes.

2.3 The Cauchy problem for the equations of linearised gravity and gauge-normalisation of initial data

In this third part of the overview we discuss a well-posedness result for the equations of linearised gravity as a Cauchy problem, thereby initiating their formal analysis. In addition, we discuss a class of solutions to the equations of linearised gravity the Cauchy

[17] Earlier works of [49] and [44] established a covariant derivation of the Regge–Wheeler and Zerilli equations. On the other hand, in [50] a non-covariant, non-modal derivation was given, albeit via a Hamiltonian formulation of the problem.

data of which has been gauge-normalised via the addition of a pure gauge solution of section 2.1.4.

This section of the overview corresponds to Chapter 5 in the main body of the thesis.

2.3.1 The Cauchy problem for the equations of linearised gravity

We begin with a discussion towards establishing the well-posedness of the equations of linearised gravity as a Cauchy problem.

This part of the overview corresponds to section 5.1 in the bulk of the thesis.

It is well known that the Einstein equations (1.1) must satisfy certain *constraints* which arise as a consequence of the Gauss–Codazzi equations restricting the embedding of a 3-manifold as a hypersurface in a 4-manifold. These constraints are of course inherited by the linearisation, although they can be understood purely within the context of the linear theory. Moreover, in regards to the *Cauchy* problem for the equations of linearised gravity, a further constraint is imposed by the generalised wave gauge condition (2.8). Therefore, any well-posedness result for the equations of linearised gravity must neccesarily include a procedure by which one generates *admissible* initial data. A definition of such admissible initial data, along with the constraints it must satisfy, is to be found in section 5.1.1.1 in the bulk of the thesis.

We confront this issue by introducing a notion of *seed data* for the equations of linearised gravity. Indeed, it is from this *freely prescribed* seed data that we are able to generate, uniquely, an admissible initial data set for the system of gravitational perturbations.

Fixing an initial Cauchy hypersurface Σ, corresponding to a level set of the time function t^* introduced in section 2.1.2, this seed data consists of a collection of freely prescribed quantities on Σ. We denote this collection of seed by $\mathscr{D}$, with corresponding admissible initial data denoted by $\mathscr{A}$. An example of the seed quantities are

- the two smooth functions $\overset{(1)}{\bar{\Phi}}$ and $\overset{(1)}{\underline{\Phi}}$ on Σ that are supported on the $l \geq 2$ spherical harmonics

See section 5.1.1.2 in the bulk of the paper for a full description of the seed data $\mathscr{D}$.

The procedure by which we extend this seed data to a full admissible initial data set exploits the existence of three explicit classes of solutions to the equations of linearised gravity. Two have been discussed already, namely the linearised Kerr and pure gauge solutions. The third class, which are parametrised by two scalar quantities satisfying the Regge–Wheeler and Zerilli equations respectively, are the class of solutions to the equations of linearised gravity that are in the modified Regge–Wheeler gauge of section

2.3.1.1 of the overview. Consequently, by (appropriately) projecting these three classes of solutions onto Σ one generates three explicit classes of solutions to the linearised constraint equations and thus prescribing seed data in such a way as to generate said solutions determines, by linearity, an admissible initial data set given as their sum.

For instance, the seed quantities $\underline{\overset{(1)}{\Phi}}$ and $\overset{(1)}{\overline{\Phi}}$ introduced above are to determine Cauchy data for the gauge-invariant quantity $\overset{(1)}{\Phi}$ of section 2.2 which satisfies the Regge–Wheeler equation. This part of the seed data thus explicitly generates a solution to the linearised constraint equations corresponding to the third class of explicit solutions. Moreover since, once certain 'gauge-considerations' are taken into account, the freely prescribed seed data contains four[18] functional degrees of freedom, the full prescription of seed data generates a class of solutions to the linearised constraint equations which agrees with the full functional degrees of freedom one associates to the equations of linearised gravity by a crude function counting arguement (see the book of Wald[22]).

In order to evolve this admissible initial data into a full solution to the equations of linearised gravity we appeal to the argument employed by Choquet-Bruhat in her celebrated[11]. Indeed, it is now a classical[19] result that solutions to the equation (2.7) arising from Cauchy data that is admissible automatically satisfy the gauge condition (2.8). Well-posedness of the equations of linearised gravity thus reduces to well-posedness of the system (2.7) in light of our previous discussion regarding the construction of admissible initial data.

Now, the well-posedess of the system (2.7) of course depends upon the regularity of the map f introduced in section 2.1.3 of the overview. Recalling therefore that f had the three constituent components $\mathring{f}$, $\overset{\mathscr{K}}{f}$ and $\overset{\diamond}{f}$ we first observe from the definition of the map $\mathring{f}$ given explicitly at the end of section 2.1.3 that the expression $\mathring{f}(\overset{(1)}{g})$ is a local expression in $\overset{(1)}{g}$ at the same level of regularity of $\overset{(1)}{g}$. The same in fact holds true[20] for the expression $\overset{\mathscr{K}}{f}(\overset{(1)}{g})$. Conversely, the map $\overset{\diamond}{f}$ when applied to $\overset{(1)}{g}$ returns a *non-local* expression in the gauge-invariant quantities $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ associated to $\overset{(1)}{g}$ (see section 3.3.1.3 and Proposition 4.1 in the bulk of the thesis). To construct a solution $\overset{(1)}{g}$ to (2.7) we therefore first invoke the fact that the corresponding quantities $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ associated to $\overset{(1)}{g}$ must then necessarily satisfy, by the discussion in section 2.2 of the overview, the (decoupled) Regge–Wheeler and Zerilli equations (2.18) and (2.19). We thus construct the quantities $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ by virtue of the well-posedness of these respective equations (see Proposition 4.2 in the bulk of the thesis) with Cauchy data in fact determined from the seed $\mathscr{D}$. One then constructs the solution $\overset{(1)}{g}$ to (2.7) by writing (2.7) as a system of inhomogeneous wave equations

[18] Corresponding exactly to Cauchy data for the gauge-invariant quantities $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$.

[19] See the book[12] of Choquet-Bruhat.

[20] Although we note the application of the map $\overset{\mathscr{K}}{f}$ to $\overset{(1)}{g}$ returns an expression in the $l = 0, 1$ spherical harmonic modes of $\overset{(1)}{g}$ (see section 3.2.5 in the bulk of the thesis for a precise definition). However, by the spherical symmetry of $(\mathcal{M}, g_M)$, the $l = 0, 1$ modes propagate orthogonally to the rest under evolution.

with inhomogeneities given by $\overset{\scriptscriptstyle(1)}{\Phi}$ and $\overset{\scriptscriptstyle(1)}{\Psi}$ and with the first order terms arising from the expressions $(\mathring{f} + \overset{\mathscr{K}}{f})(\overset{\scriptscriptstyle(1)}{g})$ 'absorbed' into the left hand side of (2.7).

This leads to the well-posedness theorem of section 5.1.2 in the bulk of the paper, which we summarise here as:

Theorem 0. *The smooth seed data set $\mathscr{D}$ prescribed on Σ gives rise to a unique, smooth solution $\mathscr{S}$ to the equations of linearised gravity on $\mathcal{M} \cap D^+(\Sigma)$.*

The boundedness and decay statements we discuss in section 2.4.1 for solutions to the equations of linearised gravity will require that the initial data from which they arise satisfy in addition a notion of *asymptotic flatness*. Importantly, we are able to provide such a notion on the freely prescribed seed data alone, which we then show propagates under the above well-posedness theorem. See Proposition 5.3 in the bulk of the thesis. Moreover, the precise definition of asymptotic flatness we require can be found in section 5.1.3 in the bulk of the thesis[21].

2.3.1.1 Gauge-normalisation of initial data

We end this section of the overview by discussing a certain class of solutions to the equations of linearised gravity the initial data of which has been gauge-normalised via the addition of a pure gauge solution. It is these 'gauge-normalised' solutions that will be subject to the decay statement discussed in section 2.4.2 in the overview.

This part of the overview corresponds to section 5.2 in the main body of the thesis.

The class of solutions under consideration are defined as follows. Indeed, given a constant $\mathfrak{m} \in \mathbb{R}$ and a smooth function $\mathfrak{a}$ on S^2 lying in the span of the $l = 1$ spherical harmonics then we say that a smooth solution $\mathscr{S}$ to the equations of linearised gravity is in the modified Regge–Wheeler gauge with parameters $\mathfrak{m}$ and $\mathfrak{a}$ iff the following conditions on $D^+(\Sigma)$[22]:

i) $\overset{\scriptscriptstyle(1)}{\not{g}}{}'_{\mathrm{e}} - \tilde{\mathrm{d}}^{\mathcal{I}}\left(r\overset{\scriptscriptstyle(1)}{\Psi}\right) + 2\tilde{\mathrm{d}}r\,\overset{\scriptscriptstyle(1)}{\Psi} = 0$

ii) $\overset{\scriptscriptstyle(1)}{\hat{\not{g}}} - r\nabla\!\!\!/\hat{\otimes}\mathcal{D}\!\!\!/_1^{\star}\left(\overset{\scriptscriptstyle(1)}{\Psi}, \overset{\scriptscriptstyle(1)}{\Phi}\right) = 0$

iii) $\mathscr{S}_{l=0,1} - \mathscr{K}_{\mathfrak{m},\mathfrak{a}} = 0$

[21]In fact, we actually impose a notion of compact support on the seed data $\mathscr{D}$ and one advantage of our method of generating the admissible initial data set $\mathscr{A}$ from $\mathscr{D}$ is that this notion of compact support is preserved.

[22]Note we define the modified Regge–Wheeler gauge differently in the bulk of the thesis but the proceeding discussion will show that both definitions are equivalent.

Here, the subscripts $l = 0, 1$ denote a projection onto the $l = 0, 1$ spherical harmonics whereas the superscript $'$ denotes a projection away from the $l = 0, 1$ spherical harmonics, each of which are to be made precise in section 5.2.1.1 in the bulk of the thesis, and we recall that $\mathscr{K}_{\mathfrak{m},\mathfrak{a}}$ is a member of the linearised Kerr family introduced in section 2.1.4. In addition, $\overset{\scriptscriptstyle(1)}{\Phi}$ and $\overset{\scriptscriptstyle(1)}{\Psi}$ are the gauge-invariant quantities of section 2.2 associated to the solution $\mathscr{S}$, $\tilde{\mathrm{d}}^{\mathcal{I}}$ is the first order derivative operator defined in section 3.2.4.2 in the bulk of the thesis and $\not{\mathcal{D}}_1^\star$ is the operator

$$\not{\mathcal{D}}_1^\star\Big(\tilde{q}_1, \tilde{q}_2\Big) := -\not{\nabla}\tilde{q}_1 - \not{\star}\not{\nabla}\tilde{q}_2$$

where $\not{\star}$ is the Hodge dual associated to the S-metric $\not{g}_M$. Finally, $\overset{\scriptscriptstyle(1)}{\not{g}}_{\mathrm{e}}$ is the unique $\mathcal{Q}$ 1-form associated to the $\mathcal{Q}\otimes S$ 1-form $\overset{\scriptscriptstyle(1)}{\not{g}}$ under the Hodge decomposition of section 3.2.5.3 in the bulk of the thesis:

$$\overset{\scriptscriptstyle(1)}{\not{g}} = \not{\mathcal{D}}_1^\star\Big(\overset{\scriptscriptstyle(1)}{\not{g}}_{\mathrm{e}}, \overset{\scriptscriptstyle(1)}{\not{g}}_{\mathrm{o}}\Big).$$

We denote such solutions by $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$.

Indeed, it is smooth solutions to the equations of linearised gravity satisfying the above, when supplemented with an appropriate asymptotic flatness condition, that will be subject to our quantitative boundedness and decay statement to be discussed in section 2.4.2 of the overview. This will follow as a consequence of the quantitative boundedness and decay bounds we are able to derive for the gauge-invariant quantities $\overset{\scriptscriptstyle(1)}{\Phi}$ and $\overset{\scriptscriptstyle(1)}{\Psi}$ satisfying the Regge–Wheeler and Zerilli equations respectively which shall in addition be discussed in section 2.4.1 of the overview. For if a solution $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$ to the equations of linearised gravity satisfies conditions i)-iii) then the projection solution $\overset{\diamond}{\mathscr{S}} := \overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}} - \mathscr{K}_{\mathfrak{m},\mathfrak{a}}$ in fact must obey on $D^+(\Sigma)$ the relations

$$\hat{\tilde{\overset{\scriptscriptstyle(1)}{g}}} = \widetilde{\nabla}\hat{\otimes}\tilde{\mathrm{d}}^{\mathcal{I}}\Big(r\overset{\scriptscriptstyle(1)}{\Psi}\Big) + 6\mu\,\mathrm{d}r\hat{\tilde{\otimes}}\not{C}^{[1]}\tilde{\mathrm{d}}\overset{\scriptscriptstyle(1)}{\Psi}, \tag{2.20}$$

$$\tilde{\overset{\scriptscriptstyle(1)}{g}} = 0, \tag{2.21}$$

$$\overset{\scriptscriptstyle(1)}{\not{g}} = \not{\mathcal{D}}_1^\star\Big(\tilde{\mathrm{d}}^{\mathcal{I}}\Big(r\overset{\scriptscriptstyle(1)}{\Psi}\Big) - 2\tilde{\mathrm{d}}r\,\overset{\scriptscriptstyle(1)}{\Psi}, \tilde{\mathrm{d}}^{\mathcal{I}}\Big(r\overset{\scriptscriptstyle(1)}{\Phi}\Big) - 2\tilde{\mathrm{d}}r\,\overset{\scriptscriptstyle(1)}{\Phi}\Big), \tag{2.22}$$

$$\hat{\overset{\scriptscriptstyle(1)}{\not{g}}} = r\not{\nabla}\hat{\otimes}\not{\mathcal{D}}_1^\star\Big(\overset{\scriptscriptstyle(1)}{\Psi}, \overset{\scriptscriptstyle(1)}{\Phi}\Big), \tag{2.23}$$

$$\not{\mathrm{tr}}\overset{\scriptscriptstyle(1)}{\not{g}} = 4\tilde{\mathrm{d}}_P^{\mathcal{I}}\overset{\scriptscriptstyle(1)}{\Psi} + 12\mu r^{-1}(1-\mu)\not{C}^{[1]}\overset{\scriptscriptstyle(1)}{\Psi}. \tag{2.24}$$

Here, $\not{C}^{[1]}$ is the operator introduced in section 2.2 of the overview.

Of course, the question remains as to whether there exist solutions that satisfy conditions i)-iii) which moreover verify the required conditions on asymptotic regularity. That such a gauge can indeed be realised is the content of Theorem 5.5 in the bulk of the thesis which states that, given the solution $\mathscr{S}$ to the equations of linearised gravity arising from the seed data set $\mathscr{D}$ courtesy of Theorem 0, one can add to it a pure gauge solution $\overset{\diamond}{\mathscr{G}}$ for which the resulting solution $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$ is in the modified Regge–Wheeler gauge, the parameters of which are to be determined explicitly from the seed data $\mathscr{D}$, part of which contains

- a smooth function $\mathfrak{a}$ on the horizon sphere $\mathcal{H}^+ \cap \Sigma$ lying in the span of the $l = 1$ spherical harmonics
- a constant $\mathfrak{m}$

The theorem further implies that if the seed data, which we now explicitly denote by $\mathscr{D}_{\mathfrak{m},\mathfrak{a}}$, moreover satisfies an appropriate notion of asymptotic flatness then this property is inherited by the pure gauge solution $\overset{\diamond}{\mathscr{G}}_{\mathfrak{m},\mathfrak{a}}$ and hence the solution $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$.

An important property of the pure gauge solution $\overset{\diamond}{\mathscr{G}}_{\mathfrak{m},\mathfrak{a}}$ that one can conclude from the proof of Theorem 5.5 is that the initial data from which it arises is actually constructed explicitly from the seed data of the solution $\mathscr{S}$ alone. In particular, the addition of the pure gauge solution $\overset{\diamond}{\mathscr{G}}$ to the solution $\mathscr{S}$ simply serves to ensure that the Cauchy data for all quantities on the left hand side of the conditions i)-iii), associated now to the solution $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$, is trivial. Indeed, the equations of linearised gravity then force the conditions i)-iii) to propagate under evolution by Theorem 0 – see section 5.2.3 in the bulk of the thesis for details. We note that this in fact entirely analogous to the propagation of the gauge condition (2.8) under evolution by (2.7) as discussed in section 2.3.1 of the overview.

Consequently, a modified Regge–Wheeler gauge is in fact realisable *purely from seed data* $\mathscr{D}_{\mathfrak{m},\mathfrak{a}}$ *alone* and the solution $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$ is thus said to be *initial-data-normalised*.

We stress that the fact that one can ensure that conditions i)–iii) (and consequently the relations (2.20)) hold simply by normalising the Cauchy data of the solution $\mathscr{S}$ is a consequence of how we defined the map f that subsequently defined the generalised wave gauge we imposed in order to define the equations of linearised gravity in section 2.1.3[23]. The motivation behind choosing the map f therefore deserves further elaboration.

Indeed, the Regge–Wheeler gauge was originally a gauge discovered by Regge and Wheeler in their pioneering study[23] of the linearised Einstein equations about Schwarzschild, i.e. the linearisation of the equations (1.1) about $\left(\mathcal{M}, g_M\right)$. There they showed that given a solution to the linearised equations then one can add to it a 'pure gauge solution' (cf.

[23] In particular, this statement does not hold true if one were to, for example, set f to be the trivial map and then define the equations of linearised gravity as those that result from linearising the Einstein vacuum equations (1.1), as expressed in a generalised 0-wave gauge with respect to g_M, around $\left(\mathcal{M}, g_M\right)$.

section 2.2) such that the resulting solution $\overset{(1)}{\mathfrak{g}}$ satisfies

$$
\begin{aligned}
\overset{(1)}{\hat{\tilde{\mathfrak{g}}}} &= \widetilde{\nabla}\hat{\otimes}\tilde{\mathrm{d}}\Big(r\overset{(1)}{\Psi}\Big) + 6\mu\,\mathrm{d}r\hat{\tilde{\otimes}}\not{\zeta}^{[1]}\tilde{\mathrm{d}}\overset{(1)}{\Psi},\\
\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{\mathfrak{g}}} &= 0,\\
\overset{(1)}{\mathfrak{g}}\!\!\!\!- &= -\not{\mathcal{D}}_1^{\star}\Big(0,\tilde{\star}\tilde{\mathrm{d}}\Big(r\overset{(1)}{\Phi}\Big)\Big),\\
\overset{(1)}{\hat{\not{\mathfrak{g}}}} &= 0,\\
\not{\mathrm{t}}\mathrm{r}\overset{(1)}{\not{\mathfrak{g}}} &= -2r\not{\Delta}\overset{(1)}{\Psi} + 4\tilde{\mathrm{d}}_P\overset{(1)}{\Psi} + 12\mu r^{-1}(1-\mu)\not{\zeta}^{[1]}\overset{(1)}{\Psi}
\end{aligned}
$$

where $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ satisfy the Regge–Wheeler and Zerilli equations respectively. Now, in our recent [5], we showed that one can in fact realise the above 'gauge' as a residual gauge choice associated to the linearisation of the Einstein vacuum equations (1.1), as expressed in a generalised $\mathring{f}$-wave gauge with respect to g_M, about $\big(\mathcal{M}, g_M\big)$. Here, $\mathring{f}$ is the map defined as in section 2.1.3 of the overview and by residual gauge we mean an analogous construction to the one discussed above for the initial-data-normalised solutions to the equations of linearised gravity. In particular, in [5] the Regge–Wheeler gauge was identified as a gauge choice at the level of initial data. We remark that this identification of the Regge–Wheeler gauge is perhaps more natural in light of the dynamical formulation of the Einstein equations afforded to us by the work of Choquet-Bruaht [11].

However, the Regge–Wheeler gauge is not asymptotically flat in the sense that the collection of quantities $\overset{(1)}{\hat{\tilde{\mathfrak{g}}}}, \mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{\mathfrak{g}}}, \overset{(1)}{\mathfrak{g}}\!\!\!\!-, \overset{(1)}{\hat{\not{\mathfrak{g}}}}$ and $\not{\mathrm{t}}\mathrm{r}\overset{(1)}{\not{\mathfrak{g}}}$ defined as above do not satisfy the r-weighted pointwise bounds satisfied by the initial-data-normalised solution $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$ to the equations of linearised gravity as in Theorem 2 in section 2.4.2 of the overview. Thus, whereas the definition of the map $\mathring{f}$ was motivated by the classical insights of Regge and Wheeler regarding the Regge–Wheeler gauge, the map $\overset{\diamond}{f}$ of section 2.1.3 of the overview was chosen so as to modify the map $\mathring{f}$ in such a way as to allow for the modified Regge–Wheeler gauge (the nomenclature now clear) to be realised as a residual gauge choice within the generalised wave gauge defined by the map f.

Finally, we emphasize that we expect that the identification of the (modified) Regge–Wheeler gauge as a residual gauge choice, at the level of initial data, within a well-posed formulation of the linearised Einstein equations around $\big(\mathcal{M}, g_M\big)$ will potentially prove key to future nonlinear applications. In particular, we stress that the linearised equations studied by Regge and Wheeler in [23] are not well-posed and thus it is not possible to make the connection between the Regge–Wheeler gauge and a normalisation of initial data within that formulation of the linearised Einstein equations about $\big(\mathcal{M}, g_M\big)$. For this reason we actually include the result of our [5] regarding the Regge–Wheeler gauge as an appendix to this thesis where the connection with the modified Regge–Wheeler gauge is in addition made manifest.

2.4 The main theorems

In this fourth part of the overview we give rough statements of the main theorems of this thesis.

This part of the overview corresponds to Chapter 6 in the main body of the thesis.

2.4.1 Theorem 1: Boundedness and decay for solutions to the Regge–Wheeler and Zerilli equations

We begin with a rough version of our first theorem which concerns a boundedness and decay statement for solutions to the Regge–Wheeler and Zerilli equations on $\left(\mathcal{M}, g_M\right)$. Such a statement will have application to the equations of linearised gravity in light of the initial-data-normalised solution discussed in section 2.3.1.1 of the overview.

The precise statement of the theorem can be found in section 6.1.3 in the main body of the thesis.

The statement in question involves both certain natural r-weighted energy norms on hypersurfaces which penetrate both the future event horizon and future null infinity and are given as the level set of a function $\tau^\star$ on $\mathcal{M}$ and certain bulk norms over the region foliated by these hypersurfaces as depicted in the following Penrose diagram:

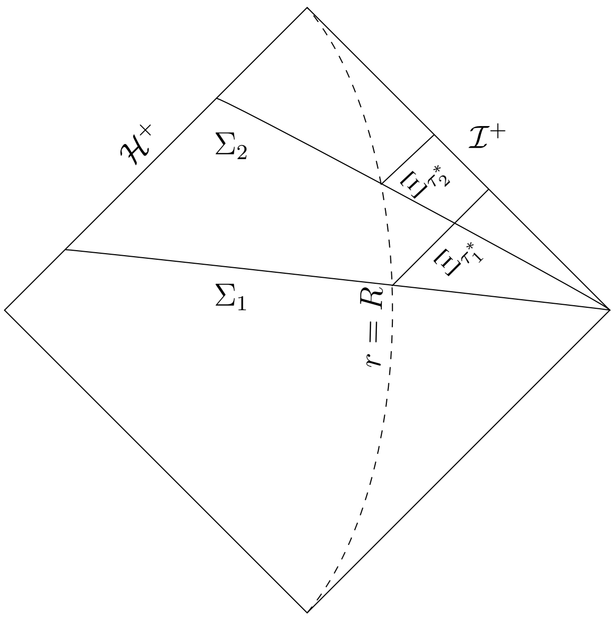

Figure 2.2: A Penrose diagram of $\left(\mathcal{M}, g_M\right)$ depicting the hypersurfaces $\Xi_{\tau^\star}$ which penetrate both $\mathcal{H}^+$ and $\mathcal{I}^+$. Here, the hypersurfaces Σ_{t^*} are level sets of the time function t^*.

However, we postpone giving a more precise description of both the foliation and the norms until we come to discuss the scalar wave equation on $\left(\mathcal{M}, g_M\right)$ in section 2.5 of the overview.

A rough formulation of Theorem 1 is then as follows.

Theorem 1. *Let Φ be a smooth solution to the Regge–Wheeler equation* (2.18) *on* $\big(\mathcal{M}, g_M\big)$ *supported on the $l \geq 2$ spherical harmonics.*

We assume that Cauchy data for Φ is compactly supported on the initial Cauchy hypersurface Σ.

Then for any integer $n \geq 0$ the following estimates hold:

i) the higher order flux and weighted bulk estimates

$$\mathbb{F}^n[r^{-1}\Phi] + \mathbb{I}^n[r^{-1}\Phi] \lesssim \mathbb{D}^n[r^{-1}\Phi]$$

ii) the higher order integrated decay estimate

$$\mathbb{M}^n[r^{-1}\Phi] \lesssim \mathbb{D}^{n+1}[r^{-1}\Phi]$$

Here, $\mathbb{D}$ is an energy norm acting on Cauchy data for Φ on Σ.

Let now Ψ be a smooth solution to the Zerilli equation (2.19) *on* $\big(\mathcal{M}, g_M\big)$ *supported on the $l \geq 2$ spherical harmonics.*

We assume that Cauchy data for Ψ is compactly supported on the initial Cauchy hypersurface Σ.

Then for any integer $n \geq 0$ the following estimates hold:

i) the higher order flux and weighted bulk estimates

$$\mathbb{F}^n[r^{-1}\Psi] + \mathbb{I}^n[r^{-1}\Psi] \lesssim \mathbb{D}^n[r^{-1}\Psi]$$

ii) the higher order integrated decay estimate

$$\mathbb{M}^n[r^{-1}\Psi] \lesssim \mathbb{D}^{n+1}[r^{-1}\Psi]$$

We note that the necessity of the r-weights in the above norms is a consequence of how the norms are defined for solutions to the scalar wave equation on $\big(\mathcal{M}, g_M\big)$.

Moreover, the norms in i) and ii) are such that now via a hierarchy of r-weighted estimates (see section 2.5) and an application of Sobolev embedding one can derive as a corollary to Theorem 1 the following pointwise decay bounds:

Corollary. *Let Φ be as in Theorem 1. Then on $D^+(\Sigma)$*[24] *one has the pointwise decay bound*

$$|\Phi| \lesssim \frac{1}{\sqrt{\tau^\star}} \cdot \mathbb{D}^4[r^{-1}\Phi].$$

Let now Ψ be as in Theorem 1. Then on $D^+(\Sigma)$ one has the pointwise decay bound

$$|\Psi| \lesssim \frac{1}{\sqrt{\tau^\star}} \cdot \mathbb{D}^4[r^{-1}\Psi].$$

We make the following remarks regarding Theorem 1.

The flux estimate associated to the norm $\mathbb{F}$ in both parts *i*) of Theorem 1 should be considered as a boundedness statement that does not lose derivatives. Conversely, the derivative loss in parts *ii*) is unavoidable and relates to the trapping effect on $\big(\mathcal{M}, g_M\big)$. Lastly, one can obtain higher pointwise bounds courtesy of Theorem 1 – see section 2.5 of the overview for a discussion towards each of these statements.

In addition, the assumption of compact support is merely a convenience for this thesis and can be readily removed – see Remark 31 in the bulk of the thesis.

Finally, we note a version of Theorem 1 regarding solutions to the Regge–Wheeler equation was originally given by Holzegel in[27] (see also[1]). Conversely, a version of Theorem 1 regarding solutions to the Zerilli equation was originally given in the independent works of the author[6] and Hung–Keller–Wang[30].

2.4.2 Theorem 2: Boundedness, decay and propagation of asymptotic flatness for solutions to the equations of linearised gravity in the modified Regge–Wheeler gauge

We continue with a rough statement of our second theorem which concerns a boundedness and decay statement for the initial-data-normalised solutions $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$ discussed in section 2.3.1.1 of the overview and which is a more precise version of the Theorem discussed in the introduction.

The precise statement of the theorem can be found in section 6.2 in the main body of the thesis.

The statement in question once again involves certain natural r-weighted energy norms on hypersurfaces which penetrate both the future event horizon and future null infinity and certain integrated decay norms over the region foliated by these hypersurfaces which

[24]Here we recall that $D^+(\Sigma)$ denotes the domain of dependence of Σ in $\big(\mathcal{M}, g_M\big)$.

are generalisations of the norms introduced in section 2.5 for scalar waves to solutions to the equations of linearised gravity (in particular, tensor fields on $\mathcal{M}$). See section 6.1 in the bulk of the paper for the full description.

A rough formulation of Theorem 2 is as follows.

In what follows, we shall employ the schematic notation of an estimate for a norm on the solution $\mathscr{S}$ to denote that said estimate holds, in that norm, for any quantity contained within the collection $\mathscr{S}$.

Theorem 2. *Let $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$ be a smooth solution to the equations of linearised gravity that is in the modified Regge–Wheeler gauge with parameters $\mathfrak{m}$ and $\mathfrak{a}$.*

We consider the projection

$$\overset{\diamond}{\mathscr{S}} := \overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}} - \mathscr{K}_{\mathfrak{m},\mathfrak{a}}$$

and assume that Cauchy data for the pure gauge and linearised Kerr invariant quantities $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ associated to the solution $\overset{\diamond}{\mathscr{S}}$ are compactly supported on Σ.

Then for any $n \geq 0$ the following estimates hold:

i) the higher order flux and weighted bulk estimates

$$\mathbb{F}^n[\overset{\diamond}{\mathscr{S}}] + \mathbb{I}^n[\overset{\diamond}{\mathscr{S}}] \lesssim \mathbb{D}^{n+2}[r^{-1}\overset{(1)}{\Phi}, r^{-1}\overset{(1)}{\Psi}].$$

ii) the higher order integrated decay estimate

$$\mathbb{M}^n[\overset{\diamond}{\mathscr{S}}] \lesssim \mathbb{D}^{n+3}[r^{-1}\overset{(1)}{\Phi}, r^{-1}\overset{(1)}{\Psi}].$$

Moreover, the norms in i) and ii) are such that now via a hierarchy of r-weighted estimates and an application of Sobolev embedding one can derive as a corollary to Theorem 2 the following pointwise decay bounds and statement of asymptotic flatness:

Corollary. *Let $\overset{\diamond}{\mathscr{S}}$ be as in Theorem 2. Then on $D^+(\Sigma)$ one has the r-weighted pointwise decay bound*

$$|r\overset{\diamond}{\mathscr{S}}| \lesssim \frac{1}{\sqrt{\tau^\star}} \cdot \mathbb{D}^6[r^{-1}\overset{(1)}{\Phi}, r^{-1}\overset{(1)}{\Psi}].$$

In particular, the solution $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$ of Theorem 2 decays inverse polynomially to the linearised Kerr solution $\mathscr{K}_{\mathfrak{m},\mathfrak{a}}$.

We make the following remarks regarding Theorem 2.

The flux estimate associated to the norm $\mathbb{F}$ in part *i*) of Theorem 2 should be considered as a boundedness statement that does not lose derivatives[25]. Conversely, the derivative loss in parts *ii*) is unavoidable and relates once more to the trapping effect on $\left(\mathcal{M}, g_M\right)$. Lastly, one can obtain higher pointwise bounds courtesy of Theorem 2 although we shall not state them explicitly in this thesis.

In addition, we note that if one is given a smooth solution $\mathscr{S}$ to the equations of linearised gravity arising from the smooth seed data set $\mathscr{D}_{\mathfrak{m},\mathfrak{a}}$, all quantities of which are compactly supported on Σ, then the initial-data-normalised solution constructed from it via the addition of the pure gauge solution $\overset{\diamond}{\mathscr{G}}$ as discussed in section 2.3.1.1 of the overview will satisfy the assumptions and hence the conclusions of Theorem 2. In particular, in light of the geometrical interpretation of pure gauge solutions given in section 2.1.4 of the overview, it follows that Theorem 2 indeed comprises the quantitative statement of linear stability for $\left(\mathcal{M}, g_M\right)$ as a solution to the Einstein vacuum equations (1.1) as expressed in a generalised f-wave gauge with respect to g_M. Here, f is the map discussed in section 2.1.3.

Moreover, we note that the assumption of compact support is merely a convenience for this thesis and can be readily removed.

Finally, we note a version of Theorem 2 regarding solutions to the system of equations that result from linearising the Einstein vacuum equations, as they are expressed in a generalised $\mathring{f}$-wave gauge with respect to g_M, around $\left(\mathcal{M}, g_M\right)$ was given by the author in[5]. Here, $\mathring{f}$ is the map of section 2.1.3. In particular, we note the analogous version of Theorem 2 proved in[5] had weaker r-weights in the norms of parts *i*) and *ii*) and as a consequence the r-weight in the pointwise norm of the above Corollary had to be weakened to $r^{-\frac{1}{2}}$.

2.5 Aside: The scalar wave equation on the Schwarzschild exterior spacetime

In this section of the overview we make an aside to discuss the scalar wave equation $\Box_{g_M}\psi = 0$ on $\left(\mathcal{M}, g_M\right)$ and the methods by which one establishes a decay statement for solutions thereof. Indeed, insights gained for this simpler problem will prove fundamental in establishing Theorems 1 and 2 of section 2.4 in the overview.

[25] In particular, note from Proposition 4.1 in the bulk of the thesis that the linearised metric is at the level of two derivatives of $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$.

2.5.1 Boundedness and decay for solutions to the scalar wave equation on $\left(\mathcal{M}, g_M\right)$

Let ψ be a smooth solution to the scalar wave equation on Schwarzschild:

$$\Box_{g_M}\psi = 0. \tag{2.25}$$

We associate to ψ the flux norms (for $R >> 10M$):

$$\begin{aligned}\mathbb{F}[\psi] := &\sup_{\tau^\star\geq\tau_0^\star}\int_{\Xi_{\tau^\star}\cap\{r\leq R\}}\left(|\partial_{t^*}\psi|^2+|\partial_r\psi|^2+|\nabla\!\!\!/\psi|^2\right)\mathrm{d}r\,\mathring{\epsilon}\\ &+\sup_{\tau^\star\geq\tau_0^\star}\int_{\Xi_{\tau^\star}\cap\{r\geq R\}}\left(r^2|D(r\psi)|^2+|\nabla\!\!\!/(r\psi)|^2\right)\mathrm{d}r\,\mathring{\epsilon},\end{aligned} \tag{2.26}$$

$$\mathbb{D}[\psi] := \int_\Sigma r^2\left(|\partial_{t^*}(r\psi)|^2+|\partial_r(r\psi)|^2+|\nabla\!\!\!/(r\psi)|^2\right)\mathrm{d}r\,\mathring{\epsilon} \tag{2.27}$$

along with the integrated decay norms (for $1>\beta_0>>0$):

$$\mathbb{I}[\psi] := \int_{\tau_0^\star}^\infty\int_{\Xi_{\tau^\star}\cap\{r\geq R\}}\left(r|D(r\psi)|^2+r^{\beta_0}|\nabla\!\!\!/(r\psi)|^2\right)\mathrm{d}\tau^\star\,\mathrm{d}r\,\mathring{\epsilon}, \tag{2.28}$$

$$\mathbb{M}[\psi] := \int_{\tau_0^\star}^\infty\int_{\Xi_{\tau^\star}} r^{-3}\left(|\partial_{t^*}(r\psi)|^2+|\partial_r(r\psi)|^2+|r\nabla\!\!\!/(r\psi)|^2+|(r\psi)|^2\right)\mathrm{d}\tau^\star\,\mathrm{d}r\,\mathring{\epsilon}. \tag{2.29}$$

Here, $\mathring{\epsilon}$ is the volume form on the unit round sphere, D is the derivative operator

$$D := \frac{1+\frac{2M}{r}}{1-\frac{2M}{r}}\partial_{t^*}+\partial_r$$

and $\Xi_{\tau^\star}$ is the level set of the function

$$\tau^\star := \begin{cases} t^* & r\leq R\\ u & r\geq R,\end{cases}$$

where u is the optical function $u := t^*-r-4M\log(r-2M)+R+4M\log(R-2M)$. In particular, D is the derivative operator associated to the null vector $L := -(\mathrm{d}u)^\sharp$ and thus the flux norms (2.26) and (2.27) denote energy norms containing all tangential and normal derivatives to $\Xi_{\tau^\star}$ and Σ.

We note that the above defined norms are the same that appear in the statement of Theorem 1 whereas they are the scalar wave prototypes of those found in the statement of Theorem 2. Indeed, one has the following theorem regarding solutions to (2.25), the complete statement of which is due to Dafermos and Rodnianski and which we note is the scalar wave analogue of Theorems 1 and 2 in section 2.4 of the overview.

Theorem (Dafermos–Rodnianski –[34],[35],[37])**.** *Let ψ be a smooth solution to* (2.25).

Then for any $n \geq 0$ the following estimates hold, provided that the fluxes on the right hand side are finite.

i) *the higher order flux and weighted bulk estimates*

$$\mathbb{F}^n[\psi] + \mathbb{I}^n[\psi] \lesssim \mathbb{D}^n[\psi]. \tag{2.30}$$

ii) *the higher order integrated decay estimate*

$$\mathbb{M}^n[\psi] \lesssim \mathbb{D}^{n+1}[\psi]. \tag{2.31}$$

iii) *the higher order pointwise decay bounds for $i + j + k + l \leq n$*

$$|\partial_{t^*}^i \partial_r^j (r \nabla\!\!\!/)^k \Big((r - 2M)D\Big)^l \Big(r\psi\Big)| \lesssim \frac{1}{\sqrt{\tau^\star}} \mathbb{D}^{n+4}[\psi].$$

Here, the above are natural higher order norms defined by replacing ψ in (2.26)-(2.29) with the appropriate derivatives (see section 6.1 in the bulk of the thesis for the precise definition).

The proof of the $n = 0$ case of the above Theorem as given by Dafermos and Rodnianski relies on the following two key estimates for solutions to (2.25) and for any $\tau_2^\star \geq \tau_1^\star$:

$$\begin{aligned}
&\int_{\Xi_{\tau_2^\star} \cap \{r \leq R\}} \Big(|\partial_{t^*}\psi|^2 + |\partial_r \psi|^2 + |\nabla\!\!\!/ \psi|^2\Big) r^2 \, \mathrm{d}r \, \mathring{\epsilon} + \int_{\Xi_{\tau_2^\star} \cap \{r \geq R\}} \Big(|D\psi|^2 + |\nabla\!\!\!/ \psi|^2\Big) r^2 \, \mathrm{d}r \, \mathring{\epsilon}, \\
\lesssim &\int_{\Xi_{\tau_1^\star} \cap \{r \leq R\}} \Big(|\partial_{t^*}\psi|^2 + |\partial_r \psi|^2 + |\nabla\!\!\!/ \psi|^2\Big) \, \mathrm{d}r \, \mathring{\epsilon} + \int_{\Xi_{\tau_1^\star} \cap \{r \geq R\}} \Big(|D\psi|^2 + |\nabla\!\!\!/ \psi|^2\Big) r^2 \, \mathrm{d}r \, \mathring{\epsilon}
\end{aligned} \tag{2.32}$$

and

$$\begin{aligned}
&\int_{\tau_1^\star}^{\infty} \int_{\Xi_{\tau^\star}} r^{-3} \Big(|\partial_{t^*}(r\psi)|^2 + |\partial_r (r\psi)|^2 + |r \nabla\!\!\!/ (r\psi)|^2 + |(r\psi)|^2\Big) \, \mathrm{d}\tau^\star \, \mathrm{d}r \, \mathring{\epsilon} \\
\lesssim &\sum_{i=0}^{1} \Bigg(\int_{\Xi_{\tau_1^\star} \cap \{r \leq R\}} \Big(|\partial_{t^*} \partial_{t^*}^i \psi|^2 + |\partial_r \partial_{t^*}^i \psi|^2 + |\nabla\!\!\!/ \partial_{t^*}^i \psi|^2\Big) \, \mathrm{d}r \, \mathring{\epsilon} \\
&+ \int_{\Xi_{\tau_1^\star} \cap \{r \geq R\}} \Big(|D \partial_{t^*}^i \psi|^2 + |\nabla\!\!\!/ \partial_{t^*}^i \psi|^2\Big) r^2 \, \mathrm{d}r \, \mathring{\epsilon} \Bigg).
\end{aligned} \tag{2.33}$$

Indeed, in[35] Dafermos and Rodnianski developed a very robust method which takes as input the estimates (2.32)-(2.33) and returns, via a hierarchy of r-weighted estimates on the r-weighted quantity $r\psi$, the estimates $i) - iii)$ of the Theorem statement, where for the latter pointwise bounds one in addition appeals to a Sobolev embedding. Consequently, we shall see in the following two sections that establishing the estimates (2.32) and (2.33) requires an intimate understanding of the geometry of $\Big(\mathcal{M}, g_M\Big)$, in particular how the celebrated red-shift effect and the presence of trapped null geodesics (see[22]) effects the propagation of waves.

The higher order estimates will then be discussed in section 2.5.4.

2.5.2 The degenerate energy and red-shift estimates

To investigate how one proves such estimates it is expedient to introduce the stress-energy tensor

$$\mathbb{T}[\psi] := \mathrm{d}\psi \otimes_{\mathrm{s}} \mathrm{d}\psi - g_M \, |\,\mathrm{d}\psi|^2_{g_M}$$

where $\otimes_{\mathrm{s}}$ denotes the symmetrised tensor product, d is the exterior derivative on $\mathcal{M}$ and $|\,\mathrm{d}\psi|^2_{g_M}$ denote the 'norm' of the 1-form $\mathrm{d}\psi$ with respect g_M. Then one has the following positivity properties at any $p \in \mathcal{M}$ and for vector fields X, Y on $\mathcal{M}$:

1) if $g_M(X,Y)\big|_p < 0$ and X, Y are future-directed[26] then $\mathbb{T}[\psi](X,Y)\big|_p$ bounds all derivatives of ψ at p
2) if $g_M(X,Y)\big|_p \leq 0$ X, Y are future-directed then $0 \leq \mathbb{T}[\psi](X,Y)\big|_p$

Moreover, if ψ in addition satisfies (2.25) then

$$\operatorname{div} \mathbb{T}[\psi] = 0.$$

where div is the divergence operator associated to g_M.

Defining the 1-form $\mathbb{J}^X[\psi] := \mathbb{T}[\psi](X, \cdot)$ where X is a causal vector field on $\mathcal{M}$ Stokes Theorem (on a manifold with corners) therefore yields, for ψ a solution to (2.25), the inequality

$$\begin{aligned} &\int_{\Xi_{\tau_2^\star}} \mathbb{J}^X[\psi](n_{\Xi_{\tau_2^\star}}) \, \mathrm{dVol}(\Xi_{\tau_2^*}) + \int_{\tau_1^\star}^{\tau_2^\star} \int_{\Xi_{\tau^\star}} \frac{1}{2} \mathcal{L}_X g_M \cdot \mathbb{T}[\psi] \, \mathrm{dVol}(\Xi_{\tau^\star}) \, \mathrm{d}\tau^\star \, \mathrm{d}r \, \mathring{\epsilon} \\ \lesssim &\int_{\Xi_{\tau_1^\star}} \mathbb{J}^X[\psi](n_{\Xi_{\tau_1^\star}}) \, \mathrm{dVol}(\Xi_{\tau_1^\star}) \, \mathrm{d}r \, \mathring{\epsilon} \end{aligned} \tag{2.34}$$

where $n_{\Xi_{\tau^\star}}$ is a suitably interpreted normal to the hypersurface $\Xi_{\tau^\star}$ and we have discarded the flux term on $\mathcal{H}^+$ as this is positive-definite by property 2).

In particular, as the vector field ∂_{t^*} is causal and Killing we have from (2.34) the

[26] With respect to the time-orientation provided by ∂_{t^*}.

degenerate energy estimate

$$
\begin{aligned}
&\int_{\Xi_{\tau_2^\star}\cap\{r\le R\}} \Big(|\partial_{t^*}\psi|^2 + \Big(1-\tfrac{2M}{r}\Big)|\partial_r\psi|^2 + |\nabla\!\!\!\!/\,\psi|^2\Big)\,\mathrm{d}r\,\mathring{\epsilon} \\
&+ \int_{\Xi_{\tau_2^\star}\cap\{r\ge R\}} \Big(|D\psi|^2 + |\nabla\!\!\!\!/\,\psi|^2\Big) r^2\,\mathrm{d}r\,\mathring{\epsilon}, \\
&\lesssim \int_{\Xi_{\tau_1^\star}\cap\{r\le R\}} \Big(|\partial_{t^*}\psi|^2 + \Big(1-\tfrac{2M}{r}\Big)|\partial_r\psi|^2 + |\nabla\!\!\!\!/\,\psi|^2\Big)\,\mathrm{d}r\,\mathring{\epsilon} \\
&+ \int_{\Xi_{\tau_1^\star}\cap\{r\ge R\}} \Big(|D\psi|^2 + |\nabla\!\!\!\!/\,\psi|^2\Big) r^2\,\mathrm{d}r\,\mathring{\epsilon}
\end{aligned}
\tag{2.35}
$$

where the degeneration in the transversal derivative ∂_r to $\mathcal{H}^+$ occurs due to the fact that ∂_{t^*} is null there (cf. property 2)). The first estimate (2.32) thus follows from the estimate (2.35) if the weights at $\mathcal{H}^+$ can be improved and this improvement was achieved by Dafermos and Rodnianski in[37] where they established the existence of a time-like vector field N which satisfies the following so-called red-shift estimate in a neighbourhood of $\mathcal{H}^+$:

$$
\mathbb{J}^N[\psi](n_{\Xi_{\tau^\star}}) \lesssim \mathcal{L}_N g_M \cdot \mathbb{T}[\psi] \tag{2.36}
$$

where we note that the existence of such a vector field N is intimately related to the celebrated red-shift effect on $\big(\mathcal{M}, g_M\big)$ – see[37]. Moreover, noting that the left hand side of (2.36) controls all derivatives of ψ by property 2) (and the fact that $\Xi_{\tau^\star}$ and the spacelike hypersurface $\Sigma_{\tau^\star}$ coincide near $\mathcal{H}^+$), the estimate (2.36) when combined with the degenerate estimate (2.35) and the integral inequality (2.34) ultimately suffices to establish the desired estimate (2.32).

2.5.3 Integrated local energy decay and the role of trapping

In order to establish the estimate (2.33) it is convenient to exploit once more the formalism of the previous section. Indeed, revisiting the integral inequality (2.34) one has the aim of choosing a vector field X so as to generate a bulk term which controls all derivatives of ψ.

Now, it turns out (see[37]) that one can use estimate (2.34) with X now being space-like[27], in conjunction with both the estimate (2.32) and the red-shift estimate (2.36), to establish

[27]The additional flux term this generates at $\mathcal{H}^+$ in the application of Stokes theorem is in fact controllable by the flux term of $\mathcal{H}^+$ associated to the $X = \partial_{t^*}$ estimate of section 2.5.2 that was discarded in the estimate (2.34).

the bound

$$\int_{\tau_1^\star}^\infty \int_{\Xi_{\tau^\star}} r^{-3}\bigg(\Big(1-\tfrac{3M}{r}\Big)^2\Big(|\partial_{t^*}(r\psi)|^2+|\partial_r(r\psi)|^2+|\nabla\!\!\!/(r\psi)|^2\Big)+|(r\psi)|^2\bigg)\,\mathrm{d}\tau^\star\,\mathrm{d}r\,\mathring{\epsilon}$$
$$\lesssim \int_{\Xi_{\tau_1^\star}\cap\{r\le R\}}\Big(|\partial_{t^*}\psi|^2+|\partial_r\psi|^2+|\nabla\!\!\!/\psi|^2\Big)\,\mathrm{d}r\,\mathring{\epsilon}+\int_{\Xi_{\tau_1^\star}\cap\{r\ge R\}}\Big(|D\psi|^2+|\nabla\!\!\!/\psi|^2\Big)r^2\,\mathrm{d}r\,\mathring{\epsilon} \tag{2.37}$$

where the degeneration at $r=3M$ is a necessary consequence of the existence of trapped[28] null geodesics at $r=3M$ on $\big(\mathcal{M},g_M\big)$ and a general result due to Sbierski[61]. However, to provide a bulk estimate that does not degenerate at $r=3M$ it in fact suffices to obtain the estimate

$$\begin{aligned}&\int_{\tau_1^\star}^\infty \int_{\Xi_{\tau^\star}} r^{-3}|(r\partial_{t^*}\psi)|^2\,\mathrm{d}\tau^\star\,\mathrm{d}r\,\mathring{\epsilon}\\ &\lesssim \int_{\Xi_{\tau_1^\star}\cap\{r\le R\}}\Big(|\partial_{t^*}\partial_{t^*}\psi|^2+|\partial_r\partial_{t^*}\psi|^2+|\nabla\!\!\!/\partial_{t^*}\psi|^2\Big)\,\mathrm{d}r\,\mathring{\epsilon}\\ &\quad+\int_{\Xi_{\tau_1^\star}\cap\{r\ge R\}}\Big(|D\partial_{t^*}\psi|^2+|\nabla\!\!\!/\partial_{t^*}\psi|^2\Big)r^2\,\mathrm{d}r\,\mathring{\epsilon}\end{aligned}$$

which follows easily from (2.37) and the fact that ∂_{t^*} is Killing and therefore commutes with the wave operator $\Box_{g_M}$. This consequently yields the estimate (2.33) and moreover explains the derivative loss in the statement of the Theorem[29] .

2.5.4 Higher order estimates

With the key ingredients for the proving the $n=0$ case of the Theorem understood we turn now to the higher order cases.

Indeed, we first observe that since ∂_{t^*} and Ω_i for $i=1,2,3$ are Killing fields of $\big(\mathcal{M},g_M\big)$, where Ω_i denote a basis of $SO(3)$, then ∂_{t^*} and each of the Ω_i commute trivially with the wave operator $\Box_{g_M}$ and thus the $n=0$ case of the Theorem holds replacing ψ by[30] $\partial_{t^*}^i(r\nabla\!\!\!/)^j\psi$ for any positive integers i,j. In addition, by writing the wave equation for ψ as an ODE in r with inhomogeneities given by derivatives of ψ containing at least one t^* or angular derivative, then the previously derived bounds on $\partial_{t^*}^i(r\nabla\!\!\!/)^j\psi$ in fact allows one to replace ψ by $\partial_{t^*}^i\Big(\big(1-\frac{2M}{r}\big)\partial_r\Big)^j(r\nabla\!\!\!/)^k\Big(\big(1-\frac{2M}{r}\big)D\Big)^l\psi$ in the $n=0$ case of the Theorem statement, for any positive integers i,j,k,l. It thus remains to remove the degeneration at $\mathcal{H}^+$ for the derivative ∂_r and the degeneration towards $\mathcal{I}^+$ for the derivative operator rD (cf. the pointwise bounds in part *iii*) of the Theorem statement).

[28]Null geodesics which remain tangent to the hypersurface $r=3M$ for all t^*.

[29]Although this derivative loss can be improved[63] some degree of loss is required due to the result of Sbierski.

[30]Here we use that $\sum_{i=0}^3|\Omega_i(\psi)|^2\lesssim|r\nabla\!\!\!/\psi|^2\lesssim\sum_{i=0}^3|\Omega_i(\psi)|^2$.

Consequently, to remove the degeneration at $\mathcal{H}^+$ one proceeds by first commuting the wave equation (2.25) with the (time-like) vector field $-\partial_r^m$ for any positive integer m, thus generating additional lower order terms as the vector field $-\partial_r$ is not Killing. However, these lower order terms are such that they are either controlled by the bounds derived on the quantities $\partial_{t^*}^i\Big(\big(1-\frac{2M}{r}\big)\partial_r\Big)^j(r\nabla\!\!\!/)^k\Big(\big(1-\frac{2M}{r}\big)D\Big)^l\psi$ in the previous step for sufficiently many i,j,k,l or they come with the correct sign. In particular, for any positive integer j, one has the higher order red-shift estimate[31]

$$\mathbb{J}^N[\partial_r^j\psi](n_{\Xi_{\tau^\star}}) \lesssim \mathcal{L}_N g_M\cdot\mathbb{T}[\partial_r^j\psi] - \{\text{controllable terms}\}.$$

Proceeding as in section 2.5.2 one thus removes the degeneration at $\mathcal{H}^+$ for the derivative ∂_r – see[37] for further details.

Similarly, to remove the degeneration towards $\mathcal{I}^+$ one proceeds by now considering the wave equation satisfied by the r-weighted commuted quantity $\big((r-2M)D\big)^m(r\psi)$ for any positive integer m. The error terms this generates are lower order in the sense that they are either controllable by the estimates derived on the quantities $\partial_{t^*}^i\partial_r^j(r\nabla\!\!\!/)^k\Big(\big(1-\frac{2M}{r}\big)D\Big)^l\psi$ in the previous two steps for sufficiently many i,j,k,l or they come with favourable weights in the sense that the hierarchy of r-weighted estimates established by Dafermos and Rodnianski for the scalar wave $r\psi$ hold with equal validity for the commuted quantity $\big((r-2M)D\big)^m(r\psi)$ – see[37] (and also[67]) for further details.

2.5.5 Outline of the proofs of the main theorems

In this final part of the overview we discuss the proofs of Theorems 1 and 2 of section 2.4 in the overview.

This part of the overview corresponds to Chapter 7 in the main body of the thesis.

2.5.5.1 Outline of the proof of Theorem 1

We begin by discussing the proof of Theorem 1.

This section of the overview corresponds to section 7.1 in the main body of the thesis.

We start with the observation that one can write the Regge–Wheeler and Zerilli equations

[31]This presentation of the higher order red-shift estimate is borrowed from the overview of[1].

(2.18) and (2.19) respectively as[32]

$$\Box_{g_M}(r^{-1}\Phi) = -\frac{8}{r^2}\frac{M}{r}r^{-1}\Phi \tag{2.38}$$

and

$$\Box_{g_M}(r^{-1}\Psi) = -\frac{8}{r^2}\frac{M}{r}r^{-1}\Psi + \frac{24}{r^3}\frac{M}{r}(r-3M)\not{\zeta}^{[1]}r^{-1}\Psi + \frac{72}{r^3}\frac{M}{r}\frac{M}{r}(r-2M)\not{\zeta}^{[2]}r^{-1}\Psi. \tag{2.39}$$

Thus, the Regge–Wheeler and Zerilli equations differ from the scalar wave equation (2.25) by an r-weight and the presence of the lower order terms on the right hand side of (2.38) and (2.39) respectively. Consequently, all insights gained for the scalar wave equation in section 2.5 of the overview enter and it remains to understand the complications, if any, provided by these additional lower order terms.

These complications can in fact be understood at the level of the ∂_{t^*}-flux estimate of section 2.5.2 in the overview. Indeed, introducing the stress-energy tensors associated to $r^{-1}\Phi$ and $r^{-1}\Psi$ as

$$\mathbb{T}[r^{-1}\Phi] = \mathrm{d}(r^{-1}\Phi)\otimes_{\mathrm{s}}\mathrm{d}(r^{-1}\Phi) - g_M\,|\,\mathrm{d}(r^{-1}\Phi)|^2_{g_M}$$

and

$$\mathbb{T}[r^{-1}\Psi] = \mathrm{d}(r^{-1}\Psi)\otimes_{\mathrm{s}}\mathrm{d}(r^{-1}\Psi) - g_M\,|\,\mathrm{d}(r^{-1}\Psi)|^2_{g_M}$$

then it follows from (2.38) and (2.39) that

$$\operatorname{div}\mathbb{T}[r^{-1}\Phi] = -\frac{8}{r^2}\frac{M}{r}r^{-1}\Phi\ \mathrm{d}(r^{-1}\Phi) \tag{2.40}$$

and

$$\begin{aligned}\operatorname{div}\mathbb{T}[r^{-1}\Psi] = \Bigg(-\frac{8}{r^2}\frac{M}{r}r^{-1}\Psi + \frac{24}{r^3}\frac{M}{r}(r-3M)\not{\zeta}^{[1]}r^{-1}\Psi \\ + \frac{72}{r^3}\frac{M}{r}\frac{M}{r}(r-2M)\not{\zeta}^{[2]}r^{-1}\Psi\Bigg)\ \mathrm{d}(r^{-1}\Psi).\end{aligned} \tag{2.41}$$

Applying Stokes theorem as in section 2.5.2 of the overview with $X = \partial_{t^*}$ (noting that the positivity properties 1) and 2) hold for $\mathbb{T}[r^{-1}\Phi]$ and $\mathbb{T}[r^{-1}\Psi]$ so that the flux term

[32] We remark that it is quite interesting that the Regge–Wheeler and Zerilli equations are so closely related to the scalar wave equation.

along $\mathcal{H}^+$ can be ignored) we therefore find

$$\begin{aligned}
&\int_{\Xi_{\tau_2^\star}\cap\{r\leq R\}}\Big(|\partial_{t^*}\Phi|^2+\big(1-\tfrac{2M}{r}\big)|\partial_r\Phi|^2+|\nabla\!\!\!/_{\Upsilon}\Phi|^2\Big)\,\mathrm{d}r\,\mathring{\epsilon}\\
+&\int_{\Xi_{\tau_2^\star}\cap\{r\geq R\}}\Big(|D\Phi|^2+|\nabla\!\!\!/_{\Upsilon}\Phi|^2\Big)\,\mathrm{d}r\,\mathring{\epsilon},\\
\lesssim&\int_{\Xi_{\tau_1^\star}\cap\{r\leq R\}}\Big(|\partial_{t^*}\Phi|^2+\big(1-\tfrac{2M}{r}\big)|\partial_r\Phi|^2+|\nabla\!\!\!/\Phi|^2\Big)\,\mathrm{d}r\,\mathring{\epsilon}+\int_{\Xi_{\tau_1^\star}\cap\{r\geq R\}}\Big(|D\Phi|^2+|\nabla\!\!\!/\Phi|^2\Big)\,\mathrm{d}r\,\mathring{\epsilon}
\end{aligned}\tag{2.42}$$

and

$$\begin{aligned}
&\int_{\Xi_{\tau_2^\star}\cap\{r\leq R\}}\Big(|\partial_{t^*}\Psi|^2+\big(1-\tfrac{2M}{r}\big)|\partial_r\Psi|^2+|\nabla\!\!\!/_{\Upsilon+\mathfrak{z}}\Psi|^2\Big)\,\mathrm{d}r\,\mathring{\epsilon}\\
+&\int_{\Xi_{\tau_2^\star}\cap\{r\geq R\}}\Big(|D\Psi|^2+|\nabla\!\!\!/_{\Upsilon+\mathfrak{z}}\Psi|^2\Big)\,\mathrm{d}r\,\mathring{\epsilon},\\
\lesssim&\int_{\Xi_{\tau_1^\star}\cap\{r\leq R\}}\Big(|\partial_{t^*}\Psi|^2+\big(1-\tfrac{2M}{r}\big)|\partial_r\Psi|^2+|\nabla\!\!\!/\Psi|^2\Big)\,\mathrm{d}r\,\mathring{\epsilon}+\int_{\Xi_{\tau_1^\star}\cap\{r\geq R\}}\Big(|D\Psi|^2+|\nabla\!\!\!/\Psi|^2\Big)\,\mathrm{d}r\,\mathring{\epsilon}.
\end{aligned}\tag{2.43}$$

Here, we have defined

$$|\nabla\!\!\!/_{\Upsilon}\Phi|^2:=|\nabla\!\!\!/\Phi|^2-\frac{1}{r^2}|\Phi|^2-\frac{6}{r^2}\frac{M}{r}|\Phi|^2$$

and

$$\begin{aligned}
|\nabla\!\!\!/_{\Upsilon+\mathfrak{z}}\Psi|^2:=|\nabla\!\!\!/\Psi|^2-\frac{1}{r^2}|\Psi|^2-\frac{6}{r^2}\frac{M}{r}|\Psi|^2&-\frac{24}{r^3}\frac{M}{r}(r-3M)|r\nabla\!\!\!/\not{\zeta}^{[1]}\Psi|^2\\
&+\frac{24}{r^4}\frac{M}{r}\Big(2(r-3M)^2+3M(r-2M)\Big)|\not{\zeta}^{[1]}\Psi|^2.
\end{aligned}$$

In addition, we have integrated by parts to write the additional bulk terms generated from (2.40) and (2.41) as a flux term. In particular, we note the integration by parts formulae associated to the operator $\not{\zeta}^{[p]}$ of Lemma 3.14 in the bulk of thesis. Positivity of the left hand side of the ∂_{t^*}-fluxes (2.42) and (2.43), which we recall was immediate for the case of the scalar wave equation (cf. estimate (2.35)), thus rests upon whether one can ensure positivity of the terms $|\nabla\!\!\!/_{\Upsilon}\Phi|^2$ and $|\nabla\!\!\!/_{\Upsilon}\Psi|^2$.

To see that this is indeed the case we invoke the fact that Φ and Ψ are supported on the $l\geq 2$ spherical harmonics. One thus has on any 2-sphere $S^2_{\tau^\star,r}\subset\mathcal{M}$ given as the intersection of the level sets of the functions $\tau^\star$ and r the Poincarè inequality (see Lemma 3.9 in the bulk of the thesis)

$$\frac{6}{r^2}\int_{S^2_{\tau^\star,r}}|\Phi|^2\,\mathring{\epsilon}\lesssim\int_{S^2_{\tau^\star,r}}|\nabla\!\!\!/\Phi|^2\,\mathring{\epsilon}\tag{2.44}$$

and

$$\frac{6}{r^2}\int_{S^2_{\tau^\star,r}}|\Psi|^2\,\mathring{\epsilon}\lesssim\int_{S^2_{\tau^\star,r}}|\nabla\!\!\!/\Psi|^2\,\mathring{\epsilon}. \tag{2.45}$$

The estimate (2.44) thus already suffices to establish positivity of the left hand side of (2.42). In particular, one has the bound

$$\int_{S^2_{\tau^\star,r}}|\nabla\!\!\!/\Phi|^2\,\mathring{\epsilon}\lesssim\int_{S^2_{\tau^\star,r}}|\nabla\!\!\!/_{\Upsilon}\Phi|^2\,\mathring{\epsilon}.$$

For the positivity of (2.43) we first note the refined estimate of Corollary 3.13 in the bulk of the thesis:

$$\frac{2}{r^2}\int_{S^2_{\tau^\star,r}}\left(2(r-3M)^2|\not{\zeta}^{[1]}\Psi|^2+r(r+9M)|r\nabla\!\!\!/\not{\zeta}^{[1]}\Psi|^2\right)\mathring{\epsilon}\lesssim\int_{S^2_{\tau^\star,r}}|\Psi|^2\,\mathring{\epsilon}.$$

This estimate combined with the estimate (2.45) thus ultimately yields positivity of the left hand side of (2.43). In particular, one has the bound

$$\int_{S^2_{\tau^\star,r}}|\nabla\!\!\!/\Psi|^2\,\mathring{\epsilon}\lesssim\int_{S^2_{\tau^\star,r}}|\nabla\!\!\!/_{\Upsilon+\mathfrak{z}}\Psi|^2\,\mathring{\epsilon}.$$

The degenerate energy estimate of section 2.5.2 for the scalar wave ψ thus holds for the r-weighted solutions to the Regge–Wheeler and Zerilli $r^{-1}\Phi$ and $r^{-1}\Psi$ respectively.

Consequently, arguing similarly as in the remainder of section 2.5.2 and sections 2.5.3-2.5.4 of the overview, in particular using the estimates (2.44)-(2.45) combined with the estimates and integration by parts formulae of sections 3.2.6.3-3.2.6.4 in the bulk of the thesis to control the lower order terms appearing in the Regge–Wheeler and Zerilli equations as was done above for the ∂_{t^*}-flux estimate, ultimately yields the Theorem of section 2.5 with ψ replaced by $r^{-1}\Phi$ and $r^{-1}\Psi$ respectively. In particular, we emphasize that the techniques developed by Dafermos and Rodnianski in[35] to derive the aforementioned hierarchy of r-weighted estimates are indeed robust enough to allow for the lower order terms appearing in the Regge–Wheeler and Zerilli equations respectively.

Indeed, the above procedure is explicitly carried out in section 7.1 in the bulk of the thesis although we note that in carrying out the proof of Theorem 1 we prefer to analyse the Regge–Wheeler and Zerilli equations directly as opposed to passing to the wave equation by weighting in r. This is due to the fact that the operator $\tilde{\Box}+\Delta\!\!\!/$ is actually more symmetric than the operator $\Box$ and this makes some of the constructions involved in the proof of Theorem 1 simpler[33].

[33] In particular, one should compare the 'multiplier' used to establish the (degenerate) integrated local energy decay estimate of Proposition 7.7 for solutions to the Regge–Wheeler and Zerilli equations in the bulk of the thesis with the 'multiplier' used to establish the analogous estimate for solutions to the scalar wave equation in[37].

2.5.5.2 Outline of the proof of Theorem 2

We finish the overview by discussing the proof of Theorem 2.

This section of the overview corresponds to section 7.2 in the main body of the thesis.

We recall from section 2.3.1.1 that the solution $\overset{\diamond}{\mathscr{S}}$ to the equations of linearised in the statement of Theorem 2 satisfies

$$
\begin{aligned}
\overset{\scriptscriptstyle(1)}{\hat{\tilde{g}}} &= \widetilde{\nabla}\hat{\otimes}\tilde{\mathrm{d}}^{\mathcal{I}}\Big(r\overset{\scriptscriptstyle(1)}{\Psi}\Big) + 6\mu\,\mathrm{d}r\hat{\tilde{\otimes}}\not{\zeta}^{[1]}\tilde{\mathrm{d}}\overset{\scriptscriptstyle(1)}{\Psi},\\
\overset{\scriptscriptstyle(1)}{\tilde{g}} &= 0,\\
\overset{\scriptscriptstyle(1)}{\not{g}} &= \not{\mathcal{D}}_1^\star\Big(\tilde{\mathrm{d}}^{\mathcal{I}}\Big(r\overset{\scriptscriptstyle(1)}{\Psi}\Big) - 2\tilde{\mathrm{d}}r\,\overset{\scriptscriptstyle(1)}{\Psi}, \tilde{\mathrm{d}}^{\mathcal{I}}\Big(r\overset{\scriptscriptstyle(1)}{\Phi}\Big) - 2\tilde{\mathrm{d}}r\,\overset{\scriptscriptstyle(1)}{\Phi}\Big),\\
\overset{\scriptscriptstyle(1)}{\hat{\not{g}}} &= r\not{\nabla}\hat{\otimes}\not{\mathcal{D}}_1^\star\Big(\overset{\scriptscriptstyle(1)}{\Psi}, \overset{\scriptscriptstyle(1)}{\Phi}\Big),\\
\not{\mathrm{tr}}\overset{\scriptscriptstyle(1)}{\not{g}} &= 4\tilde{\mathrm{d}}^{\mathcal{I}}_P\overset{\scriptscriptstyle(1)}{\Psi} + 12\mu r^{-1}(1-\mu)\not{\zeta}^{[1]}\overset{\scriptscriptstyle(1)}{\Psi}
\end{aligned}
$$

where $\overset{\scriptscriptstyle(1)}{\Phi}$ and $\overset{\scriptscriptstyle(1)}{\Psi}$ satisfy the Regge–Wheeler and Zerilli equations respectively. Since in addition $\overset{\scriptscriptstyle(1)}{\Phi}$ and $\overset{\scriptscriptstyle(1)}{\Psi}$ were assumed to have compactly supported Cauchy data on Σ then we in fact see that Theorem 2 follows almost immediately from the higher order estimates of Theorem 1. Indeed, it remains simply to verify that one obtains the correct r-weighted bounds for the solution $\overset{\diamond}{\mathscr{S}}$ as demanded by the theorem statement. However, this follows as a consequence of the special property of the derivative operator $\tilde{\mathrm{d}}^{\mathcal{I}}$ which satisfies (see Lemma 3.3 in the bulk of the thesis) for a smooth function f on $\mathcal{M}$

$$\tilde{\mathrm{d}}^{\mathcal{I}}_X f = 2g_M(X, \underline{L})\,Df$$

for a X a vector field on $\mathcal{M}$ and $\underline{L}$ the unique null vector conjugate to the null vector L of section 2.5.1 in the overview: $g_M(\underline{L}, L) = 2$. In particular, recalling from section 2.5.4 in the overview (applied now to solutions of the Regge–Wheeler and Zerilli equations) that commuting with the operator D gains a weight in r, the statement of Theorem 2 follows.

Finally, we emphasize that although the proof of Theorem 2 does indeed follow quite readily from Theorem 1 this is precisely a consequence of what preceeded it. In particular, we highlight the crucial role played by the choice of the map f that defined the equations of linearised gravity in section 2.1.3 of the overview.

So ends our detailed overview regarding the linear stability of the Schwarzschild exterior family as solutions to the Einstein vacuum solutions when expressed in a generalised wave gauge and thus the thesis shall now begin in earnest.

3

The equations of linearised gravity around Schwarzschild

This first chapter of the thesis is concerned with both the derivation of the equations of linearised gravity around Schwarzschild and the identification of two special classes of solutions to said equations.

We begin in section 3.1 by defining the notion of a generalised wave gauge on an abstract Lorentzian manifold and then present the form of the Einstein vacuum equations as expressed in such a gauge.

Then in section 3.2 we define the Schwarzschild exterior solution as well as introducing various mathematical operations and objects that shall play an important role throughout the thesis.

Then in section 3.3 we formally linearise the equations of section 3.1 about the Schwarzschild exterior solution of section 3.2. Included in particular is the generalised wave gauge that we are to impose in the equations of section 3.1.

Finally, in section 3.4 the two special classes of solutions to the equations of linearised gravity corresponding to the pure gauge and linearised Kerr solutions are introduced.

3.1 The Einstein vacuum equations in a generalised wave gauge

In this section we introduce the notion of a generalised wave gauge on an abstract Lorentzian manifold and then review the structure of the Einstein vacuum equations when expressed in such a gauge.

It is these equations that we shall linearise about a fixed Schwarzschild exterior solution in section 3.3.

3.1.1 The generalised wave gauge

In this section we provide the definition of a generalised wave gauge on an abstract Lorentzian manifold as it is found in[12].

We note that the specification of such a gauge plays a vital role in the works[39] and[42].

Let $\left(\boldsymbol{\mathcal{M}}, \boldsymbol{g}\right)$ and $\left(\boldsymbol{\mathcal{M}}, \overline{\boldsymbol{g}}\right)$ be $3+1$ Lorentzian manifolds with $\boldsymbol{f} : T^2(\boldsymbol{\mathcal{M}}) \to T^1(\boldsymbol{\mathcal{M}})$ a operator.

Then we say that $\boldsymbol{g}$ is in a generalised $\boldsymbol{f}$-wave gauge with respect to $\overline{\boldsymbol{g}}$ iff the identity map

$$\mathrm{Id} : \left(\boldsymbol{\mathcal{M}}, \boldsymbol{g}\right) \to \left(\boldsymbol{\mathcal{M}}, \overline{\boldsymbol{g}}\right)$$

is an $\boldsymbol{f}(\boldsymbol{g})$-wave map. Denoting by $\boldsymbol{C}_{g,\overline{g}}$ the connection tensor of $\boldsymbol{g}$ and $\overline{\boldsymbol{g}}$,

$$\left(\boldsymbol{C}_{g,\overline{g}}\right)^{\alpha}_{\beta\gamma} := \frac{1}{2}\left(\boldsymbol{g}^{-1}\right)^{\alpha\delta}\left(2\overline{\boldsymbol{\nabla}}_{(\beta}\boldsymbol{g}_{\gamma)\delta} - \overline{\boldsymbol{\nabla}}_{\delta}\boldsymbol{g}_{\beta\gamma}\right) \tag{3.1}$$

with $\overline{\boldsymbol{\nabla}}$ the Levi-Civita connection associated to $\overline{\boldsymbol{g}}$, the imposition of such a gauge is equivalent to the condition

$$\boldsymbol{g}^{-1} \cdot \left(\boldsymbol{C}_{g,\overline{g}}\right)_{\overline{\flat}} = \boldsymbol{f}(\boldsymbol{g}) \tag{3.2}$$

where $\overline{\flat}$ is the (lowering) musical isomorphism associated to $\overline{\boldsymbol{g}}$.

3.1.2 The Einstein vacuum equations

In this section we present the Einstein vacuum equations assuming that a generalised wave has been imposed.

Indeed, if $\boldsymbol{g}$ is in a generalised $\boldsymbol{f}$-wave gauge with respect to $\overline{\boldsymbol{g}}$ the Einstein vacuum equations for $\boldsymbol{g}$,

$$\mathbf{Ric}_{\alpha\beta}[\boldsymbol{g}] = 0,$$

take the following form:

$$\begin{aligned}\left(\boldsymbol{g}^{-1}\right)^{\gamma\delta}\overline{\boldsymbol{\nabla}}_{\gamma}\overline{\boldsymbol{\nabla}}_{\delta}\boldsymbol{g}_{\alpha\beta}+2\boldsymbol{C}^{\gamma}_{\delta\epsilon}\cdot\boldsymbol{g}_{\gamma(\alpha}\overline{\boldsymbol{\nabla}}_{\beta)}\left(\boldsymbol{g}^{-1}\right)^{\delta\epsilon}-4\boldsymbol{g}_{\delta\epsilon}\boldsymbol{C}^{\epsilon}_{\beta[\alpha}\overline{\boldsymbol{\nabla}}_{\gamma]}\left(\boldsymbol{g}^{-1}\right)^{\gamma\delta}&\\ -4\boldsymbol{C}^{\delta}_{\beta[\alpha}\boldsymbol{C}^{\gamma}_{\gamma]\delta}+2\boldsymbol{g}^{\gamma\delta}\boldsymbol{g}_{\epsilon(\alpha}\overline{\mathbf{Riem}}_{\beta)\gamma\delta}{}^{\epsilon}&\\ =2\boldsymbol{g}_{\gamma(\alpha}\overline{\boldsymbol{\nabla}}_{\beta)}\left(\overline{\boldsymbol{g}}^{\gamma\epsilon}\boldsymbol{f}_{\epsilon}(\boldsymbol{g})\right).&\end{aligned}\tag{3.3}$$

Here, $\overline{\mathbf{Riem}}$ is the Riemann tensor of $\overline{\boldsymbol{g}}$ and $\boldsymbol{C}$ is defined as in (3.1).

3.2 The exterior Schwarzschild background

In this section we define the Schwarzschild exterior spacetime as well as introducing various background objects and operations that shall prove vital throughout the remainder of the thesis.

3.2.1 The differential structure and metric of the Schwarzschild exterior spacetime

We begin in this section by defining the differential structure and metric of the Schwarzschild exterior spacetime $\left(\mathcal{M}, g_M\right)$.

Let $M>0$ be a fixed parameter.

We define the smooth manifold with boundary

$$\mathcal{M}:=(-\infty,\infty)\times[2M,\infty)\times S^2$$

and endow it with the coordinate system $\left(t^*, r, \theta, \varphi\right)$. Here S^2 is the 2-sphere with (θ,φ) coordinates on S^2. Equipping $\mathcal{M}$ with the smooth Ricci-flat Lorentzian metric

$$g_M=-\left(1-\frac{2M}{r}\right)\mathrm{d}t^{*2}+\frac{4M}{r}\,\mathrm{d}t^*\,\mathrm{d}r+\left(1+\frac{2M}{r}\right)\mathrm{d}r^2+r^2\mathring{g},\tag{3.4}$$

with $\mathring{g}$ the metric on the unit round sphere, thus defines the Schwarzschild exterior spacetime (of mass M) as the Lorenztian manifold with boundary $\left(\mathcal{M}, g_M\right)$. It is time-orientable, with time-orientation given by the hypersurface-orthogonal vector field ∂_{t^*}. Moreover, the boundary of $\mathcal{M}$, which we denote by

$$\mathcal{H}^+:=(-\infty,\infty)\times\{2M\}\times S^2,$$

is a null hypersurface termed *the future event horizon*. In addition, it is manifest that the generators $\{\Omega_i\}_{i=1,\ldots,3}$ of the rotation group $SO(3)$ and the causal vector field ∂_{t^*} are

Killing fields for g_M:

$$\mathcal{L}_\Omega g_M = 0, \qquad \mathcal{L}_T g_M = 0.$$

The Schwarzschild exterior spacetime is thus both *static* and *spherically symmetric.*

A Penrose diagram of the exterior Schwarzschild spacetime can be found in section 3.2.4.

3.2.2 The 2+2 formalism

We continue in this section by detailing the so-called $2+2$ formalism on $\mathcal{M}$.

We shall make heavy use of the enlargened mathematical toolbox this formalism provides throughout the paper.

3.2.2.1 The 2+2 decomposition of tensor fields on $\mathcal{M}$

We begin by employing this formalism to decompose tensor fields on $\mathcal{M}$ into what we term as $\mathcal{Q}$-tensors, S-tensors and $\mathcal{Q}\otimes S$-tensors respectively.

Observe that one can express the manifold $\mathcal{M}$ as

$$\mathcal{M} = \mathcal{Q} \times S^2$$

where[1]

$$\mathcal{Q} \cong \mathbb{R} \times \mathcal{H}$$

is a manifold with boundary. This leads to the following definition.

Definition 3.1. *Let $\mathfrak{T} \in \mathscr{T}^n(\mathcal{M})$ be a smooth n-covariant tensor field on $\mathcal{M}$ that is symmetric in all its indices. Then we say that $\mathfrak{T}$ is a smooth n-covariant $\mathcal{Q}$-tensor field that is symmetric in all its indices iff there exists $n+1$ smooth functions $\{f_{ij}\}_{i+j=n}$ on $\mathcal{M}$ such that*

$$\mathfrak{T} = \sum_{i+j=n} f_{ij} (\,\mathrm{d}t^*)^i (\,\mathrm{d}r)^j.$$

We denote by $\tilde{\mathscr{T}}^n(\mathcal{M})$ the space of smooth n-covariant $\mathcal{Q}$ tensors.

Conversely, we say that $\mathfrak{T}$ is a smooth n-covariant S-tensor field iff $\mathfrak{T}$ vanishes when

[1]Here $\mathcal{H}$ is the half-space, not to be confused with the event horizon.

acting on either of the vector fields ∂_{t^*} or ∂_r:

$$\mathfrak{T}\Big(\cdot, ..., \partial_{t^*}, ..., \cdot\Big) = \mathfrak{T}\Big(\cdot, ..., \partial_r, ..., \cdot\Big) = 0.$$

We denote by $\tilde{\mathscr{T}}^n(\mathcal{M})$ the space of smooth n-covariant $\mathcal{Q}$ tensors.

Finally, if $\mathfrak{T}$ is now a smooth, symmetric 2-covariant tensor field on $\mathcal{M}$ then we say that $\mathfrak{T}$ is a smooth $\mathcal{Q}\otimes S$ 1-form iff there exists two smooth S 1-forms $\not{\sigma}$ and $\not{\rho}$ such that

$$\mathfrak{T} = \mathrm{d}t^*\otimes_{\mathrm{s}}\not{\sigma} + \mathrm{d}r\otimes_{\mathrm{s}}\not{\rho}$$

where $\otimes_{\mathrm{s}}$ denotes the symmetrised tensor product.

We denote by $\tilde{\mathscr{T}}^1(\mathcal{M})\otimes_{\mathrm{s}}\not{\mathscr{T}}^1(\mathcal{M})$ the space of smooth $\mathcal{Q}\otimes S$ 1-forms.

Note by convention we set a 0-covariant $\mathcal{Q}$-tensor field and a 0-covariant S-tensor field to be simply a scalar field on $\mathcal{M}$.

Given any tensor field on $\mathcal{M}$ one projects it onto a $\mathcal{Q}$-tensor field and an S-tensor fields as follows.

First we need the notion of an S-vector field and an associated projection of vector fields on $\mathcal{M}$ onto S-vector fields.

Definition 3.2. *Let V be a smooth vector field on $\mathcal{M}$. Then we say that V is a smooth S-vector field iff V satisfies*

$$V(t^*) = V(r) = 0.$$

Conversely, given a smooth vector field V then we define its projection onto the smooth S-vector field $\not{V}$ according to

$$\not{V} := V - V(t^*)\partial_{t^*} - V(r)\partial_r.$$

This leads to the projection of n-covariant tensor fields..

Definition 3.3. *Let $\mathfrak{T}$ be a smooth n-covariant tensor field on $\mathcal{M}$. Then we define its projection onto the smooth n-covariant $\mathcal{Q}$-tensor $\tilde{\mathfrak{T}}$ according to*

$$\tilde{\mathfrak{T}}\Big(..., \partial_{t^*}, ..., \partial_r, ...\Big) = \mathfrak{T}\Big(..., \partial_{t^*}, ..., \partial_r, ...\Big).$$

Conversely, we define the projection of $\mathfrak{T}$ onto the smooth n-covariant S-tensor $\not{\mathfrak{T}}$ according to

$$\not{\mathfrak{T}}\Big(V_1, ..., V_n\Big) = \mathfrak{T}\Big(\not{V}_1, ..., \not{V}_n\Big),$$

where $V_1, ..., V_n$ are an n-tuple of vector fields on $\mathcal{M}$ with $\not{V}_1, ..., \not{V}_n$ the corresponding projections onto S-vector fields.

Finally, if $\mathfrak{T}$ is now a symmetric 2-covariant tensor field on $\mathcal{M}$ then we define the projection of $\mathfrak{T}$ onto the smooth $\mathcal{Q} \otimes S$ 1-form to according to

$$\begin{aligned}\not{\tilde{\mathfrak{T}}}\big(\partial_{t^*}, V\big) &= \mathfrak{T}\big(\partial_{t^*}, \not{V}\big),\\ \not{\tilde{\mathfrak{T}}}\big(\partial_r, V\big) &= \mathfrak{T}\big(\partial_r, \not{V}\big)\end{aligned}$$

where V is a smooth vector field on $\mathcal{M}$ with $\not{V}$ its corresponding projection onto a smooth S-vector field.

Note that a symmetric 2-covariant tensor $\mathfrak{T}$ is completely specified by the projections $\tilde{\mathfrak{T}}, \not{\tilde{\mathfrak{T}}}$ and $\not{\mathfrak{T}}$.

A particularly useful application of this decomposition is to the Schwarzschild metric g_M. Indeed

- $\tilde{g}_M$ is the smooth, symmetric 2-covariant $\mathcal{Q}$-tensor

  $$\tilde{g}_M = -\left(1 - \frac{2M}{r}\right) \mathrm{d}t^{*2} + \frac{4M}{r} \mathrm{d}t^* \, \mathrm{d}r + \left(1 + \frac{2M}{r}\right) \mathrm{d}r^2$$

  which we refer to as a $\mathcal{Q}$-metric

- $\not{\tilde{g}}_M$ is trivial

- $\not{g}_M$ is the smooth, symmetric 2-covariant S-tensor

  $$\not{g}_M = r^2 \mathring{\not{g}}$$

  which we refer to as an S-metric

3.2.2.2 Tensor analysis

We now develop a series of natural operations and differential operators on tensor fields on $\mathcal{M}$ that arise as a result of the $2+2$ formalism of the previous section, in particular the $2+2$ decomposition of the metric g_M.

In what follows, $\tilde{g}_M^{-1}$ is the 2-contravariant tensor field on $\mathcal{M}$ defined by (with $\mu := \frac{2M}{r}$)

$$\tilde{g}_M^{-1} := -(1+\mu)\, \partial_{t^*} \otimes \partial_{t^*} + \mu\, \partial_{t^*} \otimes_{\mathrm{s}} \partial_r + (1-\mu)\, \partial_r \otimes \partial_r$$

and $\not{g}_M^{-1}$ is the 2-contravariant tensor field on $\mathcal{M}$ that induces the inverse metric to $\not{g}_M$ on every 2-sphere given by the level sets of t^* and r. In addition, $\tilde{\epsilon}$ is the 2-form on $\mathcal{M}$ defined by $\tilde{\epsilon} := \mathrm{d}t^* \wedge \mathrm{d}r$ and $\not{\epsilon}$ is the 2-form on $\mathcal{M}$ which induces the volume form associated to $\not{g}_M$ on every 2-sphere given by the level sets of t^* and r. Moreover, $\mathfrak{T}$ and $\mathfrak{T}'$ denote smooth n-covariant and m-covariant tensor fields on $\mathcal{M}$ respectively, $\mathfrak{p}$ denotes a smooth n-form on $\mathcal{M}$, $\mathfrak{t}$ and $\mathfrak{t}'$ denote smooth 1-forms on $\mathcal{M}$ and f denotes a smooth function on $\mathcal{M}$. Finally, we shall employ abstract index notation.

- the index raising operators $\tilde{\sharp}$ and $\not{\sharp}$ are defined by

$$\left(\mathfrak{T}^{\tilde{\sharp}}\right)^{a_1}{}_{a_2 \dots a_n} := \left(\tilde{g}_M^{-1}\right)^{a_1 b}\mathfrak{T}_{b a_2 \dots a_n}, \qquad \left(\mathfrak{T}^{\not{\sharp}}\right)^{a_1}{}_{a_2 \dots a_n} := \left(\not{g}_M^{-1}\right)^{a_1 b}\mathfrak{T}_{b a_2 \dots a_n}$$

- the contraction operators $\tilde{\cdot}$ and $/$ are defined by

$$\begin{aligned} (\mathfrak{T}\,\tilde{\cdot}\,\mathfrak{T}')_{a_{m+1}\dots a_n} &:= \mathfrak{T}_{a_1 \dots a_m a_{m+1} \dots a_n}\mathfrak{T}'_{b_1 \dots b_m}\tilde{g}_M^{-1\,a_1 b_1} \dots \tilde{g}_M^{-1\,a_m b_m}, \\ (\mathfrak{T}\,/\,\mathfrak{T}')_{a_{m+1}\dots a_n} &:= \mathfrak{T}_{a_1 \dots a_m a_{m+1} \dots a_n}\mathfrak{T}'_{b_1 \dots b_m}\not{g}_M^{-1\,a_1 b_1} \dots \not{g}_M^{-1\,a_m b_m} \end{aligned}$$

- the operations $|\cdot|^2_{\tilde{g}_M}$ and $|\cdot|^2_{\not{g}_M}$ are defined by

$$|\mathfrak{T}|^2_{\tilde{g}_M} := \mathfrak{T}\,\tilde{\cdot}\,\mathfrak{T}, \qquad |\mathfrak{T}|^2_{\not{g}_M} := \mathfrak{T}\,/\,\mathfrak{T}$$

- the trace operators $\mathrm{tr}_{\tilde{g}_M}$ and $\not{\mathrm{tr}}$ are defined by

$$\mathrm{tr}_{\tilde{g}_M}\mathfrak{T} := \tilde{g}_M\,\tilde{\cdot}\,\mathfrak{T}, \qquad \not{\mathrm{tr}}\mathfrak{T} := \not{g}_M\,/\,\mathfrak{T}$$

- the traceless symmetrised product operators $\hat{\tilde{\otimes}}$ and $\hat{\not{\otimes}}$ acting on 1-forms are defined by

$$\mathfrak{t}\hat{\tilde{\otimes}}\mathfrak{t}' := \mathfrak{t}\otimes_{\mathrm{s}}\mathfrak{t}' - \tilde{g}_M\,\mathfrak{t}\,\tilde{\cdot}\,\mathfrak{t}', \qquad \mathfrak{t}\hat{\not{\otimes}}\mathfrak{t}' := \mathfrak{t}\otimes_{\mathrm{s}}\mathfrak{t}' - \not{g}_M\,\mathfrak{t}\,/\,\mathfrak{t}'$$

- the Hodge star operators $\tilde{\star}$ and $\not{\star}$ are defined by

$$\tilde{\star}\mathfrak{t} := \tilde{\epsilon}\,\tilde{\cdot}\,\mathfrak{T}, \qquad \not{\star}\mathfrak{t} := \not{\epsilon}\,/\,\mathfrak{T}$$

- the exterior derivative operators $\tilde{\mathrm{d}}f$ and $\not{\mathrm{d}}f$ acting on smooth vector fields V on $\mathcal{M}$ are defined by

$$\left(\tilde{\mathrm{d}}f\right)(V) := \left(\mathrm{d}f\right)\left(V(t^*)\partial_{t^*} + V(r)\partial_r\right), \qquad \left(\not{\mathrm{d}}f\right)\left(V\right) := \left(\mathrm{d}f\right)\left(\not{V}\right)$$

with $\not{V}$ the S-vector field determined from V

To continue we denote by $\widetilde{\nabla}$ the derivative operator defined according to

$$\begin{aligned}\left(\widetilde{\nabla}_V\mathfrak{T}\right)_{a_1\ldots a_n} :=& V(t^*)\Big(\partial_{t^*}\mathfrak{T}_{a_1\ldots a_n}-\sum_{j=1}^{n}\widetilde{\Gamma}^b_{t^*t^*}\mathfrak{T}_{a_1\ldots b\ldots a_n}\delta^{a_j}_{t^*}\Big)\\ &+V(t^*)\Big(\partial_{t^*}\mathfrak{T}_{a_1\ldots a_n}-\sum_{j=1}^{n}\widetilde{\Gamma}^b_{t^*r}\mathfrak{T}_{a_1\ldots b\ldots a_n}\delta^{a_j}_{r}\Big)\\ &+V(r)\Big(\partial_{r}\mathfrak{T}_{a_1\ldots a_n}-\sum_{j=1}^{n}\widetilde{\Gamma}^b_{t^*r}\mathfrak{T}_{a_1\ldots b\ldots a_n}\delta^{a_j}_{t^*}\Big)\\ &+V(r)\Big(\partial_{r}\mathfrak{T}_{a_1\ldots a_n}-\sum_{j=1}^{n}\widetilde{\Gamma}^b_{rr}\mathfrak{T}_{a_1\ldots b\ldots a_n}\delta^{a_j}_{r}\Big)\end{aligned}$$

where V is a smooth vector field, δ is the Kronecker delta symbol and $\widetilde{\Gamma}_{t^*t^*}, \widetilde{\Gamma}_{t^*r}$ and $\widetilde{\Gamma}_{rr}$ are defined by

$$\begin{aligned}\widetilde{\Gamma}_{t^*t^*} &:= \frac{\mu}{2}\frac{\mu}{r}\,\partial_{t^*}+\frac{1}{2}\frac{\mu}{r}(1-\mu)\,\partial_r,\\ \widetilde{\Gamma}_{t^*r} &:= \frac{1}{2}\frac{\mu}{r}(1+\mu)\,\partial_{t^*}+\frac{\mu}{2}\frac{\mu}{r}\,\partial_r,\\ \widetilde{\Gamma}_{rr} &:= \frac{1}{2}\frac{\mu}{2}(2+3\mu)\,\partial_{t^*}+\frac{1}{2}\frac{\mu}{r}(1-3\mu)\,\partial_r.\end{aligned}$$

In addition, we denote by $\nabla\!\!\!/$ the derivative operator on $\mathcal{M}$ that induces the covariant derivative associated to $g\!\!\!/_M$ on every 2-sphere given by the level sets of t^* and r.

- the divergence operators $\tilde{\delta}$ and $\mathrm{d}\!\!/\mathrm{iv}$ are defined by

$$\tilde{\delta}\mathfrak{T} := -\mathrm{tr}_{\tilde{g}_M}\left(\widetilde{\nabla}\mathfrak{T}\right),\qquad \mathrm{d}\!\!/\mathrm{iv}\mathfrak{T} := \mathrm{t}\!\!/\mathrm{r}\left(\nabla\!\!\!/\mathfrak{T}\right)$$

- the curl operators $\tilde{\star}\tilde{\mathrm{d}}$ and $\mathrm{c}\!\!/\mathrm{url}$ are defined by

$$\tilde{\star}\tilde{\mathrm{d}}\mathfrak{T} := \frac{1}{2}\tilde{\epsilon}\cdot\widetilde{\nabla}\mathfrak{T},\qquad \mathrm{c}\!\!/\mathrm{url}\mathfrak{T} := \frac{1}{2}\epsilon\!\!/\ /\ \nabla\!\!\!/\mathfrak{T}$$

- the wave operator $\tilde{\Box}$ and the Laplace operator $\Delta\!\!\!/$ are defined by

$$\tilde{\Box}\mathfrak{T} := -\tilde{\delta}\left(\widetilde{\nabla}\mathfrak{T}\right),\qquad \Delta\!\!\!/\mathfrak{T} := \mathrm{d}\!\!/\mathrm{iv}\left(\nabla\!\!\!/\mathfrak{T}\right)$$

Lastly, we follow[1] in introducing a family of angular on $\mathcal{M}^2$.

- the operator $\mathcal{D}\!\!\!/_1$ is defined by

$$\mathcal{D}\!\!\!/_1\mathfrak{T} = -\left(\mathrm{d}\!\!/\mathrm{iv}\mathfrak{T},\mathrm{c}\!\!/\mathrm{url}\mathfrak{T}\right)$$

[2]Note however the difference in sign convention for the operators $\mathcal{D}\!\!\!/_1$ and $\mathcal{D}\!\!\!/_1^\star$.

- the operator $\not{\mathcal{D}}_1^\star$ is defined by

$$\not{\mathcal{D}}_1^\star(\mathfrak{T}, \mathfrak{T}') = \not{\nabla}\mathfrak{T} + \not{\epsilon}\not{\nabla}\mathfrak{T}'$$

Finally, in the sequel we shall employ the following notation.

- $\widetilde{\nabla}\otimes\mathfrak{t}$ and $\not{\nabla}\otimes\mathfrak{t}$ denote the operators

$$\left(\widetilde{\nabla}\otimes\mathfrak{t}\right)_{ab} = \widetilde{\nabla}_a\mathfrak{t}_b + \widetilde{\nabla}_b\mathfrak{t}_a, \qquad \left(\not{\nabla}\otimes\mathfrak{t}\right)_{ab} = \not{\nabla}_a\mathfrak{t}_b + \not{\nabla}_b\mathfrak{t}_a$$

- $\widetilde{\nabla}\hat{\otimes}\mathfrak{t}$ and $\not{\nabla}\hat{\otimes}\mathfrak{t}$ denote the traceless Lie derivatives $\mathcal{L}_{\mathfrak{t}^\sharp}\tilde{g}_M + \tilde{g}_M\,\tilde{\cdot}\,\tilde{\delta}\mathfrak{t}$ and $\mathcal{L}_{\mathfrak{t}^\sharp}\not{g}_M - \not{g}_M \cdot \not{\mathrm{d}}\mathrm{iv}\,\mathfrak{t}$ respectively

- for an n-tuple of vector fields $V_1, ..., V_n$ we denote by $\mathfrak{T}_{V_1 ... V_n}$ the contraction $(((\mathfrak{T} \cdot V_1) \cdot V_2)...) \cdot V_n$

3.2.2.3 The $\mathcal{Q}$-frame $\{\partial_{t^*}, \partial_r\}$

We end this section by introducing a $\mathcal{Q}$-frame on $\mathcal{M}$, that is a frame in which one can evaluate $\mathcal{Q}$-tensors and $\mathcal{Q} \otimes S$ 1-forms. This moreover allows one to express the '$\mathcal{Q}$-operations' introduced in section 3.2.2.2 with respect to this frame. We will make (implicit) use of this in section 7.2.

Proposition 3.1. *The linearly independent vector fields ∂_{t^*} and ∂_r determine a spherically symmetric $\mathcal{Q}$-frame on $\mathcal{M}$ such that*

$$\tilde{g}_M(\partial_{t^*}, \partial_{t^*}) = -(1-\mu), \qquad \tilde{g}_M(\partial_{t^*}, \partial_r) = \mu, \qquad \tilde{g}_M(\partial_r, \partial_r) = 1 + \mu. \tag{3.5}$$

In addition one has the (smooth) connection coefficients

$$\begin{aligned}
\widetilde{\nabla}_{\partial_{t^*}}\partial_{t^*} &= \frac{\mu}{2}\frac{\mu}{r}\,\partial_{t^*} + \frac{1}{2}\frac{\mu}{r}(1-\mu)\,\partial_r, &\qquad \widetilde{\nabla}_{\partial_{t^*}}\partial_r &= \frac{1}{2}\frac{\mu}{r}(1+\mu)\,\partial_{t^*} - \frac{\mu}{2}\frac{\mu}{r}\,\partial_r,\\
\widetilde{\nabla}_{\partial_r}\partial_{t^*} &= -\widetilde{\nabla}_{\partial_{t^*}}\partial_r, &\qquad \widetilde{\nabla}_{\partial_r}\partial_r &= \frac{1}{2}\frac{\mu}{2}(2+\mu)\,\partial_{t^*} - \frac{1}{2}\frac{\mu}{r}(1+\mu)\,\partial_r.
\end{aligned}$$

Proof. Computation. □

3.2.3 The Cauchy hypersurface Σ

In this section we consider a foliation of $\left(\mathcal{M}, g_M\right)$ by Cauchy hypersurfaces Σ_{t^*}, thus identifying an initial Cauchy hypersurface Σ. In addition, a restricted version of the $2+2$ formalism to the hypersurface Σ is detailed.

Initial data for the equations of linearised gravity will be prescribed on Σ in section 5.1.1 with the aid of this restricted formalism.

3.2.3.1 The Cauchy hypersurface Σ

We define the manifolds with boundary

$$\Sigma_{t^*} := \{t^*\} \times [2M, \infty) \times S^2.$$

As the gradient of t^* is globally time-like on $\mathcal{M}$, the family Σ_{t^*} describe a foliation of $\mathcal{M}$ by Cauchy hypersurfaces. We henceforth fix an initial time t_0^*:

$$\Sigma := \Sigma_{t_0^*}.$$

The initial hypersurface Σ comes equipped with the Riemannian metric

$$h_M = (1+\mu)\,\mathrm{d}r^2 + r^2\mathring{g}$$

along with the associated second fundamental form

$$k := \frac{1}{2}\mathcal{L}_n h_M = \frac{1}{2}\frac{1}{\bar{h}}\frac{\mu}{r}(2+\mu)\,\mathrm{d}r^2 - \frac{\mu}{\bar{h}}r\mathring{g}.$$

Here, n is the future-pointing unit normal to Σ

$$n = \bar{h}\,\partial_{t^*} - \frac{\mu}{\bar{h}}\partial_r$$

where we have defined the *lapse* function

$$\bar{h} := \sqrt{1+\mu}.$$

3.2.3.2 S_ν-tensor analysis

We consider now a foliation of Σ by the 2-spheres S_r^2 given as the level sets of the areal function r. Associated to this foliation is the inward pointing unit normal

$$\nu = \frac{1}{\bar{h}}\partial_r.$$

This leads to the following definition.

Definition 3.4. *Let $\{\nu, e_1, e_2\}$ be a frame on Σ. Then we say that a smooth n-covariant* *tensor field H on Σ is an n-covariant S_ν-tensor field if*

$$H(\cdot, ..., \nu, ..., \cdot) = 0.$$

Given a symmetric 2-covariant tensor field H on Σ one projects it onto the function $\bar{H}$ on Σ, the S_ν 1-form $\not{H}$ and the symmetric 2-covariant S_ν-tensor field as follows:

$$\bar{H} := H(\nu, \nu), \tag{3.6}$$

$$\not{H}(e_I) := H(\nu, e_I), \tag{3.7}$$

$$\not{\not{H}}(e_I, e_J) := H(e_I, e_J). \tag{3.8}$$

It is natural to decompose the latter into its trace and trace-free parts with respect to the induced metric $\not{g}_M$ on S_r^2:

$$\not{\not{H}} = \hat{\not{\not{H}}} + \frac{1}{2}\not{g}_M \cdot \not{\mathrm{tr}}\not{\not{H}}.$$

A particularly useful application of this procedure is for the second fundamental form k.

First one decomposes k into its trace and tracefree parts with respect to h_M:

$$k = \hat{k} + \frac{1}{3}h \cdot \mathrm{tr}k, \qquad \mathrm{tr}k = -\frac{1}{2}\frac{\mu}{\bar{h}}\frac{1}{r}\frac{2+3\mu}{1+\mu}.$$

Decomposing $\hat{k}$ according to (3.6)-(3.8) then yields (supressing the hat notation):

$$\begin{aligned}\bar{k} &= \frac{1}{3}\frac{\mu}{\bar{h}}\frac{1}{r}\frac{4+3\mu}{1+\mu},\\ \not{\mathrm{tr}}\not{\not{k}} &= -\bar{k},\\ \not{k} = \hat{\not{\not{k}}} &= 0.\end{aligned}$$

Lastly, one has a natural calculus on S_ν tensor fields induced from (Σ, h_M).

Indeed, the ‘angular’ operations introduced in section 3.2.2.2 have analagous definitions for S_ν-tensor fields. In addition, for smooth S_ν-tensor fields we define the differential operator $\not{\nabla}_\nu$ as corresponding to the action of covariant differentiation (with respect to h_M) in the direction ν:

$$\begin{aligned}\not{\nabla}_\nu \bar{H} &= \mathcal{L}_\nu \bar{H},\\ \not{\nabla}_\nu \not{H} &= \mathcal{L}_\nu \not{H} - \frac{1}{\bar{h}}\frac{1}{r}\not{H},\\ \not{\nabla}_\nu \not{\not{H}} &= \mathcal{L}_\nu \not{\not{H}} - \frac{1}{\bar{h}}\frac{2}{r}\not{\not{H}}.\end{aligned}$$

3.2.4 The $\tau^\star$-foliation

In this section we define a foliation of $\mathcal{M}$ by 2-spheres which foliate Σ_{t^*} in a compact region of spacetime but which foliate a null-hypersurface in the non-compact region. A pointwise norm acting on tensor fields on $\mathcal{M}$ and a frame which are both adapted to this foliation is then defined.

It is through this foliation that we will capture the dispersive properties of solutions to the equations of linearised gravity – see the theorem statements of section 6.

3.2.4.1 A foliation of $\mathcal{M}$ that 'penetrates' both $\mathcal{H}^+$ and $\mathcal{I}^+$ and the spacetime region $D^+(\Sigma_R)$

To define this foliation we introduce the following optical function u on $\mathcal{M} - \mathcal{H}^+$

$$u := t^* - r - 4M\log(r-2M) + R + 4M\log(R-2M)$$

where $R >> 10M$ is a fixed constant. We note that u indeed solves the Eikonal equation on $\big(\mathcal{M}, g_M\big)$. The function $\tau^\star$ is then defined according to

$$\tau^\star(t^*, r, \theta^A) = \begin{cases} t^* & r \leq R \\ u & r \geq R \end{cases}$$

where θ^A is any coordinate chart on S^2.

This subsequently defines the 2-sphere $S^2_{\tau^\star,r} \subset \mathcal{M}$ given as the intersection of the level sets of $\tau^\star$ and r:

$$S^2_{\tau_1^\star, r_1} := \{t_1^\star\} \times \{r_1\} \times S^2$$

where t_1^* is the unique value of t^* such that $\tau^\star(t_1^*, r_1) = \tau_1^\star$.

The desired foliation then arises as the union of these 2-spheres:

$$\mathcal{M} = \bigcup_{\tau^\star\in\mathbb{R}} \bigcup_{r\in[2M,\infty)} S^2_{\tau^\star,r}.$$

We shall informally refer to the limiting sphere as $r \to \infty$ as a sphere on *future null infinity* $\mathcal{I}^+$. The function $\tau^\star$ thus serves to parametrise $\mathcal{I}^+$. This yields the Penrose diagram of Figure 3.

Finally, the region of interest in this paper will in fact be the spacetime region corresponding

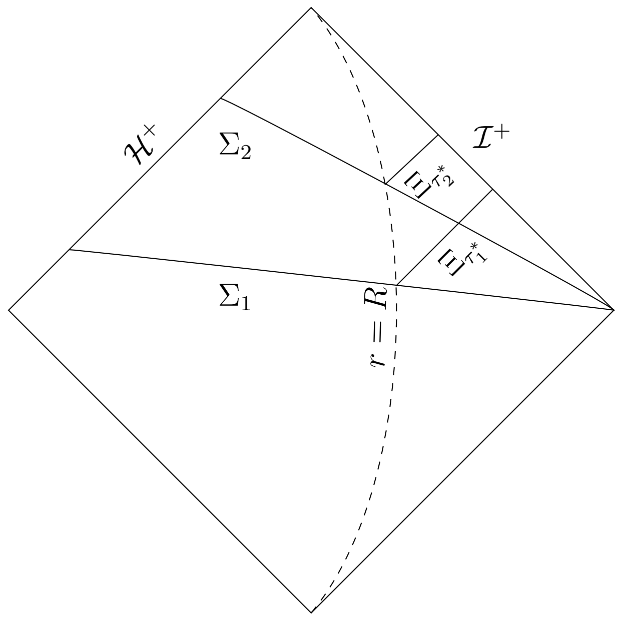

Figure 3.1: A Penrose diagram of $(\mathcal{M}, g_M)$ depicting the hypersurfaces Σ_{t^*} and $\Xi_{\tau^\star}$ given as the level sets of t^* and $\tau^\star$.

to the intersection of the causal future of a compact subset of the initial hypersurface Σ:

$$D^+(\Sigma_R) := \bigcup_{\tau^\star \in [t_0^*, \infty]} \bigcup_{r \in [2M, \infty)} S^2_{\tau^\star, r}.$$

This yields the resulting Penrose diagram of Figure 4.

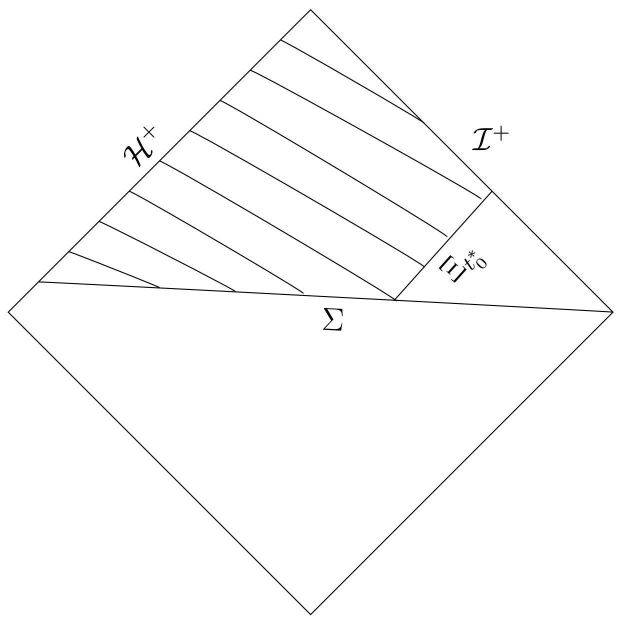

Figure 3.2: A Penrose diagram of $(\mathcal{M}, g_M)$ depicting the causal future of $\Xi_{t_0^*}$.

3.2.4.2 A pointwise norm on $\mathcal{Q}$-tensors, $\mathcal{Q} \otimes S$ 1-forms and S-tensors and the frame $\{\underline{L}, L\}$

We now introduce a pointwise norm on $\mathcal{Q}$-tensors, $\mathcal{Q} \otimes S$ 1-forms and S-tensors that is adapted to the foliation of $\mathcal{M}$ by the 2-spheres of the previous section.

Definition 3.5. *Let Q be a symmetric 2-covariant $\mathcal{Q}$-tensor, let ϖ be an $\mathcal{Q} \otimes S$ 1-form and let Θ be an n-covariant S-tensor for $n \geq 0$ an integer. Then on any 2-sphere $S^2_{\tau^\star,r}$ we define the pointwise norm $|\cdot|^2_{S^2_{\tau^\star,r}}$ according to*

$$
\begin{aligned}
|Q|^2_{S^2_{\tau^\star,r}} &:= \sup_{S^2_{\tau^\star,r}} \Big(|Q_{\partial_{t^*}\partial_{t^*}}|^2 + |Q_{\partial_{t^*}\partial_r}|^2 + |Q_{\partial_r\partial_r}|^2 \Big), \\
|\varpi|^2_{S^2_{\tau^\star,r}} &:= \sup_{S^2_{\tau^\star,r}} \Big(|\varpi_{\partial_{t^*}}|^2_{\slashed{g}_M} + |\varpi_{\partial_r}|^2_{\slashed{g}_M} \Big), \\
|\Theta|^2_{S^2_{\tau^\star,r}} &:= \sup_{S^2_{\tau^\star,r}} |\Theta|^2_{\slashed{g}_M}.
\end{aligned}
$$

Finally, we introduce a null frame $\{L, \underline{L}\}$ associated to the optical function u. Indeed, we define L to be the future-directed generator of the null hypersurfaces given by the level sets of u:

$$L = -(\,\mathrm{d}u)^\sharp$$

where $\sharp$ denotes the musical isomorphism on $\mathcal{M}$ given by g_M. Subsequently, we define $\underline{L}$ to be the unique future-directed null vector conjugate to L:

$$g_M(\underline{L}, L) = -2.$$

We have the following proposition.

Proposition 3.2. *In the $\mathcal{Q}$-frame $\{\partial_{t^*}, \partial_r\}$*

$$\underline{L} = (1-\mu)\Big(\partial_{t^*} - \partial_r\Big), \qquad L = \frac{1+\mu}{1-\mu}\partial_{t^*} + \partial_r.$$

In particular

$$\widetilde{\nabla}_{\underline{L}}\underline{L} = -\frac{\mu}{r}\underline{L}, \qquad \widetilde{\nabla}_{\underline{L}}L = \frac{\mu}{r}L, \qquad \widetilde{\nabla}_L\underline{L} = \widetilde{\nabla}_L L = 0 \tag{3.9}$$

and

$$\underline{D}r = -(1-\mu), \qquad Dr = 1 \tag{3.10}$$

where $\underline{D}$ and D denote the action of $\underline{L}$ and L on smooth functions.

Proof. Computation. □

We end this section with the following lemma which will be made use of in section 7.2.

Lemma 3.3. *Given a smooth function f on $\mathcal{M}$ we define the operator $\tilde{\mathrm{d}}^{\mathcal{I}} f$ acting on smooth vector fields according*

$$\tilde{\mathrm{d}}^{\mathcal{I}} f := \tilde{\mathrm{d}} f - \tilde{\star}\tilde{\mathrm{d}} f.$$

Then for a vector field $V = \alpha\,\underline{L} + \beta\,L$ where α and β are smooth functions on $\mathcal{M}$ it holds that

$$\tilde{\mathrm{d}}^{\mathcal{I}}_V f = 2\beta D f.$$

Proof. Computation. □

3.2.5 The projection of tensor fields on $\mathcal{M}$ onto and away from the $l = 0, 1$ spherical harmonics

In this section we provide a notion of tensor fields on $\mathcal{M}$ having support on and outside of the $l = 0, 1$ spherical harmonics and then define a projection map onto each respective space.

In section 5.2 we will establish an identification between the projection of a solution to the equations of linearised gravity onto $l = 0, 1$ and stationary solutions to said equations.

This section follows closely section 4.4 of[1].

3.2.5.1 The $l = 0, 1$ spherical harmonics and the spherical harmonic decomposition

We recall the classical spherical harmonics Y^l_m with $l \in \mathbb{N}$ and $m \in \{-l, ..., 0, ...l\}$ defined as the set of orthogonal eigenfunctions for the Laplacian $\mathring{\Delta}$ associated to the metric $\mathring{g}$ on the unit round sphere:

$$\mathring{\Delta} Y^l_m = -l(l+1) Y^l_m$$

and

$$\int_{S^2} Y^l_m \, Y^{l'}_{m'} \, \mathring{\epsilon} = \delta^{ll'} \delta_{mm'}.$$

Here, δ is the Kronecker delta symbol and $\mathring{\epsilon}$ is the volume form associated to $\mathring{g}$.

We explicitly note the form of the $l=0$ and $l=1$ modes

$$Y^{l=0}_{m=0} = \frac{1}{\sqrt{4\pi}} \tag{3.11}$$

$$Y^{l=1}_{m=-1} = \sqrt{\frac{3}{4\pi}}\sin\theta\cos\varphi, \qquad Y^{l=1}_{m=0} = \sqrt{\frac{3}{8\pi}}\cos\theta, \qquad Y^{l=1}_{m=1} = \sqrt{\frac{3}{4\pi}}\sin\theta\sin\varphi. \tag{3.12}$$

Definition 3.6. *Let f be a smooth function on S^2.*

Then we say that f is supported only on $l=0,1$ iff for $l\geq 2$

$$\int_{S^2} f\cdot Y^l_m\,\mathring{\epsilon} = 0.$$

Conversely, we say that f has vanishing projection to $l=0,1$ iff for $l=0,1$

$$\int_{S^2} f\cdot Y^l_m\,\mathring{\epsilon} = 0.$$

This leads to the classical spherical harmonic decomposition of square integrable functions on S^2.

Proposition 3.4. *Let $f\in L^2(S^2)$. Then one has the unique, orthogonal decomposition*

$$f = f_{l=0,1} + f'$$

where the function $f_{l=0,1}\in L^2(S^2)$ is supported only on $l=0,1$ and the function $f'\in L^2(S^2)$ has vanishing projection to $l=0,1$.

The remainder of this section is concerned with generalising this result to tensor fields on $\mathcal{M}$.

3.2.5.2 The $l=0,1$ spherical harmonics and $\mathcal{Q}$-tensors

We begin with $\mathcal{Q}$-tensors.

In what follows $n\geq 0$ is an integer.

Definition 3.7. *Let Q be a smooth n-covariant $\mathcal{Q}$-tensor.*

Then we say that Q is supported only on $l=0,1$ iff for $l\geq 2$ all components of Q in the $\mathcal{Q}$-frame $\{\partial_{t^*},\partial_r\}$ satisfy

$$\int_{S^2_{\tau^\star,r}} Q_{\cdots\partial_{t^*}\cdots\partial_r\cdots}\cdot Y^l_m\,\mathring{\epsilon} = 0.$$

Conversely, we say that Q has vanishing projection to $l=0,1$ iff for $l=0,1$ all *components of Q in the $\mathcal{Q}$-frame $\{\partial_{t^*}, \partial_r\}$ satisfy*

$$\int_{S^2_{\tau^\star,r}} Q_{\cdots\partial_{t^*}\cdots\partial_r\cdots} \cdot Y^l_m \, \mathring{\epsilon} = 0.$$

It will be useful in the sequel to denote by $\Lambda(\mathcal{M})$ the space of $\mathcal{Q}$-tensor fields that have vanishing projection to $l = 0, 1$.

Subsequently, applying Proposition 3.4 to each frame component yields the following.

Proposition 3.5. *Let Q be a smooth n-covariant $\mathcal{Q}$-tensor field. Then one has the unique, orthogonal decomposition*

$$Q = Q_{l=0,1} + Q'$$

where the smooth n-covariant $\mathcal{Q}$-tensor $Q_{l=0,1}$ is supported only on $l = 0, 1$ and $Q' \in \Lambda(\mathcal{M})$.

3.2.5.3 The $l = 0, 1$ spherical harmonics and $\mathcal{Q} \otimes S$ 1-forms

Next we consider $\mathcal{Q} \otimes S$ 1-forms.

First we recall the classical Hodge decomposition of smooth 1-forms ξ on S^2

$$\xi = \mathring{\nabla} f_1 + \mathring{\star}\mathring{\nabla} f_2.$$

Here, f_1 and f_2 are smooth functions on S^2 with $\mathring{\nabla}$ and $\mathring{\star}$ the Levi-Civita connection and Hodge dual associated to the metric on the unit round sphere. This decomposition is unique if one assumes that the smooth functions f_1 and f_2 have vanishing mean over S^2 and can be readily extended to smooth $\mathcal{Q} \otimes S$ 1-forms by using the frame $\{\partial_{t^*}, \partial_r\}$ and the fact that $\not{g}_M = r^2 \mathring{\not{g}}$. This leads to the following definition, an analogue of which for smooth S 1-forms was first given in[1].

Definition 3.8. *Let ϖ be a smooth $\mathcal{Q} \otimes S$ 1-form.*

Then we say that ϖ is supported only on $l = 0, 1$ iff the two smooth $\mathcal{Q}$ 1-forms q_1 and q_2 defined according to

$$\varpi = \not{\mathcal{D}}_1^\star\big(q_1, q_2\big) \tag{3.13}$$

are supported only on $l = 0, 1$ and moreover have vanishing mean on every 2-sphere $S^2_{\tau^\star,r}$.

Conversely, we say that ϖ has vanishing projection to $l = 0, 1$ iff $q_1, q_2 \in \Lambda(\mathcal{M})$.

Let now ξ be a smooth S 1-form.

Then we say that ξ is supported only on $l = 0, 1$ iff the two smooth functions f_1 and f_2 defined according to

$$\xi = \not{\mathcal{D}}_1^\star\Big(f_1, f_2\Big) \tag{3.14}$$

are supported only on $l = 0, 1$ and moreover have vanishing mean on every 2-sphere $S^2_{\tau^\star,r}$.

Conversely, we say that ξ has vanishing projection to $l = 0, 1$ iff $q_1, q_2 \in \Lambda(\mathcal{M})$.

Linearity and Proposition 3.5 then yields the desired decomposition of smooth $\mathcal{Q} \otimes S$ 1-forms.

Proposition 3.6. *Let ϖ be a smooth $\mathcal{Q} \otimes S$ 1-form. Then one has the unique, orthogonal decomposition*

$$\varpi = \varpi_{l=0,1} + \varpi'$$

where the smooth $\mathcal{Q} \otimes S$ 1-form ϖ is supported only on $l = 0, 1$ and ϖ' has vanishing projection to $l = 0, 1$.

Conversely, let ξ be a smooth S 1-form. Then one has the unique, orthogonal decomposition

$$\xi = \xi_{l=0,1} + \xi'$$

where $\xi_{l=0,1}$ is a smooth S 1-form that is supported only on $l = 0, 1$ and ξ' is a smooth S 1-form that has vanishing projection to $l = 0, 1$.

Finally, the following definition will prove useful in the sequel.

Definition 3.9. *Let ϖ be a smooth $\mathcal{Q} \otimes S$ 1-form.*

Then we define by ϖ_{e} and ϖ_{o} the two smooth $\mathcal{Q}$ 1-forms of vanishing mean on every 2-sphere $S^2_{\tau^\star,r}$ that arise from the Hodge decomposition of (3.13)*:*

$$\varpi = \not{\mathcal{D}}_1^\star\Big(\varpi_{\mathrm{e}}, \varpi_{\mathrm{o}}\Big).$$

Let now ξ be a smooth S 1-form.

Then we define by ξ_{e} and ξ_{o} the two smooth functions of vanishing mean on every 2-sphere $S^2_{\tau^\star,r}$ that arise from the Hodge decomposition of (3.14)*:*

$$\xi = \not{\mathcal{D}}_1^\star\Big(\xi_{\mathrm{e}}, \xi_{\mathrm{o}}\Big).$$

3.2.5.4 The $l = 0, 1$ spherical harmonics and symmetric, traceless 2-covariant S-tensors

Now we consider symmetric, traceless 2-covariant S-tensors.

One has the following proposition, proved in [1], which establishes the sense in which all smooth, symmetric, traceless, 2-covariant S-tensors have vanishing projection to $l = 0, 1$.

Proposition 3.7. *The operator* $\nabla\!\!\!/\hat{\otimes}\mathcal{D}\!\!\!/_1^\star$ *has trivial kernel over the space of smooth functions in* $\Lambda(\mathcal{M}) \times \Lambda(\mathcal{M})$*. Moreover, for any smooth, symmetric, traceless 2-covariant S-tensor* Θ *there exists two unique, smooth functions* $f_1, f_2 \in \Lambda(\mathcal{M})$ *such that*

$$\Theta = \nabla\!\!\!/\hat{\otimes}\mathcal{D}\!\!\!/_1^\star\Big(f_1, f_2\Big). \tag{3.15}$$

Finally, the following definition will prove useful in the sequel.

Definition 3.10. *Let* Θ *be a smooth, symmetric, traceless 2-covariant S-tensor. Then we define by* Θ_{e} *and* Θ_{o} *the two smooth functions in* $\Lambda(\mathcal{M})$ *that arise from the Hodge decomposition of* (3.15)*:*

$$\Theta = \nabla\!\!\!/\hat{\otimes}\mathcal{D}\!\!\!/_1^\star\Big(\Theta_{\mathrm{e}}, \Theta_{\mathrm{o}}\Big).$$

3.2.5.5 The $l = 0, 1$ spherical harmonics and S_ν-tensors

Our final considerations are S_ν-tensors.

Analagously to the previous three sections one has the following.

In the sequel, we denote by $\Lambda(\Sigma)$ the space of smooth functions on Σ having vanishing projection to $l = 0, 1$.

Proposition 3.8. *Let f be a smooth function on* Σ*. Then one has the unique, orthogonal decomposition*

$$f = f_{l=0,1} + f'$$

where $f_{l=0,1}$ *is a smooth function on* Σ *supported only on* $l = 0, 1$ *and* $f' \in \Lambda(\Sigma)$ *is smooth.*

Moreover, if ξ *is a smooth* S_ν *1-form then one has the unique, orthogonal decomposition*

$$\xi = \xi_{l=0,1} + \xi'$$

where $\xi_{l=0,1}$ *is a smooth* S_ν *1-form supported only on* $l = 0, 1$ *and* ξ' *is a smooth S 1-form with vanishing projection to* $l = 0, 1$ *(cf. Definition 3.8).*

Finally, if θ is a smooth, symmetric, traceless 2-covariant S_ν-tensor then θ can be uniquely represented as

$$\theta = \nabla\!\!\!/\hat{\otimes}\mathcal{D}\!\!\!/_1^\star\Big(\theta_{\mathrm{e}}, \theta_{\mathrm{o}}\Big)$$

where $\theta_{\mathrm{e}}, \theta_{\mathrm{o}} \in \Lambda(\Sigma)$ are smooth.

3.2.5.6 The projection onto and away from $l = 0, 1$

We end this section by defining a projection mapping onto and away from the space of tensor fields supported only on $l = 0, 1$.

In what follows, $n \geq 0$ is an integer.

Definition 3.11. *Let Q be a smooth n-covariant $\mathcal{Q}$-tensor field with ϖ a smooth $\mathcal{Q}\otimes S$ 1-form and ξ a smooth S 1-form.*

Then we respectively call the maps

$$\begin{aligned} Q &\to Q_{l=0,1}, \\ \varpi &\to \varpi_{l=0,1}, \\ \xi &\to \xi_{l=0,1} \end{aligned}$$

the projection of Q, ϖ and ξ onto $l = 0, 1$.

Conversely, we respectively call the maps

$$\begin{aligned} Q &\to Q', \\ \varpi &\to \varpi', \\ \xi &\to \xi' \end{aligned}$$

the projection of Q, ϖ and f, ξ away from $l = 0, 1$.

Of course analogous definitions hold for S_ν-tensors. Moreover, we note that for $Q \in \Lambda(\mathcal{M})$ one has $Q = Q'$.

3.2.6 Elliptic estimates on 2-spheres and the operator $\not{\xi}^{[p]}$

In this section we introduce an L^2 norm on the 2-spheres of section 3.2.4. A family of elliptic operators acting on tensor fields on $\mathcal{M}$ are then defined for which elliptic estimates will be derived with respect to these norms. We then finish this section by presenting various commutation relations and indentities associated to these operators.

These norms will appear in the theorem statements of section 6. Moreover, the elliptic estimates will allow us to control higher order angular derivatives of solutions to the equations of linearised gravity in section 7.2. Finally, the operator $\not{\mathcal{L}}^{[p]}$ will play an important role in this paper as it appears in the definition of the Zerilli equation in Chapter 4.

3.2.6.1 Norms on spheres

First we define the norms.

Definition 3.12. *Let Q be a symmetric 2-covariant $\mathcal{Q}$-tensor, let ϖ be an $\mathcal{Q}\otimes S$ 1-form and let Θ be an n-covariant S-tensor for $n \geq 0$ an integer. Then on any 2-sphere $S^2_{\tau^\star,r}$ we define the L^2 norm $||\cdot||^2_{S^2_{\tau^\star,r}}$ according to*

$$
\begin{aligned}
||Q||^2_{S^2_{\tau^\star,r}} &:= \int_{S^2_{\tau^\star,r}} \Big(|Q_{\partial_{t^*}\partial_{t^*}}|^2 + |Q_{\partial_{t^*}\partial_r}|^2 + |Q_{\partial_r\partial_r}|^2\Big)\mathring{\epsilon},\\
||\varpi||^2_{S^2_{\tau^\star,r}} &:= \int_{S^2_{\tau^\star,r}} \Big(|\varpi_{\partial_{t^*}}|^2_{\not{g}_M} + |\varpi_{\partial_r}|^2_{\not{g}_M}\Big)\mathring{\epsilon},\\
||\Theta||^2_{S^2_{\tau^\star,r}} &:= \int_{S^2_{\tau^\star,r}} |\Theta|^2_{\not{g}_M}\mathring{\epsilon}.
\end{aligned}
$$

Moreover, the higher order norms $||(r\not\nabla)^k\cdot||^2_{S^2_{\tau^\star,r}}$ for $k \geq 1$ are defined according to

$$
\begin{aligned}
||(r\not\nabla)^k Q||^2_{S^2_{\tau^\star,r}} &:= \int_{S^2_{\tau^\star,r}} \Big(|(r\not\nabla)^k Q_{\partial_{t^*}\partial_{t^*}}|^2_{\not{g}_M} + |(r\not\nabla)^k Q_{\partial_{t^*}\partial_r}|^2_{\not{g}_M} + |(r\not\nabla)^k Q_{\partial_r\partial_r}|^2_{\not{g}_M}\Big)\mathring{\epsilon},\\
||(r\not\nabla)^k\varpi||^2_{S^2_{\tau^\star,r}} &:= \int_{S^2_{\tau^\star,r}} \Big(|(r\not\nabla)^k\varpi_{\partial_{t^*}}|^2_{\not{g}_M} + |(r\not\nabla)^k\varpi_{\partial_r}|^2_{\not{g}_M}\Big)\mathring{\epsilon},\\
||(r\not\nabla)^k\Theta||^2_{S^2_{\tau^\star,r}} &:= \int_{S^2_{\tau^\star,r}} |(r\not\nabla)^k\Theta|^2_{\not{g}_M}\mathring{\epsilon}.
\end{aligned}
$$

Note that by convention a smooth function on $\mathcal{M}$ is a smooth 0-covariant S-tensor.

We make the following remark.

Remark 1. Note the absence of the r^2 volume weights in the above norms when compared with the analogous L^2 norms on spheres defined in [1].

3.2.6.2 The family of operators $\not{\mathcal{A}}$

We continue by introducing a family of operators on $\mathcal{M}$ which shall ultimately serve as a shorthand notation for controlling higher order angular derivatives of tensor fields on $\mathcal{M}$ measured in the norms of the previous section. Indeed, proceeding again as in [1], we define

- the operators $\mathcal{A}_f^{[i]}$ are defined inductively as

$$\mathcal{A}_f^{[2i+1]} := r\nabla\!\!\!/\,\mathcal{A}_f^{[2i]}, \qquad \mathcal{A}_f^{[2i+2]} := -r\mathrm{d}\!\!\!/\mathrm{iv}\,\mathcal{A}_f^{[2i+1]}$$

with $\mathcal{A}_f^{[1]} = r\nabla\!\!\!/$

- the operators $\mathcal{A}_\xi^{[i]}$ are defined inductively as

$$\mathcal{A}_\xi^{[2i+1]} := r\mathcal{D}\!\!\!/_1\mathcal{A}_\xi^{[2i]}, \qquad \mathcal{A}_\xi^{[2i+2]} := r\mathcal{D}\!\!\!/_1^\star\mathcal{A}_\xi^{[2i+1]}$$

with $\mathcal{A}_\xi^{[1]} = r\mathcal{D}\!\!\!/_1$

- the operators $\mathcal{A}_\theta^{[i]}$ are defined inductively as

$$\mathcal{A}_\theta^{[2i+1]} := r\mathrm{d}\!\!\!/\mathrm{iv}\,\mathcal{A}_\theta^{[2i]}, \qquad \mathcal{A}_\theta^{[2i+2]} := -r\nabla\!\!\!/\hat{\otimes}\mathcal{A}_\theta^{[2i+1]}$$

with $\mathcal{A}_\theta^{[1]} = r\mathrm{d}\!\!\!/\mathrm{iv}$

Before we derive the elliptic estimates we note the following lemma (we recall by convention that a 0-covariant $\mathcal{Q}$-tensor is a function on $\mathcal{M}$).

Lemma 3.9. *Let $Q' \in \tilde{\mathscr{T}}^n(\mathcal{M}) \cap \Lambda(\mathcal{M})$ for $n \geq 0$ an integer. Then for $i = 0, ..., 5$ and any 2-sphere $S^2_{\tau^\star,r}$ one has the estimate*

$$(6-i)\|(r\nabla\!\!\!/)^i Q'\|^2_{S^2_{\tau^\star,r}} \leq \|(r\nabla\!\!\!/)^{i+1}Q'\|^2_{S^2_{\tau^\star,r}}.$$

Proof. Recalling that $g\!\!\!/_M = r^2\mathring{g}$ where $\mathring{g}$ is the metric on the unit round sphere, the above estimate thus follows from applying the classical Poincaré inequality on the 2-spheres $S^2_{\tau^\star,r}$ to the components of Q' in the frame $\{\partial_{t^*}, \partial_r\}$. □

This leads to the subsequent elliptic estimates.

Proposition 3.10. *Let $Q' \in \Lambda(\mathcal{M})$ be a smooth, symmetric 2-covariant $\mathcal{Q}$-tensor, let $\varpi\!\!\!/$ be a smooth $\mathcal{Q}\otimes S$ 1-form and let Θ be a smooth, symmetric, traceless 2-covariant S-tensor respectively. Then for any 2-sphere $S^2_{\tau^\star,r}$ and any integer $m \geq 0$*

$$\sum_{i=0}^m \|(r\nabla\!\!\!/)^i Q'\|^2_{S^2_{\tau^\star,r}} \lesssim ||\mathcal{A}_f^{[m]}Q'||^2_{S^2_{\tau^\star,r}},$$
$$\sum_{i=0}^m \|(r\nabla\!\!\!/)^i \varpi\!\!\!/\|^2_{S^2_{\tau^\star,r}} \lesssim ||\mathcal{A}_\xi^{[m]}\varpi\!\!\!/||^2_{S^2_{\tau^\star,r}},$$
$$\sum_{i=0}^m \|(r\nabla\!\!\!/)^i \Theta\|^2_{S^2_{\tau^\star,r}} \lesssim ||\mathcal{A}_\theta^{[m]}\Theta||^2_{S^2_{\tau^\star,r}}.$$

Proof. We first note the identities

$$\mathcal{D}_1^\star \mathcal{D}_1 = -\frac{1}{r^2}\mathring{\Delta} + \frac{1}{r^2},$$
$$-\nabla\hat{\otimes}\mathcal{D}_2 = -\frac{1}{r^2}\mathring{\Delta} + \frac{2}{r^2}.$$

Computing thus in the frame $\{\partial_{t^*}, \partial_r\}$ one finds that on every 2-sphere $S^2_{\tau^\star,r}$

$$||\mathcal{A}_f^{[1]} Q'||^2_{S^2_{\tau^\star,r}} = ||r\nabla Q'||^2_{S^2_{\tau^\star,r}}, \tag{3.16}$$
$$||\mathcal{A}_\xi^{[1]} \varpi'||^2_{S^2_{\tau^\star,r}} = ||r\nabla \varpi'||^2_{S^2_{\tau^\star,r}} + ||\varpi'||^2_{S^2_{\tau^\star,r}}, \tag{3.17}$$
$$||\mathcal{A}_\theta^{[1]} \theta'||^2_{S^2_{\tau^\star,r}} = ||r\nabla \theta'||^2_{S^2_{\tau^\star,r}} + 2||\theta'||^2_{S^2_{\tau^\star,r}} \tag{3.18}$$

and

$$||\mathcal{A}_f^{[2]} Q'||^2_{S^2_{\tau^\star,r}} = ||r^2\Delta Q'||^2_{S^2_{\tau^\star,r}},$$
$$||\mathcal{A}_\xi^{[2]} \varpi'||^2_{S^2_{\tau^\star,r}} = ||r^2\Delta \varpi'||^2_{S^2_{\tau^\star,r}} + 2||r\nabla\varpi'||^2_{S^2_{\tau^\star,r}} + ||\varpi'||^2_{S^2_{\tau^\star,r}},$$
$$||\mathcal{A}_\theta^{[2]} \theta'||^2_{S^2_{\tau^\star,r}} = ||r^2\Delta \theta'||^2_{S^2_{\tau^\star,r}} + 4||r\nabla\theta'||^2_{S^2_{\tau^\star,r}} + 4||\theta'||^2_{S^2_{\tau^\star,r}}.$$

The former along with Lemma 3.9 immediately yields the $m = 1$ case of the proposition whereas the latter combined with elliptic estimates on Δ and Lemma 3.9 once more yields the $m = 2$ case.

The higher order cases then follow by an inductive procedure and Lemma 3.9, noting that commuting with higher order derivatives generates positively signed lower order terms. □

We note that analagous results hold for S_ν-tensors.

3.2.6.3 The operator $\zeta^{[p]}$

We consider now the operator

$$\Delta_\zeta := \Delta + \frac{2}{r^2}\left(1 - \frac{3M}{r}\right)\mathrm{Id}.$$

Elliptic theory (see e.g. [66]) and Lemma 3.9 leads to the following[3]:

Proposition 3.11. *The operator Δ_ζ is a bijection from $C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$ to $C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$.*

[3] In particular we note that the space $H^m \cap \Lambda(\mathcal{M})$, where H^m denotes the space of functions on S^2 whose $n = 0, ..., m$ (weak) derivatives are square integrable with respect to the measure of the unit round sphere, is a Hilbert space for any integer $m \geq 0$.

Here, $C^\infty(\mathcal{M})$ denotes the space of smooth functions on $\mathcal{M}$.

The above in particular means that the inverse operator to $\not\Delta_\zeta$ on $C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$ is well defined. Consequently

- the operator $\not\zeta^{[p]} : \Lambda(\mathcal{M}) \cap C^\infty(\mathcal{M}) \to \Lambda(\mathcal{M}) \cap C^\infty(\mathcal{M})$ is defined by

$$\not\zeta^{[p]} f = r^{2p} \not\Delta_\zeta^{-p} f.$$

We have the following proposition.

Proposition 3.12. *Let $f \in C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$. Then for any integer $p \geq 1$ and any 2-sphere $S^2_{\tau^\star,r}$ one has the elliptic estimates*

$$\sum_{i=0}^{2p} ||(r\not\nabla)^i \not\zeta^{[p]} f||^2_{S^2_{\tau^\star,r}} \lesssim ||f||^2_{S^2_{\tau^\star,r}}.$$

Proof. Integrating by parts on any 2-sphere $S^2_{\tau^\star,r}$ one finds

$$||\not\Delta_\zeta f||^2_{S^2_{\tau^\star,r}} = ||\not\nabla\not\nabla f||^2_{S^2_{\tau^\star,r}} - \frac{3}{r^3}(r-6M)||\not\nabla f||^2_{S^2_{\tau^\star,r}} + \frac{4}{r^6}(r-3M)^2||f||^2_{S^2_{\tau^\star,r}}.$$

Successively applying Lemma 3.9 therefore yields

$$\begin{aligned}\frac{4}{r^6}\Big((r-3M)^2 + r(r+18M)\Big)||f||^2_{S^2_{\tau^\star,r}} + \frac{1}{3}\frac{1}{r^3}(r+18M)||\not\nabla f||^2_{S^2_{\tau^\star,r}} + ||\not\nabla^2 f||^2_{S^2_{\tau^\star,r}} \\ \lesssim ||\not\Delta_\zeta f||^2_{S^2_{\tau^\star,r}}\end{aligned}$$

from which we conclude

$$\sum_{i=0}^{2} ||(r\not\nabla)^i f||^2_{S^2_{\tau^\star,r}} \lesssim ||r^2 \not\Delta_\zeta f||^2_{S^2_{\tau^\star,r}}.$$

Standard elliptic theory then yields

$$\sum_{i=0}^{2p} ||(r\not\nabla)^i f||^2_{S^2_{\tau^\star,r}} \lesssim ||r^2 \not\Delta_\zeta^p f||^2_{S^2_{\tau^\star,r}}.$$

The proposition then follows from the above estimate coupled with the fact that $\not\zeta^{[p]}$ is a bijection on the space $C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$. $\square$

Finally, we note the subsequent estimate which follows from the proof of Proposition 3.12 and will prove useful in the sequel.

Corollary 3.13. *Let $f \in C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$ be smooth. Then on any 2-sphere $S^2_{\tau^\star,r}$ one has the estimate*

$$\frac{4}{r^2}(r-3M)^2 ||\slashed{\mathcal{C}}^{[1]} f||^2_{S^2_{\tau^\star,r}} + \frac{2}{r}(r+9M)||(r\slashed{\nabla})\slashed{\mathcal{C}}^{[1]} f||^2_{S^2_{\tau^\star,r}} \lesssim ||f||^2_{S^2_{\tau^\star,r}}.$$

3.2.6.4 Commutation formulae and useful identities

In this final section we collect certain commutation relations and identities that will be used throughout the text.

Lemma 3.14. *Let $k, p \geq 1$ be integers.*

We denote by $\slashed{\mathcal{A}}^{[k]}$ any of the operators $\slashed{\mathcal{A}}^{[k]}_f, \slashed{\mathcal{A}}^{[k]}_\xi$ or $\slashed{\mathcal{A}}^{[k]}_\theta$. Then we have the commutation relations

$$\begin{aligned}
\left[\widetilde{\nabla}, \slashed{\nabla}\right] &= 0, \\
\left[\widetilde{\nabla}, \slashed{\mathcal{A}}^{[2k]}\right] &= 0, \\
\left[\widetilde{\nabla}, \slashed{\mathcal{A}}^{[2k-1]}\right] &= \frac{\tilde{\mathrm{d}}r}{r}\slashed{\mathcal{A}}^{[2k-1]}, \\
\left[\widetilde{\nabla}, \slashed{\mathcal{C}}^{[p]}\right] &= -3k\frac{\mu}{r}\,\tilde{\mathrm{d}}r\,\slashed{\mathcal{C}}^{[p+1]}
\end{aligned}$$

and

$$\begin{aligned}
\left[\slashed{\Delta}, \slashed{\mathcal{D}}^\star_1\right] &= \frac{1}{r^2}\slashed{\mathcal{D}}^\star_1, \\
\left[\slashed{\Delta}, \slashed{\nabla}\hat{\otimes}\slashed{\mathcal{D}}^\star_1\right] &= \frac{4}{r^2}\slashed{\nabla}\hat{\otimes}\slashed{\mathcal{D}}^\star_1.
\end{aligned}$$

Moreover, we have the identities

$$\begin{aligned}
\slashed{\mathrm{div}}\slashed{\mathcal{D}}^\star_1 &= \slashed{\Delta}, \\
\slashed{\mathrm{div}}\slashed{\nabla}\hat{\otimes}\slashed{\mathcal{D}}^\star_1 &= \slashed{\mathcal{D}}^\star_1\slashed{\Delta} + \frac{2}{r^2}\slashed{\mathcal{D}}^\star_1
\end{aligned}$$

and on any 2-sphere $S^2_{\tau^\star,r}$

$$\begin{aligned}
\int_{S^2_{\tau^\star,r}} \slashed{\mathcal{C}}^{[2p-1]} f \cdot f\,\mathring{\epsilon} &= -||(r\slashed{\nabla})\slashed{\mathcal{C}}^{[p]} f||^2_{S^2_{\tau^\star,r}} + (2-3\mu)||\slashed{\mathcal{C}}^{[p]} f||^2_{S^2_{\tau^\star,r}}, \\
\int_{S^2_{\tau^\star,r}} \slashed{\mathcal{C}}^{[2p]} f \cdot f\,\mathring{\epsilon} &= ||\slashed{\mathcal{C}}^{[p]} f||^2_{S^2_{\tau^\star,r}}
\end{aligned}$$

with $f \in C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$.

Proof. The first three commutation relations follow from the definitions of the operators in question, in particular noting the presence of the r-weights in the definitions of the

$\mathcal{A}^{[k]}$.

For the fourth one we have

$$\left[\widetilde{\nabla}, r^2 \not{\Delta}_\zeta\right] = \frac{3\mu}{r}\tilde{\mathrm{d}}r\,\mathrm{Id}.$$

and therefore the $p = 1$ case follows from the formula

$$[\widetilde{\nabla}, \not{\zeta}^{[1]}] = -\not{\zeta}^{[1]}\left[\widetilde{\nabla}, r^2 \not{\Delta}_\zeta\right]\not{\zeta}^{[1]}.$$

For general p, one applies the induction formulae

$$\left[\widetilde{\nabla}, \not{\zeta}^{[n]}\right] = \left[\widetilde{\nabla}, \not{\zeta}^{[n-1]}\right]\not{\zeta}^{[1]} + \not{\zeta}^{[n-1]}\left[\widetilde{\nabla}, \not{\zeta}^{[1]}\right].$$

For the final two we note the commutation relations

$$\begin{aligned}\left[\not{\Delta}, \not{\nabla}\right]f &= \frac{1}{r^2}f,\\ \left[\not{\Delta}, \not{\nabla}\right]\xi &= \frac{1}{r^2}\xi\end{aligned}$$

for a smooth function f on $\mathcal{M}$ and a smooth S 1-form ξ.

Turning now to the identities the first follows from the definition of $\not{\mathcal{D}}_1^\star$ whereas for the second we note the identity

$$\not{\mathrm{div}}\not{\nabla}\hat{\otimes}\xi = \not{\Delta}\xi + \frac{1}{r^2}\xi$$

on smooth S 1-forms ξ.

For the final two we perform an integration by parts on any 2-sphere $S^2_{\tau^\star,r}$ to find

$$\begin{aligned}\int_{S^2_{\tau^\star,r}} \not{\Delta}_\zeta f\cdot f\,\mathring{\epsilon} &= -||\not{\nabla}f||^2_{S^2_{\tau^\star,r}} + (2-3\mu)||f||^2_{S^2_{\tau^\star,r}},\\ \int_{S^2_{\tau^\star,r}} \not{\Delta}^2_\zeta f\cdot f\,\mathring{\epsilon} &= \int_{S^2_{\tau^\star,r}} \not{\Delta}_\zeta f\cdot\not{\Delta}_\zeta f\,\mathring{\epsilon}.\end{aligned}$$

This yields the $p = 1$ case of the desired identities after recalling that $\not{\zeta}^{[1]}$ is a bijection on the space $C^\infty(\mathcal{M})\cap\Lambda(\mathcal{M})$. The remaining cases then follow by induction. □

3.3 The equations of linearised gravity around Schwarzschild

In this section we consider the equations of section 3.1.2 for which a specific generalised wave gauge has been imposed. We then derive the system of equations that result from formally linearising this system of equations about the Schwarzschild exterior solution

$\left(\mathcal{M}, g_M\right)$ thus defining the so-called equations of linearised gravity.

It is solutions to these equations of linearised gravity that we shall study in the remainder of the paper.

3.3.1 The map f

We begin by defining an explicit map f which we shall use to determine a generalised wave gauge as in section 3.1.1 but now with $\left(\boldsymbol{\mathcal{M}}, \overline{\boldsymbol{g}}\right)$ identified with $\left(\mathcal{M}, g_M\right)$.

3.3.1.1 The map $\mathring{f}$

To define this map f it will in fact be more convenient both here and in the sequel to define three seperate maps for which the map f will then be defined as their sum.

Let us first recall the definition of $\mathscr{T}^2_{\mathrm{sym}}(\mathcal{M})$ as the space of smooth, symmetric 2-covariant tensor fields on $\mathcal{M}$. The first constituent component of the map f is then the map $\mathring{f} : \mathscr{T}^2_{\mathrm{sym}}(\mathcal{M}) \to \mathscr{T}^1(\mathcal{M})$ defined according to

$$\mathring{f}(X) := \frac{2}{r}\tilde{X}_P + \frac{2}{r}\not{X}_P - \frac{1}{r}\,\mathrm{d}r\,\not{\mathrm{tr}}\not{X}.$$

Here, $P := (\,\mathrm{d}r)^\sharp$ with $\sharp$ the musical isomorphism associated to g_M and we recall the $2+2$ formalism of section 3.2.2.

We note the following remark.

Remark 2. The map $\mathring{f}$ will reappear in section A of the Appendix.

3.3.1.2 The map $\overset{\mathscr{K}}{f}$

We continue by defining the second constituent component of the map f which is to be the map $\overset{\mathscr{K}}{f} : \mathscr{T}^2_{\mathrm{sym}}(\mathcal{M}) \to \mathscr{T}^1(\mathcal{M})$ defined according to

$$\overset{\mathscr{K}}{f}(X) := -\frac{1}{r}\left(\hat{\tilde{X}}_P\right)_{l=0,1}.$$

Here, $\hat{\tilde{X}} := \tilde{X} - \frac{1}{2}\tilde{g}_M\,\mathrm{tr}_{\tilde{g}_M}\tilde{X}$ and we recall the projection map onto the $l = 0, 1$ spherical harmonics of Definition 3.11.

We note the following remark.

Remark 3. The map $\overset{\mathscr{K}}{f}$ will reappear in section 3.4.2.

3.3.1.3 The map $\overset{\diamond}{f}$

To define the third constituent component of the map f it will be expedient to introduce first several auxiliary maps. We note that these latter maps will in fact play a significant role in the remainder of the paper.

We first introduce a triple of maps. Indeed, the first map $\tilde{\tau} : \mathscr{T}^2_{\text{sym}}(\mathcal{M}) \to \mathscr{T}^2_{\text{sym}}(\mathcal{M}) \cap \Lambda(\mathcal{M})$ is defined according to

$$\tilde{\tau}(X) := \tilde{X}' - \widetilde{\nabla}\otimes\Big(\not{X}'_{\mathrm{e}} - r^2\tilde{\mathrm{d}}\Big(r^{-2}\hat{\not{X}}_{\mathrm{e}}\Big)\Big)$$

whereas the second map $\tilde{\eta} : \mathscr{T}^2_{\text{sym}}(\mathcal{M}) \to \mathscr{T}^1(\mathcal{M}) \cap \Lambda(\mathcal{M})$ is defined according to

$$\tilde{\eta}(X) := \not{X}'_{\mathrm{o}} - r^2\tilde{\mathrm{d}}\Big(r^{-2}\hat{\not{X}}_{\mathrm{o}}\Big)$$

and finally the third map $\sigma : \mathscr{T}^2_{\text{sym}}(\mathcal{M}) \to C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$ is defined according to

$$\sigma(X) := \Big(\not{\mathrm{tr}}\not{X}\Big)' - \frac{4}{r}\Big(\not{X}'_{\mathrm{e}} - r^2\tilde{\mathrm{d}}\Big(r^{-2}\hat{\not{X}}_{\mathrm{e}}\Big)\Big) - 2\not{\Delta}\hat{\not{X}}_{\mathrm{e}}.$$

Here, we recall the space $\Lambda(\mathcal{M})$ from section 3.2.5.2, the Hodge mappings of Definition 3.9 and Definition 3.10 and finally the projection mapping $'$ away from the $l = 0, 1$ spherical harmonics of Definition 3.11.

The introduction of the above three maps subsequently allows the maps $\Phi : \mathscr{T}^2_{\text{sym}}(\mathcal{M}) \to C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$ and $\Psi : \mathscr{T}^2_{\text{sym}}(\mathcal{M}) \to C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$ to be defined according to

$$\Phi(X) := -r\not{\Delta}_2^{-1}\Big(\tilde{\star}\tilde{\mathrm{d}}\Big(r^{-2}\tilde{\eta}(X)\Big)\Big)$$

and

$$\Psi(X) := \not{\Delta}^{-1}\Big(\sigma(X)\Big) - \not{\Delta}^{-1}\Big(\not{\zeta}^{[1]}\Big(\widetilde{\nabla}_P\sigma(X) - 2r^{-1}\Big(\tilde{\tau}(X)\Big)_{PP}\Big)\Big).$$

Here, $\not{\Delta}_2 := \not{\Delta} + \frac{2}{r^2}\mathrm{Id}$ and we recall the operator $\not{\zeta}^{[p]}$ defined in section 3.2.6.3. Lastly, we note that the operators $\not{\Delta}_2$ and $\not{\Delta}$ are indeed invertible over the space $C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$ as can be shown using standard elliptic theory (see e.g.[66] and also cf. section 3.2.6.3).

Finally, the third constituent component of the map f is then to be the map $\overset{\diamond}{f} : \mathscr{T}^2_{\text{sym}}(\mathcal{M}) \to \mathscr{T}^1(\mathcal{M})$ defined according to

$$\overset{\diamond}{f}(X) := \frac{2}{r}(1-2\mu)\not{\mathcal{D}}_1^\star\Big(\Psi(X), \Phi(X)\Big) - \frac{1}{r^2}\tilde{\star}\tilde{\mathrm{d}}\Big(r^3\mathfrak{Z}\Big(\Psi(X)\Big)\Big) + r\not{\nabla}\mathfrak{Z}\Big(\Psi(X)\Big)$$

where $\mathfrak{z}$ is the operator $\mathfrak{z} := \frac{24}{r^3}\frac{M}{r}(r-3M)\not{\mathfrak{z}}^{[1]} + \frac{72}{r^3}\frac{M}{r}\frac{M}{r}(r-2M)\not{\mathfrak{z}}^{[2]}$.

We note the following remark.

Remark 4. The maps Φ and Ψ along with the operator $\mathfrak{z}$ will reappear in section 4.1.

3.3.1.4 The map f

We can now finally define the desired map f. Indeed, we define the map $f : \mathscr{T}^2_{\text{sym}}(\mathcal{M}) \to \mathscr{T}^1(\mathcal{M})$ according to

$$f(X) := \mathring{f}(X) + \overset{\mathscr{K}}{f}(X) + \overset{\diamond}{f}(X).$$

We make the following remarks.

Remark 5. We note that the domain of the map f is more restricted than the domain of the maps considered in section 3.1.1. The restriction to symmetric 2-covariant tensors is natural however in light of the fact that the case of interest is when the map is applied to a Lorentzian metric. In addition, the restriction to smoothness will be consistent with the regularity class of solutions to the resulting linearised equations we are to consider in section 5.1.

3.3.2 The formal linearisation of the equations of section 3.1.2

We continue by now establishing a formal linearisation theory for the Einstein equations on $\mathcal{M}$ as they appear in a generalised wave map gauge defined with respect to the map f and the Schwarzschild exterior metric g_M.

This section of the paper proceeds in a similar fashion to that of section 5.1 in [1].

3.3.2.1 Preliminaries

We first identify the manifold $\left(\boldsymbol{\mathcal{M}}, \overline{\boldsymbol{g}}\right)$ in section 3.1.1 with that of the Schwarzschild exterior solution $\left(\mathcal{M}, g_M\right)$.

On $\mathcal{M}$ we consider a smooth 1-parameter family of smooth Lorentzian metrics $\boldsymbol{g}(\epsilon)$ with $\boldsymbol{g}(0) = g_M$ and demand that each $\boldsymbol{g}(\epsilon)$ is in a generalised f-wave gauge with respect to g_M with f the map of section 3.3.1. We moreover assume that each pair $\left(\mathcal{M}, \boldsymbol{g}(\epsilon)\right)$ is a solution to the Einstein vacuum equations.

3.3.2.2 The linearisation procedure

Let us first immediately dispense with the ϵ notation and use the convention that bold quantities are with respect to the family of perturbed metrics and unbolded quantities are given by their background Schwarzschild value.

In particular, identifying $\overline{\boldsymbol{g}}$ with g_M in equations (3.2) and (3.3) of section 3.1, the assumptions of section 3.3.2.1 imply that each member $\boldsymbol{g}$ of the 1-parameter family must satisfy

$$\begin{aligned}\left(\boldsymbol{g}^{-1}\right)^{\gamma\delta}\nabla_\gamma\nabla_\delta\boldsymbol{g}_{\alpha\beta}+&2\boldsymbol{C}^{\gamma}_{\delta\varepsilon}\cdot\boldsymbol{g}_{\gamma(\alpha}\nabla_{\beta)}\left(\boldsymbol{g}^{-1}\right)^{\delta\varepsilon}-4\boldsymbol{g}_{\delta\varepsilon}\boldsymbol{C}^{\varepsilon}_{\beta[\alpha}\nabla_{\gamma]}\left(\boldsymbol{g}^{-1}\right)^{\gamma\delta}\\ &-4\boldsymbol{C}^{\delta}_{\beta[\alpha}\boldsymbol{C}^{\gamma}_{\gamma]\delta}+2\left(\boldsymbol{g}^{-1}\right)^{\gamma\delta}\boldsymbol{g}_{\varepsilon(\alpha}\mathrm{Riem}_{\beta)\gamma\delta}{}^{\varepsilon}\\ &=2\boldsymbol{g}_{\gamma(\alpha}\nabla_{\beta)}\left(g_M^{\gamma\varepsilon}f_\varepsilon(\boldsymbol{g})\right), \end{aligned}\tag{3.19}$$

$$\left(\boldsymbol{g}^{-1}\right)^{\beta\gamma}\boldsymbol{C}^{\alpha}_{\beta\gamma}=g_M^{\alpha\beta}f_\beta(\boldsymbol{g}),\tag{3.20}$$

Here, Riem and ∇ are the Riemann curvature tensor and Levi-Civita connection of $\left(\mathcal{M},g_M\right)$ respectively. Observe therefore that $\boldsymbol{g}=g_M$ is indeed a solution to this system of equations as by explicit computation one verifies that $f(g_M)=0$[4].

To formally linearise, we write

$$\boldsymbol{g}-g_M\equiv\overset{\scriptscriptstyle(1)}{g}$$

where $\equiv$ means equivalent to first order in ϵ. Thus, in keeping with the notation of[1], quantities with a superscript "(1)" denote linear perturbations of bolded quantities about their background Schwarzschild value. In particular,

- $\overset{\scriptscriptstyle(1)}{g}\in\mathscr{T}^2_{\mathrm{sym}}(\mathcal{M})$ denotes the linearised metric

Moreover, we write

$$f(\boldsymbol{g})\equiv Df\Big|_{g_M}(\overset{\scriptscriptstyle(1)}{g})$$

where $Df\Big|_{g_M}:\mathscr{T}^2_{\mathrm{sym}}(\mathcal{M})\to\mathscr{T}^1(\mathcal{M})$ is a linear map. In particular,

- the linear map $Df\Big|_{g_M}$ denotes the linearisation of the map f at g_M

However, in view of the fact that the map f is already linear, we thus have

$$Df\Big|_{g_M}=f.$$

[4] Note that this in particular means that g_M is in a generalised f-wave gauge with respect to g_M.

Subsequently, to derive the linearised equations one simply expands the terms appearing in equations (3.19) and (3.20) in powers of ϵ, keeping only those terms that enter to first order.

3.3.2.3 The linearisation of the Einstein vacuum equations in a generalised wave gauge around Schwarzschild

Proceeding in this manner one arrives at the following system of equations:

$$\Box \overset{\scriptscriptstyle(1)}{g}_{\alpha\beta} - 2\mathrm{Riem}^{\gamma}{}_{\alpha\beta}{}^{\delta} \overset{\scriptscriptstyle(1)}{g}_{\gamma\delta} = 2\nabla_{(\alpha} \overset{\scriptscriptstyle(1)}{f}_{\beta)}, \tag{3.21}$$

$$\nabla^{\beta} \overset{\scriptscriptstyle(1)}{g}_{\alpha\beta} - \frac{1}{2}\nabla_{\alpha} \overset{\scriptscriptstyle(1)}{g} = \overset{\scriptscriptstyle(1)}{f}_{\alpha}. \tag{3.22}$$

Here, $\Box$ is the wave operator on $\big(\mathcal{M}, g_M\big)$ and we have defined $\overset{\scriptscriptstyle(1)}{f} \in \mathscr{T}^1(\mathcal{M})$ according to

$$\overset{\scriptscriptstyle(1)}{f} := f(\overset{\scriptscriptstyle(1)}{g}).$$

The above system of equations thus describe the linearisation of the Einstein vacuum equations, as expressed in a generalised f-wave gauge with respect to g_M, about the Schwarzschild exterior solution g_M. We shall henceforth refer to this system of equations as the **equations of linearised gravity**.

We make the following remark.

Remark 6. We note if in section 3.3.2.1 one were to instead chose any smooth map $\boldsymbol{f}$ satisfying $\boldsymbol{f}(g_M) = 0$ and which has as linearisation at g_M the map f then the resulting linearised equations would be the same system of equations (3.21)-(3.22).

3.3.3 The system of gravitational perturbations

In this section we apply the $2+2$ formalism developed in section 3.2.2 to the equations of linearised gravity.

Indeed, given the symmetric two tensor $\overset{\scriptscriptstyle(1)}{g}$ on $\mathcal{M}$ we have:

- the projection onto the symmetric 2-covariant $\mathcal{Q}$-tensor field $\overset{\scriptscriptstyle(1)}{\tilde{g}}$
- the projection onto the $\mathcal{Q}\otimes S$ 1-form $\overset{\scriptscriptstyle(1)}{\not{g}}$
- the projection onto the symmetric 2-covariant S-tensor field $\overset{\scriptscriptstyle(1)}{\not{g}}$

Furthermore, given the 1-form $\overset{\scriptscriptstyle(1)}{f}$ we have:

- the projection onto the $\mathcal{Q}$ 1-form $\overset{(1)}{\tilde{f}}$
- the projection onto the S 1-form $\overset{(1)}{\not{f}}$

Moreover, decomposing $\overset{(1)}{\tilde{g}}$ and $\overset{(1)}{\not{g}}$ into their trace and trace-free parts with respect to the $\mathcal{Q}$-metric $\tilde{g}_M$ and the S-metric $\not{g}_M$

$$\overset{(1)}{\tilde{g}} = \overset{(1)}{\hat{\tilde{g}}} + \frac{1}{2}\tilde{g}_M \cdot \mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}}$$

and

$$\overset{(1)}{\not{g}} = \overset{(1)}{\hat{\not{g}}} + \frac{1}{2}\not{g}_M \cdot \not{\mathrm{tr}}\overset{(1)}{\not{g}}$$

yields the collection

$$\overset{(1)}{\hat{\tilde{g}}}, \mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}}, \overset{(1)}{\not{g}}, \overset{(1)}{\hat{\not{g}}}, \not{\mathrm{tr}}\overset{(1)}{\not{g}}, \overset{(1)}{\tilde{f}}, \overset{(1)}{\not{f}}$$

where now

- $\overset{(1)}{\hat{\tilde{g}}}$ is a symmetric, traceless 2-covariant $\mathcal{Q}$-tensor field
- $\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}}$ is a function on $\mathcal{M}$
- $\overset{(1)}{\not{g}}$ is a $\mathcal{Q} \otimes S$ 1-form
- $\overset{(1)}{\hat{\not{g}}}$ is a symmetric, traceless 2-covariant S-tensor field
- $\not{\mathrm{tr}}\overset{(1)}{\not{g}}$ is a function on $\mathcal{M}$
- $\overset{(1)}{\tilde{f}}$ is a $\mathcal{Q}$ 1-form
- $\overset{(1)}{\not{f}}$ is an S 1-form

each of which must satisfy the following *system of gravitational perturbations*[5].

[5]Here we use that, owing to the spherical symmetry of $(\mathcal{M}, g_M)$, the Christoffel symbols associated to the connection ∇ are of the form

$$\Gamma^{\alpha}_{t^*t^*} = \widetilde{\Gamma}^{\alpha}_{t^*t^*}, \qquad \Gamma^{\alpha}_{t^*r} = \widetilde{\Gamma}^{\alpha}_{t^*r}, \qquad \Gamma^{\alpha}_{rr} = \widetilde{\Gamma}^{\alpha}_{rr},$$
$$\Gamma^{\alpha}_{AB} = -\frac{1}{r}(\,\mathrm{d}r)^{\alpha}(\not{g}_M)_{AB}, \qquad \Gamma^{A}_{\alpha B} = \frac{1}{r}(\,\mathrm{d}r)_{\alpha}\,\delta^{A}_{B},$$
$$\Gamma^{A}_{BC} = \not{\Gamma}^{A}_{BC}$$

with all other Christoffel symbols vanishing. Here, $\alpha = \{t^*, r\}$, $\widetilde{\Gamma}^{\alpha}_{t^*t^*}, \widetilde{\Gamma}^{\alpha}_{t^*r}, \widetilde{\Gamma}^{\alpha}_{rr}$ are defined as in section 3.2.2.2, (A, B) denote a coordinate system on S^2 and $\not{\Gamma}$ denotes the collection of quantities which induce the Christoffel symbols associated to the connection $\not{\nabla}$ on every 2-sphere given as the intersection of the level sets of t^* and r. For a derivation of this result, see[47].

The equations for the linearised $\mathcal{Q}$-metric:

$$\widetilde{\Box}\overset{\scriptscriptstyle(1)}{\hat{\tilde{g}}} + \slashed{\Delta}\overset{\scriptscriptstyle(1)}{\hat{\tilde{g}}} + \frac{2}{r}\widetilde{\nabla}_P\overset{\scriptscriptstyle(1)}{\hat{\tilde{g}}} - \frac{2}{r}\Big(\widetilde{\nabla}\hat{\otimes}\overset{\scriptscriptstyle(1)}{\hat{\tilde{g}}}\Big)_P - \frac{4}{r}\frac{\mu}{r}\overset{\scriptscriptstyle(1)}{\hat{\tilde{g}}} = \frac{1}{r}\,\mathrm{d}r\hat{\tilde{\otimes}}\tilde{\mathrm{d}}\mathrm{tr}_{\tilde{g}_M}\overset{\scriptscriptstyle(1)}{\tilde{g}} + \frac{2}{r}\,\mathrm{d}r\hat{\tilde{\otimes}}\slashed{\mathrm{div}}\overset{\scriptscriptstyle(1)}{\not{g}} - \frac{1}{r}\,\mathrm{d}r\hat{\tilde{\otimes}}\tilde{\mathrm{d}}\slashed{\mathrm{tr}}\overset{\scriptscriptstyle(1)}{\slashed{g}} + \widetilde{\nabla}\hat{\otimes}\underset{\circ}{\overset{\scriptscriptstyle(1)}{\tilde{f}}}, \tag{3.23}$$

$$\widetilde{\Box}\mathrm{tr}_{\tilde{g}_M}\overset{\scriptscriptstyle(1)}{\tilde{g}} + \slashed{\Delta}\mathrm{tr}_{\tilde{g}_M}\overset{\scriptscriptstyle(1)}{\tilde{g}} = -2\tilde{\delta}\underset{\circ}{\overset{\scriptscriptstyle(1)}{\tilde{f}}} + \frac{2}{r}\underset{\circ}{\overset{\scriptscriptstyle(1)}{\tilde{f}}}. \tag{3.24}$$

The equations for the linearised $\mathcal{Q}\otimes S$-metric:

$$\widetilde{\Box}\overset{\scriptscriptstyle(1)}{\not{g}} + \slashed{\Delta}\overset{\scriptscriptstyle(1)}{\not{g}} - \frac{2}{r}\widetilde{\nabla}\otimes\overset{\scriptscriptstyle(1)}{\not{g}}_P - \frac{1}{r^2}(1-\mu)\overset{\scriptscriptstyle(1)}{\not{g}} + \frac{2}{r^2}\,\mathrm{d}r\otimes_{\mathrm{s}}\overset{\scriptscriptstyle(1)}{\not{g}}_P = \frac{2}{r}\tilde{\mathrm{d}}r\otimes_{\mathrm{s}}\slashed{\mathrm{div}}\overset{\scriptscriptstyle(1)}{\hat{\slashed{g}}} + \widetilde{\nabla}\otimes\underset{\circ}{\overset{\scriptscriptstyle(1)}{\slashed{f}}} + \slashed{\nabla}\otimes\underset{\circ}{\overset{\scriptscriptstyle(1)}{\tilde{f}}} - \frac{2}{r}\,\mathrm{d}r\otimes_{\mathrm{s}}\underset{\circ}{\overset{\scriptscriptstyle(1)}{\slashed{f}}}. \tag{3.25}$$

The equations for the linearised S-metric:

$$\widetilde{\Box}\overset{\scriptscriptstyle(1)}{\hat{\slashed{g}}} + \slashed{\Delta}\overset{\scriptscriptstyle(1)}{\hat{\slashed{g}}} - \frac{2}{r}\widetilde{\nabla}_P\overset{\scriptscriptstyle(1)}{\hat{\slashed{g}}} - \frac{4}{r}\frac{\mu}{r}\overset{\scriptscriptstyle(1)}{\hat{\slashed{g}}} = \slashed{\nabla}\hat{\otimes}\underset{\circ}{\overset{\scriptscriptstyle(1)}{\slashed{f}}}, \tag{3.26}$$

$$\widetilde{\Box}\slashed{\mathrm{tr}}\overset{\scriptscriptstyle(1)}{\slashed{g}} + \slashed{\Delta}\slashed{\mathrm{tr}}\overset{\scriptscriptstyle(1)}{\slashed{g}} + \frac{2}{r}\widetilde{\nabla}_P\slashed{\mathrm{tr}}\overset{\scriptscriptstyle(1)}{\slashed{g}} + \frac{2}{r^2}\slashed{\mathrm{tr}}\overset{\scriptscriptstyle(1)}{\slashed{g}} = \frac{4}{r^2}\overset{\scriptscriptstyle(1)}{\hat{\tilde{g}}}_{PP} + \frac{2}{r^2}\mathrm{tr}_{\tilde{g}_M}\overset{\scriptscriptstyle(1)}{\tilde{g}} + 2\slashed{\mathrm{div}}\underset{\circ}{\overset{\scriptscriptstyle(1)}{\slashed{f}}} + \frac{4}{r}\underset{\circ}{\overset{\scriptscriptstyle(1)}{\tilde{f}}}_P. \tag{3.27}$$

Finally, the linearised generalised wave gauge conditions:

$$-\tilde{\delta}\overset{\scriptscriptstyle(1)}{\hat{\tilde{g}}} + \slashed{\mathrm{div}}\overset{\scriptscriptstyle(1)}{\not{g}} - \frac{1}{2}\tilde{\mathrm{d}}\slashed{\mathrm{tr}}\overset{\scriptscriptstyle(1)}{\slashed{g}} = \underset{\circ}{\overset{\scriptscriptstyle(1)}{\tilde{f}}}, \tag{3.28}$$

$$-\tilde{\delta}\overset{\scriptscriptstyle(1)}{\not{g}} - \frac{1}{2}\slashed{\nabla}\mathrm{tr}_{\tilde{g}_M}\overset{\scriptscriptstyle(1)}{\tilde{g}} + \slashed{\mathrm{div}}\overset{\scriptscriptstyle(1)}{\hat{\slashed{g}}} = \underset{\circ}{\overset{\scriptscriptstyle(1)}{\slashed{f}}}. \tag{3.29}$$

Here, we have defined $\underset{\circ}{\overset{\scriptscriptstyle(1)}{\tilde{f}}} \in \tilde{\mathscr{T}}^1(\mathcal{M})$ and $\underset{\circ}{\overset{\scriptscriptstyle(1)}{\slashed{f}}} \in \slashed{\mathscr{T}}^1(\mathcal{M})$ according to

$$\underset{\circ}{\overset{\scriptscriptstyle(1)}{\tilde{f}}} := \overset{\scriptscriptstyle(1)}{\tilde{f}} - \frac{2}{r}\overset{\scriptscriptstyle(1)}{\tilde{g}}_P + \frac{1}{r}\tilde{\mathrm{d}}r\,\slashed{\mathrm{tr}}\overset{\scriptscriptstyle(1)}{\slashed{g}}$$

$$\underset{\circ}{\overset{\scriptscriptstyle(1)}{\slashed{f}}} := \overset{\scriptscriptstyle(1)}{\slashed{f}} - \frac{2}{r}\overset{\scriptscriptstyle(1)}{\not{g}}_P$$

noting that in the above the three explicit terms in $\overset{\scriptscriptstyle(1)}{g}$ correspond to the relative projections of $\mathring{f}(\overset{\scriptscriptstyle(1)}{g})$ onto $\tilde{\mathscr{T}}^1(\mathcal{M})$ and $\slashed{\mathscr{T}}^1(\mathcal{M})$ respectively.

Following[1], in the remainder of the paper we shall use the collective notation $\mathscr{S}$ with

$$\mathscr{S} := \left(\overset{\scriptscriptstyle(1)}{\hat{\tilde{g}}}, \mathrm{tr}_{\tilde{g}_M}\overset{\scriptscriptstyle(1)}{\tilde{g}}, \overset{\scriptscriptstyle(1)}{\not{g}}, \overset{\scriptscriptstyle(1)}{\hat{\slashed{g}}}, \slashed{\mathrm{tr}}\overset{\scriptscriptstyle(1)}{\slashed{g}}\right)$$

to denote a solution to the system of equations (3.23)-(3.29).

We make the following remarks.

Remark 7. In view of the fact that in section 3.3.2.1 we fixed the differential structure of $\mathcal{M}$, one could instead decompose the perturbed metric $\boldsymbol{g}$ as in section 3.2.2.1. A further decomposition of (3.2) and (3.3) would lead to a system of equations for the quantities

$$\tilde{\boldsymbol{g}}, \not{\boldsymbol{g}}, \not{\boldsymbol{g}}$$

Upon linearisation, this yields the system (3.23)-(3.29) for

$$\overset{(1)}{\tilde{g}}, \overset{(1)}{\not{g}}, \overset{(1)}{\not{g}}$$

where we now recognise the tensor fields $\overset{(1)}{\tilde{g}}, \overset{(1)}{\not{g}}, \overset{(1)}{\not{g}}$ as the linearisation of the quantities $\tilde{\boldsymbol{g}}, \not{\boldsymbol{g}}$ and $\not{\boldsymbol{g}}$:

$$\begin{aligned} \tilde{\boldsymbol{g}} - \tilde{g}_M &\equiv \overset{(1)}{\tilde{g}}, \\ \not{\boldsymbol{g}} &\equiv \overset{(1)}{\not{g}}, \\ \not{\boldsymbol{g}} - \not{g}_M &\equiv \overset{(1)}{\not{g}}. \end{aligned}$$

Since, however, the aforementioned decompositon of (3.2) and (3.3) is rather cumbersome, we prefer our more direct approach.

Remark 8. We note that given a solution $\mathscr{S}$ to the system of equations (3.23)-(3.29) one constructs a solution $\overset{(1)}{g}$ to the system of equations (3.21)-(3.22) by defining

$$\overset{(1)}{g} = \overset{(1)}{\hat{\tilde{g}}} + \frac{1}{2}\tilde{g}_M \operatorname{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}} + \overset{(1)}{\not{g}} + \overset{(1)}{\hat{\not{g}}} + \frac{1}{2}\not{g}_M \not{\mathrm{tr}}\overset{(1)}{\not{g}}.$$

Remark 9. We will interchangeably refer to the system of equations (3.23)-(3.29) as both the equations of linearised gravity and system of gravitational perturbations.

3.4 Special solutions: Pure gauge and linearised Kerr

In this section we introduce two special classes of solutions to the equations of linearised gravity, namely the pure gauge and linearised Kerr solutions.

We note that theorem 2 of section 6 will state that sufficiently regular solutions to the equations of linearised gravity will decay to a sum of a pure gauge and linearised Kerr solution.

3.4.1 Special solutions I: Pure gauge solutions $\mathscr{G}$

The first class of solutions we introduce are the pure gauge solutions which arise as the linearisation of a 1-parameter family of Lorentzian metrics on $\mathcal{M}$ given as the pullback of g_M under a 1-parameter family of diffeomorphisms on $\mathcal{M}$ which preserve the generalised wave gauge of section 3.3.2.1 to first order.

Indeed, let $v \in \mathscr{T}^1(\mathcal{M})$ be a solution to the equation

$$\Box v = f(\mathcal{L}_{v^\sharp} g_M) \tag{3.30}$$

where f is the map of section 3.3.1. Subsequently denoting by $\boldsymbol{\phi}_\epsilon$ the smooth 1-parameter family of diffeomorphisms on $\mathcal{M}$ generated by $v^\sharp$ it follows that the corresponding smooth 1-parameter family of Lorentzian metrics $(\boldsymbol{\phi}_{-\epsilon})^* g_M \in \mathscr{T}^2_{\text{sym}}(\mathcal{M})$ define a smooth 1-parameter family of solutions to the Einstein vacuum equations. Moreover, the conditions (3.30) ensure that, to first order in ϵ, the identity map

$$\text{Id} : \Big(\mathcal{M}, (\boldsymbol{\phi}_{-\epsilon})^* g_M\Big) \to \Big(\mathcal{M}, g_M\Big) \tag{3.31}$$

is an $f((\boldsymbol{\phi}_{-\epsilon})^* g_M)$-wave map. Indeed, to first order in ϵ

$$(\boldsymbol{\phi}_{-\epsilon})^* g_M - g_M \equiv \mathcal{L}_{v^\sharp} g_M$$

and so

$$\bigg(\Big((\boldsymbol{\phi}_{-\epsilon})^* g_M\Big)^{-1} \cdot C_{(\phi_{-\epsilon})^* g_M, g_M}\bigg)_\flat - f((\boldsymbol{\phi}_{-\epsilon})^* g_M) \equiv \Box v - f(\mathcal{L}_{v^\sharp} g_M).$$

Here, $C_{(\phi_{-\epsilon})^* g_M, g_M}$ is the connection tensor between $(\boldsymbol{\phi}_{-\epsilon})^* g_M$ and g_M and recall, from section 3.1.1, that the above is equivalent to the identity map (3.31) being an f-wave map to first order.

We have therefore shown the following.

Proposition 3.15. *Let $\tilde{v} \in \tilde{\mathscr{T}}^1(\mathcal{M})$ and $\not{v} \in \not{\mathscr{T}}^1(\mathcal{M})$ be solutions to the system of equations*

$$\tilde{\Box}\tilde{v} + \not{\Delta}\tilde{v} + \frac{2}{r}\widetilde{\nabla}_P \tilde{v} - \frac{2}{r^2}\tilde{\mathrm{d}}r\, \tilde{v}_P = \frac{2}{r}\tilde{\mathrm{d}}r\, \not{\mathrm{d}}\mathrm{iv}\not{v} + \tilde{f}(\mathcal{L}_{(\tilde{v}+\not{v})^\sharp} g_M), \tag{3.32}$$

$$\tilde{\Box}\not{v} + \not{\Delta}\not{v} - \frac{1}{r^2}\not{v} = -\frac{2}{r}\not{\nabla}\tilde{v}_P + \not{f}(\mathcal{L}_{(\tilde{v}+\not{v})^\sharp} g_M) \tag{3.33}$$

where $\tilde{f}(\mathcal{L}_{v^\sharp} g_M) \in \tilde{\mathscr{T}}^1(\mathcal{M})$ and $\not{f}(\mathcal{L}_{v^\sharp} g_M) \in \not{\mathscr{T}}^1(\mathcal{M})$ are the relative projections of $f(\mathcal{L}_{v^\sharp} g_M)$ onto $\tilde{\mathscr{T}}^1(\mathcal{M})$ and $\not{\mathscr{T}}^1(\mathcal{M})$ respectively.

Then the collection[6]

$$\mathscr{G} = \left(\hat{\tilde{\overset{(1)}{g}}}, \mathrm{tr}_{\tilde{g}_M}\tilde{\overset{(1)}{g}}, \overset{(1)}{\not{g}}, \hat{\overset{(1)}{\not{g}}}, \not{\mathrm{tr}}\overset{(1)}{\not{g}}\right)$$

defined by

$$\begin{aligned}
\hat{\tilde{\overset{(1)}{g}}} &= \widetilde{\nabla}\hat{\otimes}\tilde{v}, \\
\mathrm{tr}_{\tilde{g}_M}\tilde{\overset{(1)}{g}} &= -2\tilde{\delta}\tilde{v}, \\
\overset{(1)}{\not{g}} &= \not{\nabla}\tilde{v} + \tilde{\mathrm{d}}\not{v} - \frac{2}{r}\tilde{\mathrm{d}}r\otimes_{\mathrm{s}}\not{v}, \\
\hat{\overset{(1)}{\not{g}}} &= \not{\nabla}\hat{\otimes}\not{v}, \\
\not{\mathrm{tr}}\overset{(1)}{\not{g}} &= 2\not{\mathrm{d}}\mathrm{iv}\not{v} + \frac{4}{r}\tilde{v}_P
\end{aligned}$$

is a smooth solution to the equations of linearised gravity. We such a solution **a pure gauge solution** *that is generated by $\tilde{v}$ and $\not{v}$.*

Note that one can of course verify the above explicitly from the equations (3.23)-(3.29).

We make the following remark.

Remark 10. We will show in section Lemma 5.6 that there do indeed exist non-trivial solutions to the system of equations (3.32)-(3.33).

3.4.2 Special solutions II: The 4-dimensional linearised Kerr family $\mathscr{K}$

We continue with the second class of solutions that are to be introduced namely the linearised Kerr solutions which arise as the linearisation of the Kerr exterior family about $\left(\mathcal{M}, g_M\right)$.

In order to formally linearise the Kerr exterior family about $\left(\mathcal{M}, g_M\right)$ we consider a smooth 1-parameter family of functions $\boldsymbol{M} : (-\epsilon, \epsilon) \to \mathbb{R}$ given by $\boldsymbol{M}(\epsilon) = M + \epsilon \cdot \mathfrak{m}$ where $\mathfrak{m} \in \mathbb{R}$ and a smooth 1-parameter family of functions $\boldsymbol{a} : (-\epsilon, \epsilon) \to \mathbb{R}$ given by $\boldsymbol{a} = \epsilon \cdot \mathfrak{a}$ where $\mathfrak{a} \in \mathbb{R}$. This 2-parameter family subsequently generates the smooth 2-parameter family of Kerr exterior solutions to the Einstein vacuum equations on $\mathcal{M}$:

$$\begin{aligned}
g_{\boldsymbol{M},\boldsymbol{a}} := &-(1-\boldsymbol{\mu})\,\mathrm{d}t^{*2} + 2\boldsymbol{\mu}\,\mathrm{d}t^*\,\mathrm{d}r + (1+\boldsymbol{\mu})\,\mathrm{d}r^2 \\
&-2\boldsymbol{\mu}\boldsymbol{a}\sin^2\theta\,\mathrm{d}t^*\,\mathrm{d}\varphi - 2(1+\boldsymbol{\mu})\boldsymbol{a}\sin^2\theta\,\mathrm{d}r\,\mathrm{d}\varphi \\
&+r^2\mathring{\not{g}} \\
&+\mathcal{O}\left(\boldsymbol{a}^2\right).
\end{aligned} \tag{3.34}$$

[6]This notation is borrowed from [1].

Here, $\boldsymbol{\mu} = \frac{2\boldsymbol{M}}{r}$, (θ, φ) are coordinates on S^2 and we recall that $\mathring{g}$ denotes the metric of the unit round sphere. Moreover, we have dispensed with the ϵ notation and neglected to present terms that are higher than linear order in $\boldsymbol{a}$. Lastly, we have identified the so-called Kerr-star coordinate system ([36]) on $\mathcal{M}$ with that of the Schwarzschild-star coordinate system in 3.2.1 – for a full presentation of the Kerr exterior metric in this coordinate system see[36].

Subsequently, we have by explicit computation that the identity map

$$\mathrm{Id} : \big(\mathcal{M}, g_{\boldsymbol{M},\boldsymbol{a}}\big) \to \big(\mathcal{M}, g_M\big)$$

is an f-wave map to first order in ϵ:

$$\Big(\big(g_{\boldsymbol{M},\boldsymbol{a}}\big)^{-1} \cdot C_{g_{\boldsymbol{M},\boldsymbol{a}}, g_M}\Big)_\flat - f(g_{\boldsymbol{M},\boldsymbol{a}}) \equiv \mathcal{O}\big(\epsilon^2\big),$$

as can be verified from the equations (3.28)-(3.29).

We have therefore shown the $\mathfrak{a} = \mathfrak{a}_0 Y^1_0$ case of the subsequent proposition, with the full case following from explicit computation.

Proposition 3.16. *Let $\mathfrak{m} \in \mathbb{R}$ and let $\mathfrak{a}$ be a smooth function on S^2 that is given as a linear combination of the $l = 1$ spherical harmonics. Then the collection*

$$\mathscr{K}_{\mathfrak{m},\mathfrak{a}} = \Big(\overset{(1)}{\hat{\tilde{g}}}, \mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}}, \overset{(1)}{\not{g}}, \overset{(1)}{\hat{\not{g}}}, \not{\mathrm{tr}}\overset{(1)}{\not{g}}\Big)$$

defined by

$$\begin{aligned}
\overset{(1)}{\hat{\tilde{g}}} &= -\frac{1}{(1-\mu)^2}\frac{\mathfrak{m}}{r}\Big(\tilde{\star}P \hat{\otimes} \tilde{\star}P - 2\tilde{\star}P \hat{\otimes}\,\mathrm{d}r - \mathrm{d}r \hat{\otimes}\,\mathrm{d}r\Big), \\
\overset{(1)}{\not{g}} &= -\frac{1}{1-\mu}\Big(\mu\tilde{\star}P - \mathrm{d}r\Big)\not{\star}\not{\nabla}\mathfrak{a}, \\
\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}} = \overset{(1)}{\hat{\not{g}}} &= \not{\mathrm{tr}}\overset{(1)}{\not{g}} = 0
\end{aligned}$$

is a smooth (stationary) solution to the equations of linearised gravity. We call such a solution a **linearised Kerr solution** *with parameters $\mathfrak{m}$ and $\mathfrak{a}$.*

Note that one can of course verify the above explicitly from the equations (3.23)-(3.29).

We make the following remarks.

Remark 11. The above indeed defines a 4-parameter family of solutions as the function $\mathfrak{a}$ is parametrised by three real numbers $\mathfrak{a}_i$ according to

$$\mathfrak{a} = \sum_{i=-1}^{i=+1} \mathfrak{a}_i Y^1_i.$$

Remark 12. We note that changes in the axis of symmetry for the Kerr exterior metric $g_{\boldsymbol{M},\boldsymbol{a}}$ of (3.34) leads to a 4-parameter family of solutions to the Einstein vacuum equations. However, as a change of axis corresponds to a rotation of the 2-sphere, this 4-parameter family reduces to a 2-parameter family of *geometrically distinct* solutions to the Einstein vacuum equations once equivalence up to diffeomorphism is imposed. Consequently, if one linearises this 4-parameter family about Schwarzschild one arrives at a 4-parameter family of solutions to the linearised equations, a 3-parameter subfamily of which are equivalent up to a 2-parameter family of *pure gauge solutions* corresponding to rotations of the 2-sphere on $\left(\mathcal{M}, g_M\right)$. This subsequently explains the necessity for 4 distinct parameters in Proposition 3.16 as no such family of non-trivial pure gauge solutions can exist owing to the spherical symmetry of g_M.

Remark 13. Note that is the presence of the map $\overset{\mathscr{K}}{f}$ in the definition of the map f which ensures that the linearised Kerr solutions lie in the kernel of the latter (cf. Remark 8).

Remark 14. Observe that all quantities in $\mathscr{K}_{\mathfrak{m},\mathfrak{a}}$ are supported only on $l = 0, 1$ (cf. section 3.2.5.2).

4

Pure gauge and linearised Kerr invariant quantities: the Regge–Wheeler and Zerilli equations

This chapter of the thesis is concerned with the extraction of the Regge–Wheeler and Zerilli equations from the equations of linearised gravity.

Indeed, in what follows we consider certain scalar quantities associated to solutions of the equations of linearised gravity which vanish for both the special solutions of section 3.4. We then demonstrate the well-known ([23],[24]) but nevertheless remarkable phenomena in which these quantities actually decouple from the full system of gravitational perturbations into the celebrated Regge–Wheeler and Zerilli equations.

This decoupling will prove key to our analysis of the equations of linearised gravity.

4.1 Pure gauge and linearised Kerr invariant quantities

We begin in this section by constructing these *pure gauge and linearised Kerr invariant* quantities.

The quantities under consideration are then defined as in the following proposition.

In what follows, we remind the reader of the space $\Lambda(\mathcal{M})$ defined in section 3.2.5.2.

Proposition 4.1. *Let $\mathscr{S}$ be a smooth solution to the equations of linearised gravity and let $\overset{\scriptscriptstyle(1)}{g} \in \mathscr{T}^2_{sym}(\mathcal{M})$ be constructed from it in accordance with Remark 8. We define the smooth quantities*

$$\overset{\scriptscriptstyle(1)}{\Phi} := \Phi(\overset{\scriptscriptstyle(1)}{g}),$$
$$\overset{\scriptscriptstyle(1)}{\Psi} := \Psi(\overset{\scriptscriptstyle(1)}{g})$$

where Φ and Ψ are the linear maps defined as in section 3.3.1.3.

Then $\overset{\scriptscriptstyle(1)}{\Phi}$ and $\overset{\scriptscriptstyle(1)}{\Psi}$ are pure gauge and linearised Kerr invariant.

Proof. We first observe that by definition of the maps Φ and Ψ we have $\overset{\scriptscriptstyle(1)}{\Phi}, \overset{\scriptscriptstyle(1)}{\Psi} \in C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$ and thus $\overset{\scriptscriptstyle(1)}{\Phi}$ and $\overset{\scriptscriptstyle(1)}{\Psi}$ must vanish for all linearised Kerr solutions by Remark 14.

To show that $\overset{\scriptscriptstyle(1)}{\Phi}$ and $\overset{\scriptscriptstyle(1)}{\Psi}$ must in addition vanish for all pure gauge solutions we first define the quantities

$$\overset{\scriptscriptstyle(1)}{\tilde{\tau}} := \tau(\overset{\scriptscriptstyle(1)}{g}),$$
$$\overset{\scriptscriptstyle(1)}{\tilde{\eta}} := \tilde{\eta}(\overset{\scriptscriptstyle(1)}{g}),$$
$$\overset{\scriptscriptstyle(1)}{\sigma} := \sigma(\overset{\scriptscriptstyle(1)}{g})$$

where $\tilde{\tau}, \tilde{\eta}$ and σ are the linear maps defined as in section 3.3.1.3. We then observe that it suffices to establish that $\overset{\scriptscriptstyle(1)}{\tilde{\tau}}, \overset{\scriptscriptstyle(1)}{\tilde{\eta}}$ and $\overset{\scriptscriptstyle(1)}{\sigma}$ vanish for all pure gauge solutions, a fact which follows from explicitly applying the Hodge decompositions of Definition 3.9 and Definition 3.10 to the pure gauge solutions of Proposition 3.15. This completes the proposition. □

We make the following remark.

Remark 15. Observe that the above proposition holds independently of the fact that the quantities $\tilde{v}$ and $\not{\psi}$ satisfy the equations (3.32)-(3.33).

4.2 The Regge–Wheeler and Zerilli equations

We continue in this section by introducing the two scalar wave equations on $\left(\mathcal{M}, g_M\right)$ that describe the Regge–Wheeler and Zerilli equations respectively. We will demonstrate in the next section that the equations of linearised gravity force the gauge-invariant quantities $\overset{\scriptscriptstyle(1)}{\Phi}$ and $\overset{\scriptscriptstyle(1)}{\Psi}$ of the previous section to decouple into these equations.

As the Regge–Wheeler and Zerilli equations admit a formulation independent of the equations of linearised gravity we shall denote in this section solutions to such equations without the superscript (1).

The equations are defined as in the sequel.

In what follows, we remind the reader of the operators $\widetilde{\Box}$ and $\not{\Delta}$ defined in section 3.2.2.2 and the operator $\not{\mathcal{C}}^{[p]}$ defined in section 3.2.6.3.

Definition 4.1. *Let $\psi \in \Lambda(\mathcal{M})$.*

Then we say that ψ is a smooth solution to the Regge–Wheeler equation on $\big(\mathcal{M}, g_M\big)$ iff

$$\widetilde{\Box}\psi + \not{\Delta}\psi = -\frac{6}{r^2}\frac{M}{r}\psi. \tag{4.1}$$

Conversely, we say that ψ is a smooth solution to the Zerilli equation on $\mathcal{M}$ iff

$$\widetilde{\Box}\psi + \not{\Delta}\psi = -\frac{6}{r^2}\frac{M}{r}\psi + \frac{24}{r^3}\frac{M}{r}(r-3M)\not{\mathcal{C}}^{[1]}\psi + \frac{72}{r^3}\frac{M}{r}\frac{M}{r}(r-2M)\not{\mathcal{C}}^{[2]}\psi. \tag{4.2}$$

We make the following remark.

Remark 16. Note that although the Regge–Wheeler equation is well-defined outside the space $\Lambda(\mathcal{M})$ the Zerilli equation is not owing to the fact that the operator $\not{\mathcal{C}}^{[p]}$ is only defined over the space $\Lambda(\mathcal{M})$. In any case, the above definition will suffice in the context of the equations of linearised gravity.

We also in addition state the following well-posedness result for the Regge–Wheeler and Zerilli equations which can be shown, for instance, by employing a spherical harmonic decomposition and then using standard techniques to establish well-posedness of the corresponding two dimensional wave equation. Note that we shall exploit this statement of well-posedness in section 5.1.2.

In what follows, we remind the reader of the initial Cauchy hypersurface Σ and its associated normal defined as in section 3.2.3.1 and the space $\Lambda(\Sigma)$ defined as in section 3.2.5.5.

Proposition 4.2. *Let $\psi_0, \psi_1 \in C^\infty(\Sigma) \cap \Lambda(\Sigma)$.*

Then there exists a unique solution $\psi \in C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$ to the Regge–Wheeler equation on $\mathcal{M}$ such that

$$\big(\psi, n(\psi)\big)\Big|_\Sigma = \big(\psi_0, \psi_1\big).$$

In addition, there exists a unique solution $\psi \in C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$ to the Zerilli equation on $\mathcal{M}$ such that

$$\big(\psi, n(\psi)\big)\Big|_\Sigma = \big(\psi_0, \psi_1\big).$$

We make the following remarks.

Remark 17. Observe that one can write the Zerilli equation as

$$\widetilde{\Box}\psi + \not{\Delta}\psi = -\frac{6}{r^2}\frac{M}{r}\psi + \mathfrak{Z}\psi$$

where the operator $\mathfrak{Z}$ is defined as in section 3.3.1.3.

4.3 The connection to the system of gravitational perturbations

In this section we reveal the remarkable connection, as first discovered by Regge–Wheeler [23] and Zerilli [24] in the context of a mode decomposition of the linearised Einstein equations (see section 2.2 of the overview), between the Regge–Wheeler and Zerilli equations and the system of gravitational perturbations.

The following version of this result relies heavily on the paper [45] of Chaverra, Ortiz and Sarbach.

Theorem 4.3. *Let $\mathscr{S}$ be a smooth solution to the equations of linearised gravity. Then the gauge-invariant quantities $\overset{\scriptscriptstyle(1)}{\Phi}$ and $\overset{\scriptscriptstyle(1)}{\Psi}$ of Proposition 4.1 satisfy the Regge–Wheeler and Zerilli equations respectively:*

$$\widetilde{\Box}\overset{\scriptscriptstyle(1)}{\Phi} + \not{\Delta}\overset{\scriptscriptstyle(1)}{\Phi} = -\frac{6}{r^2}\frac{M}{r}\overset{\scriptscriptstyle(1)}{\Phi}$$

and

$$\widetilde{\Box}\overset{\scriptscriptstyle(1)}{\Psi} + \not{\Delta}\overset{\scriptscriptstyle(1)}{\Psi} = -\frac{6}{r^2}\frac{M}{r}\overset{\scriptscriptstyle(1)}{\Psi} + \frac{24}{r^3}\frac{M}{r}(r-3M)\not{\zeta}^{[1]}\overset{\scriptscriptstyle(1)}{\Psi} + \frac{72}{r^3}\frac{M}{r}\frac{M}{r}(r-2M)\not{\zeta}^{[2]}\overset{\scriptscriptstyle(1)}{\Psi}.$$

Proof. We first claim that there exists two unique functions $\overset{\scriptscriptstyle(1)}{\phi}, \overset{\scriptscriptstyle(1)}{\psi} \in C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$ which satisfy the Regge–Wheeler and Zerilli equation respectively and for which

$$\begin{aligned}
\overset{\scriptscriptstyle(1)}{\tilde{\tau}} &= \widetilde{\nabla}\hat{\otimes}\tilde{\mathrm{d}}\Big(r\overset{\scriptscriptstyle(1)}{\psi}\Big) + 6\mu\,\mathrm{d}r\hat{\tilde{\otimes}}\not{\zeta}^{[1]}\tilde{\mathrm{d}}\overset{\scriptscriptstyle(1)}{\psi},\\
\overset{\scriptscriptstyle(1)}{\tilde{\eta}} &= -\tilde{\star}\tilde{\mathrm{d}}\Big(r\overset{\scriptscriptstyle(1)}{\phi}\Big),\\
\overset{\scriptscriptstyle(1)}{\sigma} &= -2r\not{\Delta}\overset{\scriptscriptstyle(1)}{\psi} + 4\widetilde{\nabla}_P\overset{\scriptscriptstyle(1)}{\psi} + 12\mu r^{-1}(1-\mu)\not{\zeta}^{[1]}\overset{\scriptscriptstyle(1)}{\psi}
\end{aligned}$$

where $\overset{\scriptscriptstyle(1)}{\tilde{\tau}}, \overset{\scriptscriptstyle(1)}{\tilde{\eta}}$ and σ are as in Proposition 4.1. We then additionally claim that in fact $\overset{\scriptscriptstyle(1)}{\Phi} \equiv \overset{\scriptscriptstyle(1)}{\psi}$ and $\overset{\scriptscriptstyle(1)}{\Psi} \equiv \overset{\scriptscriptstyle(1)}{\psi}$ from which the theorem follows.

Indeed, we have by explicit computation combined with the commutation formulae of Lemma 3.14 that the equations of linearised gravity (3.23)-(3.29) force the quantities $\overset{\scriptscriptstyle(1)}{\tilde{\tau}}, \overset{\scriptscriptstyle(1)}{\tilde{\eta}}$

and σ to satisfy the following system of wave equations:

$$\widetilde{\Box}\overset{(1)}{\hat{\tilde{\tau}}} + \slashed{\Delta}\overset{(1)}{\hat{\tilde{\tau}}} + \frac{2}{r}\widetilde{\nabla}_P\overset{(1)}{\hat{\tilde{\tau}}} - \frac{2}{r}\left(\widetilde{\nabla}\hat{\otimes}\overset{(1)}{\hat{\tilde{\tau}}}\right)_P - \frac{4}{r}\frac{\mu}{r}\overset{(1)}{\hat{\tilde{\tau}}} = -\frac{1}{r}\,\mathrm{d}r\hat{\tilde{\otimes}}\tilde{\mathrm{d}}\overset{(1)}{\sigma}, \tag{4.3}$$

$$\widetilde{\Box}\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{\tau}} + \slashed{\Delta}\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{\tau}} = 0, \tag{4.4}$$

$$\widetilde{\Box}\overset{(1)}{\tilde{\eta}} + \slashed{\Delta}\overset{(1)}{\tilde{\eta}} - \frac{2}{r}\left(\widetilde{\nabla}\overset{(1)}{\tilde{\eta}}\right)_P + \frac{2}{r^2}\,\mathrm{d}r\;\overset{(1)}{\tilde{\eta}}_P = 0, \tag{4.5}$$

$$\widetilde{\Box}\overset{(1)}{\sigma} + \slashed{\Delta}\overset{(1)}{\sigma} + \frac{2}{r}\widetilde{\nabla}_P\overset{(1)}{\sigma} + \frac{2}{r^2}\overset{(1)}{\sigma} = \frac{4}{r^2}\overset{(1)}{\hat{\tilde{\tau}}}_{PP} \tag{4.6}$$

along with the divergence relations

$$\tilde{\delta}\overset{(1)}{\hat{\tilde{\tau}}} + \frac{1}{2}\tilde{\mathrm{d}}\overset{(1)}{\sigma} = 0, \tag{4.7}$$

$$\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{\tau}} = 0, \tag{4.8}$$

$$\tilde{\delta}\overset{(1)}{\tilde{\eta}} = 0. \tag{4.9}$$

Here, $\overset{(1)}{\hat{\tilde{\tau}}} := \overset{(1)}{\tilde{\tau}} - \frac{1}{2}\tilde{g}_M\,\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{\tau}} = \overset{(1)}{\tilde{\tau}}$ in light of equation (4.8). Moreover, in the above we exploit the fact that the operators $\slashed{\mathcal{D}}_1^\star$ and $\slashed{\nabla}\hat{\otimes}\slashed{\mathcal{D}}_1^\star$ have trivial kernels over the space $\Lambda(\mathcal{M})$ (cf. section 3.2.5).

Introducing now the quantity $\overset{(1)}{\tilde{\zeta}} := \overset{(1)}{\hat{\tilde{\tau}}}_P - \frac{r}{2}\,\tilde{\mathrm{d}}\overset{(1)}{\sigma}$ we re-express the above system according to

$$\widetilde{\Box}\overset{(1)}{\tilde{\eta}} + \slashed{\Delta}\overset{(1)}{\tilde{\eta}} - \frac{2}{r}\left(\widetilde{\nabla}\overset{(1)}{\tilde{\eta}}\right)_P + \frac{2}{r^2}\,\mathrm{d}r\;\overset{(1)}{\tilde{\eta}}_P = 0, \tag{4.10}$$

$$\tilde{\delta}\overset{(1)}{\tilde{\eta}} = 0 \tag{4.11}$$

and

$$-\frac{1}{r^2}\tilde{\mathrm{d}}\left(r^2\tilde{\delta}\overset{(1)}{\tilde{\zeta}} + 3M\overset{(1)}{\sigma}\right) + \slashed{\Delta}\overset{(1)}{\tilde{\zeta}} = 0, \tag{4.12}$$

$$\frac{1}{r^2}\tilde{\mathrm{d}}\left(4r\overset{(1)}{\tilde{\zeta}}_P - r^3\slashed{\Delta}_\zeta\overset{(1)}{\sigma}\right) + 2\slashed{\Delta}\overset{(1)}{\tilde{\zeta}} = 0, \tag{4.13}$$

$$-2\tilde{\delta}\overset{(1)}{\tilde{\zeta}} + r\widetilde{\Box}\overset{(1)}{\sigma} = 0, \tag{4.14}$$

$$\tilde{\mathrm{d}}\overset{(1)}{\tilde{\zeta}} = 0. \tag{4.15}$$

Here, one arrives at the relations (4.14) and (4.15) by contracting (4.7) with P and $\tilde{\star}\,\mathrm{d}r$ respectively whereas the equations (4.12) and (4.13) follow from contracting (4.3) with P and utilising (4.6) in conjunction with (4.14).

Now, arguing as in the Poincaré lemma for the simply connected manifold $\mathcal{Q} \cong \mathcal{H} \times \mathbb{R}$ of section 3.2.2.1, it follows from relations (4.11) and (4.15) that there exists two unique functions $\overset{(1)}{\phi}, \overset{(1)}{\varphi} \in C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$ such that

$$\overset{(1)}{\tilde{\eta}} = -\tilde{\star}\tilde{\mathrm{d}}\Big(r\overset{(1)}{\phi} + f'\Big) \tag{4.16}$$

and

$$\overset{(1)}{\tilde{\zeta}} = \tilde{\mathrm{d}}\Big(\overset{(1)}{\varphi} + g'\Big). \tag{4.17}$$

Here, f' and g' are smooth functions on S^2 which are supported on the $l \geq 2$ spherical harmonics. Integrating the equations (4.10) and (4.12)-(4.13) therefore yields

$$\widetilde{\Box}\overset{(1)}{\phi} + \slashed{\Delta}\overset{(1)}{\phi} = -\frac{6}{r^2}\frac{M}{r}\overset{(1)}{\phi}$$

and

$$\slashed{\Delta}_\zeta\widetilde{\Box}\overset{(1)}{\varphi} + \slashed{\Delta}\slashed{\Delta}\overset{(1)}{\varphi} + \frac{2}{r^2}\slashed{\Delta}\overset{(1)}{\varphi} - \frac{6\mu}{r^3}\widetilde{\nabla}_P\overset{(1)}{\varphi} = 0$$

where we have used the functions f' and g' to remove the constants of integration. Thus, recalling that the operator $\slashed{\mathcal{D}}^{[1]}$ is a bijection on the space $C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$, the unique functions $\overset{(1)}{\phi}, \overset{(1)}{\psi} \in C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$ with $\overset{(1)}{\psi}$ defined according

$$\slashed{\mathcal{D}}^{[1]}\overset{(1)}{\varphi} := \overset{(1)}{\Psi} \tag{4.18}$$

satisfy the Regge–Wheeler and Zerilli equations of section 4.2:

$$\begin{aligned}
\widetilde{\Box}\overset{(1)}{\phi} + \slashed{\Delta}\overset{(1)}{\phi} &= -\frac{6}{r^2}\frac{M}{r}\overset{(1)}{\phi}, \\
\widetilde{\Box}\overset{(1)}{\psi} + \slashed{\Delta}\overset{(1)}{\psi} &= -\frac{6}{r^2}\frac{M}{r}\overset{(1)}{\psi} + \frac{24}{r^3}\frac{M}{r}(r-3M)\slashed{\mathcal{D}}^{[1]}\overset{(1)}{\psi} + \frac{72}{r^5}\frac{M}{r}\frac{M}{r}(r-2M)\slashed{\mathcal{D}}^{[2]}\overset{(1)}{\psi}.
\end{aligned}$$

Here, we use the commutation formulae of Lemma 3.14. Moreover, from equation (4.13) combined with the relations (4.16)-(4.18) one finds

$$\begin{aligned}
\overset{(1)}{\tilde{\tau}} &= \widetilde{\nabla}\hat{\otimes}\tilde{\mathrm{d}}\Big(r\overset{(1)}{\psi}\Big) + 6\mu\,\mathrm{d}r\hat{\otimes}\slashed{\mathcal{D}}^{[1]}\tilde{\mathrm{d}}\overset{(1)}{\psi}, \\
\overset{(1)}{\tilde{\eta}} &= -\tilde{\star}\tilde{\mathrm{d}}\Big(r\overset{(1)}{\phi}\Big), \\
\overset{(1)}{\sigma} &= -2r\slashed{\Delta}\overset{(1)}{\psi} + 4\widetilde{\nabla}_P\overset{(1)}{\psi} + 12\mu r^{-1}(1-\mu)\slashed{\mathcal{D}}^{[1]}\overset{(1)}{\psi}
\end{aligned}$$

which yields the first claim.

Finally, to complete the theorem we note from the relations (4.16) and (4.13) the identities

$$\slashed{\Delta}\overset{(1)}{\phi} + \frac{2}{r^2}\overset{(1)}{\phi} = -r\tilde{\star}\tilde{\mathrm{d}}\Big(r^{-2}\overset{(1)}{\tilde{\eta}}\Big), \qquad \slashed{\Delta}\overset{(1)}{\psi} = \frac{2}{r}\slashed{\mathcal{D}}^{[1]}\overset{(1)}{\hat{\tilde{\tau}}}_{PP} - \frac{1}{2}\frac{1}{r}\overset{(1)}{\sigma} - \slashed{\mathcal{D}}^{[1]}\widetilde{\nabla}_P\overset{(1)}{\sigma}$$

and thus by uniqueness we must have $\overset{(1)}{\phi} = \overset{(1)}{\Phi}$ and $\overset{(1)}{\psi} = \overset{(1)}{\Psi}$ (recalling the definition of $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ from Proposition 4.1). $\square$

We make the following remark.

Remark 18. The most efficient way to derive the system of equations (4.3)-(4.9) is to utilise the Regge–Wheeler gauge of section A in the Appendix and then exploit the fact that the quantities $\overset{(1)}{\tilde{\tau}}, \overset{(1)}{\tilde{\eta}}$ and $\overset{(1)}{\sigma}$ are gauge-invariant.

5

The Cauchy problem for the equations of linearised gravity and gauge-normalisation of initial data

This chapter of the thesis is concerned with the Cauchy problem for the equations of linearised gravity and solutions to the equations of linearsed gravity the Cauchy data of which has been 'gauge-normalised'.

In section 5.1 the well-posedness of the Cauchy problem for the equations of linearised gravity is established.

Finally, in section 5.2 we consider solutions to the equations of linearised gravity the Cauchy data of which has been gauge-normalised via the addition of a pure gauge solution.

It is these gauge-normalised solutions that will be subject to the decay statement of Theorem 2.

5.1 Initial data and well-posedness for the equations of linearised gravity

In this section we establish a well-posedness theorem for the equations of linearised gravity as a Cauchy initial value problem, thereby initiating their formal analysis.

5.1.1 Initial data for the equations of linearised gravity

We begin with a specification of Cauchy initial data for the equations of linearised gravity, a process which is complicated by the existence of *constraints*.

5.1.1.1 Admissible initial data sets for the equations of linearised gravity

First we provide a definition of *admissible* initial data for the equations of linearised gravity. Such data is to be defined on the initial Cauchy hypersurface Σ of section 3.2.3.

In what follows, we remind the reader of the contents of section 3.2.3.1 on S_ν-tensor analysis, in particular the decomposition of the second fundamental form of Σ, along with the definition of the lapse quantity $\overset{(1)}{\bar{h}}$ and the future-pointing unit normal n to Σ.

Definition 5.1. *We consider the following quantities:*

- *the smooth functions* $\overset{(1)}{N}, \overset{(1)}{\bar{b}}, \overset{(1)}{\bar{h}}, \overset{(1)}{\not{\mathrm{tr}}\not{h}}, \overset{(1)}{\bar{k}}$ *and* $\left(\overset{(1)}{\mathrm{tr}k}\right)$ *on* Σ
- *the smooth* S_ν *1-forms* $\overset{(1)}{\not{b}}, \overset{(1)}{\not{h}}$ *and* $\overset{(1)}{\not{k}}$
- *the smooth, symmetric, traceless 2-covariant* S_ν*-tensor fields* $\overset{(1)}{\hat{\not{h}}}$ *and* $\overset{(1)}{\hat{\not{k}}}$

Then we say that the collection

$$\mathscr{A} := \left(\overset{(1)}{N}, \overset{(1)}{\bar{b}}, \overset{(1)}{\not{b}}, \overset{(1)}{\bar{h}}, \overset{(1)}{\not{h}}, \overset{(1)}{\hat{\not{h}}}, \overset{(1)}{\not{\mathrm{tr}}\not{h}}, \overset{(1)}{\bar{k}}, \overset{(1)}{\not{k}}, \overset{(1)}{\hat{\not{k}}}, \left(\overset{(1)}{\mathrm{tr}k}\right)\right)$$

is an admissible initial data set for the equations of linearised gravity iff the following identities hold:

$$0 = \mathscr{H}\,\overset{(1)}{\not{\mathrm{tr}}\not{h}} - \frac{2}{r^3}\not{\nabla}_\nu\left(\frac{1}{\bar{h}}r^3\,\not{\mathrm{div}}\overset{(1)}{\not{h}}\right) + \frac{2}{\bar{h}}\not{\Delta}\overset{(1)}{\bar{h}} - \not{\mathrm{div}}\,\not{\mathrm{div}}\overset{(1)}{\hat{\not{h}}} - \frac{4}{r^2}\bar{h}^2\not{\nabla}_\nu\left(\frac{r}{\bar{h}^4}\overset{(1)}{\bar{h}}\right) - 3\bar{k}\,\overset{(1)}{\bar{k}} + \frac{4}{3}\mathrm{tr}k\left(\overset{(1)}{\mathrm{tr}k}\right), \tag{5.1}$$

$$0 = \not{\mathrm{div}}\overset{(1)}{\not{k}} - \frac{2}{3}\not{\nabla}_\nu\left(\overset{(1)}{\mathrm{tr}k}\right) + \frac{1}{r^3}\not{\nabla}_\nu\left(r^3\overset{(1)}{\bar{k}}\right) + \frac{3}{4}\bar{k}\not{\nabla}_\nu\overset{(1)}{\not{\mathrm{tr}}\not{h}} - \frac{3}{2}\frac{\bar{k}}{\bar{h}}\not{\mathrm{div}}\overset{(1)}{\not{h}}, \tag{5.2}$$

$$0 = \not{\mathrm{div}}\overset{(1)}{\hat{\not{k}}} + \frac{1}{r^3}\not{\nabla}_\nu\left(r^3\overset{(1)}{\not{k}}\right) - \frac{3}{2}\not{\nabla}\overset{(1)}{\bar{k}} - \frac{3}{2}\frac{\bar{k}}{\bar{h}}\not{\nabla}\overset{(1)}{\bar{h}} - \frac{1}{2}\left(\frac{2}{3}\mathrm{tr}k - \bar{k}\right)\not{\mathrm{div}}\overset{(1)}{\hat{\not{h}}}. \tag{5.3}$$

Here, $\mathscr{H}$ *is the operator defined by*

$$\mathscr{H}\cdot = \frac{1}{r^3}\not{\nabla}_\nu\left(r^3\not{\nabla}_\nu\,\cdot\right) + \frac{1}{2}\not{\Delta}\cdot + \frac{1}{r^2}\left(1 - \frac{r}{1+\mu}\not{\nabla}_\nu\bar{h}\right)\cdot\,. \tag{5.4}$$

Given such an admissible data set, initial data on the hypersurface Σ *for the equations of linearised gravity is defined as follows.*

For the symmetric, traceless 2-covariant $\mathcal{Q}$-tensor $\overset{(1)}{\hat{\tilde{g}}}$

$$
\begin{aligned}
\overset{(1)}{\hat{\tilde{g}}}_{nn}\Big|_{\Sigma} &:= -\frac{1}{N}\overset{(1)}{N} + \frac{1}{\bar{h}}\overset{(1)}{\bar{h}},\\
\left(\mathcal{L}_n\overset{(1)}{\hat{\tilde{g}}}\right)_{nn}\Big|_{\Sigma} &:= \overset{(1)}{\underline{w}}_1,
\end{aligned}
$$

$$
\begin{aligned}
\overset{(1)}{\hat{\tilde{g}}}_{n\nu}\Big|_{\Sigma} &:= \overset{(1)}{\bar{b}} - \frac{2\mu}{\bar{h}}\overset{(1)}{\bar{h}},\\
&:= \overset{(1)}{\bar{s}},\\
\left(\mathcal{L}_n\overset{(1)}{\hat{\tilde{g}}}\right)_{n\nu}\Big|_{\Sigma} &:= \overset{(1)}{\bar{w}}
\end{aligned}
$$

and

$$
\begin{aligned}
\overset{(1)}{\hat{\tilde{g}}}_{\nu\nu}\Big|_{\Sigma} &:= -\frac{1}{N}\overset{(1)}{N} + \frac{1}{\bar{h}}\overset{(1)}{\bar{h}},\\
\left(\mathcal{L}_n\overset{(1)}{\hat{\tilde{g}}}\right)_{\nu\nu}\Big|_{\Sigma} &:= 2\overset{(1)}{\bar{k}} + \frac{2}{3}\overset{(1)}{\left(\mathrm{tr}k\right)} + \not{\nabla}_{\nu}\overset{(1)}{\bar{s}} - 2\mu\bar{h}\not{\nabla}_{\nu}\left(\frac{1}{\bar{h}}\overset{(1)}{\bar{h}}\right) + \frac{2}{\bar{h}}\left(\bar{k} + \frac{1}{3}\mathrm{tr}k\right)\overset{(1)}{\bar{h}} - \frac{1}{2}\overset{(1)}{\underline{w}}_2.
\end{aligned}
$$

For the function $\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}}$

$$
\begin{aligned}
\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}}\Big|_{\Sigma} &:= \frac{2}{N}\overset{(1)}{N} + \frac{2}{\bar{h}}\overset{(1)}{\bar{h}},\\
\left(\mathcal{L}_n\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}}\right)\Big|_{\Sigma} &:= \overset{(1)}{\underline{w}}_2.
\end{aligned}
$$

For the $\mathcal{Q}\otimes S$ 1-form $\overset{(1)}{\not{g}}$

$$
\begin{aligned}
\overset{(1)}{\not{g}}_n\Big|_{\Sigma} &:= \bar{h}\cdot\overset{(1)}{\not{b}} - \frac{\mu}{\bar{h}}\overset{(1)}{\not{h}},\\
&:= \overset{(1)}{\not{s}},\\
\left(\mathcal{L}_n\overset{(1)}{\not{g}}\right)_n\Big|_{\Sigma} &:= \overset{(1)}{\not{\psi}}
\end{aligned}
$$

and

$$
\begin{aligned}
\overset{(1)}{\not{g}}_{\nu}\Big|_{\Sigma} &:= \frac{1}{\bar{h}}\overset{(1)}{\not{h}},\\
\left(\mathcal{L}_n\overset{(1)}{\not{g}}\right)_{\nu}\Big|_{\Sigma} &:= 2\overset{(1)}{\not{k}} + \not{\nabla}_{\nu}\overset{(1)}{\not{s}} + \not{\nabla}\overset{(1)}{\bar{s}} - \frac{1}{r}\overset{(1)}{\not{s}} - \frac{1}{2}\frac{1}{\bar{h}}\left(\bar{k} - \frac{2}{3}\mathrm{tr}k\right)\overset{(1)}{\not{h}}.
\end{aligned}
$$

For the symmetric, traceless 2-covariant S-tensor $\overset{(1)}{\hat{\slashed{g}}}$

$$
\begin{aligned}
\overset{(1)}{\hat{\slashed{g}}}\Big|_{\Sigma} &:= \overset{(1)}{\hat{\slashed{h}}}, \\
\left(\mathcal{L}_n\overset{(1)}{\hat{\slashed{g}}}\right)\Big|_{\Sigma} &:= 2\overset{(1)}{\hat{\slashed{k}}} - 2\slashed{\mathcal{D}}_2^{\star}\overset{(1)}{\slashed{s}}.
\end{aligned}
$$

Finally, for the function $\slashed{\mathrm{tr}}\overset{(1)}{\slashed{g}}$

$$
\begin{aligned}
\slashed{\mathrm{tr}}\overset{(1)}{\slashed{g}}\Big|_{\Sigma} &:= \slashed{\mathrm{tr}}\overset{(1)}{\slashed{h}}, \\
\left(\mathcal{L}_n\slashed{\mathrm{tr}}\overset{(1)}{\slashed{g}}\right)\Big|_{\Sigma} &:= \frac{4}{3}\left(\overset{(1)}{\mathrm{tr}k}\right) - 2\overset{(1)}{\overline{k}} + 2\slashed{\mathrm{div}}\overset{(1)}{\slashed{s}} + \frac{1}{r}\frac{4}{\overline{h}}\overset{(1)}{\overline{s}} - \frac{2}{N}\left(\overline{k} - \frac{2}{3}\mathrm{tr}k\right)\overset{(1)}{N}.
\end{aligned}
$$

Here, $\overset{(1)}{\underline{w}}_1, \overset{(1)}{\underline{w}}_2, \overset{(1)}{\overline{w}}$ *and* $\overset{(1)}{\slashed{\psi}}$ *are chosen such that the gauge conditions* (3.28) *and* (3.29) *are satisfied on* Σ.

Note that the quantities $\overset{(1)}{\underline{w}}_1, \overset{(1)}{\underline{w}}_2, \overset{(1)}{\overline{w}}$ and $\overset{(1)}{\slashed{\psi}}$ can indeed be computed explicitly from the admissible initial data set $\mathscr{A}$.

We make the following remarks.

Remark 19. The collection $\mathscr{A}$ correspond to linearised versions of the lapse, shift and various decompositions of the induced metric and second fundamental form of Σ. In particular, the coupled system of equations (5.1)-(5.3) arise as the linearisation of the Gauss–Codazzi equations. Conversely, the four remaining constraints arise as a consequence of the divergence relation (3.22).

Remark 20. One can explicitly verify that given a smooth solution $\mathscr{S}$ to the equations of linearised gravity then the collection $\mathscr{A}$ obtained from it by reversing the above construction must satisfy the equations (5.1)-(5.3).

5.1.1.2 Seed data for linearised gravity

In view of these constraints on initial data we provide in this section a notion of freely prescribed *seed* data for the equations of linearised gravity.

In what follows, we remind the reader of the space $\Lambda(\Sigma)$ defined in section 3.2.5.5.

Definition 5.2. *A smooth seed data set for the equations of linearised gravity consists of prescribing:*

- *four smooth functions* $\overset{(1)}{\overline{\Phi}}, \overset{(1)}{\underline{\Phi}}, \overset{(1)}{\overline{\Psi}}, \overset{(1)}{\underline{\Psi}} \in \Lambda(\Sigma)$
- *four smooth functions* $\overset{(1)}{\underline{\mathfrak{v}}}, \overset{(1)}{\overline{\mathfrak{v}}}, \overset{(1)}{\underline{\mathfrak{w}}}$ *and* $\overset{(1)}{\overline{\mathfrak{w}}}$ *on* Σ

- *two smooth S_ν 1-forms* $\overset{\scriptscriptstyle (1)}{\not{p}}$ *and* $\overset{\scriptscriptstyle (1)}{\not{\mathfrak{p}}}$
- *a smooth function* $\mathfrak{a}$ *on the horizon sphere* $\mathcal{H}^+ \cap \Sigma$ *given as a linear combination of the* $l = 1$ *spherical harmonics (cf. section 3.2.5.1)*
- *a constant* $\mathfrak{m}$

We will denote such a smooth seed initial data set by the collection

$$\mathscr{D} = \left(\overset{\scriptscriptstyle (1)}{\bar{\Phi}}, \overset{\scriptscriptstyle (1)}{\underline{\Phi}}, \overset{\scriptscriptstyle (1)}{\bar{\Psi}}, \overset{\scriptscriptstyle (1)}{\underline{\Psi}}, \overset{\scriptscriptstyle (1)}{\underline{\mathfrak{v}}}, \overset{\scriptscriptstyle (1)}{\bar{\mathfrak{v}}}, \overset{\scriptscriptstyle (1)}{\not{p}}, \overset{\scriptscriptstyle (1)}{\underline{\mathfrak{w}}}, \overset{\scriptscriptstyle (1)}{\bar{\mathfrak{w}}}, \overset{\scriptscriptstyle (1)}{\not{\mathfrak{p}}}, \mathfrak{m}, \mathfrak{a} \right).$$

5.1.2 The well-posedness theorem

In this section we establish the foundational well-posedness result for the equations of linearised gravity.

In what follows, we recall the notation $D^+(\Sigma)$ for the domain of dependence of the hypersurface Σ in $\mathcal{M}$.

Theorem 5.1. *Let $\mathscr{D}$ be a smooth seed data set on Σ. Then*

i) there exists an extension of $\mathscr{D}$ to a smooth admissible initial data set $\mathscr{A}$ on Σ

ii) the initial data set $\mathscr{A}$ gives rise to a unique, smooth solution $\mathscr{S}$ to the equations of linearised gravity on $D^+(\Sigma)$.

We shall prove parts *i*) and *ii*) of Theorem 5.1 separately.

Before we proceed with the proof of part *i*) it will be useful to first note the following lemma.

Lemma 5.2. *Let $\mathscr{G}$ be a pure gauge solution as in Proposition 3.15. Then its generators $\tilde{v}$ and $\not{p}$ satisfy the system of equations*

$$\begin{aligned} \tilde{\Box}\tilde{v} + \not{\Delta}\tilde{v} - \frac{2}{r}\big(\widetilde{\nabla}\tilde{v}\big)_P + \frac{2}{r^2}\,\mathrm{d}r\,\tilde{v}_P &= -\frac{1}{r}\big(\widetilde{\nabla}\otimes\tilde{v}_{l=0,1}\big)_P, \\ \tilde{\Box}\not{p} + \not{\Delta}\not{p} - \frac{2}{r}\widetilde{\nabla}_P\not{p} + \frac{1}{r^2}(3-4\mu)\not{p} &= 0. \end{aligned}$$

Proof. We recall from Proposition 3.15 that the generators $\tilde{v}$ and $\not{p}$ of a pure gauge solution must satisfy the system

$$\begin{aligned} \tilde{\Box}\tilde{v} + \not{\Delta}\tilde{v} + \frac{2}{r}\widetilde{\nabla}_P\tilde{v} - \frac{2}{r^2}\tilde{\mathrm{d}}r\,\tilde{v}_P &= \frac{2}{r}\tilde{\mathrm{d}}r\,\not{\mathrm{d}}\mathrm{iv}\not{p} + \tilde{f}(\mathcal{L}_{(\tilde{v}+\not{p})^\sharp}g_M), \\ \tilde{\Box}\not{p} + \not{\Delta}\not{p} - \frac{1}{r^2}\not{p} &= -\frac{2}{r}\not{\nabla}\tilde{v}_P + \not{f}(\mathcal{L}_{(\tilde{v}+\not{p})^\sharp}g_M) \end{aligned}$$

where f is the map of section 3.3.1. However, recalling further that $f = \mathring{f} + \overset{\mathscr{K}}{f} + \overset{\diamond}{f}$ where $\mathring{f}, \overset{\mathscr{K}}{f}$ and $\overset{\diamond}{f}$ are defined as in sections 3.3.1.1-3.3.1.3 respectively then we note that

$$\overset{\diamond}{f}(\mathcal{L}_{(\tilde{\mathfrak{v}}+\not{\mathfrak{v}})^\sharp} g_M) = 0$$

precisely because the map $\overset{\diamond}{f}$ returns expressions in the gauge-invariant quantities $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ when applied to solutions of the equations of linearised gravity (cf. section 3.3.1.3 and the proof of Proposition 4.1). Computing the remaining expression $(\mathring{f} + \overset{\mathscr{K}}{f})(\mathcal{L}_{(\tilde{\mathfrak{v}}+\not{\mathfrak{v}})^\sharp} g_M)$ thus returns the system of equations in the statement of the lemma. $\square$

We are now in a position to prove part i) of Theorem 5.1.

We note that the proof of the following theorem exploits the existence of three explicit classes of solutions to the equations of linearised gravity each of which must necessarily generate three classes of admissible initial data. We also recall the operator $\tilde{\mathrm{d}}^{\mathcal{I}}$ defined as in section 3.2.4.2.

Proof of part i) of Theorem 5.1. Let $\mathscr{D}$ be the seed data set in question,

$$\mathscr{D} = \left(\overset{(1)}{\bar{\Phi}}, \overset{(1)}{\underline{\Phi}}, \overset{(1)}{\bar{\Psi}}, \overset{(1)}{\underline{\Psi}}, \overset{(1)}{\underline{\mathfrak{v}}}, \overset{(1)}{\bar{\mathfrak{v}}}, \overset{(1)}{\not{\mathfrak{v}}}, \overset{(1)}{\underline{\mathfrak{w}}}, \overset{(1)}{\bar{\mathfrak{w}}}, \overset{(1)}{\not{\mathfrak{w}}}, \mathfrak{m}, \mathfrak{a} \right).$$

We proceed in three steps.

1. From the subset of seed $\left(\overset{(1)}{\bar{\Psi}}, \overset{(1)}{\bar{\Phi}}, \overset{(1)}{\underline{\Psi}}, \overset{(1)}{\underline{\Phi}} \right)$ lying in the space $\Lambda(\Sigma)$ we consider two smooth and unique functions $\overset{(1)}{\Phi}, \overset{(1)}{\Psi} \in \Lambda(\mathcal{M})$ which satisfy

$$\begin{aligned} \widetilde{\Box}\overset{(1)}{\Phi} + \not{\Delta}\overset{(1)}{\Phi} &= -\frac{6}{r^2}\frac{M}{r}\overset{(1)}{\Phi}, \\ \widetilde{\Box}\overset{(1)}{\Psi} + \not{\Delta}\overset{(1)}{\Psi} &= -\frac{6}{r^2}\frac{M}{r}\overset{(1)}{\Psi} + \frac{24}{r^3}\frac{M}{r}(r-3M)\not{c}^{[1]}\overset{(1)}{\Psi} + \frac{72}{r^5}\frac{M}{r}\frac{M}{r}(r-2M)\not{c}^{[2]}\overset{(1)}{\Psi} \end{aligned} \tag{5.5}$$

with

$$\left(\overset{(1)}{\Phi}, n\overset{(1)}{\Phi} \right)\Big|_\Sigma = \left(\overset{(1)}{\bar{\Phi}}, \overset{(1)}{\underline{\Phi}} \right), \qquad \left(\overset{(1)}{\Psi}, n\overset{(1)}{\Psi} \right)\Big|_\Sigma = \left(\overset{(1)}{\bar{\Psi}}, \overset{(1)}{\underline{\Psi}} \right). \tag{5.6}$$

This in turn uniquely determines the smooth solution $\mathscr{S}$ to the equations of linearised gravity (with vanishing projection to $l = 0, 1$) defined according to:

$$
\begin{aligned}
\overset{(1)}{\tilde{g}} &= \widetilde{\nabla}\hat{\otimes}\tilde{\mathrm{d}}^{\mathcal{I}}\Big(r\overset{(1)}{\Psi}\Big) + 6\mu\,\mathrm{d}r\hat{\hat{\otimes}}\not{\zeta}^{[1]}\tilde{\mathrm{d}}\overset{(1)}{\Psi}, \\
\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}} &= 0, \\
\overset{(1)}{\not{g}} &= \not{\mathcal{D}}_1^{\star}\Big(\tilde{\mathrm{d}}^{\mathcal{I}}\Big(r\overset{(1)}{\Psi}\Big) - 2\tilde{\mathrm{d}}r\,\overset{(1)}{\Psi}, \tilde{\mathrm{d}}^{\mathcal{I}}\Big(r\overset{(1)}{\Phi}\Big) - 2\tilde{\mathrm{d}}r\,\overset{(1)}{\Phi}\Big), \\
\overset{(1)}{\hat{\not{g}}} &= r\not{\nabla}\hat{\otimes}\not{\mathcal{D}}_1^{\star}\Big(\overset{(1)}{\Psi}, \overset{(1)}{\Phi}\Big), \\
\not{\mathrm{tr}}\overset{(1)}{\not{g}} &= 4\tilde{\mathrm{d}}^{\mathcal{I}}\overset{(1)}{\Psi} + 12\mu r^{-1}(1-\mu)\not{\zeta}^{[1]}\overset{(1)}{\Psi}.
\end{aligned}
$$

Indeed, that the above collection determines a solution to the equations of linearised gravity will be verified in Corollary 5.8 (cf. Remark 29).

We subsequently denote by $\mathscr{A}_{\overset{(1)}{\bar{\Phi}},\overset{(1)}{\bar{\Psi}},\overset{(1)}{\underline{\Phi}},\overset{(1)}{\underline{\Psi}}}$ the corresponding admissible initial data set for the equations of linearised gravity determined from the collection $\mathscr{S}$ according to Definition 5.1 and the rules (5.5)-(5.6) for projecting normal derivatives onto Σ (cf. Remark 20).

2. From the subset of seed $\Big(\mathfrak{m}, \mathfrak{a}\Big)$ we determine the linearised Kerr solution $\mathscr{K}_{\mathfrak{m},\mathfrak{a}}$ according to Proposition 3.16.

   We subsequently denote by $\mathscr{A}_{\mathfrak{m},\mathfrak{a}}$ the corresponding admissible initial data set for the equations of linearised gravity determined from $\mathscr{K}_{\mathfrak{m},\mathfrak{a}}$ according to Definition 5.1.

3. From the subset of seed $\Big(\overset{(1)}{\underline{\mathfrak{v}}}, \overset{(1)}{\bar{\mathfrak{v}}}, \overset{(1)}{\not{p}}, \overset{(1)}{\underline{\mathfrak{w}}}, \overset{(1)}{\bar{\mathfrak{w}}}, \overset{(1)}{\not{\mathfrak{p}}}\Big)$ we consider the smooth $\mathcal{Q}$ 1-form $\tilde{v}$ and the smooth S 1-form $\not{p}$ which satisfy

$$
\begin{aligned}
\widetilde{\Box}\tilde{v} + \not{\Delta}\tilde{v} - \frac{2}{r}\Big(\widetilde{\nabla}\tilde{v}\Big)_P + \frac{2}{r^2}\,\mathrm{d}r\,\tilde{v}_P &= -\frac{1}{r}\Big(\widetilde{\nabla}\otimes\tilde{v}_{l=0}\Big)_P, \\
\widetilde{\Box}\not{p} + \not{\Delta}\not{p} - \frac{2}{r}\widetilde{\nabla}_P\not{p} + \frac{1}{r^2}(3-4\mu)\not{p} &= 0
\end{aligned}
\tag{5.7}
$$

   with

$$
\Big(\tilde{v}_n, \tilde{v}_\nu, (\widetilde{\nabla}_n\tilde{v})_n, (\widetilde{\nabla}_n\tilde{v})_\nu\Big)\Big|_\Sigma = \Big(\overset{(1)}{\underline{\mathfrak{v}}}, \overset{(1)}{\bar{\mathfrak{v}}}, \overset{(1)}{\underline{\mathfrak{w}}}, \overset{(1)}{\bar{\mathfrak{w}}}\Big), \qquad \Big(\not{p}, \widetilde{\nabla}_n\not{p}\Big)\Big|_\Sigma = \Big(\overset{(1)}{\not{p}}, \overset{(1)}{\not{\mathfrak{p}}}\Big) \tag{5.8}
$$

   This in turn uniquely determines the pure gauge solution $\mathscr{G}$ generated by $\tilde{v}$ and $\not{p}$ according to Proposition 3.15 and Lemma 5.2.

   We subsequently denote by $\mathscr{A}_{\overset{(1)}{\underline{\mathfrak{v}}},\overset{(1)}{\bar{\mathfrak{v}}},\overset{(1)}{\not{p}},\overset{(1)}{\underline{\mathfrak{w}}},\overset{(1)}{\bar{\mathfrak{w}}},\overset{(1)}{\not{\mathfrak{p}}}}$ the corresponding admissible initial data for the equations of linearised gravity determined from $\mathscr{G}$ according to Definition 5.1 and the rules (5.7)-(5.8) for projecting normal derivatives onto Σ.

Hence, from the full seed $\mathscr{D}$, we now determine the collection

$$\mathscr{A} := \mathscr{A}_{\overset{(1)}{\bar{\Phi}},\overset{(1)}{\bar{\Psi}},\overset{(1)}{\underline{\Phi}},\overset{(1)}{\underline{\Psi}}} + \mathscr{A}_{\mathfrak{m},\mathfrak{a}} + \mathscr{A}_{\overset{(1)}{\underline{\mathfrak{v}}},\overset{(1)}{\bar{\mathfrak{v}}},\overset{(1)}{\not{\mathfrak{v}}},\overset{(1)}{\underline{\mathfrak{w}}},\overset{(1)}{\bar{\mathfrak{w}}},\overset{(1)}{\not{\mathfrak{w}}}}$$

which, by linearity and steps 1.-3., thus determines an admissible initial data set for the system of gravitational perturbations constructed uniquely from $\mathscr{D}$. $\square$

We make the following remarks.

Remark 21. We emphasize that the admissible initial data set constructed above is determined from expressions in the seed quantities which can be written down explicitly without recourse to the consideration of global quantities. In particular, whether solutions to (5.5)-(5.8) exist on $\mathcal{M}$ is immaterial at this stage as the equations (5.5) and (5.7) are merely used as tools for computing higher order expressions in the seed quantities.

Remark 22. We note that, by appropriately reversing the above procedure, one can actually show that given a smooth admissible initial data set $\mathscr{A}$ to the equations of linearised gravity then there exists a smooth seed data set $\mathscr{D}$ that gives rise to it, thus establishing a bijection between the space of admissible initial data and seed data – we will not pursue this explicitly in the thesis however. In addition, see section 2.3.1 of the overview for comments regarding the parametrisation of the space of solutions to the constraint equations (5.1)-(5.3) by the seed data $\mathscr{D}$.

Now we prove part *ii*) of Theorem 5.1.

Proof of part ii) of Theorem 5.1. We first invoke Proposition 4.2 to construct from the seed functions $\overset{(1)}{\bar{\Phi}}, \overset{(1)}{\underline{\Phi}}, \overset{(1)}{\bar{\Psi}}, \overset{(1)}{\underline{\Psi}} \in C^\infty(\Sigma) \cap \Lambda(\Sigma)$ two unique solutions $\overset{(1)}{\Phi}, \overset{(1)}{\Psi} \in C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$ to the Regge–Wheeler and Zerilli equations respectively:

$$\begin{aligned}
\widetilde{\Box}\overset{(1)}{\Phi} + \not{\Delta}\overset{(1)}{\Phi} &= -\frac{6}{r^2}\frac{M}{r}\overset{(1)}{\Phi}, \\
\widetilde{\Box}\overset{(1)}{\Psi} + \not{\Delta}\overset{(1)}{\Psi} &= -\frac{6}{r^2}\frac{M}{r}\overset{(1)}{\Psi} + \frac{24}{r^3}\frac{M}{r}(r-3M)\not{\mathcal{C}}^{[1]}\overset{(1)}{\Psi} + \frac{72}{r^5}\frac{M}{r}\frac{M}{r}(r-2M)\not{\mathcal{C}}^{[2]}\overset{(1)}{\Psi}
\end{aligned}$$

with

$$\Big(\overset{(1)}{\Phi}, n\big(\overset{(1)}{\Phi}\big)\Big)\Big|_\Sigma = \Big(\overset{(1)}{\bar{\Phi}}, \overset{(1)}{\underline{\Phi}}\Big), \qquad \Big(\overset{(1)}{\Psi}, n\big(\overset{(1)}{\Psi}\big)\Big)\Big|_\Sigma = \Big(\overset{(1)}{\bar{\Psi}}, \overset{(1)}{\underline{\Psi}}\Big).$$

We then construct on $D^+(\Sigma)$ the unique collection of tensors fields

$$\overset{(1)}{\hat{\tilde{g}}}, \mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}}, \overset{(1)}{\not{g}}, \overset{(1)}{\hat{\not{g}}}, \not{\mathrm{tr}}\overset{(1)}{\not{g}}$$

where

- $\hat{\tilde{\overset{(1)}{g}}}$ is a smooth, symmetric, traceless 2-covariant $\mathcal{Q}$-tensor field

- $\mathrm{tr}_{\tilde{g}_M}\tilde{\overset{(1)}{g}}$ is a smooth function

- $\overset{(1)}{\not{g}}$ is a smooth $\mathcal{Q}\otimes S$ 1-form

- $\hat{\overset{(1)}{\not{g}}}$ is a smooth, symmetric, traceless 2-covariant S-tensor field

- $\not{\mathrm{tr}}\overset{(1)}{\not{g}}$ is a smooth function

by solving the following coupled system of inhomogeneous wave equations with Cauchy data given by the admissible initial data set $\mathscr{A}$ according to Definition 5.1:

$$
\begin{aligned}
\widetilde{\Box}\hat{\tilde{\overset{(1)}{g}}} + \not{\Delta}\hat{\tilde{\overset{(1)}{g}}} + \frac{2}{r}\widetilde{\nabla}_P\hat{\tilde{\overset{(1)}{g}}} - \frac{2}{r}\Big(\widetilde{\nabla}\hat{\otimes}\hat{\tilde{\overset{(1)}{g}}}\Big)_P - \frac{4}{r}\frac{\mu}{r}\hat{\tilde{\overset{(1)}{g}}} &= \frac{1}{r}\,\mathrm{d}r\hat{\tilde{\otimes}}\tilde{\mathrm{d}}\mathrm{tr}_{\tilde{g}_M}\tilde{\overset{(1)}{g}} + \frac{2}{r}\,\mathrm{d}r\hat{\tilde{\otimes}}\not{\mathrm{div}}\overset{(1)}{\not{g}} - \frac{1}{r}\,\mathrm{d}r\hat{\tilde{\otimes}}\tilde{\mathrm{d}}\not{\mathrm{tr}}\overset{(1)}{\not{g}} \\
&\quad - \widetilde{\nabla}\hat{\otimes}\Big(r^{-1}\Big(\hat{\tilde{\overset{(1)}{g}}}_{l=0,1}\Big)_P\Big) + \widetilde{\nabla}\hat{\otimes}\tilde{\overset{(1)}{F}}, \qquad (5.9)\\
\widetilde{\Box}\mathrm{tr}_{\tilde{g}_M}\tilde{\overset{(1)}{g}} + \not{\Delta}\mathrm{tr}_{\tilde{g}_M}\tilde{\overset{(1)}{g}} &= -\frac{2}{r}\tilde{\delta}\Big(\Big(\hat{\tilde{\overset{(1)}{g}}}_{l=0,1}\Big)_P\Big) + \frac{2}{r^2}\Big(\hat{\tilde{\overset{(1)}{g}}}_{l=0,1}\Big)_P, \qquad (5.10)
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\Box}\overset{(1)}{\not{g}} + \not{\Delta}\overset{(1)}{\not{g}} - \frac{2}{r}\widetilde{\nabla}\otimes\overset{(1)}{\not{g}}_P - \frac{1}{r^2}(1-\mu)\overset{(1)}{\not{g}} + \frac{2}{r^2}\,\mathrm{d}r\otimes_{\mathrm{s}}\overset{(1)}{\not{g}}_P &= \frac{2}{r}\,\mathrm{d}r\otimes_{\mathrm{s}}\not{\mathrm{div}}\hat{\overset{(1)}{\not{g}}} - \frac{1}{r}\not{\nabla}\otimes\Big(\Big(\hat{\tilde{\overset{(1)}{g}}}_{l=0,1}\Big)_P\Big) \\
&\quad + \widetilde{\nabla}\otimes\overset{(1)}{\not{F}} + \not{\nabla}\otimes\tilde{\overset{(1)}{F}} - \frac{2}{r}\tilde{\mathrm{d}}r\otimes_{\mathrm{s}}\overset{(1)}{\not{F}}, \qquad (5.11)\\
\widetilde{\Box}\hat{\overset{(1)}{\not{g}}} + \not{\Delta}\hat{\overset{(1)}{\not{g}}} - \frac{2}{r}\widetilde{\nabla}_P\hat{\overset{(1)}{\not{g}}} - \frac{4}{r}\frac{\mu}{r}\hat{\overset{(1)}{\not{g}}} &= \not{\nabla}\hat{\otimes}\overset{(1)}{\not{F}}, \qquad (5.12)\\
\widetilde{\Box}\not{\mathrm{tr}}\overset{(1)}{\not{g}} + \not{\Delta}\not{\mathrm{tr}}\overset{(1)}{\not{g}} + \frac{2}{r}\widetilde{\nabla}_P\not{\mathrm{tr}}\overset{(1)}{\not{g}} + \frac{2}{r^2}\not{\mathrm{tr}}\overset{(1)}{\not{g}} &= \frac{4}{r^2}\hat{\tilde{\overset{(1)}{g}}}_{PP} + \frac{2}{r^2}\mathrm{tr}_{\tilde{g}_M}\tilde{\overset{(1)}{g}} - \frac{4}{r^2}\Big(\hat{\tilde{\overset{(1)}{g}}}_{l=0,1}\Big)_{PP} \\
&\quad + 2\not{\mathrm{div}}\overset{(1)}{\not{F}} + \frac{4}{r}\tilde{\overset{(1)}{F}}_P. \qquad (5.13)
\end{aligned}
$$

Here, $\tilde{\overset{(1)}{F}} \in \tilde{\mathscr{T}}^1(\mathcal{M})$ and $\overset{(1)}{\not{F}} \in \not{\mathscr{T}}^1(\mathcal{M})$ are defined according to

$$
\begin{aligned}
\tilde{\overset{(1)}{F}} &:= -\frac{1}{r^2}\tilde{\star}\tilde{\mathrm{d}}\Big(r^3\mathfrak{Z}\overset{(1)}{\Psi}\Big),\\
\overset{(1)}{\not{F}} &:= \frac{2}{r}(1-2\mu)\not{\mathcal{D}}_1^\star\Big(\overset{(1)}{\Psi},\overset{(1)}{\Phi}\Big) + r\not{\nabla}\mathfrak{Z}\overset{(1)}{\Psi}
\end{aligned}
$$

where $\mathfrak{Z}$ is the operator defined as in section 3.3.1.3. We note that one can indeed solve the system (5.9)-(5.13) using standard techniques for solving systems of linear wave equations. In particular, employing a spherical harmonic decomposition as in section 3.2.5.1 then the spherical symmetry of $\big(\mathcal{M}, g_M\big)$ implies that the $l = 0,1$ modes will propagate orthogonally to the rest.

Moreover, introducing the smooth $\mathcal{Q}$ 1-form $\tilde{\overset{(1)}{w}}$ and the smooth S 1-form $\overset{(1)}{\not{\psi}}$ defined according to

$$
\begin{aligned}
\overset{(1)}{\tilde{w}} &:= -\tilde{\delta}\overset{(1)}{\hat{\tilde{g}}} + \not{\mathrm{d}}\mathrm{iv}\,\overset{(1)}{\mathrlap{-}g} - \frac{1}{2}\tilde{\mathrm{d}}\not{\mathrm{t}}\mathrm{r}\overset{(1)}{\not{g}} + \frac{1}{r}\Big(\overset{(1)}{\hat{\tilde{g}}}_{l=0}\Big)_P - \overset{(1)}{\tilde{F}},\\
\overset{(1)}{\not{\psi}} &:= -\tilde{\delta}\overset{(1)}{\mathrlap{-}g} - \frac{1}{2}\not{\nabla}\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}} + \not{\mathrm{d}}\mathrm{iv}\,\overset{(1)}{\hat{\not{g}}} - \overset{(1)}{\not{F}}
\end{aligned}
$$

it follows from equations (5.9)-(5.13) that $\overset{(1)}{\tilde{w}}$ and $\overset{(1)}{\not{\psi}}$ satisfy the following system of wave equations:

$$
\begin{aligned}
\widetilde{\Box}\overset{(1)}{\tilde{w}} + \not{\Delta}\overset{(1)}{\tilde{w}} + \frac{2}{r}\widetilde{\nabla}_P\overset{(1)}{\tilde{w}} - \frac{2}{r^2}\,\mathrm{d}r\;\overset{(1)}{\tilde{w}}_P &= \frac{2}{r}\,\mathrm{d}r\;\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\not{\psi}},\\
\widetilde{\Box}\overset{(1)}{\not{\psi}} + \not{\Delta}\overset{(1)}{\not{\psi}} - \frac{1}{r^2}\overset{(1)}{\not{\psi}} &= -\frac{2}{r}\not{\nabla}\overset{(1)}{\tilde{w}}_P.
\end{aligned}
$$

In addition, the admissibility of the initial data set $\mathscr{A}$ implies (by explicit computation) that $\overset{(1)}{\tilde{w}}$ and $\overset{(1)}{\not{\psi}}$ possess *trivial Cauchy data*. Consequently, by a standard Grönwall-type argument it must be that $\overset{(1)}{\tilde{w}}$ and $\overset{(1)}{\not{\psi}}$ vanish globally on $D^+(\Sigma)$. We therefore conclude that the collection

$$
\mathscr{S} = \left(\overset{(1)}{\hat{\tilde{g}}}, \mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}}, \overset{(1)}{\mathrlap{-}g}, \overset{(1)}{\hat{\not{g}}}, \not{\mathrm{t}}\mathrm{r}\overset{(1)}{\not{g}}\right)
$$

in fact satisfies the system of equations

$$
\begin{aligned}
\widetilde{\Box}\overset{(1)}{\hat{\tilde{g}}} + \not{\Delta}\overset{(1)}{\hat{\tilde{g}}} + \frac{2}{r}\widetilde{\nabla}_P\overset{(1)}{\hat{\tilde{g}}} - \frac{2}{r}\Big(\widetilde{\nabla}\hat{\otimes}\overset{(1)}{\hat{\tilde{g}}}\Big)_P - \frac{4}{r}\frac{\mu}{r}\overset{(1)}{\hat{\tilde{g}}} &= \frac{1}{r}\,\mathrm{d}r\hat{\otimes}\tilde{\mathrm{d}}\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}} + \frac{2}{r}\,\mathrm{d}r\hat{\otimes}\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\mathrlap{-}g} - \frac{1}{r}\,\mathrm{d}r\hat{\otimes}\tilde{\mathrm{d}}\not{\mathrm{t}}\mathrm{r}\overset{(1)}{\not{g}}\\
&\quad + \widetilde{\nabla}\hat{\otimes}\underset{\circ}{\overset{(1)}{\tilde{f}}},\\
\widetilde{\Box}\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}} + \not{\Delta}\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}} &= -2\tilde{\delta}\underset{\circ}{\overset{(1)}{\tilde{f}}} + \frac{2}{r}\underset{\circ}{\overset{(1)}{\tilde{f}}},\\
\widetilde{\Box}\overset{(1)}{\mathrlap{-}g} + \not{\Delta}\overset{(1)}{\mathrlap{-}g} - \frac{2}{r}\widetilde{\nabla}\otimes\overset{(1)}{\mathrlap{-}g}_P - \frac{1}{r^2}(1-\mu)\overset{(1)}{\mathrlap{-}g} + \frac{2}{r^2}\,\mathrm{d}r\otimes_{\mathrm{s}}\overset{(1)}{\mathrlap{-}g}_P &= \frac{2}{r}\tilde{\mathrm{d}}r\otimes_{\mathrm{s}}\not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\hat{\not{g}}} + \widetilde{\nabla}\otimes\underset{\circ}{\overset{(1)}{\not{f}}} + \not{\nabla}\otimes\underset{\circ}{\overset{(1)}{\tilde{f}}} - \frac{2}{r}\,\mathrm{d}r\otimes_{\mathrm{s}}\underset{\circ}{\overset{(1)}{\not{f}}},\\
\widetilde{\Box}\overset{(1)}{\hat{\not{g}}} + \not{\Delta}\overset{(1)}{\hat{\not{g}}} - \frac{2}{r}\widetilde{\nabla}_P\overset{(1)}{\hat{\not{g}}} - \frac{4}{r}\frac{\mu}{r}\overset{(1)}{\hat{\not{g}}} &= \not{\nabla}\hat{\otimes}\underset{\circ}{\overset{(1)}{\not{f}}},\\
\widetilde{\Box}\not{\mathrm{t}}\mathrm{r}\overset{(1)}{\not{g}} + \not{\Delta}\not{\mathrm{t}}\mathrm{r}\overset{(1)}{\not{g}} + \frac{2}{r}\widetilde{\nabla}_P\not{\mathrm{t}}\mathrm{r}\overset{(1)}{\not{g}} + \frac{2}{r^2}\not{\mathrm{t}}\mathrm{r}\overset{(1)}{\not{g}} &= \frac{4}{r^2}\overset{(1)}{\hat{\tilde{g}}}_{PP} + \frac{2}{r^2}\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}} + 2\not{\mathrm{d}}\mathrm{iv}\underset{\circ}{\overset{(1)}{\not{f}}} + \frac{4}{r}\underset{\circ}{\overset{(1)}{\tilde{f}}}_P,\\
-\tilde{\delta}\overset{(1)}{\hat{\tilde{g}}} + \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\mathrlap{-}g} - \frac{1}{2}\tilde{\mathrm{d}}\not{\mathrm{t}}\mathrm{r}\overset{(1)}{\not{g}} &= \underset{\circ}{\overset{(1)}{\tilde{f}}},\\
-\tilde{\delta}\overset{(1)}{\mathrlap{-}g} - \frac{1}{2}\not{\nabla}\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}} + \not{\mathrm{d}}\mathrm{iv}\overset{(1)}{\hat{\not{g}}} &= \underset{\circ}{\overset{(1)}{\not{f}}}
\end{aligned}
$$

where we have observed the equivalence

$$
\begin{aligned}
\underset{\circ}{\overset{(1)}{\tilde{f}}} &\equiv -\frac{1}{r}\Big(\overset{(1)}{\hat{\tilde{g}}}_P\Big)_{l=0,1} + \overset{(1)}{\tilde{F}},\\
\underset{\circ}{\overset{(1)}{\not{f}}} &\equiv \overset{(1)}{\not{F}}.
\end{aligned}
$$

Here, $\overset{(1)}{\underset{\circ}{\tilde{f}}}$ and $\overset{(1)}{\underset{\circ}{\not{f}}}$ are defined as section 3.3.3.

This completes the theorem. □

We make the following remarks.

Remark 23. The fact that the admissibility of the initial data set $\mathscr{A}$ implies that the gauge conditions (3.28)-(3.29) propagate under evolution by the system (3.23)-(3.27) is a classical result – see for instance the book of Choquet-Bruhat [12].

5.1.3 Pointwise asymptotic Kerrness

In this final section we provide a notion of regularity on the seed data for which Theorem 2 of section 6 will be most naturally formulated. We then show that this notion of regularity propagates under Theorem 5.1.

It will be convenient to give first the following definition.

In what follows, we recall the spacetime region $D^+(\Sigma_R) \subset \mathcal{M}$ defined as in section 3.2.4.1 and use the convention that functions on Σ are 0-covariant tensor fields.

Definition 5.3. *Let $S \in \mathscr{T}^n(\Sigma)$ for $n \geq 0$ an integer. Then we say that S is compactly supported on Σ_R iff the support of S is contained within $D^+(\Sigma_R) \cap \Sigma$.*

The regularity we are to impose on the seed is then defined below.

In what follows, we shall employ the notation $\mathscr{D}_{\mathfrak{m},\mathfrak{a}}$ to explicitly reference the two seed quantities $\mathfrak{m}$ and $\mathfrak{a}$ that are contained within the smooth seed data set $\mathscr{D}$.

Definition 5.4. *We say that a smooth seed data set $\mathscr{D}_{\mathfrak{m},\mathfrak{a}}$ is **asymptotically Kerr with parameters** $\mathfrak{m}$ **and** $\mathfrak{a}$ iff all seed quantities in $\mathscr{D}_{\mathfrak{m},\mathfrak{a}}$ are compactly supported on Σ_R.*

We make the following remark.

Remark 24. It is clear from the proof of part $i)$ of Theorem 5.1 that seed data sets $\mathscr{D}_{\mathfrak{m},\mathfrak{a}}$ which give rise to the linearised Kerr solutions of section Proposition 3.16 are pointwise asymptotically Kerr with parameters $\mathfrak{m}$ and $\mathfrak{a}$.

We now prove that the assumption of pointwise asymptotic Kerrness on the seed data propagates under Theorem 5.1 in the sense of Proposition 5.3.

It will be convenient to give first the following definition.

Definition 5.5. *Let $\mathfrak{T} \in \mathscr{T}^n(\mathcal{M})$ for $n \geq 0$ an integer. Then we say that $\mathfrak{T}$ is compactly supported on $D^+(\Sigma_R)$ iff the support of $\mathfrak{T}$ is contained within $D^+(\Sigma_R)$.*

The proposition we are to prove is then as below.

In what follows, we recall from the proof of part $i)$ of Theorem 5.1 that given a seed data set $\mathscr{D}_{\mathfrak{m},\mathfrak{a}}$ (recalling further the notation convention of section 5.1.3) then $\mathscr{A}_{\mathfrak{m},\mathfrak{a}}$ denotes the admissible initial data set for a linearised Kerr solution $\mathscr{K}_{\mathfrak{m},\mathfrak{a}}$ constructed from $\mathscr{D}_{\mathfrak{m},\mathfrak{a}}$.

Proposition 5.3. *Let $\mathscr{D}_{\mathfrak{m},\mathfrak{a}}$ be a smooth seed data set for the equations of linearised gravity which is asymptotically Kerr with parameters $\mathfrak{m}$ and $\mathfrak{a}$. We consider both the admissible initial data set $\mathscr{A}$ constructed from it in accordance with part $i)$ of Theorem 5.1 and the corresponding solution $\mathscr{S}$ to the equations of linearised gravity constructed from $\mathscr{A}$ in accordance with part $ii)$ of Theorem 5.1. Then*

i) all quantities in the collection $\mathscr{A}-\mathscr{A}_{\mathfrak{m},\mathfrak{a}}$ are compactly supported on Σ_R

ii) all quantities in the collection $\mathscr{S}-\mathscr{K}_{\mathfrak{m},\mathfrak{a}}$ are compactly supported on $D^+(\Sigma_R)$.

Proof. The proof of part $i)$ follows immediately from the proof of part $i)$ Theorem 5.1 after noting by linearity that the construction of $\mathscr{A}-\mathscr{A}_{\mathfrak{m},\mathfrak{a}}$ from the seed $\mathscr{D}_{\mathfrak{m},\mathfrak{a}}$ involves performing operations under which the property of being compactly supported in preserved.

To prove part $ii)$ we first note from the assumptions on the seed data that the Cauchy data for the (gauge-invariant) quantities $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ that are constructed as in first step in the proof of part $ii)$ of Theorem 5.1 is compactly supported on Σ_R. In particular, by a standard Grönwall type argument applied to linear wave equations, the quantities $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ are in addition compactly supported on $D^+(\Sigma_R)$. Moreover, from part $i)$, linearity and Definition 5.1 (cf. also 20) it follows that Cauchy data for all quantities in the collection $\mathscr{S}-\mathscr{K}_{\mathfrak{m},\mathfrak{a}}$ is in fact compactly supported on Σ_R. Consequently, applying once more a Grönwall type argument to the system of linear wave equations (5.9)-(5.13), noting that the inhomogeneous terms were just shown to compactly supported on $D^+(\Sigma_R)$, the theorem follows. □

5.2 Gauge-normalisation of initial data for the equations of linearised gravity

In this section we construct certain solutions to the equations of linearised gravity the initial data of which has been normalised via the addition of a pure gauge solution of section 3.4.1.

It is for these *gauge-normalised* solutions that the quantitative boundedness and decay statements of Theorem 2 in section 6 shall apply.

5.2.1 The modified Regge–Wheeler gauge

We shall define these gauge-normalised solutions by imposing certain *gauge* conditions. A solution to the equations of linearised gravity that satisfies these conditions is then said to be in the *modified Regge–Wheeler gauge*.

This gauge is so named as it is a modification of the Regge–Wheeler gauge of section A in the Appendix, with the precise relation the content Proposition A.3. In addition, we note that this latter gauge is an adaptation of the Regge–Wheeler gauge used by Regge and Wheeler in their original[23] – see section 2.3.1.1 of the overview.

5.2.1.1 The projection onto and away from $l = 0, 1$

To define this Regge–Wheeler gauge will first require defining the projection of a solution to the equations of linearised gravity onto and away from $l = 0, 1$.

This section of the paper is lifted almost verbatim from section 9.5.1 in[1].

First we provide a definition of a solution $\mathscr{S}$ having support only on $l = 0, 1$.

Definition 5.6. *Let $\mathscr{S}$ be a smooth solution to the equations of linearised gravity.*

Then we say that $\mathscr{S}$ is supported only on $l = 0, 1$ iff all quantities in $\mathscr{S}$ are supported only on $l = 0, 1$ (cf. Definitions 3.7 and 3.8). In particular,

$$\hat{\overset{\scriptscriptstyle(1)}{\not{g}}} = 0.$$

Conversely, we say that $\mathscr{S}$ has vanishing projection to $l = 0, 1$ iff all quantities in $\mathscr{S}$ have vanishing projection to $l = 0, 1$ (cf. Definitions 3.7 and 3.8).

One then has the following proposition, which follows easily from linearity along with Propositions 3.5 and 3.6.

Proposition 5.4. *Let $\mathscr{S}$ be a smooth solution to the equations of linearised gravity. Then one has the unique, orthogonal decomposition*

$$\mathscr{S} = \mathscr{S}_{l=0,1} + \mathscr{S}'$$

where $\mathscr{S}_{l=0,1}$ is a smooth solution to the equations of linearised gravity supported only on $l = 0, 1$ and $\mathscr{S}'$ is a smooth solution to the equations of linearised gravity that has vanishing projection to $l = 0, 1$.

Finally, the projection of $\mathscr{S}$ onto and away from $l = 0, 1$ is defined below.

Definition 5.7. *Let $\mathscr{S}$ be a smooth solution to the equations of linearised gravity. Then we call the map*

$$\mathscr{S} \to \mathscr{S}_{l=0,1}$$

the projection of $\mathscr{S}$ onto $l = 0, 1$.

Conversely, we call the map

$$\mathscr{S} \to \mathscr{S}'$$

the projection of $\mathscr{S}$ away from $l = 0, 1$.

We make the following remark.

Remark 25. Observe from Remark 14 that for the linearised Kerr solutions $\mathscr{K}_{\mathfrak{m},\mathfrak{a}}$ of Proposition 3.16 $\mathscr{K}'_{\mathfrak{m},\mathfrak{a}} = 0$.

5.2.1.2 The gauge-invariant collection $\overset{\scriptscriptstyle(1)}{\gamma}$

In this section we define a collection of gauge-invariant quantities that one associates to a smooth solution to the equations of linearised gravity. This will expedite the definition of the Regge–Wheeler gauge as well as being of later convenience.

The collection of quantities are defined below.

In what follows, we recall the operator $\tilde{\mathrm{d}}^{\mathcal{I}}$ defined as in section 3.2.4.2.

Definition 5.8. *Let $\mathscr{S}$ be a smooth solution to the equations of linearised gravity with $\overset{\scriptscriptstyle(1)}{\Phi}$ and $\overset{\scriptscriptstyle(1)}{\Psi}$ the associated gauge-invariant quantities of Proposition 4.1. Then*

i) $\overset{\scriptscriptstyle(1)}{\hat{\tilde{\gamma}}} \in \tilde{\mathscr{T}}^2_{sym}(\mathcal{M}) \cap \Lambda(\mathcal{M})$ *is defined according to*

$$\overset{\scriptscriptstyle(1)}{\hat{\tilde{\gamma}}} := \widetilde{\nabla}\hat{\otimes}\tilde{\mathrm{d}}^{\mathcal{I}}\Big(r\overset{\scriptscriptstyle(1)}{\Psi}\Big) + 6\mu\, \mathrm{d}r\hat{\tilde{\otimes}}\not{\epsilon}^{[1]}\tilde{\mathrm{d}}\overset{\scriptscriptstyle(1)}{\Psi}$$

ii) $\overset{\scriptscriptstyle(1)}{\not{\gamma}} \in \tilde{\mathscr{T}}^1(\mathcal{M})\otimes_{\mathrm{s}}\not{\mathscr{T}}^1(\mathcal{M})$ *is defined according to*

$$\overset{\scriptscriptstyle(1)}{\not{\gamma}} := \not{\mathcal{D}}_1^{\star}\Big(\tilde{\mathrm{d}}^{\mathcal{I}}\Big(r\overset{\scriptscriptstyle(1)}{\Psi}\Big) - 2\tilde{\mathrm{d}}r\,\overset{\scriptscriptstyle(1)}{\Psi}, \tilde{\mathrm{d}}^{\mathcal{I}}\Big(r\overset{\scriptscriptstyle(1)}{\Phi}\Big) - 2\tilde{\mathrm{d}}r\,\overset{\scriptscriptstyle(1)}{\Phi}\Big)$$

iii) $\overset{\scriptscriptstyle(1)}{\hat{\not{\gamma}}} \in \not{\mathscr{T}}^2_{sym}(\mathcal{M})$ *is defined according to*

$$\overset{\scriptscriptstyle(1)}{\hat{\not{\gamma}}} := r\not{\nabla}\hat{\otimes}\not{\mathcal{D}}_1^{\star}\Big(\overset{\scriptscriptstyle(1)}{\Psi}, \overset{\scriptscriptstyle(1)}{\Phi}\Big)$$

iv) $\operatorname{tr}\!\!\!/\,\overset{(1)}{\not\gamma} \in C^\infty(\mathcal{M})$ *is defined according to*

$$\operatorname{tr}\!\!\!/\,\overset{(1)}{\not\gamma} := 4\tilde{\mathrm{d}}_P^{\mathcal{I}}\overset{(1)}{\Psi} + 12\mu r^{-1}(1-\mu)\not{\mathcal{g}}^{[1]}\overset{(1)}{\Psi}$$

We note that $\overset{(1)}{\hat{\tilde{\gamma}}}$ and $\overset{(1)}{\hat{\not\gamma}}$ are traceless with respect to $\tilde{g}_M$ and $\not{g}_M$ respectively.

5.2.1.3 The modified Regge–Wheeler gauge

We are now in a position to define the modified Regge–Wheeler gauge.

First however we introduce the notation $\mathbb{C}_\Sigma f$ to denote the mapping of the quantity f onto Cauchy data on the initial hypersurface of section 3.2.3.1:

$$\mathbb{C}_\Sigma f := \Big(f, \mathcal{L}_n f\Big)\Big|_\Sigma.$$

In addition, for a collection of quantities $f = (f_1, ..., f_n)$ with $n \geq 1$ an integer we define

$$\mathbb{C}_\Sigma f := (\mathbb{C}_\Sigma f_1, ..., \mathbb{C}_\Sigma f_n)$$

and use the notation

$$\mathbb{C}_\Sigma f = 0$$

to denote that

$$(\mathbb{C}_\Sigma f_1, ..., \mathbb{C}_\Sigma f_n) = (0, ..., 0).$$

Finally, we recall the Hodge maps of Definition 3.9.

Definition 5.9. *Let* $\mathfrak{m} \in \mathbb{R}$ *and let* $\mathfrak{a}$ *be a smooth function on* S^2 *supported only on the* $l = 1$ *spherical harmonics. Then a solution* $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$ *to the equations of linearised gravity is said to be in the modified Regge–Wheeler gauge with parameters* $\mathfrak{m}$ *and* $\mathfrak{a}$ *iff the following conditions hold on* Σ*:*

- *the quantities* $\overset{(1)}{\not{g}}, \overset{(1)}{\hat{\not{g}}}, \overset{(1)}{\not\gamma}$ *and* $\overset{(1)}{\hat{\not\gamma}}$ *associated to the projection* $\Big(\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}\Big)'$ *satisfy*

$$\mathbb{C}_\Sigma\Big(\overset{(1)}{\not{g}}_{\mathrm{e}} - \overset{(1)}{\not\gamma}_{\mathrm{e}}\Big) = 0,$$
$$\mathbb{C}_\Sigma\Big(\overset{(1)}{\hat{\not{g}}} - \overset{(1)}{\hat{\not\gamma}}\Big) = 0$$

- *the projection* $\left(\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}\right)_{l=0,1}$ *satisfies*

$$\mathbb{C}_\Sigma\left(\left(\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}\right)_{l=0,1} - \mathscr{K}_{\mathfrak{m},\mathfrak{a}}\right) = 0$$

We make the following remark.

Remark 26. We stress that the gauge conditions as given above can be written explicitly in terms of initial data quantities on Σ. In particular, whether a solution to the equations of linearised gravity constructed as in Theorem 5.1 is in the modified Regge–Wheeler gauge can be verified explicitly from the initial data from which it arises.

In view of Remark 26 solutions $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$ to the equations of linearised gravity that are in the modified Regge–Wheeler gauge are thus said to be **initial-data-normalised**.

5.2.2 Achieving the modified Regge–Wheeler gauge: initial-data-normalised solutions to the equations of linearised gravity

In this section we show that, given a solution to the equations of linearised gravity arising under Theorem 5.1, one can indeed add to it a pure gauge solution for which the resulting solution is initial-data-normalised.

In the following theorem statement we recall the notation convention $\mathscr{D}_{\mathfrak{m},\mathfrak{a}}$ to explicitly reference the constant $\mathfrak{m}$ and function $\mathfrak{a}$ appearing in the seed data set $\mathscr{D}$ as per Definition 5.2. In addition, for a collection of quantities $f = (f_1, ..., f_n), g = (g_1, ..., g_n), h = (h_1, ..., h_n)$ with $n \geq 1$ we use the notation

$$f = g + h$$

to denote that

$$(f_1, ..., f_n) = (g_1 + h_1, ..., g_n + h_n).$$

Theorem 5.5. *Let $\mathscr{S}$ be the smooth solution to the equations of linearised gravity arising from the smooth seed data set $\mathscr{D}_{\mathfrak{m},\mathfrak{a}}$. Then there exists a unique pure gauge solution $\overset{\diamond}{\mathscr{G}}$ such that the resulting solution*

$$\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}} := \mathscr{S} + \overset{\diamond}{\mathscr{G}}$$

is in the modified Regge–Wheeler gauge with parameters $\mathfrak{m}$ and $\mathfrak{a}$.

Before we proceed with the proof it will be expedient to introduce the following notation:

- given $f_1, f_2, f_3, f_4 \in C^\infty(\Sigma) \cap \Lambda(\Sigma)$ we denote by $\overset{\diamond}{\mathscr{A}}[f_1, f_2, f_3, f_4]$ the smooth admissible initial data set for the equations of linearised gravity obtained from setting $\overset{(1)}{\bar{\Phi}} = f_1, \overset{(1)}{\bar{\Psi}} = f_2, \overset{(1)}{\underline{\Phi}} = f_3$ and $\overset{(1)}{\underline{\Psi}} = f_4$ in the class of admissible initial data sets constructed in step 1. of the proof of part i) of Theorem 5.1

- given a constant m and a smooth function a on S^2 supported on the $l = 1$ spherical harmonics we denote by $\overset{\mathscr{K}}{\mathscr{A}}[m, a]$ the smooth admissible initial data set for the equations of linearised gravity obtained from setting $\mathfrak{m} = m$ and $\mathfrak{a} = a$ in the class of admissible initial data sets constructed in step 2. of the proof of part i) of Theorem 5.1

- given $\underline{\mathfrak{v}}, \bar{\mathfrak{v}}, \underline{\mathfrak{w}}, \bar{\mathfrak{w}} \in C^\infty(\Sigma)$ and $\not{\mathfrak{v}}, \not{\mathfrak{w}} \in \mathscr{T}^1(\Sigma)$ we denote by $\overset{\mathscr{G}}{\mathscr{A}}[\underline{\mathfrak{v}}, \bar{\mathfrak{v}}, \not{\mathfrak{v}}, \underline{\mathfrak{w}}, \bar{\mathfrak{w}}, \not{\mathfrak{w}}]$ the smooth admissible initial data set for the equations of linearised gravity obtained from setting $\overset{(1)}{\underline{\mathfrak{v}}} = \underline{\mathfrak{v}}, \overset{(1)}{\bar{\mathfrak{v}}} = \bar{\mathfrak{v}}, \overset{(1)}{\not{\mathfrak{v}}} = \not{\mathfrak{v}}, \overset{(1)}{\underline{\mathfrak{w}}} = \underline{\mathfrak{w}}, \overset{(1)}{\bar{\mathfrak{w}}} = \bar{\mathfrak{w}}$ and $\overset{(1)}{\not{\mathfrak{w}}} = \not{\mathfrak{w}}$ in the class of admissible initial data sets constructed in step 3. of the proof of part i) of Theorem 5.1

Lastly, given a smooth solution $\mathscr{S}$ to the equations of linearised gravity we denote by $\mathscr{A}[\mathscr{S}]$ the smooth admissible initial data set constructed from it in accordance with Definition 5.1.

We further note the following lemma.

Lemma 5.6. *Let $\underline{\mathfrak{v}}, \bar{\mathfrak{v}}, \underline{\mathfrak{w}}, \bar{\mathfrak{w}} \in C^\infty(\Sigma)$ and let $\not{\mathfrak{v}}, \not{\mathfrak{w}} \in \mathscr{T}^1(\Sigma)$. Then there exists a pure gauge solution $\mathscr{G}$ such that $\mathscr{A}[\mathscr{G}] = \overset{\mathscr{G}}{\mathscr{A}}[\underline{\mathfrak{v}}, \bar{\mathfrak{v}}, \not{\mathfrak{v}}, \underline{\mathfrak{w}}, \bar{\mathfrak{w}}, \not{\mathfrak{w}}]$.*

Proof. In light of Lemma 5.2 and step 2. in the proof of part i) of Theorem 5.1 it suffices to construct $\tilde{v} \in \tilde{\mathscr{T}}^1(\mathcal{M})$ and $\not{v} \in \mathscr{T}^1(\mathcal{M})$ such that

$$\begin{aligned}
\tilde{\Box}\tilde{v} + \not{\Delta}\tilde{v} - \frac{2}{r}\Big(\widetilde{\nabla}\tilde{v}\Big)_P + \frac{2}{r^2}\,\mathrm{d}r\,\tilde{v}_P &= -\frac{1}{r}\Big(\widetilde{\nabla} \otimes \tilde{v}_{l=0}\Big)_P, \\
\tilde{\Box}\not{v} + \not{\Delta}\not{v} - \frac{2}{r}\widetilde{\nabla}_P\not{v} + \frac{1}{r^2}(3 - 4\mu)\not{v} &= 0
\end{aligned}$$

with

$$\begin{aligned}
\tilde{v}_n\Big|_\Sigma &= \underline{\mathfrak{v}}, & \tilde{v}_\nu\Big|_\Sigma &= \bar{\mathfrak{v}}, & \not{v}\Big|_\Sigma &= \not{\mathfrak{v}}, \\
\Big(\widetilde{\nabla}_n\tilde{v}_n\Big)\Big|_\Sigma &= \underline{\mathfrak{w}}, & \Big(\widetilde{\nabla}_n\tilde{v}_\nu\Big)\Big|_\Sigma &= \bar{\mathfrak{w}}, & \Big(\widetilde{\nabla}_n\not{v}\Big)\Big|_\Sigma &= \not{\mathfrak{w}}.
\end{aligned}$$

However, one can indeed solve the above system for $\tilde{v}$ and $\not{v}$ using standard techniques for solving systems of linear wave equations. In particular, employing a spherical harmonic decomposition as in section 3.2.5.1 then the spherical symmetry of $\left(\mathcal{M}, g_M\right)$ implies that the $l = 0, 1$ modes will propagate orthogonally to the rest. This completes the lemma. $\square$

The proof of Theorem 5.5 then proceeds as follows.

Proof. We first observe the following fact from the proof of part i) of Theorem 5.1: if $\mathscr{S}$ is a smooth solution to the equations of linearised gravity for which there exists $f_1, f_2, f_3, f_4 \in C^\infty(\Sigma) \cap \Lambda(\Sigma)$, a constant m and a smooth function a on S^2 supported on the $l = 1$ spherical harmonics such that

$$\mathscr{A}[\mathscr{S}] = \overset{\diamond}{\mathscr{A}}[f_1, f_2, f_3, f_4] + \overset{\mathscr{K}}{\mathscr{A}}[m, a]$$

then $\mathscr{S}$ is in the modified Regge–Wheeler gauge with parameters $\mathfrak{m} = m$ and $\mathfrak{a} = a$. In particular, the functions f_1, f_2, f_3, f_4 comprise Cauchy data for the corresponding gauge-invariant quantities $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ associated to $\mathscr{S}$.

Let now $\mathscr{S}$ be the smooth solution to the equations of linearised in the theorem statement with $\mathscr{D}_{\mathfrak{m},\mathfrak{a}}$ the corresponding seed data from which it arises:

$$\mathscr{D} = \left(\overset{(1)}{\bar{\Phi}}, \overset{(1)}{\underline{\Phi}}, \overset{(1)}{\bar{\Psi}}, \overset{(1)}{\underline{\Psi}}, \overset{(1)}{\underline{\mathfrak{v}}}, \overset{(1)}{\bar{\mathfrak{v}}}, \overset{(1)}{\not{\mathfrak{v}}}, \overset{(1)}{\underline{\mathfrak{w}}}, \overset{(1)}{\bar{\mathfrak{w}}}, \overset{(1)}{\not{\mathfrak{w}}}, \mathfrak{m}, \mathfrak{a}\right).$$

Then we have from the proof of part i) of Theorem 5.1 that

$$\mathscr{A}[\mathscr{S}] = \overset{\diamond}{\mathscr{A}}[\overset{(1)}{\bar{\Phi}}, \overset{(1)}{\bar{\Psi}}, \overset{(1)}{\underline{\Phi}}, \overset{(1)}{\underline{\Psi}}] + \overset{\mathscr{K}}{\mathscr{A}}[\mathfrak{m}, \mathfrak{a}] + \overset{\mathscr{G}}{\mathscr{A}}[\overset{(1)}{\underline{\mathfrak{v}}}, \overset{(1)}{\bar{\mathfrak{v}}}, \overset{(1)}{\not{\mathfrak{v}}}, \overset{(1)}{\underline{\mathfrak{w}}}, \overset{(1)}{\bar{\mathfrak{w}}}, \overset{(1)}{\not{\mathfrak{w}}}].$$

It therefore suffices to construct a pure gauge solution $\overset{\diamond}{\mathscr{G}}$ such that

$$\mathscr{A}[\overset{\diamond}{\mathscr{G}}] = -\overset{\mathscr{G}}{\mathscr{A}}[\overset{(1)}{\underline{\mathfrak{v}}}, \overset{(1)}{\bar{\mathfrak{v}}}, \overset{(1)}{\not{\mathfrak{v}}}, \overset{(1)}{\underline{\mathfrak{w}}}, \overset{(1)}{\bar{\mathfrak{w}}}, \overset{(1)}{\not{\mathfrak{w}}}]$$

since then by linearity the solution $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$ defined as $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}} := \mathscr{S} + \overset{\diamond}{\mathscr{G}}$ satisfies

$$\mathscr{A}[\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}] = \overset{\diamond}{\mathscr{A}}[\overset{(1)}{\bar{\Phi}}, \overset{(1)}{\bar{\Psi}}, \overset{(1)}{\underline{\Phi}}, \overset{(1)}{\underline{\Psi}}] + \overset{\mathscr{K}}{\mathscr{A}}[\mathfrak{m}, \mathfrak{a}].$$

Consequently, invoking Lemma 5.6, the theorem follows. $\square$

We make the following remark.

Remark 27. It is clear from the proof of Theorem 5.5 that if the seed data set $\mathscr{D}_{\mathfrak{m},\mathfrak{a}}$ is pointwise asymptotically Kerr then so is the seed data for the pure gauge solution $\overset{\diamond}{\mathscr{G}}$.

5.2.3 Global properties of initial-data-normalised solutions

In this final section we prove certain global properties of initial-data-normalised solutions $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$ which will be fundamental in establishing the boundedness and decay statements of Theorem 2 in section 6.

We first have the following proposition which establishes in particular that the gauge conditions associated to the modified Regge–Wheeler gauge of Definition 5.9 actually propagate under evolution.

In what follows, we denote by $\hat{\overset{\diamond}{\mathscr{S}}}$ the projection of $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$ away from $l=0,1$:

$$\hat{\overset{\diamond}{\mathscr{S}}} := \overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}} - \left(\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}\right)_{l=0,1}$$

noting that this notation scheme is consistent with the conclusions of the proposition below.

Proposition 5.7. *Let $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$ be a smooth solution to the equations of linearised gravity that is initial-data-normalised according to Definition 5.9. Then the following hold on $D^+(\Sigma)$:*

- *the quantities $\overset{(1)}{\not{g}}, \hat{\overset{(1)}{\not{g}}}, \overset{(1)}{\not{\gamma}}$ and $\hat{\overset{(1)}{\not{\gamma}}}$ associated to the projection $\hat{\overset{\diamond}{\mathscr{S}}}$ satisfy*

$$\begin{aligned} \overset{(1)}{\not{g}}{}'_{\mathrm{e}} - \overset{(1)}{\not{\gamma}}_{\mathrm{e}} &= 0, \\ \hat{\overset{(1)}{\not{g}}} - \hat{\overset{(1)}{\not{\gamma}}} &= 0 \end{aligned}$$

- *the projection $\left(\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}\right)_{l=0,1}$ satisfies*

$$\left(\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}\right)_{l=0,1} - \mathscr{K}_{\mathfrak{m},\mathfrak{a}} = 0$$

Moreover, the quantity $\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}}$ associated to the projection $\hat{\overset{\diamond}{\mathscr{S}}}$ vanishes on $D^+(\Sigma)$:

$$\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}} = 0.$$

Proof. We consider first the projection $\hat{\overset{\diamond}{\mathscr{S}}}$ which by Proposition 5.4 is a solution to the equations of linearised gravity with vanishing projection to $l=0,1$. Consequently, writing out the expression $\underset{\circ}{\overset{(1)}{\not{f}}}$ explicitly (cf. the proof of part *ii*) of Theorem 5.1) we have from equation (3.26) that the quantity $\overset{(1)}{\hat{\not g}}$ associated to the solution $\overset{\diamond}{\mathscr{S}}$ satisfies the system

$$\widetilde{\Box}\overset{(1)}{\hat{\not g}} + \not\Delta\overset{(1)}{\hat{\not g}} - \frac{2}{r}\widetilde{\nabla}_P\overset{(1)}{\hat{\not g}} - \frac{4}{r}\frac{\mu}{r}\overset{(1)}{\hat{\not g}} = \frac{2}{r^2}(1-2\mu)\overset{(1)}{\hat{\not\gamma}} + r\not\nabla\hat{\otimes}\not\nabla\mathfrak{Z}\overset{(1)}{\Psi}, \tag{5.14}$$

$$\mathbb{C}_\Sigma\overset{(1)}{\hat{\not g}} = \mathbb{C}_\Sigma\overset{(1)}{\hat{\not\gamma}}, \tag{5.15}$$

where $\overset{(1)}{\Psi}$ and $\overset{(1)}{\hat{\not\gamma}}$ are now associated to the solution $\overset{\diamond}{\mathscr{S}}$. Proceeding by direct computation we then verify that by virtue of the fact that from Theorem 4.3 the quantities $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ associated to $\overset{\diamond}{\mathscr{S}}$ satisfy the Regge–Wheeler and Zerilli equations respectively then $\overset{(1)}{\hat{\not g}} = \overset{(1)}{\hat{\not\gamma}}$ is a solution to the system (5.14)-(5.15). A standard Grönwall type argument then yields that $\overset{(1)}{\hat{\not g}} = \overset{(1)}{\hat{\not\gamma}}$ is the only solution.

Similarly, writing out now the expression $\underset{\circ}{\overset{(1)}{\not f}}$ explicitly (cf. once more the proof of Theorem 5.1) and exploiting the commutation relations of Lemma 3.14 we have from equation (3.25) that the quantity $\overset{(1)}{\not g}_{\mathrm{e}}$ associated to the solution $\overset{\diamond}{\mathscr{S}}$ satisfies a wave equation for which $\overset{(1)}{\not\gamma}_{\mathrm{e}}$ is also a solution, where now we associate the latter with the solution $\overset{\diamond}{\mathscr{S}}$. Since by definition of $\overset{\diamond}{\mathscr{S}}$ the Cauchy data of $\overset{(1)}{\not g}_{\mathrm{e}}$ agrees with that $\overset{(1)}{\not\gamma}_{\mathrm{e}}$, arguing as previously we thus conclude that $\overset{(1)}{\not g}_{\mathrm{e}} = \overset{(1)}{\not\gamma}_{\mathrm{e}}$.

Lastly, given that $\overset{(1)}{\hat{\not g}} = \overset{(1)}{\hat{\not\gamma}}$ and $\overset{(1)}{\not g}_{\mathrm{e}} = \overset{(1)}{\not\gamma}_{\mathrm{e}}$ we compute explicitly from the gauge condition (3.29) that the quantity $\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}}$ associated to the solution $\overset{\diamond}{\mathscr{S}}$ satisfies

$$\not\nabla\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}} = 0$$

from which we conclude that $\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}} = 0$ owing to the fact that all quantities in $\overset{\diamond}{\mathscr{S}}$ have vanishing projection to $l = 0, 1$.

We consider now the projection $\left(\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}\right)_{l=0,1}$ which by Proposition 5.4 is a solution to the equations of linearised gravity supported only on $l = 0, 1$. Moreover, by assumption the Cauchy data of the solution $\left(\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}\right)_{l=0,1}$ agrees with that of the linearised Kerr solution $\mathscr{K}_{\mathfrak{m},\mathfrak{a}}$ and thus by the uniqueness criterion of Theorem 5.1 we conclude that $\left(\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}\right)_{l=0,1} = \mathscr{K}_{\mathfrak{m},\mathfrak{a}}$ which completes the proposition. □

We make the following remark.

Remark 28. The reason for introducing the map $\overset{\diamond}{f}$ of section 3.3.1.3 is to ensure the validity of the above proposition – see section 2.3.1.1 of the overview for a discussion.

One consequence of Proposition 5.7 is that one can in fact fully describe the solution $\overset{\diamond}{\mathscr{S}}$ in terms of the associated quantities $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$. This shall be exploited in section 7.2 to prove Theorem 2 of section 6.

Corollary 5.8. *Let* $\overset{\diamond}{\mathscr{S}}$ *be as in Proposition 5.7. Then the following identities hold on* $D^+(\Sigma)$*:*

$$\hat{\tilde{\overset{\scriptscriptstyle(1)}{g}}} - \hat{\tilde{\overset{\scriptscriptstyle(1)}{\gamma}}} = 0,$$
$$\overset{\scriptscriptstyle(1)}{\not{g}} - \overset{\scriptscriptstyle(1)}{\not{\gamma}} = 0,$$
$$\hat{\overset{\scriptscriptstyle(1)}{\not{g}}} - \hat{\overset{\scriptscriptstyle(1)}{\not{\gamma}}} = 0,$$
$$\not{\mathrm{tr}}\overset{\scriptscriptstyle(1)}{\not{g}} - \not{\mathrm{tr}}\overset{\scriptscriptstyle(1)}{\not{\gamma}} = 0$$

and

$$\mathrm{tr}_{\tilde{g}_M}\tilde{\overset{\scriptscriptstyle(1)}{g}} = 0.$$

Proof. That $\mathrm{tr}_{\tilde{g}_M}\tilde{\overset{\scriptscriptstyle(1)}{g}}$ vanishes was shown in Proposition 5.7.

For the remaining identities, we first recall from the proof of Theorem 4.3 that the quantities $\hat{\tilde{\overset{\scriptscriptstyle(1)}{\tau}}}, \tilde{\overset{\scriptscriptstyle(1)}{\eta}}$ and $\overset{\scriptscriptstyle(1)}{\sigma}$ associated to the solution $\overset{\diamond}{\mathscr{S}}$ must satisfy

$$\hat{\tilde{\overset{\scriptscriptstyle(1)}{\tau}}} = \widetilde{\nabla}\hat{\otimes}\tilde{\mathrm{d}}\left(r\overset{\scriptscriptstyle(1)}{\Psi}\right) + 6\mu\,\mathrm{d}r\hat{\otimes}\not{\zeta}^{[1]}\tilde{\mathrm{d}}\overset{\scriptscriptstyle(1)}{\Psi},$$
$$\tilde{\overset{\scriptscriptstyle(1)}{\eta}} = -\tilde{\star}\tilde{\mathrm{d}}\left(r\overset{\scriptscriptstyle(1)}{\Phi}\right),$$
$$\overset{\scriptscriptstyle(1)}{\sigma} = -2r\not{\Delta}\overset{\scriptscriptstyle(1)}{\Psi} + 4\widetilde{\nabla}_P\overset{\scriptscriptstyle(1)}{\Psi} + 12\mu r^{-1}(1-\mu)\not{\zeta}^{[1]}\overset{\scriptscriptstyle(1)}{\Psi}$$

where $\overset{\scriptscriptstyle(1)}{\Phi}$ and $\overset{\scriptscriptstyle(1)}{\Psi}$ are the quantities of Proposition 4.1 associated now to the solution $\overset{\diamond}{\mathscr{S}}$. Subsequently, from the definition of $\hat{\tilde{\overset{\scriptscriptstyle(1)}{\tau}}}, \tilde{\overset{\scriptscriptstyle(1)}{\eta}}$ and $\overset{\scriptscriptstyle(1)}{\sigma}$ in terms of the maps $\tilde{\tau}, \tilde{\eta}$ and σ of section 3.3.1.3 it follows that

$$\hat{\tilde{\overset{\scriptscriptstyle(1)}{g}}} - \widetilde{\nabla}\hat{\otimes}\left(\overset{\scriptscriptstyle(1)}{\not{g}}_{\mathrm{e}} - r^2\tilde{\mathrm{d}}\left(r^{-2}\hat{\overset{\scriptscriptstyle(1)}{\not{g}}}_{\mathrm{e}}\right)\right) = \widetilde{\nabla}\hat{\otimes}\tilde{\mathrm{d}}\left(r\overset{\scriptscriptstyle(1)}{\Psi}\right) + 6\mu\,\mathrm{d}r\hat{\otimes}\not{\zeta}^{[1]}\tilde{\mathrm{d}}\overset{\scriptscriptstyle(1)}{\Psi},$$
$$\overset{\scriptscriptstyle(1)}{\not{g}}_{\mathrm{o}} - r^2\tilde{\mathrm{d}}\left(r^{-2}\hat{\overset{\scriptscriptstyle(1)}{\not{g}}}_{\mathrm{o}}\right) = -\tilde{\star}\tilde{\mathrm{d}}\left(r\overset{\scriptscriptstyle(1)}{\Phi}\right),$$
$$\not{\mathrm{tr}}\overset{\scriptscriptstyle(1)}{\not{g}} - \frac{4}{r}\left(\overset{\scriptscriptstyle(1)}{\not{g}}_{\mathrm{e}} - r^2\tilde{\mathrm{d}}\left(r^{-2}\hat{\overset{\scriptscriptstyle(1)}{\not{g}}}_{\mathrm{e}}\right)\right) - 2\not{\Delta}\hat{\overset{\scriptscriptstyle(1)}{\not{g}}}_{\mathrm{e}} = -2r\not{\Delta}\overset{\scriptscriptstyle(1)}{\Psi} + 4\widetilde{\nabla}_P\overset{\scriptscriptstyle(1)}{\Psi} + 12\mu r^{-1}(1-\mu)\not{\zeta}^{[1]}\overset{\scriptscriptstyle(1)}{\Psi}.$$

Invoking Proposition 5.7, the Corollary follows. □

We make the following remark.

Remark 29. Another way to interpret Corollary 5.8 is as follows: a smooth solution $\overset{\diamond}{\mathscr{S}}$ to the equations of linearised gravity that has vanishing projection to $l = 0, 1$ and which is initial-data-normalised has the ansatz

$$
\begin{aligned}
\hat{\tilde{\overset{\scriptscriptstyle(1)}{g}}} &= \widetilde{\nabla}\hat{\otimes}\tilde{\mathrm{d}}^{\mathcal{I}}\Big(r\overset{\scriptscriptstyle(1)}{\Psi}\Big) + 6\mu\,\mathrm{d}r\hat{\tilde{\otimes}}\not{\zeta}^{[1]}\tilde{\mathrm{d}}\overset{\scriptscriptstyle(1)}{\Psi},\\
\tilde{\overset{\scriptscriptstyle(1)}{g}} &= 0,\\
\not{\overset{\scriptscriptstyle(1)}{g}} &= \not{\mathcal{D}}_1^\star\Big(\tilde{\mathrm{d}}^{\mathcal{I}}\Big(r\overset{\scriptscriptstyle(1)}{\Psi}\Big) - 2\tilde{\mathrm{d}}r\,\overset{\scriptscriptstyle(1)}{\Psi}, \tilde{\mathrm{d}}^{\mathcal{I}}\Big(r\overset{\scriptscriptstyle(1)}{\Phi}\Big) - 2\tilde{\mathrm{d}}r\,\overset{\scriptscriptstyle(1)}{\Phi}\Big),\\
\hat{\not{\overset{\scriptscriptstyle(1)}{g}}} &= r\not{\nabla}\hat{\otimes}\not{\mathcal{D}}_1^\star\Big(\overset{\scriptscriptstyle(1)}{\Psi}, \overset{\scriptscriptstyle(1)}{\Phi}\Big),\\
\not{\mathrm{tr}}\not{\overset{\scriptscriptstyle(1)}{g}} &= 4\tilde{\mathrm{d}}_P^{\mathcal{I}}\overset{\scriptscriptstyle(1)}{\Psi} + 12\mu r^{-1}(1-\mu)\not{\zeta}^{[1]}\overset{\scriptscriptstyle(1)}{\Psi}
\end{aligned}
$$

where $\overset{\scriptscriptstyle(1)}{\Phi}$ and $\overset{\scriptscriptstyle(1)}{\Psi}$ satisfy the Regge–Wheeler and Zerilli equations respectively:

$$
\begin{aligned}
\widetilde{\Box}\overset{\scriptscriptstyle(1)}{\Phi} + \not{\Delta}\overset{\scriptscriptstyle(1)}{\Phi} &= -\frac{6}{r^2}\frac{M}{r}\overset{\scriptscriptstyle(1)}{\Phi},\\
\widetilde{\Box}\overset{\scriptscriptstyle(1)}{\Psi} + \not{\Delta}\overset{\scriptscriptstyle(1)}{\Psi} &= -\frac{6}{r^2}\frac{M}{r}\overset{\scriptscriptstyle(1)}{\Psi} + \frac{24}{r^3}\frac{M}{r}(r-3M)\not{\zeta}^{[1]}\overset{\scriptscriptstyle(1)}{\Psi} + \frac{72}{r^5}\frac{M}{r}\frac{M}{r}(r-2M)\not{\zeta}^{[2]}\overset{\scriptscriptstyle(1)}{\Psi}.
\end{aligned}
$$

Note that this in particular implies that there exist solutions to the equations of linearised gravity which are completely described by two scalar functions satisfying the Regge–Wheeler and Zerilli equations respectively – this fact was exploited in the proof of part *i)* of Theorem 5.1.

We end this section by noting the following result, the veracity of which is immediate from Theorem 5.5 combined with Proposition 5.7 and Corollary 5.8.

Corollary 5.9. *Let $\mathscr{S}$ be a smooth solution to the equations of linearised gravity arising from the smooth seed data $\mathscr{D}_{\mathfrak{m},\mathfrak{a}}$. Suppose further that the gauge-invariant quantities $\overset{\scriptscriptstyle(1)}{\Psi}$ and $\overset{\scriptscriptstyle(1)}{\Phi}$ associated to $\mathscr{S}$ vanish. Then $\mathscr{S}$ is the sum of the pure gauge solution $\overset{\diamond}{\mathscr{G}}$ of Theorem 5.5 and the linearised Kerr solution $\mathscr{K}_{\mathfrak{m},\mathfrak{a}}$:*

$$
\overset{\scriptscriptstyle(1)}{\Psi} = \overset{\scriptscriptstyle(1)}{\Phi} = 0 \implies \mathscr{S} = \overset{\diamond}{\mathscr{G}} + \mathscr{K}_{\mathfrak{m},\mathfrak{a}}.
$$

6

The linear stability of the Schwarzschild solution to gravitational perturbations in the generalised wave gauge: The Precise statement

This chapter of the thesis is concerned with giving precise statements of the main theorems of this thesis namely Theorem 1 which concerns quantitative boundedness and decay statements for solutions of the Regge–Wheeler and Zerilli equations and Theorem 2 which concerns quantitative boundedness and decay statements for initial-data-normalised solutions to the equations of linearised gravity.

It is this latter statement that comprises the precise, quantitative statement of linear stability for the Schwarzschild solution in a generalised wave gauge.

6.1 Flux and integrated decay norms

We begin in this section by defining the flux and integrated decay norms through which the quantitative boundedness and decay statements of Theorems 1 and 2 are to be formulated.

6.1.1 Flux and integrated decay norms on smooth functions

First we define these norms for smooth functions ψ on $\mathcal{M}$.

In what follows, we remind the reader of the function $\tau^\star$ defined in section 3.2.4 and the norms on spheres defined in section 3.2.6.1. Moreover, we recall the operator D defined as in section 3.2.4.2.

We associate to ψ the energy norm

$$\begin{aligned}\mathbb{E}[\psi](\tau^\star) := &\int_{2M}^{R} \left(||\partial_{t^*}\psi||^2_{S^2_{\tau^\star,r}} + ||\partial_r\psi||^2_{S^2_{\tau^\star,r}} + ||\nabla\!\!\!/\psi||^2_{S^2_{\tau^\star,r}} \right) \mathrm{d}r \\ &+ \int_{R}^{\infty} \left(||D(r\psi)||^2_{S^2_{\tau^\star,r}} + ||\nabla\!\!\!/(r\psi)||^2_{S^2_{\tau^\star,r}} \right) \mathrm{d}r\end{aligned}$$

and the r^p-weighted norms

$$\mathbb{F}_p[\psi](\tau^\star) := \int_{R}^{\infty} \left(r^p||D(r\psi)||^2_{S^2_{\tau^\star,r}} + ||\nabla\!\!\!/(r\psi)||^2_{S^2_{\tau^\star,r}} \right) \mathrm{d}r.$$

This leads to the weighted flux norm

$$\mathbb{F}[\psi] : = \sup_{\tau^\star\in[\tau_0^\star,\infty)} \left(\mathbb{E}[\psi](\tau^\star) + \int_{R}^{\infty} r^2||D(r\psi)||^2_{S^2_{\tau^\star,r}}\, \mathrm{d}r \right).$$

We morever define the initial flux norms along the initial Cauchy hypersurface Σ of section 3.2.3.1:

$$\mathbb{D}[\psi] := \int_{2M}^{\infty} r^2 \left(\|n(\psi)\|^2_{\Sigma,r} + \|\partial_r\psi\|^2_{\Sigma,r} + \|\nabla\!\!\!/\psi\|^2_{\Sigma,r} \right) \mathrm{d}r.$$

Here, for a smooth function f on $\mathcal{M}$

$$\|f\|^2_{\Sigma,r} := \|f\|^2_{S^2_{\tau^\star(t_0^*),r}}$$

where we note that by definition of the function $\tau^\star$ the 2-sphere $S^2_{\tau^\star(t_0^*),r}$ is the 2-sphere $\{t_0^*\} \times \{r\} \times S^2 \subset \Sigma$.

We further associate to ψ the integrated local energy decay norm

$$\mathbb{I}_{\mathrm{loc}}[\psi](\tau_1^\star) := \int_{\tau_1^\star}^{\infty} \int_{2M}^{R} \left(||\partial_{t^*}(r\psi)||^2_{S^2_{\tau^\star,r}} + ||\partial_r(r\psi)||^2_{S^2_{\tau^\star,r}} + ||\nabla\!\!\!/(r\psi)||^2_{S^2_{\tau^\star,r}} + ||r\psi||^2_{S^2_{\tau^\star,r}} \right) \mathrm{d}\tau^\star\, \mathrm{d}r$$

and the r^p-weighted bulk norms

$$\mathbb{B}_p[\psi](\tau_1^\star) := \int_{\tau_1^\star}^{\infty} \int_{R}^{\infty} r^p \left(||D(r\psi)||^2_{S^2_{\tau^\star,r}} + ||\nabla\!\!\!/(r\psi)||^2_{S^2_{\tau^\star,r}} \right) \mathrm{d}\tau^\star\, \mathrm{d}r.$$

This leads to the integrated decay norm

$$\mathbb{M}[\psi] := \int_{\tau_0^\star}^{\infty}\int_{2M}^{\infty}\frac{1}{r^3}\Big(||\partial_{t^*}(r\psi)||^2_{S^2_{\tau^\star,r}} + ||\partial_r(r\psi)||^2_{S^2_{\tau^\star,r}} + ||\nabla\!\!\!/(r\psi)||^2_{S^2_{\tau^\star,r}} + ||r\psi||^2_{S^2_{\tau^\star,r}}\Big)\,\mathrm{d}\tau^\star\,\mathrm{d}r$$

and the weighted bulk norm

$$\mathbb{I}[\psi] := \int_{\tau_0^\star}^{\infty}\int_{R}^{\infty}\Big(r||D(r\psi)||^2_{S^2_{\tau^\star,r}} + r^{\beta_0}||\nabla\!\!\!/(r\psi)||^2_{S^2_{\tau^\star,r}}\Big)\,\mathrm{d}\tau^\star\,\mathrm{d}r.$$

Here, $\beta_0 > 0$ is a fixed constant such that $1-\beta_0 << 1$.

Higher order flux norms are then defined according to, for $n \geq 1$ an integer,

$$\begin{aligned}\mathbb{F}^n[\psi] := &\sum_{i+j+k=0}^{n}\sup_{\tau^\star\in[\tau_0^\star,\infty)}\mathbb{E}[\partial_{t^*}^i\partial_r^j\nabla\!\!\!/^k\psi](\tau^\star)\\ &+\sum_{i+j+k=1}^{n}\sup_{\tau^\star\in[\tau_0^\star,\infty)}\int_R^\infty r^2||D\underline{D}^i(rD)^j(r\nabla\!\!\!/)^k(r\psi)||^2_{S^2_{\tau^\star,r}}\,\mathrm{d}r\end{aligned}$$

with analogous definitions for the higher order energy and r^p-weighted norms. Here, we recall the operator $\underline{D}$ defined as in section 3.2.4.2.

Conversely, higher order initial flux norms are defined according to

$$\mathbb{D}^n[\psi] := \sum_{i+j=0}^{n}\mathbb{D}[(r\partial_r)^i(r\nabla\!\!\!/)^k\psi].$$

Finally, higher order integrated decay norms are defined according to

$$\begin{aligned}\mathbb{M}^n[\psi] :&= \sum_{i+j+k=0}^{n}\mathbb{M}[\partial_{t^*}^i\partial_r^j\nabla\!\!\!/^k\psi],\\ \mathbb{I}^n[\psi] :&= \sum_{i+j+k=0}^{n}\mathbb{I}[\underline{D}^i(rD)^j(r\nabla\!\!\!/)^k\psi]\end{aligned}$$

with analogous definitions for the higher order integrated local energy decay norms and the r^p-weighted bulk norms.

6.1.2 Flux and integrated decay norms on smooth tensor fields

Now we upgrade the norms of the previous section to tensor fields on $\mathcal{M}$.

In fact, replacing the derivative operators $D, \underline{D}$ and n by their associated Lie derivatives in the norms of section 6.1.1 then if ψ denotes either a smooth, 2-covariant $\mathcal{Q}$-tensor, a smooth $\mathcal{Q}\otimes S$ 1-form or a smooth n-covariant S-tensor then those very same norms are equally well-defined in light of Definition 3.12 and the commutation formulae of Lemma

3.14.

6.1.2.1 Flux and integrated decay norms for collections of linearised quantities

We finish this section by introducing a concise notation for the norms of the previous section acting on both collections of quantities and collections of linearised quantities.

Indeed, if $\psi = \{\psi_1, ..., \psi_n\}$ denotes a collection of quantities for which the norms of section 6.1.1 are well defined then we define, for any integer $n \geq 0$,

$$\mathbb{F}^n[\psi] := \sum_{i=1}^{n} \mathbb{F}^n[\psi_i].$$

with analogous definitions for the remaining norms.

Similarly, if $\overset{(1)}{\psi} = \{\overset{(1)}{\psi}_1, ..., \overset{(1)}{\psi}_n\}$ denotes a collection of linearised quantities associated to a solution $\mathscr{S}$ of the equations of linearised gravity for which the norms of section 6.1.1 are well defined then we define

$$\overset{(1)}{\mathbb{F}}{}^n[\psi] := \sum_{i=1}^{n} \mathbb{F}^n[\overset{(1)}{\psi}_i]$$

with analogous definitions for the remaining norms.

In particular, if $\mathscr{S}$ is a solution to the equations of linearised gravity then we define

$$\mathbb{F}^n[\mathscr{S}] := \overset{(1)}{\mathbb{F}}{}^n[\hat{\tilde{g}}] + \overset{(1)}{\mathbb{F}}{}^n[\mathrm{tr}_{\tilde{g}_M}\tilde{g}] + \overset{(1)}{\mathbb{F}}{}^n[\not{g}] + \overset{(1)}{\mathbb{F}}{}^n[\hat{\not{g}}] + \overset{(1)}{\mathbb{F}}{}^n[\not{\mathrm{tr}}\not{g}]$$

with analogous definitions for the norms $\mathbb{E}, \mathbb{M}, \mathbb{I}$ and $\mathbb{D}$.

6.1.3 Theorem 1: Boundedness and decay for solutions to the Regge–Wheeler and Zerilli equations

In this section we state Theorem 1 which concerns both a boundedness statement for solutions to the Regge–Wheeler and Zerilli equations in the flux and integrated decay norms of section 6.1.1 and a decay statement in the pointwise norms of section 3.2.4.2.

The proof of Theorem 1 is the content of section 7.1.

The theorem statement is as below – we note that in the statement we drop the superscript (1) from all quantities under consideration as the theorem holds independently of the relation between the Regge–Wheeler and Zerilli equations and the equations of linearised gravity.

In what follows, we recall Definition 5.3 and Definition 5.5 which define the notion of a tensor field on $\mathcal{M}$ having compact support on Σ_R and $D^+(\Sigma_R)$ respectively.

Theorem 1. *Let $\Phi \in C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$ be a solution to the Regge–Wheeler equation on $\big(\mathcal{M}, g_M\big)$:*

$$\tilde{\Box}\Phi + \not{\Delta}\Phi = -\frac{6}{r^2}\frac{M}{r}\Phi.$$

We assume that Cauchy data for Φ is compactly supported on Σ_R.

Then Φ is compactly supported on $D^+(\Sigma_R)$. In addition, for any integer $n \geq 2$, the following estimates hold:

i) the higher order flux and weighted bulk estimates

$$\mathbb{F}^n[r^{-1}\Phi] + \mathbb{I}^n[r^{-1}\Phi] \lesssim \mathbb{D}^n[r^{-1}\Phi]$$

ii) the higher order integrated decay estimate

$$\mathbb{M}^n[r^{-1}\Phi] \lesssim \mathbb{D}^{n+1}[r^{-1}\Phi]$$

iii) finally, on any 2-sphere $S^2_{\tau^\star,r}$ with $\tau^\star \geq \tau^\star_0$ and any positive integers $i+j+k+l+m \geq n-2$, the pointwise decay bounds

$$|\partial^i_{t^\star}\partial^j_r\Big((r-2M)D\Big)^k(r\not{\nabla})^l\underline{D}^m\Phi|_{S^2_{\tau^\star,r}} \lesssim \frac{1}{\sqrt{\tau^\star}}\cdot\mathbb{D}^n[r^{-1}\Phi].$$

Let now $\Psi \in C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$ be a solution to the Zerilli equation on $\big(\mathcal{M}, g_M\big)$:

$$\tilde{\Box}\Psi + \not{\Delta}\Psi = -\frac{6}{r^2}\frac{M}{r}\Psi + \frac{24}{r^3}\frac{M}{r}(r-3M)\not{\mathcal{C}}^{[1]}\Psi + \frac{72}{r^3}\frac{M}{r}\frac{M}{r}(r-2M)\not{\mathcal{C}}^{[2]}\Psi.$$

We assume that Cauchy data for Ψ is compactly supported on Σ_R.

Then Ψ is compactly supported on $D^+(\Sigma_R)$. In addition, for any integer $n \geq 2$, the following estimates hold:

i) the higher order flux and weighted bulk estimates

$$\mathbb{F}^n[r^{-1}\Psi] + \mathbb{I}^n[r^{-1}\Psi] \lesssim \mathbb{D}^n[r^{-1}\Psi]$$

ii) the higher order ntegrated decay estimate

$$\mathbb{M}^n[r^{-1}\Psi] \lesssim \mathbb{D}^{n+1}[r^{-1}\Psi]$$

iii) finally, on any 2-sphere $S^2_{\tau^\star,r}$ with $\tau^\star \geq \tau^\star_0$ and any positive integers $i+j+k+l+m \geq n-2$, the pointwise decay bounds

$$|\partial^i_{t^*}\partial^j_r\Big((r-2M)D\Big)^k(r\nabla\!\!\!/)^l\underline{D}^m\Psi|_{S^2_{\tau^\star,r}} \lesssim \frac{1}{\sqrt{\tau^\star}}\cdot\mathbb{D}^n[r^{-1}\Psi].$$

We make the following remarks regarding Theorem 1.

Remark 30. The fact that the solution remains compactly supported on $D^+(\Sigma_R)$ is a standard property of waves the characteristic hypersurfaces of which are determined by the geometry of g_M (cf. the proof of Proposition 5.3).

Remark 31. The assumption of compact support is merely a convenience and can be weakened to a regularity class for which the initial data norms $\mathbb{D}^n$ are assumed only to be finite. In this latter case one would need to supplement the above with additional estimates in the semi-global region $D^+(\Sigma) - D^+(\Sigma_R)$. However, such estimates can be quite easily shown to hold using the techniques we shall employ to prove Theorem 1 and thus shall not be pursued here.

Remark 32. The $r-2M$ weight in the pointwise bounds of *iii)* is to regularise the operator D at $\mathcal{H}^+$. In particular, for $r \geq R$ one can replace $(r-2M)D$ with rD.

Remark 33. The contents of Theorem 1 regarding the Regge–Wheeler equation were originally proven by Holzegel in[27] (see also[1]). Conversely, the contents of Theorem 1 regarding the Zerilli equation were originally proven in the independent works of the author[6] and Hung–Keller–Wang[30].

Remark 34. Recalling the definition of future null infinity $\mathcal{I}^+$ in section 3.2.4.1, one can show from parts *i)* of Theorem 1 that the quantities $|\overset{\scriptscriptstyle(1)}{\Phi}|$ and $|\overset{\scriptscriptstyle(1)}{\Psi}|$ have finite limits on $\mathcal{I}^+$.

6.2 Theorem 2: Boundedness, decay and asymptotic flatness of initial-data-normalised solutions to the equations of linearised gravity

In this section we state Theorem 2 which concerns both a boundedness statement for initial-data-normalised solutions $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$ to the equations of linearised gravity in the flux and integrated decay norms of section 6.1 and a decay statement in the pointwise norms of section 3.2.4.2. In addition, we provide a statement of asymptotic flatness for the solution $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$.

The proof of Theorem 2 is the content of section 7.2.

The theorem statement is as given below.

In what follows, given a solution $\mathscr{S}$ to the equations of linearised gravity we employ the notation scheme, for any 2-sphere $S^2_{\tau^\star,r}$,

$$|r\overset{\diamond}{\mathscr{S}}|_{S^2_{\tau^\star,r}} := |r\hat{\overset{(1)}{\tilde{g}}}|_{S^2_{\tau^\star,r}} + |r\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{g}}|_{S^2_{\tau^\star,r}} + |r\overset{(1)}{\not{g}}|_{S^2_{\tau^\star,r}} + |r\hat{\overset{(1)}{\not{g}}}|_{S^2_{\tau^\star,r}} + |r\not{\mathrm{tr}}\overset{(1)}{\not{g}}|_{S^2_{\tau^\star,r}}.$$

Theorem 2. *Let $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$ be a smooth solution to the equations of linearised gravity that is initial-data-normalised according to Definition 5.9 and let $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ denote the pure gauge and linearised Kerr invariant quantities associated to $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$ in accordance with Theorem 4.3.*

We assume that Cauchy data for all quantities associated to the solution

$$\overset{\diamond}{\mathscr{S}} := \overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}} - \mathscr{K}_{\mathfrak{m},\mathfrak{a}}$$

is compactly supported on Σ_R.

Then $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ satisfy the conclusions of Theorem 1.

Moreover, all quantities associated to the solution $\overset{\diamond}{\mathscr{S}} := \overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}} - \mathscr{K}_{\mathfrak{m},\mathfrak{a}}$ are in fact compactly supported on $D^+(\Sigma_R)$. In addition, for any integer $n \geq 0$, the following estimates hold:

i) the flux and weighted bulk estimates

$$\mathbb{F}^n[\overset{\diamond}{\mathscr{S}}] + \mathbb{I}^n[\overset{\diamond}{\mathscr{S}}] \lesssim \overset{(1)}{\mathbb{D}}{}^{n+2}[r^{-1}\Phi, r^{-1}\Psi]$$

ii) the integrated decay estimates

$$\mathbb{M}^n[\overset{\diamond}{\mathscr{S}}] \lesssim \overset{(1)}{\mathbb{D}}{}^{n+3}[r^{-1}\Phi, r^{-1}\Psi]$$

iii) finally, on any 2-sphere $S^2_{\tau^\star,r}$ with $\tau^\star \geq \tau^\star_0$, the pointwise decay bounds

$$|r\overset{\diamond}{\mathscr{S}}|_{S^2_{\tau^\star,r}} \lesssim \frac{1}{\sqrt{\tau^\star}} \cdot \overset{(1)}{\mathbb{D}}{}^{4}[r^{-1}\Phi, r^{-1}\Psi].$$

We make the following remarks regarding Theorem 1.

Remark 35. If $\mathscr{S}$ is a smooth solution to the equations of linearised gravity arising from a smooth, asymptotically Kerr seed data set $\mathscr{D}_{\mathfrak{m},\mathfrak{a}}$ in accordance with Theorem 5.1 then the initial-data-normalised solution $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$ constructed from $\mathscr{S}$ as in Theorem 5.5 satisfies the assumptions and hence the conclusions of Theorem 2. In this case, the parameters of the linearised Kerr solution $\mathscr{K}_{\mathfrak{m},\mathfrak{a}}$ in the theorem statement are computed explicitly from the seed data.

Remark 36. In light of the fact that $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ vanish for all linearised Kerr solutions it follows that the $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ associated to the solution $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$ are the same quantities as the $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ associated to the solution $\overset{\diamond}{\mathscr{S}}$. In particular, that compact support of the Cauchy data for $\overset{\diamond}{\mathscr{S}}$ implies compact support of the Cauchy data for $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ is immediate from the construction of the latter two quantities from $\overset{\diamond}{\mathscr{S}}$ as in Proposition 4.1.

Remark 37. The assumption of compact support is merely for convenience and can be quite readily removed (cf. Remark 31).

Remark 38. That the assumption of compact support propagates under evolution for the solution $\overset{\diamond}{\mathscr{S}}$ in the sense of the theorem statement follows from the proof of Proposition 5.3.

Remark 39. As in parts *iii*) of Theorem 1 one can obtain higher order analogues of the pointwise decay bounds in part *iii*) of the theorem statement although we decline to state these explicitly.

Remark 40. From part *i*) of Theorem 2 one can show that the radiation fields of all quantities associated to the solution $\overset{\diamond}{\mathscr{S}}$ have finite limits on $\mathcal{I}^+$. In addition, it follows from part *iii*) that the solution $\overset{\diamond}{\mathscr{S}}_{\mathfrak{m},\mathfrak{a}}$ decays inverse polynomially to the linearised Kerr solution $\mathscr{K}_{\mathfrak{m},\mathfrak{a}}$.

7

Proof of the main Theorems

This final chapter of the thesis is concerned with the proofs of Theorems 1 and 2.

In section 7.1 we prove Theorem 1.

Finally, we finish in section 7.2 with the proof of Theorem 2.

7.1 Proof of Theorem 1

In this section we prove Theorem 1. In particular, we shall drop the superscript (1) from all quantities under consideration in this section as the results we are to prove hold independently of their relation to the equations of linearised gravity.

For the motivation behind the techniques we are to employ in this section of the thesis, see section 2.5 of the overview.

7.1.1 Formal computations

We begin in this section by deriving various identities that solutions to the Regge–Wheeler and Zerilli equations must satisfy. These identities shall then be utilised throughout the remainder of the section.

In particular, we stress that although the formal computations we are about to present appear rather daunting they shall ultimately be of great expedience in what follows.

Let us first recall for ease of reference the definition of the Regge–Wheeler and Zerilli equations from Definition 4.1 for two functions $\Phi, \Psi \in \Lambda(\mathcal{M})$ respectively:

$$\widetilde{\Box}\Phi + \not\Delta\Phi = -\frac{6}{r^2}\frac{M}{r}\Phi, \tag{7.1}$$
$$\widetilde{\Box}\Psi + \not\Delta\Psi = -\frac{6}{r^2}\frac{M}{r}\Psi + \frac{24}{r^3}\frac{M}{r}(r-3M)\not\zeta^{[1]}\Psi + \frac{72}{r^3}\frac{M}{r}\frac{M}{r}(r-2M)\not\zeta^{[2]}\Psi. \tag{7.2}$$

Here, we further recall the operator $\not\zeta^{[p]}$ defined as in section 3.2.6.3.

We then have that solutions to the above must satisfy the following set of identities.

In what follows, given two smooth functions f and g on $\mathcal{M}$ we denote by $\langle f, g\rangle_{S^2_{\tau^\star,r}}$ their L^2 product on any 2-sphere $S^2_{\tau^\star,r}$:

$$\langle f, g\rangle_{S^2_{\tau^\star,r}} := \int_{S^2_{\tau^\star,r}} f\, g\, \mathring{\epsilon}$$

where $\mathring{\epsilon}$ is the volume form for the unit round sphere. In addition, for a f a smooth function of r on $\mathcal{M}$ we define

$$f' := \partial_r f.$$

Finally, we recall the mass aspect function $\mu = \frac{2M}{r}$.

Lemma 7.1. *Let α, β and w be smooth functions of r on $\mathcal{M}$.*

Let now Φ be a smooth solution to the Regge–Wheeler equation (7.1). *Then on any 2-sphere $S^2_{\tau^\star,r}$ the following identities hold:*

$$\begin{aligned}
&\partial_{t^*}\Big[(1+\mu)\alpha\,||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}} + (1-\mu)\alpha\,||\partial_r\Phi||^2_{S^2_{\tau^\star,r}} + \alpha\,||\not\nabla_\Upsilon\Phi||^2_{S^2_{\tau^\star,r}}\Big]\\
&-\partial_r\Big[2\mu\alpha\,||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}} + 2(1-\mu)\alpha\,\langle\partial_{t^*}\Phi, \partial_r\Phi\rangle_{S^2_{\tau^\star,r}}\Big]\\
&\quad + 2\mu\alpha'\,||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}}\\
&= -\,2(1-\mu)\alpha'\,\langle\partial_{t^*}\Phi, \partial_r\Phi\rangle_{S^2_{\tau^\star,r}},
\end{aligned}$$

$$\begin{aligned}
&\partial_{t^*}\Big[2(1+\mu)\beta\,\langle\partial_{t^*}\Phi, \partial_r\Phi\rangle_{S^2_{\tau^\star,r}} - 2\mu\beta\,||\partial_r\Phi||^2_{S^2_{\tau^\star,r}}\Big]\\
&-\partial_r\Big[(1+\mu)\beta\,||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}} - (1-\mu)\beta\,||\partial_r\Phi||^2_{S^2_{\tau^\star,r}} - \beta\,||\not\nabla_\Upsilon\Phi||^2_{S^2_{\tau^\star,r}}\Big]\\
&\quad + \Big((1+\mu)\beta\Big)'||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}} + \Big((1-\mu)\beta' - \frac{\mu}{r}\beta\Big)||\partial_r\Phi||^2_{S^2_{\tau^\star,r}} - \beta'\,||\not\nabla_\Upsilon\Phi||^2_{S^2_{\tau^\star,r}}\\
&\quad - \beta\,||[\partial_r, \Upsilon]\Phi||^2_{S^2_{\tau^\star,r}}\\
&= -\,2\frac{\mu}{r}\,\beta\,\langle\partial_{t^*}\Phi, \partial_r\Phi\rangle_{S^2_{\tau^\star,r}}
\end{aligned}$$

and

$$
\begin{aligned}
&-\partial_{t^*}\Big[(1+\mu)w\,\langle\partial_{t^*}\Phi,\Phi\rangle_{S^2_{\tau^\star,r}} - 2\mu w\,\langle\partial_r\Phi,\Phi\rangle_{S^2_{\tau^\star,r}} + \frac{\mu}{r}\frac{w}{2}\,||\Phi||^2_{S^2_{\tau^\star,r}}\Big]\\
&\ +\partial_r\Big[(1-\mu)w\,\langle\partial_r\Phi,\Phi\rangle_{S^2_{\tau^\star,r}} - \frac{1}{2}\Big(\big((1-\mu)w\big)' + \frac{\mu}{r}\frac{w}{2}\Big)||\Phi||^2_{S^2_{\tau^\star,r}}\Big]\\
&\qquad + (1+\mu)w\,||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}} - (1-\mu)w\,||\partial_r\Phi||^2_{S^2_{\tau^\star,r}} - w\,||\nabla\!\!\!/_{\Upsilon}\Phi||^2_{S^2_{\tau^\star,r}} + \frac{1}{2}\big((1-\mu)w'\big)'||\Phi||^2_{S^2_{\tau^\star,r}}\\
&\quad =2\mu w\,\langle\partial_{t^*}\Phi,\partial_r\Phi\rangle_{S^2_{\tau^\star,r}}.
\end{aligned}
$$

Here, for $f\in C^\infty(\mathcal{M})$ we define

$$
||\nabla\!\!\!/_{\Upsilon}f||^2_{S^2_{\tau^\star,r}} := ||\nabla\!\!\!/ f||^2_{S^2_{\tau^\star,r}} - \frac{3}{r}\frac{\mu}{r}||f||^2_{S^2_{\tau^\star,r}}
$$

and

$$
||[\partial_r,\Upsilon]f||^2_{S^2_{\tau^\star,r}} := -\frac{2}{r}||\nabla\!\!\!/ f||^2_{S^2_{\tau^\star,r}} + \frac{9}{r^2}\frac{\mu}{r}||f||^2_{S^2_{\tau^\star,r}}.
$$

Let now Ψ be a smooth solution to the Zerilli equation (7.2)*. Then on any 2-sphere $S^2_{\tau^\star,r}$ the following identities hold:*

$$
\begin{aligned}
&\partial_{t^*}\Big[(1+\mu)\alpha\,||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}} + (1-\mu)\alpha\,||\partial_r\Psi||^2_{S^2_{\tau^\star,r}} + \alpha\,||\nabla\!\!\!/_{\Upsilon+\mathfrak{Z}}\Psi||^2_{S^2_{\tau^\star,r}}\Big]\\
&-\partial_r\Big[2\mu\alpha\,||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}} + 2(1-\mu)\alpha\,\langle\partial_{t^*}\Psi,\partial_r\Psi\rangle_{S^2_{\tau^\star,r}}\Big]\\
&\quad + 2\mu\alpha'\,||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}}\\
&\quad =-2(1-\mu)\alpha'\,\langle\partial_{t^*}\Psi,\partial_r\Psi\rangle_{S^2_{\tau^\star,r}},
\end{aligned}
$$

$$
\begin{aligned}
&\partial_{t^*}\Big[2(1+\mu)\beta\,\langle\partial_{t^*}\Psi,\partial_r\Psi\rangle_{S^2_{\tau^\star,r}} - 2\mu\beta\,||\partial_r\Psi||^2_{S^2_{\tau^\star,r}}\Big]\\
&-\partial_r\Big[(1+\mu)\beta\,||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}} - (1-\mu)\beta\,||\partial_r\Psi||^2_{S^2_{\tau^\star,r}} - \beta\,||\nabla\!\!\!/_{\Upsilon+\mathfrak{Z}}\Psi||^2_{S^2_{\tau^\star,r}}\Big]\\
&\quad + \big((1+\mu)\beta\big)'||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}} + \Big((1-\mu)\beta' - \frac{\mu}{r}\beta\Big)||\partial_r\Psi||^2_{S^2_{\tau^\star,r}}\\
&\quad - \beta'\,||\nabla\!\!\!/_{\Upsilon+\mathfrak{Z}}\Psi||^2_{S^2_{\tau^\star,r}} - \beta\,||[\partial_r,\Upsilon+\mathfrak{Z}]\Psi||^2_{S^2_{\tau^\star,r}}\\
&\quad =-2\frac{\mu}{r}\,\beta\,\langle\partial_{t^*}\Psi,\partial_r\Psi\rangle_{S^2_{\tau^\star,r}}
\end{aligned}
$$

and

$$
\begin{aligned}
&-\partial_{t^*}\left[(1+\mu)w\,\langle\partial_{t^*}\Psi,\Psi\rangle_{S^2_{\tau^\star,r}} - 2\mu w\,\langle\partial_r\Psi,\Psi\rangle_{S^2_{\tau^\star,r}} + \frac{\mu}{r}\frac{w}{2}\,||\Psi||^2_{S^2_{\tau^\star,r}}\right] \\
&+\partial_r\left[(1-\mu)w\,\langle\partial_r\Psi,\Psi\rangle_{S^2_{\tau^\star,r}} - \frac{1}{2}\Big(\big((1-\mu)w\big)' + \frac{\mu}{r}\frac{w}{2}\Big)||\Psi||^2_{S^2_{\tau^\star,r}}\right] \\
&\quad + (1+\mu)w\,||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}} - (1-\mu)w\,||\partial_r\Psi||^2_{S^2_{\tau^\star,r}} - w\,||\nabla\!\!\!/_{\Upsilon+\mathfrak{Z}}\Psi||^2_{S^2_{\tau^\star,r}} \\
&\quad + \frac{1}{2}\Big((1-\mu)w'\Big)'||\Psi||^2_{S^2_{\tau^\star,r}} \\
&=2\mu w\,\langle\partial_{t^*}\Psi,\partial_r\Psi\rangle_{S^2_{\tau^\star,r}}.
\end{aligned}
$$

Here, for $f\in C^\infty(\mathcal{M})\cap\Lambda(\mathcal{M})$ we define

$$
\begin{aligned}
||\nabla\!\!\!/_{\Upsilon+\mathfrak{Z}}f||^2_{S^2_{\tau^\star,r}} := ||\nabla\!\!\!/_{\Upsilon}f||^2_{S^2_{\tau^\star,r}} &- \frac{6}{r}\frac{\mu}{r}(2-3\mu)||(r\nabla\!\!\!/)\not{\mathfrak{z}}^{[1]}f||^2_{S^2_{\tau^\star,r}} \\
&+ \frac{6}{r}\frac{\mu}{r}\Big((2-3\mu)^2+3\mu(1-\mu)\Big)||\not{\mathfrak{z}}^{[1]}f||^2_{S^2_{\tau^\star,r}}
\end{aligned}
$$

and

$$
\begin{aligned}
||[\partial_r,\Upsilon+\mathfrak{Z}]f||^2_{S^2_{\tau^\star,r}} := ||[\partial_r,\Upsilon]f||^2_{S^2_{\tau^\star,r}} &+ \frac{36}{r^2}\frac{\mu}{r}(1-2\mu)||(r\nabla\!\!\!/)\not{\mathfrak{z}}^{[1]}f||^2_{S^2_{\tau^\star,r}} \\
&- \frac{18}{r^2}\frac{\mu}{r}\Big((2-3\mu)^2+\mu^2\Big)||\not{\mathfrak{z}}^{[1]}f||^2_{S^2_{\tau^\star,r}} \\
&+ \frac{108}{r^3}\mu^3(1-\mu)||(r\nabla\!\!\!/)\not{\mathfrak{z}}^{[2]}f||^2_{S^2_{\tau^\star,r}} \\
&- \frac{108}{r^3}\mu^3(1-\mu)(2-3\mu)||\not{\mathfrak{z}}^{[2]}f||^2_{S^2_{\tau^\star,r}}.
\end{aligned}
$$

Proof. We first observe by explicit computation that the action of the operators $\tilde{\Box}$ defined as in section 3.2.2.2 on $f\in C^\infty(\mathcal{M})$ returns

$$
\tilde{\Box}f = -(1+\mu)\partial^2_{t^*}f + 2\mu\,\partial^2_{t^*r}f + (1-\mu)\partial^2_r f - \frac{\mu}{r}\partial_{t^*}f + \frac{\mu}{r}\partial_r f.
$$

In addition, we recall that

$$
\Delta\!\!\!/ f = \frac{1}{r^2}\mathring{\Delta}
$$

where $\mathring{\Delta}$ is the Laplacian associated to the metric of the unit round sphere.

Consequently, the first half of the Lemma follows after multiplying the Regge–Wheeler equation (7.1) successively by the smooth functions

$$
\alpha\,\partial_{t^*}\Phi, \beta\,\partial_r\Phi, w\,\Phi
$$

and then integrating by parts on any 2-sphere $S^2_{\tau^\star,r}$ with respect to the measure $\mathring{\epsilon}$.

Conversely, recalling the commutation relations and integration by parts formulae for the operator $\not\zeta^{[p]}$ of Lemma 3.14, the second half of the Lemma follows after multiplying the Zerilli equation (7.2) successively by the smooth functions

$$\alpha\,\partial_{t^*}\Psi,\beta\,\partial_r\Psi,w\,\Psi\in\Lambda(\mathcal{M})$$

and then integrating by parts on any 2-sphere $S^2_{\tau^\star,r}$ with respect to the measure $\mathring{\epsilon}$.

$\square$

Given a smooth solution Φ to the Regge–Wheeler equation on $\mathcal{M}$ and three smooth radial functions α,β and w, the first three identities of Lemma 7.1 motivate introducing the 1-form $\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi]$ and function $\widetilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi]$ on $\mathcal{M}$ defined according to

$$\begin{aligned}\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi]:=\Bigg[&(1+\mu)\alpha\,||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}}+2(1+\mu)\beta\,\langle\partial_{t^*}\Phi,\partial_r\Phi\rangle_{S^2_{\tau^\star,r}}+\Big((1-\mu)\alpha-2\mu\beta\Big)||\partial_r\Phi||^2_{S^2_{\tau^\star,r}}\\&+\alpha\,||\nabla\!\!\!\!/_{\Upsilon}\Phi||^2_{S^2_{\tau^\star,r}}-(1+\mu)w\,\langle\partial_{t^*}\Phi,\Phi\rangle_{S^2_{\tau^\star,r}}+2\mu w\,\langle\partial_r\Phi,\Phi\rangle_{S^2_{\tau^\star,r}}\\&-\frac{\mu}{r}\frac{w}{2}\,||\Phi||^2_{S^2_{\tau^\star,r}}\Bigg]\mathrm{d}t^*\\-\Bigg[&\Big(2\mu\alpha+(1+\mu)\beta\Big)\,||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}}+2(1-\mu)\alpha\,\langle\partial_{t^*}\Phi,\partial_r\Phi\rangle_{S^2_{\tau^\star,r}}+(1-\mu)\beta\,||\partial_r\Phi||^2_{S^2_{\tau^\star,r}}\\&-\beta\,||\nabla\!\!\!\!/_{\Upsilon}\Phi||^2_{S^2_{\tau^\star,r}}-(1-\mu)w\,\langle\partial_r\Phi,\Phi\rangle_{S^2_{\tau^\star,r}}+\frac{1}{2}\bigg(\Big((1-\mu)w\Big)'\\&-\frac{\mu}{r}\frac{w}{2}\bigg)||\Phi||^2_{S^2_{\tau^\star,r}}\Bigg]\mathrm{d}r\end{aligned}$$

and

$$\begin{aligned}\widetilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi]=&\Big(2\mu\alpha'+\Big((1+\mu)\beta\Big)'+(1+\mu)w\Big)||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}}\\&2\bigg((1-\mu)\alpha'+\frac{\mu}{r}\,\beta-2\mu\,w\bigg)\langle\partial_{t^*}\Phi,\partial_r\Phi\rangle_{S^2_{\tau^\star,r}}\\&+\bigg((1-\mu)\beta'-\frac{\mu}{r}\beta-(1-\mu)w\bigg)||\partial_r\Phi||^2_{S^2_{\tau^\star,r}}\\&-(\beta'+w)||\nabla\!\!\!\!/_{\Upsilon}\Phi||^2_{S^2_{\tau^\star,r}}\\&-\beta\,||[\partial_r,\Upsilon]\Phi||^2_{S^2_{\tau^\star,r}}\\&+\frac{1}{2}\Big((1-\mu)w'\Big)'||\Phi||^2_{S^2_{\tau^\star,r}}.\end{aligned}$$

Indeed, given in addition three real numbers $T^*>\tau^\star_2\geq\tau^\star_1$, then summing the identities in the first half of Lemma 7.1 and integrating over the region[1]

$$\mathfrak{R}^{T^*}_{\tau_1^\star,\tau_2^\star} := \Bigg(\bigcup_{t^*\in[\tau_1^\star,T^*]} \Sigma_{t^*}\Bigg)\bigcap\Bigg(\bigcup_{\tau^\star\in[\tau_1^\star,\tau_2^\star]}\ \bigcup_{r\in[2M,\infty)} S^2_{\tau^\star,r}\Bigg)$$

with respect to the measure $\mathrm{d}\tau^\star\,\mathrm{d}r\,\mathring{\epsilon}$ yields the conservation law

$$\begin{aligned}
&\int_{2M}^{r(T^*,\tau_2^\star)} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau_2^\star,r}[\Phi](\partial_{\tau^\star})\,\mathrm{d}r + \int_{r(T^*,\tau_1^\star)}^{r(T^*,\tau_2^\star)} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star(T^*,r),r}[\Phi](\partial_{t^*})\,\mathrm{d}r + \int_{\tau_1^\star}^{\tau_2^\star}\int_{2M}^{r(T^*,\tau^\star)} \tilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi]\,\mathrm{d}\tau^\star\,\mathrm{d}r \\
&= \int_{2M}^{r(T^*,\tau_1^\star)} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau_1^\star,r}[\Phi](\partial_{\tau^\star})\,\mathrm{d}r + \int_{\tau_1^\star}^{\tau_2^\star} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Phi](\partial_r)\,\mathrm{d}\tau^\star. \qquad (7.3)
\end{aligned}$$

Here, $r(T^*,\tau_i^\star) > R$ for $i=1,2$ is the unique value of r such that $\tau^\star(T^*,r)=\tau_i^\star$ and we note that $S^2_{\tau^\star(T^*,r)}$ is the 2-sphere $\{T^*\}\times\{r\}\times S^2\subset\Sigma_{T^*}$. In addition, we have defined for $i=1,2$

$$\int_{2M}^{r(T^*,\tau_i^\star)} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau_i^\star,r}[\Phi](\partial_{\tau^\star})\,\mathrm{d}r := \int_{2M}^{r(T^*,\tau_i^\star)} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau_i^\star,r}[\Phi](\partial_{t^*})\,\mathrm{d}r - \int_{R}^{r(T^*,\tau_i^\star)} \frac{1+\mu}{1-\mu}\,\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau_i^\star,r}[\Phi](\partial_r)\,\mathrm{d}r.$$

Similarly, given a smooth solution Ψ to the Zerilli equation on $\mathcal{M}$ and three smooth radial functions α,β and w, the last three identities of Lemma 7.1 motivate introducing the 1-form $\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi]$ and function $\tilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi]$ on $\mathcal{M}$ defined according to

$$\begin{aligned}
\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi] := \Bigg[&(1+\mu)\alpha\,||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}} + 2(1+\mu)\beta\,\langle\partial_{t^*}\Psi,\partial_r\Psi\rangle_{S^2_{\tau^\star,r}} + \Big((1-\mu)\alpha - 2\mu\beta\Big)||\partial_r\Psi||^2_{S^2_{\tau^\star,r}} \\
&+\alpha\,||\nabla\!\!\!\!/\,_{\Upsilon+\mathfrak{z}}\Psi||^2_{S^2_{\tau^\star,r}} - (1+\mu)w\,\langle\partial_{t^*}\Psi,\Psi\rangle_{S^2_{\tau^\star,r}} + 2\mu w\,\langle\partial_r\Psi,\Psi\rangle_{S^2_{\tau^\star,r}} \\
&-\frac{\mu}{r}\frac{w}{2}\,||\Psi||^2_{S^2_{\tau^\star,r}}\Bigg]\mathrm{d}t^* \\
-\Bigg[&\Big(2\mu\alpha + (1+\mu)\beta\Big)\,||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}} + 2(1-\mu)\alpha\,\langle\partial_{t^*}\Psi,\partial_r\Psi\rangle_{S^2_{\tau^\star,r}} + (1-\mu)\beta\,||\partial_r\Psi||^2_{S^2_{\tau^\star,r}} \\
&-\beta\,||\nabla\!\!\!\!/\,_{\Upsilon+\mathfrak{z}}\Psi||^2_{S^2_{\tau^\star,r}} - (1-\mu)w\,\langle\partial_r\Psi,\Psi\rangle_{S^2_{\tau^\star,r}} \\
&+\frac{1}{2}\bigg(\Big((1-\mu)w\Big)' - \frac{\mu}{r}\frac{w}{2}\bigg)||\Psi||^2_{S^2_{\tau^\star,r}}\Bigg]\mathrm{d}r
\end{aligned}$$

[1] We recall from section 3.2.3.1 the definition of the hypersurfaces Σ_{t^*}.

and

$$
\begin{aligned}
\widetilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi] = \Big(2\mu\alpha' + \big((1+\mu)\beta\big)' + (1+\mu)w\Big)||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}} \\
2\Big((1-\mu)\alpha' + \frac{\mu}{r}\,\beta - 2\mu\, w\Big)\langle\partial_{t^*}\Psi, \partial_r\Psi\rangle_{S^2_{\tau^\star,r}} \\
+\Big((1-\mu)\beta' - \frac{\mu}{r}\beta - (1-\mu)w\Big)||\partial_r\Psi||^2_{S^2_{\tau^\star,r}} \\
-(\beta'+w)||\nabla\!\!\!/_{\Upsilon+\mathfrak{Z}}\Psi||^2_{S^2_{\tau^\star,r}} \\
-\beta\,||[\partial_r, \Upsilon+\mathfrak{Z}]\Psi||^2_{S^2_{\tau^\star,r}} \\
+\frac{1}{2}\big((1-\mu)w'\big)'||\Psi||^2_{S^2_{\tau^\star,r}}.
\end{aligned}
$$

Indeed, given in addition three real numbers $T^* > \tau_2^\star \geq \tau_1^\star$, then summing the identities in the second half of Lemma 7.1 and integrating over the region $\mathfrak{R}^{T^*}_{\tau_1^\star,\tau_2^\star}$ with respect to the measure $\mathrm{d}\tau^\star\,\mathrm{d}r\,\mathring{\epsilon}$ yields the conservation law

$$
\begin{aligned}
&\int_{2M}^{r(T^*,\tau_2^\star)} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau_2^\star,r}[\Psi](\partial_{\tau^\star})\,\mathrm{d}r + \int_{r(T^*,\tau_1^\star)}^{r(T^*,\tau_2^\star)} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star(T^*,r),r}[\Psi](\partial_{t^*})\,\mathrm{d}r + \int_{\tau_1^\star}^{\tau_2^\star}\int_{2M}^{r(T^*,\tau^\star)} \widetilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi]\,\mathrm{d}\tau^\star\,\mathrm{d}r \\
&= \int_{2M}^{r(T^*,\tau_1^\star)} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau_1^\star,r}[\Psi](\partial_{\tau^\star})\,\mathrm{d}r + \int_{\tau_1^\star}^{\tau_2^\star} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Psi](\partial_r)\,\mathrm{d}\tau^\star. \qquad (7.4)
\end{aligned}
$$

Here, we have defined for $i=1,2$

$$
\int_{2M}^{r(T^*,\tau_i^\star)} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau_i^\star,r}[\Psi](\partial_{\tau^\star})\,\mathrm{d}r := \int_{2M}^{r(T^*,\tau_i^\star)} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau_i^\star,r}[\Psi](\partial_{t^*})\,\mathrm{d}r - \int_{R}^{r(T^*,\tau_i^\star)} \frac{1+\mu}{1-\mu}\,\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau_i^\star,r}[\Psi](\partial_r)\,\mathrm{d}r.
$$

We end this section with the following Proposition that arises from taking the limit $T^*\to\infty$ in the conservation laws (7.3) and (7.4).

Proposition 7.2. *Let α, β and w be smooth functions of r on $\mathcal{M}$ and let $T^* > \tau_2^\star \geq \tau_1^\star$ be three real numbers.*

We suppose that Φ is a smooth solution to the Regge–Wheeler equation (7.1) on $\mathcal{M}$ for which at least one of the following conditions hold:

i) for any $(\tau^\star, r)$ the quantities $\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi](\partial_{t^*})$ and $-\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Phi](\partial_r)$ are non-negative

ii) for any $(\tau^\star, r)$ the quantity $\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi](\partial_{t^*})$ is non-negative and uniformly in T^* and $\tau_2^\star$ it holds that
$\int_{\tau_1^\star}^{\tau_2^\star} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Phi](\partial_r)\,\mathrm{d}\tau^\star \lesssim \int_{2M}^{r(T^*,\tau_1^\star)} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau_1^\star,r}[\Phi](\partial_{\tau^\star})\,\mathrm{d}r$

iii) for any $\tau^\star$ the quantity $-\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Phi](\partial_r)$ is non-negative and uniformly in T^* and $\tau_2^\star$

it holds that
$-\int_{r(T^*,\tau_1^\star)}^{r(T^*,\tau_2^\star)} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star(T^*,r),r}[\Phi](\partial_{t^*})\,\mathrm{d}r \lesssim \int_{2M}^{r(T^*,\tau_1^\star)} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau_1^\star,r}[\Phi](\partial_{\tau^\star})\,\mathrm{d}r$

iv) uniformly in T^* and $\tau_2^\star$ it holds that
$\int_{\tau_1^\star}^{\tau_2^\star} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Phi](\partial_r)\,\mathrm{d}\tau^\star - \int_{r(T^*,\tau_1^\star)}^{r(T^*,\tau_2^\star)} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star(T^*,r),r}[\Phi](\partial_{t^*})\,\mathrm{d}r \lesssim \int_{2M}^{r(T^*,\tau_1^\star)} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau_1^\star,r}[\Phi](\partial_{\tau^\star})\,\mathrm{d}r$

Then provided that the flux term on the right hand side is finite one has the estimate

$$\int_{2M}^{\infty} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau_2^\star,r}[\Phi](\partial_{\tau^\star})\,\mathrm{d}r + \int_{\tau_1^\star}^{\tau_2^\star}\int_{2M}^{\infty} \tilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi]\,\mathrm{d}\tau^\star\,\mathrm{d}r \lesssim \int_{2M}^{\infty} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau_1^\star,r}[\Phi](\partial_{\tau^\star})\,\mathrm{d}r.$$

.

We suppose now that Ψ is a smooth solution to the Zerilli equation (7.1) *on $\mathcal{M}$ for which at least one of the following two conditions hold:*

i) for any $(\tau^\star, r)$ the quantities $\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi](\partial_{t^*})$ and $-\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Psi](\partial_r)$ are non-negative

ii) for any $(\tau^\star, r)$ the quantity $\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi](\partial_{t^*})$ is non-negative and uniformly in T^* and $\tau_2^\star$ it holds that
$\int_{\tau_1^\star}^{\tau_2^\star} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Psi](\partial_r)\,\mathrm{d}\tau^\star \lesssim \int_{2M}^{r(T^*,\tau_1^\star)} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau_1^\star,r}[\Psi](\partial_{\tau^\star})\,\mathrm{d}r$

iii) for any $\tau^\star$ the quantity $-\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Psi](\partial_r)$ is non-negative and uniformly in T^* and $\tau_2^\star$ it holds that
$-\int_{r(T^*,\tau_1^\star)}^{r(T^*,\tau_2^\star)} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star(T^*,r),r}[\Psi](\partial_{t^*})\,\mathrm{d}r \lesssim \int_{2M}^{r(T^*,\tau_1^\star)} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau_1^\star,r}[\Psi](\partial_{\tau^\star})\,\mathrm{d}r$

iv) uniformly in T^* and $\tau_2^\star$ it holds that
$\int_{\tau_1^\star}^{\tau_2^\star} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Psi](\partial_r)\,\mathrm{d}\tau^\star - \int_{r(T^*,\tau_1^\star)}^{r(T^*,\tau_2^\star)} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star(T^*,r),r}[\Psi](\partial_{t^*})\,\mathrm{d}r \lesssim \int_{2M}^{r(T^*,\tau_1^\star)} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau_1^\star,r}[\Psi](\partial_{\tau^\star})\,\mathrm{d}r$

Then provided that the flux term on the right hand side is finite one has the estimate

$$\int_{2M}^{\infty} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau_2^\star,r}[\Psi](\partial_{\tau^\star})\,\mathrm{d}r + \int_{\tau_1^\star}^{\tau_2^\star}\int_{2M}^{\infty} \tilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi]\,\mathrm{d}\tau^\star\,\mathrm{d}r \lesssim \int_{2M}^{\infty} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau_1^\star,r}[\Psi](\partial_{\tau^\star})\,\mathrm{d}r.$$

7.1.2 Boundedness of the weighted energy and integrated energy decay

In this section we prove both parts *i*) and *ii*) of Theorem 1 for $n = 0$. The higher order estimates along with the proof of parts *iii*) will then be the content of the next two sections.

The proofs will proceed by applying Proposition 7.2 with appropriate choices of the functions α, β and w. In particular, we henceforth assume that the solutions to the Regge–Wheeler and Zerilli equations under consideration satisfy the assumptions of

Theorem 1 – this ensures finiteness of the initial flux estimates that arise in the application of said proposition.

In addition, we note that a common theme throughout this section will be that once certain separate preliminary estimates are obtained for solutions to the Regge–Wheeler and Zerilli equations respectively then deriving the flux and integrated decay estimates of Theorem 1 for each of the said solutions will follow in a similar manner. However, for reasons of completeness, we prefer to analyse each equation separately.

7.1.2.1 The degenerate energy and Morawetz estimates

We begin by first deriving certain degenerate and therefore preliminary estimates for solutions to the Regge–Wheeler and Zerilli equations that shall form the foundations for proving the weighted flux and integrated decay estimates of Theorem 1.

As to why this degeneration occurs, see section 2.5.2 of the overview.

The degenerate energy estimate

The first such estimate we derive is an (unweighted) energy estimate which degenerates at $\mathcal{H}^+$. This degeneration will then be removed in section 7.1.2.2 whereas the weights towards $\mathcal{I}^+$ will be improved in section 7.1.2.3 thus yielding the $(n=0)$ flux estimate of Theorem 1.

In order to derive such a flux estimate for solutions to the Regge–Wheeler and Zerilli equations which is moreover coercive will require controlling the terms that arise from the presence of the lower order terms that appears in the equations (7.1) and (7.2). This is the content of the following lemma.

Lemma 7.3. *Let $f \in C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$. Then on any 2-sphere $S^2_{\tau^\star,r}$ one has the bounds*

$$||\nabla\!\!\!/ f||^2_{S^2_{\tau^\star,r}} \lesssim ||\nabla\!\!\!/_{\Upsilon} f||^2_{S^2_{\tau^\star,r}} \lesssim ||\nabla\!\!\!/ f||^2_{S^2_{\tau^\star,r}} \tag{7.5}$$

and

$$||\nabla\!\!\!/ f||^2_{S^2_{\tau^\star,r}} \lesssim ||\nabla\!\!\!/_{\Upsilon+\mathfrak{z}} f||^2_{S^2_{\tau^\star,r}} \lesssim ||\nabla\!\!\!/ f||^2_{S^2_{\tau^\star,r}}. \tag{7.6}$$

Proof. We recall from section 7.1.1 that

$$||\nabla\!\!\!/_{\Upsilon} f||^2_{S^2_{\tau^\star,r}} = ||\nabla\!\!\!/ f||^2_{S^2_{\tau^\star,r}} - \frac{3}{r}\frac{\mu}{r}||f||^2_{S^2_{\tau^\star,r}}$$

and

$$||\not\nabla_{\Upsilon+\mathfrak{z}} f||^2_{S^2_{\tau^\star,r}} = ||\not\nabla_{\Upsilon} f||^2_{S^2_{\tau^\star,r}} - \frac{6}{r}\frac{\mu}{r}(2-3\mu)||(r\not\nabla)\not{\mathcal{D}}^{[1]} f||^2_{S^2_{\tau^\star,r}}$$
$$+\frac{6}{r}\frac{\mu}{r}\Big((2-3\mu)^2+3\mu(1-\mu)\Big)||\not{\mathcal{D}}^{[1]} f||^2_{S^2_{\tau^\star,r}}.$$

The upper bound in the (7.5) is therefore immediate[2] whereas the upper bound in (7.6) follows from the elliptic estimates on the operator $\not{\mathcal{D}}^{[p]}$ of Proposition 3.12.

For the lower bound, we further recall for $f \in C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$ the Poincaré inequality of Lemma 3.9 on any 2-sphere $S^2_{\tau^\star,r}$:

$$\frac{6}{r^2}||f||^2_{S^2_{\tau^\star,r}} \lesssim ||\not\nabla f||^2_{S^2_{\tau^\star,r}}. \tag{7.7}$$

This immediately yields the lower bound of (7.5) after noting that $\mu \le 1$ on $\mathcal{M}$. For the lower bound of (7.6) we first observe that the coefficient of the term $||\not{\mathcal{D}}^{[1]} f||^2_{S^2_{\tau^\star,r}}$ in the expression $||\not\nabla_{\Upsilon+\mathfrak{z}} f||^2_{S^2_{\tau^\star,r}}$ is non-negative on $r \ge 2M$ whereas the coefficient of the term $||(r\not\nabla)\not{\mathcal{D}}^{[1]} f||^2_{S^2_{\tau^\star,r}}$ is non-negative for $2M \le r \le 3M$. In addition, we recall from Corollary 3.13 the estimate on any 2-sphere $S^2_{\tau^\star,r}$

$$-\frac{1}{2+9\mu}||f||^2_{S^2_{\tau^\star,r}} \le -||r\not\nabla\not{\mathcal{D}}^{[1]} f||^2_{S^2_{\tau^\star,r}}.$$

Consequently, to establish the lower bound of (7.6) it suffices to establish the estimate

$$||\not\nabla f||^2_{S^2_{\tau^\star,r}} \lesssim ||\not\nabla_{\Upsilon} f||^2_{S^2_{\tau^\star,r}} - \frac{3}{4}\frac{1}{r}\frac{\mu}{r}(2-3\mu)||f||^2_{S^2_{\tau^\star,r}}$$

for $2M \le r \le 3M$ and the estimate

$$||\not\nabla f||^2_{S^2_{\tau^\star,r}} \lesssim ||\not\nabla_{\Upsilon} f||^2_{S^2_{\tau^\star,r}}$$

for $r \ge 3M$, the latter of which was shown previously whereas the former follows easily from (7.7). □

The desired energy estimate for solutions to the Regge–Wheeler and Zerilli equations is then as stated below.

In what follows, we recall the operator D of section 3.2.4.2 has the form

$$D = \frac{1+\mu}{1-\mu}\partial_{t^*} + \partial_r.$$

Proposition 7.4. *Let $\tau^\star_2, \tau^\star_1 \ge \tau^\star_0$ be two real numbers.*

[2]We recall that $r \ge 2M$ on $\mathcal{M}$.

Let now Φ be as in Theorem 1. Then one has the flux estimate

$$\begin{aligned}
&\int_{2M}^{R}\left(||\partial_{t^*}\Phi||^2_{S^2_{\tau_2^\star,r}}+(1-\mu)||\partial_r\Phi||^2_{S^2_{\tau_2^\star,r}}+||\nabla\!\!\!/\Phi||^2_{S^2_{\tau_2^\star,r}}\right)\mathrm{d}r\\
&+\int_{R}^{\infty}\left(||D\Phi||^2_{S^2_{\tau_2^\star,r}}+||\nabla\!\!\!/\Phi||^2_{S^2_{\tau_2^\star,r}}\right)\mathrm{d}r\\
&\lesssim\int_{2M}^{R}\left(||\partial_{t^*}\Phi||^2_{S^2_{\tau_1^\star,r}}+(1-\mu)||\partial_r\Phi||^2_{S^2_{\tau_1^\star,r}}+||\nabla\!\!\!/\Phi||^2_{S^2_{\tau_1^\star,r}}\right)\mathrm{d}r\\
&+\int_{R}^{\infty}\left(||D\Phi||^2_{S^2_{\tau_1^\star,r}}+||\nabla\!\!\!/\Phi||^2_{S^2_{\tau_1^\star,r}}\right)\mathrm{d}r. \qquad (7.8)
\end{aligned}$$

Let now Ψ be as in Theorem 1. Then one has the flux estimate

$$\begin{aligned}
&\int_{2M}^{R}\left(||\partial_{t^*}\Psi||^2_{S^2_{\tau_2^\star,r}}+(1-\mu)||\partial_r\Psi||^2_{S^2_{\tau_2^\star,r}}+||\nabla\!\!\!/\Psi||^2_{S^2_{\tau_2^\star,r}}\right)\mathrm{d}r\\
&+\int_{R}^{\infty}\left(||D\Psi||^2_{S^2_{\tau_2^\star,r}}+||\nabla\!\!\!/\Psi||^2_{S^2_{\tau_2^\star,r}}\right)\mathrm{d}r\\
&\lesssim\int_{2M}^{R}\left(||\partial_{t^*}\Psi||^2_{S^2_{\tau_1^\star,r}}+(1-\mu)||\partial_r\Psi||^2_{S^2_{\tau_1^\star,r}}+||\nabla\!\!\!/\Psi||^2_{S^2_{\tau_1^\star,r}}\right)\mathrm{d}r\\
&+\int_{R}^{\infty}\left(||D\Psi||^2_{S^2_{\tau_1^\star,r}}+||\nabla\!\!\!/\Psi||^2_{S^2_{\tau_1^\star,r}}\right)\mathrm{d}r. \qquad (7.9)
\end{aligned}$$

Proof. We consider the three smooth radial functions α,β and w on $\mathcal{M}$ given by

$$\begin{aligned}
\alpha&=1,\\
\beta&=0,\\
w&=0.
\end{aligned}$$

Then from section 7.1.1 we have

$$\begin{aligned}
\tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,r}[\Phi](\partial_{t^*})&=(1+\mu)||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}}+(1-\mu)||\partial_r\Phi||^2_{S^2_{\tau^\star,r}}+||\nabla\!\!\!/_\Upsilon\Phi||^2_{S^2_{\tau^\star,r}},\\
\tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,r}[\Phi](\partial_r)&=-2\mu\,||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}}-2(1-\mu)\langle\partial_{t^*}\Phi,\partial_r\Phi\rangle_{S^2_{\tau^\star,r}}
\end{aligned}$$

and

$$\widetilde{\mathbb{K}}^{1,0,0}_{\tau^\star,r}[\Phi]=0.$$

Consequently, applying the first half of Proposition 7.2 (noting that condition *i*) is satisfied by the first half of Lemma 7.3) in conjunction with the first half of Lemma 7.3 yields the estimate (7.8).

Similarly, from section 7.1.1 we have

$$\tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,r}[\Psi](\partial_{t^*}) = (1+\mu)||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}} + (1-\mu)||\partial_r\Psi||^2_{S^2_{\tau^\star,r}} + ||\nabla\!\!\!\!/_{\Upsilon+\mathfrak{z}}\Psi||^2_{S^2_{\tau^\star,r}},$$
$$\tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,r}[\Psi](\partial_r) = -2\mu\,||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}} - 2(1-\mu)\langle\partial_{t^*}\Psi,\partial_r\Psi\rangle_{S^2_{\tau^\star,r}}$$

and

$$\tilde{\mathbb{K}}^{1,0,0}_{\tau^\star,r}[\Psi] = 0.$$

Consequently, applying the second half of Proposition 7.2 (noting that condition *i*) is satisfied) in conjunction with the second half of Lemma 7.3 yields the estimate (7.9).

This completes the proposition. □

An immediate consequence of the above computations combined with the conservations laws (7.3) and (7.4) is the following pair of estimates which will prove useful in the sequel.

Corollary 7.5. *Let $T^* > \tau^\star_2 \geq \tau^\star_1 \geq \tau^\star_0$ be three real numbers.*

Let now Φ be as in Theorem 1. Then one has the flux estimate

$$\begin{aligned}
&\int_{\tau^\star_1}^{\tau^\star_2} ||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,2M}}\,\mathrm{d}\tau^\star + \int_{r(T^*,\tau^\star_1)}^{r(T^*,\tau^\star_2)} \tilde{\mathbb{J}}^{1,0,0}_{\tau^\star(T^*,r),r}[\Phi](\partial_{t^*})\,\mathrm{d}r \\
&\lesssim \int_{2M}^{R}\Big(||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star_1,r}} + (1-\mu)||\partial_r\Phi||^2_{S^2_{\tau^\star_1,r}} + ||\nabla\!\!\!\!/\Phi||^2_{S^2_{\tau^\star_1,r}}\Big)\,\mathrm{d}r \\
&+\int_R^\infty\Big(||D\Phi||^2_{S^2_{\tau^\star_1,r}} + ||\nabla\!\!\!\!/\Phi||^2_{S^2_{\tau^\star_1,r}}\Big)\,\mathrm{d}r.
\end{aligned} \tag{7.10}$$

Let now Ψ be as in Theorem 1. Then one has the flux estimate

$$\begin{aligned}
&\int_{\tau^\star_1}^{\tau^\star_2} ||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,2M}}\,\mathrm{d}\tau^\star + \int_{r(T^*,\tau^\star_1)}^{r(T^*,\tau^\star_2)} \tilde{\mathbb{J}}^{1,0,0}_{\tau^\star(T^*,r),r}[\Psi](\partial_{t^*})\,\mathrm{d}r \\
&\lesssim \int_{2M}^{R}\Big(||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star_1,r}} + (1-\mu)||\partial_r\Psi||^2_{S^2_{\tau^\star_1,r}} + ||\nabla\!\!\!\!/\Psi||^2_{S^2_{\tau^\star_1,r}}\Big)\,\mathrm{d}r \\
&+\int_R^\infty\Big(||D\Psi||^2_{S^2_{\tau^\star_1,r}} + ||\nabla\!\!\!\!/\Psi||^2_{S^2_{\tau^\star_1,r}}\Big)\,\mathrm{d}r.
\end{aligned} \tag{7.11}$$

The Morawetz estimate

The second such estimate we derive is an integrated local energy decay estimate which degenerates at both $\mathcal{H}^+$ and $r = 3M$. The former degeneration will then be removed in section 7.1.2.2 with the latter degeneration removed in section 7.1.2.4. In addition, the weights towards $\mathcal{I}^+$ will be improved in section 7.1.2.3. The former thus yields the ($n = 0$) case of the integrated decay estimates in Theorem 1 whereas the latter results in the ($n = 0$) case of the weighted bulk estimates in Theorem 1.

As previously, deriving this estimate will require controlling the terms that arise from the presence of the 'potential operator' in equations (7.1) and (7.2). This is the content of the following lemma.

In what follows, for a smooth radial function f on $\mathcal{M}$ we define $f^* := (1-\mu)f'$.

Lemma 7.6. *Let $\mathfrak{f}$ be the smooth radial function on $\mathcal{M}$ defined according to*

$$\mathfrak{f} := 4\left(1 - \frac{3M}{r}\right)\left(1 + \frac{3M}{r}\right).$$

Let now $f \in C^\infty(\mathcal{M}) \cap \Lambda(\mathcal{M})$. Then on any 2-sphere $S^2_{\tau^\star,r}$ one has the estimate

$$\begin{aligned}\frac{1}{r}(2-3\mu)^2||\nabla\!\!\!/ f||^2_{S^2_{\tau^\star,r}} + \frac{1}{r^3}||f||^2_{S^2_{\tau^\star,r}} \lesssim &-\mathfrak{f}\left(\frac{\mu}{r}||\nabla\!\!\!/_\Upsilon f||^2_{S^2_{\tau^\star,r}} + (1-\mu)||[\partial_r,\Upsilon]f||^2_{S^2_{\tau^\star,r}}\right)\\ &-\frac{1}{2}\frac{1}{1-\mu}\mathfrak{f}^{***}||f||^2_{S^2_{\tau^\star,r}}\end{aligned} \tag{7.12}$$

and

$$\begin{aligned}\frac{1}{r}(2-3\mu)^2||\nabla\!\!\!/ f||^2_{S^2_{\tau^\star,r}} + \frac{1}{r^3}||f||^2_{S^2_{\tau^\star,r}} \lesssim &-\mathfrak{f}\left(\frac{\mu}{r}||\nabla\!\!\!/_{\Upsilon+\mathfrak{Z}} f||^2_{S^2_{\tau^\star,r}} + (1-\mu)||[\partial_r,\Upsilon+\mathfrak{Z}]f||^2_{S^2_{\tau^\star,r}}\right)\\ &-\frac{1}{2}\frac{1}{1-\mu}\mathfrak{f}^{***}||f||^2_{S^2_{\tau^\star,r}}.\end{aligned} \tag{7.13}$$

Proof. We first recall from section 7.1.1 that

$$||[\partial_r,\Upsilon]f||^2_{S^2_{\tau^\star,r}} : = -\frac{2}{r}||\nabla\!\!\!/ f||^2_{S^2_{\tau^\star,r}} + \frac{9}{r^2}\frac{\mu}{r}||f||^2_{S^2_{\tau^\star,r}}.$$

and

$$\begin{aligned}||[\partial_r,\Upsilon+\mathfrak{Z}]f||^2_{S^2_{\tau^\star,r}} := ||[\partial_r,\Upsilon]f||^2_{S^2_{\tau^\star,r}} &+ \frac{36}{r^2}\frac{\mu}{r}(1-2\mu)||(r\nabla\!\!\!/)\not{\zeta}^{[1]}f||^2_{S^2_{\tau^\star,r}}\\ &-\frac{18}{r^2}\frac{\mu}{r}\left((2-3\mu)^2+\mu^2\right)||\not{\zeta}^{[1]}f||^2_{S^2_{\tau^\star,r}}\\ &+\frac{108}{r^3}\mu^3(1-\mu)||(r\nabla\!\!\!/)\not{\zeta}^{[2]}f||^2_{S^2_{\tau^\star,r}}\\ &-\frac{108}{r^3}\mu^3(1-\mu)(2-3\mu)||\not{\zeta}^{[2]}f||^2_{S^2_{\tau^\star,r}}.\end{aligned}$$

Subsequently, we compute that

$$\begin{aligned}&-\mathfrak{f}\left(\frac{\mu}{r}||\nabla\!\!\!/_\Upsilon f||^2_{S^2_{\tau^\star,r}} + (1-\mu)||[\partial_r,\Upsilon]f||^2_{S^2_{\tau^\star,r}}\right) - \frac{1}{2}\frac{1}{1-\mu}\mathfrak{f}^{***}||f||^2_{S^2_{\tau^\star,r}}\\ &= \frac{1}{r}(2+3\mu)(2-3\mu)^2||\nabla\!\!\!/ f||^2_{S^2_{\tau^\star,r}} + \frac{1}{r^3}p_0(\mu)\,||f||^2_{S^2_{\tau^\star,r}}\end{aligned}$$

and

$$
\begin{aligned}
-\mathfrak{f}\Big(\frac{\mu}{r}\,||\nabla\!\!\!/_{\Upsilon+\mathfrak{z}}f||^2_{S^2_{\tau^\star,r}} + (1-\mu)||[\partial_r,\Upsilon+\mathfrak{Z}]f||^2_{S^2_{\tau^\star,r}}\Big) - \frac{1}{2}\frac{1}{1-\mu}\mathfrak{f}^{***}||f||^2_{S^2_{\tau^\star,r}} \\
= \frac{1}{r}(2+3\mu)(2-3\mu)^2||\nabla\!\!\!/ f||^2_{S^2_{\tau^\star,r}} \\
+\frac{1}{r^3}p_0(\mu)\,||f||^2_{S^2_{\tau^\star,r}} \\
+\frac{1}{r^3}p_1(\mu)\,||(r\nabla\!\!\!/)\not{\zeta}^{[1]}f||^2_{S^2_{\tau^\star,r}} \\
+\frac{1}{r^3}p_2(\mu)\,||\not{\zeta}^{[1]}f||^2_{S^2_{\tau^\star,r}} \\
+\frac{1}{r^3}p_3(\mu)\,||(r\nabla\!\!\!/)\not{\zeta}^{[2]}f||^2_{S^2_{\tau^\star,r}} \\
+\frac{1}{r^3}p_4(\mu)\,||\not{\zeta}^{[2]}f||^2_{S^2_{\tau^\star,r}}.
\end{aligned}
$$

Here, we have defined the polynomials

$$
\begin{aligned}
p_0(x) &:= -12x\Big(3+5x-33x^2+27x^3\Big), \\
p_1(x) &:= 6x\Big(24-64x-90x^2+144x^3+81x^4\Big) \\
p_2(x) &:= 6x\Big(48-240x+264x^2+348x^3-837x^4+432x^5\Big), \\
p_3(x) &:= 108x^3\Big(4-8x-5x^2+18x^3-9x^4\Big), \\
p_4(x) &:= 108x^3\Big(8-28x+14x^2+51x^3-72x^4+25x^5\Big)
\end{aligned}
$$

which we observe are uniformly bounded on the domain $[0,1]$. Consequently, to establish the estimates (7.16) and (7.17) it thus follows from both the Poincaré inequality of Lemma 3.9 and the elliptic estimates of Proposition 3.12 that it is in fact sufficient to demonstrate on any 2-sphere $S^2_{\tau^\star,r}$ the bounds

$$
||f||^2_{S^2_{\tau^\star,r}} \lesssim q(\mu)||f||^2_{S^2_{\tau^\star,r}}
$$

and

$$
\begin{aligned}
||f||^2_{S^2_{\tau^\star,r}} \lesssim q(\mu)||f||^2_{S^2_{\tau^\star,r}} &+ p_1(\mu)\,||(r\nabla\!\!\!/)\not{\zeta}^{[1]}f||^2_{S^2_{\tau^\star,r}} \\
&+p_2(\mu)\,||\not{\zeta}^{[1]}f||^2_{S^2_{\tau^\star,r}} \\
&+p_3(\mu)\,||(r\nabla\!\!\!/)\not{\zeta}^{[2]}f||^2_{S^2_{\tau^\star,r}} \\
&+p_4(\mu)\,||\not{\zeta}^{[2]}f||^2_{S^2_{\tau^\star,r}}. \qquad (7.14)
\end{aligned}
$$

Here, $q(x)$ is the polynomial

$$q(x) := 48 - 108x - 168x^2 + 558x^3 - 324x^4.$$

Indeed, the former bound follows from the positivity of the polynomial $q_0(x)$ on the domain $[0,1]$ which is simple to verify. This does not establish the latter bound however as the polynomials p_1, p_2, p_3 and p_4 are not positive-definite on the domain $[0,1]$.

Consequently, in order to prove estimate (7.14) we first decompose f into spherical harmonics in accordance with Proposition 3.5:

$$f = \sum_{l=2}^{\infty} f_l^m Y_m^l.$$

Here, the convergence is pointwise. In particular, recalling that $\not{\zeta}^{[p]}$ is the inverse operator $r^2\not{\Delta} + 2 - 3\mu$ applied p-times, it follows that for any integers $i, j \geq 0$ each mode $f_l^m Y_m^l$ satisfies the identity

$$||(r\not{\nabla})^i \not{\zeta}^{[j]} f_l^m Y_m^l||^2_{S^2_{\tau^\star,r}} = \frac{(\lambda-2)^i}{(\lambda+3\mu)^j} ||f_l^m Y_m^l||^2_{S^2_{\tau^\star,r}}$$

where $\lambda := (l-1)(l+2)$. Thus, to establish the bound (7.14) it suffices to demonstrate positivity of the following expression over the domain $\lambda \geq 2$ and $x \in [0,1]$:

$$\frac{1}{(\lambda+3x)^3}\Big(q_4(x)\lambda^4 + q_3(x)\lambda^3 + q_2(x)\lambda^2 + q_1(x)\lambda + q_0(x)\Big). \tag{7.15}$$

where q_4, q_3, q_2, q_1 and q_0 are the polynomials

$$\begin{aligned}
q_4(x) &:= 8 - 12x - 18x^2 + 27x^3, \\
q_3(x) &:= 16 + 12x - 204x^2 + 288x^3 - 81x^4, \\
q_2(x) &:= x^2\Big(156 - 1224x + 2484x^2 - 5508x^3\Big), \\
q_1(x) &:= x^3\Big(396 - 3456x + 7614x^2 - 4662x^3\Big), \\
q_0(x) &:= 3x^3\Big(108 - 1116x + 2592x^2 - 1620x^3\Big).
\end{aligned}$$

Indeed, we first claim that on this domain it holds that

$$(2-3x)^2(2+3x) \lesssim q_4(x)\lambda^4 + q_3(x)\lambda^3 + q_2(x)\lambda^2 + q_1(x)\lambda + q_0(x).$$

To verify this, we borrow successively from each polynomial to derive the estimate

$$\begin{aligned}q_4(x)\lambda^4 + q_3(x)\lambda^3 + q_2(x)\lambda^2 + q_1(x)\lambda + q_0(x) \geq \frac{1}{2}q_4(x)\lambda^4 \\ +\Big(q_3(x) - x^3\Big)\lambda^3 \\ +\Big(q_2(x) + 3x^3 + 2q_4(x) - \tfrac{77}{4}x^4\Big)\lambda^2 \\ +\Big(q_1(x) + \tfrac{77}{2}x^4 - 13x^5\Big)\lambda \\ +q_0(x) + 26x^5.\end{aligned}$$

Noting that $q_4(r) = (2-3x)^2(2+3x)$, the claim thus follows if positivity holds on the domain $[0,1]$ for each of the polynomials

$$\begin{aligned}q_3(x) - x^3, \\ q_2(x) + 3x^3 + 2q_4(x) - \tfrac{77}{4}x^4 \\ q_1(x) + \tfrac{77}{2}x^4 - 13x^5 \\ q_0(x) + 26x^5\end{aligned}$$

which is simple to verify.

Finally, to establish positivity of the expression (7.15) it remains to verify that the expression $q_4(x)\lambda^4 + q_3(x)\lambda^3 + q_2(x)\lambda^2 + q_1(x)\lambda + q_0(x)$ is positive for $x = \frac{2}{3}$ as then continuity implies positivity on an open neighbourhood of $x = \frac{2}{3}$. However, this follows easily from an explicit computation and thus the lemma follows. □

The desired integrated local energy estimate for solutions to the Regge–Wheeler and Zerilli equations is then as follows.

Proposition 7.7. *Let Φ be as in Theorem 1. Then one has the bulk estimate*

$$\begin{aligned}&\int_{\tau_0^\star}^{\infty}\int_{2M}^{\infty}\frac{1}{r^3}\bigg((2-3\mu)^2\Big(||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}} + (1-\mu)||\partial_r\Phi||^2_{S^2_{\tau^\star,r}} + r^2||\nabla\!\!\!/\Phi||^2_{S^2_{\tau^\star,r}}\Big) + ||\Phi||^2_{S^2_{\tau^\star,r}}\bigg)\,\mathrm{d}\tau^\star\,\mathrm{d}r \\ &\quad\lesssim \int_{2M}^{R}\Big(||\partial_{t^*}\Phi||^2_{S^2_{\tau_0^\star,r}} + (1-\mu)||\partial_r\Phi||^2_{S^2_{\tau_0^\star,r}} + ||\nabla\!\!\!/\Phi||^2_{S^2_{\tau_0^\star,r}}\Big)\,\mathrm{d}r \\ &\quad + \int_{R}^{\infty}\Big(||D\Phi||^2_{S^2_{\tau_0^\star,r}} + ||\nabla\!\!\!/\Phi||^2_{S^2_{\tau_0^\star,r}}\Big)\,\mathrm{d}r. \qquad (7.16)\end{aligned}$$

Let now Ψ be as in Theorem 1. Then one has the bulk estimate

$$\int_{\tau_0^\star}^{\infty}\int_{2M}^{\infty}\frac{1}{r^3}\bigg((2-3\mu)^2\Big(||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}}+(1-\mu)||\partial_r\Psi||^2_{S^2_{\tau^\star,r}}+r^2||\nabla\!\!\!/\Psi||^2_{S^2_{\tau^\star,r}}\Big)+||\Psi||^2_{S^2_{\tau^\star,r}}\bigg)\,\mathrm{d}\tau^\star\,\mathrm{d}r$$
$$\lesssim\int_{2M}^{R}\Big(||\partial_{t^*}\Psi||^2_{S^2_{\tau_0^\star,r}}+(1-\mu)||\partial_r\Psi||^2_{S^2_{\tau_0^\star,r}}+||\nabla\!\!\!/\Psi||^2_{S^2_{\tau_0^\star,r}}\Big)\,\mathrm{d}r$$
$$+\int_{R}^{\infty}\Big(||D\Psi||^2_{S^2_{\tau_0^\star,r}}+||\nabla\!\!\!/\Psi||^2_{S^2_{\tau_0^\star,r}}\Big)\,\mathrm{d}r. \tag{7.17}$$

Proof. We consider the three smooth radial functions α,β and w on $\mathcal{M}$ given by

$$\alpha=\mu\,\mathfrak{f},$$
$$\beta=(1-\mu)\mathfrak{f},$$
$$w=-(1-\mu)\mathfrak{f}'$$

where $\mathfrak{f}$ is as in Lemma 7.6.

Then from section 7.1.1 we have

$$\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi](\partial_{t^*})=\mathfrak{f}\bigg(\mu(1+\mu)||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}}+2(1-\mu)(1+\mu)\langle\partial_{t^*}\Phi,\partial_r\Phi\rangle_{S^2_{\tau^\star,r}}-\mu(1-\mu)||\partial_r\Phi||^2_{S^2_{\tau^\star,r}}$$
$$+\mu\,||\nabla\!\!\!/_\Upsilon\Phi||^2_{S^2_{\tau^\star,r}}\bigg)$$
$$+(1-\mu)\mathfrak{f}'\bigg((1+\mu)\langle\partial_{t^*}\Phi,\Phi\rangle_{S^2_{\tau^\star,r}}-2\mu\,\langle\partial_r\Phi,\Phi\rangle_{S^2_{\tau^\star,r}}+\frac{1}{2}\frac{\mu}{r}\,||\Phi||^2_{S^2_{\tau^\star,r}}\bigg),$$
$$\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi](\partial_r)=-\mathfrak{f}\bigg(\Big(1+\mu^2\Big)\,||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}}+2\mu(1-\mu)\,\langle\partial_{t^*}\Phi,\partial_r\Phi\rangle_{S^2_{\tau^\star,r}}+(1-\mu)^2\,||\partial_r\Phi||^2_{S^2_{\tau^\star,r}}$$
$$-(1-\mu)\,||\nabla\!\!\!/_\Upsilon\Phi||^2_{S^2_{\tau^\star,r}}\bigg)$$
$$+(1-\mu)^2\mathfrak{f}\,\langle\partial_r\Phi,\Phi\rangle_{S^2_{\tau^\star,r}}-\frac{1}{2}\bigg(\Big((1-\mu)^2\mathfrak{f}'\Big)'-\frac{\mu}{r}\frac{1-\mu}{2}\mathfrak{f}'\bigg)||\Phi||^2_{S^2_{\tau^\star,r}}$$

and

$$\tilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi]=2\mathfrak{f}'\,||\mu\partial_{t^*}\Phi+(1-\mu)\partial_r\Phi||^2_{S^2_{\tau^\star,r}}-\mathfrak{f}\bigg(\frac{\mu}{r}\,||\nabla\!\!\!/_\Upsilon\Phi||^2_{S^2_{\tau^\star,r}}+(1-\mu)||[\partial_r,\Upsilon]\Phi||^2_{S^2_{\tau^\star,r}}\bigg)$$
$$-\frac{1}{2}\frac{1}{1-\mu}\mathfrak{f}^{***}||\Phi||^2_{S^2_{\tau^\star,r}}.$$

Now, as $\mathfrak{f}$ and its derivatives are uniformly bounded on $\mathcal{M}$, we have from Cauchy–Schwarz combined with the Poincaré inequality of Lemma 3.9 the estimates[3]

$$-\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi](\partial_{t^*})\lesssim\tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,r}[\Phi](\partial_{t^*}), \tag{7.18}$$
$$-\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Phi](\partial_r)\lesssim\tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,2M}[\Phi](\partial_r) \tag{7.19}$$

[3]Note in particular the $(1-\mu)$ weight that appears with the radial derivatives in the expression $\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi](\partial_{t^*})$.

and

$$\int_{2M}^{\infty} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi](\partial_{\tau^\star})\,\mathrm{d}r \lesssim \int_{2M}^{\infty} \tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,r}[\Phi](\partial_{\tau^\star})\,\mathrm{d}r. \tag{7.20}$$

Here, the 1-form $\tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,r}[\Phi]$ is as in the proof of Proposition 7.4. Consequently, applying the first half of Proposition 7.2 (noting that condition *iv*) is satisfied by estimates (7.18) and (7.19) combined with the first half of Corollary 7.5) in conjunction with the first half of Lemma 7.6 yields the estimate

$$\begin{aligned}
&\int_{\tau_0^\star}^{\infty}\int_{2M}^{\infty} \frac{1}{r^3}\Big(||\mu\partial_{t^*}\Phi + (1-\mu)\partial_r\Phi||^2_{S^2_{\tau^\star,r}} + (2-3\mu)^2 r^2||\nabla\!\!\!\!/\,\Phi||^2_{S^2_{\tau^\star,r}} + ||\Phi||^2_{S^2_{\tau^\star,r}}\Big)\,\mathrm{d}\tau^\star\,\mathrm{d}r \\
&\quad\lesssim \int_{2M}^{R}\Big(||\partial_{t^*}\Phi||^2_{S^2_{\tau_0^\star,r}} + (1-\mu)||\partial_r\Phi||^2_{S^2_{\tau_0^\star,r}} + ||\nabla\!\!\!\!/\,\Phi||^2_{S^2_{\tau_0^\star,r}}\Big)\,\mathrm{d}r \\
&\quad\quad + \int_{R}^{\infty}\Big(||D\Phi||^2_{S^2_{\tau_0^\star,r}} + ||\nabla\!\!\!\!/\,\Phi||^2_{S^2_{\tau_0^\star,r}}\Big)\,\mathrm{d}r.
\end{aligned} \tag{7.21}$$

Here, we have combined estimate (7.20) with the first half Proposition 7.4 to control the flux terms arising in the first half of Proposition 7.2.

Similarly, from section 7.1.1 we have

$$\begin{aligned}
\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi](\partial_{t^*}) &= \mathfrak{f}\Big(\mu(1+\mu)||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}} + 2(1-\mu)(1+\mu)\langle\partial_{t^*}\Psi,\partial_r\Psi\rangle_{S^2_{\tau^\star,r}} \\
&\qquad -\mu(1-\mu)||\partial_r\Psi||^2_{S^2_{\tau^\star,r}} + \mu\,||\nabla\!\!\!\!/_{\Upsilon+\mathfrak{Z}}\Psi||^2_{S^2_{\tau^\star,r}}\Big) \\
&\quad +(1-\mu)\mathfrak{f}'\Big((1+\mu)\langle\partial_{t^*}\Psi,\Psi\rangle_{S^2_{\tau^\star,r}} - 2\mu\,\langle\partial_r\Psi,\Psi\rangle_{S^2_{\tau^\star,r}} + \frac{1}{2}\frac{\mu}{r}\,||\Psi||^2_{S^2_{\tau^\star,r}}\Big), \\
\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi](\partial_r) &= -\mathfrak{f}\Big(\big(1+\mu^2\big)\,||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}} + 2\mu(1-\mu)\,\langle\partial_{t^*}\Psi,\partial_r\Psi\rangle_{S^2_{\tau^\star,r}} + (1-\mu)^2\,||\partial_r\Psi||^2_{S^2_{\tau^\star,r}} \\
&\qquad -(1-\mu)\,||\nabla\!\!\!\!/_{\Upsilon+\mathfrak{Z}}\Psi||^2_{S^2_{\tau^\star,r}}\Big) \\
&\quad +(1-\mu)^2\mathfrak{f}\,\langle\partial_r\Psi,\Psi\rangle_{S^2_{\tau^\star,r}} - \frac{1}{2}\Big(\big((1-\mu)^2\mathfrak{f}\big)' - \frac{\mu}{r}\frac{1-\mu}{2}\mathfrak{f}'\Big)||\Psi||^2_{S^2_{\tau^\star,r}}
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi] &= 2\mathfrak{f}'\,||\mu\partial_{t^*}\Psi + (1-\mu)\partial_r\Psi||^2_{S^2_{\tau^\star,r}} \\
&\quad -\mathfrak{f}\Big(\frac{\mu}{r}\,||\nabla\!\!\!\!/_{\Upsilon+\mathfrak{Z}}\Psi||^2_{S^2_{\tau^\star,r}} + (1-\mu)||[\partial_r,\Upsilon+\mathfrak{Z}]\Psi||^2_{S^2_{\tau^\star,r}}\Big) - \frac{1}{2}\frac{1}{1-\mu}\mathfrak{f}^{***}||\Psi||^2_{S^2_{\tau^\star,r}}.
\end{aligned}$$

Now, as $\mathfrak{f}$ and its derivatives are uniformly bounded on $\mathcal{M}$, we have from Cauchy–Schwarz combined with the Poincaré inequality of Lemma 3.9 the estimates

$$-\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi](\partial_{t^*}) \lesssim \tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,r}[\Psi](\partial_{t^*}), \tag{7.22}$$

$$-\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Psi](\partial_r) \lesssim \tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,2M}[\Psi](\partial_r) \tag{7.23}$$

and

$$\int_{2M}^{\infty}\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi](\partial_{\tau^\star})\,\mathrm{d}r \lesssim \int_{2M}^{\infty}\tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,r}[\Psi](\partial_{\tau^\star})\,\mathrm{d}r. \tag{7.24}$$

Here, the 1-form $\tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,r}[\Psi]$ is as in the proof of Proposition 7.4. Consequently, applying the second half of Proposition 7.2 (noting that condition *iv*) is satisfied by estimates (7.22) and (7.23) combined with the second half of Corollary 7.5) in conjunction with the second half of Lemma 7.6 yields the estimate

$$\begin{aligned}
&\int_{\tau_0^\star}^{\infty}\int_{2M}^{\infty}\frac{1}{r^3}\Big(||\mu\partial_{t^*}\Psi+(1-\mu)\partial_r\Psi||^2_{S^2_{\tau^\star,r}}+(2-3\mu)^2r^2||\nabla\!\!\!/\Psi||^2_{S^2_{\tau^\star,r}}+||\Psi||^2_{S^2_{\tau^\star,r}}\Big)\,\mathrm{d}\tau^\star\,\mathrm{d}r\\
&\quad\lesssim\int_{2M}^{R}\Big(||\partial_{t^*}\Psi||^2_{S^2_{\tau_0^\star,r}}+(1-\mu)||\partial_r\Psi||^2_{S^2_{\tau_0^\star,r}}+||\nabla\!\!\!/\Psi||^2_{S^2_{\tau_0^\star,r}}\Big)\,\mathrm{d}r\\
&\quad+\int_{R}^{\infty}\Big(||D\Psi||^2_{S^2_{\tau_0^\star,r}}+||\nabla\!\!\!/\Psi||^2_{S^2_{\tau_0^\star,r}}\Big)\,\mathrm{d}r.
\end{aligned}\tag{7.25}$$

Here, we have combined estimate (7.24) with the second half Proposition 7.4 to control the flux terms arising in the first half of Proposition 7.2.

We consider now the three smooth radial functions α,β and w on $\mathcal{M}$ given by

$$\begin{aligned}
\alpha&=\mu\,\mathfrak{g},\\
\beta&=(1-\mu)\mathfrak{g},\\
w&=0
\end{aligned}$$

where $\mathfrak{g}$ is the smooth radial function

$$\mathfrak{g}:=-\frac{2}{r^2}\Big(1-\frac{3M}{r}\Big)^3\Big(1-\frac{2M}{r}\Big).$$

Then from section 7.1.1 we have

$$\begin{aligned}
\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi](\partial_{t^*})&=\mathfrak{g}\Big(\mu(1+\mu)||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}}+2(1-\mu)(1+\mu)\langle\partial_{t^*}\Phi,\partial_r\Phi\rangle_{S^2_{\tau^\star,r}}\\
&\qquad-\mu(1-\mu)||\partial_r\Phi||^2_{S^2_{\tau^\star,r}}+\mu\,||\nabla\!\!\!/_{\Upsilon}\Phi||^2_{S^2_{\tau^\star,r}}\Big)\\
\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi](\partial_r)&=-\mathfrak{g}\Big(\Big(1+\mu^2\Big)\,||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}}+2\mu(1-\mu)\,\langle\partial_{t^*}\Phi,\partial_r\Phi\rangle_{S^2_{\tau^\star,r}}+(1-\mu)^2\,||\partial_r\Phi||^2_{S^2_{\tau^\star,r}}\\
&\qquad-(1-\mu)\,||\nabla\!\!\!/_{\Upsilon}\Phi||^2_{S^2_{\tau^\star,r}}\Big)
\end{aligned}$$

and

$$
\begin{aligned}
\widetilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi] =& \mathfrak{g}' \, ||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}} \\
& + \mathfrak{g}' \, ||\mu\partial_{t^*}\Phi + (1-\mu)\partial_r\Phi||^2_{S^2_{\tau^\star,r}} - \mathfrak{g}\left(\frac{\mu}{r} \, ||\nabla\!\!\!/_\Upsilon\Phi||^2_{S^2_{\tau^\star,r}} + (1-\mu)||[\partial_r,\Upsilon]\Phi||^2_{S^2_{\tau^\star,r}}\right) \\
-(1-\mu)\mathfrak{g}' \, ||\nabla\!\!\!/_\Upsilon\Phi||^2_{S^2_{\tau^\star,r}}.
\end{aligned}
$$

Now, as $\mathfrak{g}$ and its derivatives are uniformly bounded on $\mathcal{M}$, we have from Cauchy–Schwarz combined with the Poincaré inequality of Lemma 3.9 the estimates

$$
-\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi](\partial_{t^*}) \lesssim \tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,r}[\Phi](\partial_{t^*}), \tag{7.26}
$$

$$
-\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Phi](\partial_r) \lesssim \tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,2M}[\Phi](\partial_r) \tag{7.27}
$$

and

$$
\int_{2M}^{\infty} \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi](\partial_{\tau^\star}) \, \mathrm{d}r \lesssim \int_{2M}^{\infty} \tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,r}[\Phi](\partial_{\tau^\star}) \, \mathrm{d}r. \tag{7.28}
$$

In addition, as both $\mathfrak{g}'$ and $\mathfrak{g}$ vanish to second order at $r = 3M$ with both $\mathfrak{g}'$ and $\frac{1}{r}\mathfrak{g}$ vanishing to third order as $r \to \infty$, we have on any 2-sphere $S^2_{\tau^\star,r}$ the estimate

$$
\begin{aligned}
& \mathfrak{g}' \, ||\mu\partial_{t^*}\Phi + (1-\mu)\partial_r\Phi||^2_{S^2_{\tau^\star,r}} - \mathfrak{g}\left(\frac{\mu}{r} \, ||\nabla\!\!\!/_\Upsilon\Phi||^2_{S^2_{\tau^\star,r}} + (1-\mu)||[\partial_r,\Upsilon]\Phi||^2_{S^2_{\tau^\star,r}}\right) \\
& -(1-\mu)\mathfrak{g}' \, ||\nabla\!\!\!/_\Upsilon\Phi||^2_{S^2_{\tau^\star,r}} \\
& \lesssim \frac{1}{r^3} \, ||\mu\partial_{t^*}\Phi + (1-\mu)\partial_r\Phi||^2_{S^2_{\tau^\star,r}} + \frac{1}{r}(2-3\mu)^2 \, ||\nabla\!\!\!/\Phi||^2_{S^2_{\tau^\star,r}} + \frac{1}{r^3} \, ||\Phi||^2_{S^2_{\tau^\star,r}}.
\end{aligned} \tag{7.29}
$$

Consequently, Proposition 7.2 combined with Proposition 7.4, Corollary 7.5, estimate (7.29) and the fact that the function $\mathfrak{g}'$ is non-negative on $\mathcal{M}$ yields the improved estimate

$$
\begin{aligned}
& \int_{\tau_0^\star}^{\infty}\int_{2M}^{\infty} \frac{1}{r^3}\Bigg(||\mu\partial_{t^*}\Phi + (1-\mu)\partial_r\Phi||^2_{S^2_{\tau^\star,r}} \\
& \qquad + (2-3\mu)^2\left(||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}} + r^2||\nabla\!\!\!/\Phi||^2_{S^2_{\tau^\star,r}} + ||\Phi||^2_{S^2_{\tau^\star,r}}\right)\Bigg) \, \mathrm{d}\tau^\star \, \mathrm{d}r \\
& \qquad \lesssim \int_{2M}^{R} \left(||\partial_{t^*}\Phi||^2_{S^2_{\tau_0^\star,r}} + (1-\mu)||\partial_r\Phi||^2_{S^2_{\tau_0^\star,r}} + ||\nabla\!\!\!/\Phi||^2_{S^2_{\tau_0^\star,r}}\right) \mathrm{d}r \\
& \qquad + \int_{R}^{\infty} \left(||D\Phi||^2_{S^2_{\tau_0^\star,r}} + ||\nabla\!\!\!/\Phi||^2_{S^2_{\tau_0^\star,r}}\right) \mathrm{d}r
\end{aligned}
$$

from which estimate (7.16) follows.

Similarly, from section 7.1.1 we have

$$
\begin{aligned}
\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi](\partial_{t^*}) = \mathfrak{g}\Big(&\mu(1+\mu)||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}} + 2(1-\mu)(1+\mu)\langle\partial_{t^*}\Psi,\partial_r\Psi\rangle_{S^2_{\tau^\star,r}} \\
&-\mu(1-\mu)||\partial_r\Psi||^2_{S^2_{\tau^\star,r}} + \mu\,||\nabla\!\!\!/_{\Upsilon+\mathfrak{Z}}\Psi||^2_{S^2_{\tau^\star,r}}\Big) \\
\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi](\partial_r) = -\mathfrak{g}\Big(&\left(1+\mu^2\right)||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}} + 2\mu(1-\mu)\,\langle\partial_{t^*}\Psi,\partial_r\Psi\rangle_{S^2_{\tau^\star,r}} + (1-\mu)^2\,||\partial_r\Psi||^2_{S^2_{\tau^\star,r}} \\
&-(1-\mu)\,||\nabla\!\!\!/_{\Upsilon+\mathfrak{Z}}\Psi||^2_{S^2_{\tau^\star,r}}\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi] =& \mathfrak{g}'\,||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}} \\
&+\mathfrak{g}'\,||\mu\partial_{t^*}\Psi + (1-\mu)\partial_r\Psi||^2_{S^2_{\tau^\star,r}} \\
&-\mathfrak{g}\left(\frac{\mu}{r}\,||\nabla\!\!\!/_{\Upsilon+\mathfrak{Z}}\Psi||^2_{S^2_{\tau^\star,r}} + (1-\mu)||[\partial_r,\Upsilon+\mathfrak{Z}]\Psi||^2_{S^2_{\tau^\star,r}}\right) \\
&-(1-\mu)\mathfrak{g}'\,||\nabla\!\!\!/_{\Upsilon+\mathfrak{Z}}\Psi||^2_{S^2_{\tau^\star,r}}.
\end{aligned}
$$

Now, as $\mathfrak{g}$ and its derivatives are uniformly bounded on $\mathcal{M}$, we have from Cauchy–Schwarz combined with the Poincaré inequality of Lemma 3.9 the estimates

$$
-\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi](\partial_{t^*}) \lesssim \tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,r}[\Psi](\partial_{t^*}), \tag{7.30}
$$

$$
-\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Psi](\partial_r) \lesssim \tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,2M}[\Psi](\partial_r) \tag{7.31}
$$

and

$$
\int_{2M}^{\infty}\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi](\partial_{\tau^\star})\,\mathrm{d}r \lesssim \int_{2M}^{\infty}\tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,r}[\Psi](\partial_{\tau^\star})\,\mathrm{d}r. \tag{7.32}
$$

In addition, as both $\mathfrak{g}'$ and $\mathfrak{g}$ vanish to second order at $r = 3M$ with both $\mathfrak{g}'$ and $\frac{1}{r}\mathfrak{g}$ vanishing to third order as $r \to \infty$, we have from the elliptic estimates of Proposition 3.12 on any 2-sphere $S^2_{\tau^\star,r}$ the estimate

$$
\begin{aligned}
&\mathfrak{g}'\,||\mu\partial_{t^*}\Psi + (1-\mu)\partial_r\Psi||^2_{S^2_{\tau^\star,r}} \\
&-\mathfrak{g}\left(\frac{\mu}{r}\,||\nabla\!\!\!/_{\Upsilon+\mathfrak{Z}}\Psi||^2_{S^2_{\tau^\star,r}} + (1-\mu)||[\partial_r,\Upsilon+\mathfrak{Z}]\Psi||^2_{S^2_{\tau^\star,r}}\right) \\
&-(1-\mu)\mathfrak{g}'\,||\nabla\!\!\!/_{\Upsilon+\mathfrak{Z}}\Psi||^2_{S^2_{\tau^\star,r}} \\
&\lesssim \frac{1}{r^3}\,||\mu\partial_{t^*}\Psi + (1-\mu)\partial_r\Psi||^2_{S^2_{\tau^\star,r}} + \frac{1}{r}(2-3\mu)^2\,||\nabla\!\!\!/\Psi||^2_{S^2_{\tau^\star,r}} + \frac{1}{r^3}\,||\Psi||^2_{S^2_{\tau^\star,r}}.
\end{aligned} \tag{7.33}
$$

Consequently, Proposition 7.2 combined with Proposition 7.4, Corollary 7.5, estimate

(7.33) and the fact that the function $\mathfrak{g}'$ is non-negative on $\mathcal{M}$ yields the improved estimate

$$\begin{aligned}\int_{\tau_0^\star}^{\infty}\int_{2M}^{\infty}\frac{1}{r^3}\Big(&||\mu\partial_{t^*}\Psi+(1-\mu)\partial_r\Psi||^2_{S^2_{\tau^\star,r}}\\&+(2-3\mu)^2\Big(||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}}+r^2||\nabla\!\!\!/\Psi||^2_{S^2_{\tau^\star,r}}+||\Psi||^2_{S^2_{\tau^\star,r}}\Big)\,\mathrm{d}\tau^\star\,\mathrm{d}r\\&\lesssim\int_{2M}^{R}\Big(||\partial_{t^*}\Psi||^2_{S^2_{\tau_0^\star,r}}+(1-\mu)||\partial_r\Psi||^2_{S^2_{\tau_0^\star,r}}+||\nabla\!\!\!/\Psi||^2_{S^2_{\tau_0^\star,r}}\Big)\,\mathrm{d}r\\&+\int_{R}^{\infty}\Big(||D\Psi||^2_{S^2_{\tau_0^\star,r}}+||\nabla\!\!\!/\Psi||^2_{S^2_{\tau_0^\star,r}}\Big)\,\mathrm{d}r\end{aligned}$$

from which estimate (7.17) follows.

This completes the proposition. □

An immediate corollary of the above proof is the following estimate which shall prove useful in the sequel.

In what follows, we recall the definition of the energy norm $\mathbb{E}$ from section 6.1.

Corollary 7.8. *Let $\tau_1^\star\geq\tau_0^\star$ be a real number.*

Let now Φ be as in Theorem 1. Then one has the bulk estimate

$$\int_{\tau_1^\star}^{\infty}\int_{2M}^{R}||\mu\,\partial_{t^*}\Phi+(1-\mu)\,\partial_r\Phi||^2_{S^2_{\tau^\star,r}}\,\mathrm{d}\tau^\star\,\mathrm{d}r\lesssim\mathbb{E}[r^{-1}\Phi](\tau_1^\star).$$

Let now Ψ be as in Theorem 1. Then one has the bulk estimate

$$\int_{\tau_1^\star}^{\infty}\int_{2M}^{R}||\mu\,\partial_{t^*}\Psi+(1-\mu)\,\partial_r\Psi||^2_{S^2_{\tau^\star,r}}\,\mathrm{d}\tau^\star\,\mathrm{d}r\lesssim\mathbb{E}[r^{-1}\Psi](\tau_1^\star).$$

We make the following remarks regarding Proposition 7.7.

Remark 41. The use of the function $\mathfrak{f}$ to derive the estimates (7.12) and (7.13) of Lemma 7.6 first appeared in the work [6] of the author where it was motivated by earlier works of Holzegel [27] on the Regge–Wheeler equation. See also [30].

7.1.2.2 Improving the weights near $\mathcal{H}^+$

We continue by first removing the degeneration at $r=2M$ in the estimates of Propositions 7.4 and 7.7.

This will follow as a consequence of the following "red-shift" estimate.

Lemma 7.9. *Let α, β and w be three smooth radial functions on $\mathcal{M}$ given by*

$$\begin{aligned}
\alpha &= 2, \\
\beta &= 3 - \frac{8M}{r}, \\
w &= 0.
\end{aligned}$$

Let now Φ be as in Theorem 1. Then there exists an $r_1 \in (2M, 3M)$ such that on any 2-sphere $S^2_{\tau^\star,r}$ with $r \in [2M, r_1]$ it holds that

$$\begin{aligned}
||\partial_r \Phi||^2_{S^2_{\tau^\star,r}} \lesssim \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi](\partial_{t^*}) &\lesssim \widetilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi] + c\,||\nabla\!\!\!/ \Phi||^2_{S^2_{\tau^\star,r}} \\
&\lesssim ||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}} + ||\partial_r \Phi||^2_{S^2_{\tau^\star,r}} + ||\nabla\!\!\!/ \Phi||^2_{S^2_{\tau^\star,r}}
\end{aligned}$$

where c is a positive constant.

Let now Ψ be as in Theorem 1. Then there exists an $r_1 \in (2M, 3M)$ such that on any 2-sphere $S^2_{\tau^\star,r}$ with $r \in [2M, r_1]$ it holds that

$$\begin{aligned}
||\partial_r \Psi||^2_{S^2_{\tau^\star,r}} \lesssim \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi](\partial_{t^*}) &\lesssim \widetilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi] + c\,||\nabla\!\!\!/ \Psi||^2_{S^2_{\tau^\star,r}} \\
&\lesssim ||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}} + ||\partial_r \Psi||^2_{S^2_{\tau^\star,r}} + ||\nabla\!\!\!/ \Psi||^2_{S^2_{\tau^\star,r}}
\end{aligned}$$

where c is a positive constant.

Proof. By continuity it suffices to establish on any 2-sphere $S^2_{\tau^\star,2M}$ the bounds

$$\begin{aligned}
||\partial_r \Phi||^2_{S^2_{\tau^\star,2M}} < \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Phi](\partial_{t^*}) &< \widetilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Phi] + c\,||\nabla\!\!\!/ \Phi||^2_{S^2_{\tau^\star,2M}} \\
&< ||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}} + ||\partial_r \Phi||^2_{S^2_{\tau^\star,2M}} + ||\nabla\!\!\!/ \Phi||^2_{S^2_{\tau^\star,2M}}
\end{aligned} \tag{7.34}$$

and

$$\begin{aligned}
||\partial_r \Psi||^2_{S^2_{\tau^\star,2M}} < \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Psi](\partial_{t^*}) &< \widetilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Psi] + c\,||\nabla\!\!\!/ \Psi||^2_{S^2_{\tau^\star,2M}} \\
&< ||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,2M}} + ||\partial_r \Psi||^2_{S^2_{\tau^\star,2M}} + ||\nabla\!\!\!/ \Psi||^2_{S^2_{\tau^\star,2M}}.
\end{aligned} \tag{7.35}$$

Indeed, from section 7.1.1 we have

$$\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Phi](\partial_{t^*}) = 4\,||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,2M}} - 4\,\langle \partial_{t^*}\Phi, \partial_r \Phi\rangle_{S^2_{\tau^\star,2M}} + 2\,||\partial_r \Phi||^2_{S^2_{\tau^\star,2M}} + 2\,||\nabla\!\!\!/_{\Upsilon} \Phi||^2_{S^2_{\tau^\star,2M}}$$

and

$$\widetilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Phi] = \frac{5}{M}\,||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,2M}} - \frac{1}{M}\,\langle\partial_{t^*}\Phi,\partial_r\Phi\rangle_{S^2_{\tau^\star,2M}} + \frac{1}{2}\frac{1}{M}\,||\partial_r\Phi||^2_{S^2_{\tau^\star,2M}} \\ -\frac{2}{M}\,||\nabla\!\!\!\!/_{\Upsilon}\Phi||^2_{S^2_{\tau^\star,2M}} + ||[\partial_r,\Upsilon]\Phi||^2_{S^2_{\tau^\star,2M}}.$$

The upper and lower bounds in (7.34) then follow from applying Cauchy–Schwarz in conjunction with the first half of Lemma 7.3 and the Poincaré inequality of Lemma 3.9.

Similarly, from section 7.1.1 we have

$$\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Psi](\partial_{t^*}) = 4\,||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,2M}} - 4\,\langle\partial_{t^*}\Psi,\partial_r\Psi\rangle_{S^2_{\tau^\star,2M}} + 2\,||\partial_r\Psi||^2_{S^2_{\tau^\star,2M}} + 2\,||\nabla\!\!\!\!/_{\Upsilon+\mathfrak{Z}}\Psi||^2_{S^2_{\tau^\star,2M}}$$

and

$$\widetilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Psi] = \frac{5}{M}\,||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,2M}} - \frac{1}{M}\,\langle\partial_{t^*}\Psi,\partial_r\Psi\rangle_{S^2_{\tau^\star,2M}} + \frac{1}{2}\frac{1}{M}\,||\partial_r\Psi||^2_{S^2_{\tau^\star,2M}} \\ -\frac{2}{M}\,||\nabla\!\!\!\!/_{\Upsilon+\mathfrak{Z}}\Psi||^2_{S^2_{\tau^\star,2M}} + ||[\partial_r,\Upsilon+\mathfrak{Z}]\Psi||^2_{S^2_{\tau^\star,2M}}.$$

The upper and lower bounds in (7.35) then follow from applying Cauchy–Schwarz in conjunction with the second half of Lemma 7.3, the Poincaré inequality of Lemma 3.9 and the elliptic estimates of Proposition 3.12. □

We then have the following proposition.

Proposition 7.10. *Let $\tau_2^\star, \tau_1^\star \geq \tau_0^\star$ be two real numbers.*

Let now Φ be as in Theorem 1. Then one has the estimate

$$\mathbb{E}[r^{-1}\Phi](\tau_2^\star) \\ + \int_{\tau_1^\star}^{\tau_2^\star}\int_{2M}^{\infty}\frac{1}{r^3}\Big((2-3\mu)^2\Big(||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}} + ||\partial_r\Phi||^2_{S^2_{\tau^\star,r}} + r^2||\nabla\!\!\!\!/\Phi||^2_{S^2_{\tau^\star,r}}\Big) + ||\Phi||^2_{S^2_{\tau^\star,r}}\Big)\,\mathrm{d}\tau^\star\,\mathrm{d}r \\ \lesssim \mathbb{E}[r^{-1}\Phi](\tau_1^\star). \tag{7.36}$$

Let now Ψ be as in Theorem 1. Then one has the flux estimate

$$\mathbb{E}[r^{-1}\Psi](\tau_2^\star) \\ + \int_{\tau_1^\star}^{\tau_2^\star}\int_{2M}^{\infty}\frac{1}{r^3}\Big((2-3\mu)^2\Big(||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}} + ||\partial_r\Psi||^2_{S^2_{\tau^\star,r}} + r^2||\nabla\!\!\!\!/\Psi||^2_{S^2_{\tau^\star,r}}\Big) + ||\Psi||^2_{S^2_{\tau^\star,r}}\Big)\,\mathrm{d}\tau^\star\,\mathrm{d}r \\ \lesssim \mathbb{E}[r^{-1}\Psi](\tau_1^\star). \tag{7.37}$$

Proof. We consider the three smooth radial functions α, β and w on $\mathcal{M}$ given by

$$\alpha = 3 - \chi,$$
$$\beta = \left(3 - \frac{8M}{r}\right)\chi,$$
$$w = 0$$

where χ is a smooth cut-off function such that

$$\chi = \begin{cases} 1 & \text{for } 2M \leq r \leq r_1 \\ 0 & \text{for } r \geq r_2. \end{cases}$$

Here, r_1 is as in Lemma 7.9 and $r_2 \in (r_1, 3M)$.

Then from section 7.1.1 we have

$$\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Phi](\partial_r) = -2\,||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}} - ||\nabla\!\!\!\!/_\Upsilon\Phi||^2_{S^2_{\tau^\star,2M}}.$$

Moreover, it follows easily from the Poincaré inequality of Lemma 3.9 and the fact that χ and its derivatives are compactly supported on $\mathcal{M}$ that on any 2-sphere $S^2_{\tau^\star,r}$ one has the bounds

$$\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi](\partial_{t^*}) \lesssim \chi\,||\partial_r\Phi||^2_{S^2_{\tau^\star,r}} + (1-\chi)\,\tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,r}[\Phi](\partial_{t^*}) \tag{7.38}$$

and

$$\widetilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi] \lesssim \chi'\chi\,||\partial_r\Phi||^2_{S^2_{\tau^\star,r}} + \chi'(1-\chi)\,\tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,r}[\Phi](\partial_{t^*}). \tag{7.39}$$

Here, $\tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,r}[\Phi](\partial_{t^*})$ is as in the proof of Proposition 7.4. Consequently, applying the first half of Proposition 7.2 (noting that condition *iii*) is satisfied by estimate (7.38) combined with the first half of Corollary 7.5) in conjunction with the first half of Lemma 7.9 yields the estimate

$$\int_{2M}^{r_1} ||\partial_r\Phi||^2_{S^2_{\tau_2^\star,r}}\,\mathrm{d}r + \int_{\tau_1^\star}^{\tau_2^\star}\int_{2M}^{r_1} ||\partial_r\Phi||^2_{S^2_{\tau^\star,r}}\,\mathrm{d}\tau^\star\,\mathrm{d}r \lesssim \mathbb{E}[\Phi](\tau_1^\star). \tag{7.40}$$

Here, we have used the first bulk estimate of Proposition 7.7 to control the bulk terms arising from (7.39) in the region $r_1 \leq r \leq r_2$, recalling that $r_2 < 3M$.

The estimate (7.36) then follows after adding the first estimates of Proposition 7.4 and Proposition 7.7 to (7.40).

Similarly, from section 7.1.1 we have

$$\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,2M}[\Psi](\partial_r) = -2\,||\partial_{t^*}\Psi||^2_{S^2_{\tau^\star,r}} - ||\nabla\!\!\!\!/\,_{\Upsilon+\mathfrak{z}}\Psi||^2_{S^2_{\tau^\star,2M}}.$$

Moreover, it follows easily from the Poincaré inequality of Lemma 3.9, the elliptic estimates of Proposition 3.12 and the fact that χ and its derivatives are compactly supported on $\mathcal{M}$ that on any 2-sphere $S^2_{\tau^\star,r}$ one has the bounds

$$\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi](\partial_{t^*}) \lesssim \chi\,||\partial_r\Psi||^2_{S^2_{\tau^\star,r}} + (1-\chi)\,\tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,r}[\Psi](\partial_{t^*}) \tag{7.41}$$

and

$$\widetilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi] \lesssim \chi'\chi\,||\partial_r\Psi||^2_{S^2_{\tau^\star,r}} + \chi'(1-\chi)\,\tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,r}[\Psi](\partial_{t^*}). \tag{7.42}$$

Here, $\tilde{\mathbb{J}}^{1,0,0}_{\tau^\star,r}[\Psi](\partial_{t^*})$ is as in the proof of Proposition 7.4. Consequently, applying the second half of Proposition 7.2 (noting that condition *iii*) is satisfied by estimate (7.41) combined with the second half of Corollary 7.5) in conjunction with the second half of Lemma 7.9 yields the estimate

$$\int_{2M}^{r_1} ||\partial_r\Psi||^2_{S^2_{\tau_2^\star,r}}\,\mathrm{d}r + \int_{\tau_1^\star}^{\tau_2^\star}\int_{2M}^{r_1} ||\partial_r\Psi||^2_{S^2_{\tau^\star,r}}\,\mathrm{d}\tau^\star\,\mathrm{d}r \lesssim \mathbb{E}[r^{-1}\Psi](\tau_1^\star). \tag{7.43}$$

Here, we have used the second bulk estimate of Proposition 7.7 to control the bulk terms arising from (7.42) in the region $r_1 \le r \le r_2$.

The estimate (7.37) then follows after adding the second estimates of Proposition 7.4 and Proposition 7.7 to (7.43).

This completes the proposition. □

We make the following remark regarding Proposition 7.10.

Remark 42. One can prove the boundedness of the non-degenerate energy in the above proposition without invoking the bulk estimate of Proposition 7.7 – see[37].

7.1.2.3 Improving the weights near $\mathcal{I}^+$

Next we improve the weights near $\mathcal{I}^+$ in the flux and bulk estimates of Proposition 7.10.

In what follows, we recall the flux and bulk norms $\mathbb{F}_p$ and $\mathbb{B}_p$ defined as in section 6.1.

Proposition 7.11. *Let $\tau_2^\star \ge \tau_1^\star \ge \tau_0^\star$ and $1 \le p \le 2$ be three real numbers.*

Let now Φ be as in Theorem 1. Then the following hierarchy of estimates hold:

$$\mathbb{F}_p[r^{-1}\Phi](\tau_2^\star) + \mathbb{B}_p[r^{-1}\Phi](\tau_1^\star) \lesssim \mathbb{F}_p[r^{-1}\Phi](\tau_1^\star) + \mathbb{E}[r^{-1}\Phi](\tau_1^\star). \tag{7.44}$$

Let now Ψ be as in Theorem 1. Then the following hierarchy of estimates hold:

$$\mathbb{F}_p[r^{-1}\Psi](\tau_2^\star) + \mathbb{B}_p[r^{-1}\Psi](\tau_1^\star) \lesssim \mathbb{F}_p[r^{-1}\Psi](\tau_1^\star) + \mathbb{E}[r^{-1}\Psi](\tau_1^\star). \tag{7.45}$$

Proof. We consider the three smooth radial functions α, β and w on $\mathcal{M}$ given by

$$\begin{aligned}
\alpha &= \frac{1+\mu}{1-\mu}\chi\, r^p, \\
\beta &= \chi\, r^p, \\
w &= 0
\end{aligned}$$

where χ is a smooth cut-off function such that

$$\chi = \begin{cases} 0 & \text{for } 2M \leq r \leq R \\ 1 & \text{for } r \geq 2R. \end{cases}$$

Then from section 7.1.1 we have

$$\begin{aligned}
\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi](\partial_{t^*}) &= (1-\mu)^2\chi r^p\,||D\Phi||^2_{S^2_{\tau^\star,r}} + \frac{1+\mu}{1-\mu}\chi r^p\,||\nabla\!\!\!/_\Upsilon\Phi||^2_{S^2_{\tau^\star,r}} \\
\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi](\partial_r) &= -(1-\mu)^2\chi r^p\,||D\Phi||^2_{S^2_{\tau^\star,r}} + \chi r^p\,||\nabla\!\!\!/_\Upsilon\Phi||^2_{S^2_{\tau^\star,r}}
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi] =& \chi r^{p-1}\Big(\big(p(1-\mu)-\mu\big)\,||D\Phi||^2_{S^2_{\tau^\star,r}} - p\,||\nabla\!\!\!/_\Upsilon\Phi||^2_{S^2_{\tau^\star,r}} - r\,||[\partial_r,\Upsilon]\Phi||^2_{S^2_{\tau^\star,r}}\Big) \\
&+\chi' r^p\left(||D\Phi||^2_{S^2_{\tau^\star,r}} - ||\nabla\!\!\!/_\Upsilon\Phi||^2_{S^2_{\tau^\star,r}}\right).
\end{aligned}$$

Now, as $p \geq 1$ and $\mu \geq \frac{1}{3}$ for $r \geq R$, we have the bound

$$p(1-\mu)-\mu \geq \frac{1}{3}. \tag{7.46}$$

In addition, by the Poincaré inequality of Lemma 3.9 we have on any 2-sphere $S^2_{\tau^\star,r}$ for $p < 2$ the estimate

$$(2-p)r^p\,||\nabla\!\!\!/\Phi||^2_{S^2_{\tau^\star,r}} \lesssim -p\,||\nabla\!\!\!/_\Upsilon\Phi||^2_{S^2_{\tau^\star,r}} - r\,||[\partial_r,\Upsilon]\Phi||^2_{S^2_{\tau^\star,r}} \tag{7.47}$$

whereas for $p = 2$

$$-p\,||\nabla\!\!\!/_\Upsilon\Phi||^2_{S^2_{\tau^\star,r}} - r\,||[\partial_r,\Upsilon]\Phi||^2_{S^2_{\tau^\star,r}} \lesssim ||\nabla\!\!\!/\Phi||^2_{S^2_{\tau^\star,r}}. \tag{7.48}$$

Consequently, applying the first half of Proposition 7.2 (noting that condition *i*) is satisfied by the first half of Lemma 7.3 combined with the fact that χ vanishes for $r = 2M$)

in conjunction with the first half of Lemma 7.3 yields for $p<2$ the estimate

$$\begin{aligned}&\int_R^\infty r^p||D\Phi||^2_{S^2_{\tau_2^\star,r}}\,\mathrm{d}r+\int_{\tau_1^\star}^{\tau_2^\star}\int_R^\infty\left(r^{p-1}||D\Phi||^2_{S^2_{\tau^\star,r}}+(2-p)r^{p-1}||\nabla\!\!\!\!/\,\Phi||^2_{S^2_{\tau^\star,r}}\right)\mathrm{d}\tau^\star\,\mathrm{d}r\\ \lesssim&\,\mathbb{E}[r^{-1}\Phi](\tau_1^\star)+\int_R^\infty r^p||D\Phi||^2_{S^2_{\tau_1^\star,r}}\,\mathrm{d}r.\end{aligned}\tag{7.49}$$

Here, we have used the first estimate of Proposition 7.7 to control the bulk terms multiplying χ' in $\widetilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,r}[\Phi]$ (noting that χ' vanishes for $r\geq R$) along with the fact that $|\chi|\leq 1$ on $\mathcal{M}$.

The estimate (7.44) for $p=2$ then follows after utilising Proposition 7.2 once more in conjunction with estimates (7.48) and (7.47) with $p=1$.

Similarly, from section 7.1.1 we have

$$\begin{aligned}\tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi](\partial_{t^*})&=(1-\mu)^2\chi r^p\,||D\Psi||^2_{S^2_{\tau^\star,r}}+\frac{1+\mu}{1-\mu}\chi r^p\,||\nabla\!\!\!\!/\,_{\Upsilon+\mathfrak{z}}\Psi||^2_{S^2_{\tau^\star,r}}\\ \tilde{\mathbb{J}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi](\partial_r)&=-(1-\mu)^2\chi r^p\,||D\Psi||^2_{S^2_{\tau^\star,r}}+\chi r^p\,||\nabla\!\!\!\!/\,_{\Upsilon+\mathfrak{z}}\Psi||^2_{S^2_{\tau^\star,r}}\end{aligned}$$

and

$$\begin{aligned}\widetilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi]=&\chi r^{p-1}\Big(\big(p(1-\mu)-\mu\big)\,||D\Psi||^2_{S^2_{\tau^\star,r}}-p\,||\nabla\!\!\!\!/\,_{\Upsilon+\mathfrak{z}}\Psi||^2_{S^2_{\tau^\star,r}}-r\,||[\partial_r,\Upsilon+\mathfrak{Z}]\Psi||^2_{S^2_{\tau^\star,r}}\Big)\\ &+\chi' r^p\left(||D\Psi||^2_{S^2_{\tau^\star,r}}-||\nabla\!\!\!\!/\,_{\Upsilon+\mathfrak{z}}\Psi||^2_{S^2_{\tau^\star,r}}\right).\end{aligned}$$

In addition, by the Poincaré inequality of Lemma 3.9 and the elliptic estimates of Proposition 3.12 we have on any 2-sphere $S^2_{\tau^\star,r}$ for $p<2$ the estimate

$$(2-p)r^p\,||\nabla\!\!\!\!/\,\Psi||^2_{S^2_{\tau^\star,r}}\lesssim -p\,||\nabla\!\!\!\!/\,_{\Upsilon+\mathfrak{z}}\Psi||^2_{S^2_{\tau^\star,r}}-r\,||[\partial_r,\Upsilon+\mathfrak{Z}]\Psi||^2_{S^2_{\tau^\star,r}}\tag{7.50}$$

whereas for $p=2$

$$-p\,||\nabla\!\!\!\!/\,_{\Upsilon+\mathfrak{z}}\Psi||^2_{S^2_{\tau^\star,r}}-r\,||[\partial_r,\Upsilon+\mathfrak{Z}]\Psi||^2_{S^2_{\tau^\star,r}}\lesssim ||\nabla\!\!\!\!/\,\Psi||^2_{S^2_{\tau^\star,r}}.\tag{7.51}$$

Consequently, applying the second half of Proposition 7.2 (noting that condition *i*) is satisfied by the second half of Lemma 7.3 combined with the fact that χ vanishes for $r=2M$) in conjunction with the second half of Lemma 7.3 yields for $p<2$ the estimate

$$\begin{aligned}&\int_R^\infty r^p||D\Psi||^2_{S^2_{\tau_2^\star,r}}\,\mathrm{d}r+\int_{\tau_1^\star}^{\tau_2^\star}\int_R^\infty\left(r^{p-1}||D\Psi||^2_{S^2_{\tau^\star,r}}+(2-p)r^{p-1}||\nabla\!\!\!\!/\,\Psi||^2_{S^2_{\tau^\star,r}}\right)\mathrm{d}\tau^\star\,\mathrm{d}r\\ \lesssim&\,\mathbb{E}[r^{-1}\Psi](\tau_1^\star)+\int_R^\infty r^p||D\Psi||^2_{S^2_{\tau_1^\star,r}}\,\mathrm{d}r.\end{aligned}\tag{7.52}$$

Here, we have used the second estimate of Proposition 7.7 to control the bulk terms multiplying χ' in $\widetilde{\mathbb{K}}^{\alpha,\beta,w}_{\tau^\star,r}[\Psi]$.

The estimate (7.45) for $p = 2$ then follows after utilising Proposition 7.2 once more in conjunction with estimates (7.51) and (7.50) with $p = 1$.

This completes the proposition. $\square$

An immediate corollary of the above proposition is the boundedness of smooth and compactly supported solutions to the Regge–Wheeler and Zerilli equations in the norms $\mathbb{F}$ and $\mathbb{I}$ of section 6.1. This in particular yields the ($n = 0$ case) of parts i) in Theorem 1.

In what follows, we recall that since the solutions Φ and Ψ to the Regge–Wheeler and Zerilli equations of Theorem 1 are compactly supported on Σ_R (cf. Definition 5.3) then

$$\mathbb{E}[r^{-1}\Phi](\tau_0^\star) + \int_R^\infty r^2 ||D\Phi||^2_{S^2_{\tau_0^\star,r}} \, \mathrm{d}r \lesssim \mathbb{D}[r^{-1}\Phi]$$

and

$$\mathbb{E}[r^{-1}\Psi](\tau_0^\star) + \int_R^\infty r^2 ||D\Psi||^2_{S^2_{\tau_0^\star,r}} \, \mathrm{d}r \lesssim \mathbb{D}[r^{-1}\Psi]$$

where $\mathbb{D}$ is the initial data norm of section 6.1.

Corollary 7.12. *Let Φ be as in Theorem 1. Then one has the estimate*

$$\mathbb{F}[r^{-1}\Phi] + \mathbb{I}[r^{-1}\Phi] \lesssim \mathbb{D}[r^{-1}\Phi]. \tag{7.53}$$

Let now Ψ be as in Theorem 1. Then one has the estimate

$$\mathbb{F}[r^{-1}\Psi] + \mathbb{I}[r^{-1}\Psi] \lesssim \mathbb{D}[r^{-1}\Psi]. \tag{7.54}$$

7.1.2.4 Improving the weights near $r = 3M$

It remains now to remove the degeneration at $r = 3M$ in the bulk estimates (7.36) and (7.37) of Proposition 7.10.

For this we will need the following Lemma which states that the Killing fields of $\left(\mathcal{M}, g_M\right)$ commute with the Regge–Wheeler and Zerilli equations.

Lemma 7.13. *Let $\{\Omega_i\}_{i=1,2,3}$ be a basis for $SO(3)$.*

Let now Φ be a smooth solution to the Regge–Wheeler equation on $\mathcal{M}$ the Cauchy data of which is compactly supported on Σ_R. Then for $i = 1, 2, 3$ $\partial_{t^*}\Phi$ and $\Omega_i(\Phi)$ are also smooth solutions to the Regge–Wheeler equation on $\mathcal{M}$ with Cauchy data that is compactly supported on Σ_R.

Let now Ψ be a smooth solution to the Zerilli equation on $\mathcal{M}$ the Cauchy data of which is compactly supported on Σ_R. Then for $i=1,2,3$ $\partial_{t^*}\Psi$ and $\Omega_i(\Psi)$ are also smooth solutions to the Regge–Wheeler equation on $\mathcal{M}$ with Cauchy data that is compactly supported on Σ_R.

Proof. Direct computation, in particular noting that the operators ∂_{t^*} and Ω_i commute with the operator $\not\Delta$ and hence with the operator $\not\zeta^{[p]}$ for any p. □

Utilising the above lemma thus allows one to remove the degeneration at $r=3M$ as follows.

In what follows, we recall the norm $\mathbb{I}_{\text{loc}}$ defined as in section 6.1.

Proposition 7.14. *Let $\tau_1^\star \geq \tau_0^\star$ be a real number.*

Let now Φ be as in Theorem 1. Then one has the integrated local energy decay estimate

$$\mathbb{I}_{\text{loc}}[r^{-1}\Phi](\tau_1^\star) \lesssim \mathbb{E}[r^{-1}\Phi](\tau_1^\star) + \mathbb{E}[r^{-1}\partial_{t^*}\Phi](\tau_1^\star) + \sum_{i=1}^{3}\mathbb{E}[\Omega_i(r^{-1}\Phi)](\tau_1^\star). \tag{7.55}$$

Let now Ψ be as in Theorem 1. Then one has the integrated local energy decay estimate

$$\mathbb{I}_{\text{loc}}[r^{-1}\Psi](\tau_1^\star) \lesssim \mathbb{E}[r^{-1}\Psi](\tau_1^\star) + \mathbb{E}[r^{-1}\partial_{t^*}\Psi](\tau_1^\star) + \sum_{i=1}^{3}\mathbb{E}[\Omega_i(r^{-1}\Psi)](\tau_1^\star). \tag{7.56}$$

Proof. We recall from the first half of Corollary 7.8 the estimate

$$\int_{\tau_1^\star}^{\infty}\int_{2M}^{R} ||\mu\,\partial_{t^*}\Phi + (1-\mu)\,\partial_r\Phi||^2_{S^2_{\tau^\star,r}}\,\mathrm{d}\tau^\star\,\mathrm{d}r \lesssim \mathbb{E}[\Phi](\tau_1^\star). \tag{7.57}$$

Recalling further from the first half of Proposition 7.10 the bound

$$\int_{\tau_1^\star}^{\infty}\int_{2M}^{R}\Big((r-3M)^2||\partial_r\Phi||^2_{S^2_{\tau^\star,r}} + ||\Phi||^2_{S^2_{\tau^\star,r}}\Big)\,\mathrm{d}\tau^\star\,\mathrm{d}r \lesssim \mathbb{E}[r^{-1}\Phi](\tau_1^\star), \tag{7.58}$$

we thus have from the first half of Lemma 7.13 the estimate

$$\int_{\tau_1^\star}^{\infty}\int_{2M}^{R}\Big(||\partial_{t^*}\Phi||^2_{S^2_{\tau^\star,r}} + ||\not\nabla\Phi||^2_{S^2_{\tau^\star,r}}\Big)\,\mathrm{d}\tau^\star\,\mathrm{d}r \lesssim \mathbb{E}[r^{-1}\partial_{t^*}\Phi](\tau_1^\star) + \sum_{i=1}^{3}\mathbb{E}[r^{-1}\Omega_i(\Phi)](\tau_1^\star). \tag{7.59}$$

The estimate (7.55) then follows from summing estimates (7.57)-(7.59).

Finally, the estimate (7.56) follows in a similar fashion and this subsequently completes the proposition. □

An immediate corollary of the above proposition and Proposition 7.7 is the boundedness of smooth and compactly supported solutions to the Regge–Wheeler and Zerilli equations in the norm $\mathbb{M}$ of section 6.1. This in particular yields, when combined with Corollary 7.12, the ($n = 0$ case) of parts *ii*) in Theorem 1.

In what follows, we recall the higher order norms defined in section 6.1. In addition, we note that for smooth functions f on $\mathcal{M}$ one has $|r\nabla\!\!\!/ f|^2_{\slashed{g}_M} \lesssim \sum_{i=1}^3 |\Omega_i(f)|^2_{\slashed{g}_M} \lesssim |r\nabla\!\!\!/ f|^2_{\slashed{g}_M}$.

Corollary 7.15. *Let Φ be as in Theorem 1. Then one has the estimate*

$$\mathbb{M}[r^{-1}\Phi] \lesssim \mathbb{D}^1[r^{-1}\Phi].$$

Let now Ψ be as in Theorem 1. Then one has the estimate

$$\mathbb{M}[r^{-1}\Psi] \lesssim \mathbb{D}^1[r^{-1}\Psi].$$

Remark 43. To prove Proposition 7.14 it in fact suffices to commute only with the operator ∂_{t^*}. Indeed, once $\partial_{t^*}\Phi$ and $\partial_{t^*}\Psi$ are controlled non-degenerately, choosing a sufficiently regular $\mathfrak{g}$ in the proof of Proposition 7.7 allows one to in addition remove the degeneration on the angular terms. This in particular allows one to weaken the norm that appears on the right hand side of estimates (7.55) and (7.56) and therefore also in the above.

7.1.3 Higher order estimates

In this section we prove the higher order cases of both parts *i*) and *ii*) in Theorem 1.

In fact, with the bounds of section 7.1.2 now understood, their higher order versions follow entirely analgously as to how one derives the higher order estimates for the scalar wave as discussed in section 2.5.4 of the overview. For this reason we shall not give an explicit proof of the following proposition explicitly in this thesis.

Proposition 7.16. *Let $\tau_2^\star \geq \tau_1^\star \geq \tau_0^\star$ and $1 \leq p \leq 2$ be three real numbers with $n \geq 0$ an integer.*

Let now Φ be as in Theorem 1. Then one has the energy estimate

$$\mathbb{E}^n[r^{-1}\Phi](\tau_2^\star) \lesssim \mathbb{E}^n[r^{-1}\Phi](\tau_1^\star), \tag{7.60}$$

the integrated local energy decay estimate

$$\mathbb{I}^n_{\mathrm{loc}}[r^{-1}\Phi](\tau_1^\star) \lesssim \mathbb{E}^n[r^{-1}\Phi](\tau_1^\star) + \mathbb{E}^n[r^{-1}\partial_{t^*}\Phi](\tau_1^\star) + \sum_{i=1}^{3} \mathbb{E}^n[r^{-1}\Omega_i(\Phi)](\tau_1^\star) \tag{7.61}$$

and the p-weighted hierarchy

$$\mathbb{F}^n_p[r^{-1}\Phi](\tau^\star_2)+\mathbb{B}^n_p[r^{-1}\Phi](\tau^\star_1)\lesssim\mathbb{E}^n[r^{-1}\Phi](\tau^\star_1)+\mathbb{F}^n_p[r^{-1}\Phi](\tau^\star_1).\tag{7.62}$$

In addition, parts i) and ii) in the first half of Theorem 1 hold true.

Let now Ψ be as in Theorem 1. Then one has the energy estimate

$$\mathbb{E}^n[r^{-1}\Psi](\tau^\star_2)\lesssim\mathbb{E}^n[r^{-1}\Psi](\tau^\star_1),\tag{7.63}$$

the integrated local energy decay estimate

$$\mathbb{I}^n_{\mathrm{loc}}[r^{-1}\Psi](\tau^\star_1)\lesssim\mathbb{E}^n[r^{-1}\Psi](\tau^\star_1)+\mathbb{E}^n[r^{-1}\partial_{t^*}\Psi](\tau^\star_1)+\sum_{i=1}^{3}\mathbb{E}^n[r^{-1}\Omega_i(\Psi)](\tau^\star_1)\tag{7.64}$$

and the p-weighted hierarchy

$$\mathbb{F}^n_p[r^{-1}\Psi](\tau^\star_2)+\mathbb{B}^n_p[r^{-1}\Psi](\tau^\star_1)\lesssim\mathbb{E}^n[r^{-1}\Psi](\tau^\star_1)+\mathbb{F}^n_p[r^{-1}\Psi](\tau^\star_1).\tag{7.65}$$

In addition, parts i) and ii) in the first half of Theorem 1 hold true.

7.1.4 Pointwise decay bounds

In this section we prove both parts *iii*) of Theorem 1 using the higher order estimates of Proposition 7.16.

Indeed, the desired pointwise bounds follow as an immediate consequence of the following proposition utilised in conjuncton with a Sobolev embedding on the 2-spheres $S^2_{\tau^\star,r}$ and an application of the fundamental theorem of calculus.

Proposition 7.17. *Let $\tau^\star\geq\tau^\star_0$ be a real number with $n\geq 0$ an integer.*

Let now Φ be as in Theorem 1. Then one has the decay estimate

$$\mathbb{E}^n[\Phi](\tau^\star)\lesssim\frac{1}{\tau^{\star 2}}\,\mathbb{D}^{n+2}[\Phi].\tag{7.66}$$

Let now Ψ be as in Theorem 1. Then one has the decay estimate

$$\mathbb{E}^n[\Psi](\tau^\star)\lesssim\frac{1}{\tau^{\star 2}}\,\mathbb{D}^{n+2}[\Psi].\tag{7.67}$$

Proof. The proof proceeds entirely analogously as to the proof of energy decay for solutions to the scalar wave equation on $\left(\mathcal{M},g_M\right)$ established by Dafermos and Rodnianski in[35]. For this reason we will not provide a plethora of details in what is follow.

We first observe that the estimates (7.62) and (7.61) in the first half of Proposition 7.16

imply that there exists a dyadic[4] sequence of times $(\tau^\star_n)_{n\in\mathbb{N}}$ such that

$$\mathbb{E}^n[\Phi](\tau^\star_n) \lesssim \frac{1}{(\tau^\star_n)^2}\,\mathbb{D}^n[\Phi] + \frac{1}{\tau^\star_n}\Big(\mathbb{E}^n[\partial_{t^*}\Phi](\tau^\star_n) + \sum_{i=1}^{3}\mathbb{E}^n[\Omega_i(\Phi)](\tau^\star_n)\Big).$$

Thus, combining the first half of Lemma 7.13 with the estimate (7.60) in the first half of Proposition 7.16 we therefore conclude that for each $\tau^\star_n$ one in fact has the bound

$$\mathbb{E}^n[\Phi](\tau^\star_n)\,\mathrm{d}\tau\,\mathrm{d}r \lesssim \frac{1}{(\tau^\star_n)^2}\,\mathbb{D}^{n+2}[\Phi].$$

The estimate (7.66) then follows from estimate (7.60) of Proposition 7.10 and the fact that the sequence $(\tau^\star_n)$ is dyadic.

Finally, the estimate (7.67) follows in a similar fashion and this subsequently completes the proposition. □

This completes the proof of Theorem 1.

7.2 Proof of Theorem 2

In this section we prove Theorem 2.

The proof in fact essentially follows from Theorem 1 combined with Proposition 5.7 and Corollary 5.8.

7.2.1 Boundedness and decay for the pure gauge and linearised Kerr invariant quantities

We begin in this section by first applying Theorem 1 to the gauge-invariant quantities $\overset{\scriptscriptstyle(1)}{\Phi}$ and $\overset{\scriptscriptstyle(1)}{\Psi}$ associated to the solution $\overset{\diamond}{\mathscr{S}}$ of Theorem 2. Indeed, this is immediately applicable courtesy of Theorem 4.3 and the fact that $\overset{\scriptscriptstyle(1)}{\Phi}$ and $\overset{\scriptscriptstyle(1)}{\Psi}$ were assumed to have compactly supported Cauchy data. We thus have:

Proposition 7.18. *Let $\overset{\diamond}{\mathscr{S}}$ be as in the statement of Theorem 1. Then the quantities $\overset{\scriptscriptstyle(1)}{\Phi}$ and $\overset{\scriptscriptstyle(1)}{\Psi}$ assoicated to $\overset{\diamond}{\mathscr{S}}$ satisfy the assumptions and hence the conclusions of Theorem 1.*

7.2.2 Completing the proof of Theorem 2

In this section we complete the proof of Theorem 2 with the aid of Corollary 5.8.

[4] A sequence such that $\tau^\star_n \leq \tau^\star_{n+1} \leq 2^k\tau^\star_n$ for some fixed integer k.

Proof of Theorem 2. We have from Corollary 5.8 that the solution $\overset{\diamond}{\mathscr{S}}$ satisfies

$$
\begin{aligned}
\hat{\tilde{\overset{\scriptscriptstyle(1)}{g}}} &= \widetilde{\nabla}\hat{\otimes}\tilde{\mathrm{d}}^{\mathcal{I}}\Big(r\overset{\scriptscriptstyle(1)}{\Psi}\Big) + 6\mu\,\mathrm{d}r\hat{\tilde{\otimes}}\not{\zeta}^{[1]}\tilde{\mathrm{d}}\overset{\scriptscriptstyle(1)}{\Psi}, \\
\tilde{\overset{\scriptscriptstyle(1)}{g}} &= 0, \\
\overset{\scriptscriptstyle(1)}{\not{g}} &= \not{\mathcal{D}}_1^{\star}\Big(\tilde{\mathrm{d}}^{\mathcal{I}}\Big(r\overset{\scriptscriptstyle(1)}{\Psi}\Big) - 2\tilde{\mathrm{d}}r\,\overset{\scriptscriptstyle(1)}{\Psi}, \tilde{\mathrm{d}}^{\mathcal{I}}\Big(r\overset{\scriptscriptstyle(1)}{\Phi}\Big) - 2\tilde{\mathrm{d}}r\,\overset{\scriptscriptstyle(1)}{\Phi}\Big), \\
\hat{\overset{\scriptscriptstyle(1)}{\not{g}}} &= r\not{\nabla}\hat{\otimes}\not{\mathcal{D}}_1^{\star}\Big(\overset{\scriptscriptstyle(1)}{\Psi}, \overset{\scriptscriptstyle(1)}{\Phi}\Big), \\
\not{\mathrm{tr}}\overset{\scriptscriptstyle(1)}{\not{g}} &= 4\tilde{\mathrm{d}}_P^{\mathcal{I}}\overset{\scriptscriptstyle(1)}{\Psi} + 12\mu r^{-1}(1-\mu)\not{\zeta}^{[1]}\overset{\scriptscriptstyle(1)}{\Psi}
\end{aligned}
$$

where $\overset{\scriptscriptstyle(1)}{\Phi}$ and $\overset{\scriptscriptstyle(1)}{\Psi}$ are as in Proposition 7.18. All the estimates that were derived for solutions to the Regge–Wheeler and Zerilli equations in section 7.1 can thus be shown to hold for the solution $\overset{\diamond}{\mathscr{S}}$ (but with an additional r weight placed on $\overset{\diamond}{\mathscr{S}}$) by dilligently commuting and evaluating the above expressions in the frame $\{\partial_{t^*}, \partial_r\}$ of section 3.2.2.3, keeping careful track of r-weights, and then applying the higher order estimates of Proposition 7.16. This in particular yields the pojntwise decay bounds of part *iii*) in the statement of Theorem 2 courtesy of the Sobolev embedding on 2-spheres. Since however this would be rather cumbersome to carry out in practice we only note the key points:

- Propositions 3.1 and 3.2 allows one to perfom all necessary computations in the $\{\partial_{t^*}, \partial_r\}$ frame. In particular, the connection coefficients in this frame are of order $O(r^{-2})$ and hence play no role when evaluating the (commuted) tensorial expressions
- to control higher order angular derivatives of the solution $\overset{\diamond}{\mathscr{S}}$ one commutes with the family of angular operators $\not{\mathcal{A}}_f^{[k]}, \not{\mathcal{A}}_\xi^{[k]}$ and $\not{\mathcal{A}}_\theta^{[k]}$ of section 3.2.6.2, noting the commutation relations of Lemma 3.14, and then apply the elliptic estimates of Proposition 3.10
- by definition of the flux and integrated decay norms the derivatives D and $\not{\nabla}$ always appear with an additional r-weight, thus gaining in regularity towards $\mathcal{I}^+$
- by Lemma 3.3 'contracting' the operator $\tilde{\mathrm{d}}^{\mathcal{I}}$ in the frame $\{\partial_{t^*}, \partial_r\}$ always returns a D derivative which gains an r-weight by the previous point
- to bound the (commuted) terms involving the operator $\not{\zeta}^{[1]}$ one applies the commutation relations of Lemma 3.14 along with the estimates of Proposition 3.12

$\square$

We make the following remark.

Remark 44. We emphasize the role played by the operator $\tilde{\mathrm{d}}^{\mathcal{I}}$ in preserving the estimates as one transfroms from $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ to the r-weighted collection given by $r\overset{\diamond}{\mathscr{S}}$ (cf. section A.2.2 in the appendix).

A

The Regge–Wheeler gauge

In this chapter of the appendix we consider the so called Regge–Wheeler gauge and show how it connects to the modified Regge–Wheeler gauge associated to the equations of linearised gravity as defined in section 5.2.1. In particular, we recall from the discussion at the end of section 2.3.1.1 in the overview that this former gauge played an important role in the definition of the map f of section 3.3.1.

We note that this section is a summarised version of our recent [5].

A.1 The linearised equations

The Regge–Wheeler gauge is to be realised as a 'residual gauge' choice associated to the system of equations that result from linearising the Einstein vacuum equations, as they are expressed in a generalised $\mathring{f}$-wave gauge with respect to g_M, about $\left(\mathcal{M}, g_M\right)$. Here, $\mathring{f}$ is the map of section 3.3.1.1.

Definition A.1. *Let $\mathfrak{S}$ denote the collection*

$$\mathfrak{S} := \left(\overset{(1)}{\hat{\tilde{\mathfrak{g}}}}, \mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{\mathfrak{g}}}, \overset{(1)}{\not{\mathfrak{g}}}, \overset{(1)}{\hat{\not{\mathfrak{g}}}}, \not{\mathrm{t}}\mathrm{r}\overset{(1)}{\not{\mathfrak{g}}}\right)$$

where

- $\overset{(1)}{\hat{\tilde{\mathfrak{g}}}}$ *is a smooth, symmetric, traceless 2-covariant $\mathcal{Q}$-tensor field*
- $\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{\mathfrak{g}}}$ *is a smooth function on $\mathcal{M}$*
- $\overset{(1)}{\not{\mathfrak{g}}}$ *is a smooth $\mathcal{Q} \otimes S$ 1-form*

- $\overset{(1)}{\hat{\slashed{\mathfrak{g}}}}$ *is a smooth symmetric, traceless 2-covariant S-tensor field*
- $\slashed{\mathrm{tr}}\overset{(1)}{\slashed{\mathfrak{g}}}$ *is a smooth function on* $\mathcal{M}$

Then we say that $\mathfrak{S}$ *is a smooth solution to the linearised Einstein equations on* $\left(\mathcal{M}, g_M\right)$ *in a generalised* $\mathring{f}$*-wave gauge with respect to* g_M *iff each quantity in* $\mathfrak{S}$ *satisfies the following system of equations:*

$$\widetilde{\Box}\overset{(1)}{\hat{\tilde{\mathfrak{g}}}} + \slashed{\Delta}\overset{(1)}{\hat{\tilde{\mathfrak{g}}}} + \frac{2}{r}\widetilde{\nabla}_P\overset{(1)}{\hat{\tilde{\mathfrak{g}}}} - \frac{2}{r}\Big(\widetilde{\nabla}\hat{\otimes}\overset{(1)}{\hat{\tilde{\mathfrak{g}}}}\Big)_P - \frac{4}{r}\frac{\mu}{r}\overset{(1)}{\hat{\tilde{\mathfrak{g}}}} = \frac{1}{r}\,\mathrm{d}r\hat{\tilde{\otimes}}\tilde{\mathrm{d}}\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{\mathfrak{g}}} + \frac{2}{r}\,\mathrm{d}r\hat{\tilde{\otimes}}\slashed{\mathrm{div}}\overset{(1)}{\not{\mathfrak{g}}} - \frac{1}{r}\,\mathrm{d}r\hat{\tilde{\otimes}}\tilde{\mathrm{d}}\slashed{\mathrm{tr}}\overset{(1)}{\slashed{\mathfrak{g}}}, \tag{A.1}$$

$$\widetilde{\Box}\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{\mathfrak{g}}} + \slashed{\Delta}\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{\mathfrak{g}}} = 0. \tag{A.2}$$

$$\widetilde{\Box}\overset{(1)}{\not{\mathfrak{g}}} + \slashed{\Delta}\overset{(1)}{\not{\mathfrak{g}}} - \frac{2}{r}\widetilde{\nabla}\otimes\overset{(1)}{\not{\mathfrak{g}}}_P - \frac{1}{r^2}(1-\mu)\overset{(1)}{\not{\mathfrak{g}}} + \frac{2}{r^2}\,\mathrm{d}r\otimes_{\mathrm{s}}\overset{(1)}{\not{\mathfrak{g}}}_P = \frac{2}{r}\tilde{\mathrm{d}}r\otimes_{\mathrm{s}}\slashed{\mathrm{div}}\overset{(1)}{\hat{\slashed{\mathfrak{g}}}}. \tag{A.3}$$

$$\widetilde{\Box}\overset{(1)}{\hat{\slashed{\mathfrak{g}}}} + \slashed{\Delta}\overset{(1)}{\hat{\slashed{\mathfrak{g}}}} - \frac{2}{r}\widetilde{\nabla}_P\overset{(1)}{\hat{\slashed{\mathfrak{g}}}} - \frac{4}{r}\frac{\mu}{r}\overset{(1)}{\hat{\slashed{\mathfrak{g}}}} = 0, \tag{A.4}$$

$$\widetilde{\Box}\slashed{\mathrm{tr}}\overset{(1)}{\slashed{\mathfrak{g}}} + \slashed{\Delta}\slashed{\mathrm{tr}}\overset{(1)}{\slashed{\mathfrak{g}}} + \frac{2}{r}\widetilde{\nabla}_P\slashed{\mathrm{tr}}\overset{(1)}{\slashed{\mathfrak{g}}} + \frac{2}{r^2}\slashed{\mathrm{tr}}\overset{(1)}{\slashed{\mathfrak{g}}} = \frac{4}{r^2}\overset{(1)}{\hat{\tilde{\mathfrak{g}}}}_{PP} + \frac{2}{r^2}\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{\mathfrak{g}}}. \tag{A.5}$$

$$-\tilde{\delta}\overset{(1)}{\hat{\tilde{\mathfrak{g}}}} + \slashed{\mathrm{div}}\overset{(1)}{\not{\mathfrak{g}}} - \frac{1}{2}\tilde{\mathrm{d}}\slashed{\mathrm{tr}}\overset{(1)}{\slashed{\mathfrak{g}}} = 0, \tag{A.6}$$

$$-\tilde{\delta}\overset{(1)}{\not{\mathfrak{g}}} - \frac{1}{2}\slashed{\nabla}\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{\mathfrak{g}}} + \slashed{\mathrm{div}}\overset{(1)}{\hat{\slashed{\mathfrak{g}}}} = 0. \tag{A.7}$$

Note that one can indeed verify from the formal linearisation procedure of section 3.3 that the system of equations (A.1)-(A.7) define the system of equations that result from linearising the Einstein vacuum equations, as they are expressed in a generalised $\mathring{f}$-wave gauge with respect to g_M, about $\left(\mathcal{M}, g_M\right)$. In particular, we note that the map $\mathring{f}$ satisfies $\mathring{f}(g_M) = 0$. Moreover, the motivation behind choosing such a map is made manifest by the observation that this choice of map forces the pair of quantities $\overset{(1)}{\not{\mathfrak{g}}}$ and $\overset{(1)}{\hat{\slashed{\mathfrak{g}}}}$ to actually decouple from the rest of the system.

Finally, we note that there exists analogous definitions of pure gauge and linearised Kerr solutions associated to the system (A.1)-(A.7) just as in section 3.4. In addition, one can formulate a well-posedness theory for the system (A.1)-(A.7) analogously to the one established in section 5.1 for the equations of linearised gravity. See[5] for specifics.

A.2 The Regge–Wheeler gauge

In this section we define what it means for a solution $\mathfrak{S}$ to the linearised Einstein equations on $\left(\mathcal{M}, g_M\right)$ in a generalised $\mathring{f}$-wave gauge with respect to g_M to be in the Regge–Wheeler gauge.

A.2.1 The Regge–Wheeler gauge

The definition is as follows.

In what follows, we recall the notation $\mathbb{C}_\Sigma f$ from section 5.2.1 to denote the mapping onto Cauchy data of f on Σ. Note also that the projection maps of section 5.2.1.1 apply readily to the collection $\mathfrak{S}$.

Definition A.2. *Let $\mathfrak{S}$ be a solution to the linearised Einstein equations on $\left(\mathcal{M}, g_M\right)$ in a generalised $\mathring{f}$-wave gauge with respect to g_M. Then we say that $\mathfrak{S}$ is in the Regge–Wheeler gauge iff the following conditions hold on Σ:*

- *the quantities $\overset{(1)}{\mathfrak{g}\!\!\!-}$ and $\overset{(1)}{\hat{\not{\mathfrak{g}}}}$ associated to the projection $\mathfrak{S}'$ satisfy*

$$
\mathbb{C}_\Sigma \overset{(1)}{\mathfrak{g}\!\!\!-}_{\mathrm{e}} = 0,
$$
$$
\mathbb{C}_\Sigma \overset{(1)}{\hat{\not{\mathfrak{g}}}} = 0
$$

Note that since the condition for the collection $\mathfrak{S}$ being in the Regge–Wheeler gauge is a condition on Cauchy data associated to $\mathfrak{S}$, as in section 5.2.1 one can say that a solution $\mathfrak{S}$ to the linearised Einstein equations on $\left(\mathcal{M}, g_M\right)$ in a generalised $\mathring{f}$-wave gauge with respect to g_M that is in the Regge–Wheeler gauge is initial-data-normalised.

We further note that one can prove an analogous result to Theorem 5.5 in that given any smooth solution $\mathfrak{S}$ to the linearised Einstein equations on $\left(\mathcal{M}, g_M\right)$ in a generalised $\mathring{f}$-wave gauge with respect to g_M then one can add to it a pure gauge solution to the linearised Einstein equations on $\left(\mathcal{M}, g_M\right)$ in a generalised $\mathring{f}$-wave gauge with respect to g_M such that the resulting solution is in the Regge–Wheeler gauge – see[5].

A.2.2 Global properties of solutions in the Regge–Wheeler gauge

As in section 5.2.3 we have that the conditions associated to a solution $\mathfrak{S}$ to the linearised Einstein equations on $\left(\mathcal{M}, g_M\right)$ in a generalised $\mathring{f}$-wave gauge with respect to g_M being in the Regge–Wheeler gauge actually propagate under evolution by the equations (A.1)-(A.7).

Proposition A.1. *Let $\mathfrak{S}$ be a smooth solution to the linearised Einstein equations on $\left(\mathcal{M}, g_M\right)$ in a generalised $\mathring{f}$-wave gauge with respect to g_M that is in the Regge–Wheeler gauge. Then the following hold on $D^+(\Sigma)$:*

- *the quantities $\overset{(1)}{\mathfrak{g}}$ and $\hat{\overset{(1)}{\not{\mathfrak{g}}}}$ associated to the projection $\mathfrak{S}'$ satisfy*

$$
\begin{aligned}
\overset{(1)}{\mathfrak{g}}_{\mathrm{e}} &= 0,\\
\hat{\overset{(1)}{\not{\mathfrak{g}}}} &= 0
\end{aligned}
$$

Moreover, the quantity $\mathrm{tr}_{\tilde{g}_M}\tilde{\overset{(1)}{\mathfrak{g}}}$ associated to the projection $\mathfrak{S}'$ vanishes on $D^+(\Sigma)$:

$$
\mathrm{tr}_{\tilde{g}_M}\tilde{\overset{(1)}{\mathfrak{g}}} = 0.
$$

Proof. Proceed analogously as in the proof of Proposition A.1. See[5] for details. □

As previously, we have the following corollary to Proposition A.1.

Corollary A.2. *Let $\mathfrak{S}$ be as in Proposition 5.7. Then the following identities hold on $D^+(\Sigma)$:*

$$
\begin{aligned}
\hat{\tilde{\overset{(1)}{\mathfrak{g}}}} &= \widetilde{\nabla}\hat{\otimes}\tilde{\mathrm{d}}\Big(r\overset{(1)}{\Psi}\Big) + 6\mu\,\mathrm{d}r\hat{\tilde{\otimes}}\not{C}^{[1]}\tilde{\mathrm{d}}\overset{(1)}{\Psi},\\
\mathrm{tr}_{\tilde{g}_M}\tilde{\overset{(1)}{\mathfrak{g}}} &= 0,\\
\overset{(1)}{\mathfrak{g}} &= -\not{\mathcal{D}}_1^\star\Big(0, \tilde{\star}\tilde{\mathrm{d}}\Big(r\overset{(1)}{\Phi}\Big)\Big),\\
\hat{\overset{(1)}{\not{\mathfrak{g}}}} &= 0,\\
\not{\mathrm{tr}}\overset{(1)}{\not{\mathfrak{g}}} &= -2r\not{\Delta}\overset{(1)}{\Psi} + 4\tilde{\mathrm{d}}_P\overset{(1)}{\Psi} + 12\mu r^{-1}(1-\mu)\not{C}^{[1]}\overset{(1)}{\Psi}
\end{aligned}
$$

where $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ satisfy the Regge–Wheeler and Zerilli equations respectively.

Proof. Proceed analogously as in the proof of Corollary 5.8. See[5] for details. □

We note that one can use Corollary A.2 to show that sufficiently regular solutions to the linearised Einstein equations on $\left(\mathcal{M}, g_M\right)$ in a generalised $\mathring{f}$-wave gauge with respect to g_M that are in the Regge–Wheeler gauge actually decay to a member of the linearised Kerr family, with the the caveat that the solutions are not asymptotically flat as in Theorem 2 (cf. Remark 40.). See[5] for the full proof.

We make the following remark.

Remark 45. We note that the ansatz given above for the collection of quantities $\hat{\tilde{\overset{(1)}{\mathfrak{g}}}}, \mathrm{tr}_{\tilde{g}_M}\tilde{\overset{(1)}{\mathfrak{g}}}, \overset{(1)}{\mathfrak{g}}, \hat{\overset{(1)}{\not{\mathfrak{g}}}}$ and $\not{\mathrm{tr}}\overset{(1)}{\not{\mathfrak{g}}}$ was the original definition of the Regge–Wheeler gauge for solutions to the linearised Einstein equations given by Regge and Wheeler in[23]. In particular, it is important to realise the distinction between Definition A.2 – see the discussion at the end of section 2.3.1.1 of the overview.

A.3 The connection with the modified Regge–Wheeler gauge

Finally, we now reveal the connection between solutions to the linearised Einstein equations on $\left(\mathcal{M}, g_M\right)$ in a generalised $\mathring{f}$-wave gauge with respect to g_M that are in the Regge–Wheeler gauge and solutions to the equations of linearised gravity that are in the modified Regge–Wheeler gauge.

In what follows, we note that one can prove the same versions of Proposition 4.1 and Theorem 4.3 for the linearised Einstein equations on $\left(\mathcal{M}, g_M\right)$ in a generalised $\mathring{f}$-wave gauge with respect to g_M (cf. in particular Remark 15). See[5] for specifics.

Proposition A.3. *Let $\mathfrak{S}$ be a smooth solution to the linearised Einstein equations on $\left(\mathcal{M}, g_M\right)$ in a generalised $\mathring{f}$-wave gauge with respect to g_M that has vanishing projection to $l = 0, 1$, that is moreover in the Regge–Wheeler gauge, and let $\overset{(1)}{\Phi}$ and $\overset{(1)}{\Psi}$ be the gauge-invariant quantities associated to it in accordance with Proposition 4.1. Moreover, let*

$$\mathfrak{G} := \left(\overset{(1)}{\hat{\tilde{\mathfrak{g}}}}, \mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{\mathfrak{g}}}, \overset{(1)}{\mathfrak{g}\!\!\!-}, \overset{(1)}{\hat{\not{\mathfrak{g}}}}, \not{\mathrm{t}}\mathrm{r}\overset{(1)}{\not{\mathfrak{g}}}\right)$$

denote the collection of quantities

$$\begin{aligned}
\overset{(1)}{\hat{\tilde{\mathfrak{g}}}} &= \widetilde{\nabla}\hat{\otimes}\tilde{\mathfrak{v}}, \\
\mathrm{tr}_{\tilde{g}_M}\overset{(1)}{\tilde{\mathfrak{g}}} &= -2\tilde{\delta}\tilde{\mathfrak{v}}, \\
\overset{(1)}{\mathfrak{g}\!\!\!-} &= \not{\nabla}\tilde{\mathfrak{v}} + \tilde{\mathrm{d}}\not{\mathfrak{v}} - \frac{2}{r}\tilde{\mathrm{d}}r\otimes_{\mathrm{s}}\not{\mathfrak{v}}, \\
\overset{(1)}{\hat{\not{\mathfrak{g}}}} &= \not{\nabla}\hat{\otimes}\not{\mathfrak{v}}, \\
\not{\mathrm{t}}\mathrm{r}\overset{(1)}{\not{\mathfrak{g}}} &= 2\mathrm{d}\not{\mathrm{i}}\mathrm{v}\not{\mathfrak{v}} + \frac{4}{r}\tilde{\mathfrak{v}}_P
\end{aligned}$$

where $\tilde{\mathfrak{v}}$ and $\not{\mathfrak{v}}$ are the smooth $\mathcal{Q}$ 1-form and smooth S 1-form defined according to

$$\begin{aligned}
\tilde{\mathfrak{v}} &: = -\tilde{\star}\tilde{\mathrm{d}}\left(r\overset{(1)}{\Psi}\right), \\
\not{\mathfrak{v}} &: = r\not{\mathcal{D}}_1^\star\left(\overset{(1)}{\Psi}, \overset{(1)}{\Phi}\right).
\end{aligned}$$

Then the collection $\mathring{\mathscr{S}}$ defined according to

$$\mathring{\mathscr{S}} := \mathfrak{S} + \mathfrak{G}$$

is a smooth solution to the equations of linearised gravity that is in a modified Regge–Wheeler gauge with trivial parameters.

In addition, the quantities $\tilde{v}$ and $\not{\mathfrak{p}}$ satisfy the system of equations

$$\begin{aligned}
\widetilde{\Box}\tilde{\mathfrak{v}} + \not{\Delta}\tilde{\mathfrak{v}} - \frac{2}{r}\Big(\widetilde{\nabla}\tilde{\mathfrak{v}}\Big)_P + \frac{2}{r^2}\,\mathrm{d}r\,\tilde{\mathfrak{v}}_P &= -\frac{1}{r^2}\tilde{\star}\tilde{\mathrm{d}}\Big(r^3\mathfrak{Z}\Big(\overset{\scriptscriptstyle(1)}{\Psi}\Big)\Big),\\
\widetilde{\Box}\not{\mathfrak{p}} + \not{\Delta}\not{\mathfrak{p}} - \frac{2}{r}\widetilde{\nabla}_P\not{\mathfrak{p}} + \frac{1}{r^2}(3-4\mu)\not{\mathfrak{p}} &= \frac{2}{r}(1-2\mu)\not{\mathcal{D}}_1^\star\Big(\overset{\scriptscriptstyle(1)}{\Psi},\overset{\scriptscriptstyle(1)}{\Phi}\Big) + r\not{\nabla}\mathfrak{Z}\Big(\overset{\scriptscriptstyle(1)}{\Psi}\Big)
\end{aligned}$$

where $\mathfrak{Z}$ is the operator defined as in section 3.3.1.3.

Proof. Direct computation, noting in particular that the gauge-invariant quantities $\overset{\scriptscriptstyle(1)}{\Phi}$ and $\overset{\scriptscriptstyle(1)}{\Psi}$ associated to $\mathfrak{S}$ and $\mathring{\mathscr{S}}$ agree from Remark 15. $\square$

In particular, we see that solutions to the equations of linearised gravity that are in the modified Regge–Wheeler gauge are derived from solutions to the linearised Einstein equations on $\Big(\mathcal{M}, g_M\Big)$ in a generalised $\mathring{f}$-wave gauge with respect to g_M that are in the Regge–Wheeler gauge via a collection of quantities that almost define a pure gauge solution of Proposition 3.15, and it is their failure to do so which defines the map $\mathring{f}$ of section 3.3.1.3.

Finally, the reason for considering the pair $\tilde{\mathfrak{v}}$ and $\not{\mathfrak{p}}$ as potential generators for a pure gauge solution is due to the good asymptotic properties of the operator $\tilde{\mathrm{d}}^{\mathcal{I}}$ (cf. Lemma 3.3).

B

The restricted nonlinear stability conjecture for the Schwarzschild exterior family of Dafermos, Holzegel and Rodnianski

In this section of the appendix we state the restricted nonlinear stability conjecture of Dafermos, Holzegel and Rodnianski regarding the Schwarzschild exterior family which should in principle be possible to resolve in the affirmative by utilising a generalised wave gauge combined with the insights of this thesis.

The conjecture, lifted verbatim from [1], is as follows.

Conjecture (Dafermos–Holzegel–Rodnianski)**.** Let $(\Sigma_M, \bar{g}_M, K_M)$ be the induced data on a spacelike asymptotically flat slice of the Schwarzschild solution of mass M crossing the future horizon and bounded by a trapped surface. Then in the space of all nearby vacuum data $(\Sigma, \bar{g}, K)$, in a suitable norm, there exists a codimension-3 subfamily for which the corresponding maximal vacuum Cauchy development $(\mathcal{M}, g)$ contains a black-hole exterior region (characterized as the past $J^-(\mathcal{I}^+)$ of a complete future null infinity $\mathcal{I}^+$), bounded by a non-empty future affine-complete event horizon $\mathcal{H}^+$, such that in $J^-(\mathcal{I}^+)$ (a) the metric remains close to g_M and moreover (b) asymptotically settles down to a nearby Schwarzschild metric $g_{\tilde{M}}$ at suitable inverse polynomial rates.

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