Abstract of thesis entitled

MEAN FIELD GAMES WITH IMPERFECT INFORMATION

submitted by

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for the Degree of Joint Doctor of Philosophy
at Imperial College London and The University of Hong Kong
in October 2017

In this thesis, three topics in Mean Field Games in the absence of complete information have been studied.

The first part of the thesis focus on Mean Field Stackelberg Games between a large group of followers and a leader, in such a way that each follower is subject to a delay effect inherited from the leader. The case with delays being identical among the followers in the population is first considered. Under mild assumptions of regular enough coefficients, the whole Stackelberg game problem is solved via stochastic maximum principle. The solution could be represented by a system of six coupled forward backward stochastic differential equations. A comprehensive study on the particular Linear Quadratic case has been provided. By considering the corresponding linear functional, the time-independent sufficient condition which warrants the unique existence of the solution of the whole Stackelberg game is obtained. Several numerical examples are also demonstrated.

The second work studies another class of Stackelberg games, under a Linear Quadratic setting, in the presence with an additional leader. Given the trajectories of the mean field term and two leaders, the follower's optimal control problem is

first solved. Depending on whether or not the leaders cooperate, the solutions of the respective Pareto and Nash games between the leaders are obtained, which can be represented by systems of forward backward stochastic functional differential equations. To numerically implement the obtained results, explicit expression of solutions of the whole problem: Mean Field Game among the followers and Nash (and Pareto) Game between the leaders, are provided. Finally, several examples are given to study the impact of different games on the cost functionals of the followers. An interesting example shows that the population are worse off as the leaders cooperate.

The last part of the thesis studies discrete time partially observable mean field systems in the presence of a common noise. Each player makes decision solely based on the observable processes but not the common noise. Both the mean field game and the associated mean field type stochastic control problem are formulated. The mean field type control problem is solved by adopting the classical discrete time Kalman filter with notable modifications; indeed, the unique existence of the resulting forward-backward stochastic difference system is then established by Separation Principle. The mean field game problem is also solved via an application of stochastic maximum principle, while the existence of the mean field equilibrium is shown by the Schauder's fixed point theorem.

MEAN FIELD GAMES WITH IMPERFECT INFORMATION

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A thesis submitted in partial fulfilment of the requirements for the Degree of Joint Doctor of Philosophy at Imperial College London and The University of Hong Kong

October 2017

DECLARATION

I declare that this thesis represents my own work under the supervision of Prof. Hailiang Yang and Dr Thomas Cass during the period 2014-2017 for the degree of Joint Doctor of Philosophy at Imperial College London and the University of Hong Kong. The work submitted has not been previously included in any thesis, dissertation or report submitted to these Universities or to any other institution for a degree, diploma or other qualification.

CHAU, MAN HO

ACKNOWLEDGEMENTS

The author would like to express his deepest gratitude to his supervisors, Prof. Hailiang Yang and Dr Thomas Cass, for their continual guidance and constant encouragement throughout the period of the author's postgraduate studies and in the preparation of this thesis. I am very grateful to Prof. Phillip Sheung Chi Yam, former supervisor in my Master degree studies, for his dedication in guiding me during my entire postgraduate study, and for introducing me into research frontiers in mathematical finance and mean field games, including the subject of this thesis. Many ideas in this research evolved from countless discussions with him.

I also want to thank Prof. Alain Bensoussan, Dr Chi Chung Siu, Dr Zheng Zhang, Dr Kwok Chuen Wong, Dr Yingying Lai, Dr Alfred Chong and Dr Yiqun Li, for their supports and many helpful discussions.

I am deeply grateful to my parents, for their precious love and supports and, last but not least, to my wife, Francesca, for her love, care and long time support.

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Chapter 1

Introduction

To study and analyze the collective behavior of a dynamical system involving multiple decision makers, one can model the population by stochastic differential games (SDGs). Under this framework, evolutions of individuals' states are described by stochastic differential equations (SDEs), each of which aims at optimizing certain objective functionals (e.g. cost minimization). The interactions between each individual are explicitly given in terms governing the SDEs and the cost functionals. For example, see Varaiya [47], Bensoussan and Frehse [7] for details.

The computational complexity of the solution, for example, to solve for a Nash equilibrium, increases dramatically as the number of particles in the system raises. As an alternative macroscopic approach, Huang, Caines and Malhamé [29, 30] first studied the limiting case - SDGs with infinitely many players. Independently, Lasry and Lions [33, 34, 35] first developed the theory of Mean Field Games (MFGs), which combines SDGs with the *mean field theory* originated in physics describing phase transitions.

Unlike SDGs, which explicitly state the interactions between any two players, each individual in MFGs interacts with the community through a common medium - the *mean field term*. By properly choosing it as a functional of the probability distribution of the population, one can show that the empirical interacting system converges to the analogical mean field system, as the number of particles increases, see for example [10]. In the simplest MFGs, each agent in the population interacts indirectly through the deterministic mean field term only. Individuals can hence be regarded as independent of each other, which significantly simplifies the analysis of the whole dynamical system. To solve a canonical MFG, one first solves the individual's optimal control problem, considering the mean field term as exogenous and given. The second step is to tackle the fixed point problem, by equating function of the probability distribution of an individual, who adopts the optimal control solved in the first step (which is clearly a functional fo the given mean field term), with the mean field term.

Since the celebrated work by Lasry and Lions [33, 34, 35], theories and applications of mean field games - the combination of mean field theory and SDG, has enjoyed a rapid development. Carmona, Delarue and Lacker [19] studies Mean Field Games with a common noise on top of the whole population. Unlike the canonical case with a deterministic mean field term, the presence of a common noise introduces randomness in that common medium. From modeling perspective, the common noise could be regarded as an external economic factor affecting simultaneously the whole population. From the application perspective, Carmona, Fouque and Sun [20] considers an interbank borrowing-lending model under the mean field game framework. The log-monetary reserve of each bank is affected not only by individual independent noises but also by a common noise, which introduces a drastic effect on the dynamics and the value function of each bank, and hence results in systemic risk. For other works in MFGs, also see Andersson and Djehiche [1], Bardi [2], Bensoussan, Frehse and Yam[8], Buckdahn, Djehiche and Li [13], Buckdahn, Djehiche, Li and Peng [14], Cardaliaguet [15], Carmona and Delarue [16], Garnier, Papanicolaou and Yang [23], Guéant, Lasry and Lions [24], Huang [27], Kolokoltsov, Troeva and Yang [32], Meyer-Brandis, Øksendal and Zhou [37], Nourian and Caines [39], and the references therein.

Mean Field Type Control Problem is another topic that is connected to, but fundamentally different from MFGs. In MFGs, the mean field term is exogenous to individual's optimal control problem - perturbing individual's decision would not affect the mean field term as it is fixed at the first place. Intuitively, it means that individual behavior has a negligible effect on the whole community as his noise is averaged out when the number of agents in the system gets huge. For mean field type control problems, on the other hand, the mean field term is endogenous and it is also a state to be controlled. It is also called the control problem of McKean-Vlasov type in the literature, see the recent work by Pham and Wei [41], where they consider a continuous time model with full observation in the presence of a common noise.

In this thesis, three topics in Mean Field Games and Mean Field Type Control Problems in the absence of complete information will be investigated. In the first work, we consider MFGs between a leader and a group of followers, where each follower makes decision based on the delayed information received from the leader. The second work introduces one additional leader. Depending on whether or not the leaders cooperate, we investigate the impact on the cost of followers. The last part of this thesis consider a discrete time mean field game and mean field type control problems under a partial observation setting. Each player does not have the information of his own state.

1.1 Mean Field Stackelberg Games with Heterogeneous Followers

Stackelberg game is first introduced by Heinrich von Stackelberg [44] to solve for an equilibrium in hierarchical markets with a leader and follower. Based on the policies issued by the leader, the follower makes his optimal decision based on his performance index and his choice has negligible impact on the leader. Assuming the follower is rational, the leader then makes his decision based on his objective functional. Mathematically, the follower would consider the leader as exogenous factor in the follower's stochastic control problem; while the leader would consider the follower as a functional of the leader's decision in his stochastic control problem.

As a hierarchical model, Huang [28] first investigates mean field games between a large group of small players and a big player under the Linear Quadratic setting and later Nourian and Caines [39] generalize it to the general Lipschitz coefficients case. In their model, the mean field term is *exogenous* to both the small players and the big player; that is changes in decisions of the big player would not affect the mean field term immediately. On the other hand, Bensoussan, Chau and Yam [5] consider mean field games between a large group of followers and a leader, where the mean field term is *endogenous* to the control problem of the leader. Mathematically, the mean field term itself becomes a functional of the leader's control. Decisions of the leader would now directly affect the whole population through the mean field term and this matches the philosophy of Stackelberg games. Under a Linear Quadratic setting, our previous work [5] discuss the Mean Field Stackelberg game with no terminal costs, where each follower is subject to a delay affect from the leader. By choosing a suitable linear functional, a *time dependent* sufficient condition is given to warrant the unique existence of the solution to the Stackelberg game.

Chapter 2 generalizes our work [5] in certain aspects. We considers a general Lipschitz Stackelberg game with terminal costs, where the dynamics and cost functional of each follower is subject to an identical delay impact from the leader. Interaction between followers are explicitly given as empirical measures. Assuming the coefficients are sufficiently smooth, we argue that the empirical system would converge to the limiting mean field one as the number of followers increase. We then tackle the Stackelberg game by splitting it into three subproblems and they are solved in order: 1) Follower's Control Problem, 2) Mean Field Equilibrium and 3) Leader's Control Problem.

Given a pair of exogenous mean field term and leader's state processes, applying the stochastic maximum principle, we derive the necessary condition for follower's optimality in Section 2.2. The mean field equilibrium is then obtained by equating the given mean field term and the density function of the optimal state of the follower, which yields a pair of Hamilton-Jacobi-Bellman and Fokker-Planck equation (HJB-FP). Hence, the optimal control problem for the leader involves three controlled states - his original state and the HJB-FP pair. Applying the stochastic maximum principle again, we obtain three adjoint equations corresponding to the three controlled states. The necessary condition for the leader is given in Theorem 2.2.2, which can be represented by a system of six forward backward stochastic differential equations and forward backward stochastic partial differential equations (2.2.19) and (2.2.20). The results are then generalized to the case with multiple classes of delays in Subsection 2.2.3.

A comprehensive study of the particular LQ case is given in Section 2.3. By modifying the linear functional in [5], we can represent the original system of six forward backward stochastic differential equation by a forward backward stochastic functional differential equation. We provide a set of sufficient conditions, which is *independent of time length*, to guarantee the unique existence of the Stackelberg equilibrium in Section 2.4. Several numerical examples are demonstrated in Section 2.5.

1.2 Two-party Governance: Cooperation versus Competition

In game theory, there are certain notions of solution concepts in a multiple players game. Pareto equilibrium (or Pareto efficiency), named after the Italian economist Vilfredo Pareto, is an optimal solution in allocation of resources such that no individual could be better off without hurting other players' benefits if he moves away from the equilibrium. Under the SDGs framework, it is a cooperative solution among all players. For example in a two person cost minimization SDG, one could obtain the Pareto equilibrium by minimizing the sum of individual's cost functionals, such that infinitesimal changes in the decision of one player would affect the choice of another player right away. See Chapter 5.2 in Yeung and Petrosjan [48] and Reddy and Engwerda [42] for details.

Two persons would not always cooperate, see the prisoner's dilemma, perhaps the most famous layman's example. Nash Equilibrium, first introduced by Nash's celebrated work [38] in 1950, plays a crucial role in a non-cooperative environment. In an interactive game, individual's objective function depends not only on his own decision, but also other players'. One should hence consider the choices of all players in making decision, even in the non-cooperative setting. A Nash equilibrium is a solution with the property that, assuming other players stand still, one would be worse off if his decision moves away from the equilibrium point. Nash equilibrium raises many applications in analyzing system with multiple interactive decision makers in economics theory (Maskin [36]), evolutionary biology ((mathematical biology murray) Taylor [46]) and engineering (Ferris and Pang [22]).

Chapter 2 considers first a SDG between a leader and a group of followers. By mean field approximation, we formulate and solve the corresponding Mean Field Stackelberg game. In reality, for example in an oligopoly market, there might be multiple leaders announcing policies. Different interactions between them might lead to a significant impact on the population. We consider a model with two leaders over a group of followers. By the very definition of a Pareto game, as the leaders cooperate, the total cost among them reduces comparing with the noncooperative Nash game. It is however nontrivial that whether such cooperation would benefit the community and that is the rationale behind this study.

In Section 3.1, we consider another class of Stackelberg games under a Linear Quadratic setting - two leaders over a group of followers. Each individual follower has a negligible impact on two leaders as in the previous work and they are significantly influenced by the policies set by the leaders. We assume that the homogeneous followers would not cooperate. Given the evolutions of two leaders and the population average (mean field term), we solve for the follower's optimal control and the ϵ-Nash equilibrium for the community in Section 3.2.1. Depending on whether or not the leaders cooperate, we define and solve for the corresponding Pareto and Nash games between them in Theorem 3.2.9 and 3.2.8. The solutions can be represented by two systems of Forward Backward Stochastic Functional Differential Equation. We provide the explicit expression of solutions to the whole problem: Mean Field Game among the followers and Nash (and Pareto) Game between the leaders in Section 3.3. Finally, several numerical examples are given in Section 3.4 to study the impact of different games on the cost functionals of the followers.

1.3 Discrete-time Mean Field Partially Observable Controlled Systems Subject to Common Noise

Classical stochastic control problem assumes that each decision maker has the knowledge of the state evolution or even the underlying driving noises. Normally, the control process is chosen based on this set of full information. In the real world, however, neither of them are applicable, this raises the importance of studying the problem under a partially observable framework. In particular, one could only choose his control based on an intermediate observable process. See the comprehensive introduction in Bensoussan [3] and the references therein.

To incorporate the mean field game theory with the partial observation feature, Huang and Wang [26] solves the continuous time linear quadratic mean field game with a common noise subject to partial observation; the decisions of the player are assumed to be made based on the knowledge of **both** the observable process and the common noise. Şen and Caines [43] studies general non-linear mean field games with a major agent, in which the state of the major agent is partially observed by individuals.

In Chapter 4, we consider a discrete time mean field game in a linear quadratic setting, where each player makes decision based **solely** on the observable process, in the absence of the knowledge of the common noise. The control in Huang and Wang [26], by contrast, is still based on an observable common noise drives the unobservable individual state. And their model can be solved by classical Kalman filter as the evolution of the mean field process could be obtained easily. In particular, to derive the evolution of the mean field process, they take expectation on both sides conditional on the common noise to eliminate all negligible individuals effects. If the control is adapted to both the filtration generated by the common noise and the observable process, thanks to the tower property of conditional expectation, the expectation of the control term conditional only on the common noise could be greatly simplified. In our case, however, in which the control process only depends on the observation but nothing more, the tower property could not be applied and the mean field evolution is hardly obtained. Our model brings in new mathematical challenges due to this implicit connection between the filtrations generated by the common noise and that by the observable process. Instead of finding the explicit evolution of the mean field process, we establish the existence of the equilibrium solution of the resulted forward backward stochastic difference equation by an application of Schauder's fixed point theorem.

Apart from the newly proposed mean field game, one may also be interested in the mean field type stochastic control problem. While individuals have negligible effect on the mean field term in mean field game, the mean field term is indeed endogenous to the decision of the agent and is a functional of the density of the agent's state. To demonstrate the differences arising from the partial observation feature, we also study mean field type stochastic control problems subject to only the observable process. We anticipate that even under the same set of coefficients in the evolution and cost functional, due to the difference of the characters playing in the mean field term, the very existence of the solutions of the mean field type control problem and that of the mean field game control problem are not the same.

We formulate the discrete time mean field partially observable systems subject to a common noise in Section 4.1. Both the associated mean field game and mean field type stochastic control problem are defined. We demonstrate the fundamental and structural differences of the mean field term between these two problems due to the very definition: the mean field term is not affected by individual's decision in mean field game; while it is not the case in the corresponding mean field type control problem. In Section 4.2, we first solve the mean field type stochastic control problem, whose state is augmented with the mean field term, using classical discrete time Kalman Filter with subtle modifications. We then decompose the resulting forward backward stochastic difference equation into two parts. The first system is decoupled whose unique existence is immediately guaranteed; while the second part is a fully coupled forward backward system, in which the solution of the backward equation can be written as an affine transformation of the solution of the forward equation; and the unique existence of the second part is therefore established by verifying with an *Ansatz*. For the mean field game, we first solve for the individual optimal control in Section 4.3.1 by assuming the existence of the mean field term. In Section 4.3.2, we seek for the equilibrium solution by considering both the equilibrium condition and the optimal forward backward system obtained in Section 4.3.1. Schauder's fixed point is then utilized to established the existence of the equilibrium forward backward system.

1.4 Hilbert Calculus and Mean Field Games

As shown in Chapter 2, using the tools in (stochastic) partial differential equation, the solution of a (stochastic) mean field game can be represented by a system of (Stochastic) Hamilton-Jacobi-Bellman Equation and Fokker Planck Equation. The unique existence of a solution to this system is however difficult to be clarified due to 1) The forward-backward nature of the system and 2) The infinite dimensionality of PDE. This Chapter provides another direction of future researches in probabilistic analysis of mean field games. We aim at establishing the general unique and existence results of a system of a non-linear Forward Backward Stochastic Differential Equation resulted from Mean Field Games, by interpreting the McKean Vlasov type equation in an appropriate Hilbert space. The sufficient condition we assumed is likely to be independent of the time horizon. The SHJB-FP pair in Chapter 2 is connected to the FBSDE of McKean Vlasov through a "Master Equation", see [9], [17] and [18] for details.

Chapter 2

Mean Field Stackelberg Games with Heterogeneous Followers

In Section 2.1, we first formulate the Stackelberg game with finitely many players. Each follower is subject to an identical delay impact from the leader. Interactions between followers are explicitly given in this empirical system. Providing the coefficients are sufficiently smooth, the empirical system would converge to the mean field counterpart as the number of followers increases. In Section 2.2, by applying the stochastic maximum principle, the necessary conditions for the optimal control problems of the follower and leader are given in Theorem 2.2.1 and 2.2.2 respectively. The equilibrium solution of the Stackelberg game can then be represented by a system of six forward-backward stochastic differential equations (2.2.19) and (2.2.20). The results are then generalized to the case with multiple classes of delays in Subsection 2.2.3. A comprehensive study of the case with linear dynamics and quadratics costs is given in Section 2.3. We provide a set of sufficient conditions, which is independent of time horizon, to guarantee the unique existence of the Stackelberg equilibrium in Section 2.4. Several numerical examples are demonstrated in Section 2.5.

2.1 Problem Setting

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We first consider a fixed delay $\theta \in (0, +\infty)$ and a finite time horizon $[-\theta, T]$. We model a population consisting of two types of individuals: a leader whose *empirical dynamics* are described by

$$\begin{cases} dy_0(t) = g_0(y_0(t), \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \delta_{y_1^j(t)}, v_0(t))dt + \sigma_0(y_0(t))dW_0(t), \quad t \in (0, T], \\ y_0(t) = \xi_0(t), \quad t \in [-\theta, 0]; \end{cases} \tag{2.1.1}$$

and N followers whose *empirical dynamics* are given by

$$\begin{cases} dy_1^i(t) = g_1(y_1^i(t), y_0(t-\theta), \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \delta_{y_1^j(t)}, v_1^i(t))dt + \sigma(y_1^i(t))dW_1^i(t), \quad t \in (0, T], \\ y_1^i(0) = \xi_1^i, \end{cases} \tag{2.1.2}$$

where $i \in \{1, 2, \ldots, N\}$. The (empirical) objective functional is defined as:

$$\mathcal{J}_0^N(v_0) := \mathbb{E}\Big[\int_0^T f_0(y_0(t), \tfrac{1}{N}\textstyle\sum_{j=1}^N \delta_{y_1^j(t)}, v_0(t))dt + h_0(y_0(T), \tfrac{1}{N}\sum_{j=1}^N \delta_{y_1^j(T)})\Big],$$

under control $v_0(t)$. The corresponding (empirical) objective functional of the i-th follower is given by

$$\begin{aligned} \mathcal{J}_1^{N,i}(\mathbf{v_1}) := \mathbb{E}\Big[\int_0^T & f_1(y_1^i(t), y_0(t-\theta), \tfrac{1}{N}\textstyle\sum_{j=1}^N \delta_{y_1^j(t)}, v_0(t))dt \\ & + h_1(y_1^i(T), y_0(T-\theta), \tfrac{1}{N}\textstyle\sum_{j=1}^N \delta_{y_1^j(T)})\Big], \end{aligned}$$

where

$$\mathbf{v_1} = (v_1^1, v_1^2, \ldots, v_1^N)$$

with its i-th component being the control taken by the i-th follower. The initial condition ξ_0 for the leader and the d_0-dimensional Wiener process W_0 are independent, and together they generate the filtration

$$\mathcal{F}_t^0 = \sigma(\xi_0(s \wedge 0), W_0(s \vee 0) : s \in [-\theta, t]), \quad t \in [-\theta, T].$$

Similarly the initial random variables i-th ($i \in \{1, 2, \ldots, N\}$) for the followers and the d_1-dimensional Wiener process W_1^i are independent, that together generate the filtration

$$\mathcal{F}_t^{1,i} = \sigma(\xi_1^i, W_1^i(s) : s \in [0,t]).$$

We also assume that $\{\xi_1^i\}_i$ (and $\{W_1^i\}_i$) are identically and independently distributed among different followers. The filtrations generated by the leader $\mathcal{F}^0$ and any i-th follower $\mathcal{F}^{1,i}$ are also assumed to be independent.

Apart from the empirical system, we consider the analogical *mean field system* for the leader and N followers, whose evolutions are respectively described by

$$\left\{\begin{array}{rcl} dx_0(t) &=& g_0(x_0(t), m(t), v_0(t))dt + \sigma_0(x_0(t))dW_0(t), \qquad t \in (0,T], \\ x_0(t) &=& \xi_0(t), \quad t \in [-\theta, 0], \end{array}\right. \tag{2.1.3}$$

and

$$\left\{\begin{array}{rcl} dx_1^i(t) &=& g_1(x_1^i(t), x_0(t-\theta), m(t), v_1^i(t))dt + \sigma_1(x_1^i(t))dW_1^i(t), \qquad t \in (0,T], \\ x_1^i(0) &=& \xi_1^i. \end{array}\right. \tag{2.1.4}$$

The cost functionals in the mean field system are given by:

$$J_0(v_0) := \mathbb{E}\Big[\int_0^T f_0(x_0(t), m(t), v_0(t))dt + h_0(x_0(T), m(T))\Big].$$

and

$$J_1^i(v_1^i) := \mathbb{E}\Big[\int_0^T f_1(x_1^i(t), x_0(t-\theta), m(t), v_1^i(t))dt + h_1(x_1^i(T), x_0(T-\theta), m(T))\Big]. \tag{2.1.5}$$

In the mean field system, $m(t)$ is a probability density process on $\mathbb{R}^{n_1}$ which is $\mathcal{F}_{t-\theta}^0$ adapted and to be determined. We now define our admissible set of controls for the followers in the mean field formulation:

Definition 2.1.1. *The control for the i-th follower, conditioning on $\mathcal{F}_{\cdot-\theta}^0$, is in feedback form. That is $v_1^i(t) = \mathcal{V}_1^i(x_1^i(t), t)$ and $\mathcal{V}$ satisfies the following two conditions*

1. $\mathcal{V}_1^i(\cdot,\cdot) \in L^2([0,T], L^2_{loc}(\mathbb{R}^{n_1}))$;

2. $\mathcal{V}_1^i(\cdot,t)$ *is a* $\mathcal{F}^0_{t-\theta}$ *measurable random field on* $\mathbb{R}^{n_1}$.

Without loss of generality, suppose that the first follower picks an arbitrary admissible control v_1^1 while other i-th followers adopt the same (functional form described above) control u_1^i.

$$dx_1^1(t) = g_1(x_1^1(t), x_0(t-\theta), m(t), v_1^1(t))dt + \sigma_1(x_1^1(t))dW_1^1(t), \quad x_1^1(0) = \xi_1^1.$$
$$dx_1^i(t) = g_1(x_1^i(t), x_0(t-\theta), m(t), u_1^i(t))dt + \sigma_1(x_1^i(t))dW_1^i(t), \quad x_1^i(0) = \xi_1^i; \quad i \neq 1.$$

Let $k \in \{0,1\}$, before we move on, we first impose the following assumptions on the coefficient functions:

1. g_k, f_k and h_k are continuously differentiable in all Euclidean arguments with bounded derivatives; σ_k is twice continuously differentiable with bounded derivatives.

2. g_k is Lipschitz, uniformly on (x_1, x_0, v_0), in the measure argument with respect to the 2^{nd}-Wasserstein metric. In particular, there exists $L > 0$ such that

$$\begin{aligned}&|g_0(x_0, m, v_0) - g_0(x_0, m', v_0)|\\ &\qquad + |g_1(x_1, x_0, m, v_0) - g_1(x_1, x_0, m', v_0)| \leq LW_2(m, m').\end{aligned}$$

Here,
$$W_2(m, m') = \Big(\inf_{\gamma \in \Gamma(m,m')} \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_1}} |x-y|^2 d\gamma(x,y)\Big)^{\frac{1}{2}},$$
where Γ is the collection of all measures with marginals m and m'.

3. f_1, h_1 satisfy the locally Lipschitz property

$$\begin{aligned}&|f_1(x_1, x_0, m, v_1) - f_1(x_1', x_0', m', v_1')| + |h_1(x_1, x_0, m) - h_1(x_1', x_0', m')|\\ &\quad \leq L\big[1 + |x_1| + |x_1'| + |x_0| + |x_0'| + W_2(m, 0) + W_2(m', 0) + |v_1| + |v_1'|\big]\\ &\qquad \cdot\big[|x_1 - x_1'| + |x_0 - x_0'| + W_2(m, m') + |v_1 - v_1'|\big].\end{aligned}$$

Under these regularity assumptions, if we choose $m(t)$ to be the conditional probability density function of $x_1^j (j \neq 1)$, conditioning on $\mathcal{F}_{t-\theta}^0$, such that the *empirical system* $(y_0, y_1^1, \{y_1^i\}_i)$ converge to the *mean field system* $(x_0, x_1^1, \{x_1^i\}_i)$ in the sense that

$$\begin{aligned}&\mathbb{E}\sup_{t\leq T}|y_0(t)-x_0(t)|^2\\&\quad+\mathbb{E}\sup_{t\leq T}|y_1^1(t)-x_1^1(t)|^2+\sup_{i\in\{2,\ldots,N\}}\mathbb{E}\sup_{t\leq T}|y_1^i(t)-x_1^i(t)|^2=o(1),\end{aligned} \tag{2.1.6}$$

see Appendix in Bensoussan, Chau and Yam [6]. We call the measure m the mean field term and its thorough description will be given in later sections. Recall that $m(t)$ is a probability density on $\mathbb{R}^{n_1}$, we may write $m(t) = m(x_1, t)$ if necessary. We also have the convergence for cost functionals:

$$|\mathcal{J}_1^{N,i}(\mathbf{v_1}) - J_1^i(v_1^i)| = o(1), \text{ where } \mathbf{v_1} = (v_1^1, u_1^2, \ldots, u_1^N).$$

Hence, by tackling the optimal control problem described by the mean field dynamics and objective functional (2.1.4,2.1.5), an ϵ-Nash equilibrium for the empirical system can be obtained:

Theorem 2.1.2. *Suppose that* $\mathbf{v_1} = (v_1^1, u_1^2, \ldots, u_1^N)$. *Then* $\mathbf{u_1} = (u_1^1, u_1^2, \ldots, u_1^N)$ *is an* ϵ*-Nash equilibrium for the empirical system, where* u_1^i *is optimal in the stochastic control problem given by (2.1.4,2.1.5). In particular, we have*

$$\mathcal{J}^{N,i}(\mathbf{u_1}) \leq \mathcal{J}^{N,i}(\mathbf{v_1}) + o(1). \tag{2.1.7}$$

Remark 2.1.3. *In some mean field game literatures, one would directly consider the limiting case with infinitely many players and no index* i *is involved in the evolutions and cost functionals. As in Bensoussan [5], we here keep the indexes and consider the finite* N*-player counterpart in order to obtain the rate of convergence. In particular, if the second assumption is relaxed:*

$$\begin{aligned}&|g_0(x_0, m, v_0) - g_0(x_0, m', v_0)|\\&\qquad+|g_1(x_1, x_0, m, v_0) - g_1(x_1, x_0, m', v_0)| \leq \int_{\mathbb{R}^{n_1}} x^2 m(x)dx,\end{aligned}$$

then we would obtain the respective rate of convergence in (2.1.6) *and* (2.1.7)*:*

$$\begin{aligned}&\mathbb{E}\sup_{t\le T}|y_0(t)-x_0(t)|^2\\&\quad+\mathbb{E}\sup_{t\le T}|y_1^1(t)-x_1^1(t)|^2+\sup_{i\in\{2,\dots,N\}}\mathbb{E}\sup_{t\le T}|y_1^i(t)-x_1^i(t)|^2=O(\tfrac{1}{N}),\end{aligned}\tag{2.1.8}$$

and

$$\mathcal{J}^{N,i}(\mathbf{u_1})\le\mathcal{J}^{N,i}(\mathbf{v_1})+O(\frac{1}{\sqrt{N}}).\tag{2.1.9}$$

Nonetheless, the evolution (2.1.4) *and cost functional* (2.1.5) *of the i-th follower are in fact symmetric and independent of the index i. We shall call the i-th follower the representative follower and for simplicity, we skip the index i if no ambiguity is caused. For details, please refer to Section 2 in [5].*

To motivate more results in the later development, we also need the following additional assumptions:

4. g_i, f_i and h_i are Gâteaux differentiable in the density argument. In particular, given $m\in L^2(\mathbb{R}^n)$, $m\mapsto f(m)$ is said to be Gâteaux differentiable if there uniquely exists $\frac{\partial f}{\partial m}(m)(x)\in L^2(\mathbb{R}^n)$ such that

$$\lim_{\epsilon\to 0}\frac{f(m+\epsilon\tilde{m})-f(m)}{\epsilon}=\int_{\mathbb{R}^n}\frac{\partial f}{\partial m}(m)(x)\tilde{m}(x)dx,$$

   for any $\tilde{m}\in L^2(\mathbb{R}^n)$. We call $\frac{\partial f}{\partial m}(m)$ as the Gâteaux derivative.

Example 2.1.4. *Let $G(m)=\int g(x)m(x)dx$, where $g\in L^2(\mathbb{R}^n)$. We have*

$$\lim_{\epsilon\to 0}\frac{G(m+\epsilon\tilde{m})-G(m)}{\epsilon}=\int g(x)\tilde{m}(x)dx,$$

and hence g is the Gâteaux derivative of G.

5. σ_i is uniformly elliptic.

Define the Lagrangian

$$L_1(x_1,x_0,m,v_1,\lambda):=f_1(x_1,x_0,m,v_1)+\lambda\cdot g_1(x_1,x_0,m,v_1),\tag{2.1.10}$$

and the Hamiltonian

$$H_1(x_1, x_0, m, \lambda) := \inf_{v_1} L_1(x_1, x_0, m, v_1, \lambda).$$

We assume that the infimum is uniquely attained at $u_1(x_1, x_0, m, \lambda)$. Also define the second order operator

$$\mathcal{A}\varphi(x) := -\tfrac{1}{2}\mathrm{tr}\Big[(\sigma_1\sigma_1^*)(x)\mathcal{D}^2\varphi(x)\Big],$$

and its adjoint is denoted by $\mathcal{A}^*$.

2.2 General Setting

2.2.1 Optimal Control for the Follower

In canonical mean field games without common noise (see Bensoussan et al. [10]), the mean field term is simply the law (or probability measure) of individual's state, which is **deterministic**. If one introduce an additional common noise to the population, then the mean field term is no longer deterministic. To be precise, as in the recent work by Pham et al. [41], the mean field term becomes the law of individual's state **conditional on the filtration generated by the common noise**, which is a random process.

In the present setting with the presence of a leader, the mean field term is the law of individual's state **conditional on the leader** (the filtration generated by the Brownian noise which drives the evolution of the leader, to be precise). That is, the mean field term itself is **random** and depends on the leader. To be precise, let $p^{v_1}(x_1, t)$ be the probability density function of $x_1(t)$ with control v_1, conditioning on $\mathcal{F}^0_{t-\theta}$. Clearly, p^{v_1} satisfies the Fokker Planck equation:

$$\begin{aligned}
&\partial_t p^{v_1}(x_1, t) \\
&\qquad + \Big(\mathcal{A}^* p^{v_1}(x_1, t) + \mathrm{div}\Big(g_1(x_1, x_0(t-\theta), m(t), v_1(x_1, t))p^{v_1}(x_1, t)\Big)\Big)dt = 0, \\
&p^{v_1}(x_1, 0) = \omega(x_1);
\end{aligned} \tag{2.2.11}$$

where ω is the density function of the initial random variable ξ_1 of the follower. Note that the coefficients in (2.2.11) depends upon $x_0(t-\theta)$ and $m(t)$, which are both $\mathcal{F}^0_{t-\theta}$ measurable. With our regularity assumptions on the coefficients, the Fokker Planck equation (2.2.11) admits a unique solution $p^{v_1} \in L^\infty([0,T], L^2(\mathbb{R}^{n_1}))$ providing $\omega(x) \in L^2 \cap L^\infty(\mathbb{R}^{n_1})$. See Le Bris and Lions [12] for details.

Proposition 2.2.1 (Necessary Condition for Follower)**.** *Given x_0 and m as exogenous process, which are adapted to $\mathcal{F}^0_\cdot$ and $\mathcal{F}^0_{\cdot-\theta}$ respectively. The control for the representative follower is optimal only if $v_1(t) = u_1(x_1(t), x_0(t-\theta), m(t), \mathcal{D}\psi(x_1(t), t))$, where u_1 is the unique minimizer of the Lagrangian* (2.1.10)*; ψ satisfies the Stochastic Hamilton Jacobi Bellman (SHJB) equation (see Peng [40] for details):*

$$\begin{cases} -\partial_t \psi(x_1, t) &= \Big(H_1(x_1, x_0(t-\theta), m(t), \mathcal{D}\psi(x_1,t)) - \mathcal{A}\psi(x_1,t)\Big)dt \\ & \qquad\qquad\qquad\qquad\qquad -K_\psi(x_1,t)dW_0(t-\theta), \\ \psi(x_1, T) &= h_1(x_1, x_0(T-\theta), m(T)). \end{cases} \tag{2.2.12}$$

Proof. Suppose that ψ solves for the backward stochastic partial differential equation (BSPDE):

$$\begin{cases} -\partial_t \psi(x_1, t) &= \Big(L_1(x_1, x_0(t-\theta), m(t), v_1(x_1,t), \mathcal{D}\psi(x_1,t)) - \mathcal{A}\psi(x_1,t)\Big)dt \\ & \qquad\qquad\qquad\qquad\qquad -K_\psi(x_1,t)dW_0(t-\theta), \\ \psi(x_1, T) &= h_1(x_1, x_0(T-\theta), m(T)); \end{cases}$$

Recall that $x_0(\cdot-\theta)$ and $m(\cdot)$ are $\mathcal{F}^0_{\cdot-\theta}$ measurable, we then have the expression

$$\begin{aligned} J_1(v_1) &= \mathbb{E}\Big[\int_0^T \mathbb{E}^{\mathcal{F}^0_{t-\theta}} f_1(x_1(t), x_0(t-\theta), m(t), v_1(x_1(t),t))dt \\ &\qquad + \mathbb{E}^{\mathcal{F}^0_{T-\theta}} h_1(x_1(T), x_0(T-\theta), m(T))\Big] \\ &= \mathbb{E}\Big[\int_0^T \int_{\mathbb{R}^{n_1}} f_1(x_1, x_0(t-\theta), m(t), v_1(x_1,t))p^{v_1}(x_1,t)dx_1 dt \\ &\qquad + \int_{\mathbb{R}^{n_1}} h_1(x_1, x_0(T-\theta), m(T))p^{v_1}(x_1,T)dx_1\Big]. \end{aligned}$$

Let $\epsilon \in \mathbb{R}$, $\tilde{v}_1$ be an arbitrary admissible control, where clearly $v + \epsilon\tilde{v}$ stays in the admissible set. Consider the first order condition

$$\begin{aligned} 0 &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} J_1(v_1+\epsilon\tilde{v}_1) \\ &= \mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^{n_1}}[f_{1,v_1}(x_1,x_0(t-\theta),m(t),v_1(x_1,t))\tilde{v}_1(x_1,t)p^{v_1}(x_1,t) \\ &\qquad\qquad +f_1(x_1,x_0(t-\theta),m(t),v_1(x_1,t))\tilde{p}(x_1,t)]dx_1dt \\ &\qquad +\int_{\mathbb{R}^{n_1}} h_1(x_1,x_0(T-\theta),m(T))\tilde{p}(x_1,T)dx_1\Big], \end{aligned} \tag{2.2.13}$$

where $\tilde{p} = \frac{d}{d\epsilon}\Big|_{\epsilon=0} p^{v_1+\epsilon\tilde{v}_1}$, satisfies

$$\begin{cases} \partial_t\tilde{p}(x_1,t) + \Big(\mathcal{A}^*\tilde{p}(x_1,t) + \mathrm{div}\Big(g_{1,v_1}(x_1,x_0(t-\theta),m(t),v_1(x_1,t))\tilde{v}_1(x_1,t)p^{v_1}(x_1,t)\Big) \\ \qquad\qquad +\mathrm{div}\Big(g_1(x_1,x_0(t-\theta),m(t),v_1(x_1,t))\tilde{p}(x_1,t)\Big)\Big)dt = 0, \\ \tilde{p}(x_1,0)=0. \end{cases}$$

We thus have

$$\begin{aligned} & d\int_{\mathbb{R}^{n_1}} \tilde{p}(x_1,t)\psi(x_1,t)dx_1 \\ = & \int_{\mathbb{R}^{n_1}} -\tilde{p}(x_1,t)\Big[f_1(x_1,x_0(t-\theta),m(t),v_1(x_1,t)) \\ & \qquad +\mathcal{D}\psi(x_1,t)\cdot g_1(x_1,x_0(t-\theta),m(t),v_1(x_1,t)) - \mathcal{A}\psi(x_1,t)\Big] \\ & -\Big[\mathcal{A}^*\tilde{p}(x_1,t) + \mathrm{div}\Big(g_{1,v_1}(x_1,x_0(t-\theta),m(t),v_1(x_1,t))\tilde{v}_1(x_1,t)p^{v_1}(x_1,t)\Big) \\ & \qquad +\mathrm{div}\Big(g_1(x_1,x_0(t-\theta),m(t),v_1(x_1,t))\tilde{p}(x_1,t)\Big)\Big]\psi(x_1,t)dx_1dt \\ & +\{\dots\}dW_0(t-\theta). \end{aligned} \tag{2.2.14}$$

We take integration over $[0,T]$ and then expectation on both sides of (2.2.14). A further application of integration by parts yields

$$\begin{aligned} & \mathbb{E}\Big[\int_{\mathbb{R}^{n_1}} h_1(x_1,x_0(T-\theta),m(T))\tilde{p}(x_1,T)dx_1 \\ & \qquad +\int_0^T\int_{\mathbb{R}^{n_1}} f_1(x_1,x_0(t-\theta),m(t),v_1(x_1,t))\tilde{p}(x_1,t)dx_1dt\Big] \\ = & \mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^{n_1}} g_{1,v_1}(x_1,x_0(t-\theta),m(t),v_1(x_1,t))\tilde{v}_1(x_1,t)p^{v_1}(x_1,t)\cdot\mathcal{D}\psi(x_1,t)dx_1dt\Big]. \end{aligned}$$

Together with the first order condition (2.2.13), we have

$$\begin{aligned} 0 = \mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^{n_1}} &\Big[f_{1,v_1}(x_1,x_0(t-\theta),m(t),v_1(x_1,t)) \\ & +\mathcal{D}\psi(x_1,t)\cdot g_{1,v_1}(x_1,x_0(t-\theta),m(t),v_1(x_1,t))\Big]\cdot\tilde{v}_1(x_1,t)p^{v_1}(x_1,t)dx_1dt\Big]. \end{aligned}$$

Since p^{v_1} is a probability density and hence always non-negative, and $\tilde{v}_1$ is arbitrary, we have the necessary condition for the optimal control

$$
\begin{aligned}
&f_{1,v_1}(x_1, x_0(t-\theta), m(t), v_1(x_1,t)) \\
&\qquad\qquad + \mathcal{D}\psi(x_1,t)\cdot g_{1,v_1}(x_1, x_0(t-\theta), m(t), v_1(x_1,t)) = 0, \quad a.e.(\omega, x_1, t);
\end{aligned}
$$

(ω, x_1, t)-almost everywhere on the support of p^{v_1}. To obtain the necessary condition for the optimality, we choose v_1 to be the minimizer of the Lagrangian and hence we obtain the SHJB (2.2.12) from (2.2.1). $\square$

To obtain the equilibrium condition, i.e. the mean field term and the probability density function of x_1 conditioning on $\mathcal{F}^0_{t-\theta}$ coincides, we have the following coupled SHJB-FP equations

$$
\begin{cases}
\partial_t m(x_1,t) = \Big(-\mathcal{A}^* m(x_1,t) - \operatorname{div}\Big(G_1(x_1, x_0(t-\theta), m(t), \mathcal{D}\psi(x_1,t)) m(x_1,t)\Big)\Big) dt, \\
\quad m(x_1, 0) = \omega(x_1); \\
-\partial_t \psi(x_1,t) = \Big(H_1(x_1, x_0(t-\theta), m(t), \mathcal{D}\psi(x_1,t)) - \mathcal{A}\psi(x_1,t)\Big) dt \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - K_\psi(x_1,t) dW_0(t-\theta), \\
\quad \psi(x_1, T) = h_1(x_1, x_0(T-\theta), m(T));
\end{cases}
$$

where

$$
G_1(x_1, x_0(t-\theta), m(t), \mathcal{D}\psi(x_1,t)) = g_1(x_1, x_0(t-\theta), m(t), u_1(x_1, x_0(t-\theta), m(t), \mathcal{D}\psi(x_1,t))).
$$

Apparently, the backward equation in (2.2.1) is not classical SHJB as considered in Peng [40] as it involves the delay $x_0(t-\theta)$, which might make K_ψ ill-posed as we are working on the delayed filtration. That is not the case, as the whole system (m, ψ) are $\mathcal{F}^0_{\cdot-\theta}$ adapted and the time argument t inside are just served as indexes. Indeed, one can define and solve $(m^*(x_1,t), \psi^*(x_1,t)) := (m(x_1, t+\theta), \psi(x_1, t+\theta))$, where the illusion of delays are eliminated in the new system.

2.2.2 Optimal Control for the Leader

We next proceed to solve the control problem for the leader. Since the mean field term is endogenous to the leader, he should consider the interaction with (m, ψ) when making decision (control). We may write the three state processes $(x_0, m, \psi) = (x_0^{v_0}, m^{v_0}, \psi^{v_0})$ to indicate the direct influence of v_0. When we apply the stochastic maximum principle to obtain the necessary condition for the leader's optimal control, we expect that there would be three adjoint equations as outlined in the next theorem. We again define the Lagrangian

$$L_0(x_0, m, v_0, \lambda) = f_0(x_0, m, v_0) + \lambda \cdot g_0(x_0, m, v_0)$$

and the Hamiltonian

$$H_0(x_0, m, \lambda) = \inf_{v_0} L_0(x_0, m, v_0, \lambda).$$

We also assume that the infimum is uniquely attained at $u_0(x_0, m, \lambda)$.

Proposition 2.2.2 (Necessary Condition of Leader)**.** *The control for the leader is optimal only if $v_0(t) = u_0(x_0(t), m(t), p(t))$, where p satisfies*

$$\begin{aligned}
-dp(t) &= H_{0,x_0}(x_0(t), m(t), p(t))dt \\
&\qquad - \sum_{l=1}^{d_0} K_p^l(t)dW_0^l(t) + \sum_{l=1}^{d_0} \sigma_{0,x_0}^{l*}(x_0(t))K_p^l(t)dt, t \in (T-\theta, T); \\
-dp(t) &= \Big(H_{0,x_0}(x_0(t), m(t), p(t)) \\
&\qquad + \int_{\mathbb{R}^{n_1}} \mathcal{D}\zeta(x_1, t+\theta)G_{1,x_0}(x_1, x_0(t), m(t+\theta), \mathcal{D}\psi(x_1, t+\theta))m(x_1, t+\theta)dx_1 \\
&\qquad + \int_{\mathbb{R}^{n_1}} \eta(x_1, t+\theta)H_{1,x_0}(x_1, x_0(t), m(t+\theta), \mathcal{D}\psi(x_1, t+\theta))dx_1\Big)dt \\
&\qquad - \sum_{l=1}^{d_0} K_p^l(t)dW_0^l(t) + \sum_{l=1}^{d_0} \sigma_{0,x_0}^{l*}(x_0(t))K_p^l(t)dt, t \in (0, T-\theta); \\
p(T) &= h_{0,x_0}(x_0(T), m(T)); \\
p(T-\theta) &= p((T-\theta)-) - \int_{\mathbb{R}^{n_1}} \eta(x_1, T)h_{1,x_0}(x_1, x_0(T-\theta), m(T))dx_1.
\end{aligned} \tag{2.2.15}$$

$$
\begin{aligned}
-\partial_t \zeta(x_1,t) = \Big(& -\mathcal{A}\zeta(x_1,t) + \mathbb{E}^{\mathcal{F}^0_{t-\theta}} \frac{\partial H_0}{\partial m}(x_0(t),m(t),p(t))(x_1) \\
&+ \mathcal{D}\zeta(x_1,t) G_1(x_1,x_0(t-\theta),m(t),\mathcal{D}\psi(x_1,t)) \\
&+ \int_{\mathbb{R}^{n_1}} \mathcal{D}\zeta(\xi,t)\frac{\partial G_1}{\partial m}(\xi,x_0(t-\theta),m(t),\mathcal{D}\psi(\xi,t))(x_1) m(\xi,t)d\xi \\
&+ \int_{\mathbb{R}^{n_1}} \eta(\xi,t)\frac{\partial H_1}{\partial m}(\xi,x_0(t-\theta),m(t),\mathcal{D}\psi(\xi,t))(x_1) d\xi\Big) dt \\
&- K_\zeta(x_1,t) dW_0(t-\theta), \\
\zeta(x_1,T) = \mathbb{E}^{\mathcal{F}^0_{T-\theta}}& \frac{\partial h_0}{\partial m}(x_0(T),m(T))(x_1) + \int_{\mathbb{R}^{n_1}} \eta(\xi,T)\frac{\partial h_1}{\partial m}(\xi,x_0(T-\theta),m(T))(x_1)d\xi.
\end{aligned}
\tag{2.2.16}
$$

$$
\begin{aligned}
\partial_t \eta(x_1,t) = \Big(& -\mathcal{A}^*\eta(x_1,t) - \mathrm{div}\Big(\eta(x_1,t)H_{1,\lambda}(x_1,x_0(t-\theta),m(t),\mathcal{D}\psi(x_1,t)) \\
&+ G_{1,\lambda}(x_1,x_0(t-\theta),m(t),\mathcal{D}\psi(x_1,t))\mathcal{D}\zeta(x_1,t)m(x_1,t)\Big)\Big)dt, \\
\eta(x_1,0) = 0&.
\end{aligned}
\tag{2.2.17}
$$

Proof. Let p satisfies the backward stochastic differential equation (with jump at $t = T - \theta$)

$$
\begin{array}{rcll}
-dp(t) & = & \Big(f_{0,x_0}(x_0(t),m(t),v_0(t)) + g_{0,x_0}(x_0(t),m(t),v_0(t))p(t)\Big)dt & \\
& & -\sum_{l=1}^{d_0} K_p^l(t)dW_0^l(t) + \sum_{l=1}^{d_0}\sigma_{0,x_0}^{l*}(x_0(t))K_p^l(t)dt, & t\in(T-\theta,T); \\
-dp(t) & = & \Big(f_{0,x_0}(x_0(t),m(t),v_0(t)) + g_{0,x_0}(x_0(t),m(t),v_0(t))p(t)dt & \\
& & +\int_{\mathbb{R}^{n_1}}\mathcal{D}\zeta(x_1,t+\theta)G_{1,x_0}(x_1,x_0(t),m(t+\theta),\mathcal{D}\psi(x_1,t+\theta))m(x_1,t+\theta)dx_1 & \\
& & +\int_{\mathbb{R}^{n_1}}\eta(x_1,t+\theta)H_{1,x_0}(x_1,x_0(t),m(t+\theta),\mathcal{D}\psi(x_1,t+\theta))dx_1\Big)dt & \\
& & -\sum_{l=1}^{d_0} K_p^l(t)dW_0^l(t) + \sum_{l=1}^{d_0}\sigma_{0,x_0}^{l*}(x_0(t))K_p^l(t)dt, & t\in(0,T-\theta); \\
p(T) & = & h_{0,x_0}(x_0(T),m(T)); & \\
p(T-\theta) & = & p((T-\theta)-) - \int_{\mathbb{R}^{n_1}}\eta(x_1,T)h_{1,x_0}(x_1,x_0(T-\theta),m(T))dx_1. &
\end{array}
$$

Here ζ and η are defined as in (2.2.16) and (2.2.17). We again consider the first order condition

$$
\begin{aligned}
0 &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} J_0(v_0+\epsilon\tilde{v}_0) \\
&= \mathbb{E}\Big[\int_0^T [f_{0,x_0}(x_0(t),m(t),v_0(t))\tilde{x}_0(t) \\
&\qquad\qquad + \int_{\mathbb{R}^{n_1}} \frac{\partial f_0}{\partial m}(x_0(t),m(t),v_0(t))(\xi)\tilde{m}(\xi,t)d\xi \\
&\qquad\qquad + f_{0,v_0}(x_0(t),m(t),v_0(t))\tilde{v}_0(t)]dt \\
&\quad + h_{0,x_0}(x_0(T),m(T))\tilde{x}_0(T) \\
&\qquad\qquad + \int_{\mathbb{R}^{n_1}} \frac{\partial h_0}{\partial m}(x_0(T),m(T))(\xi)\tilde{m}(\xi,T)d\xi\Big].
\end{aligned}
\tag{2.2.18}
$$

where $\tilde{x}_0 = \frac{d}{d\epsilon}\Big|_{\epsilon=0} x_0^{v_0+\epsilon\tilde{v}_0}$, $\tilde{m} = \frac{d}{d\epsilon}\Big|_{\epsilon=0} m^{v_0+\epsilon\tilde{v}_0}$, $\tilde{\psi} = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \psi^{v_0+\epsilon\tilde{v}_0}$. We have

$$
\begin{aligned}
d\tilde{x}_0(t) &= \Big(g_{0,x_0}(x_0(t),m(t),v_0(t))\tilde{x}_0(t) \\
&\qquad + \int_{\mathbb{R}^{n_1}} \frac{\partial g_0}{\partial m}(x_0(t),m(t),v_0(t))(\xi)\tilde{m}(\xi,t)d\xi \\
&\qquad + g_{0,v_0}(x_0(t),m(t),v_0(t))\tilde{v}_0(t)\Big)dt \\
&\quad + \sum_{l=1}^{d_0} \sigma_{0,x_0}^l(x_0(t))\tilde{x}_0(t)dW_0^l(t), \\
\tilde{x}_0(t) &= 0, \quad t\in[-\theta,0].
\end{aligned}
$$

$$
\begin{aligned}
\partial_t\tilde{m}(x_1,t) &= \Big(-\mathcal{A}^*\tilde{m}(x_1,t) \\
&\quad -\mathrm{div}\Big([G_{1,x_0}(x_1,x_0(t-\theta),m(t),\mathcal{D}\psi(x_1,t))\tilde{x}_0(t-\theta) \\
&\qquad + \int_{\mathbb{R}^{n_1}} \frac{\partial G_1}{\partial m}(x_1,x_0(t-\theta),m(t),\mathcal{D}\psi(x_1,t))(\xi)\tilde{m}(\xi,t)d\xi \\
&\qquad + G_{1,\lambda}(x_1,x_0(t-\theta),m(t),\mathcal{D}\psi(x_1,t))\mathcal{D}\tilde{\psi}(x_1,t)]m(x_1,t)\Big) \\
&\quad -\mathrm{div}\Big(G_1(x_1,x_0(t-\theta),m(t),\mathcal{D}\psi(x_1,t))\tilde{m}(x_1,t)\Big)\Big)dt, \\
\tilde{m}(x_1,0) &= 0.
\end{aligned}
$$

$$
\begin{aligned}
-\partial_t\tilde{\psi}(x_1,t) &= \Big(H_{1,x_0}(x_1,x_0(t-\theta),m(t),\mathcal{D}\psi(x_1,t))\tilde{x}_0(t-\theta) \\
&\qquad + \int_{\mathbb{R}^{n_1}} \frac{\partial H_1}{\partial m}(x_1,x_0(t-\theta),m(t),\mathcal{D}\psi(x_1,t))(\xi)\tilde{m}(\xi,t)d\xi \\
&\qquad + H_{1,\lambda}(x_1,x_0(t-\theta),m(t),\mathcal{D}\psi(x_1,t))\mathcal{D}\tilde{\psi}(x_1,t) \\
&\quad -\mathcal{A}\tilde{\psi}(x_1,t)\Big)dt - K_{\tilde{\psi}}(x_1,t)dW_0(t-\theta), \\
\tilde{\psi}(x_1,T) &= h_{1,x_0}(x_1,x_0(T-\theta),m(T))\tilde{x}_0(T-\theta) \\
&\qquad + \int_{\mathbb{R}^{n_1}} \frac{\partial h_1}{\partial m}(x_1,x_0(T-\theta),m(T))(\xi)\tilde{m}(\xi,T)d\xi.
\end{aligned}
$$

Applying Itô's formula, we have the inner products

$$\begin{aligned}
& \Big(\mathbb{E}[p(T)\tilde{x}_0(T)] - \mathbb{E}[p(T-\theta)\tilde{x}_0(T-\theta)]\Big) + \Big(\mathbb{E}[p((T-\theta)-)\tilde{x}_0(T-\theta) - 0]\Big) \\
= \;& \mathbb{E}\Big[\int_0^T p(t)\int_{\mathbb{R}^{n_1}} \tfrac{\partial g_0}{\partial m}(x_0(t), m(t), v_0(t))(\xi)\tilde{m}(\xi,t)d\xi dt\Big] \\
& +\mathbb{E}\Big[\int_0^T p(t)g_{0,v_0}(x_0(t), m(t), v_0(t))\tilde{v}_0(t)dt\Big] \\
& -\mathbb{E}\Big[\int_0^T f_{0,x_0}(x_0(t), m(t), v_0(t))\tilde{x}_0(t)dt\Big] \\
& -\mathbb{E}\Big[\int_0^{T-\theta}\int_{\mathbb{R}^{n_1}} \mathcal{D}\zeta(x_1,t+\theta)G_{1,x_0}(x_1,x_0(t),m(t+\theta),\mathcal{D}\psi(x_1,t+\theta))m(x_1,t+\theta)dx_1\tilde{x}_0(t)dt\Big] \\
& -\mathbb{E}\Big[\int_0^{T-\theta}\int_{\mathbb{R}^{n_1}} \eta(x_1,t+\theta)H_{1,x_0}(x_1,x_0(t),m(t+\theta),\mathcal{D}\psi(x_1,t+\theta))dx_1\tilde{x}_0(t)dt\Big].
\end{aligned}$$

$$\begin{aligned}
& \mathbb{E}\Big[\int_{\mathbb{R}^{n_1}} \zeta(x_1,T)\tilde{m}(x_1,T)dx_1\Big] \\
= \;& \mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^{n_1}} \mathcal{D}\zeta(x_1,t)G_{1,x_0}(x_1,x_0(t-\theta),m(t),\mathcal{D}\psi(x_1,t))\tilde{x}_0(t-\theta)m(x_1,t)dx_1dt\Big] \\
& +\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^{n_1}} \mathcal{D}\zeta(x_1,t)G_{1,\lambda}(x_1,x_0(t-\theta),m(t),\mathcal{D}\psi(x_1,t))\mathcal{D}\tilde{\psi}(x_1,t)m(x_1,t)dx_1dt\Big] \\
& -\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^{n_1}} \tfrac{\partial f_0}{\partial m}(x_0(t),m(t),v_0(t))(x_1)\tilde{m}(x_1,t)dx_1dt\Big] \\
& -\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^{n_1}} \tfrac{\partial g_0}{\partial m}(x_0(t),m(t),v_0(t))(x_1)\tilde{m}(x_1,t)dx_1p(t)dt\Big] \\
& -\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^{n_1}} \Big(\int_{\mathbb{R}^{n_1}} \eta(\xi,t)\tfrac{\partial H_1}{\partial m}(\xi,x_0(t-\theta),m(t),\mathcal{D}\psi(\xi,t))(x_1)d\xi\Big)\tilde{m}(x_1,t)dx_1dt\Big].
\end{aligned}$$

$$\begin{aligned}
& -\mathbb{E}\Big[\int_{\mathbb{R}^{n_1}} \eta(x_1,T)\tilde{\psi}(x_1,T)dx_1\Big] \\
= \;& \mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^{n_1}} \eta(x_1,t)H_{1,x_0}(x_1,x_0(t-\theta),m(t),\mathcal{D}\psi(x_1,t))\tilde{x}_0(t-\theta)dx_1dt\Big] \\
& +\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^{n_1}} \eta(x_1,t)\Big(\int_{\mathbb{R}^{n_1}} \tfrac{\partial H_1}{\partial m}(x_1,x_0(t-\theta),m(t),\mathcal{D}\psi(x_1,t))(\xi)\tilde{m}(\xi,t)d\xi\Big)dx_1dt\Big] \\
& -\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^{n_1}} G_{1,\lambda}(x_1,x_0(t-\theta),m(t),\mathcal{D}\psi(x_1,t))\mathcal{D}\zeta(x_1,t)m(x_1,t)\mathcal{D}\tilde{\psi}(x_1,t)dx_1dt\Big].
\end{aligned}$$

Summing three equations and putting the terminal conditions, we have

$$\begin{aligned}
& \mathbb{E}[h_{0,x_0}(x_0(T),m(T))\tilde{x}_0(T)] \\
& \quad +\mathbb{E}[\textstyle\int_{\mathbb{R}^{n_1}} \eta(x,T)h_{1,x_0}(x_1,x_0(T-\theta),m(T))dx_1\tilde{x}_0(T-\theta)] \\
& \quad +\mathbb{E}\Big[\int_{\mathbb{R}^{n_1}} \mathbb{E}^{\mathcal{F}^0_{T-\theta}}\Big(\tfrac{\partial h_0}{\partial m}(x_0(T),m(T))(x_1) \\
& \qquad\qquad + \int_{\mathbb{R}^{n_1}} \eta(\xi,T)\tfrac{\partial h_1}{\partial m}(\xi,x_0(T-\theta),m(T))(x_1)d\xi\Big)\tilde{m}(x_1,T)dx_1\Big] \\
& \quad -\mathbb{E}\Big[\int_{\mathbb{R}^{n_1}} \eta(x_1,T)\Big(h_{1,x_0}(x_1,x_0(T-\theta),m(T))\tilde{x}_0(T-\theta) \\
& \qquad\qquad + \int_{\mathbb{R}^{n_1}} \tfrac{\partial h_1}{\partial m}(x_1,x_0(T-\theta),m(T))(\xi)\tilde{m}(\xi,T)d\xi\Big)dx_1\Big] \\
= \;& -\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^{n_1}} \tfrac{\partial f_0}{\partial m}(x_0(t),m(t),v_0(t))(x_1)\tilde{m}(x_1,t)dx_1dt\Big] \\
& \quad +\mathbb{E}\Big[\int_0^T p(t)g_{0,v_0}(x_0(t),m(t),v_0(t))\tilde{v}_0(t)dt\Big] - \mathbb{E}\Big[\int_0^T f_{0,x_0}(x_0(t),m(t),v_0(t))\tilde{x}_0(t)dt\Big].
\end{aligned}$$

Or we can rearrange and cancel the terms to obtain

$$\begin{aligned}
& \mathbb{E}[h_{0,x_0}(x_0(T),m(T))\tilde{x}_0(T)] + \mathbb{E}\Big[\int_{\mathbb{R}^{n_1}} \frac{\partial h_0}{\partial m}(x_0(T),m(T))(x_1)\tilde{m}(x_1,T)dx_1\Big] \\
& +\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}^{n_1}} \frac{\partial f_0}{\partial m}(x_0(t),m(t),v_0(t))(x_1)\tilde{m}(x_1,t)dx_1 dt\Big] \\
& +\mathbb{E}\Big[\int_0^T f_{0,x_0}(x_0(t),m(t),v_0(t))\tilde{x}_0(t)dt\Big] \\
= \; & \mathbb{E}\Big[\int_0^T p(t)g_{0,v_0}(x_0(t),m(t),v_0(t))\tilde{v}_0(t)dt\Big].
\end{aligned}$$

Combining with the first order condition (2.2.18), we finally obtain

$$0 = \mathbb{E}\Big[\int_0^T \Big(f_{0,v_0}(x_0(t),m(t),v_0(t)) + p(t)g_{0,v_0}(x_0(t),m(t),v_0(t))\Big)\tilde{v}_0(t)dt\Big].$$

Since v_0 is arbitrary, we have the necessary condition for the optimal control

$$f_{0,v_0}(x_0(t),m(t),v_0(t)) + p(t)\cdot g_{0,v_0}(x_0(t),m(t),v_0(t)) = 0 \quad a.s.$$

We choose v_0 to be the minimizer of the Lagrangian and we obtain (2.2.15). $\square$

Let $G_0(x_0(t),m(t),p(t)) = g_0(x_0(t),m(t),u_0(x_0(t),m(t),p(t)))$. The full set of solutions is represented by the system of six equations:

$$\begin{cases}
dx_0(t) &= G_0(x_0(t),m(t),p(t))dt + \sigma_0(x_0(t))dW_0(t), \\
x_0(t) &= \xi_0(t), \quad t\in[-\theta,0]; \\
\partial_t m(x_1,t) &= \Big(-\mathcal{A}^* m(x_1,t) - \operatorname{div}\Big(G_1(x_1,x_0(t-\theta),m(t),\mathcal{D}\psi(x_1,t))m(x_1,t)\Big)\Big)dt, \\
m(x_1,0) &= \omega(x_1); \\
-\partial_t\psi(x_1,t) &= \Big(H_1(x_1,x_0(t-\theta),m(t),\mathcal{D}\psi(x_1,t)) - \mathcal{A}\psi(x_1,t)\Big)dt - K_\psi(x_1,t)dW_0(t-\theta) \\
\psi(x_1,T) &= h_1(x_1,x_0(T-\theta),m(T)).
\end{cases} \tag{2.2.19}$$

$$
\begin{cases}
-dp(t) &= H_{0,x_0}(x_0(t), m(t), p(t))dt \\
&\quad - \sum_{l=1}^{d_0} K_p^l(t)dW_0^l(t) + \sum_{l=1}^{d_0} \sigma_{0,x_0}^{l*}(x_0(t))K_p^l(t)dt, \qquad t \in (T-\theta, T); \\
-dp(t) &= \Big(H_{0,x_0}(x_0(t), m(t), p(t)) \\
&\quad + \int_{\mathbb{R}^{n_1}} \mathcal{D}\zeta(x_1, t+\theta)G_{1,x_0}(x_1, x_0(t), m(t+\theta), \mathcal{D}\psi(x_1, t+\theta))m(x_1, t+\theta)dx_1 \\
&\quad + \int_{\mathbb{R}^{n_1}} \eta(x_1, t+\theta)H_{1,x_0}(x_1, x_0(t), m(t+\theta), \mathcal{D}\psi(x_1, t+\theta))dx_1\Big)dt \\
&\quad - \sum_{l=1}^{d_0} K_p^l(t)dW_0^l(t) + \sum_{l=1}^{d_0} \sigma_{0,x_0}^{l*}(x_0(t))K_p^l(t)dt, \qquad t \in (0, T-\theta); \\
p(T) &= h_{0,x_0}(x_0(T), m(T)); \\
p(T-\theta) &= p((T-\theta)-) - \int_{\mathbb{R}^{n_1}} \eta(x_1, T)h_{1,x_0}(x_1, x_0(T-\theta), m(T))dx_1. \\
-\partial_t \zeta(x_1, t) &= \Big(- A\zeta(x_1, t) + \mathbb{E}^{\mathcal{F}^0_{t-\theta}} \frac{\partial H_0}{\partial m}(x_0(t), m(t), p(t))(x_1) \\
&\quad + \mathcal{D}\zeta(x_1, t)G_1(x_1, x_0(t-\theta), m(t), \mathcal{D}\psi(x_1, t)) \\
&\quad + \int_{\mathbb{R}^{n_1}} \mathcal{D}\zeta(\xi, t)\frac{\partial G_1}{\partial m}(\xi, x_0(t-\theta), m(t), \mathcal{D}\psi(\xi, t))(x_1)m(\xi, t)d\xi \\
&\quad + \int_{\mathbb{R}^{n_1}} \eta(\xi, t)\frac{\partial H_1}{\partial m}(\xi, x_0(t-\theta), m(t), \mathcal{D}\psi(\xi, t))(x_1)d\xi\Big)dt \\
&\quad - K_\zeta(x_1, t)dW_0(t-\theta), \\
\zeta(x_1, T) &= \mathbb{E}^{\mathcal{F}^0_{T-\theta}} \frac{\partial h_0}{\partial m}(x_0(T), m(T))(x_1) + \int_{\mathbb{R}^{n_1}} \eta(\xi, T)\frac{\partial h_1}{\partial m}(\xi, x_0(T-\theta), m(T))(x_1)d\xi. \\
\partial_t \eta(x_1, t) &= \Big(- \mathcal{A}^*\eta(x_1, t) - \mathrm{div}\Big(\eta(x_1, t)H_{1,\lambda}(x_1, x_0(t-\theta), m(t), \mathcal{D}\psi(x_1, t)) \\
&\quad + G_{1,\lambda}(x_1, x_0(t-\theta), m(t), \mathcal{D}\psi(x_1, t))\mathcal{D}\zeta(x_1, t)m(x_1, t)\Big)\Big)dt, \\
\eta(x_1, 0) &= 0.
\end{cases}
\tag{2.2.20}
$$

The unique existence of the system of SPDEs and SDEs (2.2.19,2.2.20) is difficult to clarify and remains an open problem to literature. Apart from the partial differential equation approach introduced in this chapter, a pure probabilistic framework is recently gaining its popularity and we will discuss some results in this direction in Chapter 5. Besides, we will apply the obtained results in the PDE framework under a linear quadratic setting in Section 2.3. The existence for the corresponding system is provided in Section 2.4.

2.2.3 Multiple Classes

Consider now θ is discretely distributed in $\{a = \theta_1 < \theta_2 < \cdots < \theta_n = b\}$ with probability distribution $\{p^1, p^2, \ldots, p^n\}$. Applying Proposition 2.2.1 analogically, we have the necessary condition for the optimal control for the representative follower with delay θ_k:

Proposition 2.2.3 (Necessary Condition). *Consider x_0 and m as exogenous, the control for the representative follower with delay δ_k is optimal only if $v_1^k(t) = u_1^k(x_1^k(t), x_0(t-\theta_k), m(t), \mathcal{D}\psi^k(x_1^k(t), t))$, where ψ satisfies the Stochastic Hamilton Jacobi Bellman equation:*

$$\begin{aligned} -\partial_t \psi^k(x_1, t) &= \Big(H_1(x_1, x_0(t-\theta_k), m(t), \mathcal{D}\psi^k(x_1, t)) - \mathcal{A}\psi^k(x_1, t)\Big)dt \\ &\qquad -K_{\psi^k}(x_1, t)dW_0(t-a); \\ \psi^k(x_1, T) &= h_1(x_1, x_0(T-\theta_k), m(T)). \end{aligned}$$

We can write down the FP equation for the follower with delay δ_k:

$$\begin{aligned} \partial_t m^k(x_1, t) &= \Big(-\mathcal{A}^* m^k(x_1, t) - \operatorname{div}\Big(G_1(x_1, x_0(t-\theta_k), m(t), \mathcal{D}\psi^k(x_1, t))m^k(x_1, t)\Big)\Big)dt, \\ m^k(x_1, 0) &= \omega(x_1). \end{aligned}$$

The fixed point problem for the equilibrium condition is now $m = \sum_{k=1}^n p^k m^k$ (see Bensoussan et al. [5]), or we can write the pair

$$\begin{cases} \partial_t m^k(x_1, t) = \Big(-\mathcal{A}^* m^k(x_1, t) \\ \qquad\qquad - \operatorname{div}\Big(G_1(x_1, x_0(t-\theta_k), \sum_{k=1}^n p^k m^k(t), \mathcal{D}\psi^k(x_1, t))m^k(x_1, t)\Big)\Big)dt, \\ m^k(x_1, 0) = \omega(x_1); \\ -\partial_t \psi^k(x_1, t) = \Big(H_1(x_1, x_0(t-\theta_k), \sum_{k=1}^n p^k m^k(t), \mathcal{D}\psi^k(x_1, t)) - \mathcal{A}\psi^k(x_1, t)\Big)dt \\ \qquad\qquad - K_{\psi^k}(x_1, t)dW_0(t-a), \\ \psi^k(x_1, T) = h_1(x_1, x_0(T-\theta_k), \sum_{k=1}^n p^k m^k(T)). \end{cases}$$

Similar to the proof of Proposition 2.2.2, we now state the main results in this section:

Proposition 2.2.4 (Necessary Condition of Leader)**.** *The control for the leader is optimal only if* $v_0(t) = u_0(x_0(t), m(t), p(t))$, *where* p *satisfies*

$$\begin{aligned}
-dp(t) &= H_{0,x_0}(x_0(t), m(t), p(t))dt \\
&\qquad - \textstyle\sum_{l=1}^{d_0} K_p^l(t)dW_0^l(t) + \sum_{l=1}^{d_0} \sigma_{0,x_0}^{l*}(x_0(t))K_p^l(t)dt, \ t \in (T-a, T); \\
-dp(t) &= \Big(H_{0,x_0}(x_0(t), m(t), p(t)) \\
&\quad + \textstyle\sum_{k=1}^n p^k \mathbb{I}_{[0,T-\theta_k]}(t) \int_{\mathbb{R}^{n_1}} \mathcal{D}\zeta^k(x_1, t+\theta_k) \cdot \\
&\qquad\quad G_{1,x_0}(x_1, x_0(t), m(t+\theta_k), \mathcal{D}\psi^k(x_1, t+\theta_k))m^k(x_1, t+\theta_k)dx_1 \\
&\quad + \textstyle\sum_{k=1}^n p^k \mathbb{I}_{[0,T-\theta_k]}(t) \int_{\mathbb{R}^{n_1}} \eta^k(x_1, t+\theta_k) \cdot \\
&\qquad\quad H_{1,x_0}(x_1, x_0(t), m(t+\theta_k), \mathcal{D}\psi(x_1, t+\theta_k))dx_1 \Big) dt \\
&\quad - \textstyle\sum_{l=1}^{d_0} K_p^l(t)dW_0^l(t) + \sum_{l=1}^{d_0} \sigma_{0,x_0}^{l*}(x_0(t))K_p^l(t)dt, \qquad t \in (0, T-a); \\
p(T) &= h_{0,x_0}(x_0(T), m(T)); \\
p(T-\theta_k) &= p((T-\theta_k)-) \\
&\qquad -p^k \textstyle\int_{\mathbb{R}^{n_1}} \eta^k(x_1, T)h_{1,x_0}(x_1, x_0(T-\theta_k), m(T))dx_1, \quad k = 1, \ldots, n; \\
-\partial_t \zeta^k(x_1, t) &= \Big(-\mathcal{A}\zeta^k(x_1, t) + \mathbb{E}^{\mathcal{F}^0_{t-a}} \tfrac{\partial H_0}{\partial m}(x_0(t), m(t), p(t))(x_1) \\
&\qquad + \mathcal{D}\zeta^k(x_1, t)G_1(x_1, x_0(t-\theta_k), m(t), \mathcal{D}\psi^k(x_1, t)) \\
&\qquad + \textstyle\sum_{j=1}^n p^j \mathbb{I}_{[0,T-\theta_j]}(t) \cdot \\
&\qquad\qquad \textstyle\int_{\mathbb{R}^{n_1}} \mathcal{D}\zeta^j(\xi, t) \tfrac{\partial G_1}{\partial m}(\xi, x_0(t-\theta_j), m(t), \mathcal{D}\psi^j(\xi, t))(x_1)m^j(\xi, t)d\xi \\
&\qquad + \textstyle\sum_{j=1}^n p^j \mathbb{I}_{[0,T-\theta_j]}(t) \cdot \\
&\qquad\qquad \textstyle\int_{\mathbb{R}^{n_1}} \eta^j(\xi, t) \tfrac{\partial H_1}{\partial m}(\xi, x_0(t-\theta_j), m(t), \mathcal{D}\psi^j(\xi, t))(x_1)d\xi \Big) dt \\
&\qquad - K_{\zeta^k}(x_1, t)dW_0(t-a), \\
\zeta^k(x_1, T) &= \mathbb{E}^{\mathcal{F}^0_{T-a}} \tfrac{\partial h_0}{\partial m}(x_0(T), m(T))(x_1) \\
&\qquad\qquad + \textstyle\sum_{j=1}^n p^j \int_{\mathbb{R}^{n_1}} \eta^j(\xi, T) \tfrac{\partial h_1}{\partial m}(\xi, x_0(T-\theta_j), m(T))(x_1)d\xi. \\
\partial_t \eta^k(x_1, t) &= \Big(-\mathcal{A}^*\eta^k(x_1, t) - \operatorname{div}\Big(\eta^k(x_1, t)H_{1,\lambda}(x_1, x_0(t-\theta_k), m(t), \mathcal{D}\psi^k(x_1, t)) \\
&\quad + G_{1,\lambda}(x_1, x_0(t-\theta_k), m(t), \mathcal{D}\psi^k(x_1, t))\mathcal{D}\zeta^k(x_1, t)m(x_1, t) \Big) \Big) dt, \\
\eta^k(x_1, 0) &= 0.
\end{aligned}$$

2.3 Linear Quadratic Case

In this section we derive the solution for the linear quadratic problem from the six equations in (2.2.19) and (2.2.20) directly. Let

$$
\begin{aligned}
g_0(x_0, m, v_0) &= A_0 x_0 + B_0 \int \xi m(\xi) d\xi + C_0 v_0; \\
\sigma_0(x_0) &= \sigma_0; \\
g_1(x_1, x_0, m, v_1) &= A_1 x_1 + B_1 \int \xi m(\xi) d\xi + C_1 v_1 + D_1 x_0; \\
\sigma_1(x_1) &= \sigma_1; \\
f_0(x_0, m, v_0) &= \tfrac{1}{2}[|x_0 - E_0 \int \xi m(\xi) d\xi - G_0|^2_{Q_0} + v_0^* R_0 v_0]; \\
f_1(x_1, x_0, m, v_1) &= \tfrac{1}{2}[|x_1 - E_1 \int \xi m(\xi) d\xi - F_1 x_0 - G_1|^2_{Q_1} + v_1^* R_1 v_1]; \\
h_0(x_0, m) &= \tfrac{1}{2}[|x_0 - \bar{E}_0 \int \xi m(\xi) d\xi - \bar{G}_0|^2_{\bar{Q}_0}] \\
h_1(x_1, x_0, m) &= \tfrac{1}{2}[|x_1 - \bar{E}_1 \int \xi m(\xi) d\xi - \bar{F}_1 x_0 - \bar{G}_1|^2_{\bar{Q}_1}].
\end{aligned}
$$

2.3.1 Optimal Control for the Follower

For simplicity, we may write $z = \int \xi m(\xi) d\xi$. We can write down the Lagrangian for the representative follower

$$
\begin{aligned}
L_1(x_1, x_0, m, v_1, \lambda) &= \tfrac{1}{2}|x_1 - E_1 z - F_1 x_0 - G_1|^2_{Q_1} \\
&\quad + \lambda \cdot (A_1 x_1 + B_1 z + D_1 x_0) + \tfrac{1}{2} v_1^* R_1 v_1 + \lambda \cdot C_1 v_1.
\end{aligned}
$$

and the Hamiltonian

$$
\begin{aligned}
H_1(x_1, x_0, m, \lambda) &= \tfrac{1}{2}|x_1 - E_1 z - F_1 x_0 - G_1|^2_{Q_1} + \lambda \cdot (A_1 x_1 + B_1 z + D_1 x_0) \\
&\quad - \tfrac{1}{2} \lambda^* C_1 R_1^{-1} C_1^* \lambda
\end{aligned}
$$

in which the minimum is evaluated at

$$
u_1 = -R_1^{-1} C_1^* \lambda.
$$

Let $P(t)$, $\beta(t)$, $\alpha(t)$, $K_\beta(t)$ and $K_\alpha(t)$ be $\mathcal{F}^0_{t-\theta}$ adapted, and $P(t) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_1}$ is a symmetric matrix for all $t \in [0, T]$. We give the *Ansatz* that

$$
\psi(x_1, t) = \tfrac{1}{2} x_1^* P(t) x_1 + \beta^*(t) x_1 + \alpha(t)
$$

and

$$K_\psi(x_1,t) = K_\beta(t)x_1 + K_\alpha(t).$$

We then have $\mathcal{D}\psi(x_1,t) = P(t)x_1 + \beta(t)$ and $\mathcal{D}^2\psi(x_1,t) = P(t)$, and hence the optimal control for the representative follower is

$$u_1(t) = -R_1^{-1}C_1^*(P(t)x_1(t) + \beta(t)).$$

Then the coupled SHJB-FP becomes

$$\begin{cases} \partial_t m(x_1,t) &= \Big(-\mathcal{A}^* m(x_1,t) - \mathrm{div}[(A_1x_1 + B_1z(t) - C_1R_1^{-1}C_1^*(P(t)x_1+\beta(t)) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad + D_1x_0(t-\theta))m(x_1,t)]\Big)dt, \\ m(x_1,0) &= \omega(x_1); \end{cases}$$

$$\begin{cases} -\partial_t\psi(x_1,t) &= \Big[\frac{1}{2}|x_1 - E_1z(t) - F_1x_0(t-\theta) - G_1|^2_{Q_1} \\ & \qquad + (P(t)x_1+\beta(t))^*(A_1x_1 + B_1z(t) + D_1x_0(t-\theta)) \\ & \qquad - \frac{1}{2}(P(t)x_1+\beta(t))^*C_1R_1^{-1}C_1^*(P(t)x_1+\beta(t)) + \frac{1}{2}\mathrm{tr}[\sigma\sigma^*P(t)]\Big]dt \\ & \qquad - (K_\beta(t)x_1 + K_\alpha(t))dW_0(t-\theta); \\ \psi(x_1,T) &= \frac{1}{2}[|x_1 - \bar{E}_1z(T) - \bar{F}_1x_0(T-\theta) - \bar{G}_1|^2_{\bar{Q}_1}]. \end{cases}$$

Recall that $\partial_t\psi(x_1,t) = \frac{1}{2}x_1^*(dP(t))x_1 + d\beta^*(t)x_1 + d\alpha(t)$. Comparing coefficients yields

$$\begin{aligned} -dP(t) &= (Q_1 + P(t)A_1 + A_1^*P(t) - P(t)C_1R_1^{-1}C_1^*P(t))dt, \\ P(T) &= \bar{Q}. \end{aligned}$$

$$\begin{aligned} -d\beta(t) &= \Big[-Q_1(E_1z(t) + F_1x_0(t-\theta) + G_1) \\ & \qquad + P(t)(B_1z(t) + D_1x_0(t-\theta)) + A_1^*\beta(t) \\ & \qquad - P(t)C_1R_1^{-1}C_1^*\beta(t)\Big]dt - K_\beta(t)dW_0(t-\theta) \\ &= \Big[(A_1^* - P(t)C_1R_1^{-1}C_1^*)\beta(t) + (P(t)B_1 - Q_1E_1)z(t) \\ & \qquad + (P(t)D_1 - Q_1F_1)x_0(t-\theta) - Q_1G_1\Big]dt - K_\beta(t)dW_0(t-\theta), \\ \beta(T) &= -\bar{Q}_1(\bar{E}_1z(T) + \bar{F}_1x_0(T-\theta) - \bar{G}_1). \end{aligned}$$

$$\begin{aligned}
-d\alpha(t) &= \Big[\tfrac{1}{2}|E_1 z(t) + F_1 x_0(t-\theta) + G_1|^2_{Q_1} \\
&\qquad +\beta^*(t)(B_1 z(t) + D_1 x_0(t-\theta)) \\
&\qquad -\tfrac{1}{2}\beta^*(t) C_1 R_1^{-1} C_1^* \beta(t) + \tfrac{1}{2}\mathrm{tr}[\sigma\sigma^* P(t)]\Big] dt - K_\alpha(t) dW_0(t-\theta), \\
\alpha(T) &= \tfrac{1}{2}|\bar{E}_1 z(t) + \bar{F}_1 x_0(t-\theta) + \bar{G}_1|^2_{\bar{Q}_1}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
dz(t) &= d\int_{\mathbb{R}^{n_1}} x_1 m(x_1,t) dx_1 \\
&= \int_{\mathbb{R}^{n_1}} x_1 \Big(- A^* m(x_1,t) \\
&\qquad -\mathrm{div}[(A_1 x_1 + B_1 z(t) - C_1 R_1^{-1} C_1^* (P(t)x_1 + \beta(t)) + D_1 x_0(t-\theta)) m(x_1,t)]\Big) dx_1 dt \\
&= ((A_1 + B_1 - C_1 R_1^{-1} C_1^* P(t)) z(t) - C_1 R_1^{-1} C_1^* \beta(t) + D_1 x_0(t-\theta)) dt; \\
z(0) &= \mathbb{E}[\xi_1].
\end{aligned}$$

Observe that P satisfies a symmetric Riccati equation which guarantees a solution on $[0,T]$; on the other hand, the BSDE α always admits a solution once we solve for z and β. Hence the solvability of the mean field equilibrium, that is the coupled SHJB-FP, reduce to the solvability of the following FBSDE:

$$\begin{cases}
dz(t) = ((A_1 + B_1 - C_1 R_1^{-1} C_1^* P(t)) z(t) - C_1 R_1^{-1} C_1^* \beta(t) + D_1 x_0(t-\theta)) dt; \\
z(0) = \mathbb{E}[\xi_1].
\end{cases}$$
$$\begin{cases}
-d\beta(t) = \Big[(A_1 - C_1 R_1^{-1} C_1^* P(t))^* \beta(t) + (P(t) B_1 - Q_1 E_1) z(t) \\
\qquad\qquad +(P(t) D_1 - Q_1 F_1) x_0(t-\theta) - Q_1 G_1\Big] dt - K_\beta(t) dW_0(t-\theta), \\
\beta(T) = -\bar{Q}_1(\bar{E}_1 z(T) + \bar{F}_1 x_0(T-\theta) - \bar{G}_1).
\end{cases} \tag{2.3.21}$$

2.3.2 Optimal Control for the Leader

In this subsection we investigate the six equations (2.2.19,2.2.20) derived in the optimal control problem for the leader. Again we can write down the Lagrangian

$$L_0(x_0, m, v_0, \lambda) = \tfrac{1}{2}|x_0 - E_0 z - G_0|^2_{Q_0} + \lambda \cdot (A_0 x_0 + B_0 z) + \tfrac{1}{2} v_0^* R_0 v_0 + \lambda \cdot C_0 v_0.$$

Similarly, the Hamiltonian is given by

$$H_0(x_0, m, \lambda) = \tfrac{1}{2}|x_0 - E_0 z - G_0|^2_{Q_0} + \lambda \cdot (A_0 x_0 + B_0 z) - \tfrac{1}{2}\lambda^* C_0 R_0^{-1} C_0^* \lambda$$

in which the minimum point if evaluated at

$$u_0 = -R_0^{-1} C_0^* \lambda.$$

Similarly, for some $\mathcal{F}^0_{t-\theta}$ adapted q, γ, K_q and K_γ, we give the *Ansatz*:

$$\zeta(x_1, t) = q^*(t) x_1 + \gamma(t) \tag{2.3.22}$$

and

$$K_\zeta(x_1, t) = K_q(t) x_1 + K_\gamma(t).$$

Clearly we have $\mathcal{D}\zeta(x_1,t) = q(t)$ and $\mathcal{D}^2\zeta(x_1,t) = 0$. We write $r(t) = \int_{\mathbb{R}^{n_1}} x_1 \eta(x_1,t) dx_1$. To compute the Gâteaux derivative, for example, $\frac{\partial G_1}{\partial m}(x_1, x_0(t-\theta), m(t), \mathcal{D}\psi(x_1,t))(\cdot)$, we first have

$$\begin{aligned} & G_1(x_1, x_0(t-\theta), m(t), \mathcal{D}\psi(x_1,t)) \\ = \ & A_1 x_1 + B_1 \int_{\mathbb{R}^{n_1}} \xi m(\xi,t) d\xi - C_1 R_1^{-1} C_1^* \mathcal{D}\psi(x_1,t) + D_1 x_0(t-\theta). \end{aligned}$$

Hence

$$\begin{aligned} & \lim_{\epsilon \to 0} \tfrac{G_1(\cdot, m(t)+\epsilon \tilde{m}(t), \cdot) - G_1(\cdot, m(t), \cdot)}{\epsilon} \\ = \ & \int_{\mathbb{R}^{n_1}} (B_1 \xi)(\tilde{m}(\xi,t)) d\xi, \end{aligned}$$

in which we have $\frac{\partial G_1}{\partial m}(x_1, x_0(t-\theta), m(t), \mathcal{D}\psi(x_1,t))(\xi) = B_1\xi$. Other derivatives can be obtained by similar arguments. The six equations (2.2.19), (2.2.20) become

$$\left\{ \begin{aligned} dx_0(t) &= (A_0 x_0(t) + B_0 z(t) - C_0 R_0^{-1} C_0^* p(t)) dt + \sigma_0 dW_0(t), \\ x_0(t) &= \xi_0(t), \quad t \in [-\theta, 0]; \\ \partial_t m(x_1,t) &= \Big(-\mathcal{A}^* m(x_1,t) - \mathrm{div}[(A_1 x_1 + B_1 z(t) - C_1 R_1^{-1} C_1^* (P(t) x_1 + \beta(t)) \\ &\qquad\qquad\qquad\qquad + D_1 x_0(t-\theta)) m(x_1,t)] \Big) dt, \\ m(x_1, 0) &= \omega(x_1); \\ -\partial_t \psi(x_1,t) &= \Big[\tfrac{1}{2} |x_1 - E_1 z(t) - F_1 x_0(t-\theta) - G_1|^2_{Q_1} \\ &\quad + (P(t)x_1 + \beta(t))^* (A_1 x_1 + B_1 z(t) + D_1 x_0(t-\theta)) \\ &\quad - \tfrac{1}{2} (P(t) x_1 + \beta(t))^* C_1 R_1^{-1} C_1^* (P(t) x_1 + \beta(t)) + \tfrac{1}{2} \mathrm{tr}[\sigma\sigma^* P(t)] \Big] dt \\ &\quad - (K_\beta(t) x_1 + K_\alpha(t)) dW_0(t-\theta); \\ \psi(x_1, T) &= \tfrac{1}{2} [|x_1 - \bar{E}_1 z(T) - \bar{F}_1 x_0(T-\theta) - \bar{G}_1|^2_{\bar{Q}_1}]. \end{aligned} \right. \tag{2.3.23}$$

$$
\left\{
\begin{array}{rcl}
-dp(t) & = & \Big(A_0^*p(t) + Q_0(x_0(t) - E_0z(t) - G_0)\Big)dt - K_p(t)dW_0(t), \qquad t \in (T-\theta, T); \\
-dp(t) & = & \Big(A_0^*p(t) + Q_0(x_0(t) - E_0z(t) - G_0) \\
& & + \int_{\mathbb{R}^{n_1}} D_1^*q(t+\theta)m(x_1, t+\theta)dx_1 \\
& & + \int_{\mathbb{R}^{n_1}} \eta(x_1, t+\theta)[-(Q_1F_1)^*(x_1 - E_1z(t+\theta) - F_1x_0(t) - G_1) \\
& & \qquad +D_1^*(P(t+\theta)x_1 + \beta(t+\theta))]dx_1\Big)dt - K_p(t)dW_0(t), \\
& & \qquad t \in (0, T-\theta); \\
p(T) & = & \bar{Q}_0(x_0(T) - \bar{E}_0z(T) - \bar{G}_0); \\
p(T-\theta) & = & p((T-\theta)-) + \int_{\mathbb{R}^{n_1}} \eta(x_1, T)[(\bar{Q}_1\bar{F}_1)^*(x_1 - \bar{E}_1z(T) - \bar{F}_1x_0(T-\theta) - \bar{G}_1)]dx_1. \\
-\partial_t\zeta(x_1, t) & = & \Big(\mathbb{E}^{\mathcal{F}^0_{t-\theta}}[-(x_0(t) - E_0z(t) - G_0)^*(Q_0E_0)x_1 + p^*(t)B_0x_1] \\
& & +q^*(t)[A_1x_1 + B_1z(t) - C_1R_1^{-1}C_1^*(P(t)x_1 + \beta(t)) + D_1x_0(t-\theta)] \\
& & + \int_{\mathbb{R}^{n_1}} q^*(t)B_1x_1m(\xi, t)d\xi \\
& & + \int_{\mathbb{R}^{n_1}} \eta(\xi, t)[-(\xi - E_1z(t) - F_1x_0(t-\theta) - G_1)^*(Q_1E_1)x_1 \\
& & \qquad +(P(t)\xi + \beta(t))B_1x_1]d\xi\Big)dt \\
& & -(K_q(t)x_1 + K_\gamma(t))dW_0(t-\theta), \\
\zeta(x_1, T) & = & \mathbb{E}^{\mathcal{F}^0_{T-\theta}}[-(x_0(T) - \bar{E}_0z(T) - \bar{G}_0)^*(\bar{Q}_0\bar{E}_0)x_1] \\
& & \quad + \int_{\mathbb{R}^{n_1}} \eta(\xi, T)[-(\zeta - \bar{E}_1z(T) - \bar{F}_1x_0(T-\theta) - \bar{G}_1)^*(\bar{Q}_1\bar{E}_1)x_1]d\xi. \\
\partial_t\eta(x_1, t) & = & \Big(-\mathcal{A}^*\eta(x_1, t) \\
& & \quad -\mathrm{div}\Big(\eta(x_1, t)[A_1x_1 + B_1z(t) - C_1R_1^{-1}C_1^*(P(t)x_1 + \beta(t)) + D_1x_0(t-\theta)] \\
& & \qquad -C_1R_1^{-1}C_1^*q(t)m(x_1, t)\Big)\Big)dt, \\
\eta(x_1, 0) & = & 0.
\end{array}
\right.
\tag{2.3.24}
$$

From (2.3.22), we have $\partial_t\zeta(x_1, t) = dq^*(t)x_1 + (d\gamma(t))$; while $\int_{\mathbb{R}^{n_1}} \eta(x_1, t)dx_1 = 0$. Comparing coefficients yields

$$
\begin{array}{rcl}
dq(t) & = & \Big((A_1 + B_1 - C_1R_1^{-1}C_1^*P(t))^*q(t) + B_0^*\mathbb{E}^{\mathcal{F}^0_{t-\theta}}p(t) + (P(t)B_1 - Q_1E_1)^*r(t) \\
& & \quad -(Q_0E_0)^*(\mathbb{E}^{\mathcal{F}^0_{t-\theta}}x_0(t) - E_0z(t) - G_0)\Big)dt - K_q(t)dW_0(t-\theta), \\
q(T) & = & -(\bar{Q}_0\bar{E}_0)^*(\mathbb{E}^{\mathcal{F}^0_{T-\theta}}x_0(T) - \bar{E}_0z(T) - \bar{G}_0) - (\bar{Q}_1\bar{E}_1)^*r(T).
\end{array}
$$

$$\begin{aligned}
d\gamma(t) &= q(t)^*[B_1 z(t) - C_1 R_1^{-1} C_1^* \beta(t) + D_1 x_0(t-\theta)]dt - Z_\gamma(t) dW_0(t-\theta), \\
\gamma(T) &= 0.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
dr(t) &= \int_{\mathbb{R}^{n_1}} x_1 (\partial_t \eta(x_1, t)) dx_1 \\
&= \int_{\mathbb{R}^{n_1}} x_1 \Big(-\mathcal{A}^* \eta(x_1, t) - \mathrm{div}\Big(\eta(x_1, t)[A_1 x_1 + B_1 z(t) - C_1 R_1^{-1} C_1^* (P(t) x_1 + \beta(t)) \\
&\qquad + D_1 x_0(t-\theta)] - C_1 R_1^{-1} C_1^* q(t) m(x_1, t) \Big) \Big) dx_1 dt \\
&= (A_1 - C_1 R_1^{-1} C_1^* P(t)) r(t) - C_1 R_1^{-1} C_1^* q(t), \\
r(0) &= 0.
\end{aligned}$$

Note that the BSDE γ always admits a solution once we can solve for q, z, β and x_0. Hence, under the LQ setting, the solvability of the original six equations (2.3.23, 2.3.24) reduced to the solvability of the following FBSDEs:

$$\left\{ \begin{aligned}
dx_0(t) &= (A_0 x_0(t) + B_0 z(t) - C_0 R_0^{-1} C_0^* p(t)) dt + \sigma_0 dW_0(t), \\
x_0(t) &= \xi_0(t), \quad t \in [-\theta, 0]; \\
dz(t) &= ((A_1 + B_1 - C_1 R_1^{-1} C_1^* P(t)) z(t) - C_1 R_1^{-1} C_1^* \beta(t) + D_1 x_0(t-\theta)) dt; \\
z(0) &= \mathbb{E}[\xi_1]. \\
-d\beta(t) &= \Big[(A_1 - C_1 R_1^{-1} C_1^* P(t))^* \beta(t) + (P(t) B_1 - Q_1 E_1) z(t) \\
&\qquad + (P(t) D_1 - Q_1 F_1) x_0(t-\theta) - Q_1 G_1 \Big] dt - K_\beta(t) dW_0(t-\theta), \\
\beta(T) &= -\bar{Q}_1 (\bar{E}_1 z(T) + \bar{F}_1 x_0(T-\theta) - \bar{G}_1).
\end{aligned} \right. \tag{2.3.25}$$

$$\begin{cases} -dp(t) &= \Big(A_0^* p(t) + Q_0(x_0(t) - E_0 z(t) - G_0)\Big)dt \\ & \quad -K_p(t)dW_0(t), \qquad\qquad t \in (T-\theta, T); \\ -dp(t) &= \Big(A_0^* p(t) + Q_0(x_0(t) - E_0 z(t) - G_0) \\ & \quad +D_1^* q(t+\theta) + (P(t+\theta)D_1 - Q_1 F_1)^* r(t+\theta)\Big)dt \\ & \quad -K_p(t)dW_0(t), \qquad\qquad t \in (0, T-\theta); \\ p(T) &= \bar{Q}_0(x_0(T) - \bar{E}_0 z(T) - \bar{G}_0); \\ p(T-\theta) &= p((T-\theta)-) + (\bar{Q}_1 \bar{F}_1)^* r(T). \\ -dq(t) &= \Big((A_1 + B_1 - C_1 R_1^{-1} C_1^* P(t))^* q(t) + B_0^* \mathbb{E}^{\mathcal{F}^0_{t-\theta}} p(t) \\ & \quad +(P(t)B_1 - Q_1 E_1)^* r(t) \\ & \quad -(Q_0 E_0)^* (\mathbb{E}^{\mathcal{F}^0_{t-\theta}} x_0(t) - E_0 z(t) - G_0)\Big)dt - K_q(t)dW_0(t-\theta), \\ q(T) &= -(\bar{Q}_0 \bar{E}_0)^* (\mathbb{E}^{\mathcal{F}^0_{T-\theta}} x_0(T) - \bar{E}_0 z(T) - \bar{G}_0) - (\bar{Q}_1 \bar{E}_1)^* r(T). \\ dr(t) &= ((A_1 - C_1 R_1^{-1} C_1^* P(t)) r(t) - C_1 R_1^{-1} C_1^* q(t))dt, \\ r(0) &= 0. \end{cases} \tag{2.3.26}$$

2.4 Sufficient Condition for Unique Existence

In this section we demonstrate the sufficient condition which guarantees the existence of a unique solution to the forward backward stochastic differential equation (2.3.25), (2.3.26). We first show the following lemma regarding the solvability of system (2.3.21):

Lemma 2.4.1. *Let x_0 be a given square integrable $\mathcal{F}^0$ adapted process. Suppose that the following conditions are satisfied:*

$$\begin{cases} \lambda_{\min}(Q_1(I - E_1)) - \dfrac{\|B_1^* B_1\|}{2\lambda_{\min}(C_1 R_1^{-1} C_1^*)} := K_1 > 0, \\ \lambda_{\min}(\bar{Q}_1(I - \bar{E}_1)) := K_2 > 0; \end{cases} \tag{2.4.27}$$

then the forward backward system (2.3.21) *admits a unique solution.*

Proof. To obtain the condition (2.4.27), which is independent of time, we first denote

$$\beta'(t) = P(t)z(t) + \beta(t). \tag{2.4.28}$$

We argue that the drift coefficients of z and β' is then independent of t. In particular, a simple application of the Itô's formula shows that (z, β') satisfies:

$$\begin{cases} dz(t) = ((A_1 + B_1)z(t) - C_1 R_1^{-1} C_1^* \beta'(t) + D_1 x_0(t-\theta))dt; \\ z(0) = \mathbb{E}[\xi_1]; \\ -d\beta'(t) = \Big[A_1^* \beta'(t) + Q_1(I - E_1)z(t) - Q_1 F_1 x_0(t-\theta) - Q_1 G_1\Big]dt - K_{\beta'}(t)dW_0(t-\theta), \\ \beta'(T) = \bar{Q}_1(I - \bar{E}_1)z(T) - \bar{Q}_1 \bar{F}_1 x_0(T-\theta) - \bar{Q}_1 \bar{G} 6_1. \end{cases} \tag{2.4.29}$$

Define two linear operators

$$\mathbb{A} : \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_1} \to \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_1},$$

and

$$\bar{\mathbb{A}} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}.$$

In particular, $\mathbb{A}$ is defined through the drift coefficients in system (2.4.29)

$$\mathbb{A} \begin{pmatrix} z \\ \beta' \end{pmatrix} = \begin{pmatrix} -A_1^* \beta' - Q_1(I - E_1)z \\ (A_1 + B_1)z - C_1 R_1^{-1} C_1^* \beta' \end{pmatrix};$$

while $\bar{\mathbb{A}}$ associates with the terminal condition in (2.4.29)

$$\bar{\mathbb{A}}(z) = \bar{Q}_1(I - \bar{E}_1)z.$$

To obtain a unique solution of (2.4.29), it suffices to check the monotonicity condition proposed in Hu and Peng [25] is satisfied. Providing there is no ambiguity,

$\langle\cdot\rangle$ represents the usual inner product on Euclidean space. To be precise, we have,

$$\begin{aligned}\left\langle \mathbb{A}\begin{pmatrix} z \\ \beta' \end{pmatrix}, \begin{pmatrix} z \\ \beta' \end{pmatrix} \right\rangle &= \langle B_1 z, \beta' \rangle - \langle Q_1(I-E_1)z, z\rangle - \langle \beta', C_1 R_1^{-1} C_1^* \beta' \rangle \\ &\leq \frac{\|B_1^* B_1\|}{4\lambda_{\min}(C_1 R_1^{-1} C_1^*)}\|z\|^2 + \lambda_{\min}(C_1 R_1^{-1} C_1^*)\|\beta'\|^2 \\ &\quad - \lambda_{\min}(Q_1(I-E_1))\|z\|^2 - \lambda_{\min}(C_1 R_1^{-1} C_1^*)\|\beta'\|^2 \\ &\leq \Big(\frac{\|B_1^* B_1\|}{4\lambda_{\min}(C_1 R_1^{-1} C_1^*)} - \lambda_{\min}(Q_1(I-E_1))\Big)\|z\|^2;\end{aligned}$$

where we apply the Young's inequality in the second row. On the other hand,

$$\langle \bar{\mathbb{A}}(z_0), z_0 \rangle = \langle \bar{Q}_1(I-\bar{E}_1)z_0, z_0\rangle \geq \lambda_{\min}(\bar{Q}_1(I-\bar{E}_1))\|z_0\|^2.$$

We conclude that the monotonicity condition is satisfied providing the statement hypothesis holds. After obtaining the unique existence of the pair (z, β'), we can recover the original (z, β) by reading equation (2.4.28) from right hand side. $\square$

Remark 2.4.2. *Recall that Q_1, $\bar{Q}_1$ and $C_1 R_1^{-1} C_1^*$ are positive definite matrices. Lemma 2.4.1 suggests that providing the influences generated by the mean field term (or the magnitude of B_1, E_1 and $\bar{E}_1$) on the followers are small, the mean field term system* (2.4.1) *admits a unique solution for any given state process of the leader (x_0).*

We assume that the two sufficient conditions (2.4.27) in Lemma 2.4.1 are satisfied in the rest of this work. As in Bensoussan et al. [5] and Bensoussan et al. [4], we can decompose forward backward system (2.3.21) such that $(z, \beta, K_\beta) = (z_0, \beta_0, K_{\beta_0}) + (z_c, \beta_c, 0)$, where $(z_0, \beta_0, K_{\beta_0})$ is linear to x_0 and $(z_c, \beta_c, 0)$ is deterministic (and hence the third element vanishes).

$$\begin{cases} dz_0(t) &= ((A_1 + B_1 - C_1 R_1^{-1} C_1^* P(t))z_0(t) - C_1 R_1^{-1} C_1^* \beta_0(t) + D_1 x_0(t-\theta))dt; \\ z_0(0) &= 0. \\ -d\beta_0(t) &= \Big[(A_1 - C_1 R_1^{-1} C_1^* P(t))^* \beta_0(t) + (P(t)B_1 - Q_1 E_1)z_0(t) \\ & \qquad + (P(t)D_1 - Q_1 F_1)x_0(t-\theta)\Big]dt - K_{\beta_0}(t)dW_0(t-\theta), \\ \beta_0(T) &= -\bar{Q}_1(\bar{E}_1 z_0(T) + \bar{F}_1 x_0(T-\theta)). \end{cases}$$

$$\begin{cases} dz_c(t) &= ((A_1 + B_1 - C_1R_1^{-1}C_1^*P(t))z_c(t) - C_1R_1^{-1}C_1^*\beta_c(t))dt; \\ z_c(0) &= \mathbb{E}[\xi_1]. \\ -d\beta_c(t) &= \Big[(A_1 - C_1R_1^{-1}C_1^*P(t))^*\beta_c(t) + (P(t)B_1 - Q_1E_1)z_c(t) - Q_1G_1\Big]dt, \\ \beta_c(T) &= -\bar{Q}_1(\bar{E}_1 z_c(T) - \bar{G}_1). \end{cases}$$

Let $\kappa \geq 0$ and $n \geq 1$, consider the Hilbert Space:

$\mathcal{H}(\kappa, n) = \{\{f(t)\}_{t\in[-\kappa, T-\kappa]} : f$ is progressively measurable $(w.r.t.\mathcal{B}([-\kappa, t])\otimes\mathcal{F}^0_{t-\theta+\kappa})$ in $\mathbb{R}^n.\}$

with inner product

$$\langle f_1, f_2\rangle_{\mathcal{H}(\kappa,n)} = \mathbb{E}\Big[\int_{-\kappa}^{T-\kappa} f_1(t)\cdot f_2(t)dt + f_1(T-\kappa)\cdot f_2(T-\kappa)\Big].$$

Consider the linear operator

$$\mathcal{L} : \mathcal{H}(\theta, n_0) \to \mathcal{H}(0, n_1)$$

defined by

$$\mathcal{L}(x_0)(t) = z_0(t). \tag{2.4.30}$$

The linear operator $\mathcal{L}$ is well defined as shown in Lemma 2.4.1; we argue in the following that it is also bounded:

Lemma 2.4.3. *The linear operator $\mathcal{L}$ defined in* (2.4.30) *is bounded.*

Proof. Similar to the proof in Lemma 2.4.1, we define

$$\beta_0'(t) = P(t)z(t) + \beta_0(t),$$

where (z_0, β_0') satisfies the forward backward stochastic differential equation:

$$\begin{cases} dz_0(t) = ((A_1 + B_1)z_0(t) - C_1R_1^{-1}C_1^*\beta_0'(t) + D_1x_0(t-\theta))dt; \\ z(0) = 0; \\ -d\beta_0'(t) = \Big[A_1^*\beta_0'(t) + Q_1(I - E_1)z_0(t) - Q_1F_1x_0(t-\theta)\Big]dt - K_{\beta'}(t)dW_0(t-\theta), \\ \beta_0'(T) = \bar{Q}_1(I - \bar{E}_1)z_0(T) - \bar{Q}_1\bar{F}_1x_0(T-\theta). \end{cases}$$

Applying Itô's formula on the inner product $\langle z_0(t), \beta_0'(t)\rangle$ yields

$$\begin{aligned}\mathbb{E}[\langle z_0(T), \beta_0'(T)\rangle] = \mathbb{E}\int_0^T \langle B_1 z_0(t), \beta_0'(t)\rangle dt \\ - \mathbb{E}\int_0^T \langle C_1 R_1^{-1} C_1^* \beta_0'(t), \beta_0'(t)\rangle dt - \mathbb{E}\int_0^T \langle Q_1(I-E_1) z_0(t), z_0(t)\rangle dt \\ + \mathbb{E}\int_0^T \langle D_1 x_0(t-\theta), \beta_0'(t)\rangle dt + \mathbb{E}\int_0^T \langle Q_1 F_1 x_0(t-\theta), z_0(t)\rangle dt.\end{aligned}$$

Together with the terminal condition of β_0', we have

$$\begin{aligned}&\mathbb{E}[\langle z_0(T), \bar{Q}_1(I-\bar{E}_1) z_0(T)\rangle] \\ &= \mathbb{E}[\langle z_0(T), \bar{Q}_1 \bar{F}_1 x_0(T-\theta)\rangle] + \mathbb{E}\int_0^T \langle B_1 z_0(t), \beta_0'(t)\rangle dt \\ &\quad - \mathbb{E}\int_0^T \langle C_1 R_1^{-1} C_1^* \beta_0'(t), \beta_0'(t)\rangle dt - \mathbb{E}\int_0^T \langle Q_1(I-E_1) z_0(t), z_0(t)\rangle dt \\ &\quad + \mathbb{E}\int_0^T \langle D_1 x_0(t-\theta), \beta_0'(t)\rangle dt + \mathbb{E}\int_0^T \langle Q_1 F_1 x_0(t-\theta), z_0(t)\rangle dt.\end{aligned}$$

Applying the Young's inequality as before gives

$$\begin{aligned}&\lambda_{\min}(\bar{Q}_1(I-\bar{E}_1))\|z_0(T)\|_{L^2}^2 \\ &\le \frac{\lambda_{\min}(\bar{Q}_1(I-\bar{E}_1))}{2}\|z_0(T)\|_{L^2}^2 + \frac{\|\bar{F}_1^* \bar{Q}_1 \bar{Q}_1 \bar{F}_1\|}{2\lambda_{\min}(\bar{Q}_1(I-\bar{E}_1))}\|x_0(T-\theta)\|_{L^2}^2 \\ &\quad + \frac{\|B_1^* B_1\|}{2\lambda_{\min}(C_1 R_1^{-1} C_1^*)}\|z_0\|_{L^2([0,T])}^2 + \frac{\lambda_{\min}(C_1 R_1^{-1} C_1^*)}{2}\|\beta_0'\|_{L^2([0,T])}^2 \\ &\quad - \lambda_{\min}(C_1 R_1^{-1} C_1^*)\|\beta_0'\|_{L^2([0,T])}^2 - \lambda_{\min}(Q_1(I-E_1))\|z_0\|_{L^2([0,T])}^2 \\ &\quad + \frac{\|D_1^* D_1\|}{2\lambda_{\min}(C_1 R_1^{-1} C_1^*)}\|x_0\|_{L^2[-\theta,T-\theta]}^2 + \frac{\lambda_{\min}(C_1 R_1^{-1} C_1^*)}{2}\|\beta_0'\|_{L^2([0,T])}^2 \\ &\quad + \frac{\|F_1^* Q_1 Q_1 F_1\|}{2K_1}\|x_0\|_{L^2[-\theta,T-\theta]}^2 + \frac{K_1}{2}\|z_0\|_{L^2([0,T])}^2.\end{aligned}$$

We can hence obtain the estimate

$$\begin{aligned}&K_2\|z_0(T)\|_{L^2}^2 + K_1\|z_0\|_{L^2([0,T])}^2 \\ &\le \frac{\|\bar{F}_1^* \bar{Q}_1 \bar{Q}_1 \bar{F}_1\|}{\lambda_{\min}(\bar{Q}_1(I-\bar{E}_1))}\|x_0(T-\theta)\|_{L^2}^2 \\ &\qquad + \Big[\frac{\|D_1^* D_1\|}{\lambda_{\min}(C_1 R_1^{-1} C_1^*)} + \frac{\|F_1^* Q_1 Q_1 F_1\|}{K_1}\Big]\|x_0\|_{L^2[-\theta,T-\theta]}^2, \\ &=: K_3\|x_0(T-\theta)\|_{L^2}^2 + K_4\|x_0\|_{L^2[-\theta,T-\theta]}^2\end{aligned}$$

which concludes that L is bounded. $\square$

Denote

$$K_5 = \frac{K_3 \vee K_4}{K_1 \wedge K_2}, \tag{2.4.31}$$

we easily have $\|\mathcal{L}\|^2 \leq K_5$. Once the boundedness of $\mathcal{L}$ is guaranteed, by the Risez representation theorem, the adjoint $\mathcal{L}^*$ uniquely exists. In particular, for any $f \in \mathcal{H}(0, n_1)$ and $g \in \mathcal{H}(\theta, n_0)$, we have $\langle f, \mathcal{L}(g)\rangle_{\mathcal{H}(0,n_1)} = \langle \mathcal{L}^*(f), g\rangle_{\mathcal{H}(\theta,n_0)}$. That is:.

$$\begin{aligned} &\mathbb{E}\Big[\textstyle\int_0^T f(t) \cdot \mathcal{L}(g)(t)dt + f(T) \cdot \mathcal{L}(g)(T)\Big] \\ &\qquad = \mathbb{E}\Big[\textstyle\int_{-\theta}^{T-\theta} \mathcal{L}^*(f)(t) \cdot g(t)dt + \mathcal{L}^*(f)(T-\theta) \cdot g(T-\theta)\Big]. \end{aligned} \tag{2.4.32}$$

The explicit form of $\mathcal{L}^*$ is also given by the next theorem:

Theorem 2.4.4.

$$\begin{aligned} \mathcal{L}^*(f)(t) &= D_1^* q(t+\theta) + (P(t+\theta)D_1 - Q_1F_1)^* r(t+\theta), \quad t \in [-\theta, T-\theta), \\ \mathcal{L}^*(f)(T-\theta) &= -(\bar{Q}_1\bar{F}_1)^* r(T); \end{aligned}$$

where

$$\begin{cases} -dq(t) &= \Big((A_1 + B_1 - C_1R_1^{-1}C_1^*P(t))^*q(t) + (P(t)B_1 - Q_1E_1)^*r(t) + f(t)\Big)dt \\ & \qquad -K_q(t)dW_0(t-\theta), \\ q(T) &= f(T) - (\bar{Q}_1\bar{E}_1)^*r(T). \\ dr(t) &= (A_1 - C_1R_1^{-1}C_1^*P(t))r(t) - C_1R_1^{-1}C_1^*q(t), \\ r(0) &= 0. \end{cases}$$

Proof. Consider the difference of the inner products

$$\begin{aligned} & d(\langle q, z_0\rangle - \langle r, \beta_0\rangle) \\ = \ & \Big[q^*(t)D_1x_0(t-\theta) - f^*(t)z_0(t) + r^*(t)(P(t)D_1 - Q_1F_1)x_0(t-\theta)\Big]dt \\ & \quad + z_0^*(t)K_q(t)dW_0(t-\theta) - r^*(t)K_{\beta_0}(t)dW_0(t-\theta). \end{aligned}$$

Taking integration on $[0, T]$ and expectation on both sides yields

$$\begin{aligned} & \mathbb{E}\Big[(f^*(T) - r^*(T)(\bar{Q}_1\bar{E}_1))z_0(T) + r^*(T)\bar{Q}_1(\bar{E}_1z_0(T) + \bar{F}_1x_0(T-\theta))\Big] \\ = \ & \mathbb{E}\textstyle\int_0^T \Big[q^*(t)D_1x_0(t-\theta) - f^*(t)z_0(t) + r^*(t)(P(t)D_1 - Q_1F_1)x_0(t-\theta)\Big]dt. \end{aligned}$$

After rearranging, and by definition $z_0(t) = \mathcal{L}(x_0)(t)$, we have

$$\begin{aligned}
& \mathbb{E}\Big[\int_0^T f(t)\cdot\mathcal{L}(x_0)(t)dt + f(T)\cdot\mathcal{L}(x_0)(T)\Big] \\
= \ & \mathbb{E}\Big[\int_{-\theta}^{T-\theta}(D_1^* q(t+\theta) + (P(t+\theta)D_1 - Q_1F_1)^* r(t+\theta))\cdot x_0(t)dt \\
& \quad -(\bar{Q}_1\bar{F}_1)^* r(T)\cdot x_0(T-\theta)\Big].
\end{aligned}$$

□

By putting

$$\begin{cases}
f(t) = B_0^*\mathbb{E}^{\mathcal{F}^0_{t-\theta}} p(t) \\
\qquad\qquad -(Q_0E_0)^*(\mathbb{E}^{\mathcal{F}^0_{t-\theta}} x_0(t) - E_0(\mathcal{L}(x_0)(t) + z_c(t)) - G_0), \quad t\in[0,T); \\
f(T) = -(\bar{Q}_0\bar{E}_0)^*(\mathbb{E}^{\mathcal{F}^0_{T-\theta}} x_0(T) - \bar{E}_0(\mathcal{L}(x_0)(T) + z_c(T)) - \bar{G}_0),
\end{cases}$$

in the explicit adjoint operator $\mathcal{L}^*$ given in Theorem 3.2.7, we can represent the original six equations derived in (2.3.25) and (2.3.26) in the following functional form:

$$\begin{cases}
dx_0(t) = (A_0x_0(t) + B_0(\mathcal{L}(x_0)(t) + z_c(t)) - C_0R_0^{-1}C_0^*p(t))dt + \sigma_0 dW_0(t), \\
x_0(t) = \xi_0(t), \quad t\in[-\theta,0]; \\
\\
-dp(t) = \Big(A_0^*p(t) + Q_0(x_0(t) - E_0(\mathcal{L}(x_0)(t) + z_c(t)) - G_0)\Big)dt \\
\qquad\qquad -K_p(t)dW_0(t), \qquad\qquad t\in(T-\theta,T); \\
-dp(t) = \Big(A_0^*p(t) + Q_0(x_0(t) - E_0(\mathcal{L}(x_0)(t) + z_c(t)) - G_0) + \mathcal{L}^*(f)(t)\Big)dt \\
\qquad\qquad -K_p(t)dW_0(t), \qquad\qquad t\in(0,T-\theta); \\
p(T) = \bar{Q}_0(x_0(T) - \bar{E}_0(\mathcal{L}(x_0)(T) + z_c(T)) - \bar{G}_0); \\
p(T-\theta) = p((T-\theta)-) - \mathcal{L}^*(f)(T-\theta).
\end{cases} \tag{2.4.33}$$

The next theorem provides time independent sufficient conditions which guarantee the unique existence of a solution to the forward backward stochastic functional differential equation (2.4.33).

Theorem 2.4.5. *Define the constants* $K_6 = \lambda_{\min}(\bar{Q}_0) \wedge \lambda_{\min}(Q_0)$, $K_7 = \lambda_{\min}(C_0 R_0^{-1} C_0^*)$, *and* K_5 *is given in* (2.4.31). *In addition to the assumptions in Lemma 2.4.1, suppose that the following conditions are satisfied:*

$$\begin{cases} 2K_5\Big[\dfrac{\|B_0^* B_0\|}{2K_7(K_6 \wedge K_7)} \\ \qquad + \dfrac{2(\|\bar{E}_0^* \bar{Q}_0 \bar{Q}_0 \bar{E}_0\| \vee \|E_0^* Q_0 Q_0 E_0\|) + \|\bar{E}_0^* \bar{Q}_0 \bar{E}_0\|^2 \vee \|E_0^* Q_0 E_0\|^2}{K_6(K_6 \wedge K_7)}\Big] < 1; \\ \dfrac{2K_5 \|B_0^* B_0\|}{K_6(K_6 \wedge K_7)} < 1. \end{cases} \tag{2.4.34}$$

Then the Forward Backward Stochastic Functional Differential Equation (2.4.33) *admits a unique solution.*

Proof. Consider the Hilbert spaces

$$\mathbb{H}_1 = \{\{f\}_{t\in[-\theta,T]} : f \text{ is progressively measurable } (w.r.t. \mathcal{B}([-\theta,t]) \otimes \mathcal{F}_t^0) \text{ in } \mathbb{R}^{n_0}; \\ f(t) = \xi_0(t), t \in [-\theta, 0].\}$$

and

$$\mathbb{H}_2 = \{\{f\}_{t\in[0,T]} : f \text{ is progressively measurable } (w.r.t. \mathcal{B}([0,t]) \otimes \mathcal{F}_t^0) \text{ in } \mathbb{R}^{n_0}.\}$$

with corresponding norms

$$\|f\|_{\mathbb{H}_1}^2 = \|f(T)\|_{L^2}^2 + \|f\|_{L^2([0,T])}^2,$$

and

$$\|f\|_{\mathbb{H}_2}^2 = \|f\|_{L^2([0,T])}^2.$$

Let $\mathbb{X}_0 \in \mathbb{H}_1$, $\mathbb{P} \in \mathbb{H}_2$. Consider the mapping $\mathbb{T} : (\mathbb{X}_0, \mathbb{P}) \in \mathbb{H}_1 \times \mathbb{H}_2 \mapsto (x, p) \in$

$\mathbb{H}_1 \times \mathbb{H}_2$ defined by

$$\left\{\begin{array}{rcl} dx_0(t) &=& (A_0 x_0(t) + B_0(\mathcal{L}(\mathbb{X}_0)(t) + z_c(t)) - C_0 R_0^{-1} C_0^* p(t))dt + \sigma_0 dW_0(t), \\ x_0(t) &=& \xi_0(t), \quad t \in [-\theta, 0]; \\ \\ -dp(t) &=& \Big(A_0^* p(t) + Q_0(x_0(t) - E_0(\mathcal{L}(\mathbb{X}_0)(t) + z_c(t)) - G_0)\Big)dt \\ && \quad -K_p(t)dW_0(t), \hfill t \in (T-\theta, T); \\ -dp(t) &=& \Big(A_0^* p(t) + Q_0(x_0(t) - E_0(\mathcal{L}(\mathbb{X}_0)(t) + z_c(t)) - G_0) + \mathcal{L}^*(\mathbb{F})(t)\Big)dt \\ && \quad -K_p(t)dW_0(t), \hfill t \in (0, T-\theta); \\ p(T) &=& \bar{Q}_0(x_0(T) - \bar{E}_0(\mathcal{L}(\mathbb{X}_0)(T) + z_c(T)) - \bar{G}_0); \\ p(T-\theta) &=& p((T-\theta)-) - \mathcal{L}^*(\mathbb{F})(T-\theta). \end{array}\right. \tag{2.4.35}$$

Here $\mathbb{F} \in \mathcal{H}(0, n_1)$ is defined by

$$\left\{\begin{array}{rcl} \mathbb{F}(t) &=& B_0^* \mathbb{E}^{\mathcal{F}^0_{t-\theta}} \mathbb{P}(t) \\ && \quad -(Q_0 E_0)^* (\mathbb{E}^{\mathcal{F}^0_{t-\theta}} \mathbb{X}_0(t) - E_0(\mathcal{L}(\mathbb{X}_0)(t) + z_c(t)) - G_0), \quad t \in [0, T). \\ \mathbb{F}(T) &=& -(\bar{Q}_0 \bar{E}_0)^* (\mathbb{E}^{\mathcal{F}^0_{T-\theta}} \mathbb{X}_0(T) - \bar{E}_0(\mathcal{L}(\mathbb{X}_0)(T) + z_c(T)) - \bar{G}_0). \end{array}\right. \tag{2.4.36}$$

We first argue that the mapping $\mathbb{T}$ is well defined. In particular, for any given $\mathbb{X}_0, \mathbb{P}$, system (2.4.35) is a classical linear forward backward stochastic differential equation. The monotonicity condition suggested in Hu and Peng [25], which guarantees the existence of a unique solution to (2.4.35), can easily be checked.

It remains to show that $\mathbb{T}$ is a contraction under the conditions (2.4.34). Denote $(\mathbb{X}_0, \mathbb{P})$ and $(\mathbb{X}_0', \mathbb{P}')$ the two inputs into $\mathbb{T}$, with corresponding outputs (x_0, p)

and (x_0', p'). The difference $(\hat{x}_0, \hat{p}) := (x_0 - x_0', p - p')$ satisfies:

$$\begin{cases} d\hat{x}_0(t) &=& (A_0\hat{x}_0(t) + B_0\mathcal{L}(\hat{\mathbb{X}}_0)(t) - C_0R_0^{-1}C_0^*\hat{p}(t))dt, \\ \hat{x}_0(t) &=& 0, \quad t \in [-\theta, 0]; \\ \\ -d\hat{p}(t) &=& \Big(A_0^*\hat{p}(t) + Q_0(\hat{x}_0(t) - E_0\mathcal{L}(\hat{\mathbb{X}}_0)(t))\Big)dt \\ & & \qquad -K_{\hat{p}}(t)dW_0(t), \qquad\qquad t \in (T-\theta, T); \\ -d\hat{p}(t) &=& \Big(A_0^*\hat{p}(t) + Q_0(\hat{x}_0(t) - E_0\mathcal{L}(\hat{\mathbb{X}}_0)(t)) + \mathcal{L}^*(\hat{\mathbb{F}})(t)\Big)dt \\ & & \qquad -K_{\hat{p}}(t)dW_0(t), \qquad\qquad t \in (0, T-\theta); \\ \hat{p}(T) &=& \bar{Q}_0(\hat{x}_0(T) - \bar{E}_0\mathcal{L}(\hat{\mathbb{X}}_0)(T)); \\ \hat{p}(T-\theta) &=& \hat{p}((T-\theta)-) - \mathcal{L}^*(\hat{\mathbb{F}})(T-\theta). \end{cases}$$

$\hat{\mathbb{X}}_0$ and $\hat{\mathbb{P}}$ are defined similarly as the difference between the inputs. We also have

$$\begin{cases} \hat{\mathbb{F}}(t) &=& B_0^*\mathbb{E}^{\mathcal{F}^0_{t-\theta}}\hat{\mathbb{P}}(t) - (Q_0E_0)^*(\mathbb{E}^{\mathcal{F}^0_{t-\theta}}\hat{\mathbb{X}}_0(t) - E_0\mathcal{L}(\hat{\mathbb{X}}_0)(t)), \quad t \in [0, T) \\ \hat{\mathbb{F}}(T) &=& -(\bar{Q}_0\bar{E}_0)^*(\mathbb{E}^{\mathcal{F}^0_{T-\theta}}\hat{\mathbb{X}}_0(T) - \bar{E}_0\mathcal{L}(\hat{\mathbb{X}}_0)(T)). \end{cases}$$

Applying Itô's formula on the inner product $\langle \hat{x}_0, \hat{p} \rangle$, we have

$$\begin{aligned} &\mathbb{E}[\langle \hat{x}_0(T), \bar{Q}\hat{x}_0(T)\rangle] - \mathbb{E}[\langle \hat{x}_0(T), \bar{Q}_0\bar{E}_0\mathcal{L}(\hat{\mathbb{X}}_0)(T)\rangle] + \mathbb{E}[\langle \hat{x}_0(T-\theta), \mathcal{L}^*(\hat{\mathbb{F}})(T-\theta)\rangle] \\ = &- \mathbb{E}[\int_0^T \langle \hat{p}(t), C_0R_0^{-1}C_0^*\hat{p}(t)\rangle dt] - \mathbb{E}[\int_0^T \langle \hat{x}(t), Q_0\hat{x}(t)\rangle dt] \\ &+ \mathbb{E}[\int_0^T \langle B_0\mathcal{L}(\hat{\mathbb{X}})(t), \hat{p}(t)\rangle dt] + \mathbb{E}[\int_0^T \langle Q_0E_0\mathcal{L}(\hat{\mathbb{X}})(t), \hat{x}_0(t)\rangle dt] \\ &- \mathbb{E}[\int_0^{T-\theta} \langle \mathcal{L}^*(\hat{\mathbb{F}})(t), \hat{x}_0(t)\rangle dt]. \end{aligned}$$

After rearranging and some algebra, we have

$$\begin{aligned} &\mathbb{E}[\langle \hat{x}_0(T), \bar{Q}\hat{x}_0(T)\rangle] + \mathbb{E}[\int_0^T \langle \hat{p}(t), C_0R_0^{-1}C_0^*\hat{p}(t)\rangle dt] + \mathbb{E}[\int_0^T \langle \hat{x}(t), Q_0\hat{x}(t)\rangle dt] \\ =&\mathbb{E}[\langle \hat{x}_0(T), \bar{Q}_0\bar{E}_0\mathcal{L}(\hat{\mathbb{X}}_0)(T)\rangle] \\ &+ \mathbb{E}[\int_0^T \langle B_0\mathcal{L}(\hat{\mathbb{X}})(t), \hat{p}(t)\rangle dt] + \mathbb{E}[\int_0^T \langle Q_0E_0\mathcal{L}(\hat{\mathbb{X}})(t), \hat{x}_0(t)\rangle dt] \\ &- \mathbb{E}[\langle \hat{x}_0(T-\theta), \mathcal{L}^*(\hat{\mathbb{F}})(T-\theta)\rangle] - \mathbb{E}[\int_0^{T-\theta} \langle \mathcal{L}^*(\hat{\mathbb{F}})(t), \hat{x}_0(t)\rangle dt]. \end{aligned}$$

Using the fact that $\hat{x}_0 = 0$ on $[-\theta, 0]$, together with the property of the adjoint operator (2.4.32), it becomes

$$\begin{aligned}
&\mathbb{E}[\langle \hat{x}_0(T), \bar{Q}\hat{x}_0(T)\rangle] + \mathbb{E}[\int_0^T \langle \hat{p}(t), C_0 R_0^{-1} C_0^* \hat{p}(t)\rangle dt] + \mathbb{E}[\int_0^T \langle \hat{x}(t), Q_0 \hat{x}(t)\rangle dt] \\
=&\mathbb{E}[\langle \hat{x}_0(T), \bar{Q}_0 \bar{E}_0 \mathcal{L}(\hat{\mathbb{X}}_0)(T)\rangle] \\
&\quad + \mathbb{E}[\int_0^T \langle B_0 \mathcal{L}(\hat{\mathbb{X}})(t), \hat{p}(t)\rangle dt] + \mathbb{E}[\int_0^T \langle Q_0 E_0 \mathcal{L}(\hat{\mathbb{X}})(t), \hat{x}_0(t)\rangle dt] \\
&\quad - \mathbb{E}[\langle \mathcal{L}(\hat{x}_0)(T), \hat{\mathbb{F}}(T)\rangle] - \mathbb{E}[\int_0^T \langle \mathcal{L}(\hat{x}_0)(t), \hat{\mathbb{F}}(t)\rangle dt].
\end{aligned}$$

One could get the following estimates by several applications of the Young's inequality

$$\begin{aligned}
&K_6(\|\hat{x}_0(T)\|_{L^2}^2 + \|\hat{x}_0\|_{L^2([0,T])}^2) + K_7\|\hat{p}\|_{L^2([0,T])} \\
\leq&\frac{K_6}{4}\|\hat{x}_0(T)\|_{L^2}^2 + \frac{1}{K_6}\|\bar{E}_0^*\bar{Q}_0\bar{Q}_0\bar{E}_0\|\|\mathcal{L}(\hat{\mathbb{X}}_0)(T)\|_{L^2}^2 \\
&+ \frac{K_7}{2}\|\hat{p}\|_{L^2([0,T])}^2 + \frac{1}{2K_7}\|B_0^* B_0\|\|\mathcal{L}(\hat{\mathbb{X}}_0)(T)\|_{L^2([0,T])}^2 \\
&+ \frac{K_6}{4}\|\hat{x}_0\|_{L^2([0,T])}^2 + \frac{1}{K_6}\|E_0^* Q_0 Q_0 E_0\|\|\mathcal{L}(\hat{\mathbb{X}}_0)\|_{L^2([0,T])}^2 \\
&+ \frac{K_6}{4K_5}\|\mathcal{L}(\hat{x}_0)(T)\|_{L^2}^2 + \frac{K_5}{K_6}\|\hat{\mathbb{F}}(T)\|_{L^2}^2 \\
&+ \frac{K_6}{4K_5}\|\mathcal{L}(\hat{x}_0)\|_{L^2([0,T])}^2 + \frac{K_5}{K_6}\|\hat{\mathbb{F}}\|_{L^2([0,T])}^2.
\end{aligned}$$

Using the operator bound obtained in (2.4.31), further simplification yields

$$\begin{aligned}
&\frac{K_6}{2}(\|\hat{x}_0(T)\|_{L^2}^2 + \|\hat{x}_0\|_{L^2([0,T])}^2) + \frac{K_7}{2}\|\hat{p}\|_{L^2([0,T])} \\
\leq&\frac{1}{K_6}\|\bar{E}_0^*\bar{Q}_0\bar{Q}_0\bar{E}_0\|\|\mathcal{L}(\hat{\mathbb{X}}_0)(T)\|_{L^2}^2 + \frac{1}{2K_7}\|B_0^* B_0\|\|\mathcal{L}(\hat{\mathbb{X}}_0)(T)\|_{L^2([0,T])}^2 \\
&+ \frac{1}{K_6}\|E_0^* Q_0 Q_0 E_0\|\|\mathcal{L}(\hat{\mathbb{X}}_0)\|_{L^2([0,T])}^2 + \frac{K_5}{K_6}\|\hat{\mathbb{F}}(T)\|_{L^2}^2 + \frac{K_5}{K_6}\|\hat{\mathbb{F}}\|_{L^2([0,T])}^2 \\
\leq&K_5\Big[\frac{2(\|\bar{E}_0^*\bar{Q}_0\bar{Q}_0\bar{E}_0\| \vee \|E_0^* Q_0 Q_0 E_0\|) + \|\bar{E}_0^*\bar{Q}_0\bar{E}_0\|^2 \vee \|E_0^* Q_0 E_0\|^2}{K_6} + \frac{\|B_0^* B_0\|}{2K_7}\Big] \\
&\quad \cdot (\|\hat{\mathbb{X}}_0(T)\|_{L^2}^2 + \|\hat{\mathbb{X}}_0\|_{L^2([0,T])}^2) + \frac{K_5}{K_6}\|B_0^* B_0\|\|\hat{\mathbb{P}}\|_{L^2([0,T])}^2
\end{aligned}$$

Thus,

$$
\begin{aligned}
&(\|\hat{x}_0(T)\|_{L^2}^2 + \|\hat{x}_0\|_{L^2([0,T])}^2) + \|\hat{p}\|_{L^2([0,T])} \\
\leq & 2K_5 \Big[\frac{2(\|\bar{E}_0^*\bar{Q}_0\bar{Q}_0\bar{E}_0\| \vee \|E_0^* Q_0 Q_0 E_0\|) + \|\bar{E}_0^*\bar{Q}_0\bar{E}_0\|^2 \vee \|E_0^* Q_0 E_0\|^2}{K_6(K_6 \wedge K_7)} + \frac{\|B_0^* B_0\|}{2K_7(K_6 \wedge K_7)} \Big] \\
& \cdot (\|\hat{\mathbb{X}}_0(T)\|_{L^2}^2 + \|\hat{\mathbb{X}}_0\|_{L^2([0,T])}^2) + \frac{2K_5}{K_6(K_6 \wedge K_7)} \|B_0^* B_0\| \|\hat{\mathbb{P}}\|_{L^2([0,T])}^2 .
\end{aligned}
$$

We conclude that $\mathbb{T}$ is a contraction providing the conditions in Theorem statement hold. □

Remark 2.4.6. *Providing the mean field component is removed from the dynamics and cost functional of the leader* ($B_0 = 0$ *and* $E_0 = \bar{E}_0 = 0$)*, the stochastic optimal control problem for the leader is always solvable as in the classical case. In other words, the conditions* (2.4.34) *in Theorem 2.4.5 state that if the mean field effects are sufficiently small (magnitude of* B_0, E_0 *and* $\bar{E}_0$*) in the control problem for the leader, then the stochastic functional differential equation* (2.4.33) *admits a unique solution. The whole mean field Stackelberg game is hence uniquely solvable.*

2.5 Numerical Examples

We assume that $\sigma_0 \equiv 0$ in this section, i.e. the control problem for the leader is deterministic; while we keep the followers' randomness in their state evolutions. Consider the following example:

Follower's Problem

$$\begin{cases} dx_1(t) = \Big[\Big(0.2x_1(t) + 0.2z(t) - 0.4x_0(t-\theta)\Big) + 0.4v_1(t)\Big]dt + 0.1dW(t); \\ x_1(0) = 1. \\ J_1(v_1) = \mathbb{E}\Big[\int_0^T \Big[10|x_1(t) - x_0(t-\theta)|^2 + 0.1|v_1(t)|^2\Big]dt + 10|x_1(T) - x_0(T-\theta)|^2\Big] \end{cases}$$

Leader's Problem

$$\begin{cases} dx_0(t) = \Big[\Big(0.1x_0(t) - 0.1z(t)\Big) + 0.2v_1(t)\Big]dt; \\ x_0(0) = 1. \\ J_0(v_0) = \mathbb{E}\Big[\int_0^T \Big[10|x_0(t) - 1.5|^2 + 0.1|v_1(t)|^2\Big]dt + 10|x_0(T) - 1.5|^2\Big] \end{cases} \tag{2.5.37}$$

It is easy to verify that the conditions in Theorem 2.4.5 are satisfied. The history of x_0 before $t = 0$ is assumed to be sinusoidal. The follower would like to minimize the squared distance between his own state $(x_1(t))$ and the delayed leader's state $(x_0(t-\theta))$; while the leader would like to minimize the squared distance between his own state $x_0(t)$ and the desired level 1.5.

As in Theorem 2.4.5 regarding the uniqueness and existence of the solution of Stackelberg game, for any given $\mathbb{X}_0$ and $\mathbb{P}$, we first numerically solve for $\mathcal{L}(\mathbb{X}_0)$ and $\mathcal{L}^*(\mathbb{F})$ by finite difference method, where $\mathbb{F}$ is given in (2.4.36).

For $T = 1$, Figure 2.1 and 2.2 respectively show the simulation results for the case with $\theta = 0$ (no delay) and $\theta = 1$. The smooth red and blue lines respectively represent the leader's state and the mean field term; while we simulate five paths of followers. As there is no randomness in the leader, trajectory of the mean field term is in fact the average of individuals'. Starting from time 0, the leader's state

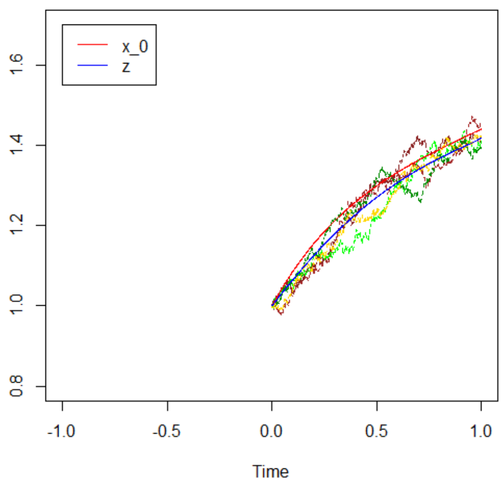

Figure 2.1: Trajectories of leader, mean field term and followers ($T = 1, \theta = 0$).

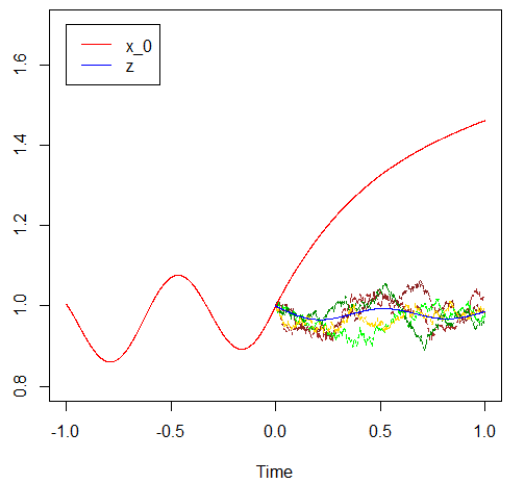

Figure 2.2: Trajectories of leader, mean field term and followers ($T = 1, \theta = 1$).

in both graphs move towards the level 1.5 stated in the cost functional (2.5.37). Nonetheless, follower's state (and the mean field term) evolutes differently in different cases. The increase of x_0 instantly affects the follower's evolution in Figure 2.1 ($\theta = 0$); while the followers are influenced by the delayed sinusoidal pattern of x_0 in Figure 2.2.

Note that the sufficient conditions assuring the unique existence of the Stackelberg solution given in Theorem 2.4.5 is independent of T. It is interesting to investigate numerically solution of the Stackelberg game under a longer time horizon. Figure 2.3 demonstrates the simulated evolutions of the leader and followers with $\theta = 1$ and $T = 5$. The leader's state surges from $T = 0$, and becomes steady as it approaches 1.5. While the followers' trajectories appear to be the leader's path shifted by a time length of 1.

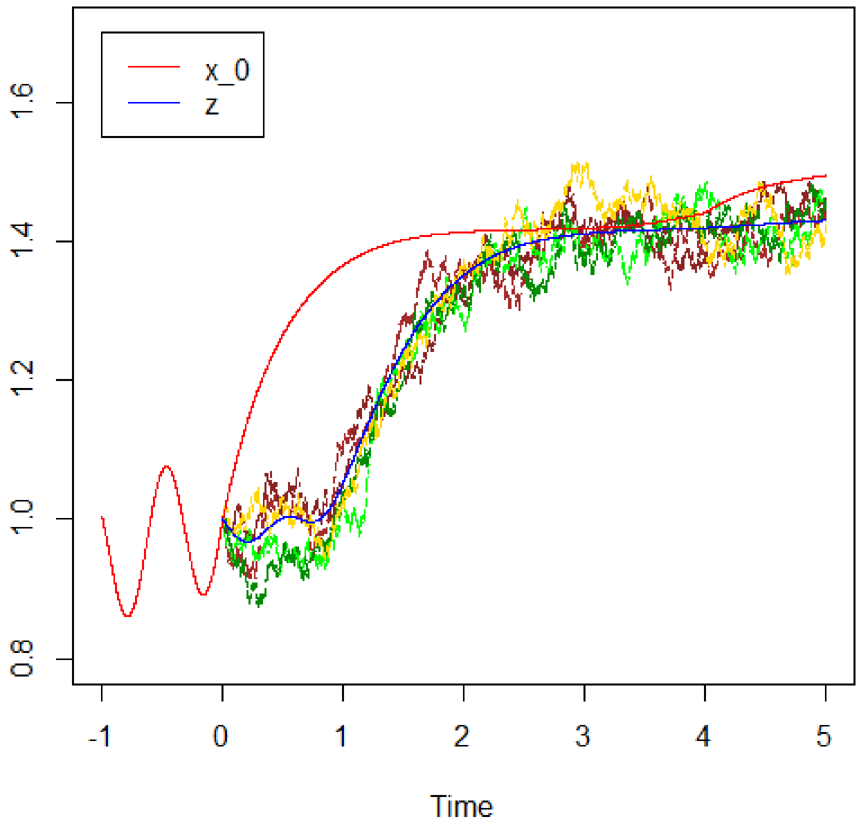

Figure 2.3: Trajectories of leader, mean field term and followers ($T = 5, \theta = 1$).

2.6 Conclusion

Mean field Stackelberg games under a linear quadratic setting with no terminal cost has been studied previously in our previous work [5]. The sufficient condition to guarantee the unique existence of a solution in [6] is also time horizon dependent. In this chapter, we consider Mean field Stackelberg game with Lipschitz coefficients and the presence of terminal costs. We provide the necessary conditions of optimality under sufficiently smooth functional coefficients. We showed that the resulting system of six forward backward stochastic differential equations reduced to those in [5] under the Linear Quadratic setting. Choosing an appropriate Hilbert space and a linear functional, the system of six equations is equivalent to a forward backward stochastic functional differential equation. A set of sufficient conditions, which is independent of the time horizon, is given to guarantee the unique existence of the Stackelberg solution. This algorithm is then numerically implemented in the example given in Section 2.5, which demonstrates impact of different delay magnitudes and time horizon on the state evolutions of the both leader, followers and the mean field term.

Chapter 3

Two-party Governance: Cooperation versus Competition

In this chapter we consider another class of Stackelberg games under a Linear Quadratic setting - two leaders over a group of followers. Depending on whether or not the leaders cooperate, we solve for the respective Pareto and Nash game between the leaders in Theorem 3.2.9 and 3.2.8. For the ease of studying the whole Stackelberg game numerically, we provide the explicit expression of solutions to the whole problem: Mean Field Game among the followers and Nash (and Pareto) Game between the leaders in Section 3.3. Finally, several numerical examples are given in Section 3.4 to study the impact of different games on the cost functionals of the followers.

3.1 Problem Setting

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $T \in \mathbb{R}^+$ be the fixed terminal time, also let d_α, d_β and $d_1 \in \mathbb{N}^+$. Assume that W^α, W^β and $\{W_1^i\}_{i\in\{1,\ldots,N\}}$ are independent Wiener processes over $\mathbb{R}^{d_\alpha}$, $\mathbb{R}^{d_\beta}$ and $\mathbb{R}^{d_1}$ respectively. Suppose that the random variables ξ_α and ξ_β, representing the initial states of the two leaders, are square integrable and are independent of each other and the mentioned

Wiener processes. Also let the random variables $\{\xi_1^i\}_{i\in\{1,\dots,N\}}$, representing the initial states of the followers, be square integrable, identically and independently distributed and they are also independent of ξ_α and ξ_β; again they are assumed to be independent of all the mentioned Wiener processes. Define the following filtrations,

$$\begin{aligned}\mathcal{F}^\alpha &:= \sigma(\xi_\alpha, W^\alpha(s) : s \le t), \quad t > 0;\\ \mathcal{F}^\beta &:= \sigma(\xi_\beta, W^\beta(s) : s \le t), \quad t > 0;\\ \mathcal{F}^i &:= \sigma(\xi_1^i, W_1^i(s) : s \le t), \quad t > 0.\end{aligned}$$

We now introduce the dynamical system of two leaders over N small players. The empirical states for the leaders are described by the following stochastic differential equations on $\mathbb{R}^{n_\alpha}$ and $\mathbb{R}^{n_\beta}$ respectively:

$$\begin{cases} dy_\alpha = \Big(A_\alpha y_\alpha(t) + B_\alpha y_\beta(t) + C_\alpha \dfrac{\sum_{j=1,}^N y_1^j(t)}{N} + D_\alpha v_\alpha(t)\Big)dt + \sigma_\alpha dW^\alpha(t);\\ y_\alpha(0) = \xi_\alpha, \end{cases} \tag{3.1.1}$$

$$\begin{cases} dy_\beta = \Big(A_\beta y_\alpha(t) + B_\beta y_\beta(t) + C_\beta \dfrac{\sum_{j=1,}^N y_1^j(t)}{N} + D_\beta v_\beta(t)\Big)dt + \sigma_\beta dW^\beta(t);\\ y_\beta(0) = \xi_\beta. \end{cases} \tag{3.1.2}$$

The followers are homogeneous, and the empirical state for the i-th player is given by the SDE on $\mathbb{R}^{n_1}$:

$$\begin{cases} dy_1^i = \Big(A_1 y_1^i(t) + B_1 \dfrac{\sum_{j=1,j\neq i}^N y_1^j(t)}{N-1} + C_1 y_\alpha(t) + D_1 y_\beta(t) + E_1 v_1^i(t)\Big)dt + \sigma_1 dW_1^i(t);\\ y_1^i(0) = \xi_1^i. \end{cases} \tag{3.1.3}$$

Here v_α, v_β and v_1^i represent the controls for two leaders and the i-th follower correspondingly. The matrices $A_\alpha, B_\alpha, C_\alpha, D_\alpha, \sigma_\alpha, A_\beta, B_\beta, C_\beta, D_\beta, \sigma_\beta, A_1, B_1, C_1, D_1, E_1$ and σ_1 are assumed to be constant with appropriate dimensions. It is natural to assume that $v_\alpha, v_\beta \in \mathcal{F}^\alpha \vee \mathcal{F}^\beta$ and $v_1^i \in \mathcal{F}^\alpha \vee \mathcal{F}^\beta \vee \mathcal{F}^i$. The two leaders and the i-th player aim to minimize the following cost functionals respectively:

$$\begin{aligned}
\mathcal{J}_\alpha(v_\alpha; v_\beta) &= \mathbb{E}\int_0^T \left| y_\alpha(t) - F_\alpha \frac{\sum_{j=1}^N y_1^j(t)}{N} - G_\alpha y_\beta(t) - M_\alpha \right|^2_{Q_\alpha} + |v_\alpha(t)|^2_{R_\alpha} dt; \\
\mathcal{J}_\beta(v_\alpha, v_\beta) &= \mathbb{E}\int_0^T \left| y_\beta(t) - F_\beta \frac{\sum_{j=1}^N y_1^j(t)}{N} - G_\beta y_\alpha(t) - M_\beta \right|^2_{Q_\beta} + |v_\beta(t)|^2_{R_\beta} dt; \\
\mathcal{J}_1^i(\mathbf{v}_1^i) &= \mathbb{E}\int_0^T \left| y_1^i(t) - F_1 \frac{\sum_{j=1, j\neq i}^N y_1^j(t)}{N-1} - G_1 y_\alpha(t) - H_1 y_\beta(t) - M_1 \right|^2_{Q_1} + |v_1^i(t)|_{R_1} dt.
\end{aligned} \tag{3.1.4}$$

where $\mathbf{v}_1^i = (v_1^1, \ldots, v_1^N)$, $|\cdot|_{\mathcal{Q}} := \langle \cdot, \mathcal{Q}\cdot \rangle$ for any positive definite matrix $\mathcal{Q}$ and $\langle \cdot, \star \rangle$ is the usual Euclidean inner product. The matrices $F_\alpha, G_\alpha, M_\alpha Q_\alpha, R_\alpha, F_\beta, G_\beta, M_\beta$, $Q_\beta, R_\beta, F_1, G_1, H_1, Q_1$ and R_1 are assumed to be constant with appropriate dimensions; while $Q_\alpha, R_\alpha, Q_\beta, R_\beta, Q_1$ and R_1 are positive definite.

Solving this stochastic differential games (either in the sense of a Nash game or a Pareto game to be described in Definition 3.1.2 and 3.1.3) is rather complicated as N becomes very large. On the other hand, we can transform the original problem under the context of mean field game. Consider the limiting mean field evolutions for the two leaders and the i-th player respectively(to be justified in Theorem 3.1.1):

$$\begin{cases} dx_\alpha &= \Big(A_\alpha x_\alpha(t) + B_\alpha x_\beta(t) + C_\alpha z(t) + D_\alpha v_\alpha(t)\Big)dt + \sigma_\alpha dW^\alpha(t); \\ x_\alpha(0) &= \xi_\alpha, \end{cases} \tag{3.1.5}$$

$$\begin{cases} dx_\beta &= \Big(A_\beta x_\alpha(t) + B_\beta x_\beta(t) + C_\beta z(t) + D_\beta v_\beta(t)\Big)dt + \sigma_\beta dW^\beta(t); \\ x_\beta(0) &= \xi_\beta, \end{cases} \tag{3.1.6}$$

$$\begin{cases} dx_1^i &= \Big(A_1 x_1^i(t) + B_1 z(t) + C_1 x_\alpha(t) + D_1 x_\beta(t) + E_1 v_1^i(t)\Big)dt + \sigma_1 dW_1^i(t); \\ x_1^i(0) &= \xi_1^i. \end{cases} \tag{3.1.7}$$

Their cost functionals are respectively given by

$$\begin{aligned} J_\alpha(v_\alpha, v_\beta) &= \mathbb{E}\int_0^T |x_\alpha(t) - F_\alpha z(t) - G_\alpha x_\beta(t) - M_\alpha|^2_{Q_\alpha} + |v_\alpha(t)|^2_{R_\alpha} dt; \\ J_\beta(v_\alpha, v_\beta) &= \mathbb{E}\int_0^T |x_\beta(t) - F_\beta z(t) - G_\beta x_\alpha(t) - M_\beta|^2_{Q_\beta} + |v_\beta(t)|^2_{R_\beta} dt; \end{aligned} \tag{3.1.8}$$

$$J_1^i(v_1^i; x_\alpha, x_\beta, z) = \mathbb{E}\int_0^T |x_1^i(t) - F_1 z(t) - G_1 x_\alpha(t) - H_1 x_\beta(t) - M_1|^2_{Q_1} + |v_1^i(t)|^2_{R_1} dt. \tag{3.1.9}$$

Here z is called the mean field term which is a stochastic process adapted to $\mathcal{F}^\alpha \vee \mathcal{F}^\beta$ to be introduced later in Theorem 3.1.1. Given z, x_α and x_β, the i-th follower aims at solving the optimal control problem defined by (3.1.7) and (3.1.9), that is

$$u_1^i = \arg\min_{v_1^i} J_1^i(v_1^i; x_\alpha, x_\beta, z). \tag{3.1.10}$$

Motivated by the results in [5] and [10], we can easily obtain the following result, and the proof is omitted here.

Theorem 3.1.1. *Given z, x_α and x_β, suppose that the i^{th} player adopts the optimal control u_1^i, and we denote $x_1^i(z, x_\alpha, x_\beta) = x_1^i$ and y_1^i the corresponding trajectory of* (3.1.7) *and* (3.1.3) *under u_1^i. If z is chosen such that the fixed point $z(t) = \mathbb{E}^{\mathcal{F}_t^\alpha \vee \mathcal{F}_t^\beta} x_1^1(t)$ holds, then*

$$\mathbb{E}\Big[\sup_t |y_\alpha - x_\alpha|^2(t) + \sup_t |y_\beta - x_\beta|^2(t) + \sup_i \sup_t |y_1^i - x_1^i|^2(t)\Big] = O\left(\frac{1}{N}\right). \tag{3.1.11}$$

Moreover, $(u_1^1, u_1^2, \ldots, u_1^N)$ served as an ϵ-Nash equilibrium (of order $O\left(\frac{1}{\sqrt{N}}\right)$) among the followers for the original empirical problem. That is for arbitrary v_1^i, we have

$$J_1(u_1^1, \ldots, u_1^{i-1}, u_1^i, u_1^{i+1}, \ldots, u_1^N) \leq J_1(u_1^1, \ldots, u_1^{i-1}, v_1^i, u_1^{i+1}, \ldots, u_1^N) + O\left(\frac{1}{\sqrt{N}}\right). \tag{3.1.12}$$

The above theorem allows us to consider the much simpler mean field dynamical system. Note that, given z, x_α and x_β, $\{x_1^i\}_{i=1}^n$ described by (3.1.7) are i.i.d..

We therefore drop the index i in (3.1.7), (3.1.9) and call x_1 the (representative) follower throughout this chapter.

Depending on whether or not the two leaders cooperate, we consider two kinds of stochastic differential games. In particular, we make the following definitions:

Definition 3.1.2 (Nash Game). *The optimal control $u_\alpha^{\mathcal{N}}$ and $u_\beta^{\mathcal{N}}$ for the non-cooperative Nash game between the two leaders are define as follows:*

$$u_\alpha^{\mathcal{N}} := \arg\min_{v_\alpha} J_\alpha(v_\alpha, u_\beta^{\mathcal{N}}); \tag{3.1.13}$$
$$u_\beta^{\mathcal{N}} := \arg\min_{v_\beta} J_\beta(u_\alpha^{\mathcal{N}}, v_\beta).$$

Definition 3.1.3 (Pareto Game). *The optimal control $u_\alpha^{\mathcal{P}}$ and $u_\beta^{\mathcal{P}}$ for the cooperative Pareto game between the two leaders are define as follows:*

$$(u_\alpha^{\mathcal{P}}, u_\beta^{\mathcal{P}}) = \arg\min_{v_\alpha, v_\beta} \Big(J_\alpha(v_\alpha, v_\beta) + J_\beta(v_\alpha, v_\beta) \Big). \tag{3.1.14}$$

The notion of ϵ-Nash equilibrium among the group of followers introduced in Theorem 3.1.1 should not be confused with the Nash (or Pareto) game in Definition 3.1.2 (or 3.1.3) between the two leaders. We solve the whole leader-follower problem by introducing three sub-problems in order:

Problem 3.1.4. *Given x_α, x_β and z, find a control u_1 such that*

$$J_1(u_1; x_\alpha, x_\beta, z) = \min_{v_1} J_1(v_1; x_\alpha, x_\beta, z).$$

Problem 3.1.5. *Find the process z such that the fixed point property is satisfied:*

$$z(t) = \mathbb{E}^{\mathcal{F}_t^\alpha \vee \mathcal{F}_t^\beta} x_1(t) \tag{3.1.15}$$

where x_1 is the optimal trajectory given by the solution of Problem 3.1.4, which clearly also depends on z.

Problem 3.1.6. *Find the optimal control for the Nash and Pareto Games defined in Definition 3.1.2 and 3.1.3 respectively, where z is the solution in Problem 3.1.5.*

3.2 Solution

3.2.1 Optimal Control for the Follower

Similar to our previous work [5] and [6], we can state the optimal control of the follower:

Theorem 3.2.1. *Given x_α, x_β and z, Problem 3.1.4 is uniquely solvable and the optimal control is given by $u_1(t) = -R_1^{-1}E_1 n(t)$, where $n(t)$ satisfies the following backward stochastic differential equation:*

$$\begin{cases} -dn = \Big(A_1^* n(t) + Q_1\Big(x_1(t) - F_1 z(t) - G_1 x_\alpha(t) - H_1 x_\beta(t) - M_1\Big)\Big)dt - Z_{n,\alpha} dW^\alpha(t) \\ \qquad\qquad - Z_{n,\beta} dW^\beta(t) - Z_{n,1} dW^1(t), \\ n(T) = 0. \end{cases} \tag{3.2.16}$$

Substituting the optimal control of the follower given in Theorem 3.2.1 into equation (3.1.7), the solution of Problem 3.1.4 is completely characterized by the following forward backward stochastic differential equation:

$$\begin{cases} dx_1 = \Big(A_1 x_1(t) + B_1 z(t) + C_1 x_\alpha(t) + D_1 x_\beta(t) - E_1 R_1^{-1} E_1^* n(t)\Big)dt + \sigma_1 dW_1(t), \\ x_1(0) = \xi_1; \\ -dn = \Big(A_1^* n(t) + Q_1\Big(x_1(t) - F_1 z(t) - G_1 x_\alpha(t) - H_1 x_\beta(t) - M_1\Big)\Big)dt - Z_{n,\alpha} dW^\alpha(t) \\ \qquad\qquad - Z_{n,\beta} dW^\beta(t) - Z_{n,1} dW^1(t), \\ n(T) = 0. \end{cases} \tag{3.2.17}$$

Theorem 3.2.2. *The FBSDEs* (3.2.17) *admits a unique solution.*

Proof. Given z, x_α and x_β, the system (3.2.17) satisfies certain monotonicity condition proposed in Hu and Peng [25]. The unique existence of a L^2-solution is hence ensured. One can refer to Lemma 2.4.1 for details. □

To obtain the mean field equilibrium stated in Problem 3.1.5, we take conditional expectation on $\mathcal{F}_t^\alpha \vee \mathcal{F}_t^\beta$ on both sides of (3.2.17), which yields

$$\begin{cases} dz = \Big((A_1 + B_1)z(t) + C_1 x_\alpha(t) + D_1 x_\beta(t) - E_1 R_1^{-1} E_1^* m(t)\Big)dt, \\ z(0) = \mathbb{E}[\xi_1]; \\ -dm = \Big(A_1^* m(t) + Q_1(I - F_1)z(t) - Q_1 G_1 x_\alpha(t) - Q_1 H_1 x_\beta(t) - Q_1 M_1\Big)dt \\ \qquad\qquad - Z_{m,\alpha} dW^\alpha(t) - Z_{m,\beta} dW^\beta(t), \\ m(T) = 0. \end{cases} \tag{3.2.18}$$

Define the constant

$$K_1 := \lambda_{\min}(Q_1(I - F_1)) - \frac{\|B_1^* B_1\|}{2\lambda_{\min}(E_1 R_1^{-1} E_1)} \tag{3.2.19}$$

Theorem 3.2.3. *Under the condition that* $K_1 > 0$*, the system* (3.2.18) *has a unique solution.*

Proof. See the proof of Theorem 2.4.1 in Chapter 2 for details. □

Remark 3.2.4. *The condition in Theorem 3.2.3 is satisfied when the coefficients of the mean filed term,* B_1 *and* F_1*, have small magnitudes. In particular, if* $B_1 = 0 = F_1$*, then* (3.2.18) *always admit a unique solution, which agrees with the classical Linear Quadratic control problem.*

3.2.2 Optimal Control for the Leader

From now on, we assume the condition in Theorem 3.2.3 holds. Before we proceed to the optimal control problem for the two leaders, we first introduce two linear operators:

$$\mathcal{L}_\alpha : x_\alpha \in L^2([0,T]; \mathbb{R}^{n_\alpha}) \mapsto z_\alpha \in L^2([0,T]; \mathbb{R}^{n_1}) \tag{3.2.20}$$

given by

$$\begin{cases} \quad dz_\alpha = \Big((A_1 + B_1)z_\alpha(t) + C_1 x_\alpha(t) - E_1 R_1^{-1} E_1^* m_\alpha(t)\Big)dt, \\ z_\alpha(0) = 0; \\ -dm_\alpha = \Big(A_1^* m_\alpha(t) + Q_1(I - F_1)z_\alpha(t) - Q_1 G_1 x_\alpha(t)\Big)dt - Z_{m_\alpha,\alpha} dW^\alpha(t), \\ m_\alpha(T) = 0; \end{cases} \tag{3.2.21}$$

and

$$\mathcal{L}_\beta : x_\beta \in L^2([0,T]; \mathbb{R}^{n_\beta}) \mapsto z_\beta \in L^2([0,T]; \mathbb{R}^{n_1}) \tag{3.2.22}$$

given by

$$\begin{cases} \quad dz_\beta = \Big((A_1 + B_1)z_\beta(t) + D_1 x_\beta(t) - E_1 R_1^{-1} E_1^* m_\beta(t)\Big)dt, \\ z_\beta(0) = 0; \\ -dm_\beta = \Big(A_1^* m_\beta(t) + Q_1(I - F_1)z_\beta(t) - Q_1 H_1 x_\beta(t)\Big)dt - Z_{m_\beta,\beta} dW^\beta(t), \\ m_\beta(T) = 0. \end{cases} \tag{3.2.23}$$

Lemma 3.2.5. *Given that the condition in Theorem 3.2.3 holds, then $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ are bounded. In particular*

$$\begin{aligned} \|\mathcal{L}_\alpha\|^2 &\le \frac{1}{K_1}\Big(\frac{\|C_1^* C_1\|}{\lambda_{\min}(E_1 R_1^{-1} E_1^*)} + \frac{\|G_1^* Q_1 Q_1 G_1\|}{\lambda_{\min}(Q_1(I - F_1))}\Big) \\ \|\mathcal{L}_\beta\|^2 &\le \frac{1}{K_1}\Big(\frac{\|D_1^* D_1\|}{\lambda_{\min}(E_1 R_1^{-1} E_1^*)} + \frac{\|H_1^* Q_1 Q_1 H_1\|}{\lambda_{\min}(Q_1(I - F_1))}\Big) \end{aligned} \tag{3.2.24}$$

Proof. We consider $\mathcal{L}_\alpha$ and the bound estimate for $\mathcal{L}_\beta$ can be obtained similarly. Applying Itô;s formula to the inner product $\langle z_\alpha, m_\alpha \rangle$, then taking integration on

$[0, T]$ and also taking expectation, we can get

$$\begin{aligned}
0 =& \mathbb{E}\int_0^T \langle B_1 z_\alpha(t), m_\alpha(t)\rangle dt + \mathbb{E}\int_0^T \langle C_1 x_\alpha(t), m_\alpha(t)\rangle dt + \mathbb{E}\int_0^T \langle Q_1 G_1 x_\alpha(t), z_\alpha(t)\rangle dt \\
& - \mathbb{E}\int_0^T \langle E_1 R_1^{-1} E_1^* m_\alpha(t), m_\alpha(t)\rangle dt - \mathbb{E}\int_0^T \langle Q_1(I - F_1) z_\alpha(t), z_\alpha(t)\rangle dt \\
\leq & \frac{\|B_1^* B_1|}{2\lambda_{\min}(E_1 R_1^{-1} E_1^*)}\|z_\alpha\|^2 + \frac{1}{2}\lambda_{\min}(E_1 R_1^{-1} E_1^*)\|m_\alpha\|^2 \\
& + \frac{\|C_1^* C_1\|}{2\lambda_{\min}(E_1 R_1^{-1} E_1^*)}\|x_\alpha\|^2 + \frac{1}{2}\lambda_{\min}(E_1 R_1^{-1} E_1^*)\|m_\alpha\|^2 \\
& + \frac{\|G_1^* Q_1 Q_1 G_1\|}{2\lambda_{\min}(Q_1(I - F_1))}\|x_\alpha\|^2 + \frac{1}{2}\lambda_{\min}(Q_1(I - F_1))\|z_\alpha\|^2 \\
& - \lambda_{\min}(E_1 R_1^{-1} E_1^*)\|m_\alpha\|^2 - \lambda_{\min}(Q_1(I - F_1))\|z_\alpha\|^2,
\end{aligned}$$

Under the condition that $K_1 > 0$, we have

$$\|z_\alpha\|^2 \leq \frac{1}{K_1}\Big(\frac{\|C_1^* C_1\|}{\lambda_{\min}(E_1 R_1^{-1} E_1^*)} + \frac{\|G_1^* Q_1 Q_1 G_1\|}{\lambda_{\min}(Q_1(I - F_1))}\Big)\|x_\alpha\|^2,$$

which show that $\mathcal{L}_\alpha$ is bounded. □

Remark 3.2.6. *We observe that the norms of the operators are small providing that the mean field effects (B_1 and F_1) and leaders effects (C_1 and D_1)on the followers are small.*

By Riesz Representation Theorem, the adjoint operators of L_α and L_β uniquely exist such that for all $f \in L^2([0, T]; \mathbb{R}^{n_\alpha})$, $g \in L^2([0, T]; \mathbb{R}^{n_1})$, $h \in L^2([0, T]; \mathbb{R}^{n_\beta})$, we have

$$\begin{aligned}
\mathbb{E}\int_0^T \langle \mathcal{L}_\alpha(f)(t), g(t)\rangle dt &= \mathbb{E}\int_0^T \langle f(t), \mathcal{L}_\alpha^*(g)(t)\rangle dt; \\
\mathbb{E}\int_0^T \langle \mathcal{L}_\beta(h)(t), g(t)\rangle dt &= \mathbb{E}\int_0^T \langle h(t), \mathcal{L}_\beta^*(g)(t)\rangle dt.
\end{aligned}$$

The explicit form of the adjoint operators are given by the following theorem.

Theorem 3.2.7. *The adjoints $\mathcal{L}_\alpha^* : L^2([0, T]; \mathbb{R}^{n_1}) \to L^2([0, T]; \mathbb{R}^{n_\alpha})$ and $\mathcal{L}_\beta^* : L^2([0, T]; \mathbb{R}^{n_1}) \to L^2([0, T]; \mathbb{R}^{n_\beta})$ of $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ respectively, defined by* (3.2.20) *and* (3.2.22), *are given by*

$$\mathcal{L}_\alpha^*(g_1)(t) := C_1^* r_\alpha(t) - (Q_1 G_1)^* s_\alpha(t); \tag{3.2.25}$$

$$\mathcal{L}_\beta^*(g_2)(t) := D_1^* r_\beta(t) - (Q_1 H_1)^* s_\beta(t); \tag{3.2.26}$$

for any $g_1, g_2 \in L^2([0,T];\mathbb{R}^{n_1})$, *where*

$$\begin{cases} -dr_\alpha & = (A_1+B_1)^* r_\alpha(t) + (Q_1(I-F_1))^* s_\alpha(t) + g_1(t) - Z_{r_\alpha,\alpha} dW^\alpha(t) - Z_{r_\alpha,\beta} dW^\beta(t), \\ r_\alpha(T) & = 0; \\ ds_\alpha & = A_1 s_\beta(t) - E_1 R_1^{-1} E_1^* r_\beta(t), \\ s_\alpha(0) & = 0; \end{cases}$$

$$\begin{cases} -dr_\beta & = (A_1+B_1)^* r_\beta(t) + (Q_1(I-F_1))^* s_\beta(t) + g_2(t) - Z_{r_\beta,\alpha} dW^\alpha(t) - Z_{r_\beta,\beta} dW^\beta(t), \\ r_\beta(T) & = 0; \\ ds_\beta & = A_1 s_\beta(t) - E_1 R_1^{-1} E_1^* r_\beta(t), \\ s_\beta(0) & = 0. \end{cases}$$

Proof. Clearly, $\mathcal{L}_\alpha^*$ and $\mathcal{L}_\beta^*$ as defined above in (3.2.25) and (3.2.26) respectively are linear. Applying Itô's formula to $\langle r_\alpha, z_\alpha\rangle - \langle s_\alpha, m_\alpha\rangle$, we can get that

$$\begin{aligned} \mathbb{E}\int_0^T \langle z_\alpha(t), g(t)\rangle dt &= \mathbb{E}\int_0^T \langle r_\alpha(t), c_1 x_\alpha(t)\rangle dt - \mathbb{E}\int_0^T \langle s_\alpha(t), Q_1 G_1 x_\alpha(t)\rangle dt \\ &= \mathbb{E}\int_0^T \langle \mathcal{L}_\alpha^*(g)(t), x_\alpha(t)\rangle dt, \end{aligned}$$

since g is arbitrary, the claim for $\mathcal{L}_\alpha^*$ follows. Similarly, if we consider $\langle r_\beta, z_\beta\rangle - \langle s_\beta, m_\beta\rangle$, the result for $\mathcal{L}_\beta^*$ also follows. □

Similar to Chapter 2.4 and [5], we can decompose the forward equation in (3.2.18) into sum of three parts given by

$$z(t) = z_\alpha(t) + z_\beta(t) + z_c(t) = \mathcal{L}_\alpha(x_\alpha)(t) + \mathcal{L}_\beta(x_\beta)(t) + z_c(t), \tag{3.2.27}$$

where z_c is satisfies the deterministic system:

$$\begin{cases} dz_c = \Big((A_1 + B_1)z_c(t) - E_1 R_1^{-1} E_1^* m_c(t)\Big)dt, \\ z(0) = \mathbb{E}[\xi_1]; \\ -dm_c = \Big(A_1^* m_c(t) + Q_1(I - F_1)z_c(t) - Q_1 M_1\Big)dt, \\ m(T) = 0. \end{cases} \tag{3.2.28}$$

We can express the evolutions and objective functionals for the leaders in the following functional form

$$\begin{cases} dx_\alpha = \Big(A_\alpha x_\alpha(t) + B_\alpha x_\beta(t) + C_\alpha\Big(\mathcal{L}_\alpha(x_\alpha)(t) + \mathcal{L}_\beta(x_\beta)(t) + z_c(t)\Big) + D_\alpha u_\alpha(t)\Big)dt \\ \qquad\qquad + \sigma_\alpha dW^\alpha(t), \\ x_\alpha(0) = \xi_\alpha; \end{cases} \tag{3.2.29}$$

$$\begin{cases} dx_\beta = \Big(A_\beta x_\alpha(t) + B_\beta x_\beta(t) + C_\beta\Big(\mathcal{L}_\alpha(x_\alpha)(t) + \mathcal{L}_\beta(x_\beta)(t) + z_c(t)\Big) + D_\beta u_\beta(t)\Big)dt \\ \qquad\qquad + \sigma_\beta dW^\beta(t), \\ x_\beta(0) = \xi_\beta; \end{cases} \tag{3.2.30}$$

and

$$\begin{aligned} J_\alpha(u_\alpha, u_\beta) &= \mathbb{E}\int_0^T \Big|x_\alpha(t) - F_\alpha\Big(\mathcal{L}_\alpha(x_\alpha)(t) + \mathcal{L}_\beta(x_\beta)(t) + z_c(t)\Big) - G_\alpha x_\beta(t) - M_\alpha\Big|^2_{Q_\alpha} \\ &\qquad + |u_\alpha(t)|^2_{R_\alpha} dt; \\ J_\beta(u_\alpha, u_\beta) &= \mathbb{E}\int_0^T \Big|x_\beta(t) - F_\beta\Big(\mathcal{L}_\alpha(x_\alpha)(t) + \mathcal{L}_\beta(x_\beta)(t) + z_c(t)\Big) - G_\beta x_\alpha(t) - M_\beta\Big|^2_{Q_\beta} \\ &\qquad + |u_\beta(t)|^2_{R_\beta} dt. \end{aligned}$$

3.2.3 Nash Game

In this section we solve the Nash game introduced in Definition 3.1.2 via the operator approach.

Theorem 3.2.8. *The solution of the Nash Game is*

$$u_\alpha^{\mathcal{N}}(t) = -R_\alpha^{-1} D_\alpha^* p_\alpha^{\mathcal{N}}(t),$$
$$u_\beta^{\mathcal{N}}(t) = -R_\beta^{-1} D_\beta^* p_\beta^{\mathcal{N}}(t);$$

where $p_\alpha^{\mathcal{N}}$ and $p_\beta^{\mathcal{N}}$ are given by the backward stochastic functional differential equations:

$$\begin{cases} -dp_\alpha^{\mathcal{N}} = \Big(A_\alpha^* p_\alpha^{\mathcal{N}}(t) + \mathcal{L}_\alpha^*(C_\alpha^* p_\alpha^{\mathcal{N}})(t) \\ \qquad + Q_\alpha\Big(x_\alpha(t) - F_\alpha\Big(\mathcal{L}_\alpha(x_\alpha)(t) + \mathcal{L}_\beta(x_\beta)(t) + z_c(t)\Big) - G_\alpha x_\beta(t) - M_\alpha\Big) \\ \qquad - \mathcal{L}_\alpha^*\Big[(Q_\alpha F_\alpha)^*\Big(x_\alpha - F_\alpha\Big(\mathcal{L}_\alpha(x_\alpha) + \mathcal{L}_\beta(x_\beta) + z_c\Big) - G_\alpha x_\beta - M_\alpha\Big)\Big](t)\Big)dt \\ \qquad - Z_{p_\alpha^{\mathcal{N}},\alpha} dW^\alpha(t) - Z_{p_\alpha^{\mathcal{N}},\beta} dW^\beta(t), \\ p_\alpha^{\mathcal{N}}(T) = 0; \end{cases} \tag{3.2.31}$$

and

$$\begin{cases} -dp_\beta^{\mathcal{N}} = \Big(B_\beta^* p_\beta^{\mathcal{N}}(t) + \mathcal{L}_\beta^*(C_\beta^* p_\beta^{\mathcal{N}})(t) \\ \qquad - \mathcal{L}_\beta^*\Big[(Q_\beta F_\beta)^*\Big(x_\beta - F_\beta\Big(\mathcal{L}_\alpha(x_\alpha) + \mathcal{L}_\beta(x_\beta) + z_c\Big) - G_\beta x_\alpha - M_\beta\Big)\Big](t) \\ \qquad + Q_\beta\Big(x_\beta(t) - F_\beta\Big(\mathcal{L}_\alpha(x_\alpha)(t) + \mathcal{L}_\beta(x_\beta)(t) + z_c(t)\Big) - G_\beta x_\alpha(t) - M_\beta\Big)\Big)dt \\ \qquad - Z_{p_\beta^{\mathcal{N}},\alpha} dW^\alpha(t) - Z_{p_\beta^{\mathcal{N}},\beta} dW^\beta(t), \\ p_\beta^{\mathcal{N}}(T) = 0. \end{cases} \tag{3.2.32}$$

Proof. We first consider the optimal control problem for the α-leader. Under the Nash Game setting, the α-leader assumes that the β-leader uses an optimal strategy and hence x_β is considered unchanged and exogenous. Due to the convexity and coerciveness of the quadratic cost functional, we can directly apply standard stochastic maximum principle. Consider a perturbation of the optimal control $u_\alpha^{\mathcal{N}}(t) + \tau\tilde{u}_\alpha(t)$, where $\tilde{u}_\alpha$ is arbitrarily chosen adapted to $\mathcal{F}^\alpha \vee \mathcal{F}^\beta$. The state for the α-leader becomes $x_\alpha + \tau\tilde{x}_\alpha$, where

$$\begin{cases} d\tilde{x}_\alpha = \Big(A_\alpha \tilde{x}_\alpha(t) + C_{x_\alpha}\mathcal{L}_\alpha(\tilde{x}_\alpha)(t)\Big)dt, \\ \tilde{x}_\alpha(0) = 0. \end{cases}$$

The optimality of $u_\alpha^{\mathcal{N}}$ yields the Euler condition:

$$\begin{aligned} 0 &= \frac{d}{d\tau}\bigg|_{\tau=0} J_\alpha(u_\alpha^{\mathcal{N}}(t) + \tau\tilde{u}_\alpha(t), u_\beta(t)) \\ &= 2\mathbb{E}\int_0^T \Big\{\Big\langle \tilde{x}_\alpha(t) - F_\alpha\mathcal{L}_\alpha(\tilde{x}_\alpha)(t) \\ &\qquad\qquad , Q_\alpha\Big(x_\alpha(t) - F_\alpha\Big(\mathcal{L}_\alpha(x_\alpha)(t) + \mathcal{L}_\beta(x_\beta)(t) + z_c(t)\Big) - G_\alpha x_\beta(t) - M_\alpha\Big)\Big\rangle \\ &\qquad + \langle \tilde{u}_\alpha(t), R_\alpha u_\alpha^{\mathcal{N}}(t)\rangle\Big\}dt. \end{aligned} \tag{3.2.33}$$

On the other hand, applying Itô's formula to the inner product $\langle p_\alpha^{\mathcal{N}}, \tilde{x}_\alpha\rangle$ and combining with (3.2.33), following similar arguments found in Chapter 2 and [5], we get

$$0 = \mathbb{E}\int_0^T \langle \tilde{u}_\alpha(t), R_\alpha u_\alpha^{\mathcal{N}}(t) + D_\alpha^* p_\alpha^{\mathcal{N}}(t)\rangle dt.$$

Since $\tilde{u}_\alpha(t)$ is arbitrary, we obtain the optimal control $u_\alpha^{\mathcal{N}} = -R_\alpha^{-1}D_\alpha^* p_\alpha^{\mathcal{N}}(t)$. The optimal control $u_\beta^{\mathcal{N}}$ can be obtained similarly. □

The full solution of the Nash game in operator form is given by the following system of forward backward stochastic functional differential equations:

$$
\begin{cases}
\begin{aligned}
dx_\alpha^{\mathcal{N}} = &\Big(A_\alpha x_\alpha^{\mathcal{N}}(t) + B_\alpha x_\beta^{\mathcal{N}}(t) + C_\alpha\Big(\mathcal{L}_\alpha(x_\alpha^{\mathcal{N}})(t) + \mathcal{L}_\beta(x_\beta^{\mathcal{N}})(t) + z_c(t)\Big) \\
&- D_\alpha R_\alpha^{-1} D_\alpha^* p_\alpha^{\mathcal{N}}(t)\Big)dt + \sigma_\alpha dW^\alpha(t),
\end{aligned} \\
x_\alpha^{\mathcal{N}}(0) = \xi_\alpha; \\
\begin{aligned}
dx_\beta^{\mathcal{N}} = &\Big(A_\beta x_\alpha^{\mathcal{N}}(t) + B_\beta x_\beta^{\mathcal{N}}(t) + C_\beta\Big(\mathcal{L}_\alpha(x_\alpha^{\mathcal{N}})(t) + \mathcal{L}_\beta(x_\beta^{\mathcal{N}})(t) + z_c(t)\Big) \\
&- D_\beta R_\beta^{-1} D_\beta^* p_\beta^{\mathcal{N}}(t)\Big)dt + \sigma_\beta dW^\beta(t),
\end{aligned} \\
x_\beta^{\mathcal{N}}(0) = \xi_\beta; \\
\begin{aligned}
-dp_\alpha^{\mathcal{N}} = &\Big(A_\alpha^* p_\alpha^{\mathcal{N}}(t) + \mathcal{L}_\alpha^*(C_\alpha^* p_\alpha^{\mathcal{N}})(t) \\
&+ Q_\alpha\Big(x_\alpha^{\mathcal{N}}(t) - F_\alpha\Big(\mathcal{L}_\alpha(x_\alpha^{\mathcal{N}})(t) + \mathcal{L}_\beta(x_\beta^{\mathcal{N}})(t) + z_c(t)\Big) - G_\alpha x_\beta^{\mathcal{N}}(t) - M_\alpha\Big) \\
&- \mathcal{L}_\alpha^*\Big[(Q_\alpha F_\alpha)^*\Big(x_\alpha^{\mathcal{N}} - F_\alpha\Big(\mathcal{L}_\alpha(x_\alpha^{\mathcal{N}}) + \mathcal{L}_\beta(x_\beta^{\mathcal{N}}) + z_c\Big) - G_\alpha x_\beta^{\mathcal{N}} - M_\alpha\Big)\Big](t)\Big)dt \\
&- Z_{p_\alpha^{\mathcal{N}},\alpha} dW^\alpha(t) - Z_{p_\alpha^{\mathcal{N}},\beta} dW^\beta(t),
\end{aligned} \\
p_\alpha^{\mathcal{N}}(T) = 0; \\
\begin{aligned}
-dp_\beta^{\mathcal{N}} = &\Big(B_\beta^* p_\beta^{\mathcal{N}}(t) + \mathcal{L}_\beta^*(C_\beta^* p_\beta^{\mathcal{N}})(t) \\
&- \mathcal{L}_\beta^*\Big[(Q_\beta F_\beta)^*\Big(x_\beta^{\mathcal{N}} - F_\beta\Big(\mathcal{L}_\alpha(x_\alpha^{\mathcal{N}}) + \mathcal{L}_\beta(x_\beta^{\mathcal{N}}) + z_c\Big) - G_\beta x_\alpha^{\mathcal{N}} - M_\beta\Big)\Big](t) \\
&+ Q_\beta\Big(x_\beta^{\mathcal{N}}(t) - F_\beta\Big(\mathcal{L}_\alpha(x_\alpha^{\mathcal{N}})(t) + \mathcal{L}_\beta(x_\beta^{\mathcal{N}})(t) + z_c(t)\Big) - G_\beta x_\alpha^{\mathcal{N}}(t) - M_\beta\Big)\Big)dt \\
&- Z_{p_\beta^{\mathcal{N}},\alpha} dW^\alpha(t) - Z_{p_\beta^{\mathcal{N}},\beta} dW^\beta(t),
\end{aligned} \\
p_\beta^{\mathcal{N}}(T) = 0.
\end{cases}
\tag{3.2.34}
$$

3.2.4 Pareto Game

In this section we solve the Pareto game introduced in Definition 3.1.3 via the operator approach.

Theorem 3.2.9. *The solution of the Pareto Game is*

$$(u_\alpha^{\mathcal{P}}(t), u_\beta^{\mathcal{P}}(t)) = (-R_\alpha^{-1} D_\alpha^* p_\alpha^{\mathcal{P}}(t), -R_\beta^{-1} D_\beta^* p_\beta^{\mathcal{P}}(t)),$$

where $p_\alpha^{\mathcal{P}}$ and $p_\beta^{\mathcal{P}}$ are given by the backward stochastic functional differential equations:

$$\begin{cases} -dp_\alpha^{\mathcal{P}} = \Big(A_\alpha^* p_\alpha^{\mathcal{P}}(t) + A_\beta^* p_\beta^{\mathcal{P}}(t) + \mathcal{L}_\alpha^*(C_\alpha^* p_\alpha^{\mathcal{P}})(t) + \mathcal{L}_\alpha^*(C_\beta^* p_\beta^{\mathcal{P}})(t) \\ \qquad + Q_\alpha\Big(x_\alpha(t) - F_\alpha\Big(\mathcal{L}_\alpha(x_\alpha)(t) + \mathcal{L}_\beta(x_\beta)(t) + z_c(t)\Big) - G_\alpha x_\beta(t) - M_\alpha\Big) \\ \qquad - \mathcal{L}_\alpha^*\Big[(Q_\alpha F_\alpha)^*\Big(x_\alpha - F_\alpha\Big(\mathcal{L}_\alpha(x_\alpha) + \mathcal{L}_\beta(x_\beta) + z_c(t)\Big) - G_\alpha x_\beta - M_\alpha\Big)\Big](t) \\ \qquad - (Q_\beta G_\beta)^*\Big(x_\beta(t) - F_\beta\Big(\mathcal{L}_\alpha(x_\alpha)(t) + \mathcal{L}_\beta(x_\beta)(t) + z_c(t)\Big) - G_\beta x_\alpha(t) - M_\beta\Big) \\ \qquad - \mathcal{L}_\alpha^*\Big[(Q_\beta F_\beta)^*\Big(x_\beta - F_\beta\Big(\mathcal{L}_\alpha(x_\alpha) + \mathcal{L}_\beta(x_\beta) + z_c\Big) - G_\beta x_\alpha - M_\beta\Big)\Big](t)\Big)dt \\ \qquad - Z_{p_\alpha^{\mathcal{P}},\alpha} dW^\alpha(t) - Z_{p_\alpha^{\mathcal{P}},\beta} dW^\beta(t), \\ p_\alpha^{\mathcal{P}}(T) = 0; \end{cases}$$

and

$$\begin{cases} -dp_\beta^{\mathcal{P}} = \Big(B_\beta^* p_\beta^{\mathcal{P}}(t) + B_\alpha^* p_\alpha^{\mathcal{P}}(t) + \mathcal{L}_\beta^*(C_\alpha^* p_\alpha^{\mathcal{P}})(t) + \mathcal{L}_\beta^*(C_\beta^* p_\beta^{\mathcal{P}})(t) \\ \qquad + Q_\beta\Big(x_\beta(t) - F_\beta\Big(\mathcal{L}_\alpha(x_\alpha)(t) + \mathcal{L}_\beta(x_\beta)(t) + z_c(t)\Big) - G_\beta x_\alpha(t) - M_\beta\Big) \\ \qquad - \mathcal{L}_\beta^*\Big[(Q_\beta F_\beta)^*\Big(x_\beta - F_\beta\Big(\mathcal{L}_\alpha(x_\alpha) + \mathcal{L}_\beta(x_\beta) + z_c(t)\Big) - G_\beta x_\alpha - M_\beta\Big)\Big](t) \\ \qquad - (Q_\alpha G_\alpha)^*\Big(x_\alpha(t) - F_\alpha\Big(\mathcal{L}_\alpha(x_\alpha)(t) + \mathcal{L}_\beta(x_\beta)(t) + z_c(t)\Big) - G_\alpha x_\beta(t) - M_\alpha\Big) \\ \qquad - \mathcal{L}_\beta^*\Big[(Q_\alpha F_\alpha)^*\Big(x_\alpha - F_\alpha\Big(\mathcal{L}_\alpha(x_\alpha) + \mathcal{L}_\beta(x_\beta) + z_c\Big) - G_\alpha x_\beta - M_\alpha\Big)\Big](t)\Big)dt \\ \qquad - Z_{p_\beta^{\mathcal{P}},\alpha} dW^\alpha(t) - Z_{p_\beta^{\mathcal{P}},\beta} dW^\beta(t), \\ p_\beta^{\mathcal{P}}(T) = 0. \end{cases}$$

Proof. Similar to the proof of Theorem 3.2.8, we apply the standard stochastic maximum principle. Note that under this Pareto game setting, controls of the leaders would both be perturbed simultaneously. In particular, consider the perturbation of the optimal controls $(u_\alpha^{\mathcal{P}}(t), u_\beta^{\mathcal{P}}(t)) + \tau(\tilde{u}_\alpha(t), \tilde{u}_\beta(t))$, where $\tilde{u}_\alpha$ and $\tilde{u}_\beta$ are arbitrarily square integrable processes adapted to $\mathcal{F}^\alpha \vee \mathcal{F}^\beta$. The states of two leaders become $x_\alpha + \tau\tilde{x}_\alpha$ and $x_\beta + \tau\tilde{x}_\beta$ respectively, where

$$
\begin{aligned}
d\tilde{x}_\alpha &= \Big(A_\alpha\tilde{x}_\alpha(t) + B_\alpha\tilde{x}_\beta(t) + C_\alpha\Big(\mathcal{L}_\alpha(\tilde{x}_\alpha)(t)\mathcal{L}_\beta(\tilde{x}_\beta)(t)\Big) + D_\alpha\tilde{u}_\alpha(t)\Big)dt, \quad \tilde{x}_\alpha(0) = 0;\\
d\tilde{x}_\beta &= \Big(A_\beta\tilde{x}_\alpha(t) + B_\beta\tilde{x}_\beta(t) + C_\beta\Big(\mathcal{L}_\alpha(\tilde{x}_\alpha)(t)\mathcal{L}_\beta(\tilde{x}_\beta)(t)\Big) + D_\beta\tilde{u}_\beta(t)\Big)dt, \quad \tilde{x}_\beta(0) = 0.
\end{aligned}
$$

We consider the first order condition

$$
\begin{aligned}
0 &= \frac{d}{d\tau}\bigg|_{\tau=0}\Big(J_\alpha(u_\alpha^{\mathcal{P}}(t) + \tau\tilde{u}_\alpha(t), u_\beta^{\mathcal{P}}(t) + \tau\tilde{u}_\beta(t)) + J_\beta(u_\alpha^{\mathcal{P}}(t) + \tau\tilde{u}_\alpha(t), u_\beta^{\mathcal{P}}(t) + \tau\tilde{u}_\beta(t))\Big)\\
&= 2\mathbb{E}\int_0^T \Big\{\Big\langle\tilde{x}_\alpha(t) - F_\alpha\Big(\mathcal{L}_\alpha(\tilde{x}_\alpha)(t) + \mathcal{L}_\beta(\tilde{x}_\beta)(t)\Big)\\
&\qquad\qquad , Q_\alpha\Big(x_\alpha(t) - F_\alpha\Big(\mathcal{L}_\alpha(x_\alpha)(t) + \mathcal{L}_\beta(x_\beta)(t) + z_c(t)\Big) - G_\alpha x_\beta(t) - M_\alpha\Big)\Big\rangle\\
&\qquad + \langle\tilde{u}_\alpha(t), R_\alpha u_\alpha^{\mathcal{P}}(t)\rangle\\
&\qquad + \Big\langle\tilde{x}_\alpha(t) - F_\beta\Big(\mathcal{L}_\alpha(\tilde{x}_\alpha)(t) + \mathcal{L}_\beta(\tilde{x}_\beta)(t)\Big)\\
&\qquad\qquad , Q_\beta\Big(x_\beta(t) - F_\beta\Big(\mathcal{L}_\alpha(x_\alpha)(t) + \mathcal{L}_\beta(x_\beta)(t) + z_c(t)\Big) - G_\beta x_\alpha(t) - M_\beta\Big)\Big\rangle\\
&\qquad + \langle\tilde{u}_\beta(t), R_\beta u_\beta^{\mathcal{P}}(t)\rangle\Big\}dt.
\end{aligned} \tag{3.2.35}
$$

On the other hand, applying Itô's formula to the inner products $\langle p_\alpha^{\mathcal{P}}, \tilde{x}_\alpha\rangle$, $\langle p_\beta^{\mathcal{P}}, \tilde{x}_\beta\rangle$ and combining with the first order condition (3.2.35), following similar arguments found in Chapter 2 and [5], we get

$$
0 = \mathbb{E}\int_0^T \langle\tilde{u}_\alpha(t), R_\alpha u_\alpha^{\mathcal{P}}(t) + D_\alpha^* p_\alpha^{\mathcal{P}}(t)\rangle + \langle\tilde{u}_\beta(t), R_\beta u_\beta^{\mathcal{P}}(t) + D_\beta^* p_\beta^{\mathcal{P}}(t)\rangle dt.
$$

Since $\tilde{u}_\alpha(t), \tilde{u}_\beta(t)$ are arbitrary, we obtain the optimal control

$$
(u_\alpha^{\mathcal{P}}(t), u_\beta^{\mathcal{P}}(t)) = (-R_\alpha^{-1}D_\alpha^* p_\alpha^{\mathcal{P}}(t), -R_\beta^{-1}D_\beta^* p_\beta^{\mathcal{P}}(t)),
$$

$\square$

The full solution of the Pareto game in operator form is given by the following forward backward stochastic functional differential equations:

$$
\begin{cases}
dx_\alpha^{\mathcal{P}} = \Big(A_\alpha x_\alpha^{\mathcal{P}}(t) + B_\alpha x_\beta^{\mathcal{P}}(t) + C_\alpha\Big(\mathcal{L}_\alpha(x_\alpha^{\mathcal{P}})(t) + \mathcal{L}_\beta(x_\beta^{\mathcal{P}})(t) + z_c(t)\Big) \\
\qquad - D_\alpha R_\alpha^{-1} D_\alpha^* p_\alpha^{\mathcal{P}}(t)\Big)dt + \sigma_\alpha dW^\alpha(t), \\
x_\alpha^{\mathcal{P}}(0) = \xi_\alpha; \\
dx_\beta^{\mathcal{P}} = \Big(A_\beta x_\alpha^{\mathcal{P}}(t) + B_\beta x_\beta^{\mathcal{P}}(t) + C_\beta\Big(\mathcal{L}_\alpha(x_\alpha^{\mathcal{P}})(t) + \mathcal{L}_\beta(x_\beta^{\mathcal{P}})(t) + z_c(t)\Big) \\
\qquad - D_\beta R_\beta^{-1} D_\beta^* p_\beta^{\mathcal{P}}(t)\Big)dt + \sigma_\beta dW^\beta(t), \\
x_\beta^{\mathcal{P}}(0) = \xi_\beta; \\
-dp_\alpha^{\mathcal{P}} = \Bigg(A_\alpha^* p_\alpha^{\mathcal{P}}(t) + A_\beta^* p_\beta^{\mathcal{P}}(t) + \mathcal{L}_\alpha^*(C_\alpha^* p_\alpha^{\mathcal{P}})(t) + \mathcal{L}_\alpha^*(C_\beta^* p_\beta^{\mathcal{P}})(t) \\
\qquad + Q_\alpha\Big(x_\alpha^{\mathcal{P}}(t) - F_\alpha\Big(\mathcal{L}_\alpha(x_\alpha^{\mathcal{P}})(t) + \mathcal{L}_\beta(x_\beta^{\mathcal{P}})(t) + z_c(t)\Big) - G_\alpha x_\beta^{\mathcal{P}}(t) - M_\alpha\Big) \\
\qquad - \mathcal{L}_\alpha^*\Big[(Q_\alpha F_\alpha)^*\Big(x_\alpha^{\mathcal{P}} - F_\alpha\Big(\mathcal{L}_\alpha(x_\alpha^{\mathcal{P}}) + \mathcal{L}_\beta(x_\beta^{\mathcal{P}}) + z_c(t)\Big) - G_\alpha x_\beta^{\mathcal{P}} - M_\alpha\Big)\Big](t) \\
\qquad - (Q_\beta G_\beta)^*\Big(x_\beta^{\mathcal{P}}(t) - F_\beta\Big(\mathcal{L}_\alpha(x_\alpha^{\mathcal{P}})(t) + \mathcal{L}_\beta(x_\beta^{\mathcal{P}})(t) + z_c(t)\Big) - G_\beta x_\alpha^{\mathcal{P}}(t) - M_\beta\Big) \\
\qquad - \mathcal{L}_\alpha^*\Big[(Q_\beta F_\beta)^*\Big(x_\beta^{\mathcal{P}} - F_\beta\Big(\mathcal{L}_\alpha(x_\alpha^{\mathcal{P}}) + \mathcal{L}_\beta(x_\beta^{\mathcal{P}}) + z_c\Big) - G_\beta x_\alpha^{\mathcal{P}} - M_\beta\Big)\Big](t)\Bigg)dt \\
\qquad - Z_{p_\alpha^{\mathcal{P}},\alpha} dW^\alpha(t) - Z_{p_\alpha^{\mathcal{P}},\beta} dW^\beta(t), \\
p_\alpha^{\mathcal{P}}(T) = 0; \\
-dp_\beta^{\mathcal{P}} = \Bigg(B_\beta^* p_\beta^{\mathcal{P}}(t) + B_\alpha^* p_\alpha^{\mathcal{P}}(t) + \mathcal{L}_\beta^*(C_\alpha^* p_\alpha^{\mathcal{P}})(t) + \mathcal{L}_\beta^*(C_\beta^* p_\beta^{\mathcal{P}})(t) \\
\qquad + Q_\beta\Big(x_\beta^{\mathcal{P}}(t) - F_\beta\Big(\mathcal{L}_\alpha(x_\alpha^{\mathcal{P}})(t) + \mathcal{L}_\beta(x_\beta^{\mathcal{P}})(t) + z_c(t)\Big) - G_\beta x_\alpha^{\mathcal{P}}(t) - M_\beta\Big) \\
\qquad - \mathcal{L}_\beta^*\Big[(Q_\beta F_\beta)^*\Big(x_\beta^{\mathcal{P}} - F_\beta\Big(\mathcal{L}_\alpha(x_\alpha^{\mathcal{P}}) + \mathcal{L}_\beta(x_\beta^{\mathcal{P}}) + z_c(t)\Big) - G_\beta x_\alpha^{\mathcal{P}} - M_\beta\Big)\Big](t) \\
\qquad - (Q_\alpha G_\alpha)^*\Big(x_\alpha^{\mathcal{P}}(t) - F_\alpha\Big(\mathcal{L}_\alpha(x_\alpha^{\mathcal{P}})(t) + \mathcal{L}_\beta(x_\beta^{\mathcal{P}})(t) + z_c(t)\Big) - G_\alpha x_\beta^{\mathcal{P}}(t) - M_\alpha\Big) \\
\qquad - \mathcal{L}_\beta^*\Big[(Q_\alpha F_\alpha)^*\Big(x_\alpha^{\mathcal{P}} - F_\alpha\Big(\mathcal{L}_\alpha(x_\alpha^{\mathcal{P}}) + \mathcal{L}_\beta(x_\beta^{\mathcal{P}}) + z_c\Big) - G_\alpha x_\beta^{\mathcal{P}} - M_\alpha\Big)\Big](t)\Bigg)dt \\
\qquad - Z_{p_\beta^{\mathcal{P}},\alpha} dW^\alpha(t) - Z_{p_\beta^{\mathcal{P}},\beta} dW^\beta(t), \\
p_\beta^{\mathcal{P}}(T) = 0.
\end{cases}
\tag{3.2.36}
$$

3.3 Explicit Solutions

3.3.1 Explicit solution for the Nash Game

In lights of the explicit form of $\mathcal{L}_\alpha^*$ and $\mathcal{L}_\beta^*$ derived in Theorem 3.2.7, we can get a full solution of the Nash Game for both the leader and follower by substituting

$$g_1 = C_\alpha^* p_\alpha^{\mathcal{N}} - (Q_\alpha F_\alpha)^* \big(x_\alpha^{\mathcal{N}} - F_\alpha z - G_\alpha x_\beta^{\mathcal{N}} - M_\alpha \big)$$

and

$$g_2 = C_\beta^* p_\beta^{\mathcal{N}} - (Q_\beta F_\beta)^* \big(x_\beta^{\mathcal{N}} - F_\beta z - G_\beta x_\alpha^{\mathcal{N}} - M_\beta \big)$$

into (3.2.25) and (3.2.26) respectively:

$$\begin{cases} dx_1 = \Big(A_1 x_1(t) + B_1 z(t) + C_1 x_\alpha^{\mathcal{N}}(t) + D_1 x_\beta^{\mathcal{N}}(t) - E_1 R_1^{-1} E_1^* n(t) \Big) dt + \sigma_1 dW_1(t), \\ x_1^{\mathcal{N}}(0) = \xi_1; \\ dx_\alpha^{\mathcal{N}} = \Big(A_\alpha x_\alpha^{\mathcal{N}}(t) + B_\alpha x_\beta^{\mathcal{N}}(t) + C_\alpha z(t) - D_\alpha R_\alpha^{-1} D_\alpha^* p_\alpha^{\mathcal{N}}(t) \Big) dt + \sigma_\alpha dW^\alpha(t), \\ x_\alpha^{\mathcal{N}}(0) = \xi_\alpha; \\ dx_\beta^{\mathcal{N}} = \Big(A_\beta x_\alpha^{\mathcal{N}}(t) + B_\beta x_\beta^{\mathcal{N}}(t) + C_\beta z(t) - D_\beta R_\beta^{-1} D_\beta^* p_\beta^{\mathcal{N}}(t) \Big) dt + \sigma_\beta dW^\beta(t), \\ x_\beta^{\mathcal{N}}(0) = \xi_\beta; \\ dz = \Big((A_1 + B_1) z(t) + C_1 x_\alpha^{\mathcal{N}}(t) + D_1 x_\beta^{\mathcal{N}}(t) - E_1 R_1^{-1} E_1^* m(t) \Big) dt, \\ m(0) = \mathbb{E}[\xi_1]; \\ ds_\alpha = \Big(A_1^* s_\alpha(t) - E_1 R_1^{-1} E_1^* r_\alpha(t) \Big) dt, \\ s_\alpha(0) = 0; \\ ds_\beta = \Big(A_1^* s_\beta(t) - E_1 R_1^{-1} E_1^* r_\beta(t) \Big) dt, \\ s_\beta(0) = 0. \end{cases} \tag{3.3.37}$$

$$
\begin{cases}
-dn = \Big(A_1^* n(t) + Q_1(x_1(t) - F_1 z(t) - G_1 x_\alpha^{\mathcal{N}}(t) - H_1 x_\beta^{\mathcal{N}}(t) - M_1)\Big)dt \\
\qquad\qquad - Z_{n,\alpha} dW^\alpha(t) - Z_{n,\beta} dW^\beta(t) - Z_{n,1} dW^1(t) \\
n(T) = 0; \\
-dp_\alpha^{\mathcal{N}} = \Big(A_\alpha^* p_\alpha^{\mathcal{N}}(t) + C_1^* r_\alpha(t) - (Q_1 G_1)^* s_\alpha(t) + Q_\alpha(x_\alpha^{\mathcal{N}}(t) - F_\alpha z(t) - G_\alpha x_\beta^{\mathcal{N}}(t) - M_\alpha)\Big)dt \\
\qquad\qquad - Z_{p_\alpha^{\mathcal{N}},\alpha} dW^\alpha(t) - Z_{p_\alpha^{\mathcal{N}},\beta} dW^\beta(t), \\
p_\alpha^{\mathcal{N}}(T) = 0; \\
-dp_\beta^{\mathcal{N}} = \Big(B_\beta^* p_\beta^{\mathcal{N}}(t) + D_1^* r_\beta(t) - (Q_1 H_1)^* s_\beta(t) + Q_\beta(x_\beta^{\mathcal{N}}(t) - F_\beta z(t) - G_\beta x_\alpha^{\mathcal{N}}(t) - M_\beta)\Big)dt \\
\qquad\qquad - Z_{p_\beta^{\mathcal{N}},\alpha} dW^\alpha(t) - Z_{p_\beta^{\mathcal{N}},\beta} dW^\beta(t), \\
p_\beta^{\mathcal{N}}(T) = 0; \\
-dm = \Big(A_1^* m(t) + Q_1(I - F_1) z(t) - Q_1 G_1 x_\alpha^{\mathcal{N}}(t) - Q_1 H_1 x_\beta^{\mathcal{N}}(t) - Q_1 M_1\Big)dt \\
\qquad\qquad - Z_{m,\alpha} dW^\alpha(t) - Z_{m,\beta} dW^\beta(t) \\
m(T) = 0; \\
-dr_\alpha = \Big((A_1 + B_1)^* r_\alpha(t) + C_\alpha p_\alpha^{\mathcal{N}}(t) + (Q_1(I - F_1))^* s_\alpha(t) \\
\qquad\qquad - (Q_\alpha F_\alpha)^*(x_\alpha^{\mathcal{N}}(t) - F_\alpha z(t) - G_\alpha x_\beta^{\mathcal{N}}(t) - M_\alpha)\Big) - Z_{r_\alpha,\alpha} dW^\alpha(t) - Z_{r_\alpha,\beta} dW^\beta(t), \\
r_\alpha(T) = 0; \\
-dr_\beta = \Big((A_1 + B_1)^* r_\beta(t) + C_\beta p_\beta^{\mathcal{N}}(t) + (Q_1(I - F_1))^* s_\beta(t) \\
\qquad\qquad - (Q_\beta F_\beta)^*(x_\beta^{\mathcal{N}}(t) - F_\beta z(t) - G_\beta x_\alpha^{\mathcal{N}}(t) - M_\beta)\Big) - Z_{r_\beta,\alpha} dW^\alpha(t) - Z_{r_\beta,\beta} dW^\beta(t), \\
r_\beta(T) = 0.
\end{cases}
\tag{3.3.38}
$$

We can express this linear system in matrix form:

$$\begin{cases} d\mathbf{x}^{\mathcal{N}} = \Big(\mathbf{A}^{\mathcal{N}}\mathbf{x}^{\mathcal{N}}(t) - \mathbf{B}^{\mathcal{N}}\mathbf{p}^{\mathcal{N}}(t)\Big)dt + \mathbf{\Sigma}^{\mathcal{N}} d\mathbf{W}(t), \\ \mathbf{x}^{\mathcal{N}}(0) = \Xi; \\ -d\mathbf{p}^{\mathcal{N}} = \Big(\mathbf{C}^{\mathcal{N}}\mathbf{p}^{\mathcal{N}}(t) + \mathbf{D}^{\mathcal{N}}\mathbf{x}^{\mathcal{N}}(t) + \mathbf{M}^{\mathcal{N}}\Big)dt - \mathbf{Z}^{\mathcal{N}}(t)d\mathbf{W}(t), \\ \mathbf{p}^{\mathcal{N}}(T) = 0, \end{cases} \tag{3.3.39}$$

where

$$\mathbf{x}^{\mathcal{N}} := \begin{pmatrix} x_1 \\ x_\alpha^{\mathcal{N}} \\ x_\beta^{\mathcal{N}} \\ z \\ s_\alpha \\ s_\beta \end{pmatrix}, \qquad \mathbf{p}^{\mathcal{N}} := \begin{pmatrix} n \\ p_\alpha^{\mathcal{N}} \\ p_\beta^{\mathcal{N}} \\ m \\ r_\alpha \\ r_\beta \end{pmatrix}, \qquad \Xi := \begin{pmatrix} \xi_1 \\ \xi_\alpha \\ \xi_\beta \\ \mathbb{E}[\xi_1] \\ 0 \end{pmatrix},$$

and

$$\mathbf{A}^{\mathcal{N}} := \begin{pmatrix} A_1 & C_1 & D_1 & B_1 & 0 & 0 \\ 0 & A_\alpha & B_\alpha & C_\alpha & 0 & 0 \\ 0 & A_\beta & B_\beta & C_\beta & 0 & 0 \\ 0 & C_1 & D_1 & A_1+B_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_1 \end{pmatrix},$$

$$\mathbf{B}^{\mathcal{N}} := \begin{pmatrix} E_1R_1^{-1}E_1^* & 0 & 0 & 0 & 0 & 0 \\ 0 & D_\alpha R_\alpha^{-1} D_\alpha^* & 0 & 0 & 0 & 0 \\ 0 & 0 & D_\beta R_\beta^{-1} D_\beta^* & 0 & 0 & 0 \\ 0 & 0 & 0 & E_1R_1^{-1}E_1^* & 0 & 0 \\ 0 & 0 & 0 & 0 & E_1R_1^{-1}E_1^* & 0 \\ 0 & 0 & 0 & 0 & 0 & E_1R_1^{-1}E_1^* \end{pmatrix},$$

$$\mathbf{\Sigma}^{\mathcal{N}} := \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_\alpha & 0 \\ 0 & 0 & \sigma_\beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad d\mathbf{W}(t) := \begin{pmatrix} dW_1(t) \\ dW^\alpha(t) \\ dW^\beta(t) \end{pmatrix},$$

$$\mathbf{C}^{\mathcal{N}} := \begin{pmatrix} A_1^* & 0 & 0 & 0 & 0 & 0 \\ 0 & A_\alpha^* & 0 & 0 & C_1^* & 0 \\ 0 & 0 & B_\beta^* & 0 & 0 & D_1^* \\ 0 & 0 & 0 & A_1^* & 0 & 0 \\ 0 & C_\alpha & 0 & 0 & (A_1+B_1)^* & 0 \\ 0 & 0 & C_\beta & 0 & 0 & (A_1+B_1)^* \end{pmatrix},$$

$$\mathbf{D}^{\mathcal{N}} := \begin{pmatrix} Q_1 & -Q_1G_1 & -Q_1H_1 & -Q_1F_1 & 0 & 0 \\ 0 & Q_\alpha & -Q_\alpha G_\alpha & -Q_\alpha F_\alpha & -(Q_1G_1)^* & 0 \\ 0 & -Q_\beta G_\beta & Q_\beta & -Q_\beta F_\beta & 0 & -(Q_1H_1)^* \\ 0 & -(Q_1G_1) & -(Q_1H_1) & Q_1(I-F_1) & 0 & 0 \\ 0 & -(Q_\alpha F_\alpha)^* & (Q_\alpha F_\alpha)^* G_\alpha & (Q_\alpha F_\alpha)^* F_\alpha & Q_1(I-F_1)^* & 0 \\ 0 & (Q_\beta F_\beta)^* G_\beta & -(Q_\beta F_\beta)^* & (Q_\beta F_\beta)^* F_\beta & 0 & Q_1(I-F_1)^* \end{pmatrix},$$

$$\mathbf{M}^{\mathcal{N}} := \begin{pmatrix} -Q_1M_1 \\ -Q_\alpha M_\alpha \\ -Q_\beta M)_\beta \\ -Q_1M_1 \\ (Q_\alpha F_\alpha)^* M_\alpha \\ (Q_\beta F_\beta)^* M_\beta \end{pmatrix}.$$

Theorem 3.3.1. *Given any square integrable process* $\mathbf{x}$*, suppose that the following non-symmetric Riccati equation*

$$d\Gamma_t^{\mathcal{N}} + \Gamma_t^{\mathcal{N}}\mathbf{A}^{\mathcal{N}} + \mathbf{C}^{\mathcal{N}}\Gamma_t^{\mathcal{N}} - \Gamma_t^{\mathcal{N}}\mathbf{B}^{\mathcal{N}}\Gamma_t^{\mathcal{N}} + \mathbf{D}^{\mathcal{N}} = 0, \quad \Gamma^{\mathcal{N}}(T) = 0 \tag{3.3.40}$$

admits a unique solution on $[0,T]$*, then there is a unique solution to* (3.3.39).

Proof. It is easy to check $\mathbf{p}^{\mathcal{N}}(t) = \Gamma_t^{\mathcal{N}}\mathbf{x}^{\mathcal{N}}(t) + \mathbf{g}^{\mathcal{N}}(t)$, where

$$\begin{cases} -d\mathbf{g}^{\mathcal{N}} = \Big((\mathbf{C}^{\mathcal{N}} - \Gamma_t^{\mathcal{N}}\mathbf{B}^{\mathcal{N}})\mathbf{g}^{\mathcal{N}}(t) + \mathbf{M}^{\mathcal{N}}\Big)dt, \\ \mathbf{g}^{\mathcal{N}}(T) = 0. \end{cases}$$

The existence of the forward equation $\mathbf{x}$ is then immediate. The uniqueness is clear. $\square$

Denote

$$\mathcal{M}^{\mathcal{N}} := \begin{pmatrix} \mathbf{A}^{\mathcal{N}} & -\mathbf{B}^{\mathcal{N}} \\ -\mathbf{D}^{\mathcal{N}} & -\mathbf{C}^{\mathcal{N}} \end{pmatrix} \tag{3.3.41}$$

then the solution of (3.3.40) is given by:

$$\Gamma_t^{\mathcal{N}} = -\left[\begin{pmatrix} 0 & I \end{pmatrix} e^{\mathcal{M}^{\mathcal{N}}(T-t)} \begin{pmatrix} 0 \\ I \end{pmatrix}\right]^{-1}\left[\begin{pmatrix} 0 & I \end{pmatrix} e^{\mathcal{M}^{\mathcal{N}}(T-t)} \begin{pmatrix} I \\ 0 \end{pmatrix}\right].$$

We then rewrite equation (3.3.39) in the following decoupled form:

$$\begin{cases} d\mathbf{x}^{\mathcal{N}} = \Big((\mathbf{A}^{\mathcal{N}} - \mathbf{B}^{\mathcal{N}}\Gamma_t^{\mathcal{N}})\mathbf{x}^{\mathcal{N}}(t) - \mathbf{B}^{\mathcal{N}}\mathbf{g}^{\mathcal{N}}(t)\Big)dt + \mathbf{\Sigma}^{\mathcal{N}}d\mathbf{W}(t), \\ \mathbf{x}^{\mathcal{N}}(0) = \Xi; \\ -d\mathbf{g}^{\mathcal{N}} = \Big((\mathbf{C}^{\mathcal{N}} - \Gamma_t^{\mathcal{N}}\mathbf{B}^{\mathcal{N}})\mathbf{g}^{\mathcal{N}}(t) + \mathbf{M}^{\mathcal{N}}\Big)dt, \\ \mathbf{g}^{\mathcal{N}}(T) = 0. \end{cases} \tag{3.3.42}$$

3.3.2 Explicit solution for the Pareto Game

Similarly, by putting

$$\begin{aligned} g_1 = C_\alpha^* p_\alpha^{\mathcal{P}} + C_\beta^* p_\beta^{\mathcal{P}} - (Q_\alpha F_\alpha)^*\big(x_\alpha^{\mathcal{P}} - F_\alpha z - G_\alpha x_\beta^{\mathcal{P}} - M_\alpha\big) \\ - (Q_\beta F_\beta)^*\big(x_\beta^{\mathcal{P}} - F_\beta z - G_\beta x_\alpha^{\mathcal{P}} - M_\beta\big) = g_2, \end{aligned}$$

the full solution of the Pareto game for both the leader and follower can be characterized by the following system of FBSDEs:

$$
\begin{cases}
dx_1 = \Big(A_1 x_1(t) + B_1 z(t) + C_1 x_\alpha^{\mathcal{P}}(t) + D_1 x_\beta^{\mathcal{P}}(t) - E_1 R_1^{-1} E_1^* n(t)\Big) dt + \sigma_1 dW_1(t), \\
x_1(0) = \xi_1; \\
dx_\alpha^{\mathcal{P}} = \Big(A_\alpha x_\alpha^{\mathcal{P}}(t) + B_\alpha x_\beta^{\mathcal{P}}(t) + C_\alpha z(t) - D_\alpha R_\alpha^{-1} D_\alpha^* p_\alpha^{\mathcal{P}}(t)\Big) dt + \sigma_\alpha dW^\alpha(t), \\
x_\alpha^{\mathcal{P}}(0) = \xi_\alpha; \\
dx_\beta^{\mathcal{P}} = \Big(A_\beta x_\alpha^{\mathcal{P}}(t) + B_\beta x_\beta^{\mathcal{P}}(t) + C_\beta z(t) - D_\beta R_\beta^{-1} D_\beta^* p_\beta^{\mathcal{P}}(t\Big) dt + \sigma_\beta dW^\beta(t) \\
x_\beta^{\mathcal{P}}(0) = \xi_\beta; \\
dz = \Big((A_1 + B_1) z(t) + C_1 x_\alpha^{\mathcal{P}}(t) + D_1 x_\beta^{\mathcal{P}}(t) - E_1 R_1^{-1} E_1^* m(t)\Big) dt, \\
z(0) = \mathbb{E}[\xi_1]; \\
ds = \Big(A_1^* s(t) - E_1 R_1^{-1} E_1^* r(t)\Big) dt, \\
s(0) = 0.
\end{cases}
\tag{3.3.43}
$$

$$
\begin{cases}
-dn = \Big(A_1^* n(t) + Q_1(x_1(t) - F_1 z(t) - G_1 x_\alpha^{\mathcal{P}}(t) - H_1 x_\beta^{\mathcal{P}}(t) - M_1)\Big)dt \\
\qquad - Z_{n,\alpha} dW^\alpha(t) - Z_{n,\beta} dW^\beta(t) - Z_{n,1} dW^1(t), \\
n(T) = 0; \\
-dm = \Big(A_1^* m(t) + Q_1(I - F_1)z(t) - Q_1 G_1 x_\alpha^{\mathcal{P}}(t) - Q_1 H_1 x_\beta^{\mathcal{P}}(t) - Q_1 M_1\Big)dt \\
\qquad - Z_{m,\alpha} dW^\alpha(t) - Z_{m,\beta} dW^\beta(t), \\
m(T) = 0; \\
-dp_\alpha^{\mathcal{P}} = \Big(A_\alpha^* p_\alpha^{\mathcal{P}}(t) + A_\beta^* p_\beta^{\mathcal{P}}(t) + C_1^* r(t) - (Q_1 G_1)^* s(t) + Q_\alpha(x_\alpha^{\mathcal{P}}(t) - F_\alpha z(t) - G_\alpha x_\beta^{\mathcal{P}}(t) - M_\alpha) \\
\qquad + (Q_\beta G_\beta)^*(x_\beta^{\mathcal{P}}(t) - F_\beta z(t) - G_\beta x_\alpha^{\mathcal{P}}(t) - M_\beta)\Big)dt - Z_{p_\alpha^{\mathcal{P}},\alpha} dW^\alpha(t) - Z_{p_\alpha^{\mathcal{P}},\beta} dW^\beta(t), \\
p_\alpha^{\mathcal{P}}(T) = 0; \\
-dp_\beta^{\mathcal{P}} = \Big(B_\beta^* p_\beta^{\mathcal{P}}(t) + B_\alpha^* p_\alpha^{\mathcal{P}}(t) + D_1^* r(t) - (Q_1 H_1)^* s(t) + Q_\beta(x_\beta^{\mathcal{P}}(t) - F_\beta z(t) - G_\beta x_\alpha^{\mathcal{P}}(t) - M_\beta) \\
\qquad + (Q_\alpha G_\alpha)^*(x_\alpha^{\mathcal{P}}(t) - F_\alpha z(t) - G_\alpha x_\beta^{\mathcal{P}}(t) - M_\alpha)\Big)dt - Z_{p_\beta^{\mathcal{P}},\alpha} dW^\alpha(t) - Z_{p_\beta^{\mathcal{P}},\beta} dW^\beta(t), \\
p_\beta^{\mathcal{P}}(T) = 0; \\
-dr = \Big((A_1 + B_1)^* r(t) + C_\alpha p_\alpha^{\mathcal{P}}(t) + C_\beta p_\beta^{\mathcal{P}}(t) + (Q_1(I - F_1))^* s(t) \\
\qquad - (Q_\alpha F_\alpha)^*(x_\alpha^{\mathcal{P}}(t) - F_\alpha z(t) - G_\alpha x_\beta^{\mathcal{P}}(t) - M_\alpha) \\
\qquad - (Q_\beta F_\beta)^*(x_\beta^{\mathcal{P}}(t) - F_\beta z(t) - G_\beta x_\alpha^{\mathcal{P}}(t) - M_\beta)\Big)dt \\
\qquad - Z_{r_\alpha,\alpha} dW^\alpha(t) - Z_{r_\alpha,\beta} dW^\beta(t).
\end{cases}
\tag{3.3.44}
$$

Similar to Section 3.3.1, the matrix form of system (3.3.43) is as follows:

$$
\begin{cases}
d\mathbf{x}^{\mathcal{P}} = \Big(\mathbf{A}^{\mathcal{P}} \mathbf{x}\mathcal{P}(t) - \mathbf{B}^{\mathcal{P}} \mathbf{p}^{\mathcal{P}}(t)\Big)dt + \mathbf{\Sigma}^{\mathcal{P}} d\mathbf{W}(t), \\
\mathbf{x}^{\mathcal{P}}(0) = \Xi; \\
-d\mathbf{g}^{\mathcal{N}} = \Big((\mathbf{C}^{\mathcal{N}} - \Gamma_t^{\mathcal{N}} \mathbf{B}^{\mathcal{N}})\mathbf{g}^{\mathcal{N}}(t) + \mathbf{M}^{\mathcal{N}}\Big)dt, \\
\mathbf{g}^{\mathcal{N}}(T) = 0.
\end{cases}
\tag{3.3.45}
$$

where

$$
\mathbf{x}^{\mathcal{P}} := \begin{pmatrix} x_1 \\ x_\alpha^{\mathcal{P}} \\ x_\beta^{\mathcal{P}} \\ z \\ s \end{pmatrix}, \qquad \mathbf{p}^{\mathcal{P}} := \begin{pmatrix} n \\ p_\alpha^{\mathcal{P}} \\ p_\beta^{\mathcal{P}} \\ m \\ r \end{pmatrix}, \qquad \Xi := \begin{pmatrix} \xi_1 \\ \xi_\alpha \\ \xi_\beta \\ \mathbb{E}[\xi_1] \\ 0 \end{pmatrix},
$$

and

$$\mathbf{A}^{\mathcal{P}} := \begin{pmatrix} A_1 & C_1 & D_1 & B_1 & 0 \\ 0 & A_\alpha & B_\alpha & C_\alpha & 0 \\ 0 & A_\beta & B_\beta & C_\beta & 0 \\ 0 & C_1 & D_1 & A_1+B_1 & 0 \\ 0 & 0 & 0 & 0 & A_1 \end{pmatrix},$$

$$\mathbf{B}^{\mathcal{P}} := \begin{pmatrix} E_1R_1^{-1}E_1^* & 0 & 0 & 0 & 0 \\ 0 & D_\alpha R_\alpha^{-1}D_\alpha^* & 0 & 0 & 0 \\ 0 & 0 & D_\beta R_\beta^{-1}D_\beta^* & 0 & 0 \\ 0 & 0 & 0 & E_1R_1^{-1}E_1^* & 0 \\ 0 & 0 & 0 & 0 & E_1R_1^{-1}E_1^* \end{pmatrix},$$

$$\mathbf{\Sigma}^{\mathcal{P}} := \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_\alpha & 0 \\ 0 & 0 & \sigma_\beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad d\mathbf{W}(t) := \begin{pmatrix} dW_1(t) \\ dW^\alpha(t) \\ dW^\beta(t) \end{pmatrix},$$

$$\mathbf{C}^{\mathcal{P}} := \begin{pmatrix} A_1^* & 0 & 0 & 0 & 0 \\ 0 & A_\alpha^* & A_\beta^* & 0 & C_1^* \\ 0 & B_\alpha^* & B_\beta^* & 0 & D_1^* \\ 0 & 0 & 0 & A_1^* & 0 \\ 0 & C_\alpha^* & C_\beta^* & 0 & (A_1+B_1)^* \end{pmatrix},$$

$$\mathbf{D}^{\mathcal{P}} := \begin{pmatrix} Q_1 & -Q_1G_1 & -Q_1H_1 & -Q_1F_1 & 0 \\ 0 & Q_\alpha+(Q_\beta G_\beta)^*G_\beta & -Q_\alpha G_\alpha-(Q_\beta G_\beta)^* & -Q_\alpha F_\alpha+(Q_\beta G_\beta)^*F_\beta & -(Q_1G_1)^* \\ 0 & -Q_\beta G_\beta-(Q_\alpha G_\alpha)^* & Q_\beta+(Q_\alpha G_\alpha)^*G_\alpha & -Q_\beta F_\beta+(Q_\alpha G_\alpha)^*F_\alpha & -(Q_1H_1)^* \\ 0 & -(Q_1G_1) & -(Q_1H_1) & Q_1(I-F_1) & 0 \\ 0 & -(Q_\alpha F_\alpha)^*+(Q_\beta F_\beta)^*G_\beta & -(Q_\beta F_\beta)^*+(Q_\alpha F_\alpha)^*G_\alpha & (Q_\alpha F_\alpha)^*F_\alpha+(Q_\beta F_\beta)^*F_\beta & Q_1(I-F_1)^* \end{pmatrix}.$$

$$\mathbf{M}^{\mathcal{P}} := \begin{pmatrix} -Q_1M_1 \\ -Q_\alpha M_\alpha \\ -Q_\beta M)_\beta \\ -Q_1M_1 \\ (Q_\alpha F_\alpha)^*M_\alpha+(Q_\beta F_\beta)^*M_\beta \end{pmatrix}.$$

Similar to Section 3.3.1, we define another non-symmetric Riccati equation

$$d\Gamma_t^{\mathcal{P}} + \Gamma_t^{\mathcal{P}}\mathbf{A}^{\mathcal{P}} + \mathbf{C}^{\mathcal{P}}\Gamma_t^{\mathcal{P}} - \Gamma_t^{\mathcal{P}}\mathbf{B}^{\mathcal{P}}\Gamma_t^{\mathcal{P}} + \mathbf{D}^{\mathcal{P}} = 0, \quad \Gamma^{\mathcal{P}}(T) = 0. \tag{3.3.46}$$

We then rewrite equation (3.3.45) in the following decoupled form:

$$\begin{cases} \quad d\mathbf{x}^{\mathcal{P}} = \Big((\mathbf{A}^{\mathcal{P}} - \mathbf{B}^{\mathcal{P}}\Gamma_t^{\mathcal{P}})\mathbf{x}^{\mathcal{P}}(t) - \mathbf{B}^{\mathcal{P}}\mathbf{g}^{\mathcal{P}}(t)\Big)dt + \mathbf{\Sigma}^{\mathcal{P}}d\mathbf{W}(t), \\ \mathbf{x}^{\mathcal{P}}(0) = \Xi; \\ -d\mathbf{g}^{\mathcal{P}} = \Big((\mathbf{C}^{\mathcal{P}} - \Gamma_t^{\mathcal{P}}\mathbf{B}^{\mathcal{P}})\mathbf{g}^{\mathcal{P}}(t) + \mathbf{M}^{\mathcal{P}}\Big)dt, \\ \mathbf{g}^{\mathcal{P}}(T) = 0. \end{cases} \tag{3.3.47}$$

3.4 Numerical Results

Using the explicit form obtained in Section 3.3, we can easily simulate the Stackelberg Game under both the Nash and Pareto setting.

3.4.1 Simple Example

Consider the following simple example, where we do not introduce mean field effect here. Evolutions and cost functionals of the leaders and followers are respectively given by

$$dx_\alpha = v_\alpha(t)dt + 0.1dW^\alpha(t); \qquad x_\alpha(0) = 1, \tag{3.4.48}$$

$$dx_\beta = v_\beta(t)dt + 0.1dW^\beta(t); \qquad x_\beta(0) = 1, \tag{3.4.49}$$

$$dx_1^i = v_1^i(t)dt + 0.1dW_1^i(t); \qquad x_1^i(0) = 1; \tag{3.4.50}$$

and

$$J_\alpha(v_\alpha, v_\beta) = \mathbb{E}\int_0^T 10|x_\alpha(t) - 1.5|^2 + |v_\alpha(t)|^2 dt; \tag{3.4.51}$$

$$J_\beta(v_\alpha, v_\beta) = \mathbb{E}\int_0^T 10|x_\beta(t) - x_\alpha(t)|^2 + |v_\beta(t)|^2 dt; \tag{3.4.52}$$

$$J_1^i(v_1^i; x_\alpha, x_\beta, z) = \mathbb{E}\int_0^T |x_1^i(t) - x_\beta(t)|^2 + |v_1^i(t)|^2 dt. \tag{3.4.53}$$

Both the states of two leaders and followers start at the same level 1, and the drift coefficients in three stochastic differential equations (3.4.48)-(3.4.50) involve respective control variables only. The objective of α-leader (3.4.51) is to minimize the squared distance between his state and the level 1.5; β-leader on the other hand tries to minimize his distance from the α-leader in (3.4.52). Finally, the homogeneous individual followers aim at minimizing their distance from the β-leader.

Figure 3.1 shows the simulation results of two leaders. The rough lines denote one of the simulated scenario, while the (relatively) smooth lines represent the average of 1,000 paths. The evolutions of x_α and x_β are in black and grey respectively. The solid lines indicate the Nash setting, while the dashed lines demonstrate the Pareto case. To qualitatively study the simulated results of this simple model, we focus on the average smooth lines in Figure 3.1. x_α surges sharply at first and becomes steady as it approaches the level 1.5 stated in the cost functional (3.4.51). To reduce the squared distance from the α-leader, x_β mimics his evolution - increases fast initially and becomes flattish later on.

One can also observe that, for both leaders, the smooth solid (Nash) lines lie above the dashed (Pareto) lines. In the cooperative Pareto game, α-leader not only tries to minimize his own cost functional J_α, but also J_β - the cost of β-leader. Intuitively, α-leader scarifies himself by not getting close to the level 1.5 as in the Nash game, and tries to reduce his distance from β-leader. On the other hand, β-leader benefits from the cooperative game as he can now adopts a smaller value of control v_β to minimize his distance from the α-leader. Figure 3.2 shows the average of 1,000 simulated cost of α-leader in black, β-leader in grey and their sum in blue under different games. The total cost of the two leaders reduces through cooperation by comparing the blue solid and dashed lines. While β-leader's cost becomes smaller in the Pareto setting, α-leader's terminal cost is higher. Finally, as the trajectory of β-leader moves downward in the Pareto game as shown in Figure 3.2, we expect that the follower would use a small control process to reduce the distance between his state and the β-leader. Figure 3.3 shows the average cost of 1,000 simulation for follower under different game setting, which clearly shows that the follower benefits from the cooperation between leaders in this simple example.

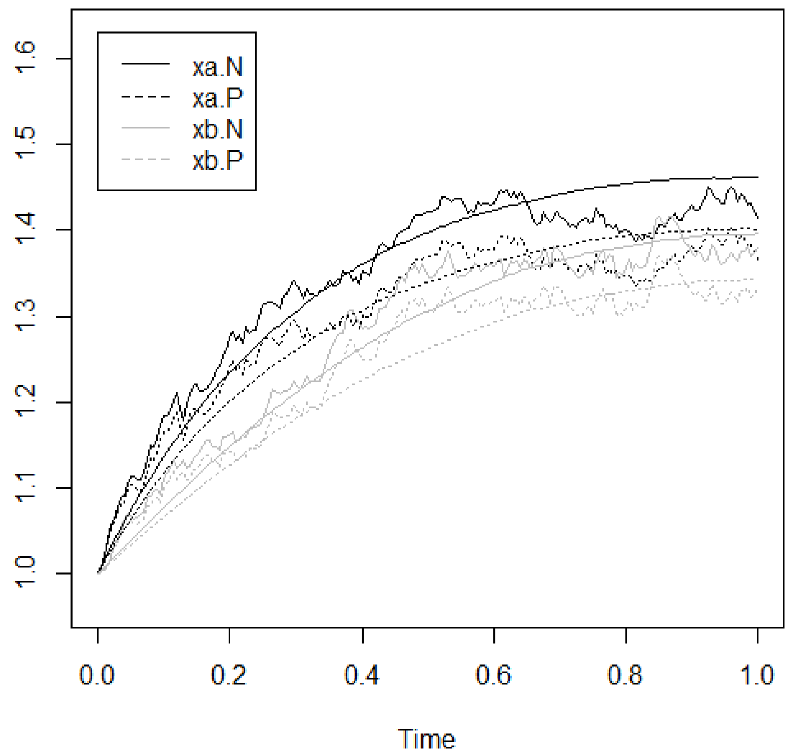

Figure 3.1: Evolution of leaders in two games.

3.4.2 Study on mean field effect

Next, we introduce a mean field effect into the simple example in Section 3.4.1. In particular, we change the cost functional of the follower:

$$J_1^i(v_1^i; x_\alpha, x_\beta, z) = \mathbb{E}\int_0^T |x_1^i(t) - (1-c)z(t) - cx_\beta(t)|^2 + |v_1^i(t)|^2 dt, \tag{3.4.54}$$

where $c \in [0,1]$. That is, the follower aims at minimizing the distance between its own states and

$$(1-c)z(t) + cx_\beta(t),$$

the convex combination of the mean field term and the state of β-leader. As the value of c approaches 1, the impact of mean field term on each follower increase;

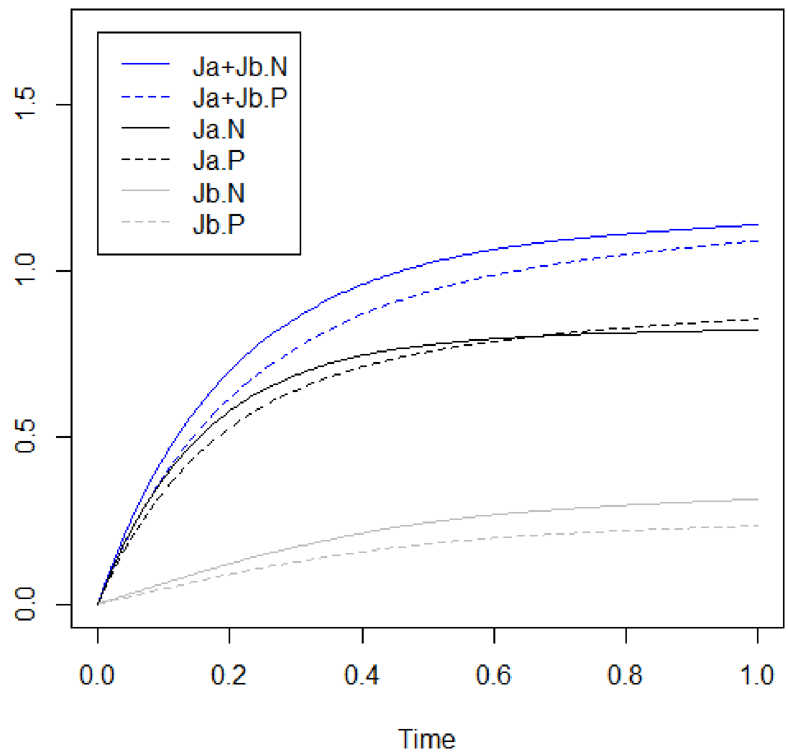

Figure 3.2: Cost of leaders in two games.

while the influence of β-leader diminishes. Figure 3.4 shows the average costs of 1,000 simulation with different value of c in both Nash game (solid lines) and Pareto game (dashed lines). As the value of c increases, the cost difference between two games drops. That is the benefit follower received through cooperation between leaders shrink.

3.5 Conclusion

Mean field Stackelberg games with one leader under a linear quadratic setting has been studied previously in our previous work [5] and Chapter 2. In this Chapter, we consider an interesting class extension by considering two leaders

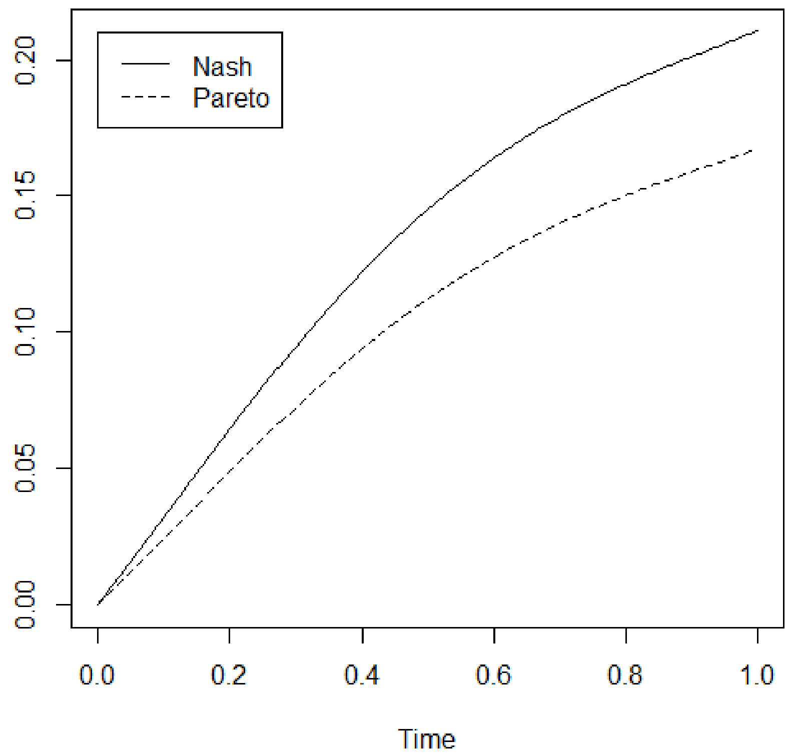

Figure 3.3: Cost of followers in two games.

over a group of followers. Depending on whether or not they cooperate, we solve for the respective Nash and Pareto game. Under the simple model proposed in Section 3.4, the group of followers benefits as the leaders cooperate. Due to the large number of parameters in the original model, the general sufficient condition for followers gaining (or losing) due to the cooperation of leaders is not discussed in this chapter and will be explored in future works.

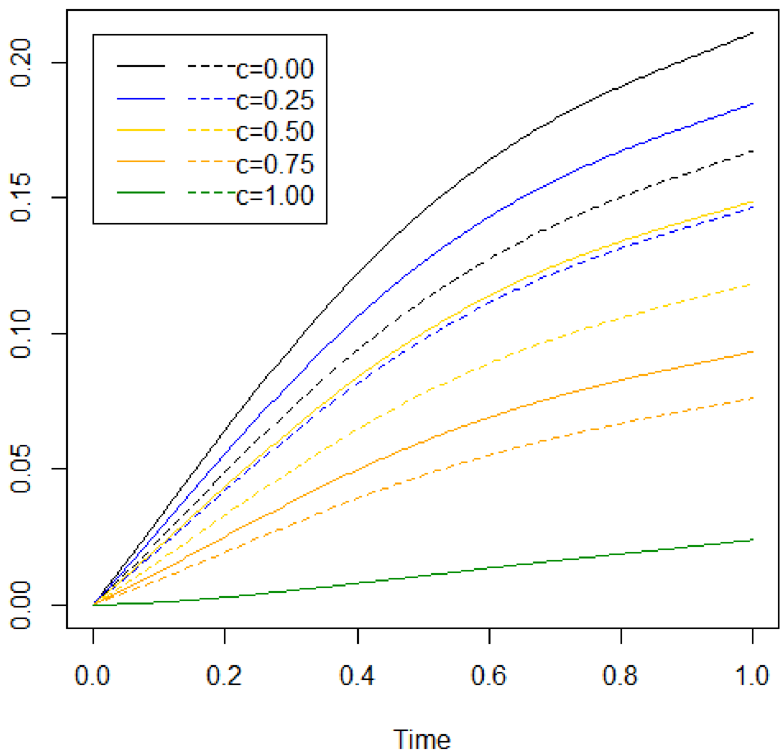

Figure 3.4: Cost of followers in two games.

Chapter 4

Discrete-time Mean Field Partially Observable Controlled Systems Subject to Common Noise

This chapter provides a systemic study on discrete time partially observable mean field systems in the presence of a common noise. Each player makes decision solely based on the observable process. Both the mean field games and the associated mean field type stochastic control problem are formulated in Section 4.1. We first solve the mean field type control problem using classical discrete time Kalman filter with notable modifications in Section 4.2. The unique existence of the resulted forward backward stochastic difference system is then established by Separation Principle. The mean field game problem is also solved via an application of stochastic maximum principle, while the existence of the mean field equilibrium is shown by the Schauder's fixed point theorem in Section 4.3.

4.1 Problem Setting

4.1.1 Mean Field Game

Let $P \in \mathbb{N}$. Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with independent discrete standardized Gaussian white noise processes $W^i, \tilde{W} \in \mathbb{R}^{d_x}$ and $V^i \in \mathbb{R}^{d_y}$, for $i = 1, \ldots, P$; their means are all zero vector, while their covariance matrices are identity; to avoid unnecessary technical details, we assume that all the volatilities in whatever dynamics appeared in the rest of this article are standardized to be equal to one. Here the dimensions d_x and d_y stand for that of the unobservable state space and observable state space respectively. Also define square integrable Gaussian random vectors $\xi^i \in \mathbb{R}^{d_x}$, for $i = 1, \ldots, P$, which are independently and identically distributed, and they are also assumed to be independent of the mentioned white noise processes. Next, let $k \leq N \in \mathbb{N}$, define the following filtrations

$$\begin{aligned}
\mathcal{F}_k &:= \sigma\{\xi, W_r^1, \ldots W_r^P, \tilde{W}_r, V_{r-1}^1, \ldots, V_{r-1}^P : 1 \leq r \leq k, \};\\
\mathcal{F}_k^{W^i} &:= \sigma\{W_r^i : 1 \leq r \leq k\};\\
\mathcal{F}_k^{\tilde{W}} &:= \sigma\{\tilde{W}_r : 1 \leq r \leq k\};\\
\mathcal{F}_k^{V^i} &:= \sigma\{V_r^i : 0 \leq r \leq k\}.
\end{aligned}$$

Here, $\mathcal{F}_k$ represents the flow of history of all the information up to time k; $\mathcal{F}_k^{W^i}$ and $\mathcal{F}_k^{\tilde{W}}$ stand for that caused by the i^{th} individual noise and the common noise respectively; $\mathcal{F}_k^{V^i}$ is the filtration induced by the i^{th} observational noise.

We first consider a finite number of P player system in which the unobservable state evolution for the i^{th} player is modelled by the following difference equations, for $i = 1, \ldots, P$,

$$\begin{cases}
\mathcal{X}_{k+1}^i &= A_k \mathcal{X}_k^i + \bar{A}_k \frac{\sum_{j=1}^P \mathcal{X}_k^j}{P} + B_k v_k^i + W_{k+1}^i + \tilde{W}_{k+1}, \\
& \qquad\qquad\qquad\qquad k \in \{0, 1, \ldots, N-1\}, \\
\mathcal{X}_0^i &= \xi^i.
\end{cases} \tag{4.1.1}$$

In the case of any potential ambiguity, we prefer to adopt the self-explaining notation $\mathcal{X}^{i,v}$ with superscript v indicating that the dynamics being governed by the specific control v; otherwise, we omit this extra superscript if the underlying context is clear. However, the information from $\mathcal{X}$ could be distorted in any common economic modelling, more specifically, due to the technological limitations encountered by the agents, the i^{th} player can only make his own decision based on another observable process $\mathcal{Y}^i$ which is described by another difference equation:

$$\mathcal{Y}_k^i = H_k\mathcal{X}_k^i + \bar{H}_k\frac{\sum_{j=1}^P \mathcal{X}_k^j}{P} + V_k^i, \qquad k \in \{0,1,\dots,N-1\}. \tag{4.1.2}$$

Again, to avoid unnecessary technical details, we assume that all matrices $A_k, \bar{A}_k, B_k, H_k, \bar{H}_k$, for $k = 0,\dots,N-1$, are constant with appropriate dimensions. In the sequel, M^* denotes the transpose of an arbitrary matrix M.

The i^{th} player aims at minimizing the following quadratic cost functional:

$$\begin{aligned}\mathcal{J}^{i,P}(\mathbf{v}) := \sum_{k=0}^{N-1}\mathbb{E}\Bigg(\|\mathcal{X}_k^i\|_{Q_k}^2 + \|v_k^i\|_{R_k}^2 + \left\|\mathcal{X}_k^i - S_k\frac{\sum_{j=1}^P \mathcal{X}_k^j}{P}\right\|_{\bar{Q}_k}^2\Bigg)\\ + \mathbb{E}\Bigg(\|\mathcal{X}_N^i\|_{Q_N}^2 + \left\|\mathcal{X}_N^i - S_N\frac{\sum_{j=1}^P \mathcal{X}_N^j}{P}\right\|_{\bar{Q}_N}^2\Bigg),\end{aligned}$$

where $\mathbf{v} = (v^1, v^2, \dots, v^P)$, $|\cdot|_{\mathcal{Q}} := \langle \cdot, \mathcal{Q}\cdot\rangle$ for any positive definite matrix $\mathcal{Q}$ and $\langle \cdot, \star\rangle$ is the usual Euclidean inner product. The matrices $Q_k, R_k, S_k, \bar{Q}_k$ for $k = 0,\dots,N-1$ and $Q_N, S_N, \bar{Q}_N$ are again assumed to be constant with appropriate dimensions; while $Q_k, R_k, \bar{Q}_k$ for $k = 0,\dots,N$ are positive definite.

On the other hand, we now consider the limiting mean field system in which the unobservable individual state evolution is given by

$$\begin{cases} x_{k+1}^i &= A_k x_k^i + \bar{A}_k z_k + B_k v_k^i + W_{k+1}^i + \tilde{W}_{k+1}, \qquad k \in \{0,1,\dots,N-1\},\\ x_0^i &= \xi^i; \end{cases} \tag{4.1.3}$$

with the corresponding observable state process

$$y_k^i := H_k x_k^i + \bar{H}_k z_k + V_k^i, \qquad k \in \{0,1,\dots,N\}; \tag{4.1.4}$$

while the cost functional for the i^{th} player is given by:

$$J^i(v) = \sum_{k=0}^{N-1} \mathbb{E}\Big(\|x_k^i\|_{Q_k}^2 + \|v_k^i\|_{R_k}^2 + \|x_k^i - S_k z_k\|_{\bar{Q}_k}^2\Big) \qquad (4.1.5)$$
$$+ \mathbb{E}\Big(\|x_N^i\|_{Q_N}^2 + \|x_N^i - S_N z_N\|_{\bar{Q}_N}^2\Big),$$

where z is the mean field term to be determined as follows. Same as above, we adopt the custom of using the notation $x^{i,v}$ when the underlying particular control v has to be specified.

Denote

$$\mathcal{F}_k^{y^i} := \sigma\{y_r^i : 0 \leq r \leq k\}$$

to be the filtration generated by the observable state process of the i^{th} individual in the mean field framework. As for the finite counter part, the i^{th} player makes his own decision based on his observations only. Therefore, the only admissible controls in this problem are those v_k^i's adapting to the filtration $\mathcal{F}_k^{y^i}$, or mathematically, v_k^i is a functional of $y_0^i, \ldots, y_k^i$, i.e., $v_k^i = v_k^i(y_0^i, \ldots, y_k^i)$. The present model is fundamentally different from that in [10] in which the decision making of each player bases directly on his own criteria and certain summary statistics (i.e., the mean field term) about the community as the optimal control was assumed to adapt to the filtration generated by the individual noise. In contrast, our model here assumes that each player's optimal control could only rely on his own observable state whose dynamics possesses the community information just implicitly.

In order to solve for an equilibrium solution of the mean field game, the first step is to establish the optimal control of the representative agent subject to an arbitrary assignment of z in the agent's dynamics. An equilibrium solution can then be resolved by choosing a desired z so that it is a $\mathcal{F}^{\tilde{W}}$-adapted process. Indeed, in light of previous works [5] and [10], we can state the following theorem:

Theorem 4.1.1. *Suppose that the i^{th} player adopts the optimal control u^i defined in the system (4.1.3), (4.1.4) and (4.1.5). We denote x^{i,u^i} and $\mathcal{X}^{i,u^i}$ the corresponding trajectory of (4.1.3) and (4.1.1) under $u^i(y_0^i,\dots,y_k^i)$ and $u^i(\mathcal{Y}_0^i,\dots,\mathcal{Y}_k^i)$ respectively. If z is chosen such that the fixed point property*

$$z_k = \mathbb{E}\Big[x_k^{1,u^1}\big|\mathcal{F}_k^{\tilde{W}}\Big], \tag{4.1.6}$$

holds, then

$$\sup_{i\in\{1,\dots,P\}} \mathbb{E}\sup_{k\le N}|x_k^{i,u^i}-\mathcal{X}_k^{i,u^i}|^2 = o(1). \tag{4.1.7}$$

Moreover, $(u^1,u^2,\dots,u^P)$ served as an ϵ-Nash equilibrium for the original empirical problem. That is for arbitrary v^i, we have

$$J^{i,P}(u^1,\dots,u^{i-1},u^i,u^{i+1},\dots,u^P) \le J^{i,P}(u^1,\dots,u^{i-1},v^i,u^{i+1},\dots,u^P)+o(1). \tag{4.1.8}$$

This convergence result suggests that, in the mean field limit, both the state process and the cost functional of each individual are identical to that of each other; we then drop off the index i for the sake of notational simplicity. In this article, the limiting mean field problem of interest can be summarized as follows:

Problem 4.1.2. *(a) Let z_k be an $\mathcal{F}_k^{\tilde{W}}$-adapted process, for $k=0,\dots,N$. Find the optimal admissible control u_k^z which minimizes the cost functional*

$$\begin{aligned} J(v) := &\sum_{k=0}^{N-1}\mathbb{E}\Big(\|x_k\|_{Q_k}^2+\|v_k\|_{R_k}^2+\|x_k-S_kz_k\|_{\bar{Q}_k}^2\Big)\\ &+\mathbb{E}\Big(\|x_N\|_{Q_N}^2+\|x_N-S_Nz_N\|_{\bar{Q}_N}^2\Big), \end{aligned}$$

where the unobservable and observable states are respectively described by the following difference equations

$$\begin{cases} x_{k+1} &= A_kx_k+\bar{A}_kz_k+B_kv_k+W_{k+1}+\tilde{W}_{k+1}, \qquad k\in\{0,1,\dots,N-1\},\\ x_0 &= \xi; \end{cases} \tag{4.1.9}$$

and

$$y_k = H_k x_k + \bar{H}_k z_k + V_k, \qquad k \in \{0, 1, \ldots, N\}. \tag{4.1.10}$$

(b) *Find the equilibrium solution by searching for a* z_k *such that the fixed point property*

$$z_k = \mathbb{E}\left(x_k | \mathcal{F}_k^{\tilde{W}}\right) \tag{4.1.11}$$

is satisfied. Here, x_k *on the right hand side of (4.1.11) is the trajectory induced by the optimal control* u_k^z *which is further a functional of* z.

The resolution of Problem 4.1.2 will be elaborated in detail in Section 4.3.

4.1.2 Mean Field Type Control Problem

Up to the moment, we have only considered the mean field game as stated in equation (4.1.9)-(4.1.11) in Problem 4.1.2. In mean field theory, one also finds interest in another framework namely, mean field type stochastic control problems. The fundamental difference between these two frameworks is that, *in mean field games, the mean field term is now exogenous to every agent's optimal control problem; while the mean field term is endogenous in mean field type control problem.*

The mean field type control problem is described by:

Problem 4.1.3. *Find the optimal admissible control* u_k *which minimizes the cost functional*

$$\begin{aligned} J(v) := \sum_{k=0}^{N-1} \mathbb{E}\Big(\|x_k\|_{Q_k}^2 + \|v_k\|_{R_k}^2 + \Big\| x_k - S_k \mathbb{E}\Big[x_k | \mathcal{F}_k^{\tilde{W}} \vee \mathcal{F}_{k-1}^y\Big] \Big\|_{\bar{Q}_k}^2 \Big) \\ + \mathbb{E}\Big(\|x_N\|_{Q_N}^2 + \Big\| x_N - S_N \mathbb{E}\Big[x_N | \mathcal{F}_N^{\tilde{W}} \vee \mathcal{F}_{N-1}^y\Big] \Big\|_{\bar{Q}_N}^2 \Big), \end{aligned}$$

where the individual unobservable and observable states are respectively given by:

$$\begin{cases} x_{k+1} = A_k x_k + \bar{A}_k \mathbb{E}\left[x_k | \mathcal{F}_k^{\tilde{W}} \vee \mathcal{F}_{k-1}^{y}\right] + B_k v_k + W_{k+1} + \tilde{W}_{k+1}, k \in \{0, 1, \ldots, N-1\}, \\ x_0 = \xi; \end{cases} \tag{4.1.12}$$

and

$$y_k = H_k x_k + \bar{H}_k \mathbb{E}\left[x_k | \mathcal{F}_k^{\tilde{W}} \vee \mathcal{F}_{k-1}^{y}\right] + V_k, \qquad k \in \{0, 1, \ldots, N\}. \tag{4.1.13}$$

Observe that the mean field term

$$z_k = \mathbb{E}\Big[x_k | \mathcal{F}_k^{\tilde{W}} \vee \mathcal{F}_{k-1}^{y}\Big]$$

is assumed to be adapted to $\mathcal{F}_k^{\tilde{W}} \vee \mathcal{F}_{k-1}^{y}$, where we justify this form in the following.

4.1.2.1 Full Observation - in the Absence of a Common Noise

Suppose that the state process x_k is observable to the agent, the control v_k depends on $x_0, \ldots, x_k$, that is $v_k = v_k(x_0, \ldots, x_k)$. Let m_k be the probability density function of x_k (at time k) on $\mathbb{R}^{d_x}$; also denote the probability density of an arbitrary random vector η by f_η. We can rewrite Equation (4.1.12) (without the common noise $\tilde{W}$) into the following form:

$$W_{k+1} = x_{k+1} - A_k x_k - \bar{A}_k z_k - B_k v_k(x_0, \ldots, x_k).$$

In our linear quadratic setting, the mean field term z_k, as a functional of the density m_k, is the first moment, i.e. $z_k = \int_{\mathbb{R}^{d_x}} x_k m_k(x_k) dx_k$, and m_k is recursively defined as follows. In particular, the density function m_1 is then given by

$$\begin{aligned} m_1(x_1) &= \int_{\mathbb{R}^{d_x}} f_{W_1}(x_1 - A_0 x_0 - \bar{A}_0 z_0 - B_0 v_0(x_0)) m_0(x_0) dx_0 \\ &= \int_{\mathbb{R}^{d_x}} f_{W_1}(x_1 - A_0 x_0 - \bar{A}_0 z_0 - B_0 v_0(x_0)) f_\xi(x_0) dx_0. \end{aligned}$$

Inductively, we have

$$
\begin{aligned}
& m_{k+1}(x_{k+1}) \\
= & \int_{\mathbb{R}^{d_x}} \cdots \int_{\mathbb{R}^{d_x}} f_{W_{k+1}}(x_{k+1} - A_k x_k - \bar{A}_k z_k - B_k v_k(x_0, \ldots, x_k)) \\
& \qquad \cdot f_{W_k}(x_k - A_{k-1} x_{k-1} - \bar{A}_{k-1} z_{k-1} - B_{k-1} v_{k-1}(x_0, \ldots, x_{k-1})) \\
& \qquad \ldots f_\xi(x_0) dx_k \ldots dx_0.
\end{aligned}
\tag{4.1.14}
$$

We notice that the right hand side of Equation (4.1.14) is deterministic.

4.1.2.2 Full Observation - in the Presence of a Common Noise

In this case, the state process x_k is again observable to the agent, and the control can also be expressed as $v_k = v_k(x_0, \ldots, x_k)$. Let m_k be the density function of x_k (at time k) on $\mathbb{R}^{d_x}$ conditional on a filtration to be identified. Similar to the argument in previous case, we can rewrite Equation (4.1.12) with the common noise into the following form:

$$
W_{k+1} = x_{k+1} - A_k x_k - \bar{A}_k z_k - B_k v_k(x_0, \ldots, x_k) - \tilde{W}_{k+1},
$$

where, again, the mean field term z_k is the first moment under the density m_k, which is recursively defined as follows:

$$
\begin{aligned}
m_1(x_1) &= \int_{\mathbb{R}^{d_x}} f_{W_1}(x_1 - A_0 x_0 - \bar{A}_0 z_0 - B_0 v_0(x_0) - \tilde{W}_1) m_0(x_0) dx_0 \\
&= \int_{\mathbb{R}^{d_x}} f_{W_1}(x_1 - A_0 x_0 - \bar{A}_0 z_0 - B_0 v_0(x_0) - \tilde{W}_1) f_\xi(x_0) dx_0.
\end{aligned}
$$

Inductively, we obtain

$$
\begin{aligned}
& m_{k+1}(x_{k+1}) \\
= & \int_{\mathbb{R}^{d_x}} \cdots \int_{\mathbb{R}^{d_x}} f_{W_{k+1}}(x_{k+1} - A_k x_k - \bar{A}_k z_k - B_k v_k(x_0, \ldots, x_k) - \tilde{W}_{k+1}) \\
& \qquad \cdot f_{W_k}(x_k - A_{k-1} x_{k-1} - \bar{A}_{k-1} z_{k-1} - B_{k-1} v_{k-1}(x_0, \ldots, x_{k-1}) - \tilde{W}_k) \\
& \qquad \ldots f_\xi(x_0) dx_k \ldots dx_0.
\end{aligned}
\tag{4.1.15}
$$

The right hand side of Equation (4.1.15) is no longer deterministic. Indeed, the density function m_{k+1} depends on the evolution of the common noise $\{\tilde{W}_1, \tilde{W}_2, \ldots, \tilde{W}_{k+1}\}$, and hence m_{k+1} is in fact the density function of x_{k+1} conditional on $\{\tilde{W}_1, \tilde{W}_2, \ldots, \tilde{W}_{k+1}\}$, or $\mathcal{F}_k^{\tilde{W}}$. We conclude that $z_k = \mathbb{E}[x_k | \mathcal{F}_k^{\tilde{W}}]$.

4.1.2.3 Partial Observation - in the Presence of a Common Noise

Under the partial observation case, the agent makes his own decision based on his observations only, and hence the only admissible controls in this problem are those v_k's adapted to the filtration $\mathcal{F}_k^y$, or mathematically, v_k is a function of $y_0, \ldots, y_k$, i.e., $v_k = v_k(y_0, \ldots, y_k)$. Again, as in Subsection 4.1.2.2, we denote m_k the density function of x_k (at time k) on $\mathbb{R}^{d_x}$ conditional on a filtration to be determined. In this case, Equation (4.1.12) becomes

$$W_{k+1} = x_{k+1} - A_k x_k - \bar{A}_k z_k - B_k v_k(y_0, \ldots, y_k) - \tilde{W}_{k+1},$$

Similar to (4.1.14) and (4.1.15), we have

$$\begin{aligned}
&m_{k+1}(x_{k+1}) \\
&= \int_{\mathbb{R}^{d_x}} \cdots \int_{\mathbb{R}^{d_x}} f_{W_{k+1}}(x_{k+1} - A_k x_k - \bar{A}_k z_k - B_k v_k(y_0, \ldots, y_k) - \tilde{W}_{k+1}) \\
&\qquad \cdot f_{W_k}(x_k - A_{k-1} x_{k-1} - \bar{A}_{k-1} z_{k-1} - B_{k-1} v_{k-1}(y_0, \ldots, y_{k-1}) - \tilde{W}_k) \\
&\qquad \ldots f_\xi(x_0) dx_k \ldots dx_0.
\end{aligned} \tag{4.1.16}$$

The randomness in the right hand side of (4.1.16) clearly comes from both the common noise $\{\tilde{W}_1, \ldots, \tilde{W}_{k+1}\}$ and the observations $\{y_0, \ldots y_k\}$, or $\mathcal{F}_{k+1}^{\tilde{W}} \vee \mathcal{F}_k^y$. Therefore, we have

$$z_k = \int_{\mathbb{R}^{d_x}} x_k m_k(x_k) dx_k = \mathbb{E}[x_k | \mathcal{F}_k^{\tilde{W}} \vee \mathcal{F}_{k-1}^y]. \tag{4.1.17}$$

To conclude, by an alternative expression of the individual noise $\{W_1, W_2, \ldots, W_k\}$ in the density m_{k+1} in a sequence of three cases, the form of mean field term in our mean field type control problem 4.1.3 is justified.

Due to the different characters playing in the mean field term, the very existence of the respective solutions of the mean field type control problem and mean field games control problem are not the same; details will be given in Section 4.2 and 4.3 respectively.

4.2 Solution of Mean Field Type Control Problem

Due to the relatively simpler in nature, we first provide a comprehensive study in this section on Problem 4.1.3 under the mean field type setting.

We first establish the recursive form of the mean field term $\mathbb{E}\left[x_k|\mathcal{F}_k^{\tilde{W}} \vee \mathcal{F}_{k-1}^y\right]$. For $j \leq k \leq N$, denote $\hat{x}_{k|j} := \mathbb{E}[x_k|\mathcal{F}_{j+1}^{\tilde{W}} \vee \mathcal{F}_j^y]$ and the "covariance matrix" $P_{k|j} := \mathbb{E}[(x_k - \hat{x}_{k|j})(x_k - \hat{x}_{k|j})^*]$. Also denote the initial point $\hat{x}_{0|-1} := \mathbb{E}[x_0] = \mathbb{E}[\xi]$.

Taking conditional expectation on both sides of (4.1.12), we obtain that

$$\hat{x}_{k+1|k} = A_k \hat{x}_{k|k} + \bar{A}_k \hat{x}_{k|k-1} + B_k v_k + \tilde{W}_{k+1}. \tag{4.2.18}$$

Further, consider the difference of (4.2.18) from (4.1.12), and by a simple calculation, we then have

$$P_{k+1|k} = A_k P_{k|k} A_k^* + I. \tag{4.2.19}$$

Assume that $\hat{x}_{k|k}$ is in the following form with K_k' and K_k to be determined:

$$\hat{x}_{k|k} = K_k' \hat{x}_{k|k-1} + K_k y_k. \tag{4.2.20}$$

The unbiased condition, that is $\mathbb{E}[\hat{x}_{k|k-1}] = \mathbb{E}[x_k]$, yields that

$$K_k' \mathbb{E}[x_k] + K_k \mathbb{E}[y_k] = \mathbb{E}[x_k],$$

by using (4.1.13), we can get that

$$K_k' \mathbb{E}[x_k] + K_k (H_k + \bar{H}_k) \mathbb{E}[x_k] = \mathbb{E}[x_k]. \tag{4.2.21}$$

As (4.2.21) holds for an arbitrary value of $\mathbb{E}[x_k]$, we have, for any $k = 0, \ldots, N-1$,

$$K_k' = I - K_k (H_k + \bar{H}_k). \tag{4.2.22}$$

On the other hand, by using (4.1.13), (4.2.20) and (4.2.22), we also have,

$$x_k - \hat{x}_{k|k} = (I - K_k H_k)(x_k - \hat{x}_{k|k-1}) - K_k V_k. \tag{4.2.23}$$

Hence,

$$P_{k|k} = (I - K_k H_k) P_{k|k-1} (I - K_k H_k)^* + K_k K_k^*. \tag{4.2.24}$$

K_k is chosen such that the L^2 error, $\mathbb{E}\|x_k - \hat{x}_{k|k}\|^2$, is minimized, in particular,

$$\min_{K_k} \mathbb{E}\|x_k - \hat{x}_{k|k}\|^2 = \min_{K_k} \mathbb{E}[\mathrm{tr}(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^*] = \min_{K_k} \mathrm{tr}(P_{k|k}), \tag{4.2.25}$$

where this minimizer is

$$K_k = P_{k|k-1} H_k^* (H_k P_{k|k-1} H_k^* + I)^{-1}, \tag{4.2.26}$$

which is independent of the choice of the control variable v. To conclude, using (4.2.19), we have

$$P_{k+1|k} = A_k (I - K_k H_k) P_{k|k-1} (I - K_k H_k)^* A_k^* + A_k K_k K_k^* A_k^* + I, \tag{4.2.27}$$

where K_k is given by (4.2.26).

Finally, the mean field term has the expression as follows:

$$\begin{cases} \hat{x}_{k+1|k} = A_k\Big(\Big(I - K_k(H_k + \bar{H}_k)\Big)\hat{x}_{k|k-1} + K_k y_k\Big) + \bar{A}_k \hat{x}_{k|k-1} + B_k v_k + \tilde{W}_{k+1} \\ \qquad\quad = \Big(A_k\Big(I - K_k(H_k + \bar{H}_k)\Big) + \bar{A}_k\Big)\hat{x}_{k|k-1} + B_k v_k + A_k K_k y_k + \tilde{W}_{k+1}, \\ \hat{x}_{0|-1} = \mathbb{E}[\xi]. \end{cases} \tag{4.2.28}$$

We can rewrite the original mean field optimal control problem in terms of the augmented state $(x_\cdot, \hat{x}_{\cdot|\cdot-1})$ as:

Minimize

$$J(v) = \sum_{k=0}^{N-1} \mathbb{E}\Big(\|\mathbb{V}_1 \cdot \mathbb{X}_k\|^2_{Q_k} + \|v_k\|^2_{R_k} + \|\mathbb{V}_{2,k} \cdot \mathbb{X}_k\|^2_{\bar{Q}_k}\Big) + \mathbb{E}\Big(\|\mathbb{V}_1 \cdot \mathbb{X}_N\|^2_{Q_N} + \|\mathbb{V}_{2,N} \cdot \mathbb{X}_N\|^2_{\bar{Q}_N}\Big), \tag{4.2.29}$$

subject to the unobservable process

$$\begin{cases} \mathbb{X}_{k+1} &= \mathbb{A}_k \mathbb{X}_k + \mathbb{B}_k v_k + \mathbb{C}_k y_k + \Sigma_k \mathbb{W}_{k+1}, \\ \mathbb{X}_0 &= \Xi; \end{cases} \tag{4.2.30}$$

and the observable process

$$y_k = \mathbb{H}_k \mathbb{X}_k + V_k; \tag{4.2.31}$$

where

$$\begin{aligned} &\mathbb{X}_k = \begin{pmatrix} x_k \\ \hat{x}_{k|k-1} \end{pmatrix}; \quad \Xi = \begin{pmatrix} \xi \\ \mathbb{E}[\xi] \end{pmatrix}; \quad \mathbb{W}_{k+1} = \begin{pmatrix} W_{k+1} \\ \tilde{W}_{k+1} \end{pmatrix}; \\ &\mathbb{V}_1 = \begin{pmatrix} I, & 0 \end{pmatrix}; \quad \mathbb{V}_{2,k} = \begin{pmatrix} I, & -S_k \end{pmatrix}; \\ &\mathbb{A}_k = \begin{pmatrix} A_k & \bar{A}_k \\ 0 & A_k\Big(I - K_k(H_k + \bar{H}_k)\Big) + \bar{A}_k \end{pmatrix}; \quad \mathbb{B}_k = \begin{pmatrix} B_k \\ B_k \end{pmatrix}; \quad \mathbb{C}_k = \begin{pmatrix} 0 \\ A_k K_k \end{pmatrix}; \\ &\Sigma_k = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}; \quad \mathbb{H}_k = \begin{pmatrix} H_k \\ \bar{H}_k \end{pmatrix}. \end{aligned} \tag{4.2.32}$$

Note that the augmented unobservable state $\mathbb{X}$ now depends explicitly on the observation y. We adopt the ideas in Section 2.4 in [3] in the present discrete time setting with notable modifications. In particular, we first propose the following decomposition, $(\mathbb{X}, y) = (\mathbb{X}^0, y^0) + (\mathbb{X}^1, y^1)$:

$$\begin{cases} \mathbb{X}^0_{k+1} &= \mathbb{A}_k \mathbb{X}^0_k + \Sigma_k \mathbb{W}_{k+1}, \\ \mathbb{X}^0_0 &= \Xi; \\ y^0_k &= \mathbb{H}_k \mathbb{X}^0_k + V_k. \end{cases} \tag{4.2.33}$$

$$\begin{cases} \mathbb{X}^1_{k+1} &= \mathbb{A}_k \mathbb{X}^1_k + \mathbb{B}_k v_k + \mathbb{C}_k (y^0_k + y^1_k), \\ \mathbb{X}^1_0 &= 0; \\ y^1_k &= \mathbb{H}_k \mathbb{X}^1_k. \end{cases} \tag{4.2.34}$$

Note that, since v_k is adapted to $\mathcal{F}_k^y$, we have that $(\mathbb{X}_{k+1}^1, y_k^1)$ is $\mathcal{F}_k^y$-measurable; therefore, $y_k^0 = y_k - y_k^1$ is also $\mathcal{F}_k^y$-measurable. We then have

$$\widehat{\mathbb{X}}_{k+1|k} := \mathbb{E}[\mathbb{X}_{k+1}|\mathcal{F}_k^y] = \mathbb{E}[\mathbb{X}_{k+1}^0|\mathcal{F}_k^y] + \mathbb{X}_{k+1}^1, \text{ for } k = 0, \dots, N-1. \quad (4.2.35)$$

Lemma 4.2.1.

$$\mathbb{E}[\mathbb{X}_{k+1}^0|\mathcal{F}_k^y] = \mathbb{E}[\mathbb{X}_{k+1}^0|\mathcal{F}_k^{y^0}] \quad \textit{for} \quad k = 0, \dots, N-1. \quad (4.2.36)$$

Proof. For a fixed k, define

$$\lambda_k := \Pi_{i=1}^k \exp(\mathbb{X}_i^* \mathbb{H}_i^* y_i - \frac{1}{2} \mathbb{X}_i^* \mathbb{H}_i^* \mathbb{H}_i \mathbb{X}_i), \quad \text{and} \quad \lambda_1 := 1; \quad (4.2.37)$$

and

$$\frac{d\mathbb{Q}_k}{d\mathbb{P}} := \lambda_k^{-1}. \quad (4.2.38)$$

We first show that λ_k^{-1} is a $\mathcal{F}_{k+1}$-martingale under $\mathbb{P}$.

$$\begin{aligned}
&\mathbb{E}[\lambda_k^{-1} \mid \mathcal{F}_k] \\
=&\mathbb{E}\Big[\Pi_{i=1}^k \exp(-\mathbb{X}_i^* \mathbb{H}_i^* y_i + \frac{1}{2} \mathbb{X}_i^* \mathbb{H}_i^* \mathbb{H}_i \mathbb{X}_i) \mid \mathcal{F}_k\Big] \\
=&\Pi_{i=1}^{k-1} \exp(-\mathbb{X}_i^* \mathbb{H}_i^* y_i + \frac{1}{2} \mathbb{X}_i^* \mathbb{H}_i^* \mathbb{H}_i \mathbb{X}_i) \mathbb{E}\Big[\exp(-\mathbb{X}_k^* \mathbb{H}_k^* y_k + \frac{1}{2} \mathbb{X}_k^* \mathbb{H}_k^* \mathbb{H}_k \mathbb{X}_k) \mid \mathcal{F}_k\Big] \\
=&\Pi_{i=1}^{k-1} \exp(-\mathbb{X}_i^* \mathbb{H}_i^* y_i + \frac{1}{2} \mathbb{X}_i^* \mathbb{H}_i^* \mathbb{H}_i \mathbb{X}_i) \mathbb{E}\Big[\exp(-\mathbb{X}_k^* \mathbb{H}_k^* (\mathbb{H}_k \mathbb{X}_k + V_k) + \frac{1}{2} \mathbb{X}_k^* \mathbb{H}_k^* \mathbb{H}_k \mathbb{X}_k) \mid \mathcal{F}_k\Big] \\
=&\Pi_{i=1}^{k-1} \exp(-\mathbb{X}_i^* \mathbb{H}_i^* y_i + \frac{1}{2} \mathbb{X}_i^* \mathbb{H}_i^* \mathbb{H}_i \mathbb{X}_i) \mathbb{E}\Big[\exp(-\mathbb{X}_k^* \mathbb{H}_k^* V_k - \frac{1}{2} \mathbb{X}_k^* \mathbb{H}_k^* \mathbb{H}_k \mathbb{X}_k) \mid \mathcal{F}_k\Big] \\
=&\Pi_{i=1}^{k-1} \exp(\mathbb{X}_i^* \mathbb{H}_i^* y_i - \frac{1}{2} \mathbb{X}_i^* \mathbb{H}_i^* \mathbb{H}_i \mathbb{X}_i) \exp(-\frac{1}{2} \mathbb{X}_k^* \mathbb{H}_k^* \mathbb{H}_k \mathbb{X}_k) \mathbb{E}\Big[\exp(-\mathbb{X}_k^* \mathbb{H}_k^* V_k) \mid \mathcal{F}_k\Big] \\
=&\Pi_{i=1}^{k-1} \exp(\mathbb{X}_i^* \mathbb{H}_i^* y_i - \frac{1}{2} \mathbb{X}_i^* \mathbb{H}_i^* \mathbb{H}_i \mathbb{X}_i) \exp(-\frac{1}{2} \mathbb{X}_k^* \mathbb{H}_k^* \mathbb{H}_k \mathbb{X}_k) \exp(\frac{1}{2} \mathbb{X}_k^* \mathbb{H}_k^* \mathbb{H}_k \mathbb{X}_k) \\
=&\Pi_{i=1}^{k-1} \exp(\mathbb{X}_i^* \mathbb{H}_i^* y_i - \frac{1}{2} \mathbb{X}_i^* \mathbb{H}_i^* \mathbb{H}_i \mathbb{X}_i) = \lambda_{k-1}^{-1}.
\end{aligned}$$

And hence $\mathbb{Q}_k$ is a probability measure by noting that

$$\mathbb{E}[\lambda_k^{-1}] = \lambda_1^{-1} = 1.$$

Now, note that

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}_k}[\exp(i\alpha y_k) \mid \mathcal{F}_k] \\
&= \mathbb{E}[\exp(-\mathbb{X}_k^*\mathbb{H}_k^* y_k + \frac{1}{2}\mathbb{X}_k^*\mathbb{H}_k^*\mathbb{H}_k\mathbb{X}_k)\exp(i\alpha y_k) \mid \mathcal{F}_k] \\
&= \exp(\frac{1}{2}\mathbb{X}_k^*\mathbb{H}_k^*\mathbb{H}_k\mathbb{X}_k)\mathbb{E}[\exp(-\mathbb{X}_k^*\mathbb{H}_k^* y_k)\exp(i\alpha y_k) \mid \mathcal{F}_k] \\
&= \exp(\frac{1}{2}\mathbb{X}_k^*\mathbb{H}_k^*\mathbb{H}_k\mathbb{X}_k)\mathbb{E}[\exp((-\mathbb{X}_k^*\mathbb{H}_k^* + i\alpha)(\mathbb{H}\mathbb{X}_k + V_k)) \mid \mathcal{F}_k] \\
&= \exp(\frac{1}{2}\mathbb{X}_k^*\mathbb{H}_k^*\mathbb{H}_k\mathbb{X}_k)\exp((-\mathbb{X}_k^*\mathbb{H}_k^* + i\alpha)(\mathbb{H}\mathbb{X}_k))\mathbb{E}[\exp((-\mathbb{X}_k^*\mathbb{H}_k^* + i\alpha)(V_k)) \mid \mathcal{F}_k] \\
&= \exp(\frac{1}{2}\mathbb{X}_k^*\mathbb{H}_k^*\mathbb{H}_k\mathbb{X}_k)\exp((-\mathbb{X}_k^*\mathbb{H}_k^* + i\alpha)(\mathbb{H}\mathbb{X}_k))\exp(\frac{1}{2}(-\mathbb{X}_k^*\mathbb{H}_k^* + i\alpha)^2) \\
&= \exp(-\frac{1}{2}\alpha^2),
\end{aligned}
$$

which shows that under $\mathbb{Q}_k$, y_k is a standardized Gaussian white noise and is independent of $W_1, \ldots, W_k, \tilde{W}_1, \ldots, \tilde{W}_k,$ and $V_1, \ldots, V_{k-1}$. Next, using Bayes rule, we have

$$
\begin{aligned}
&\mathbb{E}[\mathbb{X}_{k+1}^0 | \mathcal{F}_k^y] \\
=&\frac{\mathbb{E}^{\mathbb{Q}_k}[\lambda_k^{-1}\mathbb{X}_{k+1}^0 | \mathcal{F}_k^y]}{\mathbb{E}^{\mathbb{Q}_k}[\lambda_k^{-1} | \mathcal{F}_k^y]} \\
=&\frac{\mathbb{E}^{\mathbb{Q}_k}[\Pi_{i=1}^k \exp(-\mathbb{X}_i^*\mathbb{H}_i^* y_i + \frac{1}{2}\mathbb{X}_i^*\mathbb{H}_i^*\mathbb{H}_i\mathbb{X}_i)\mathbb{X}_{k+1}^0 | \mathcal{F}_k^y]}{\mathbb{E}^{\mathbb{Q}_k}[\Pi_{i=1}^k \exp(-\mathbb{X}_i^*\mathbb{H}_i^* y_i + \frac{1}{2}\mathbb{X}_i^*\mathbb{H}_i^*\mathbb{H}_i\mathbb{X}_i) | \mathcal{F}_k^y]} \\
=&\frac{\mathbb{E}^{\mathbb{Q}_k}[\Pi_{i=1}^k \exp(-(\mathbb{X}_i^0 + \mathbb{X}_i^1)^*\mathbb{H}_i^* y_i + \frac{1}{2}(\mathbb{X}_i^0 + \mathbb{X}_i^1)^*\mathbb{H}_i^*\mathbb{H}_i(\mathbb{X}_i^0 + \mathbb{X}_i^1))\mathbb{X}_{k+1}^0 | \mathcal{F}_k^y]}{\mathbb{E}^{\mathbb{Q}_k}[\Pi_{i=1}^k \exp(-(\mathbb{X}_i^0 + \mathbb{X}_i^1)^*\mathbb{H}_i^* y_i + \frac{1}{2}(\mathbb{X}_i^0 + \mathbb{X}_i^1)^*\mathbb{H}_i^*\mathbb{H}_i(\mathbb{X}_i^0 + \mathbb{X}_i^1)) | \mathcal{F}_k^y]} \\
=&\frac{\mathbb{E}^{\mathbb{Q}_k}[\Pi_{i=1}^k \exp(-\mathbb{X}_i^{0*}\mathbb{H}_i^* y_i + \frac{1}{2}\mathbb{X}_i^{0*}\mathbb{H}_i^*\mathbb{H}_i\mathbb{X}_i^0 + \mathbb{X}_i^{0*}\mathbb{H}_i^*\mathbb{H}_i\mathbb{X}_i^1)\mathbb{X}_{k+1}^0 | \mathcal{F}_k^y]}{\mathbb{E}^{\mathbb{Q}_k}[\Pi_{i=1}^k \exp(-\mathbb{X}_i^{0*}\mathbb{H}_i^* y_i + \frac{1}{2}\mathbb{X}_i^{0*}\mathbb{H}_i^*\mathbb{H}_i\mathbb{X}_i^0 + \mathbb{X}_i^{0*}\mathbb{H}_i^*\mathbb{H}_i\mathbb{X}_i^1) | \mathcal{F}_k^y]},
\end{aligned}
\tag{4.2.39}
$$

where the last equality holds by noting that the same term $\Pi_{i=1}^k \exp(-(\mathbb{X}_i^1)^*\mathbb{H}_i^* y_i + \frac{1}{2}(\mathbb{X}_i^1)^*\mathbb{H}_i^*\mathbb{H}_i(\mathbb{X}_i^1))$ being adapted to $\mathcal{F}_k^y$ in both numerator and denominator can be cancelled off. Plugging the relation $y_i = y_i^0 + y_i^1 = y_i^0 + \mathbb{H}\mathbb{X}_i^1$ into (4.2.39), we have

$$
\mathbb{E}[\mathbb{X}_{k+1}^0 | \mathcal{F}_k^y] = \frac{\mathbb{E}^{\mathbb{Q}_k}[\Pi_{i=1}^k \exp(-\mathbb{X}_i^{0*}\mathbb{H}_i^* y_i^0 + \frac{1}{2}\mathbb{X}_i^{0*}\mathbb{H}_i^*\mathbb{H}_i\mathbb{X}_i^0)\mathbb{X}_{k+1}^0 | \mathcal{F}_k^y]}{\mathbb{E}^{\mathbb{Q}_k}[\Pi_{i=1}^k \exp(-\mathbb{X}_i^{0*}\mathbb{H}_i^* y_i^0 + \frac{1}{2}\mathbb{X}_i^{0*}\mathbb{H}_i^*\mathbb{H}_i\mathbb{X}_i^0) | \mathcal{F}_k^y]}. \tag{4.2.40}
$$

We note that y_i^0 is $\mathcal{F}_k^y$-measurable and $\mathbb{X}_i^0$ is independent of $\mathcal{F}_k^y$ for $i \leq k$; also $\mathbb{X}_{i+1}^0$ is independent of $\mathcal{F}_k^y$. We then have

$$\mathbb{E}[\mathbb{X}_{k+1}^0|\mathcal{F}_k^y] = \frac{\mathbb{E}^{\mathbb{Q}_k}[\Pi_{i=1}^k \exp(-\mathbb{X}_i^{0*}\mathbb{H}_i^*\zeta_i + \frac{1}{2}\mathbb{X}_i^{0*}\mathbb{H}_i^*\mathbb{H}_i\mathbb{X}_i^0)\mathbb{X}_{k+1}^0]}{\mathbb{E}^{\mathbb{Q}_k}[\Pi_{i=1}^k \exp(-\mathbb{X}_i^{0*}\mathbb{H}_i^*\zeta_i + \frac{1}{2}\mathbb{X}_i^{0*}\mathbb{H}_i^*\mathbb{H}_i\mathbb{X}_i^0)]}\Bigg|_{\zeta_1=y_1^0,\ldots,\zeta_k=y_k^0}. \tag{4.2.41}$$

Therefore, we conclude that the conditional expectation $\mathbb{E}[\mathbb{X}_{k+1}^0|\mathcal{F}_k^y]$ is actually $\mathcal{F}_k^{y^0}$ measurable, and hence with a simple application of tower property,

$$\mathbb{E}[\mathbb{X}_{k+1}^0|\mathcal{F}_k^y] = \mathbb{E}\left[\mathbb{E}[\mathbb{X}_{k+1}^0|\mathcal{F}_k^y]|\mathcal{F}_k^{y^0}\right] = \mathbb{E}[\mathbb{X}_{k+1}^0|\mathcal{F}_k^{y^0}],$$

which concludes the claim. □ □

To proceed further, we now use the standard procedure to obtain the Kalman filter of $\mathbb{X}_{k+1}^0$, $\widehat{\mathbb{X}}_{k+1|k}^0 := \mathbb{E}[\mathbb{X}_{k+1}^0|\mathcal{F}_k^{y^0}]$. In particular, taking the conditional expectation of both sides of (4.2.33) given $\mathcal{F}_k^{y^0}$, similar derivation leading to (4.2.28) can be applied to get the recursive relation:

$$\begin{cases} \widehat{\mathbb{X}}_{k+1|k}^0 = \mathbb{A}_k(I - \mathbb{K}_k\mathbb{H}_k)\widehat{\mathbb{X}}_{k|k-1}^0 + \mathbb{A}_k\mathbb{K}_k y_k^0, \\ \widehat{\mathbb{X}}_{0|-1}^0 = \mathbb{E}[\Xi], \end{cases} \tag{4.2.42}$$

where

$$\begin{cases} \mathbb{K}_k = \mathbb{P}_{k|k-1}\mathbb{H}_k^*(\mathbb{H}_k\mathbb{P}_{k|k-1}\mathbb{H}_k^* + I)^{-1}, \\ \mathbb{P}_{k+1|k} = \mathbb{A}_k(I - \mathbb{K}_k\mathbb{H}_k)\mathbb{P}_{k|k-1}(I - \mathbb{K}_k\mathbb{H}_k)^*\mathbb{A}_k^* + \mathbb{A}_k\mathbb{K}_k\mathbb{K}_k^*\mathbb{A}_k^* + \Sigma\Sigma^*. \end{cases} \tag{4.2.43}$$

Let $\varepsilon_{k+1} := \mathbb{X}_{k+1} - \widehat{\mathbb{X}}_{k+1|k} = \mathbb{X}_{k+1}^0 - \widehat{\mathbb{X}}_{k+1|k}^0$, which follows from (4.2.35). According to (4.2.33), we know that $\mathbb{X}_{k+1}^0$ and y_k^0 would not be affected by the control v_k. Therefore, Lemma 4.2.1 also implies that v_k would not affect ε_k. Note that for any matrix $\mathcal{A}$,

$$\begin{aligned} \mathbb{E}\|\mathcal{A}\mathbb{X}_{k+1}\|^2 =& \mathbb{E}\|\mathcal{A}\widehat{\mathbb{X}}_{k+1|k} + \mathcal{A}\varepsilon_{k+1}\|^2 \\ =& \mathbb{E}\|\mathcal{A}\widehat{\mathbb{X}}_{k+1|k}\|^2 + \mathbb{E}\|\mathcal{A}\varepsilon_{k+1}\|^2 + 2\mathbb{E}\left[\widehat{\mathbb{X}}_{k+1|k}^*\mathcal{A}^*\mathcal{A}\mathbb{E}[\varepsilon_{k+1} \mid \mathcal{F}_k^y]\right] \\ =& \mathbb{E}\|\mathcal{A}\widehat{\mathbb{X}}_{k+1|k}\|^2 + \mathbb{E}\|\mathcal{A}\varepsilon_{k+1}\|^2. \end{aligned}$$

Therefore, we have

$$
\begin{aligned}
J(v) = &\sum_{k=0}^{N-1} \mathbb{E}\Big(\|\mathbb{V}_1 \cdot \mathbb{X}_k\|^2_{Q_k} + \|v_k\|^2_{R_k} + \|\mathbb{V}_{2,k} \cdot \mathbb{X}_k\|^2_{\bar{Q}_k}\Big) \\
&\quad + \mathbb{E}\Big(\|\mathbb{V}_1 \cdot \mathbb{X}_N\|^2_{Q_N} + \|\mathbb{V}_{2,N} \cdot \mathbb{X}_N\|^2_{\bar{Q}_N}\Big) \\
= &\sum_{k=0}^{N-1} \mathbb{E}\Big(\|\mathbb{V}_1 \cdot \widehat{\mathbb{X}}_{k|k-1}\|^2_{Q_k} + \|v_k\|^2_{R_k} + \|\mathbb{V}_{2,k} \cdot \widehat{\mathbb{X}}_{k|k-1}\|^2_{\bar{Q}_k}\Big) \\
&\quad + \mathbb{E}\Big(\|\mathbb{V}_1 \cdot \widehat{\mathbb{X}}_{N|N-1}\|^2_{Q_N} + \|\mathbb{V}_{2,N} \cdot \widehat{\mathbb{X}}_{N|N-1}\|^2_{\bar{Q}_N}\Big) \qquad (4.2.44) \\
+ &\sum_{k=0}^{N-1} \mathbb{E}\Big(\|\mathbb{V}_1 \cdot \varepsilon_k\|^2_{Q_k} + \|\mathbb{V}_{2,k} \cdot \varepsilon_k\|^2_{\bar{Q}_k}\Big) + \mathbb{E}\Big(\|\mathbb{V}_1 \cdot \varepsilon_N\|^2_{Q_N} + \|\mathbb{V}_{2,N} \cdot \varepsilon_N\|^2_{\bar{Q}_N}\Big),
\end{aligned}
$$
$$(4.2.45)$$

where (4.2.45) is invariant for whatever choices of the control v.'s. Therefore, the mean field type optimal control problem 4.1.3 can be reduced to the one under the standard full observation setting by considering the objective function (4.2.44) only:

Problem 4.2.2. Minimize

$$
\begin{aligned}
\hat{J}(v) := &\sum_{k=0}^{N-1} \mathbb{E}\Big(\|\mathbb{V}_1 \cdot \widehat{\mathbb{X}}_{k|k-1}\|^2_{Q_k} + \|v_k\|^2_{R_k} + \|\mathbb{V}_{2,k} \cdot \widehat{\mathbb{X}}_{k|k-1}\|^2_{\bar{Q}_k}\Big) \qquad (4.2.46) \\
&+ \mathbb{E}\Big(\|\mathbb{V}_1 \cdot \widehat{\mathbb{X}}_{N|N-1}\|^2_{Q_N} + \|\mathbb{V}_{2,N} \cdot \widehat{\mathbb{X}}_{N|N-1}\|^2_{\bar{Q}_N}\Big),
\end{aligned}
$$

subject to

$$
\begin{cases}
\widehat{\mathbb{X}}_{k+1|k} &= \widehat{\mathbb{X}}^0_{k+1|k} + \mathbb{X}^1_{k+1}, \\
\mathbb{X}^1_{k+1} &= \mathbb{A}_k \mathbb{X}^1_k + \mathbb{B}_k v_k + \mathbb{C}_k (y^0_k + y^1_k), \\
\mathbb{X}^1_0 &= 0, \\
\widehat{\mathbb{X}}^0_{k+1|k} &= \mathbb{A}_k (I - \mathbb{K}_k \mathbb{H}_k) \widehat{\mathbb{X}}^0_{k|k-1} + \mathbb{A}_k \mathbb{K}_k y^0_k, \\
\widehat{\mathbb{X}}^0_{0|-1} &= \mathbb{E}[\Xi],
\end{cases}
\qquad (4.2.47)
$$

where y^0_k and y^1_k satisfy the system (4.2.33) *and* (4.2.34) *respectively.*

Theorem 4.2.3. *The optimal control of fully observable Problem 4.2.2 is given by*

$$u_k = -R_k^{-1}\mathbb{B}_k^*\hat{p}_{k+1}, \ \textit{for } k = 0, 1, ..., N-1, \tag{4.2.48}$$

such that $\hat{p}_k$ exists and satisfies the following Backward Stochastic Difference Equation:

$$\begin{cases} \hat{p}_k &= (\mathbb{A}_k + \mathbb{C}_k\mathbb{H}_k)^*\hat{p}_{k+1} + (\mathbb{V}_1^*Q_k\mathbb{V}_1 + \mathbb{V}_{2,k}^*\bar{Q}_k\mathbb{V}_{2,k})\widehat{\mathbb{X}}_{k|k-1} - \Delta M_k^{\hat{p}}, \\ \hat{p}_N &= (\mathbb{V}_1^*Q_N\mathbb{V}_1 + \mathbb{V}_{2,N}^*\bar{Q}_N\mathbb{V}_{2,N})\widehat{\mathbb{X}}_{N|N-1}, \end{cases} \tag{4.2.49}$$

where $\Delta M_k^{\hat{p}}$ is a martingale difference and the generated martingale is adapted to the filtration $\mathcal{F}_k^y$.

Proof. Consider a perturbation of the optimal control $u + \tau\tilde{u}$, where $\tau \in \mathbb{R}$ is arbitrary and $\tilde{u} := (\tilde{u}_0, ..., \tilde{u}_{N-1})$ and $\tilde{u}_k$ is adapted to the filtration $\mathcal{F}_k^y$, for $k = 1, ..., N-1$. The original state $\widehat{\mathbb{X}}_{k+1|k}$ becomes $\widehat{\mathbb{X}}_{k+1|k} + \tau\widetilde{\mathbb{X}}_{k+1|k}$ with

$$\begin{cases} \widetilde{\mathbb{X}}_{k+1|k} &= \widetilde{\mathbb{X}}_{k+1}^1, \\ \widetilde{\mathbb{X}}_{k+1}^1 &= \mathbb{A}_k\widetilde{\mathbb{X}}_k^1 + \mathbb{B}_k\tilde{u}_k + \mathbb{C}_k\tilde{y}_k^1, \\ \tilde{y}_k^1 &= \mathbb{H}_k\widetilde{\mathbb{X}}_k^1, \\ \widetilde{\mathbb{X}}_0^1 &= 0. \end{cases} \tag{4.2.50}$$

The optimality of u would satisfy the Euler's condition:

$$\begin{aligned} 0 &= \frac{d}{d\tau}\bigg|_{\tau=0} \hat{J}(u + \tau\tilde{u}) \\ &= 2\mathbb{E}\Big[\langle u_0, \tilde{u}_0\rangle_{R_0} \\ &+ \sum_{k=1}^{N-1}\Big(\langle \mathbb{V}_1\widehat{\mathbb{X}}_{k|k-1}, \mathbb{V}_1\widetilde{\mathbb{X}}_{k|k-1}\rangle_{Q_k} + \langle u_k, \tilde{u}_k\rangle_{R_k} + \langle \mathbb{V}_{2,k}\widehat{\mathbb{X}}_{k|k-1}, \mathbb{V}_{2,k}\widetilde{\mathbb{X}}_{k|k-1}\rangle_{\bar{Q}_k}\Big) \\ &+ \langle \mathbb{V}_1\widehat{\mathbb{X}}_{N|N-1}, \mathbb{V}_1\widetilde{\mathbb{X}}_{N|N-1}\rangle_{Q_N} + \langle \mathbb{V}_{2,N}\widehat{\mathbb{X}}_{N|N-1}, \mathbb{V}_{2,N}\widetilde{\mathbb{X}}_{N|N-1}\rangle_{\bar{Q}_N}\Big]. \end{aligned} \tag{4.2.51}$$

Define the adjoint process satisfying (4.2.49). Consider

$$
\begin{aligned}
&\langle \widetilde{\mathbb{X}}_{k+1|k}, \hat{p}_{k+1}\rangle - \langle \widetilde{\mathbb{X}}_{k|k-1}, \hat{p}_k\rangle \\
=&\langle \widetilde{\mathbb{X}}_{k+1|k} - \widetilde{\mathbb{X}}_{k|k-1}, \hat{p}_{k+1}\rangle + \langle \widetilde{\mathbb{X}}_{k|k-1}, \hat{p}_{k+1} - \hat{p}_k\rangle \\
=&\langle (\mathbb{A}_k + \mathbb{C}_k\mathbb{H}_k)\widetilde{\mathbb{X}}_{k|k-1} + \mathbb{B}_k\tilde{u}_k - \widetilde{\mathbb{X}}_{k|k-1}, \hat{p}_{k+1}\rangle \\
&+ \langle \widetilde{\mathbb{X}}_{k|k-1}, \hat{p}_{k+1} - (\mathbb{A}_k + \mathbb{C}_k\mathbb{H}_k)^*\hat{p}_{k+1} - (\mathbb{V}_1^* Q_k \mathbb{V}_1 + \mathbb{V}_{2,k}^* \bar{Q}_k \mathbb{V}_{2,k})\widehat{\mathbb{X}}_{k|k-1}\rangle \\
&+ \langle \widetilde{\mathbb{X}}_{k|k-1}, \Delta M_k^{\hat{p}}\rangle \\
=&\langle \widetilde{\mathbb{X}}_{k|k-1}, -(\mathbb{V}_1^* Q_k \mathbb{V}_1 + \mathbb{V}_{2,k}^* \bar{Q}_k \mathbb{V}_{2,k})\widehat{\mathbb{X}}_{k|k-1}\rangle + \langle \mathbb{B}_k\tilde{u}_k, \hat{p}_{k+1}\rangle + \langle \widetilde{\mathbb{X}}_{k|k-1}, \Delta M_k^{\hat{p}}\rangle,
\end{aligned}
$$

summing up from $k = 0$ to $N - 1$ and taking expectation, we have

$$
\begin{aligned}
0 =&\mathbb{E}\Big[- \big\langle \widetilde{\mathbb{X}}_{N|N-1}, (\mathbb{V}_1^* Q_N \mathbb{V}_1 + \mathbb{V}_{2,N}^* \bar{Q}_N \mathbb{V}_{2,N})\widehat{\mathbb{X}}_{N|N-1}\big\rangle \\
&\qquad + \sum_{k=0}^{N-1} \Big(\langle \widetilde{\mathbb{X}}_{k|k-1}, -(\mathbb{V}_1^* Q_k \mathbb{V}_1 + \mathbb{V}_{2,k}^* \bar{Q}_k \mathbb{V}_{2,k})\widehat{\mathbb{X}}_{k|k-1}\rangle + \langle \mathbb{B}_k\tilde{u}_k, \hat{p}_{k+1}\rangle \Big)\Big].
\end{aligned}
$$

Using (4.2.51), we can get

$$
0 = \mathbb{E}\Big[\sum_{k=0}^{N-1} \langle R_k u_k + \mathbb{B}_k^* \hat{p}_{k+1}, \tilde{u}_k\rangle\Big].
$$

Note that u_k and $\tilde{u}_k$ is adapted to $\mathcal{F}_k^y$, since $\tilde{u}$ is arbitrary, we deduce the optimal control $u_k = -R_k^{-1}\mathbb{B}_k^*\hat{p}_{k+1}$ for $k = 0, 1, \ldots, N-1$. For the very existence of $\hat{p}_k$, we shall discuss in detail in the rest of this section. □ □

In the remaining part of this section, we shall establish the unique existence of the following forward backward stochastic differential equation $(\hat{\mathbb{X}}, \hat{p})$:

$$\widehat{\mathbb{X}}_{k+1|k} = \widehat{\mathbb{X}}^0_{k+1|k} + \mathbb{X}^1_{k+1}, \tag{4.2.52}$$
$$\begin{cases} \mathbb{X}^1_{k+1} &= \mathbb{A}_k \mathbb{X}^1_k - \mathbb{B}_k R_k^{-1} \mathbb{B}_k^* \hat{p}_{k+1} + \mathbb{C}_k (y^0_k + y^1_k), \\ \mathbb{X}^1_0 &= 0, \end{cases}$$
$$\begin{cases} \widehat{\mathbb{X}}^0_{k+1|k} &= \mathbb{A}_k (I - \mathbb{K}_k \mathbb{H}_k) \widehat{\mathbb{X}}^0_{k|k-1} + \mathbb{A}_k \mathbb{K}_k y^0_k, \\ \widehat{\mathbb{X}}^0_{0|-1} &= \mathbb{E}[\Xi], \end{cases}$$
$$\begin{cases} \hat{p}_k &= (\mathbb{A}_k + \mathbb{C}_k \mathbb{H}_k)^* \hat{p}_{k+1} + (\mathbb{V}_1^* Q_k \mathbb{V}_1 + \mathbb{V}_{2,k}^* \bar{Q}_k \mathbb{V}_{2,k}) \widehat{\mathbb{X}}_{k|k-1} - \Delta M_k^{\hat{p}}, \\ \hat{p}_N &= (\mathbb{V}_1^* Q_N \mathbb{V}_1 + \mathbb{V}_{2,N}^* \bar{Q}_N \mathbb{V}_{2,N}) \widehat{\mathbb{X}}_{N|N-1}, \end{cases}$$

where y^0 and y^1 are defined in Equations (4.2.33) and (4.2.34) respectively. Also note that y^0 is well defined by its own right in Equation (4.2.33) as it is independent of the control process v; while $y^1 = \mathbb{H}\mathbb{X}^1$ in Equation (4.2.34). Hence, we can rewrite (4.2.52) as follows:

$$\begin{cases} \widehat{\mathbb{X}}_{k+1|k} &= \widehat{\mathbb{X}}^0_{k+1|k} + \mathbb{X}^1_{k+1}, \\ \hat{p}_k &= \hat{p}^0_k + \hat{p}^1_k, \end{cases} \tag{4.2.53}$$

such that

$$\begin{cases} \widehat{\mathbb{X}}^0_{k+1|k} &= \mathbb{A}_k(I - \mathbb{K}_k\mathbb{H}_k)\widehat{\mathbb{X}}^0_{k|k-1} + \mathbb{A}_k\mathbb{K}_k y^0_k, \\ \widehat{\mathbb{X}}_{0|-1} &= \mathbb{E}[\Xi]; \\ \hat{p}^0_k &= (\mathbb{A}_k + \mathbb{C}_k\mathbb{H}_k)\hat{p}^0_{k+1} + (\mathbb{V}_1^* Q_k \mathbb{V}_1 + \mathbb{V}^*_{2,k}\bar{Q}_k\mathbb{V}_{2,k})\widehat{\mathbb{X}}^0_{k|k-1} - \Delta M^{\hat{p}^0}_k, \\ \hat{p}^0_N &= (\mathbb{V}_1^* Q_N \mathbb{V}_1 + \mathbb{V}^*_{2,N}\bar{Q}_N\mathbb{V}_{2,N})\widehat{\mathbb{X}}^0_{N|N-1}, \end{cases} \tag{4.2.54}$$

$$\begin{cases} \mathbb{X}^1_{k+1} &= \mathbb{A}_k\mathbb{X}^1_k - \mathbb{B}_k R_k^{-1}\mathbb{B}^*_k\hat{p}_{k+1} + \mathbb{C}_k(y^0_k + \mathbb{H}_k\mathbb{X}^1_k), \\ \mathbb{X}^1_0 &= 0; \\ \hat{p}^1_k &= (\mathbb{A}_k + \mathbb{C}_k\mathbb{H}_k)\hat{p}^1_{k+1} + (\mathbb{V}_1^* Q_k \mathbb{V}_1 + \mathbb{V}^*_{2,k}\bar{Q}_k\mathbb{V}_{2,k})\mathbb{X}^1_k - \Delta M^{\hat{p}^1}_k, \\ \hat{p}^1_N &= (\mathbb{V}_1^* Q_N \mathbb{V}_1 + \mathbb{V}^*_{2,N}\bar{Q}_N\mathbb{V}_{2,N})\mathbb{X}^1_N, \end{cases} \tag{4.2.55}$$

where $\Delta M^{\hat{p}^0}_k$ is a martingale difference adapted to $\mathcal{F}^{y^0}_k$, and $\Delta M^{\hat{p}^1}_k$ is another martingale difference adapted to $\mathcal{F}^y_k$. Since y^0 is well-defined in (4.2.33), $\widehat{\mathbb{X}}^0_{k+1|k}$ exists as a solution of the forward stochastic difference equation in (4.2.54); and the readiness of the existence of $\widehat{\mathbb{X}}^0$ and y^0 guarantees the existence of the backward stochastic difference equation $\hat{p}^0$ in (4.2.54) too. So the existence issue only comes with the forward backward stochastic difference equations system (4.2.55).

Assume the following *Ansatz*:

$$\hat{p}^1_k = \Gamma_k\mathbb{X}^1_k + g_k, \tag{4.2.56}$$

for some g_k adapted to $\mathcal{F}^y_{k-1}$, for $k = 1, \dots, N$, and positive definite deterministic matrix $\{\Gamma_k\}_{k=1}^N$ to be confirmed. Clearly,

$$\mathbb{X}^1_{k+1} = (I + \mathbb{B}_k R_k \mathbb{B}^*_k \Gamma_{k+1})^{-1}\Big((\mathbb{A}_k + \mathbb{C}_k\mathbb{H}_k)\mathbb{X}^1_k - \mathbb{B}_k R_k \mathbb{B}^*_k g_{k+1} + \mathbb{C}_k y^0_k\Big), \tag{4.2.57}$$

where the matrix inverse $(I+\mathbb{B}_kR_k\mathbb{B}_k^*\Gamma_{k+1})^{-1}$ is well defined; indeed,

$$\begin{aligned}(I+\mathbb{B}_kR_k\mathbb{B}_k^*\Gamma_{k+1})^{-1} &= \Big((\Gamma_{k+1}^{-1}+\mathbb{B}_kR_k\mathbb{B}_k^*)\Gamma_{k+1}\Big)^{-1} \\ &= \Gamma_{k+1}^{-1}(\Gamma_{k+1}^{-1}+\mathbb{B}_kR_k\mathbb{B}_k^*)^{-1},\end{aligned} \tag{4.2.58}$$

where Γ_{k+1}^{-1} is positive definite and $\mathbb{B}_kR_k\mathbb{B}_k^*$ is non-negative definite as $R_k>0$. Hence, $\Gamma_{k+1}^{-1}+\mathbb{B}_kR_k\mathbb{B}_k^*$ is positive definite and so invertible, and we conclude the claim. Taking conditional expectation on both sides of the backward difference equation in Equation (4.2.55), and then substitute back the *Ansatz* (4.2.56), we obtain

$$\begin{aligned}\hat{p}_k^1 &= (\mathbb{A}_k+\mathbb{C}_k\mathbb{H}_k)^*\mathbb{E}[\hat{p}_{k+1}^1|\mathcal{F}_{k-1}^y] + (\mathbb{V}_1^*Q_k\mathbb{V}_1+\mathbb{V}_{2,k}^*\bar{Q}_k\mathbb{V}_{2,k})\mathbb{X}_k^1 \\ &= (\mathbb{A}_k+\mathbb{C}_k\mathbb{H}_k)^*\mathbb{E}[\Gamma_{k+1}\mathbb{X}_{k+1}^1+g_{k+1}|\mathcal{F}_{k-1}^y] + (\mathbb{V}_1^*Q_k\mathbb{V}_1+\mathbb{V}_{2,k}^*\bar{Q}_k\mathbb{V}_{2,k})\mathbb{X}_k^1 \\ &= (\mathbb{A}_k+\mathbb{C}_k\mathbb{H}_k)^*\Gamma_{k+1}\mathbb{E}\Big[(I+\mathbb{B}_kR_k\mathbb{B}_k^*\Gamma_{k+1})^{-1}\Big((\mathbb{A}_k+\mathbb{C}_k\mathbb{H}_k)\mathbb{X}_k^1-\mathbb{B}_kR_k\mathbb{B}_k^*g_{k+1}+\mathbb{C}_ky_k^0\Big)|\mathcal{F}_{k-1}^y\Big] \\ &\quad + (\mathbb{A}_k+\mathbb{C}_k\mathbb{H}_k)^*\mathbb{E}[g_{k+1}|\mathcal{F}_{k-1}^y] + (\mathbb{V}_1^*Q_k\mathbb{V}_1+\mathbb{V}_{2,k}^*\bar{Q}_k\mathbb{V}_{2,k})\mathbb{X}_k^1 \\ &= (\mathbb{A}_k+\mathbb{C}_k\mathbb{H}_k)^*\Gamma_{k+1}(I+\mathbb{B}_kR_k\mathbb{B}_k^*\Gamma_{k+1})^{-1}(\mathbb{A}_k+\mathbb{C}_k\mathbb{H}_k)\mathbb{X}_k^1 + (\mathbb{V}_1^*Q_k\mathbb{V}_1+\mathbb{V}_{2,k}^*\bar{Q}_k\mathbb{V}_{2,k})\mathbb{X}_k^1 \\ &\quad - (\mathbb{A}_k+\mathbb{C}_k\mathbb{H}_k)^*\Gamma_{k+1}\mathbb{E}\Big[(I+\mathbb{B}_kR_k\mathbb{B}_k^*\Gamma_{k+1})^{-1}\Big(-\mathbb{B}_kR_k\mathbb{B}_k^*g_{k+1}+\mathbb{C}_ky_k^0\Big)|\mathcal{F}_{k-1}^y\Big] \\ &\quad + (\mathbb{A}_k+\mathbb{C}_k\mathbb{H}_k)^*\mathbb{E}[g_{k+1}|\mathcal{F}_{k-1}^y].\end{aligned} \tag{4.2.59}$$

Comparing coefficients of (4.2.59) with the *Ansatz* (4.2.56), it yields that

$$\begin{cases}\Gamma_k &= (\mathbb{A}_k+\mathbb{C}_k\mathbb{H}_k)^*\Gamma_{k+1}(I+\mathbb{B}_kR_k\mathbb{B}_k^*\Gamma_{k+1})^{-1}(\mathbb{A}_k+\mathbb{C}_k\mathbb{H}_k) \\ &\qquad +(\mathbb{V}_1^*Q_k\mathbb{V}_1+\mathbb{V}_{2,k}^*\bar{Q}_k\mathbb{V}_{2,k}), \\ \Gamma_N &= (\mathbb{V}_1^*Q_N\mathbb{V}_1+\mathbb{V}_{2,N}^*\bar{Q}_N\mathbb{V}_{2,N});\end{cases} \tag{4.2.60}$$

$$\begin{cases}g_k &= (\mathbb{A}_k+\mathbb{C}_k\mathbb{H}_k)^*\mathbb{E}\Big[\Gamma_{k+1}(I+\mathbb{B}_kR_k\mathbb{B}_k^*\Gamma_{k+1})^{-1}(-\mathbb{B}_kR_k\mathbb{B}_k^*g_{k+1}+\mathbb{C}_ky_k^0)+g_{k+1}|\mathcal{F}_{k-1}^y\Big], \\ g_N &= 0.\end{cases} \tag{4.2.61}$$

It remains to show that Γ and g indeed satisfy the requirement as demanded in the *Ansatz* (4.2.56). Note that that filtration $\mathcal{F}_{k-1}^y$, depending on the past unobservable states $\mathbb{X}_0, \ldots, \mathbb{X}_{k-1}$, also depends on $g_1, \ldots, g_{k-1}$ through the controls; meanwhile, g_k defined in Equation (4.2.61) relies on its future evolution and hence the existence of g_k cannot be readily concluded due to this forward-backward structure. And therefore the forward-backward system (4.2.55) is fully coupled.

Lemma 4.2.4. *Suppose that Γ_{k+1} is positive definite. Then Γ_k is also positive definite.*

Proof. Since $\mathbb{V}_1, \mathbb{V}_{2,k}$ are of full rank, and $Q_k, \bar{Q}_k$ are positive definite, we have $\mathbb{V}_1^* Q_k \mathbb{V}_1 + \mathbb{V}_{2,k}^* \bar{Q}_k \mathbb{V}_{2,k} > 0$. It remains to show that

$$(\mathbb{A}_k + \mathbb{C}_k \mathbb{H}_k)^* \Gamma_{k+1} (I + \mathbb{B}_k R_k \mathbb{B}_k^* \Gamma_{k+1})^{-1} (\mathbb{A}_k + \mathbb{C}_k \mathbb{H}_k)$$

is non-negative definite. We have

$$(I + \mathbb{B}_k R_k \mathbb{B}_k^* \Gamma_{k+1})^{-1} = \Gamma_{k+1}^{-1} (\Gamma_{k+1}^{-1} + \mathbb{B}_k R_k \mathbb{B}_k^*)^{-1},$$

from Equation (4.2.58), which is well defined as Γ_{k+1} is invertible by assumption. Finally, we have

$$\begin{aligned}
&(\mathbb{A}_k + \mathbb{C}_k \mathbb{H}_k)^* \Gamma_{k+1} (I + \mathbb{B}_k R_k \mathbb{B}_k^* \Gamma_{k+1})^{-1} (\mathbb{A}_k + \mathbb{C}_k \mathbb{H}_k) \\
=&(\mathbb{A}_k + \mathbb{C}_k \mathbb{H}_k)^* (\Gamma_{k+1}^{-1} + \mathbb{B}_k R_k \mathbb{B}_k^*)^{-1} (\mathbb{A}_k + \mathbb{C}_k \mathbb{H}_k) \geq 0,
\end{aligned}$$

since $\Gamma_{k+1}^{-1} > 0$ and $\mathbb{B}_k R_k \mathbb{B}_k^* \geq 0$, so our claim follows. □

Clearly, Γ_N is positive definite and we conclude that $\{\Gamma_k\}_{k=1}^N$ is positive definite and deterministic by Lemma 4.2.4.

Lemma 4.2.5. *The sequence $\{g_k\}_{k=1}^N$ satisfying* (4.2.61) *exists, i.e. fulfilling the adaptedness requirement.*

Proof. For notational simplicity, we rewrite Equation (4.2.61):

$$g_k = \mathbb{E}[h_k g_{k+1} + \tilde{h}_k y_k^0 | F_{k-1}^y], \qquad g_N = 0.$$

Hence

$$
\begin{aligned}
g_k &= \mathbb{E}[h_k g_{k+1} + \tilde{h}_k y_k^0 | F_{k-1}^y] \\
&= \mathbb{E}\Big[h_k \mathbb{E}[h_{k+1} g_{k+2} + \tilde{h}_{k+1} y_{k+1}^0 | F_k^y] + \tilde{h}_k y_k^0 | F_{k-1}^y\Big] \\
&= \mathbb{E}\Big[h_k h_{k+1} g_{k+2} + h_k \tilde{h}_{k+1} y_{k+1}^0 + \tilde{h}_k y_k^0 | F_{k-1}^y\Big] \\
&= \mathbb{E}\Big[\tilde{h}_k y_k^0 + \sum_{i=k+1}^{N-1} h_k h_{k+1} \cdots h_{i-1} \tilde{h}_i y_i^0 | F_{k-1}^y\Big].
\end{aligned}
$$

We first claim that g_1 exists; indeed,

$$
g_1 = \mathbb{E}\Big[\tilde{h}_1 y_1^0 + \sum_{i=2}^{N-1} h_1 h_2 \cdots h_{i-1} \tilde{h}_i y_i^0 \Big| y_0\Big], \tag{4.2.62}
$$

and hence g_1 exists as $y_\cdot^0$ is well defined in (4.2.33) and $y_0 = \mathbb{H}_0 \xi + V_0$. The ready existences of Γ_1 and g_1 guarantee that $\mathbb{X}_1^1$ exists by applying Equation (4.2.57). Next, consider

$$
g_2 = \mathbb{E}\Big[\tilde{h}_2 y_2^0 + \sum_{i=3}^{N-1} h_2 h_3 \cdots h_{i-1} \tilde{h}_i y_i^0 \Big| y_0, y_1\Big], \tag{4.2.63}
$$

$y_1 = \mathbb{H}_1 \mathbb{X}_1 + V_1 = \mathbb{H}_1(\mathbb{X}_1^0 + \mathbb{X}_1^1) + V_1$ exists as $\mathbb{X}_1^0$ is well defined in Equation (4.2.33). Therefore, g_2 exists since $y_\cdot^0$ and y_0, y_1 exist, so does $\mathbb{X}_2^1$ by Equation (4.2.57). In general, we could obtain g_k by inductive argument. □

Due to the convexity and coerciveness of the quadratic cost functional, the Euler's condition is also a sufficient condition for the oprimal control. In conclusion, the solution for the original mean field type optimal control problem is given as follows:

Corollary 4.2.6. *The optimal control of Problem 4.1.3 is given by* (4.2.48).

4.3 Solution of Mean Field Game

4.3.1 Individual's Optimal Control Problem

Theorem 4.3.1. *Let $z_\cdot$ be an arbitrary $\mathcal{F}_\cdot^{\tilde{W}}$-adapted process.*

1. *Suppose that for $k = 0, 1, ..., N-1$, the optimal control u_k of Problem 4.1.2(a) exists. Then*

$$u_k = -R_k^{-1} B_k^* q_k, \tag{4.3.64}$$

where q_k can be constructed so that q_k satisfies the following Backward Stochastic Difference Equation:

$$\begin{cases} q_k & = A_{k+1}^* q_{k+1} + (Q_{k+1} + \bar{Q}_{k+1})\mathbb{E}(x_{k+1}|\mathcal{F}_k^y) - (\bar{Q}_{k+1} S_{k+1})\mathbb{E}(z_{k+1}|\mathcal{F}_k^y) - \Delta M_k^q, \\ q_{N-1} & = (Q_N + \bar{Q}_N)\mathbb{E}(x_N|\mathcal{F}_{N-1}^y) - (\bar{Q}_N S_N)\mathbb{E}(z_N|\mathcal{F}_{N-1}^y), \end{cases} \tag{4.3.65}$$

where ΔM_k^q is a martingale difference and the corresponding martingale is adapted to the filtration $\mathcal{F}_k^y$.

2. *Conversely, suppose that the solution $q.$ of system (4.3.65) exists. Then $u_k = -R_k^{-1} B_k^* q_k$ is the optimal control for Problem 4.1.2(a).*

Remark 4.3.2. *From this theorem, we know that the solution of Problem 4.1.2(a) corresponds to the following forward backward difference equation system:*

$$\begin{cases} x_{k+1} & = A_k x_k + \bar{A}_k z_k - B_k R_k^{-1} B_k^* q_k + W_{k+1} + \tilde{W}_{k+1}, \\ x_0 & = \xi; \\ q_k & = A_{k+1}^* q_{k+1} + (Q_{k+1} + \bar{Q}_{k+1})\mathbb{E}(x_{k+1}|\mathcal{F}_k^y) - (\bar{Q}_{k+1} S_{k+1})\mathbb{E}[z_{k+1}|\mathcal{F}_k^y] - \Delta M_k^q, \\ q_N & = (Q_N + \bar{Q}_N)\mathbb{E}(x_N|\mathcal{F}_{N-1}^y) - (\bar{Q}_N S_N)\mathbb{E}(z_N|\mathcal{F}_{N-1}^y), \\ \Delta M_k^q & = A_{k+1}^* \big(q_{k+1} - \mathbb{E}(q_{k+1}|\mathcal{F}_k^y)\big). \end{cases} \tag{4.3.66}$$

Therefore, the solution $(x., q.)$ of (4.3.66) can be regarded as a functional of $(z_0, \ldots, z_N)$. For the sake of convenience in the later part of our article, we define the mapping $\mathcal{L} : (z_0, \ldots, z_N) \mapsto (x_0, \ldots, x_N)$ through the system (4.3.66).

Proof. 1. Consider a perturbation of the optimal control $u+\tau\tilde{u}$, where $\tau\in\mathbb{R}$ is arbitrary and $\tilde{u}=(\tilde{u}_0,...,\tilde{u}_{N-1})$ and $\tilde{u}_k$ is adapted to the filtration $\mathcal{F}_k^y$, for $i=1,...,N-1$. The original state x_k becomes $x_k+\tau\tilde{x}_k$ with

$$\begin{cases}\tilde{x}_{k+1} &= A_k\tilde{x}_k+B_k\tilde{u}_k,\\ \tilde{x}_0 &= 0.\end{cases}$$

The optimality of u would satisfy the Euler's condition:

$$\begin{aligned}0&=\frac{d}{d\tau}\bigg|_{\tau=0}J(u+\tau\tilde{u})\\ &=2\mathbb{E}\Big[\langle u_0,\tilde{u}_0\rangle_{R_0}+\sum_{k=1}^{N-1}\Big(\langle x_k,\tilde{x}_k\rangle_{Q_k}+\langle u_k,\tilde{u}_k\rangle_{R_k}+\langle x_k-S_kz_k,\tilde{x}_k\rangle_{\bar{Q}_k}\Big)\\ &\qquad+\langle x_N,\tilde{x}_N\rangle_{Q_N}+\langle x_N-S_Nz_N,\tilde{x}_N\rangle_{\bar{Q}_N}\Big].\end{aligned}\tag{4.3.67}$$

Define the adjoint process by the following backward stochastic difference equation, for $k=1,...,N-1$,

$$\begin{cases}p_k &= A_k^*p_{k+1}+(Q_k+\bar{Q}_k)x_k-(\bar{Q}_kS_k)z_k-\Delta M_k^p,\\ p_N &= (Q_N+\bar{Q}_N)x_N-(\bar{Q}_NS_N)z_N.\end{cases}\tag{4.3.68}$$

where ΔM_k^p is the following $\mathcal{F}_{k+1}$-measurable martingale difference

$$\Delta M_k^p=A_k^*(p_{k+1}-\mathbb{E}(p_{k+1}|\mathcal{F}_k)).\tag{4.3.69}$$

Note that the very existence of $p_k=(Q_k+\bar{Q}_k)x_k-(\bar{Q}_kS_k)z_k+A_k^*\mathbb{E}(p_{k+1}|\mathcal{F}_k)\in\mathcal{F}_k$, since $x_k,z_k\in\mathcal{F}_k$, is ensured by that of u_k which in turn warrants the existence of x_N, and then establishing those p_k backwards.

Consider

$$\begin{aligned}
&\langle \tilde{x}_{k+1}, p_{k+1}\rangle - \langle \tilde{x}_k, p_k\rangle = \langle \tilde{x}_{k+1} - \tilde{x}_k, p_{k+1}\rangle + \langle \tilde{x}_k, p_{k+1} - p_k\rangle \\
=&\langle \tilde{x}_k, p_{k+1} - A_k^* p_{k+1} - (Q_k + \bar{Q}_k)x_k + (\bar{Q}_k S_k)z_k\rangle \\
&+ \langle A_k\tilde{x}_k + B_k\tilde{u}_k - \tilde{x}_k, p_{k+1}\rangle + \langle \tilde{x}_k, \Delta M_k^p\rangle \\
=&\langle \tilde{x}_k, (I - A_k)^* p_{k+1}\rangle - \langle (I - A_k)\tilde{x}_k, p_{k+1}\rangle \\
&+ \langle \tilde{x}_k, -(Q_k + \bar{Q}_k)x_k + (\bar{Q}_k S_k)z_k\rangle + \langle B_k\tilde{u}_k, p_{k+1}\rangle + \langle \tilde{x}_k, \Delta M_k^p\rangle,
\end{aligned}$$

summing up from $k = 0$ to $N - 1$ and taking expectation, we have

$$\begin{aligned}
0 =&\mathbb{E}\Big[- \big\langle \tilde{x}_N, (Q_N + \bar{Q}_N)x_N - (\bar{Q}_N S_N)z_N \big\rangle \\
&+ \sum_{k=0}^{N-1} \Big(\langle \tilde{x}_k, -(Q_k + \bar{Q}_k)x_k + (\bar{Q}_k S_k)z_k\rangle + \langle B_k\tilde{u}_k, p_{k+1}\rangle \Big)\Big].
\end{aligned}$$

Using (4.3.67) and noting that u_k and $\tilde{u}_k$ is adapted to $\mathcal{F}_k^y$, we can get

$$0 = \mathbb{E}\Big[\sum_{k=0}^{N-1} \langle R_k u_k + B_k^* p_{k+1}, \tilde{u}_k\rangle \Big] = \mathbb{E}\Big[\sum_{k=0}^{N-1} \big\langle R_k u_k + \mathbb{E}(B_k^* p_{k+1}|\mathcal{F}_k^y), \tilde{u}_k \big\rangle \Big].$$

Since $\tilde{u}$ is arbitrary, we get the optimal control $u_k = -R_k^{-1} B_k^* \mathbb{E}(p_{k+1}|\mathcal{F}_k^y)$ for $k = 1, 2, ..., N - 1$.

Let $q_k := \mathbb{E}(p_{k+1}|\mathcal{F}_k^y)$, taking conditional expectation given $\mathcal{F}_k^y$ on both sides of (4.3.68) yields that

$$\begin{aligned}
q_k &= A_{k+1}^* \mathbb{E}(p_{k+2}|\mathcal{F}_k^y) + (Q_{k+1} + \bar{Q}_{k+1})\mathbb{E}(x_{k+1}|\mathcal{F}_k^y) - (\bar{Q}_{k+1} S_{k+1})\mathbb{E}(z_{k+1}|\mathcal{F}_k^y) \\
&= A_{k+1}^* q_{k+1} + (Q_{k+1} + \bar{Q}_{k+1})\mathbb{E}(x_{k+1}|\mathcal{F}_k^y) - (\bar{Q}_{k+1} S_{k+1})\mathbb{E}(z_{k+1}|\mathcal{F}_k^y) - \Delta M_k^q,
\end{aligned}$$

where

$$\begin{aligned}
\Delta M_k^q &= A_{k+1}^* q_{k+1} - A_{k+1}^* \mathbb{E}(p_{k+2}|\mathcal{F}_k^y) = A_{k+1}^* q_{k+1} - A_{k+1}^* \mathbb{E}(\mathbb{E}(p_{k+2}|\mathcal{F}_{k+1}^y)|\mathcal{F}_k^y) \\
&= A_{k+1}^* q_{k+1} - A_{k+1}^* \mathbb{E}(q_{k+1}|\mathcal{F}_k^y),
\end{aligned}$$

ΔM_k^q is the new martingale difference measurable with respect to $\mathcal{F}_{k+1}^y$. We can easily see that $\Delta M_k^q \neq \mathbb{E}(\Delta M_k^p|\mathcal{F}_k^y)$.

Therefore, the optimal control is given by

$$u_k = -R_k^{-1} B_k^* q_k,$$

where q_k satisfies (4.3.65).

2. Conversely, suppose that the solution $q.$ of system (4.3.65) exists. Then taking $u_k = -R_k^{-1} B_k^* q_k$, we can work backward in the part 1, as the existence of the process $p.$ can be constructed and ensured by using x^u, and we can see that the Euler's condition (4.3.67) can be satisfied.

By the convexity and coerciveness of the quadratic cost functional, the optimal control for the individual follower is uniquely defined and this necessary condition is automatically a sufficient one. □

4.3.2 Existence of an Equilibrium Solution

In order to look for the equilibrium solution, we have to seek for a $z.$ such that it satisfies the fixed point property (4.1.11) in Problem 4.1.2(b):

$$z_k = \mathbb{E}(\mathcal{L}(z_0, \dots, z_N)_k | \mathcal{F}_k^{\tilde{W}}), \text{ for } k = 0, \dots, N. \tag{4.3.70}$$

so that the limiting system with which the finite player system should converge to.

A control u_k is said to be an equilibrium control of the mean field game problem, if $u_k = u_k^z$, where u_k^z is itself the optimal control for the control problem in Theorem (4.3.1) when $z_k^u = \mathbb{E}(x_k^u | \mathcal{F}_k^{\tilde{W}})$. Therefore, the solution approaches to Problem 4.1.2 (a) and (b) can be combined by considering the fixed point problem (4.3.70) and the system (4.3.66) together, i.e. the solution of Problem 4.1.2

is given by the following forward backward stochastic difference equation:

$$\begin{cases} x_{k+1} &= A_k x_k + \bar{A}_k \mathbb{E}(x_k|\mathcal{F}_k^{\tilde{W}}) - B_k R_k^{-1} B_k^* q_k + W_{k+1} + \tilde{W}_{k+1}, \\ x_0 &= \xi; \\ q_k &= A_{k+1}^* q_{k+1} + (Q_{k+1} + \bar{Q}_{k+1}) \mathbb{E}(x_{k+1}|\mathcal{F}_k^y) \\ & \quad -(\bar{Q}_{k+1} S_{k+1}) \mathbb{E}[\mathbb{E}(x_{k+1}|\mathcal{F}_{k+1}^{\tilde{W}})|\mathcal{F}_k^y] - \Delta M_k^q, \\ q_N &= (Q_N + \bar{Q}_N) \mathbb{E}(x_N|\mathcal{F}_{N-1}^y) - (\bar{Q}_N S_N) \mathbb{E}(\mathbb{E}(x_N|\mathcal{F}_N^{\tilde{W}})|\mathcal{F}_{N-1}^y), \\ \Delta M_k^q &= A_{k+1}^* \big(q_{k+1} - \mathbb{E}(q_{k+1}|\mathcal{F}_k^y)\big), \end{cases} \tag{4.3.71}$$

with the corresponding observable process

$$y_k = H_k x_k + \bar{H}_k \mathbb{E}(x_k|\mathcal{F}_k^{\tilde{W}}) + V_k.$$

Before we proceed to establish the existence of the system (4.3.71), we would like to make the following comments. Firstly, in the continuous time setting, the interesting work [26] studied the case where all admissible controls for each player adapt not only to his observable process, but also to the common noise ($\tilde{W}$), this setting makes their work to have a limited use in the usual economic context as the common noise for the whole community can hardly be observed directly. Putting their framework in our own problem, if we assume that all the admissible controls adapt to $\mathcal{F}_k^y \vee \mathcal{F}_{k+1}^{\tilde{W}}$, since the optimal control $u_k = -R_k^{-1} B_k^* q_k$, q_k now also adapts to $\mathcal{F}_k^y \vee \mathcal{F}_{k+1}^{\tilde{W}}$, and the iterated conditional expectation $\mathbb{E}[\mathbb{E}(x_{k+1}|\mathcal{F}_{k+1}^{\tilde{W}})|\mathcal{F}_k^y]$ in the backward equation of system (4.3.71) becomes

$$\mathbb{E}[\mathbb{E}(x_{k+1}|\mathcal{F}_{k+1}^{\tilde{W}})|\mathcal{F}_k^y \vee \mathcal{F}_{k+1}^{\tilde{W}}] = \mathbb{E}(x_{k+1}|\mathcal{F}_{k+1}^{\tilde{W}}), \tag{4.3.72}$$

so we do not need to consider the complicated dependence structure between $\mathcal{F}^y$ and $\mathcal{F}^{\tilde{W}}$, and the existence result is rather immediate. The following section demonstrates how this implicit dependence structure makes the existence result of system (4.3.71) subtle.

Secondly, we have an alternative expression for the solution of the backward stochastic difference equation in system (4.3.71), which will facilitate the establishment of the existence result of system (4.3.71). By taking conditional expectation given $\mathcal{F}_k^y$ on both sides of the backward equation in (4.3.71), we get

$$\begin{cases} q_k & = A_{k+1}^* \mathbb{E}(q_{k+1}|\mathcal{F}_k^y) + (Q_{k+1} + \bar{Q}_{k+1})\mathbb{E}(x_{k+1}|\mathcal{F}_k^y) \\ & \quad -\bar{Q}_{k+1}S_{k+1}\mathbb{E}(\mathbb{E}(x_{k+1}|\mathcal{F}_{k+1}^{\tilde{W}})|\mathcal{F}_k^y), \\ q_{N-1} & = (Q_N + \bar{Q}_N)\mathbb{E}(x_N|\mathcal{F}_{N-1}^y) - (\bar{Q}_N S_N)\mathbb{E}(\mathbb{E}(x_N|\mathcal{F}_N^{\tilde{W}})|\mathcal{F}_{N-1}^y). \end{cases}$$

Inductively, we then obtain:

$$\begin{aligned} q_k =& (Q_{k+1} + \bar{Q}_{k+1})\mathbb{E}(x_{k+1}|\mathcal{F}_k^y) - \bar{Q}_{k+1}S_{k+1}\mathbb{E}(\mathbb{E}(x_{k+1}|\mathcal{F}_{k+1}^{\tilde{W}})|\mathcal{F}_k^y) \\ & + \sum_{r=k+2}^{N} A_{k+1}^* A_{k+2}^* \cdots A_{r-1}^* \Big((Q_r + \bar{Q}_r)\mathbb{E}(x_r|\mathcal{F}_k^y) - \bar{Q}_r S_r \mathbb{E}(\mathbb{E}(x_r|\mathcal{F}_r^{\tilde{W}})|\mathcal{F}_k^y) \Big). \end{aligned} \tag{4.3.73}$$

Therefore, the existence of the solution of system (4.3.71) is equivalent to the solvability of the following particular forward system:

$$\begin{cases} x_{k+1} & = A_k x_k + \bar{A}_k \mathbb{E}(x_k|\mathcal{F}_k^{\tilde{W}}) \\ & \quad -B_k R_k^{-1} B_k^* \Big((Q_{k+1} + \bar{Q}_{k+1})\mathbb{E}(x_{k+1}|\mathcal{F}_k^y) - \bar{Q}_{k+1}S_{k+1}\mathbb{E}\big(\mathbb{E}(x_{k+1}|\mathcal{F}_{k+1}^{\tilde{W}})|\mathcal{F}_k^y\big) \\ & \quad + \sum_{r=k+2}^N A_{k+1}^* \cdots A_{r-1}^* \Big((Q_r + \bar{Q}_r)\mathbb{E}(x_r|\mathcal{F}_k^y) - \bar{Q}_r S_r \mathbb{E}\big(\mathbb{E}(x_r|\mathcal{F}_r^{\tilde{W}})|\mathcal{F}_k^y\big) \Big) \Big) \\ & \quad + W_{k+1} + \tilde{W}_{k+1}, \\ y_k & = H_k x_k + \bar{H}_k \mathbb{E}(x_k|\mathcal{F}_k^{\tilde{W}}) + V_k. \end{cases} \tag{4.3.74}$$

It is now ready to establish the existence of the solution of the system (4.3.71).

Define the Gaussian space

$$\begin{aligned} G := \{(\mathbf{x}, \alpha, \beta, \gamma) | \quad & \mathbf{x} = (x_1, \cdots, x_N)^*, \alpha = (\alpha_1, \cdots, \alpha_N)^*, \beta = (\beta_1, \cdots, \beta_N)^*, \\ & \gamma = (\gamma_0, \cdots, \gamma_{N-1})^* \text{ are jointly Gaussian, with } x_k \in \mathcal{F}_k \text{ in } \mathbb{R}^{d_x}, \\ & \alpha_k \in \mathcal{F}_k^W \text{ in } \mathbb{R}^{d_x}, \beta_k \in \mathcal{F}_k^{\tilde{W}} \text{ in } \mathbb{R}^{d_x}, \gamma_k \in \mathcal{F}_k^V \text{ in } \mathbb{R}^{d_y}; \\ & \text{for } k = 1, \cdots, N\}, \end{aligned}$$

which is equipped with a canonical L^2 norm:

$$\|(\mathbf{x},\alpha,\beta,\gamma)\|_{L^2}^2 := \sum_{k=1}^{N}\mathbb{E}\|x_k\|^2 + \sum_{k=1}^{N}\mathbb{E}\|\alpha_k\|^2 + \sum_{k=1}^{N}\mathbb{E}\|\beta_k\|^2 + \sum_{k=0}^{N-1}\mathbb{E}\|\gamma_k\|^2. \quad (4.3.75)$$

Clearly, (G, L^2) is a vector space. In fact it is also complete. Let $(\mathbf{x}^n, \alpha^n, \beta^n, \gamma^n)$ be a Cauchy sequence in (G, L^2). By the completeness of finite sequence of square integrable random variables under L^2, the Cauchy sequence converges to $(\mathbf{x},\alpha,\beta,\gamma)$. It remains to check that: 1. $(\mathbf{x},\alpha,\beta,\gamma)$ preserves the adaptedness; 2. $(\mathbf{x},\alpha,\beta,\gamma)$ are jointly Gaussian. The definition (4.3.75) implies that, componentwisely, for each k, $\{x_k^n\}_n$ is again a Cauchy sequence. Using Riesz-Fischer theorem (see, for example, [45] or [31]), x_k^n converges a.e. and L^2 to x_k. Since the sequence $\{x_k^n\}_n$ is $\mathcal{F}_k$ measurable, so does the a.e. limit x_k. Similarly, we have $\alpha_k \in \mathcal{F}_k^W, \beta_k \in \mathcal{F}_k^{\tilde{W}}, \gamma_k \in \mathcal{F}_k^V$. For the second point, note that the L^2 convergence of Gaussian random variables implies the mean and variance converge (and hence the characteristic function). The L^2 limit remains Gaussian is then immediate.

For any $K > 0$, we define the nonempty subset C_K of G by

$$\begin{aligned} C_K := \{(\mathbf{x},\alpha,\beta,\gamma)| \quad & (\mathbf{x},\alpha,\beta,\gamma) \in G, \\ & 1. \quad \sup_{\|c\|\le 1}\mathbb{E}|c^*\mathbf{x}|^2 \le K; \\ & 2. \quad x_k = a\xi + \sum_{i=1}^{k} b_i W_i + \sum_{i=0}^{k-1} c_i V_i + \sum_{i=1}^{k} d_i \tilde{W}_k + e, \quad k = 1,\ldots,N; \\ & 3. \quad \alpha = \mathbf{W} := (W_1,\cdots,W_N), \beta = \tilde{\mathbf{W}} := (\tilde{W}_1,\cdots,\tilde{W}_N), \\ & \qquad \gamma = \mathbf{V} := (V_0,\cdots,V_{N-1}), \}; \end{aligned}$$

where a, b_i, c_i, d_i, e are constant matrices with appropriate dimensions. We have shown that the L^2 limit preserves the adaptedness, Gaussian structure and hence C_K is closed. We argue that C_K is also convex. For any $(\mathbf{x},\alpha,\beta,\gamma), (\mathbf{x}',\alpha',\beta',\gamma') \in$

C_K, we can see that

$$\begin{aligned}&\mathbb{E}|c^*(\lambda\mathbf{x}+(1-\lambda)\mathbf{x}')|^2\\&=\lambda^2\mathbb{E}|c^*\mathbf{x}|^2+2\lambda(1-\lambda)\mathbb{E}|(c^*\mathbf{x})^*(c^*\mathbf{x}')|+(1-\lambda)^2\mathbb{E}|c^*\mathbf{x}'|^2\\&\le\lambda^2\mathbb{E}|c^*\mathbf{x}|^2+2\lambda(1-\lambda)(\mathbb{E}|c^*\mathbf{x}|^2\mathbb{E}|c^*\mathbf{x}'|^2)^{\frac{1}{2}}+(1-\lambda)^2\mathbb{E}|c^*\mathbf{x}'|^2\le K.\end{aligned}$$

Due to linearity in $\mathbf{x}$, it is clear that C_K preserves the adaptedness structure. That is for any element $(\mathbf{x},\alpha,\beta,\gamma)\in C_K$, we have $x_k\in\mathcal{F}_k$.

In light of (4.3.74), define the mapping $T:(\mathbf{x},\mathbf{W},\tilde{\mathbf{W}},\mathbf{V})\in C_K\mapsto(\mathbf{X},\mathbf{W},\tilde{\mathbf{W}},\mathbf{V})\in G$, such that:

$$\begin{cases}X_{k+1} & =A_kx_k+\bar{A}_k\mathbb{E}(x_k|\mathcal{F}_k^{\tilde{W}})\\ & -B_kR_k^{-1}B_k^*\Big((Q_{k+1}+\bar{Q}_{k+1})\mathbb{E}(x_{k+1}|\mathcal{F}_k^y)-\bar{Q}_{k+1}S_{k+1}\mathbb{E}\big(\mathbb{E}(x_{k+1}|\mathcal{F}_{k+1}^{\tilde{W}})|\mathcal{F}_k^y\big)\\ & +\sum_{r=k+2}^N A_{k+1}^*\cdots A_{r-1}^*\Big((Q_r+\bar{Q}_r)\mathbb{E}(x_r|\mathcal{F}_k^y)-\bar{Q}_rS_r\mathbb{E}\big(\mathbb{E}(x_r|\mathcal{F}_r^{\tilde{W}})|\mathcal{F}_k^y\big)\Big)\Big)\\ & +W_{k+1}+\tilde{W}_{k+1},\\ y_k & =H_kx_k+\bar{H}_k\mathbb{E}(x_k|\mathcal{F}_k^{\tilde{W}})+V_k,\end{cases}\tag{4.3.76}$$

where $k=0,\cdots,N-1$ and $x_0=\xi$ is a Gaussian random variable. Clearly, the image X_k is Gaussian since any conditional expectations of jointly Gaussian random variables remain Gaussian and are adapted to $\mathcal{F}_k$, and hence the mapping T is well-posed. We next have the following lemmas.

Lemma 4.3.3. *T is a continuous mapping.*

Proof. Suppose that $\{(\mathbf{x}^n,\mathbf{W},\tilde{\mathbf{W}},\mathbf{V})\}_n\subset C_K$ is a sequence converging to $(\mathbf{x},\mathbf{W},\tilde{\mathbf{W}},\mathbf{V})\in C_K$ in L^2. It suffices to check that the corresponding images under T, $(\mathbf{X}^n,\mathbf{W},\tilde{\mathbf{W}},\mathbf{V})$, converges to $(\mathbf{X},\mathbf{W},\tilde{\mathbf{W}},\mathbf{V})$ in L^2.

Let $\tilde{\mathbf{W}}_k:=(\tilde{W}_1,\ldots,\tilde{W}_k)$, $\mathbf{y}_k^n:=(y_0^n,y_1^n,\ldots,y_k^n)$. The following conditional expectations of Gaussian random variables can be expressed as:

i) $\mathbb{E}(x_k^n|\mathcal{F}_k^{\tilde{W}}) = \mathbb{E}(x_k^n) - \mathrm{Cov}(x_k^n, \tilde{\mathbf{W}}_k)\mathrm{Var}^{-1}(\tilde{\mathbf{W}}_k)(\tilde{\mathbf{W}}_k - \mathbb{E}(\tilde{\mathbf{W}}_k))$;

ii) $\mathbb{E}(x_r^n|\mathcal{F}_k^{y^n}) = \mathbb{E}(x_r^n) - \mathrm{Cov}(x_r^n, \mathbf{y}_k^n)\mathrm{Var}^{-1}(\mathbf{y}_k^n)(\mathbf{y}_k^n - \mathbb{E}(\mathbf{y}_k^n))$, for $r = k+1, \ldots, N$;

iii)

$$\begin{aligned}
&\mathbb{E}(\mathbb{E}(x_r^n|\mathcal{F}_r^{\tilde{W}})|\mathcal{F}_k^{y^n}) = \mathbb{E}(x_r^n) - \mathrm{Cov}(x_r^n, \tilde{\mathbf{W}}_r)\mathrm{Var}^{-1}(\tilde{\mathbf{W}}_r)\mathbb{E}(\tilde{\mathbf{W}}_r|\mathcal{F}_k^{y^n})\\
=&\mathbb{E}(x_r^n) - \mathrm{Cov}(x_r^n, \tilde{\mathbf{W}}_r)\mathrm{Var}^{-1}(\tilde{\mathbf{W}}_r)\Big(\mathbb{E}(\tilde{\mathbf{W}}_r) - \mathrm{Cov}(\mathbf{y}_k^n, \tilde{\mathbf{W}}_r)\mathrm{Var}^{-1}(\mathbf{y}_k^n)(\mathbf{y}_k^n - \mathbb{E}(\mathbf{y}_k^n))\Big);
\end{aligned}$$

and similarly, in the limiting case,

i′) $\mathbb{E}(x_k|\mathcal{F}_k^{\tilde{W}}) = \mathbb{E}(x_k) - \mathrm{Cov}(x_k, \tilde{\mathbf{W}}_k)\mathrm{Var}^{-1}(\tilde{\mathbf{W}}_k)(\tilde{\mathbf{W}}_k - \mathbb{E}(\tilde{\mathbf{W}}_k))$;

ii′) $\mathbb{E}(x_r|\mathcal{F}_k^{y}) = \mathbb{E}(x_r) - \mathrm{Cov}(x_r, \mathbf{y}_k)\mathrm{Var}^{-1}(\mathbf{y}_k)(\mathbf{y}_k - \mathbb{E}(\mathbf{y}_k))$, for $r = k+1, \ldots, N$;

iii′)

$$\begin{aligned}
&\mathbb{E}(\mathbb{E}(x_r|\mathcal{F}_r^{\tilde{W}})|\mathcal{F}_k^{y}) = \mathbb{E}(x_r) - \mathrm{Cov}(x_r, \tilde{\mathbf{W}}_r)\mathrm{Var}^{-1}(\tilde{\mathbf{W}}_r)\mathbb{E}(\tilde{\mathbf{W}}_r|\mathcal{F}_k^{y})\\
=&\mathbb{E}(x_r) - \mathrm{Cov}(x_r, \tilde{\mathbf{W}}_r)\mathrm{Var}^{-1}(\tilde{\mathbf{W}}_r)\Big(\mathbb{E}(\tilde{\mathbf{W}}_r) - \mathrm{Cov}(\mathbf{y}_k, \tilde{\mathbf{W}}_r)\mathrm{Var}^{-1}(\mathbf{y}_k)(\mathbf{y}_k - \mathbb{E}(\mathbf{y}_k))\Big).
\end{aligned}$$

Note that the assumption of L^2 convergence $(\mathbf{x}^n, \mathbf{W}, \tilde{\mathbf{W}}, \mathbf{V}) \to (\mathbf{x}, \mathbf{W}, \tilde{\mathbf{W}}, \mathbf{V})$ implies the mean and variance convergence of the jointly Gaussian random variables. In otherwors, we clearly have $\mathbb{E}(x_k^n) \to \mathbb{E}(x_k)$, $\mathrm{Cov}(x_k^n, \tilde{\mathbf{W}}_k) \to \mathrm{Cov}(x_k, \tilde{\mathbf{W}}_k)$ and $\mathrm{Cov}(\mathbf{y}_k^n, \tilde{\mathbf{W}}_k) \to \mathrm{Cov}(\mathbf{y}_k, \tilde{\mathbf{W}}_k)$.

So according to (4.3.76) if we rewrite

$$X_{k+1}^n = (a_k^n)^*\mathbf{x}^n + (b_k^n)^*\mathbb{E}(\mathbf{x}^n) + (c_k^n)^*\mathbf{W} + (d_k^n)^*\tilde{\mathbf{W}} + (e_k^n)^*\mathbf{V},$$

and

$$X_{k+1} = (a_k)^*\mathbf{x} + (b_k)^*\mathbb{E}(\mathbf{x}) + (c_k)^*\mathbf{W} + (d_k)^*\tilde{\mathbf{W}} + (e_k)^*\mathbf{V},$$

we see that all the coefficients $a_k^n \to a_k$, $b_k^n \to b_k$, $c_k^n \to c_k$, $d_k^n \to d_k$, $e_k^n \to e_k$ converges as real vectors and $\mathbf{x}^n \to \mathbf{x}$ in L^2 as $n \to \infty$. Therefore, $(\mathbf{X}^n, \mathbf{W}, \tilde{\mathbf{W}}, \mathbf{V})$

converges to $(\mathbf{X}, \mathbf{W}, \tilde{\mathbf{W}}, \mathbf{V})$ in L^2 by triangle inequality. In particular,

$$\begin{aligned}
&\mathbb{E}|X^n_{k+1} - X_{k+1}|^2 \\
\leq &5\Big(\mathbb{E}|(a^n_k)^*\mathbf{x}^n - (a_k)^*\mathbf{x}|^2 + \mathbb{E}|(b^n_k)^*\mathbb{E}(\mathbf{x}^n) - (b_k)^*\mathbb{E}(\mathbf{x})|^2 \\
&\qquad + \mathbb{E}|(c^n_k - c_k)^*(\mathbf{W})|^2 + \mathbb{E}|(d^n_k - d_k)^*(\tilde{\mathbf{W}})|^2 + \mathbb{E}|(e^n_k - e_k)^*(\mathbf{V})|^2\Big) \\
\leq &5\Big(2|a^n_k - a_k|^2\mathbb{E}|\mathbf{x}^n|^2 + 2|a_k|^2\mathbb{E}|\mathbf{x}^n - \mathbf{x}|^2 \\
&\qquad + 2|b^n_k - b_k|^2|\mathbb{E}(\mathbf{x}^n)|^2 + 2|b_k|^2|\mathbb{E}(\mathbf{x}^n) - \mathbb{E}(\mathbf{x})|^2 \\
&\qquad + |c^n_k - c_k|^2\mathbb{E}|\mathbf{W}|^2 + |d^n_k - d_k)|^2\mathbb{E}|\tilde{\mathbf{W}}|^2 + |e^n_k - e_k|^2\mathbb{E}|\mathbf{V}|^2\Big) \\
\leq &5\Big(2K|a^n_k - a_k|^2 + 2|a_k|^2\mathbb{E}|\mathbf{x}^n - \mathbf{x}|^2 + 2K|b^n_k - b_k|^2 + 2|b_k|^2|\mathbb{E}(\mathbf{x}^n) - \mathbb{E}(\mathbf{x})|^2 \\
&\qquad + |c^n_k - c_k|^2\mathbb{E}|\mathbf{W}|^2 + |d^n_k - d_k)|^2\mathbb{E}|\tilde{\mathbf{W}}|^2 + |e^n_k - e_k|^2\mathbb{E}|\mathbf{V}|^2\Big),
\end{aligned} \tag{4.3.77}$$

where the right hand side clearly goes to zero as $n \to \infty$.

□

Lemma 4.3.4. *Denote*

$$\begin{aligned}
\eta := &8\sum_{k=1}^{N}\Big\{(\|A_k\|^2 + \|\bar{A}_k\|^2\|) + \|B_{k-1}R^{-1}_{k-1}B^*_{k-1}\|^2(\|(Q_k + \bar{Q}_k)\|^2 + \|\bar{Q}_kS_k\|^2) \\
&\qquad + N\sum_{r=0}^{k-2}\|B_rR_r^{-1}B^*_r\|^2\|A^*_{r+1}\cdots A^*_{k-1}\|^2\big(\|(Q_k + \bar{Q}_k)\|^2 + \|\bar{Q}_kS_k\|^2\big)\Big\}.
\end{aligned}$$

Suppose $\eta < 1$, and we choose M such that

$$\frac{8(\|A_0\|^2 + \|\bar{A}_0\|^2)\mathbb{E}\|\xi\|^2 + 8\sum_{k=0}^{N-1} Tr(\sigma^2_{k+1} + \tilde{\sigma}^2_{k+1})}{1-\eta} < M, \tag{4.3.78}$$

then, for any $\mathbf{x} \in C_M$, $\mathbf{X} = T(\mathbf{x}) \in C_M$, and hence T is a self- mapping in C_M.

Remark 4.3.5. *Our model is mimicking a continuous time model setting, in which the coefficients are commonly proportional to the length of each time interval of the partition. Hence, it is common that the coefficients A_k, B_k, R_k, Q_k, σ^2_k and $\tilde{\sigma}^2_k$ are of order $O(\frac{1}{N})$; while S_k and $\mathbb{E}\|\xi\|^2$ are of order $O(1)$ as S_k is the weight of mean field term and $\mathbb{E}\|\xi\|^2$ is the second moment of the initial random variable. Hence*

$\eta = O(\frac{1}{N})$ and the first condition $\eta < 1$ can be easily satisfied. Moreover, the left hand side of (4.3.78) *is of order $O(1)$, and we can then pick up a sufficiently large M so that the second condition* (4.3.78) *holds.*

Proof. By (4.3.76), we have

$$
\begin{aligned}
&\|X_{k+1}\|^2 \\
\leq & 8\Big\{ \|A_k\|^2\|x_k\|^2 + \|\bar{A}_k\|^2\|\mathbb{E}(x_k|\mathcal{F}_k^{\tilde{W}})\|^2 \\
&\quad + \|B_k R_k^{-1} B_k^*\|^2 \Big[\|(Q_{k+1}+\bar{Q}_{k+1})\|^2\|\mathbb{E}(x_{k+1}|\mathcal{F}_k^y)\|^2 + \|\bar{Q}_{k+1}S_{k+1}\|^2 \big\|\mathbb{E}\big(\mathbb{E}(x_{k+1}|\mathcal{F}_{k+1}^{\tilde{W}})|\mathcal{F}_k^y\big)\big\|^2 \\
&\qquad + (N-k-1)\sum_{r=k+2}^{N} \|A_{k+1}^* \cdots A_{r-1}^*\|^2 \big(\|(Q_r+\bar{Q}_r)\|^2\|\mathbb{E}(x_r|\mathcal{F}_k^y)\|^2 \\
&\qquad + \|\bar{Q}_r S_r\|^2 \|\mathbb{E}\big(\mathbb{E}(x_r|\mathcal{F}_r^{\tilde{W}})|\mathcal{F}_k^y\big)\|^2\big)\Big] + \|W_{k+1}\|^2 + \|\tilde{W}_{k+1}\|^2 \Big\},
\end{aligned}
$$

summing up from $k=0$ to $N-1$ on both sides, then taking expectation, we have

$$
\begin{aligned}
&\mathbb{E}\sum_{k=0}^{N-1}\|X_{k+1}\|^2\\
\leq& 8\sum_{k=0}^{N-1}\Big\{\|A_k\|^2\mathbb{E}\|x_k\|^2+\|\bar{A}_k\|^2\mathbb{E}\|\mathbb{E}(x_k|\mathcal{F}_k^{\tilde{W}})\|^2\\
&\qquad+\|B_kR_k^{-1}B_k^*\|^2\Big[\|(Q_{k+1}+\bar{Q}_{k+1})\|^2\mathbb{E}\|\mathbb{E}(x_{k+1}|\mathcal{F}_k^y)\|^2\\
&\qquad\quad+\|\bar{Q}_{k+1}S_{k+1}\|^2\mathbb{E}\big\|\mathbb{E}\big(\mathbb{E}(x_{k+1}|\mathcal{F}_{k+1}^{\tilde{W}})|\mathcal{F}_k^y\big)\big\|^2\\
&\qquad\quad+(N-k-1)\sum_{r=k+2}^{N}\|A_{k+1}^*\cdots A_{r-1}^*\|^2\big(\|(Q_r+\bar{Q}_r)\|^2\mathbb{E}\|\mathbb{E}(x_r|\mathcal{F}_k^y)\|^2\\
&\qquad\qquad+\|\bar{Q}_rS_r\|^2\mathbb{E}\|\mathbb{E}\big(\mathbb{E}(x_r|\mathcal{F}_r^{\tilde{W}})|\mathcal{F}_k^y\big)\|^2\big)\Big]\\
&\qquad+\mathbb{E}\|W_{k+1}\|^2+\mathbb{E}\|\tilde{W}_{k+1}\|^2\Big\}\\
\leq& 8\sum_{k=0}^{N-1}\Big\{(\|A_k\|^2+\|\bar{A}_k\|^2\|)\mathbb{E}\|x_k\|^2\\
&\quad+\|B_kR_k^{-1}B_k^*\|^2\Big[(\|(Q_{k+1}+\bar{Q}_{k+1})\|^2+\|\bar{Q}_{k+1}S_{k+1}\|^2)\mathbb{E}\|x_{k+1}\|^2\\
&\qquad+N\sum_{r=k+2}^{N}\|A_{k+1}^*\cdots A_{r-1}^*\|^2\big(\|(Q_r+\bar{Q}_r)\|^2+\|\bar{Q}_rS_r\|^2\big)\mathbb{E}\|x_r\|^2\Big]\Big\}\\
&+8\sum_{k=0}^{N-1}\mathrm{Tr}(\sigma_{k+1}^2+\tilde{\sigma}_{k+1}^2)\\
\leq& 8(\|A_0\|^2+\|\bar{A}_0\|^2)\mathbb{E}\|\xi\|^2\\
&+8\sum_{k=1}^{N}\Big\{(\|A_k\|^2+\|\bar{A}_k\|^2\|)+\|B_{k-1}R_{k-1}^{-1}B_{k-1}^*\|^2(\|(Q_k+\bar{Q}_k)\|^2+\|\bar{Q}_kS_k\|^2)\\
&\quad+N\sum_{r=0}^{k-2}\|B_rR_r^{-1}B_r^*\|^2\|A_{r+1}^*\cdots A_{k-1}^*\|^2\big(\|(Q_k+\bar{Q}_k)\|^2+\|\bar{Q}_kS_k\|^2\big)\Big\}\mathbb{E}\|x_k\|^2\\
&+8\sum_{k=0}^{N-1}\mathrm{Tr}(\sigma_{k+1}^2+\tilde{\sigma}_{k+1}^2)\leq M.
\end{aligned}
$$

□

Lemma 4.3.6. *The complete metric space (C_M, L_2) is a compact subset in G.*

Proof. For any sequence $\{\mathbf{X}^n\} \subset C_M$, $\mathbf{X}^n = (\mathbf{x}^n, \mathbf{W}, \tilde{\mathbf{W}}, \mathbf{V})$ with $x_k^n = a^n\xi + \sum_{i=1}^k b_i^n W_i + \sum_{i=0}^{k-1} c_i^n V_i + \sum_{i=1}^k d_i^n \tilde{W}_k + e^n$ and $\mathbf{X}^n$ is Gaussian and $\mathbb{E}|\mathbf{X}^n|^2 \leq M$. That is, the means and variances of $\mathbf{X}^n$'s are uniformly bounded. Hence, $a^n, b_i^n, c_i^n, d_i^n, e^n$ are also bounded for $i = 1, \ldots, k$, $k = 1, \ldots, N$ and all $n \in \mathbb{N}$. By the Bolzano-Weierstrass theorem applying to $a^n, b_i^n, c_i^n, d_i^n, e^n$, we can find a subsequence $a^{n_j}, b_i^{n_j}, c_i^{n_j}, d_i^{n_j}, e^{n_j}$ such that $a^{n_j}, b_i^{n_j}, c_i^{n_j}, d_i^{n_j}, e^{n_j}$ all converge to limits a, b_i, c_i, d_i, e. Let $\mathbf{X} = (\mathbf{x}, \mathbf{W}, \tilde{\mathbf{W}}, \mathbf{V})$ with $x_k = a\xi + \sum_{i=1}^k b_i W_i + \sum_{i=0}^{k-1} c_i V_i + \sum_{i=1}^k d_i \tilde{W}_k + e$. It's easy to see that $\mathbf{X} \in C_M$ and, by applying triangle inequality similar to (4.3.77), $\mathbf{X}^{n_j}$ converge to $\mathbf{X}$ in L^2. It is clear that the limit preserves adaptivity, that is $x_k \in \mathcal{F}_k$. $\square$

We next recall a standard result:

Theorem 4.3.7 (Schauder's fixed point theorem)**.** *(see Theorem 7, p.219, in [11]) Let $\mathcal{A}$ be a (non-empty) closed convex subset of a normed space X and let $f : \mathcal{A} \to \mathcal{A}$ be a continuous map such that $K = \overline{f(\mathcal{A})}$ is compact in X. Then f has a fixed point.*

Theorem 4.3.8. *Suppose the condition* (4.3.78) *in Lemma 4.3.4 holds, then the system* (4.3.71) *admits a solution.*

Proof. Now G is a Hausdorff topological vector space and C_M is a nonempty, closed and convex subset of G. If the condition (4.3.78) in Lemma 4.3.4 holds, T is a continuous convex map from C_M into C_M and C_M is compact in G. By Theorem 4.3.7, T has a fixed point in C_M and hence the solution of system (4.3.74) exists. Therefore, the solution of system (4.3.71) exists by the equivalence of the systems (4.3.74) and (4.3.71). $\square$

4.4 Conclusion

Under the discrete time partial observation setting in which individual only makes decision based on the observable processes, the mean field type control problem is always uniquely solvable. Nonetheless, due to a mixture information flows mentioned in this chapter, a similar result is not ready for mean field games. By applying the Schauder's fixed point theorem and introducing several conditions suggested in Lemma 4.3.4, the existence of a solution to mean field game is established. A generalization to the continuous time setting is rather difficult, as our proof is based heavily on the preservation of Gaussian structure in the difference equation, where we lost such property in the continuous time case. The results in this chapter is published in [21].

Chapter 5

Hilbert Calculus and Mean Field Games

This chapter introduces a probabilistic approach to obtain general unique and existence results of a non-linear Forward Backward Stochastic Differential Equation (FBSDE) related to Mean Field Games, by interpreting McKean Vlasov type equations in an appropriate Hilbert space. The SHJB-FP pair in Chapter 2 is connected to the FBSDE of McKean Vlasov type introduced in this Chapter through a "Master Equation", see [9], [17] and [18] for details, which is beyond the scope of the present thesis.

The sufficient condition we demonstrated is likely to be independent of time horizon. In the abstract sense, the individual state evolution satisfies the following stochastic differential equation:

$$\begin{cases} dx_s = f(x_s, \mathbb{L}_s, u_s)ds + \sigma dW_s, \\ x_t = \xi. \end{cases} \tag{5.0.1}$$

The cost functional is given by

$$J(U) = \int_t^T \mathbb{E}[g(x_s, \mathbb{L}_s, u_s)]ds + \mathbb{E}[h(x_T, \mathbb{L}_T)]. \tag{5.0.2}$$

In mean field game, the mean field term $\mathbb{L}_s$ is exogenous to the control problem at the first place. To solve this mean field game, given the mean field term, one first solves the stochastic optimal control problem described by (5.0.1) and (5.0.2). The mean field term $\mathbb{L}$ would then be replaced by $\mathbb{L}_{x_s}$, the measure of the optimal trajectory of the state variable obtained in the first step.

The domain and image of the functional coefficients are specified as follows

$$\begin{aligned}
&f : \mathbb{R}^{n_x} \oplus \mathcal{P}^2(\mathbb{R}^{n_x}) \oplus \mathbb{R}^{n_u} \to \mathbb{R}^{n_x};\\
&g : \mathbb{R}^{n_x} \oplus \mathcal{P}^2(\mathbb{R}^{n_x}) \oplus \mathbb{R}^{n_u} \to \mathbb{R};\\
&h : \mathbb{R}^{n_x} \oplus \mathcal{P}^2(\mathbb{R}^{n_x}) \to \mathbb{R};\\
&\sigma \in \mathcal{L}(\mathbb{R}^{n_w}; \mathbb{R}^{n_x}).
\end{aligned} \tag{5.0.3}$$

Here $\mathcal{P}^2$ is the space of probability measure of finite second moment in $\mathbb{R}^{n_x}$ equipped with the 2^{nd}-Wasserstein metric:

$$W_2(X,Y) = \inf_{\pi_{X,Y}} \int_{\mathbb{R}^{n_x}} |x-y|^2 d\pi_{X,Y}(x,y), \quad X, Y \in L^2(\Omega, \mathbb{R}^{n_x}); \tag{5.0.4}$$

where the infimum is taking over all joint measures for the random variables X, Y. As remark, the convergence of random variables in $L^2(\Omega, \mathbb{R}^{n_x})$ implies the convergence of the associated measures in W_2. Assume that the coefficients are Lipschitz and differentiable. Applying the stochastic maximum principle, we obtain the following classical FBSDE for a mean field game:

$$\begin{cases}
dx_s = f(x_s, \mathbb{L}_s, u)|_{u=u(x_s,\mathbb{L}_s,p_s)} ds + \sigma dW_s,\\
x_t = \xi;\\
-dp_s = \Big[\langle D_x f^*(x_s, \mathbb{L}_s, u)|_{u=u(x_s,\mathbb{L}_s,p_s)}, p_s\rangle_{\mathbb{R}^{n_x}} + D_x g(x_s, \mathbb{L}_s, u)|_{u=u(x_s,\mathbb{L}_s,p_s)}\Big] ds\\
\qquad\qquad - Z_s dW_s,\\
p_T = D_x h(x_T, \mathbb{L}_T),
\end{cases} \tag{5.0.5}$$

where D_x denotes the gradient with respect to the spatial variable. Here $u(x, \mathbb{L}, p)$ is the unique minimizer of the Lagrangian:

$$u(x, \mathbb{L}, p) := \operatorname{argmin}_{u \in \mathbb{R}^{n_u}} \{\langle f(x, \mathbb{L}, u), p\rangle_{\mathbb{R}^{n_x}} + g(x, \mathbb{L}, u)\}, \tag{5.0.6}$$

which implies the first order condition:

$$D_u\Big(\langle f(x,\mathbb{L},u)|_{u=u(x,\mathbb{L},p)},p\rangle_{\mathbb{R}^{n_x}}+g(x,\mathbb{L},u)|_{u=u(x,\mathbb{L},p)}\Big)=0,\quad \forall x,p\in\mathbb{R}^{n_x},\mathbb{L}\in\mathcal{P}^2(\mathbb{R}^{n_x}). \tag{5.0.7}$$

In mean field games, after solving the control problem, we will then replace $\mathbb{L}_s$ by $\mathbb{L}_{x_s}$, the law of x_s. (5.0.5) becomes a system of Forward Backward Stochastic Differential Equation of McKean Vlasov type.

$$\begin{cases} dx_s = f(x_s,\mathbb{L}_{x_s},u)|_{u=u(x_s,\mathbb{L}_s,p_s)}ds+\sigma dW_s, \\ x_t=\xi; \\ -dp_s = \Big[\langle D_x f^*(x_s,\mathbb{L}_{x_s},u)|_{u=u(x_s,\mathbb{L}_s,p_s)},p_s\rangle_{\mathbb{R}^{n_x}}+D_x g(x_s,\mathbb{L}_{x_s},u)|_{u=u(x_s,\mathbb{L}_s,p_s)}\Big]ds - Z_s dW_s, \\ p_T = D_x h(x_T,\mathbb{L}_{x_T}). \end{cases} \tag{5.0.8}$$

The first order condition (5.0.7), after putting $\mathbb{L}=\mathbb{L}_x$, becomes

$$D_u\Big(\langle f(x,\mathbb{L}_x,u)|_{u=u(x,\mathbb{L}_x,p)},p\rangle_{\mathbb{R}^{n_x}}+g(x,\mathbb{L}_x,u)|_{u=u(x,\mathbb{L}_x,p)}\Big)=0,\quad \forall x,p\in\mathbb{R}^{n_x},\mathbb{L}_x\in\mathcal{P}^2(\mathbb{R}^{n_x}). \tag{5.0.9}$$

The aim of this chapter is to establish the unique existence of a global (in time) solution of the system (5.0.8). The mean field term $\mathbb{L}_{x_s}$ is a probability measures in $\mathcal{P}^2$, which is clearly not a vector space. As suggested by in [9], we can interpret the whole equation in terms of an appropriate Hilbert space $\mathcal{H}$.

5.1 Preliminaries - Calculus in $\mathcal{H}$

We introduce the notion of calculus in $\mathcal{H}$ we used throughout this work and make a connection with other (differential) operators commonly found in the literature.

5.1.1 Functional of law

For any positive integer n, let $\mathcal{H}^n = L^2(\Omega,\mathbb{R}^n)$ be the canonical Hilbert space for square integrable random variables on $\mathbb{R}^n$ equipped with the standard inner product

$$\langle x, y\rangle_{\mathcal{H}^n} = \mathbb{E}[x \cdot y] = \mathbb{E}[\langle x, y\rangle_{\mathbb{R}^n}]$$

Let $F : \mathcal{H}^n \to \mathbb{R}$ and f depends on $X \in \mathcal{H}^n$ only through its law, i.e. $F(X) = F(Y)$ whenever $\mathbb{L}_X = \mathbb{L}_Y$ on their support. We have $F(X)$ is a deterministic number for any input $X \in \mathcal{H}^n$ and $\omega \in \Omega$. Whenever it exists, the (Gâteaux) derivative of F with respect to X is denoted by $D_X F(X)$, which is given by

$$\lim_{\theta \to 0} \frac{F(X + \theta Y) - F(X)}{\theta} = \langle D_X F(X), Y\rangle_{\mathcal{H}^n}, \quad Y \in \mathcal{H}^n. \tag{5.1.10}$$

Clearly $D_X F$ is an operator specified by $D_X F : \mathcal{H}^n \to \mathcal{H}^n$. On the other hand, since F depends on X only through its law, we can write

$$F(X) = f(\mathbb{L}_X), \tag{5.1.11}$$

where $f : \mathcal{P}^2(\mathbb{R}^n) \to \mathbb{R}$. Suppose that a random variable X admits a smooth L^2 density π_X on $\mathbb{R}^n$. If there is no ambiguity, we may interchange the usage of law and density of X and we write

$$F(X) = f(\mathbb{L}_X) = f(\pi_X). \tag{5.1.12}$$

We now make a connection between this derivative and Wasserstein gradient. (see [9])

Proposition 5.1.1. *$D_X F(X)$ agrees with the Wasserstein gradient:*

$$D_X F(X) = D_x \frac{\partial f}{\partial m}(\pi_X)(x)\Big|_{x=X}. \tag{5.1.13}$$

Proof. Let Y be another random variable in $\mathcal{H}^n$ possesses together with X a smooth joint L^2 density $\pi_{X,Y}(x, y)$. By the definition of the left hand side of (5.1.13), we have

$$\lim_{\theta \to 0} \frac{F(X + \theta Y) - F(X)}{\theta} =: \langle D_X F(X), Y\rangle_{\mathcal{H}^n}. \tag{5.1.14}$$

On the other hand, the density $\pi_{X+\theta Y}$of $X+\theta Y$ is given by the convolution

$$\pi_{X+\theta Y}(x) = \int_{\mathbb{R}^n} \pi_{X,Y}(x-\theta y, y)dy, \tag{5.1.15}$$

with partial derivative

$$\frac{\partial \pi_{X+\theta Y}}{\partial \theta} = -\mathrm{div}_x \int_{\mathbb{R}^n} \pi_{X,Y}(z-\theta y, y)ydy. \tag{5.1.16}$$

Hence

$$\begin{aligned}
\lim_{\theta\to 0}\frac{F(X+\theta Y)-F(X)}{\theta} &= \lim_{\theta\to 0}\frac{f(\pi_{X+\theta Y})-f(\pi_X)}{\theta} \\
&= \int_{\mathbb{R}^n} \frac{\partial f}{\partial m}(\pi_X)(x)\frac{\partial \pi_{X+\theta Y}}{\partial \theta}(x)dx \\
&= -\int_{\mathbb{R}^n} \frac{\partial f}{\partial m}(\pi_X)(x)\mathrm{div}_x \int_{\mathbb{R}^n} \pi_{X,Y}(x,y)ydydx \\
&= \int_{\mathbb{R}^n} D_x\frac{\partial f}{\partial m}(\pi_X)(x)\int_{\mathbb{R}^n} \pi_{X,Y}(x,y)ydydx \\
&= \mathbb{E}\Big[D_x\frac{\partial f}{\partial m}(\pi_X)(x)\Big|_{x=X}Y\Big] \\
&=: \Big\langle D_x\frac{\partial f}{\partial m}(\pi_X)(x)\Big|_{x=X}, Y\Big\rangle_{\mathcal{H}^n}.
\end{aligned} \tag{5.1.17}$$

Comparing (5.1.14) and (5.1.17) concludes the result. □

Proposition 5.1.1 suggests that, even in the simplest case that F depends on its argument $X \in \mathcal{H}^n$ only through its law (or density), the derivative $D_X F$ depends on both the law and state (or realization) of the random variable. To proceed on studying higher order derivative, we have to extend the result obtained in Proposition 5.1.1.

5.1.2 Functional of both state and law

Let $F : \mathcal{H}^n \to \mathcal{H}^m$ and F depends on $X \in \mathcal{H}^n$ both through its state and law. We note that $F(X)$ is a random variable in $\mathcal{H}^m$ for any $X \in \mathcal{H}^n$, instead of a deterministic number as in the previous case. The (Gâteaux) derivative of F with respect to X is an operator specified by

$$D_X F : \mathcal{H}^n \to \mathcal{L}(\mathcal{H}^n; \mathcal{H}^m),$$

which is define through (in weak sense)

$$\lim_{\theta\to 0}\frac{\langle F(X+\theta Y), Z\rangle_{\mathcal{H}^m}-\langle F(X), Z\rangle_{\mathcal{H}^m}}{\theta}=\Big\langle D_X F(X)(Y), Z\Big\rangle_{\mathcal{H}^m},\quad Y\in\mathcal{H}^n; Z\in\mathcal{H}^m. \tag{5.1.18}$$

Since F depends on $X\in\mathcal{H}^n$ both through its state and law, we may write

$$F(X)=f(X,\mathbb{L}_X)=f(x,\mathbb{L}_X)|_{x=X},$$

where $f:\mathbb{R}^n\oplus\mathcal{P}^2(\mathbb{R}^n)\to\mathbb{R}^m$. Similar to Section 5.1.1, suppose that a random variable X admits a smooth L^2 density π_X on $\mathbb{R}^n$, we have

$$F(X)=f(X,\pi_X)=f(x,\pi_X)|_{x=X}. \tag{5.1.19}$$

Proposition 5.1.2. *If Y is another random variable in $\mathcal{H}^n$ with smooth L^2 density in $\mathbb{R}^n$, then $D_XF(X)(Y)$ is given by*

$$D_XF(X)(Y)=D_xf(x,\pi_X)|_{x=X}\cdot Y+\mathbb{E}_{\tilde{X}\tilde{Y}}[D_{\tilde{x}}\frac{\partial}{\partial m}f(X,\pi_X)(\tilde{x})|_{\tilde{x}=\tilde{X}}(\tilde{Y})], \tag{5.1.20}$$

where $(\tilde{X},\tilde{Y})$ is a pair of independent copy of (X,Y), and $\mathbb{E}_{\tilde{X}Y}$ takes expectation of the random variable in the bracket by integrating with the joint density of $\tilde{X}$ and $\tilde{Y}$ only.

Proof. Let $Z\in\mathcal{H}^m$ with smooth density in $\mathbb{R}^m$. Using (5.1.19), we have

$$\begin{aligned}&\lim_{\theta\to 0}\frac{\langle F(X+\theta Y), Z\rangle_{\mathcal{H}^m}-\langle F(X), Z\rangle_{\mathcal{H}^m}}{\theta}\\
=&\lim_{\theta\to 0}\frac{\langle f(X+\theta Y,\pi_{X+\theta Y}), Z\rangle_{\mathcal{H}^m}-\langle f(X,\pi_X), Z\rangle_{\mathcal{H}^m}}{\theta}\\
=&\Big\langle D_xf(x,\pi_X)|_{x=X}\cdot Y, Z\Big\rangle_{\mathcal{H}^m}+\lim_{\theta\to 0}\frac{\mathbb{E}[f(X,\pi_{X+\theta Y})\cdot Z]-\mathbb{E}[f(X,\pi_X)\cdot Z]}{\theta}\\
=&\Big\langle D_xf(x,\pi_X)|_{x=X}\cdot Y, Z\Big\rangle_{\mathcal{H}^m}+\lim_{\theta\to 0}\frac{\mathbb{E}[f(X,\pi_{\tilde{X}+\theta\tilde{Y}})\cdot Z]-\mathbb{E}[f(X,\pi_{\tilde{X}})\cdot Z]}{\theta}.\end{aligned} \tag{5.1.21}$$

Note that the numerator in the second term has the expression

$$\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}f(x,\pi_{\tilde{X}+\theta\tilde{Y}})\cdot z\pi_{X,Z}(x,z)dxdz-\int_{\mathbb{R}^m}\int_{\mathbb{R}^m}f(x,\pi_{\tilde{X}})\cdot z\pi_{X,Z}(x,z)dxdz, \tag{5.1.22}$$

which is a deterministic number. Applying Proposition 5.1.1, (5.1.21) becomes

$$\Big\langle D_x f(x,\pi_X)|_{x=X}\cdot Y, Z\Big\rangle_{\mathcal{H}^m} + \Big\langle D_{\tilde{x}}\frac{\partial}{\partial m}\mathbb{E}_{XZ}[f(X,\pi_X)\cdot Z](\tilde{x})|_{\tilde{x}=\tilde{X}}, \tilde{Y}\Big\rangle_{\mathcal{H}^n}. \tag{5.1.23}$$

Since the operators in the second term of (5.1.23) are commutative, we have

$$\begin{aligned}&\lim_{\theta\to 0}\frac{\langle F(X+\theta Y), Z\rangle_{\mathcal{H}^m} - \langle F(X), Z\rangle_{\mathcal{H}^m}}{\theta}\\ =&\Big\langle D_x f(x,\pi_X)|_{x=X}\cdot Y, Z\Big\rangle_{\mathcal{H}^m} + \mathbb{E}_{XZ}\Big[\mathbb{E}_{\tilde{X}\tilde{Y}}[D_{\tilde{x}}\frac{\partial}{\partial m}f(X,\pi_X)(\tilde{x})|_{\tilde{x}=\tilde{X}}(\tilde{Y})]\cdot Z\Big],\end{aligned} \tag{5.1.24}$$

which concludes the proof. □

For $F:\mathcal{H}^n\to\mathcal{H}^m$ being a function depends on $X\in\mathcal{H}^n$ only through its state and law, we define the following operator

Definition 5.1.3.

$$\begin{aligned}&D_{\mathbb{L}}F:\mathcal{H}^n\to\mathcal{L}(\mathcal{H}^n;\mathcal{H}^m);\\ &D_{\mathbb{L}}F(X) = D_X F(X) - D_x f(x,\mathbb{L}_X)|_{x=X}.\end{aligned} \tag{5.1.25}$$

By Proposition 5.1.2, providing that X possess a smooth L^2 density on $\mathbb{R}^n$, we have

$$D_{\mathbb{L}}F(X)(Y) = \mathbb{E}_{\tilde{X}\tilde{Y}}[D_{\tilde{x}}\frac{\partial}{\partial m}f(X,\pi_X)(\tilde{x})|_{\tilde{x}=\tilde{X}}(\tilde{Y})]. \tag{5.1.26}$$

We have the following immediate application of Proposition 5.1.2. Suppose that $G:\mathcal{H}^n\to\mathbb{R}$, where G depends on $X\in\mathcal{H}^n$ only through its law. The first and second order derivatives are specified by

$$\begin{aligned}&D_X G:\mathcal{H}^n\to\mathcal{H}^n;\\ &D_{XX}G:\mathcal{H}^n\to\mathcal{L}(\mathcal{H}^n;\mathcal{H}^n).X\end{aligned}$$

To connect the second order derivative with other differential operators, we further assume that X has a smooth density π_X on $\mathbb{R}^n$. Since G only depends on the law of the argument, by Proposition 5.1.1, we put $F(X) = D_X G(X) = D_x \frac{\partial g}{\partial m}(\pi_X)(x)|_{x=X}$ into (5.1.20):

$$\begin{aligned} & D_{XX}G(X)(Y) \\ =& D_X(D_X G(X))(Y) \\ =& D_{xx}\frac{\partial}{\partial m}\Big(g(\pi_X)\Big)(x)|_{x=X} \cdot Y + \mathbb{E}_{\tilde{X}\tilde{Y}}[D_{\tilde{x}}\frac{\partial}{\partial m}\left(D_x\frac{\partial}{\partial m}\Big(g(\pi_X)\Big)(x)|_{x=X}\right)(\tilde{x})|_{\tilde{x}=\tilde{X}}(\tilde{Y})]. \end{aligned} \tag{5.1.27}$$

We further define the following derivatives, whose connections with Wasserstein gradient are omitted here. They can be obtained using similar arguments as in Proposition 5.1.1 and 5.1.2. Let $F(X_1, X_2)$ be a function in $F : \mathcal{H}^{n_1} \oplus \mathcal{H}^{n_2} \to \mathcal{H}^m$.

1. We define the partial derivative

$$D_{X_1}F : \mathcal{H}^{n_1} \oplus \mathcal{H}^{n_2} \to \mathcal{L}(\mathcal{H}^{n_1}; \mathcal{H}^m)$$

through

$$\lim_{\theta\to 0} \frac{\langle F(X_1 + \theta Y, X_2), Z\rangle_{\mathcal{H}^m} - \langle F(X_1, X_2), Z\rangle_{\mathcal{H}^m}}{\theta} = \Big\langle D_{X_1}F(X_1, X_2)(Y), Z\Big\rangle_{\mathcal{H}^m}, \tag{5.1.28}$$

for all $Y \in \mathcal{H}^n; Z \in \mathcal{H}^m$.

2. We define the cross derivative

$$D_{X_2X_1}F : \mathcal{H}^{n_1} \oplus \mathcal{H}^{n_2} \to \mathcal{L}(\mathcal{H}^{n_2} \otimes \mathcal{H}^{n_1}; \mathcal{H}^m)$$

through

$$\begin{aligned}
&\lim_{\theta,\gamma\to 0}\frac{\langle F(X_1+\theta Y_1, X_2+\gamma Y_2), Z\rangle_{\mathcal{H}^m}-\langle F(X_1, X_2+\gamma Y_2), Z\rangle_{\mathcal{H}^m}}{\theta\gamma}\\
&\quad+\lim_{\theta,\gamma\to 0}\frac{\langle F(X_1, X_2), Z\rangle_{\mathcal{H}^m}-\langle F(X_1+\theta Y_1, X_2), Z\rangle_{\mathcal{H}^m}}{\theta\gamma}\\
&=\lim_{\gamma\to 0}\frac{\Big\langle D_{X_1}F(X_1, X_2+\gamma Y_2)(Y_1), Z\Big\rangle_{\mathcal{H}^m}-\Big\langle D_{X_1}F(X_1, X_2)(Y_1), Z\Big\rangle_{\mathcal{H}^m}}{\gamma}\\
&=\Big\langle D_{X_2}\Big(D_{X_1}F(X_1, X_2)(Y_1)\Big)(Y_2), Z\Big\rangle_{\mathcal{H}^m}\\
&=:\Big\langle D_{X_2X_1}F(X_1, X_2)(Y_2, Y_1), Z\Big\rangle_{\mathcal{H}^m}
\end{aligned} \tag{5.1.29}$$

for all $Y_1, Y_2 \in \mathcal{H}^n; Z \in \mathcal{H}^m$. Clearly, we have

$$D_{X_2X_1}F(X_1, X_2)(Y_2, Y_1) = D_{X_1X_2}F(X_1, X_2)(Y_1, Y_2) \in \mathcal{H}^m \tag{5.1.30}$$

by symmetry.

5.2 A Hilbert Space Interpretation

As in the previous section, denote $\mathcal{H}^k = L^2(\Omega, \mathbb{R}^k)$ the Hilbert space of square integrable random variables on $\mathbb{R}^k$. We first interpret the functional coefficients in system (5.0.8):

$$\begin{aligned}
F(X,U) &:= f(x,\mathbb{L},u)|_{x=X,\mathbb{L}=\mathbb{L}_X,u=U},\\
G(X,U) &:= g(x,\mathbb{L},u)|_{x=X,\mathbb{L}=\mathbb{L}_X,u=U},\\
H(X) &:= h(x,\mathbb{L})|_{x=X,\mathbb{L}=\mathbb{L}_X};
\end{aligned} \tag{5.2.31}$$

where $F : \mathcal{H}^{n_x} \oplus \mathcal{H}^{n_u} \to \mathcal{H}^{n_x}$, $G : \mathcal{H}^{n_x} \oplus \mathcal{H}^{n_u} \to \mathcal{H}^1$ and $H : \mathcal{H}^{n_x} \to \mathcal{H}^1$. We thus have

$$\begin{aligned}
D_X F &: \mathcal{H}^{n_x} \oplus \mathcal{H}^{n_u} \to \mathcal{L}(\mathcal{H}^{n_x}; \mathcal{H}^{n_x}) \quad ; D_X F : \mathcal{H}^{n_x} \oplus \mathcal{H}^{n_u} \to \mathcal{L}(\mathcal{H}^{n_u}; \mathcal{H}^{n_x});\\
D_X G &: \mathcal{H}^{n_x} \oplus \mathcal{H}^{n_u} \to \mathcal{L}(\mathcal{H}^{n_x}; \mathcal{H}^1) \quad ; D_U G : \mathcal{H}^{n_x} \oplus \mathcal{H}^{n_u} \to \mathcal{L}(\mathcal{H}^{n_u}; \mathcal{H}^1);\\
D_X H &: \mathcal{H}^{n_x} \to \mathcal{L}(\mathcal{H}^{n_x}; \mathcal{H}^1).
\end{aligned} \tag{5.2.32}$$

The transpose $D_X F^* : \mathcal{H}^{n_x} \oplus \mathcal{H}^{n_u} \to \mathcal{L}(\mathcal{H}^{n_x}; \mathcal{H}^{n_x})$ operator satisfies

$$\langle D_X F(X,U)(\phi_x), \gamma_x \rangle_{\mathcal{H}^{n_x}} = \langle \phi_x, D_X F^*(X,U)\gamma_x \rangle_{\mathcal{H}^{n_x}}, \forall \phi_x, \gamma_x \in \mathcal{H}^{n_x}.$$

Similarly, we have $D_U F^* : \mathcal{H}^{n_x} \oplus \mathcal{H}^{n_u} \to \mathcal{L}(\mathcal{H}^{n_x}; \mathcal{H}^{n_u})$, which satisfies

$$\langle D_U F(X,U)(\phi_u), \gamma_x \rangle_{\mathcal{H}^{n_x}} = \langle \phi_u, D_X F^*(X,U)\gamma_x \rangle_{\mathcal{H}^{n_u}}, \forall \phi_u \in \mathcal{H}^{n_u}, \gamma_x \in \mathcal{H}^{n_x}.$$

We recall that the minimizer of the Lagrangian is given by

$$u(x, \mathbb{L}, p) := \operatorname{argmin}_{u \in \mathbb{R}^{n_u}} \{ \langle f(x, \mathbb{L}, u), p \rangle_{\mathbb{R}^{n_x}} + g(x, \mathbb{L}, u) \}, \tag{5.2.33}$$

and the first order condition

$$D_u \Big(\langle f(x, \mathbb{L}, u)|_{u=u(x,\mathbb{L},p)}, p \rangle_{\mathbb{R}^{n_x}} + g(x, \mathbb{L}, u)|_{u=u(x,\mathbb{L},p)} \Big) = 0, \qquad \forall x, p \in \mathbb{R}^{n_x}, \mathbb{L} \in \mathcal{P}^2(\mathbb{R}^{n_x}). \tag{5.2.34}$$

Their $\mathcal{H}$ interpretation are respectively given by

$$U(X, P) = u(x, \mathbb{L}_X, p)|_{x=X, p=P} \tag{5.2.35}$$

and

$$D_U \langle F(X,U)|_{U=U(X,P)}, P \rangle_{\mathbb{R}^{n_x}} + D_U G(X,U)|_{U=U(X,P)} = 0, \qquad \forall X, P \in \mathcal{H}^x. \tag{5.2.36}$$

The system (5.0.8) can be written as:

$$\begin{cases} dX_s = F\Big(X_s, U(X_s, P_s)\Big) ds + \sigma dW_s, \\ X_t = \xi; \\ -dP_s = \Big[D_X \langle F^*(X_s, U(X_s, P_s)), P_s \rangle_{\mathbb{R}^{n_x}} + D_X G(X_s, U(X_s, P_s)) \Big] ds \\ \qquad\qquad - \Big[D_{\mathbb{L}} \langle F^*(X_s, U(X_s, P_s)), P_s \rangle_{\mathbb{R}^{n_x}} + D_{\mathbb{L}} G(X_s, U(X_s, P_s)) \Big] ds - Z_s dW_s, \\ P_T = D_X H(X_T) - D_{\mathbb{L}} H(X_T); \end{cases} \tag{5.2.37}$$

where $D_{\mathbb{L}}$ is defined in Definition 5.1.3. As we will explain in later sections, it is stimulating to regard the second bracket in the backward equation as the asymmetry arising from the very definition of mean field games.

5.3 Revisit the First Order Condition

We adopt the notations for second order operators: $D_{ij}F(X,U) = D_i(D_jF(X,U))$. We have the following second order operators:

$$
\begin{aligned}
&D_{XX}F^* &&: \mathcal{H}^{n_x} \oplus \mathcal{H}^{n_u} \to \mathcal{L}(\mathcal{H}^{n_x} \otimes \mathcal{H}^{n_x}; \mathcal{H}^{n_x}); \quad &&D_{UX}F^* &&: \mathcal{H}^{n_x} \oplus \mathcal{H}^{n_u} \to \mathcal{L}(\mathcal{H}^{n_u} \otimes \mathcal{H}^{n_x}; \mathcal{H}^{n_x});\\
&D_{XU}F^* &&: \mathcal{H}^{n_x} \oplus \mathcal{H}^{n_u} \to \mathcal{L}(\mathcal{H}^{n_x} \otimes \mathcal{H}^{n_x}; \mathcal{H}^{n_u}); \quad &&D_{UU}F^* &&: \mathcal{H}^{n_u} \oplus \mathcal{H}^{n_u} \to \mathcal{L}(\mathcal{H}^{n_u} \otimes \mathcal{H}^{n_x}; \mathcal{H}^{n_u});\\
&D_{XX}G &&: \mathcal{H}^{n_x} \oplus \mathcal{H}^{n_u} \to \mathcal{L}(\mathcal{H}^{n_x} \otimes \mathcal{H}^{n_x}; \mathbb{R}); \quad &&D_{UX}G &&: \mathcal{H}^{n_x} \oplus \mathcal{H}^{n_u} \to \mathcal{L}(\mathcal{H}^{n_u}; \mathcal{H}^{n_x});\\
&D_{XU}G &&: \mathcal{H}^{n_x} \oplus \mathcal{H}^{n_u} \to \mathcal{L}(\mathcal{H}^{n_x}; \mathcal{H}^{n_u}); \quad &&D_{UU}G &&: \mathcal{H}^{n_x} \oplus \mathcal{H}^{n_u} \to \mathcal{L}(\mathcal{H}^{n_u}; \mathcal{H}^{n_u});\\
&D_{XX}H &&: \mathcal{H}^{n_x} \to \mathcal{L}(\mathcal{H}^{n_x}; \mathcal{H}^{n_x}).
\end{aligned}
\tag{5.3.38}
$$

We argue that both $D_{XX}F^*, D_{UU}F^*, D_{XX}G$ and $D_{UU}G$ are symmetric, in the sense that, taking $D_{UU}f^*$ as an illustrative example:

$$
D_{UU}F^*(X,U)(\phi_u,\phi_x)(\gamma_u) = D_{UU}F^*(X,U)(\gamma_u,\phi_x)(\phi_u), \qquad \phi_x \in \mathcal{H}^{n_x}; \phi_u, \gamma_u \in \mathcal{H}^{n_u}. \tag{5.3.39}
$$

In particular,

$$
\begin{aligned}
&D_{UU}F^*(X,U)(\phi_u,\phi_x)(\gamma_u)\\
&= \langle D_{UU}F^*(X,U)(\phi_u,\phi_x), \gamma_u \rangle_{\mathcal{H}^{n_u}}\\
&= \lim_{\theta\to 0} \langle \frac{D_UF^*(X,U+\theta\phi_u)(\phi_x) - D_UF^*(X,U)(\phi_x)}{\theta}, \gamma_u \rangle_{\mathcal{H}^{n_u}}\\
&= \lim_{\theta\to 0} \langle \phi_x, \frac{D_UF(X,U+\theta\phi_u)(\gamma_u) - D_UF(X,U)(\gamma_u)}{\theta} \rangle_{\mathcal{H}^{n_x}}\\
&=: \langle \phi_x, D_{UU}F(X,U)(\phi_u,\gamma_u) \rangle_{\mathcal{H}^{n_x}}\\
&=: \langle \phi_x, D_{UU}F(X,U)(\gamma_u,\phi_u) \rangle_{\mathcal{H}^{n_x}}\\
&= D_{UU}F^*(X,U)(\gamma_u,\phi_x)(\phi_u),
\end{aligned}
\tag{5.3.40}
$$

where we use (5.1.30) in the second last equality and the last row follows by reversing the steps. If there is no ambiguity in the partial derivative, we may factor out the differential operator. For example,

$$
D_X(F^*(P)+G) := D_XF^*(P) + D_XG := D_XF^*(X,U)(P) + D_XG(X,U). \tag{5.3.41}
$$

For notational simplicity, we may omit arguments (X, U) in the functional coefficients.

We now revisit the first order condition implied in Equation (5.2.36):

$$D_U F(X, U(X,P))(P) + D_U G(X, U(X,P)) = 0, \qquad \forall X, P \in \mathcal{H}^x.$$

Property 5.3.1 (Differentiate (5.2.36) w.r.t. X)**.**

$$\begin{aligned}
0 &= D_X\Big[D_U F^*(X, U(X,P))(P) + D_U G(X, U(X,P))\Big](\phi_x) \\
&= D_{XU} F^*(X, U(X,P))(\phi_x, P) + D_{XU} G(X, U(X,P))(\phi_x) \\
&\qquad + D_{UU} F^*(X, U(X,P))\Big(D_X U(X,P)(\phi_x), P\Big) + D_{UU} G(X, U(X,P))\Big(D_X U(X,P)(\phi_x)\Big).
\end{aligned} \tag{5.3.42}$$

Property 5.3.2 (Differentiate (5.2.36) w.r.t. P)**.**

$$\begin{aligned}
0 &= D_P\Big[D_U F^*(X, U(X,P))(P) + D_U G(X, U(X,P))\Big](\gamma_x) \\
&= D_U F^*(X, U(X,P))(\gamma_x) \\
&\qquad + D_{UU} F^*(X, U(X,P))\Big(D_P U(X,P)(\gamma_x), P\Big) + D_{UU} G(X, U(X,P))\Big(D_P U(X,P)(\gamma_x)\Big).
\end{aligned} \tag{5.3.43}$$

Property 5.3.3 (Combine (5.3.42) and (5.3.43))**.** *Applying $D_P U(X,P)(\gamma_x) \in \mathcal{H}^{n_u}$ in* (5.3.42)*, we have*

$$\begin{aligned}
0 =& \Big\langle D_{XU} F^*(X, U(X,P))(\phi_x, P) + D_{XU} G(X, U(X,P))(\phi_x) \\
&\qquad + D_{UU} F^*(X, U(X,P))\Big(D_X U(X,P)(\phi_x), P\Big) \\
&\qquad + D_{UU} G(X, U(X,P))\Big(D_X U(X,P)(\phi_x)\Big), D_P U(X,P)(\gamma_x)\Big\rangle_{\mathcal{H}^{n_u}} \\
=& \Big\langle D_{XU} F^*(X, U(X,P))(\phi_x, P) + D_{XU} G(X, U(X,P))(\phi_x), D_P U(X,P)(\gamma_x)\Big\rangle_{\mathcal{H}^{n_u}} \\
&\qquad + \Big\langle D_{UU} F^*(X, U(X,P))\Big(D_P U(X,P)(\gamma_x), P\Big) \\
&\qquad\quad + D_{UU} G(X, U(X,P))\Big(D_P U(X,P)(\gamma_x)\Big), D_X U(X,P)(\phi_x)\Big\rangle_{\mathcal{H}^{n_u}},
\end{aligned} \tag{5.3.44}$$

where we used the symmetric property of $D_{UU}F^*$ and $D_{UU}g$ shown in (5.3.40).

Similarly, applying $D_X U(X,P)(\phi_x) \in \mathcal{H}^{n_u}$ in (5.3.43), *we have*

$$\begin{aligned}0 =&\Big\langle D_U F^*(X,U(X,P))(\gamma_x), D_X U(X,P)(\phi_x)\Big\rangle_{\mathcal{H}^{n_u}}\\&+\Big\langle D_{UU}F^*(X,U(X,P))\Big(D_P U(X,P)(\gamma_x),P\Big)\\&\quad+D_{UU}G(X,U(X,P))\Big(D_P U(X,P)(\gamma_x)\Big), D_X U(X,P)(\phi_x)\Big\rangle_{\mathcal{H}^{n_u}}.\end{aligned} \tag{5.3.45}$$

Combining Equation (5.3.44) *and* (5.3.45) *yields*

$$\begin{aligned}&\Big\langle D_{XU}F^*(X,U(X,P))(\phi_x,P)+D_{XU}G(X,U(X,P))(\phi_x), D_P U(X,P)(\gamma_x)\Big\rangle_{\mathcal{H}^{n_u}}\\=&\Big\langle D_U F^*(X,U(X,P))(\gamma_x), D_X U(X,P)(\phi_x)\Big\rangle_{\mathcal{H}^{n_u}}.\end{aligned} \tag{5.3.46}$$

Property 5.3.4. *Multiplying* (5.3.42) *on both sides with $D_X U(X,P)(\phi_x)$, we have*

$$\begin{aligned}0 =& \Big\langle D_{XU}F^*(X,U(X,P))(\phi_x,P)+D_{XU}G(X,U(X,P))(\phi_x), D_X U(X,P)(\phi_x)\Big\rangle_{\mathcal{H}^{n_u}}\\&+\Big\langle D_{UU}F^*(X,U(X,P))\Big(D_X U(X,P)(\phi_x),P\Big)\\&\quad+D_{UU}G(X,U(X,P))\Big(D_X U(X,P)(\phi_x)\Big), D_X U(X,P)(\phi_x)\Big\rangle_{\mathcal{H}^{n_u}}.\end{aligned} \tag{5.3.47}$$

Property 5.3.5. *Multiplying* (5.3.43) *on both sides with $D_P U(X,P)(\gamma_x)$, we have*

$$\begin{aligned}0 =&\Big\langle D_U F^*(X,U(X,P))(\gamma_x), D_P U(X,P)(\gamma_x)\Big\rangle_{\mathcal{H}^{n_u}}\\&+\Big\langle D_{UU}F^*(X,U(X,P))\Big(D_P U(X,P)(\gamma_x),P\Big)\\&\quad+D_{UU}G(X,U(X,P))\Big(D_P U(X,P)(\gamma_x)\Big), D_P U(X,P)(\gamma_x)\Big\rangle_{\mathcal{H}^{n_u}}.\end{aligned} \tag{5.3.48}$$

We introduce the following assumptions

$(A.1)$ The Hessian of the Hamiltonian is positive:

$$\begin{aligned}
&\begin{pmatrix} D_{XX} & D_{XU} \\ D_{UX} & D_{UU} \end{pmatrix} (F^*(P) + G)\Big(\phi_x, \phi_u\Big)^{\otimes 2} \\
:=&\begin{pmatrix} D_{XX} & D_{XU} \\ D_{UX} & D_{UU} \end{pmatrix} (F^*(X, U(X,P))(P) + G(X, U(X,P)))\Big(\phi_x, \phi_u\Big)^{\otimes 2} \\
\geq&\lambda\Big(\|\phi_x\|^2_{\mathcal{H}^{n_x}} + \|\phi_u\|^2_{\mathcal{H}^{n_u}}\Big), \quad \forall X, P \in \mathcal{H}^{n_x}.
\end{aligned} \tag{5.3.49}$$

$(A.2)$ The second order derivative in the control of the Hamiltonian is invertible. That is

$$\begin{aligned}
&\Big[D^{-1}_{UU} D_{UU}\Big](F^*(P) + G)(\phi_u) \\
:=&D^{-1}_{UU}(F^*(P) + G) D_{UU}(F^*(P) + G)(\phi_u) = \phi_u,
\end{aligned} \tag{5.3.50}$$

where $D^{-1}_{UU}(F^*(P) + G) := D^{-1}_{UU}\Big(F^*(X, U(X,P))(P) + G(X, U(X,P))\Big)$ is the inverse operator of

$$D_{UU}(F^*(P) + G) := D_{UU}\Big(F^*(X, U(X,P))(P) + G(X, U(X,P))\Big).$$

$(A.3)$ The second order derivative in the terminal condition is positive:

$$\langle D_{XX} h(\phi_x), \phi_x\rangle_{\mathcal{H}^{n_x}} \geq \lambda\|\phi_x\|^2_{\mathcal{H}^{n_x}}, \quad \forall X \in \mathcal{H}^{n_x}. \tag{5.3.51}$$

Lemma 5.3.6. *Suppose that the assumptions* $(A.1)$ *and* $(A.2)$ *hold, then the Schur complement*

$$\begin{aligned}
&\Big[D_{XX} - D_{UX} D^{-1}_{UU} D_{XU}\Big](F^*(P) + G) \\
&:= D_{XX}\Big(F^*(P) + G\Big) - D_{UX}\Big(F^*(P) + G\Big) D^{-1}_{UU}\Big(F^*(P) + G\Big) D_{XU}\Big(F^*(P) + G\Big)
\end{aligned}$$

is semi-positive, for all $X, P \in \mathcal{H}^{n_x}$.

Proof.

$$
\begin{aligned}
&\begin{pmatrix} D_{XX} & D_{XU} \\ D_{UX} & D_{UU} \end{pmatrix} (F^*(P)+G)\Big(\phi_x,\phi_u\Big)^{\otimes 2} \\
=&D_{XX}(F^*(P)+G)(\phi_x,\phi_x)+2D_{XU}(F^*(P)+G)(\phi_x,\phi_u)+D_{UU}(F^*(P)+G)(\phi_u,\phi_u) \\
=&D_{XX}(F^*(P)+G)(\phi_x,\phi_x)+\langle 2D_{XU}(F^*(P)+G)(\phi_x),\phi_u\rangle_{\mathcal{H}^{n_u}}+\langle D_{UU}(F^*(P)+G)(\phi_u),\phi_u\rangle_{\mathcal{H}^{n_u}} \\
=&D_{XX}(F^*(P)+G)(\phi_x,\phi_x)+\langle 2D_{XU}(F^*(P)+G)(\phi_x)+D_{UU}(F^*(P)+G)(\phi_u),\phi_u\rangle_{\mathcal{H}^{n_u}} \\
=&\Big\langle D_{XX}(F^*(P)+G)(\phi_x),\phi_x\Big\rangle_{\mathcal{H}^{n_x}} \\
&\quad+\Big\langle D_{UU}(F^*(P)+G)\Big(\phi_u+\Big[D_{UU}^{-1}D_{XU}\Big](F^*(P)+G)(\phi_x)\Big) \\
&\quad\quad,\phi_u+\Big[D_{UU}^{-1}D_{XU}\Big](F^*(P)+G)(\phi_x)\Big\rangle_{\mathcal{H}^{n_u}}-\Big\langle\phi_x,\Big[D_{UX}D_{UU}^{-1}D_{XU}\Big](F^*(P)+G)(\phi_x)\Big\rangle_{\mathcal{H}^{n_x}} \\
=&\Big\langle\Big[D_{XX}-D_{UX}D_{UU}^{-1}D_{XU}\Big](F^*(P)+G)(\phi_x),\phi_x\Big\rangle_{\mathcal{H}^{n_x}} \\
&\quad+\Big\langle D_{UU}(F^*(P)+G)\Big(\phi_u+\Big[D_{UU}^{-1}D_{XU}\Big](F^*(P)+G)(\phi_x)\Big) \\
&\quad\quad,\phi_u+\Big[D_{UU}^{-1}D_{XU}\Big](F^*(P)+G)(\phi_x)\Big\rangle_{\mathcal{H}^{n_u}}
\end{aligned}
\tag{5.3.52}
$$

Recall that $D_{UU}(F^*(P)+G)$ is positive by $(A.2)$. We can choose

$$
\phi_u=-[D_{UU}^{-1}D_{XU}](F^*(P)+G)(\phi_x)
$$

and the second term on the right hand side attains its minimum and vanishes; while the left hand side is positive by $(A.1)$. Since ϕ_x is arbitrary, we conclude that the Schur complement is semi-positive. $\square$

$(A.4)$ The bilinear operator $D_U f D_U F^*$ on $\mathcal{H}^{n_x}$ is positive and bounded:

$$
\lambda\|\phi_x\|^2_{\mathcal{H}^{n_x}}\le\langle D_UF(X,U)D_UF^*(X,U)(\phi_x),\phi_x\rangle_{\mathcal{H}^{n_x}}\le\Lambda\|\phi_x\|^2_{\mathcal{H}^{n_x}},\quad\forall X\in\mathcal{H}^{n_x},U\in\mathcal{H}^{n_u}.
\tag{5.3.53}
$$

With assumption $(A.2)$, we can rewrite Equations (5.3.42) and (5.3.43) respectively:

$$
0=\Big[D_{UU}^{-1}D_{XU}\Big]\Big(F^*(P)+G\Big)(\phi_x)+D_XU(X,P)(\phi_x).
\tag{5.3.54}
$$

and

$$0 = D_{UU}^{-1}\Big(F^*(P) + G\Big)D_U F^*(\gamma_x) + D_P U(X,P)(\gamma_x). \tag{5.3.55}$$

5.4 Boundedness of Jacobian Flow

Recall that $\xi \in \mathcal{H}^{n_x} \mapsto X_s, P_s \in \mathcal{H}^{n_x}$. Let $\phi_x, \gamma_x \in \mathcal{H}^{n_x}$ be test functions. We first consider the case with $D_{\mathbb{L}}F \equiv 0 \equiv D_{\mathbb{L}}G$. By differentiate (5.2.37) with respect to the initial random variable ξ, we obtain the following Jacobian flow system specified by $\xi \in \mathcal{H}^{n_x} \to DX_s, DP_s \in \mathcal{L}(\mathcal{H}^{n_x}, \mathcal{H}^{n_x})$:

$$\begin{cases} dDX_s(\phi_x) = \Big(D_X F_s\Big(DX_s(\phi_x)\Big) + D_U F_s\Big(D_X U_s\Big(DX_s(\phi_x)\Big)\Big) + D_U F_s\Big(D_P U_s\Big(DP_s(\phi_x)\Big)\Big)\Big)ds, \\ DX_t(\phi_x) = \phi_x; \\ -dDP_s(\gamma_x) = \Big[D_{XX}\Big(F_s^*(P_s) + G_s\Big)\Big(DX_s(\gamma_x)\Big) + D_{UX}\Big(F_s^*(P_s) + G_s\Big)\Big(D_X U_s\Big(DX_s(\gamma_x)\Big)\Big) \\ \qquad + D_{UX}\Big(F_s^*(P_s) + G_s\Big)\Big(D_P U_s\Big(DP_s(\gamma_x)\Big)\Big) + D_X F_s^*(DP_s(\gamma_x))\Big]ds \\ \qquad - \Big[D_{X\mathbb{L}}\Big(F_s^*(P_s) + G_s\Big)\Big(DX_s(\gamma_x)\Big) + D_{U\mathbb{L}}\Big(F_s^*(P_s) + G_s\Big)\Big(D_X U_s\Big(DX_s(\gamma_x)\Big)\Big) \\ \qquad + D_{U\mathbb{L}}\Big(F_s^*(P_s) + G_s\Big)\Big(D_P U_s\Big(DP_s(\gamma_x)\Big)\Big) + D_{\mathbb{L}} F_s^*(DP_s(\gamma_x))\Big] - DZ_s(\gamma_x)dW_s, \\ DP_T(\gamma_x) = D_{XX}h_T\Big(DX_T(\gamma_x)\Big) - D_{X\mathbb{L}}h_T\Big(DX_T(\gamma_x)\Big). \end{cases} \tag{5.4.56}$$

The following lemma is crucial to the main result in this section.

Lemma 5.4.1. *Suppose that the assumptions (A.1), (A.2) and (A.3) hold, then*

$$\|DP_t(\phi_x)\|_{\mathcal{H}^{n_x}} \geq \int_t^T \lambda^2 \|DP_s(\phi_x)\|^2_{\mathcal{H}^{n_x}} ds, \quad \|\phi_x\|_{\mathcal{H}^{n_x}} \leq 1. \tag{5.4.57}$$

Proof. Consider the inner product process:

$$
\begin{aligned}
& d\langle DX_s(\phi_x), DP_s(\phi_x)\rangle_{\mathcal{H}^{n_x}} \\
=& \Big\langle D_X F_s\Big(DX_s(\phi_x)\Big) + D_U F_s\Big(D_X U_s\Big(DX_s(\phi_x)\Big)\Big) + D_U F_s\Big(D_P U_s\Big(DP_s(\phi_x)\Big)\Big), DP_s(\phi_x)\Big\rangle_{\mathcal{H}^{n_x}} ds \\
& - \Big\langle DX_s(\phi_x), \Big[D_{XX} F_s^*\Big(DX_s(\phi_x), P_s\Big) + D_{XX} G_s\Big(DX_s(\phi_x)\Big) \\
& \qquad + D_{UX} F_s^*\Big(D_X U_s\Big(DX_s(\phi_x)\Big), P_s\Big) + D_{UX} G_s\Big(D_X U_s\Big(DX_s(\phi_x)\Big)\Big) \\
& \qquad + D_{UX} F_s^*\Big(D_P U_s\Big(DP_s(\phi_x)\Big), P_s\Big) + D_{UX} G_s\Big(D_P U_s\Big(DP_s(\phi_x)\Big)\Big) \\
& \qquad + D_X F_s^*(DP_s(\phi_x))\Big]\Big\rangle_{\mathcal{H}^{n_x}} ds \\
& - \Big\langle DX_s(\phi_x), DZ_s(\phi_x) dW_s\Big\rangle_{\mathcal{H}^{n_x}} \\
=& \Big\langle D_X U_s\Big(DX_s(\phi_x)\Big) + D_P U_s\Big(DP_s(\phi_x)\Big), D_U F_s^*(DP_s(\phi_x))\Big\rangle_{\mathcal{H}^{n_u}} ds \\
& - \Big\langle DX_s(\phi_x), \Big[D_{XX} F_s^*\Big(DX_s(\phi_x), P_s\Big) + D_{XX} G_s\Big(DX_s(\phi_x)\Big) \\
& \qquad + D_{UX} F_s^*\Big(D_X U_s\Big(DX_s(\phi_x)\Big), P_s\Big) + D_{UX} G_s\Big(D_X U_s\Big(DX_s(\phi_x)\Big)\Big)\Big]\Big\rangle_{\mathcal{H}^{n_x}} ds \\
& - \Big\langle D_{XU} F_s^*\Big(DX_s(\phi_x), P_s\Big) + D_{XU} G_s\Big(DX_s(\phi_x)\Big), D_P U_s\Big(DP_s(\phi_x)\Big)\Big\rangle_{\mathcal{H}^{n_u}} ds
\end{aligned}
\tag{5.4.58}
$$

We can now replace respectively ϕ_x and γ_x by $DX_s(\phi_x)$ and $DP_s(\phi_x)$ in Equation (5.3.46), we have

$$
\begin{aligned}
& d\langle DX_s(\phi_x), DP_s(\phi_x)\rangle_{\mathcal{H}^{n_x}} \\
=& \Big\langle D_P U_s\Big(DP_s(\phi_x)\Big), D_U F_s^*(DP_s(\phi_x))\Big\rangle_{\mathcal{H}^{n_u}} ds \\
& - \Big\langle DX_s(\phi_x), D_{XX} F_s^*\Big(DX_s(\phi_x), P_s\Big) + D_{XX} G_s\Big(DX_s(\phi_x)\Big)\Big\rangle_{\mathcal{H}^{n_x}} ds \\
& - \Big\langle D_{XU} F_s^*\Big(DX_s(\phi_x), P_s\Big) + D_{XU} G_s\Big(DX_s(\phi_x)\Big), D_X U_s\Big(DX_s(\phi_x)\Big)\Big\rangle_{\mathcal{H}^{n_u}} ds.
\end{aligned}
\tag{5.4.59}
$$

Now we replace respectively ϕ_x, γ_x with $DX_s(\phi_x)$, $DP_s(\phi_x)$ in (5.3.47) and (5.3.48).

Equation (5.4.59) becomes

$$\begin{aligned}
&d\langle DX_s(\phi_x), DP_s(\phi_x)\rangle_{\mathcal{H}^{n_x}} \\
=& -\Big\langle D_{UU}F_s^*\Big(D_P U_s(DP_s(\phi_x)), P_s\Big) + D_{UU}G_s\Big(D_P U_s(DP_s(\phi_x))\Big), D_P U_s\Big(DP_s(\phi_x)\Big)\Big\rangle_{\mathcal{H}^{n_u}} ds \\
&- \Big\langle D_{XX}F_s^*\Big(DX_s(\phi_x), P_s\Big) + D_{XX}G_s\Big(DX_s(\phi_x)\Big), DX_s(\phi_x)\Big\rangle_{\mathcal{H}^{n_x}} ds \\
&+ \Big\langle D_{UU}F_s^*\Big(D_X U_s(DX_s(\phi_x)), P_s\Big) + D_{UU}G_s\Big(D_X U_s(DX_s(\phi_x))\Big), D_X U_s(DX_s(\phi_x))\Big\rangle_{\mathcal{H}^{n_u}} ds.
\end{aligned} \tag{5.4.60}$$

Finally, using Equation (5.3.54) and (5.3.55), (5.4.60) becomes

$$\begin{aligned}
&d\langle DX_s(\phi_x), DP_s(\phi_x)\rangle_{\mathcal{H}^{n_x}} \\
=& -\Big\langle D_2F_s^*\Big(DP_s(\phi_x)\Big), D_{UU}^{-1}\Big(F_s^*(P_s) + G_s\Big) D_2F_s^*\Big(DP_s(\phi_x)\Big)\Big\rangle_{\mathcal{H}^{n_u}} ds \\
&- \Big\langle D_{XX}\Big(F_s^*(P_s) + G_s\Big)\Big(DX_s(\phi_x)\Big), DX_s(\phi_x)\Big\rangle_{\mathcal{H}^{n_x}} ds \\
&+ \Big\langle D_{XU}\Big(F_s^*(P_s) + G_s\Big)\Big(DX_s(\phi_x)\Big), D_{UU}^{-1}\Big(F_s^*(P_s) + G_s\Big) D_{XU}\Big(F_s^*(P_s) + G_s\Big)\Big(DX_s(\phi_x)\Big)\Big\rangle_{\mathcal{H}^{n_u}} ds \\
=& -\Big\langle D_{UU}^{-1}\Big(F_s^*(P_s) + G_s\Big) D_U F_s^*\Big(DP_s(\phi_x)\Big), D_U F_s^*\Big(DP_s(\phi_x)\Big)\Big\rangle_{\mathcal{H}^{n_u}} ds \\
&- \Big\langle \Big[D_{XX} - D_{UX}D_{UU}^{-1}D_{XU}\Big]\Big(F_s^*(P_s) + G_s\Big)\Big(DX_s(\phi_x)\Big), DX_s(\phi_x)\Big\rangle_{\mathcal{H}^{n_x}} ds.
\end{aligned} \tag{5.4.61}$$

Applying Lemma 5.3.6 and $(A.2)$, we have

$$d\langle DX_s(\phi_x), DP_s(\phi_x)\rangle_{\mathcal{H}^{n_x}} \leq -\lambda \|D_U F_s^*\Big(DP_s(\phi_x)\Big)\|^2_{\mathcal{H}^{n_u}} ds. \tag{5.4.62}$$

Using $(A.4)$ yields

$$d\langle DX_s(\phi_x), DP_s(\phi_x)\rangle_{\mathcal{H}^{n_x}} \leq -\lambda^2 \|DP_s(\phi_x)\|^2_{\mathcal{H}^{n_x}} ds. \tag{5.4.63}$$

Integrate both sides on $[t, T]$, together with $(A.3)$, we have

$$\begin{aligned}
\langle \phi_x, DP_t(\phi_x)\rangle_{\mathcal{H}^{n_x}} &\geq \langle DX_T(\phi_x), D_{XX}h_T(DX_T(\phi_x))\rangle_{\mathcal{H}^{n_x}} + \int_t^T \lambda^2 \|DP_s(\phi_x)\|^2_{\mathcal{H}^{n_x}} ds \\
&\geq \int_t^T \lambda^2 \|DP_s(\phi_x)\|^2_{\mathcal{H}^{n_x}} ds.
\end{aligned} \tag{5.4.64}$$

Choose ϕ_x such that $\|\phi_x\|_{\mathcal{H}^{n_x}} \leq 1$,

$$\|DP_t(\phi_x)\|_{\mathcal{H}^{n_x}} \geq \int_t^T \lambda^2 \|DP_s(\phi_x)\|^2_{\mathcal{H}^{n_x}} ds, \quad \|\phi_x\|_{\mathcal{H}^{n_x}} \leq 1. \tag{5.4.65}$$

$\square$

We prove the main result in this section:

Theorem 5.4.2. *Suppose that the assumptions* $(A.1 - A.4)$ *hold, then* DP_s *is bounded.*

Proof.

$$\begin{aligned}
&d\|DX_s(\phi_x)\|^2_{\mathcal{H}^{n_x}} \\
=&d\langle DX_s(\phi_x), DX_s(\phi_x)\rangle_{\mathcal{H}^{n_x}} \\
=&2\Big\langle DX_s(\phi_x), D_X F_s\Big(DX_s(\phi_x)\Big) - D_U F_s\Big(\Big[D_{UU}^{-1} D_{XU}\Big]\Big(F_s^*(P_s) + G_s\Big)\Big(DX_s(\phi_x)\Big)\Big)\Big\rangle_{\mathcal{H}^{n_x}} ds \\
&\quad + 2\Big\langle DX_s(\phi_x), -D_U F_s\Big(D_{UU}^{-1}\Big(F_s^*(P_s) + G_s\Big)D_2 F_s^*\Big(DP_s(\phi_x)\Big)\Big)\Big\rangle_{\mathcal{H}^{n_x}} ds \\
\leq&2\Big(\|D_X f\| + \|D_U f[D_{UU}^{-1} D_{XU}](F^*(P) + G)\|\Big)\|DX_s(\phi_x)\|^2_{\mathcal{H}^{n_x}} ds \\
&\quad + \Big(\|D_U f D_{UU}^{-1}(F^*(P) + G) D_U F^*\|\Big)\|DX_s(\phi_x)\|^2_{\mathcal{H}^{n_x}} ds \\
&\quad + \Big(\|D_U f D_{UU}^{-1}(F^*(P) + G) D_U F^*\|\Big)\|DP_s(\phi_x)\|^2_{\mathcal{H}^{n_x}} ds
\end{aligned} \tag{5.4.66}$$

Using Gronwall's inequality, we have

$$\begin{aligned}
\|DX_s(\phi_x)\|^2_{\mathcal{H}^{n_x}} \leq& \Big[\|\phi_x\|^2_{\mathcal{H}^{n_x}} + \Big(\|D_U f D_{UU}^{-1}(F^*(P) + G) D_U F^*\|\Big)\int_t^s \|DP_u(\phi_x)\|^2_{\mathcal{H}^{n_x}} du\Big] \\
&\cdot \exp\Big\{2\Big(\|D_X f\| + \|D_U f[D_{UU}^{-1} D_{XU}](F^*(P) + G)\|\Big)(s-t) \\
&\quad + \Big(\|D_U f D_{UU}^{-1}(F^*(P) + G) D_U F^*\|\Big)(s-t)\Big\} \\
\leq& \Big[\|\phi_x\|^2_{\mathcal{H}^{n_x}} + B\int_t^s \|DP_u(\phi_x)\|^2_{\mathcal{H}^{n_x}} du\Big] e^{(2A+B)(s-t)}
\end{aligned} \tag{5.4.67}$$

for $t \leq s \leq T$. On the other hand,

$$
\begin{aligned}
& d\|DP_s(\phi_x)\|^2_{\mathcal{H}^{n_x}} \\
=& d\langle DP_s(\phi_x), DP_s(\phi_x)\rangle_{\mathcal{H}^{n_x}} \\
=& -2\Big\langle DP_s(\phi_x), \Big[D_{XX} - D_{UX}D_{UU}^{-1}D_{XU}\Big]\Big(F_s^*(P_s) + G_s\Big)\Big(DX_s(\phi_x)\Big) \\
& \qquad - \Big[D_{UX}D_{UU}^{-1}\Big]\Big(F_s^*(P_s) + G_s\Big)D_2F_s^*\Big(DP_s(\phi_x)\Big)\Big) \\
& \qquad + D_XF_s^*(DP_s(\phi_x))\Big\rangle_{\mathcal{H}^{n_x}} ds \\
& + \Big\langle DZ_s(\phi_x)dW_s, DZ_s(\phi_x)dW_s\Big\rangle_{\mathcal{H}^{n_x}}
\end{aligned}
\tag{5.4.68}
$$

We have

$$
\begin{aligned}
& \|DP_s(\phi_x)\|^2_{\mathcal{H}^{n_x}} + \Big\langle DZ_s(\phi_x)dW_s, DZ_s(\phi_x)dW_s\Big\rangle_{\mathcal{H}^{n_x}} \\
=& \|DP_T(\phi_x)\|^2_{\mathcal{H}^{n_x}} + 2\int_s^T \Big\langle DP_u(\phi_x), \Big[D_{XX} - D_{UX}D_{UU}^{-1}D_{XU}\Big]\Big(f_u^*(P_u) + G_u\Big)\Big(DX_u(\phi_x)\Big)\Big\rangle_{\mathcal{H}^{n_x}} du \\
& + 2\int_s^T \Big\langle DP_u(\phi_x), \Big[D_{UX}D_{UU}^{-1}\Big]\Big(f_u^*(P_u) + G_u\Big)D_2f_u^*\Big(DP_u(\phi_x)\Big)\Big\rangle_{\mathcal{H}^{n_x}} du \\
& - 2\int_s^T \Big\langle DP_u(\phi_x), D_XF_u^*(DP_u(\phi_x))\Big\rangle_{\mathcal{H}^{n_x}} ds
\end{aligned}
\tag{5.4.69}
$$

Hence

$$
\begin{aligned}
& \|DP_s(\phi_x)\|^2_{\mathcal{H}^{n_x}} \\
\leq& \|D_{XX}h_T\Big(DX_T(\phi_x)\Big)\|^2_{\mathcal{H}^{n_x}} + \int_s^T C\|DX_u(\phi_x)\|^2_{\mathcal{H}^{n_x}} + (C+2D)\|DP_u(\phi_x)\|^2_{\mathcal{H}^{n_x}} du.
\end{aligned}
\tag{5.4.70}
$$

Again, we use the Gronwall's inequality:

$$
\|DP_s(\phi_x)\|^2_{\mathcal{H}^{n_x}} \leq \Big\{\|D_{XX}h_T\Big(DX_T(\phi_x)\Big)\|^2_{\mathcal{H}^{n_x}} + \int_s^T C\|DX_u(\phi_x)\|^2_{\mathcal{H}^{n_x}} du\Big\}e^{(C+2D)(T-s)}.
\tag{5.4.71}
$$

We consider the following estimates:

$$
\|DP_t(\phi_x)\|_{\mathcal{H}^{n_x}} \geq \int_t^T \lambda^2\|DP_s(\phi_x)\|^2_{\mathcal{H}^{n_x}} ds
\tag{5.4.72}
$$

$$\|DX_s(\phi_x)\|^2_{\mathcal{H}^{n_x}} \leq \Big[\|\phi_x\|^2_{\mathcal{H}^{n_x}} + B\int_t^s \|DP_u(\phi_x)\|^2_{\mathcal{H}^{n_x}} du\Big] e^{(2A+B)(s-t)}, \tag{5.4.73}$$

$$\|DP_s(\phi_x)\|^2_{\mathcal{H}^{n_x}} \leq \Big\{\|D_{XX}h_T\Big(DX_T(\phi_x)\Big)\|^2_{\mathcal{H}^{n_x}} + \int_s^T C\|DX_u(\phi_x)\|^2_{\mathcal{H}^{n_x}} du\Big\} e^{(C+2D)(T-s)}. \tag{5.4.74}$$

Combining (5.4.72) and (5.4.73), we have

$$\|DX_s(\phi_x)\|^2_{\mathcal{H}^{n_x}} \leq \Big[\|\phi_x\|^2_{\mathcal{H}^{n_x}} + \frac{B}{\lambda^2}\|DP_t(\phi_x)\|_{\mathcal{H}^{n_x}}\Big] e^{(2A+B)(s-t)} \tag{5.4.75}$$

Combing (5.4.74) and (5.4.75), we have

$$\begin{aligned}
&\|DP_s(\phi_x)\|^2_{\mathcal{H}^{n_x}} \\
\leq& \Big\{\|D_{XX}h_T\|\Big[\|\phi_x\|^2_{\mathcal{H}^{n_x}} + \frac{B}{\lambda^2}\|DP_t(\phi_x)\|_{\mathcal{H}^{n_x}}\Big] e^{(2A+B)(T-t)} \\
&\qquad + C\Big[\|\phi_x\|^2_{\mathcal{H}^{n_x}} + \frac{B}{\lambda^2}\|DP_t(\phi_x)\|_{\mathcal{H}^{n_x}}\Big] \int_s^T e^{(2A+B)(u-t)} du\Big\} e^{(C+2D)(T-s)} \\
\leq& K\Big[\|\phi_x\|^2_{\mathcal{H}^{n_x}} + \|DP_t(\phi_x)\|_{\mathcal{H}^{n_x}}\Big] e^{2(T-t)}.
\end{aligned} \tag{5.4.76}$$

Finally,

$$\|DP_s(\phi_x)\|_{\mathcal{H}^{n_x}} \leq K\Big[\frac{\|\phi_x\|^2_{\mathcal{H}^{n_x}}}{\|DP_s(\phi_x)\|_{\mathcal{H}^{n_x}}} + 1\Big] e^{2(T-t)}. \tag{5.4.77}$$

Clearly,

$$\|DP_s(\phi_x)\|_{\mathcal{H}^{n_x}} \leq K\Big[\|\phi_x\|^2_{\mathcal{H}^{n_x}} + 1\Big] e^{2(T-t)} \vee 1. \tag{5.4.78}$$

□

5.5 Future Extension

The main result in this chapter is to give a global (in time) bound of the Jacobian flow of the McKean Vlasov Forward Backward Stochastic Differential Equation resulted from Mean Field Games. This bound is crucial in constructing the global solution of the FBSDE piece-wisely and backwardly from the terminal time T, as it stabilize the estimates in the induction argument.

The steps to complete the proof of uniqueness and existence of system (5.2.37) are outlined as below

1. Revise the bound of the Jacobian Flow under the relaxed condition that $D_{\mathbb{L}}F, D_{\mathbb{L}}G \neq 0$

2. Show that the Jacobian Flow (DX, DP) in (5.4.56) admits a unique (global) solution for any given X, P;

3. Find a small time Δ, such that for all initial ξ and $T < \Delta$, (X, P) in (5.2.37) admits a unique (local) solution;

4. Introduce a time partition $\{t_j\}_{j=1}^{n}$ on $[0, T]$, such that $|t_j - t_{j-1}| < \Delta$, prove inductively that if (X, P) admit a unique solution on $[t_{j^*}, T]$, then so does $[t_{j^*-1}, T]$.

The first task can be accomplished by controlling the norm of $D_{\mathbb{L}}F$ and $D_{\mathbb{L}}G$. Thanks to the monotonicity condition implicitly proved in (5.4.63), the second step can be done using classical results in Forward Backward Stochastic Differential Equation. The third step is relatively standard. Finally, with the revised bound of the Jacobian flow, the induction in the final step is valid.

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