Applications Of Mirror Symmetry To The Classification of Fano Varieties

A thesis presented for the degree of
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by

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Declaration of Originality

I certify that the contents of this dissertation are the product of my own work. Ideas or quotations from the work of others are appropriately acknowledged.

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Abstract

In this dissertation we discuss two new constructions of Fano varieties, each directly inspired by ideas in Mirror Symmetry. The first recasts the Fanosearch programme for surfaces laid out in [**1**, **21**] in terms of a construction related to the SYZ conjecture. In particular we construct $\mathbb{Q}$-Gorenstein smoothings of toric varieties via an application of the Gross–Siebert algorithm [**53**, **59**] to certain *affine manifolds*. We recover the theory of *combinatorial mutation*, which plays a central role in [**21**], from these affine manifolds.

Combining this construction and the work of Gross–Hacking–Keel [**54**, **55**] on log Calabi–Yau surfaces we produce a cluster structure on the mirror to a log del Pezzo surface proposed in [**1**, **21**]. We exploit the cluster structure, and the connection to toric degenerations, to prove two classification results for Fano polygons.

The cluster variety is equipped with a superpotential defined on each chart by a *maximally mutable Laurent polynomial* of [**1**]. We study an enumerative interpretation of this superpotential in terms of tropical disc counting in the example of the projective plane (with a general boundary divisor).

In the second part we develop a new construction of Fano toric complete intersections in higher dimensions. We first consider the problem of finding torus charts on the Hori–Vafa/Givental model, adapting the approach taken in [**96**]. We exploit this to identify 527 new families of four-dimensional Fano manifolds.

We then develop an inverse algorithm, *Laurent Inversion*, which decorates a Fano polytope P with additional information used to construct a candidate ambient space for a complete intersection model of the toric variety defined by P. Moving in the linear system defining this complete intersection allows us to construct new models of known Fano manifolds, and also to construct new examples of Fano manifolds from conjectured mirror Laurent polynomials.

We use this algorithm to produce families simultaneously realising certain collections of 'commuting' mutations, extending the connection between polytope mutation and deformations of toric varieties.

Ad Deum qui lætificat juventutem meam

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Contents

CHAPTER 1

Introduction

1.1. Objectives

In this dissertation we develop a program which incorporates various techniques from Mirror Symmetry and apply this program to problems related to the classification of Fano manifolds.

This program builds on recent advances made by the Fanosearch group (Coates, Corti, Kasprzyk *et al.*) at Imperial College, which have led to a completely new approach to Fano classification. This perspective, laid out in [**21**], is based on a formulation of Mirror Symmetry for Fano manifolds explored in [**21**, **50**, **96**]. In these papers the authors propose that Mirror Symmetry for Fano manifolds can be understood in terms collections of mirror-dual *Laurent polynomials*. While a good deal of progress has been made in this area [**1**, **4**, **22**, **30**, **93**], there are many important open questions which require a deeper understanding of the situation. We shall particularly focus on two of these questions.

- What class of Laurent polynomials is mirror to Fano varieties?
- Given a Laurent polynomial f conjecturally mirror to X, how can one *construct* the variety X?

Robust solutions to these would have far reaching consequences. Indeed, given that candidate mirror-dual Laurent polynomials can be generated relatively easily, these solutions would provide powerful tools to approach the classification of Fano 4-folds.

1.2. The Main Actors: Fano manifolds and Mirror Symmetry

Fano varieties are basic building blocks in algebraic geometry, both in the sense of the Minimal Model Program [**15**, **97**], and as the source of many explicit constructions. In dimension one there is only one Fano manifold, the Riemann sphere $\mathbb{P}^1$. In dimension two there are the famous *del Pezzo surfaces*: $\mathbb{P}^1 \times \mathbb{P}^1$ and the blow up of the projective plane $\mathbb{P}^2$ in $0 \leq k < 9$ general points. In dimension three there are 105 deformation families of Fano manifolds; the classification here was completed by Mori–Mukai in the 1990s, building on work by Fano in the 1930s and Iskovskikh in the 1970s. It is known there are only finitely many deformation families of Fano manifolds in every dimension, but their classification is well beyond the reach of traditional methods.

Mirror Symmetry, on the other hand, is a very modern area of mathematics, incorporating many areas of active research. In its original physical formulation Mirror Symmetry proposes the equivalence of the A-twist and B-twist of the superconformal field theory attached to a non-linear sigma model with a Kähler manifold target X with the B-twist and A-twist respectively from a Landau–Ginzburg model with a *mirror* target $\check{X}$. Reconstructing rigorous mathematics from this deep physical phenomenon has produced a wealth of remarkable results. Perhaps most famously was the prediction, via Mirror Symmetry, of the virtual number of rational curves of arbitrary degree on a quintic Calabi–Yau threefold by Candelas–de la Ossa–Green–Parkes [**17**], later proved to be correct by Givental [**48**], as well as in the series of papers of Lian–Liu–Yau [**82**–**85**].

Since then, the great industry to formulate the Mirror Symmetry correspondence in a precise mathematical framework has led to the pursuit of many new fields of research. The most dramatic of these is the Homological Mirror Symmetry conjecture of Kontsevich [**77**] which 'lifts' mirror symmetry from a conjecture concerning variations of Hodge structure to a conjecture between A_∞ categories (a derived category of sheaves on one side, and the Fukaya category on the other). Also prominent is the conjectural formulation of Mirror Symmetry as a duality of *special Lagrangian torus fibrations*, first made precise in the *Strominger–Yau–Zaslow (SYZ) conjecture* [**105**]. This is a very geometric interpretation of Mirror Symmetry, and while its naïve versions have been shown to be too much to hope for [**69**], attempts to prove 'asymptotic' versions of the conjecture have led to a great deal of beautiful mathematics. In particular the central technical tools used in this dissertation were developed by Gross–Siebert [**58**,**59**], building on work of Kontsevich–Soibelman [**79**] and Fukaya [**42**], to establish exactly such a geometric Mirror Symmetry conjecture.

While string theory predicts that Mirror Symmetry should have important consequences for Calabi–Yau manifolds there has also been a great interest in formulations of Mirror Symmetry in a more general setting. Most importantly for us, Mirror Symmetry was extended to the case of Fano manifolds by Givental [**45**–**47**], Hori–Vafa [**62**] and Kontsevich [**76**], as well as in the work of Eguchi–Hori–Xiong [**31**], and Batyrev [**12**]. Developing this deep conjecture, a view of Mirror Symmetry between Fano manifolds and Landau–Ginzburg models has been developed in [**49**] and in [**95**]. The results and examples obtained from this perspective form the starting point for our study. Having made a Hodge theoretic mirror conjecture (in this case a comparison of the Picard–Fuchs operator and quantum differential operator) it is natural to ask what the analogues of these more sophisticated formulations of Mirror Symmetry are.

There is a version of the Homological Mirror Symmetry conjecture for Fano manifolds, in which the Fukaya category associated to the Landau–Ginzburg model is now the *Fukaya–Seidel* category (see, [**101**]). However this is only defined if the superpotential is sufficently non-degenerate, that is, it defines a Lefschetz fibration. If this holds, the Fukaya–Seidel category is conjecturally equivalent to the derived category of the mirror-dual Fano variety. This has been achieved in some interesting examples, for example [**8**,**9**], building on [**99**,**100**]. However, in the vast majority of the cases we consider, we are *not* permitted to assume the superpotential defines a Lefschetz fibration, making the correct conjecture hard to formulate. In the other direction, the derived category of matrix factorizations is conjecturally equivalent to the Fukaya category of the mirror-dual Fano variety, here there is also interesting progress [**103**].

For the SYZ conjecture the picture is more optimistic, and in one sense is the subject of this dissertation. Auroux studies the case of a special Lagrangian torus fibration in the complement of an anti-canonical divisor directly [**6**, **7**]. Though this too is inaccessible in general, the large programmes used to establish asymptotic versions of the SYZ conjecture do have versions in this setting. In particular we shall study Fano manifolds and their mirrors via the Gross–Siebert programme, taking particular inspiration from [**18**, **53**, **59**].

1.3. Overview

1.3.1. The Surface Case. Chapters 3, 4 and 5 of this dissertation are devoted to studying Mirror Symmetry for log del Pezzo surfaces which admit a toric degeneration. Even in this relatively simple setting there are many unanswered questions; log del Pezzo surfaces have only been classified for Gorenstein index ≤ 3 [**5**, **37**]. In Chapter 3 we lay out the full form of our program for surfaces, taking advantage of the simpler formulations of the Gross–Siebert algorithm in this setting. In particular we prove the following result, constructing $\mathbb{Q}$-Gorenstein deformations of toric Fano surfaces by perturbing the affine structure of a polygon and applying the Gross–Siebert algorithm to this family of affine structures. This theorem involves a number of definitions which we introduce in Chapter 3.

THEOREM 1.3.1. *Given a Fano polygon P denote its polar polygon $Q := P^\circ$. Let $\mathscr{P}$ be the polygonal decomposition via the spanning fan of Q and let s be trivial gluing data. From this data we may form a flat family $\mathcal{X}_Q \to \operatorname{Spec}\mathbb{C}[\alpha][\![t]\!]$ such that:*

- *Fixing $t = 0$, the restriction of $\mathcal{X}_Q$ over $\operatorname{Spec}\mathbb{C}[\alpha]$ is $X_0(Q, \mathscr{P}, s) \times \operatorname{Spec}\mathbb{C}[\alpha]$ a union of toric varieties defined in Section 3.4.*
- *Fixing $\alpha = 0$ the restriction of $\mathcal{X}_Q$ over $\operatorname{Spec}\mathbb{C}[\![t]\!]$ is the Mumford degeneration of the pair $(Q, \mathscr{P})$*[1]*.*
- *For each boundary zero-stratum p of $X_0(Q, \mathscr{P}, s)$ there is neighbourhood U_p in $\mathcal{X}_Q$ isomorphic to a family $\mathcal{Y} \to \operatorname{Spec}\mathbb{C}[\alpha][\![t]\!]$ obtained by first taking a one-parameter $\mathbb{Q}$-Gorenstein smoothing of the singularity of X_Q at p and taking a simultaneous maximal degeneration of every fiber in a formal parameter t.*

The family $\mathcal{Y}$ appearing in the fourth point in fact has a simple general form. We see in Lemma 2.3.1 that given a cyclic quotient surface sinuglarity $\frac{1}{n}(1, q)$ any $\mathbb{Q}$-Gorenstein deformation has the form

$$\{xy + z^{w_0} f_m(z^r, \alpha) = 0\} \subset \frac{1}{r}(1, wa - 1, a) \times \operatorname{Spec}\mathbb{C}[\alpha],$$

[1]This is a flat family with general fiber X_Q and special fiber the union of toric varieties with moment polyopes given by the decomposition $\mathscr{P}$, see Definition 3.4.10.

where $f_m \in \mathbb{C}[z,\alpha]$ has z-degree m and the integers w, a, r are defined in Section 2.3. The family $\mathcal{Y}$ is obtained by adding a formal parameter t,

$$\{xy + tz^{w_0} f_m(z^r, \alpha) = 0\} \subset \frac{1}{r}(1, wa-1, a) \times \operatorname{Spec}\mathbb{C}[\alpha][\![t]\!].$$

The formality of the parameter t is obviously not required here, but appears in the construction of the global family. Indeed, the use of an order-by-order scattering process means that, outside of certain specific cases, we are unable to write down explicit expressions for the general fibers of the toric degenerations we consider. One particularly striking case in which this is possible is the case there is only a single singularity in the affine structure; analysing this case leads us to recover a theorem of Ilten ([**64**]):

Theorem 1.3.2. *For any combinatorial mutation from P to P' there is a polygonal decomposition of the polar polygons Q and Q' given by the domains of linearity of the mutation between Q and Q'. There is a flat family $\mathcal{X} \to \mathbb{C}^2$ such that restricting to either co-ordinate line produces the Mumford degeneration of X_P and $X_{P'}$ determined by these decompositions respectively.*

We construct this from a family of affine manifolds in which a single singularity traverses its mondromy invariant line. We refer to this family of affine manifolds as the *tropical Ilten family*. The Ilten pencil, which has base $\mathbb{P}^1$, is obtained from the family in Theorem 1.3.2, which has base $\mathbb{C}^2$, by taking the quotient by radial rescaling.

Following the ideas of [**18**] we expect mirror Laurent polynomials to appear from tropical disc counts. Chapter 4 is devoted to a computation of all the Laurent polynomial mirrors to $\mathbb{P}^2$ (known to be in bijection with integral solutions to the Markov equation) as tropical disc counts by considering an infinite collection of *scattering diagrams* on an affine manifold. In particular we prove the following result.

Theorem 1.3.3. *Let Q be the moment polygon for $\mathbb{P}^2$ polarised by $-K_{\mathbb{P}^2}$. Consider the toric degeneration given by fixing a non-zero value of α in the family produced by Theorem 1.3.1 applied to Q, corresponding to a particular choice of log structure on $X_0(Q, \mathscr{P}, s)$ and affine manifold B_Q. Passing to a* Legendre dual *affine manifold $B_Q^\vee$ we can construct a* compatible structure *$\mathscr{S}$ via the construction of Gross–Siebert in Section 3.5. There is a domain $U = \bigcup_{k\geq 0} U_k \subset B_{\mathbb{P}^2}^\vee$ such that:*

- *For all $j \geq k$ $\mathscr{P}_j$ restricted to U_k is a constant union of chambers $\mathfrak{u} \in \mathrm{Chambers}(\mathscr{S}, k)$.*[2]
- *Each $\mathfrak{u} \in \mathrm{Chambers}(\mathscr{S}, k)$ such that $\mathfrak{u} \subset U_k$ for some $k \geq 0$ is a triangle similar to a Fano polygon $P_\mathfrak{u}$ defined by the spanning fan of the central fiber of a toric degeneration of $\mathbb{P}^2$.*
- *The union of the support of rays of $\mathscr{S}^k$ for all $k \geq 0$ restricted to $B_Q^\vee \backslash U$ is dense.*

[2]The decompositions $\mathscr{P}_k$ and the set of chambers $\mathrm{Chambers}(\mathscr{S}, k)$ are defined using the compatible structure $\mathscr{S}$.

Combining this with the results of [**18**] we can recover all the Laurent polynomial mirrors to $\mathbb{P}^2$.

THEOREM 1.3.4. *The* tropical superpotential *(defined in Section 4.4)* $W^k_{\omega,\tau,\mathfrak{u}}$ *is* manifestly algebraic*, in sense of [**18**], and may be identified with a maximally mutable Laurent polynomial with Newton polygon* $P_{\mathfrak{u}}$*.*

In Chapter 5 we make precise the connection between cluster algebras and mutations of polygons. In particular given a Fano polygon P we define a quiver Q_P and cluster algebra $\mathcal{C}_P$ such that passing from P to Q_P commutes with mutation, while a mutation of a seed in $\mathcal{C}_P$ determines a mutation of P. We also see that Laurent polynomials admitting all possible algebraic mutations (as defined in Chapter 2) are elements of the upper cluster algebra of $\mathcal{C}_P$. We use these results to prove a finite type classification for Fano polygons.

THEOREM 1.3.5. *The mutation class of* P *is finite if and only if* Q_P *is mutation equivalent to one of the following types:*

- $(A_1)^n$*, which we refer to as type* I_n*.*
- A_2*, which we refer to as type* II*.*
- A_3*, which we refer to as type* III*.*
- D_4 *which we refer to as type* IV*.*

These connections with cluster algebras provide concrete connections to the progam of [**54**, **55**] and we expect completing this connection will settle an important conjecture (Conjecture A) from [**1**] concerning the boundary of the moduli stack of del Pezzo orbifolds.

1.3.2. The Complete Intersection Case. Most of the constructions and results in the chapters on the surface case do not generalise directly to higher dimensions. However, many of definitions do extend and produce analogous, if more complicated, structures. In particular, the notion of *maximally mutable* Laurent polynomial defined in the surface case in [**1**] extends, following [**71**], to higher dimensions. Using this we can already easily produce conjectural mirror-dual Landau–Ginzburg models to Fano varieties. The natural question which emerges is then,

Given a Laurent polynomial f*, conjecturally mirror-dual to a Fano variety* X*, how can one construct the variety* X *from* f*?*

Given that X should be provably mirror-dual to f it is logical to look first among the toric complete intersections. Indeed, this is a rare setting in which mirror-duality between f and X may actually be proved[3], using the Quantum Lefschetz Theorem ([**27**]).

To begin to answer this question we first define a technique, the *Przyjalkowski Method*, for obtaining a torus chart on the mirror-dual *Givental/Hori–Vafa* Landau–Ginzburg model for a Fano complete intersection X. Using this we can classify Fano 4-fold complete intersections

[3]For example, in the sense defined in Section 2.1

in smooth Fano 8-folds, in particular we have the result (from the work [**20**] joint with T. Coates and A. Kasprzyk).

THEOREM 1.3.6. *There are 738 four dimensional Fano manifolds which appear as complete intersections in smooth toric manifolds Y of dimension ≤ 8 by nef line bundles $L_1, \cdots L_c$ such that $-K_Y - \sum_{i=1}^{k} L_i$ is ample.*

Of these 738 there are 527 'new' Fano fourfolds, in particular, those of Fano index 1 which are neither smooth toric varieties nor products. We also consider how, via degenerations of the ambient space, this technique may be extended to complete intersections in homogeneous spaces. We then present a simple form of an inverse algorithm, *Laurent Inversion*, from which one may attempt to reconstruct a Fano variety X starting from a Laurent polynomial f. In particular we define a combinatorial decoration of a polytope P called a *scaffolding* in Chapter 6. From a scaffolding one can produce a complete intersection model for X_P in a toric ambient space, we exhibit an example where the general fibre of this linear system is smooth, and thus show that this technique produces potentially unknown Fano fourfolds. In particular all 738 examples from Theorem 1.3.6 arise in such a way.

CHAPTER 2

Fano Manifolds and Mirror Symmetry

In this chapter we provide the background material on Mirror Symmetry required for the later sections. There are a wealth of constructions and conjectures in Mirror Symmetry, and we shall focus on two specific programmes within Mirror Symmetry.

The first of these is the surprising conjecture, made in [**21**], that a Fano manifold is mirror-dual to a collection of Laurent polynomials f. We begin this chapter by describing the sense in which [**21**] conjectures that a mirror correspondence holds. Based on a large quantity of experimental data, as well as some general observations about the invariants Mirror Symmetry is conjectured to equate, a theory of *mutations* of Laurent polynomials has developed around this conjectural correspondence. We shall recall a number of the definitions and basic results of this theory from the papers [**4**, **21**].

As discussed in the Introduction, the program we lay out in this dissertation places these observations in a geometric framework. This framework is built from techniques and constructions derived from the study of another Mirror Symmetry conjecture - the Strominger–Yau–Zaslow (SYZ) conjecture [**105**]. We very briefly outline the main ideas appearing in this conjecture, how one might formulate such a conjecture for Fano manifolds and how it might be explored using the Gross–Siebert programme.

2.1. Mirror Symmetry for Fano Manifolds

The central observation in [**21**] is that Mirror Symmetry for Fano manifolds may be understood very concretely by restricting the mirror-dual Landau–Ginzburg model to a *Laurent Polynomial*. The existence of such a torus chart is not guaranteed, but experimentally and conjectually these charts are in bijection with toric degenerations of the given Fano manifold. Indeed the Newton polytope of the corresponding Laurent polynomial determines a toric variety which is the candidate for the central fiber of this degeneration. In this section we formulate what it means for a Fano manifold X and a Laurent polynomial f to be *mirror-dual*. Given the rich structures present there are are number of ways of phrasing this mirror conjecture, and we shall adopt the simple approach taken in [**21**]. Indeed, in order to state the mirror correspondence of [**21**] we shall only require a pair of local systems, defined by the Gromov–Witten theory X and periods of the fibration defined f respectively.

2.1.1. A pair of local systems. First we consider a 'B-model' local system for f. Recall from [**21**] that given a Laurent polynomial f, its *classical period* is

$$\pi_f(t) := \left(\frac{1}{2\pi i}\right)^n \int_\Gamma \frac{1}{1-tf}\frac{dx_1}{x_1}\wedge\cdots\wedge\frac{dx_n}{x_n}$$

where Γ is the torus $\{|x_i| = 1 | 1 \leq i \leq n\}$.

THEOREM 2.1.1 ([**21**, Theorem 3.2]). *The classical period π_f is in the kernel of a polynomial differential operator $L \in \mathbb{C}\langle t, D\rangle$ where $D = t\frac{d}{dt}$.*

DEFINITION 2.1.2. For operators L such that $L\pi_f \cong 0$ write

$$L = \sum_{j=0}^{k} p_j(t)D^j$$

The *Picard–Fuchs operator* L_f is the unique (up to multiplication by a constant) operator such that k is as small as possible and having fixed k the degree of p_k is as small as possible.

REMARK 2.1.3. Functions in the kernel of L_f form a local system, a summand of the variation of Hodge structure $R^{n-1}f_!\mathbb{Z}_{(\mathbb{C}^\star)^n}$.

REMARK 2.1.4. Given a Laurent polynomial f the power series expansion of π_f (the *period sequence*) is easily computed:

$$\pi_f(t) = \sum_{m\geq 0} c_m t^m$$

Where $c_m = \mathrm{coeff}_1(f^m)$. In practice the compution of L_f involves using the above formula to compute tems of $\pi_f(t)$, guessing a recursion relation of the form $\sum P_k(m-k)c_{m-k} = 0$ and reconstructing $L_f = \sum t^k P_k(D)$.

Having fixed the B-model D-module over a disc, we turn to a summary of the A-model D-module. This is the local system of solutions of the Fourier–Laplace transform (or *regularisation*) $\widehat{Q}_X$ of the *quantum differential operator* Q_X. We summarize the construction of Q_X, following [**21**], from the Gromov–Witten invariants of X in the following steps.

(1) The Gromov–Witten invariants of X define a deformation of the cup product on cohomology, the *quantum cohomology*.
(2) This product induces a connection, the *Dubrovin* connection, on the trivial $H^{ev}(X, \mathbb{C})$ bundle over $\mathbb{C}^\star \otimes H^2(X, \mathbb{Z})$. The WDVV equations imply this connection is flat.
(3) This flat connection defines the quantum D-module of sections of this bundle, and hence a local system of solutions.
(4) Tautologically these are the solutions of a cohomology valued PDE. Restricting to the degree zero component and to the line generated by $[-K_X] \in H^2(X)$, we obtain scalar-valued functions that are annihilated by an algebraic ODE Q_X. In particular the restriction of the degree zero part of the J-function, denoted G_X, is annihilated by Q_X.

We can now state the formulation of *mirror-duality* from [**21**]

DEFINITION 2.1.5. The Laurent polynomial f is *mirror-dual* to the Fano manifold X if $\widehat{Q}_X = L_f$. Here $\widehat{Q}_X$ is the differential operator of lowest order such that $\widehat{Q}_X\widehat{G}_X = 0$ where $\widehat{G}_X$ is the Fourier-Laplace transform of G_X. This regularization is required since Q_X and L_f do not have the same singularities.

EXAMPLE 2.1.6. The prototypical example is that of $\mathbb{P}^2$ which will be studied in much more detail in Chapter 4. There is a well known mirror model due to Givental/Hori–Vafa given by $f(x, y) = x + y + \frac{1}{xy}$ and

$$\pi_f(t) = \sum_{m\geq 0} \frac{(3m)!}{(m!)^3} t^{3m}$$

On the other hand the matrix M of quantum multiplication by $-K_X$ for $\mathbb{P}^2$ is given by

$$\begin{pmatrix} 0 & 0 & 27t^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

From the discussion of the A-model local system, they satisfy a cohomology-valued differential equation, $D(s_0, s_1, s_2) = (s_0, s_1, s_2)M$. Looking at the degree-zero component, we see that s_0 is annihilated by $Q_X := D^3 - 27t^3$ and so $G_X(t) = \sum_{m\geq 0} \frac{1}{(m!)^3} t^{3m}$, which is related to $\pi_f(t)$ by Fourier–Laplace transform.

Having formulated this notion of Mirror Symmetry it is natural to ask if we can find characterisation the local systems $\operatorname{Sol}\widehat{Q}_X$ given by the quantum cohomology of a Fano manifold. One approach to this is to consider the *ramification* of the local system [**50**].

DEFINITION 2.1.7. Let $\mathbb{V}$ be a local system over $\mathbb{P}^1\backslash S$, for a finite set $S \subset \mathbb{P}^1$. Given a point $s \in S$ there is a monodromy operator $T_s \in \operatorname{Aut}\mathbb{V}_x$, for a base point x. The ramification of $\mathbb{V}$ is defined to be,

$$\operatorname{rf}\mathbb{V} = \sum \dim(\mathbb{V}_x/\mathbb{V}_x^{T_s})$$

REMARK 2.1.8. It follows from Euler–Poincaré that,

$$\operatorname{rf}\mathbb{V} - 2\operatorname{rk}\mathbb{V} = h^1(\mathbb{P}^1, j_\star\mathbb{V}) \geq 0$$

where j is the inclusion $j\colon \mathbb{P}^1\backslash S \hookrightarrow \mathbb{P}^1$.

DEFINITION 2.1.9. A local system $\mathbb{V}$ over $\mathbb{P}^1\backslash S$ is *extremal* if it is irreducible, non-trivial and $\operatorname{rf}\mathbb{V} = 2\operatorname{rk}\mathbb{V}$.

Thus we can define an *extremal Laurent polynomial* to be a Laurent polynomial such that $\operatorname{Sol} L_f$ is an extremal local system. In Chapter 6 we shall see examples of four dimensional Fano manifolds and mirror-dual Laurent polynomials; as we shall see, these examples have low ramification.

EXAMPLE 2.1.10. For the eight del Pezzo surfaces with very ample anti-canonical bundle one can obtain a mirror Laurent polynomial with reflexive Newton polygon. Of these six are extremal and two (mirror-dual to $\mathbb{F}_1$ and dP_7) have $\operatorname{rf}\mathbb{V} - 2\operatorname{rk}\mathbb{V} = 1$. See [**21**] for more details.

2.1.2. Mutations of Laurent Polynomials. Observe that according to Definition 2.1.5 Laurent polynomials with the same classical period are regarded as mirror dual to the *same* (if any) Fano manifold. The key ingredient for understanding this many-to-one relation is the notion of *mutation* introduced to explain it, following the notion of *mutation of potential* in [**44**].

DEFINITION 2.1.11. Fix a lattice N, $w \in M := \operatorname{Hom}(N, \mathbb{Z})$ and $F \in \mathbb{C}[w^\perp]$. An *(algebraic) mutation* (or *symplectomorphism of cluster type [**72**]*) is a birational transformation $\theta_{w,F} \colon T_M \dashrightarrow T_M$ is defined by

$$z^n \mapsto z^n \cdot F^{\langle w,n\rangle},$$

where $T_M := \operatorname{Spec}\mathbb{C}[N]$. Given $f \in \mathbb{C}[N]$ such that $\theta^\star(f) \in \mathbb{C}[N]$ we define $\operatorname{mut}_w(f, F) := \theta^\star(f)$ the *mutation of f with weight vector w and factor F*.

REMARK 2.1.12. There are striking parallels to the wall-crossing formulas appearing, for example, in the work of Gross–Siebert [**59**], and with the mutations (in their coordinate free form) of seeds in cluster algebras defined by Fomin–Zelevinsky [**35**].

Since we insist in Definition 2.1.11 that an algebraic mutation maps a Laurent polynomial f to another Laurent polynomial by considering $\operatorname{Newt}(f)$ and $\operatorname{Newt}(\operatorname{mut}_w(f, F))$ mutations define an action on the Newton polygons, leading to a purely combinatorial definition of mutation, to which we now turn.

2.2. Mutation of Fano polygons

In this section we introduce the fundamental combinatorial notions of the Fanosearch program, following the treatment in the joint work [**70**]. The definitions in this section originate from the papers [**3**, **4**].

DEFINITION 2.2.1. A *Fano polygon* P is a convex polytope in $N_\mathbb{Q} := N \otimes_\mathbb{Z} \mathbb{Q}$, where N is a rank-two lattice, with primitive vertices $\mathcal{V}(P)$ in N such that the origin is contained in its strict interior, $\mathbf{0} \in P^\circ$.

A Fano polygon defines a toric surface X_P given by the *spanning fan* of P; that is, X_P is defined by the fan whose cones are spanned by the faces of P. The toric surface X_P has cyclic quotient singularities (corresponding to the cones over the edges of P) and the anti-canonical divisor $-K_X$ is $\mathbb{Q}$-Cartier and ample. Consequently X_P is a toric *del Pezzo surface*.

In [**4**, §3] the concept of mutation for a lattice polytope was introduced. We state it here in the simplified case of a Fano polygon $P \subset N_\mathbb{Q}$ and refer to [**4**] for the general definitions.

2.2.1. Mutation in N. Fixing a lattice N and a Fano polygon P, we now set up the notation required to define a *mutation*. As discussed, this is a combinatorial analogue of the birational map defined in Section 2.1.

Let $w \in M := \mathrm{Hom}(N, \mathbb{Z})$ be a primitive inner normal vector for an edge E of P, so $w : N \to \mathbb{Z}$ induces a grading on $N_{\mathbb{Q}}$ and $w(v) = -\ell_E$ for all $v \in E$, where ℓ_E is the lattice height of E. Define

$$h_{max} := \max\{w(v) \mid v \in P\} > 0 \quad \text{and} \quad h_{min} := -\ell_E = \min\{w(v) \mid v \in P\} < 0.$$

For each $h \in \mathbb{Z}$ we define $w_h(P)$ to be the (possibly empty) convex hull of the lattice points in P at height h,

$$w_h(P) := \mathrm{conv}\{v \in P \cap N \mid w(v) = h\}.$$

By definition $w_{h_{min}}(P) = E$ and $w_{h_{max}}(P)$ is either a vertex or an edge of P. Let $v_E \in N$ be a primitive lattice element of N such that $w(v_E) = 0$, and define $F := \mathrm{conv}\{\mathbf{0}, v_E\}$, a line-segment of unit lattice length parallel to E at height 0. Observe that v_E, and hence F, is uniquely defined up to sign.

Definition 2.2.2. Suppose that for each negative height $h_{min} \leq h < 0$ there exists a (possibly empty) lattice polytope $G_h \subset N_{\mathbb{Q}}$ satisfying

$$\{v \in \mathcal{V}(P) \mid w(v) = h\} \subseteq G_h + |h|F \subseteq w_h(P). \tag{2.2.1}$$

where '+' denotes the Minkowski sum, and we define $\varnothing + Q = \varnothing$ for any polygon Q. We call F a *factor* of P with respect to w, and define the *mutation* given by the primitive normal vector w, factor F, and polytopes $\{G_h\}$ to be:

$$\mathrm{mut}_w(P, F) := \mathrm{conv}\left(\bigcup_{h=h_{min}}^{-1} G_h \cup \bigcup_{h=0}^{h_{max}} (w_h(P) + hF) \right) \subset N_{\mathbb{Q}}.$$

Although not immediately obvious from the definition, the resulting mutation is independent of the choices of $\{G_h\}$ [**4**, Proposition 1]. Furthermore, up to isomorphism, mutation does not depend on the choice of v_E: we have that $\mathrm{mut}_w(P, F) \cong \mathrm{mut}_w(P, -F)$. Since we consider a polygon to be defined only up to $\mathrm{GL}_2(\mathbb{Z})$-equivalence, mutation is well-defined and unique. Any mutation can be inverted by inverting the sign of w: if $Q := \mathrm{mut}_w(P, F)$ then $P = \mathrm{mut}_{-w}(Q, F)$ [**4**, Lemma 2]. Finally, we note that P is a Fano polygon if and only if the mutation Q is a Fano polygon [**4**, Proposition 2].

We call two polygons P and $Q \subset N_{\mathbb{Q}}$ *mutation-equivalent* if there exists a finite sequence of mutations between the two polygons (considered up to $\mathrm{GL}_2(\mathbb{Z})$-equivalence). That is, if there exists polygons $P_0, P_1, \ldots, P_n$ with $P \cong P_0$, $P_{i+1} = \mathrm{mut}_{w_i}(P_i, F_i)$, and $Q \cong P_n$, for some $n \in \mathbb{Z}_{\geq 0}$.

In two dimensions, mutations are completely determined by the edges of P:

Lemma 2.2.3. *Let E be an edge of P with primitive inner normal vector $w \in M$. Then P admits a mutation with respect to w if and only if $|E \cap N| - 1 \geq \ell_E$.*

PROOF. Let $k := |E \cap N| - 1$ be the lattice length of E. At height $h = h_{min} = -\ell_E$, condition (2.2.1) becomes $E = G_{h_{min}} + \ell_E F$. Hence this condition can be satisfied if and only if $k \geq \ell_E$. Suppose that $k \geq \ell_E$ and consider the cone $C := \mathrm{cone}(E)$ generated by E. At height $h_{min} < h < 0$, $h \in \mathbb{Z}$, the line-segment $C_h := \{v \in C \mid w(v) = h\} \subset N_{\mathbb{Q}}$ (with rational end-points) has lattice length $|h|k/\ell_E \geq |h|$. Hence $w_h(C) \subset w_h(P)$ has lattice length at least $|h| - 1$. Suppose that there exists some $v \in \mathcal{V}(P)$ such that $w(v) = h$. Since $v \notin w_h(C)$ we conclude that $w_h(P)$ has lattice length at least $|h|$. Hence condition (2.2.1) can be satisfied. If $\{v \in \mathcal{V}(P) \mid w(v) = h\} = \varnothing$ then we can simply take $G_h = \varnothing$ to satisfy condition (2.2.1). □

2.2.2. Mutation in M. Given a Fano polygon $P \subset N_{\mathbb{Q}}$ we define the *dual polygon*

$$P^* := \{u \in M_{\mathbb{Q}} \mid u(v) \geq -1 \text{ for all } v \in P\} \subset M_{\mathbb{Q}}.$$

In general this has rational-valued vertices and necessarily contains the origin in its strict interior. Define $\varphi : M_{\mathbb{Q}} \to M_{\mathbb{Q}}$ by $u \mapsto u - u_{\min} w$, where $u_{\min} := \min\{u(v) \mid v \in F\}$. Since $F = \mathrm{conv}\{\mathbf{0}, v_E\}$, this is equivalent to

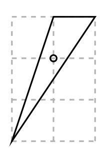
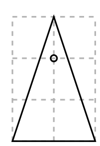
$$\varphi(u) = \begin{cases} u, & \text{if } u(v_E) \geq 0; \\ u - u(v_E)w, & \text{if } u(v_E) < 0. \end{cases}$$

This is a piecewise-$\mathrm{GL}_2(\mathbb{Z})$ map, partitioning $M_{\mathbb{Q}}$ into two half-spaces whose common boundary is generated by w. Crucially [**4**, Proposition 4]:

$$\varphi(P^*) = Q^*, \qquad \text{where } Q := \mathrm{mut}_w(P, F).$$

An immediate consequence of this is that the volume and Ehrhart series of the dual polygons are preserved under mutation: $\mathrm{Vol}(P^*) = \mathrm{Vol}(Q^*)$ and $\mathrm{Ehr}_{P^*}(t) = \mathrm{Ehr}_{Q^*}(t)$. Equivalently, mutation preserves the anti-canonical degree and Hilbert series of the corresponding toric varieties: $(-K_{X_P})^2 = (-K_{X_Q})^2$ and $\mathrm{Hilb}(X_P, -K_{X_P}) = \mathrm{Hilb}\big(X_Q, -K_{X_Q}\big)$.

EXAMPLE 2.2.4. Consider the polygon $P_{(1,1,1)} := \mathrm{conv}\{(1,1), (0,1), (-1,-2)\} \subset N_{\mathbb{Q}}$. The toric variety corresponding to $P_{(1,1,1)}$ is $\mathbb{P}^2$. Let $w = (0,-1) \in M$, so that $h_{min} = -1$ and $h_{max} = 2$, and set $F = \mathrm{conv}\{\mathbf{0}, (1,0)\} \subset N_{\mathbb{Q}}$. Then F is a factor of $P_{(1,1,1)}$ with respect to w, giving the mutation $P_{(1,1,2)} := \mathrm{mut}_w(P_{(1,1,1)}, F)$ with vertices $(0,1)$, $(-1,-2)$, $(1,-2)$ as depicted below. The toric variety corresponding to $P_{(1,1,2)}$ is $\mathbb{P}(1,1,4)$.

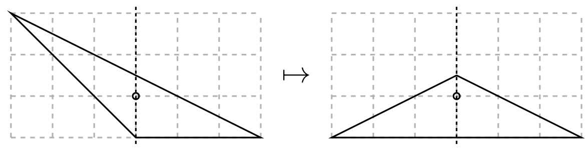
$N_{\mathbb{Q}}$:  $\mapsto$ 

In $M_{\mathbb{Q}}$ we see the mutation as a piecewise-$\mathrm{GL}_2(\mathbb{Z})$ transformation. This acts on the left-hand half-space $\{(u_1, u_2) \in M_{\mathbb{Q}} \mid u_1 < 0\}$ via the transformation

$$(u_1, u_2) \mapsto (u_1, u_2) \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and on the right-hand half-space via the identity.

$M_{\mathbb{Q}}$:



2.2.3. Singularity content. An important mutation invariant of Fano polygons is its *singularity content*, [**3**]. This is a pair $(n, \mathcal{B})$ where n is an integer and $\mathcal{B}$ is the *basket of residual singularities*. Geometrically, these are invariants of the surface X obtained from toric surface X_P by a generic $\mathbb{Q}$-Gorenstein deformation. Under this interpretation $n = e(X^0)$, the topological euler number of the smooth locus of X, and $\mathcal{B}$ is the collection of locally $\mathbb{Q}$-Gorenstein rigid singularities of X. This will be discussed more in the next section. For now we provide definitions directly from the polygon P. First we recall that a $\frac{1}{n}(1, q)$ singularity is the quotient of $\mathbb{C}^2$ by the action of μ_n determined by fixing the action of ω, a primitive n-th root of unity, to be $(x, y) \mapsto (\omega x, \omega^q y)$.

Definition 2.2.5. Given a $\frac{1}{n}(1, q)$ singularity, write $p = 1+q$, $w = \mathrm{hcf}(n, p)$, $n = wr$, $p = wa$ and $w = mr + w_0$, m is called the *singularity content of* $\frac{1}{n}(1, q)$. Any singularity with $m = 0$ is said to be *residual.* A singularity with $w_0 = 0$ is a *T-singularity* and a singularity with $m = 1$ and $w_0 = 0$ is called a *primitive T-singularity.* The *residual part* of $\frac{1}{n}(1, q)$ is defined to be $\frac{1}{w_0 r}(1, w_0 a - 1)$, a residual singularity.

Remark 2.2.6. We recall the value r is the *Gorenstein* or *local index* of the singularity $\frac{1}{n}(1, q)$. In general for a variety X this is the smallest r such that rK_X is Cartier. For a Fano polygon P the Gorenstein index of X_P is the lowest common multiple of the local indices.

T-singularities and residual singularities have an explicit form, which we also recall.

Lemma 2.2.7. *Any T-singularity has form* $\frac{1}{nl^2}(1, nlc - 1)$ *where* $k = nl$*, the primitive T-singularities are precisely those with* $n = 1$*. A residual singularity has the general form* $\frac{1}{kl}(1, kc - 1)$ *with* $k < l$*.*

Remark 2.2.8. T-singularities appear the work of Wahl [**107**] and Kollár–Shepherd-Barron [**74**]. T-singularities are precisely those which admit a $\mathbb{Q}$-Gorenstein smoothing. In comparison, as mentioned above, residual singularities are those which are $\mathbb{Q}$-Gorenstein rigid.

Definition 2.2.9. The *singularity content* $(n, \mathcal{B})$ is the pair consisting of the total singularity content over the edges of P, $\mathcal{B}$ is a cyclically ordered list of the residual parts of the cyclic quotient singularities $\mathrm{Cone}(E)$ for E an edge of P.

Remark 2.2.10. The singularity content n will appear later as the *number of focus-focus singularities in an affine structure.*

2.3. Classifying Orbifold del Pezzo surfaces

Before moving onto the perspective on the program of [**21**] afforded by open-string Mirror Symmetry we summarize some of the important aspects of the joint work [**1**]. In particular we shall describe one of the central constructions and one conjecture of [**1**] which will provide a precise context for Chapter 3.

First we recall that given a variety X with at worst quotient singularities and with Gorenstein index r a $\mathbb{Q}$-Gorenstein deformation $\mathcal{X} \to S$ is a flat deformation such that $rK_{\mathcal{X}/S}$ is a relative Cartier divisor. Given any cyclic quotient surface singularity $\frac{1}{n}(1,q)$ one can describe its $\mathbb{Q}$-Gorenstein deformation functor $\mathrm{Def}^{qG}(\frac{1}{n}(1,q))$:

Lemma 2.3.1 ([**74**]). *Given the singularity $\frac{1}{n}(1,q)$, define $p := 1+q$, $w := \mathrm{hcf}(n,p)$, $n = wr$, $p = wa$, and $w = mr + w_0$, then $\mathrm{Def}^{qG}(\frac{1}{n}(1,q)) \cong \mathbb{C}^m$ with universal family*

$$\left\{xy + z^{w_0}\left(z^{rm} + a_1 z^{r(m-1)} + \cdots a_m\right) = 0\right\} \subset \frac{1}{r}(1, wa-1, a)$$

The conjecture from [**1**] we are primarily concerned with, 'Conjecture A', compares certain $\mathbb{Q}$-Gorenstein deformation classes with of mutation equivalence classes of Fano polygons.

Conjecture 2.3.2 (Conjecture A). *There is a one-to-one correspondence between:*

- *Mutation equivalence classes of Fano polygons; and*
- *$\mathbb{Q}$-Gorenstein deformation equivalence classes of* locally *$\mathbb{Q}$-Gorenstein rigid del Pezzo surfaces X which:*
  - *admit a $\mathbb{Q}$-Gorenstein degeneration to a toric variety; and*
  - *have at worst cyclic quotient singularities.*

This bijection sends a representative Fano polygon P to a generic $\mathbb{Q}$-Gorenstein deformation of X_P.

This conjecture characterizes, solely in terms of algebraic geometry, of the objects appearing on the A-model (Fano manifold) side of the proposed Mirror correspondence. Having established the A-model objects we are led to ask the natural question of what objects appear on the mirror-dual side of the correspondence, that is:

Which class of Laurent polynomials are mirror-dual to Fano varieties?

The construction given in [**1**] which conjecturally answers this for orbifold del Pezzo surfaces (and, with modification, in greater generality) is that of *maximally mutable Laurent polynomials.*

Remark 2.3.3. Note that we implicity consider Mirror Symmetry for families of complex surfaces. This parameter space of complex structures should also appear in the form of Kähler moduli of the mirror. We propose that toric degenerations of the complex structure of a log del Pezzo surface correspond under Mirror Symmetry to choosing torus charts on the mirror-dual Landau–Ginzburg model and speculate that these represent points of a Kähler moduli space at which some symplectic inflation has occured at the boundary. However we do not rigourously explore this possibility further in this dissertation.

Definition 2.3.4. Given a Fano polygon P and let g be a Laurent polynomial supported on P, i.e. of the form

$$g = \sum_{\gamma \in P \cap N} a_\gamma z^\gamma$$

Then g is said to be *maximally mutable* if and only if for each sequence of mutations

$$P = P_0 \longrightarrow P_1 \longrightarrow \cdots \longrightarrow P_n$$

the sequence of rational functions g_i where $g_0 = g$ and g_{i+1} is the result of the mutation $P_i \to P_{i+1}$ applied to g_i is a sequence of Laurent polynomials[1].

Remark 2.3.5. As we describe in Chapter 5, the definition of maximally mutable Laurent polynomial may be made in terms of an appropriate *upper cluster algebra*. Combined with Conjecture A, this would establish a bijection between the seed tori on a certain cluster variety and the $\mathbb{Q}$-Gorenstein toric degenerations of X.

Remark 2.3.6. We shall see in Chapter 4 that in the case of $\mathbb{P}^2$ the maximally mutable Laurent polynomials are precisely those coming from certain *tropical disc counts*. This is conjectured to be more generally true, and is closely related to the work of Carl–Pumperla–Siebert [**18**].

A more precise version of the conjecture that the maximally mutable Laurent Polynomials 'are' the mirror-dual objects to $\mathbb{Q}$-Gorenstein deformation classes of orbifold del Pezzo surfaces is formulated in [**1**] as 'Conjecture B'. The assertion there is that there is an affine-linear isomorphism $\phi : L^T_P \to H^{ts}_X$ such that $\widehat{G}_X \circ \phi = \pi_f$. The respective objects appearing in this conjecture are,

(1) the affine space L^T_P of maximally mutable Laurent polynomials supported on P.
(2) the sum

$$H^{ts}_X := \bigoplus_{1 \le i \le r} \mathbb{C}\mathbf{u}_i$$

of certain twisted sectors of the Chen–Ruan cohomology of X.

For a full discussion of this conjecture see [**1**].

2.4. The SYZ conjecture and Fano manifolds

Up to this point we have only regarded formulations of Mirror Symmetry in terms of *closed-string* sectors, that is, as an identification between the structures obtained from the Gromov–Witten invariants of X, and oscillating integrals on a Landau–Ginzburg model. However, there are other, 'open-string', formulations of Mirror Symmetry which have been the subject of intense research in the last 20 years. One of the most prominent is the Strominger–Yau–Zaslow

[1] recalling that this makes sense in two dimensions as there is a canonical choice of factor polynomial

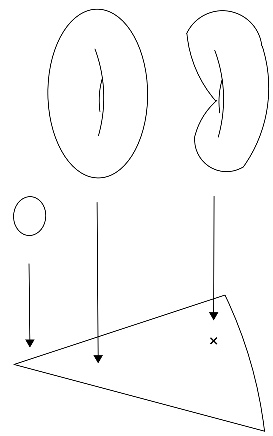


FIGURE 2.4.1. A torus fibration over a manifold with boundary

conjecture [**105**], which roughly states that a mirror pair of Calabi–Yau varieties carry a dual pair of *special Lagrangian torus fibrations*. Intensive studies of this conjecture [**58**, **59**, **78**, **79**] have revealed a beautiful, geometric, interpretation of Mirror Symmetry in terms of "quantum corrected T-duality".

Whilst the varieties we consider are not Calabi–Yau, by taking the complement of an anti-canonical divisor, one can recover a Calabi–Yau on which to formulate the conjecture. Following the treatment in [**6**] we may formulate a a naïve version of this conjecture as follows.

CONJECTURE 2.4.1. *Given a compact Kähler manifold (X, J, ω) let D be an anti-canonical divisor and Ω a holomorphic volume form over $X \backslash D$. The mirror dual M is a moduli space of special Lagrangian tori with flat $U(1)$ connections. The superpotential $W : M \to \mathbb{C}$ is given by the $\mathfrak{m}_0$ obstruction of Fukaya–Oh–Ohta–Ono [43].*

We recall that $\mathfrak{m}_0$ obstruction is, defined to be a virtual count of Maslov index two discs β with boundary on a given Lagrangian torus fiber L of X

$$\mathfrak{m}_0(L, \nabla) = \sum_{\beta \in H_2(X,L)} n_\beta(L) e^{-\int_\beta \omega} \operatorname{hol}_\nabla(\partial\beta)$$

That is, a series whose coefficents are given by counts $n_\beta(L)$ of discs which intersect the boundary divisor D in a single point (see [**6**]). We recall that this formula may also undergo certain wall crossing discontinuities as L varies, and thus only defines a global function on the proposed SYZ mirror-dual manifold after certain corrections.

REMARK 2.4.2. As well as the open Calabi–Yau in the complement of an anti-canonical divisor, it is natural to restrict to the anti-canonical hypersurface. The interaction between Mirror Symmetry for the hypersurface and for its complement is explored in [**6**, **7**].

REMARK 2.4.3. Another important variant on this theory is obtained by forgetting the anti-canonical divisor entirely. Consequently there are no Maslov index two discs intersecting boundary divisors and no superpotential. This *log Calabi–Yau* setting is intensively studied by Gross–Hacking–Keel in [**54**, **55**] and they provide a rich geometric picture of the underlying mirror object to Fano surfaces. We explore this connection further in Chapter 5.

EXAMPLE 2.4.4. An important class of examples where the SYZ conjecture can be checked comes from toric geometry. Indeed, we recall that in toric geometry a polygon $P^\vee$ is the base of a *special Lagrangian torus fibration* given by the moment map for X_P. The Maslov index two discs in a fiber are in bijection with the *facets* of the moment polytope $P^\vee$, for more details of this case see [**6**, **19**].

EXAMPLE 2.4.5. The most important non-toric example of a special Lagrangian torus fibration for this dissertation is constructed by Auroux in [**6**] and we summarize it here, adapting it somewhat to our context. Define the pair (X, D) to be $(\mathbb{P}^2, C + L)$ where C is a plane conic and L is a line. The complement of an anti-canonical divisor is then simply $\{xy - \epsilon = 0\} \subset \mathbb{C}^2_{x,y}$ the affine chart $\mathbb{C}^2 \cong \mathbb{P}^2 \backslash L$. Taking the holomorphic volume form

$$\Omega = \frac{dx \wedge dy}{xy - \epsilon}$$

and the standard Fubini-Study form on $\mathbb{P}^2$ Auroux constructs a special Lagrangian torus fibration via the following observations:

(1) The function $f(x, y) = xy$ defines a conic fibration on X, which admits a fiber-wise S^1 action $\theta\colon (x, y) \mapsto (e^{i\theta}x, e^{-i\theta}y)$.
(2) The moment map δ of this S^1 action is preserved by symplectic parallel transport in the fibration f.
(3) Considering circles $\Gamma = \{|xy - \epsilon| = r\}$ in the base of the conic fibration f one defines tori $T_{r,\lambda} := f^{-1}(\Gamma) \cap \delta^{-1}(\lambda)$.

The pair (r, λ) define a special Lagrangian torus fibration (see Proposition 5.2 of [**6**]) with one singular fiber, induced by the singular conic $xy = 0$. There is a one dimensional subspace of the base, a 'wall', for which the torus fibers bound discs of Maslov index zero, this divides the base into two regions. The mirror M is constructed from two charts related by a birational transformation, i.e. taking the dual fibration on each region and 'inflating' one finds a pair of coordinate charts related by a wall-crossing formula. To see what happens as we vary ϵ consider $X_\epsilon \cong \mathbb{P}^2$ embedded in $\mathbb{P}(1, 1, 1, 2)$ with coordinates (x_0, x_1, x_2, u) via

$$\{x_0x_2 = \epsilon x_1^2 + u\}$$

Further, set D_ϵ to be $\{x_1 u = 0\} \cap X_\epsilon$. On the mirror M, the superpotential on each coordinate chart defines a pair of Newton polygons such that their spanning fans define the surfaces $\mathbb{P}^2$ and $\mathbb{P}(1,1,4)$ respectively. In fact this example admits many of the features we study in a more general setting in Chapter 3:

(1) As one varies ϵ the respective size of the regions in the base changes, with one region 'filling out' the total space as $\epsilon \to 0$ or ∞ respectively.
(2) As $\epsilon \to 0$ the fibration converges to the moment map on $\mathbb{P}^2$.
(3) Change variables so that the above family as $\epsilon \to \infty$ is replaced by the following family as $\epsilon' \to 0$
$$\{x_0 x_2 = x_1^2 + \epsilon' u\}$$
Over $\epsilon' = 0$, the conic component C of D persists as a smooth conic, but the line L breaks into two $(x_0 x_2 = 0)$. The limiting variety is $\mathbb{P}(1,1,4)$ with its toric boundary, and after an appropriate rescaling of (r, λ) the moment map of $\mathbb{P}(1,1,4)$ is the limiting fibration as $\epsilon' \to 0$.

These important examples not withstanding, we recall that such naïve versions of the SYZ conjecture are unable to hold in general. Indeed, even if special Lagrangian torus fibrations could be found in any generality, such a fibration is expected to have singular fibers (see Figure 2.4.1) which create *quantum corrections* to the mirror geometry. Roughly, these corrections may be encoded as a *wall-and-chamber* structure in X, which make $\mathfrak{m}_0$ multi-valued. In [**6**] Auroux investigates the symplectic geometry around this conjecture in detail. Rather than exploring these details we shall observe striking parallels between the picture suggested by the symplectic geometry and the polygons appearing in the program outlined in [**21**] which we heavily exploit in the next chapter.

In order to navigate some of the difficulties involved in proving the SYZ conjecture various alternative constructions have been proposed [**59, 79**]. The construction we will make most use of in subsequent chapters is that of the *Gross–Siebert algorithm*. As described in [**51, 58**] the starting point of the program is the idea to replace the special Lagrangian fibration with a *toric degeneration*. A toric degeneration is, roughly, a degeneration of (X, D) to a union of toric varieties identified along toric strata with conditions imposed on the singularities of the total space of the degenerating family. In particular from this union of toric varieties one can construct a candidate *base* manifold B, without knowing a torus fibration, the *(dual) intersection complex*.

If B was in fact known to be the base of a special Lagrangian torus fibration it would carry a dual pair of *affine structures*[2] induced by the Kähler and holomorphic volume forms respectively. Thus, the Gross–Siebert reformulation of the SYZ conjecture states that for a mirror Calabi–Yau pair X, $\breve{X}$, X admits a toric degeneration to X_0, which induces a pair of

[2]Here, since X is Fano rather than Calabi–Yau, the affine manifold B will have non-empty boundary and its Legendre dual $\breve{B}$ will be non-compact.

affine structures on a manifold B, and $\check{X}$ admits a toric degeneration to $\check{X}_0$ which exchanges the induced affine structures on B.

In fact the Gross–Siebert algorithm produces a *constructive* formulation of Mirror Symmetry: $\check{X}_0$ is easily recovered from the pair of affine structures induced by a toric degeneration of X. The mirror $\check{X}$ is formally recovered from $\check{X}_0$ using additonal data (a log structure) which forms the input for an algorithm which produces a formal smoothing of $\check{X}$. Similarly, the starting point in Chapter 3 is also an affine manifold (with boundary) and the combinatorial aspects of the program outlined in [**21**] are incorporated in the geometry of these affine manifolds. In fact, we will consider a certain *family* of log-structures, at the central fiber of this family the reconstruction algorithm will produce a Fano *toric* variety and varying the log-structure will produce a smoothing. We can in fact describe neighbourhoods of the toric fixed points explicitly, comparing the local pieces produced by the Gross–Siebert algorithm with a standard model for the $\mathbb{Q}$-Gorenstein smoothing of a cyclic quotient singularity. In this way we pass between algebraic smoothings of toric Fano varieties and perturbations of an affine structure in a tropical manifold.

The 'perturbations' of a polygon (to an affine manifold with singularities) that we consider have been explored in the symplectic category in [**106**] and [**81**]. In [**81**] Leung–Symington provide a classification up to diffeomorphism of *almost-toric fourfolds*, which by definition are those symplectic fourfolds which admit a Lagrangian fibration with an affine structure which contains a number of focus-focus singularities (the singularities we consider). The central operation on the affine manifold we introduce to recover the notion of combinatorial mutation ('smoothing corners' and 'sliding slingularities') appears in this work as *nodal slide* and *nodal trade*, as well in the work of Kontsevich–Soibelman as *worm deformations*. In [**7**] Auroux exploited these affine manifolds to produce special Lagrangian torus fibrations and in the next chapter we use it as the central tool connecting the combinatorics of mutations of polygons with $\mathbb{Q}$-Gorenstein families of del Pezzo surfaces with cyclic quotient singularities.

CHAPTER 3

Smoothing Toric Fano Surfaces

3.1. Introduction

This chapter is devoted to a proof of Theorem 1.3.1 and proceeds in the following three steps.

- First we recast the theory of combinatorial mutation for surfaces in the language of affine manifolds with singularities.
- Second, we recover statements in algebraic geometry by applying the *Gross–Siebert algorithm* to pass from the 'base' manifold (if there were a genuine special Lagrangian fibration) to the 'total space'.
- Finally we show that these deformations of affine structures lift to $\mathbb{Q}$-*Gorenstein* deformations via the Gross–Siebert algorithm.

Phrasing the last point differently, the polygons P of toric surfaces X_P to which a del Pezzo surface X degenerates appear as *zero-strata* in a space of affine manifolds; lifting neighbourhoods around these zero-strata to deformations of the corresponding toric surface we obtain the $\mathbb{Q}$-Gorenstein deformation space.

3.2. Affine Manifolds With Singularities

In this section we shall introduce affine manifolds with singularities. From our point of view these are tropical or combinatorial avatars of algebraic varieties. We shall briefly discuss the connection to the SYZ conjecture, which also offers a first justification for this point of view: the base of a special Lagrangian torus fibration naturally has the structure of an affine manifold. By way of example: given a toric variety we can form an affine manifold via its moment map, isomorphic to a polygon Q. We shall then consider a suitable notion of families of these objects and specifically how one can 'smooth' the corners of a polygon by replacing them with singularities in the interior. In particular, starting with a Fano polygon Q this will form a combinatorial analogue of the $\mathbb{Q}$-Gorenstein deformations of the associated del Pezzo surface: indeed, the bulk of the later sections is devoted to reconstructing such an algebraic deformation from this combinatorial data.

Definition 3.2.1. An *affine manifold with singularities* is a piecewise linear (PL) manifold B together with a dense open set $B_0 \subset B$ and a maximal atlas on B_0 that is compatible with the topological manifold structure on B and which makes B_0 a manifold with transition functions in $\mathrm{GL}_n(\mathbb{Z}) \rtimes \mathbb{R}^n$.

REMARK 3.2.2. To give a maximal atlas on B_0 with transition functions in $\mathrm{GL}_n(\mathbb{Z}) \rtimes \mathbb{R}^n$ is the same as to give the structure of a smooth manifold on B_0 together with a flat, torsion-free connection on TB_0 and covariant lattice Γ in TB_0.

Following Kontsevich–Soibelman [**79**] we can reinterpret this definition in terms of the sheaf of affine functions:

DEFINITION 3.2.3. The *sheaf of affine functions* $\mathrm{Aff}_{\mathbb{Z},X}$ on an affine manifold X is the sheaf of functions which, on restriction to any affine chart, give $\mathbb{Z}$-affine functions[1].

LEMMA 3.2.4 ([**79**]). *Given a Hausdorff topological space X, an affine structure on X is uniquely determined by a subsheaf $\mathit{Aff}_{\mathbb{Z},X}$ of the sheaf of continuous functions on X, such that locally $(X, \mathit{Aff}_{\mathbb{Z},X})$ is isomorphic to $(\mathbb{R}^n, \mathit{Aff}_{\mathbb{Z},\mathbb{R}^n})$.*

REMARK 3.2.5. $\mathrm{Aff}_{\mathbb{Z},X}$ is a sheaf of $\mathbb{R}$-vector spaces, but as the product of two affine functions is not in general affine, it is not a sheaf of rings. There is a subspace analogous to the maximal ideal of a local ring, given by the kernel of the evaulation map $\mathrm{ev}\colon \mathrm{Aff}_{\mathbb{Z},B_p} \to \mathbb{R}$.

DEFINITION 3.2.6. A morphism of affine manifolds is a continuous map $f\colon B \to B'$ that is compatible with the affine structures on B and B'.

DEFINITION 3.2.7. In [**59**] the authors refer to our affine manifolds as *tropical* affine manifolds, and refer to atlases with transition functions in $\mathrm{GL}_n(\mathbb{R}) \rtimes \mathbb{R}^n$ as affine strcutures. If the transition functions lie in $\mathrm{GL}_n(\mathbb{Z}) \rtimes \mathbb{Z}^n$ then the affine manifold is called *integral*; this is equivalent to insisting that there there is a lattice in B_0 preserved by the transition functions.

NOTATION 3.2.8. Define $\Delta := B \setminus B_0$, and refer to it as the *singular locus* of the affine structure. If $\Delta = \varnothing$ then the corresponding affine manifold is called *smooth*. Since we always assume that transition functions lie in $\mathrm{GL}_n(\mathbb{Z}) \rtimes \mathbb{R}^n$ there is a covariant lattice in TB_0 which we denote $\Lambda_x \subseteq T_x B_0$.

The relevance of affine manifolds to mirror symmetry comes from the SYZ conjecture [**105**], which roughly speaking states that a pair of mirror manifolds should carry special Lagrangian torus fibrations that are dual to each other. If one is in such a favourable setting, the base of this fibration carries a pair of (smooth) affine structures, and, in this so-called *semi-flat* setting, one can reconstruct the original pair of manifolds, X, $\breve{X}$ from the affine structures. Indeed from a given smooth tropical affine manifold B one may construct a pair of manifolds $X = TB/\Lambda$, $\breve{X} = T^*B/\breve{\Lambda}$ where Λ is the covariant lattice in TB defined by the affine structure and $\breve{\Lambda} \subset T^*B$ is the dual lattice. The manifold X carries a canonical complex structure and the manifold $\breve{X}$ carries a canonical symplectic structure [**51**]. To endow X with a symplectic structure, respectively $\breve{X}$ with a complex structure, we need to attach to B a (multivalued, strictly) convex function $\varphi\colon B \to \mathbb{R}$. Here there is a canonical choice for φ: the Kähler potential for the McLean metric on B [**51, 87**]. The convex function φ allows us to define the

[1]That is, functions of the form, $f = \sum_j a_j x_j + b$ with $a_j \in \mathbb{Z}$ for each j, $b \in \mathbb{R}$ and local coordinates x_j

Legendre dual $\breve{B}$ of the affine manifold B, and one can show that Legendre duality $B \leftrightarrow \breve{B}$ interchanges the pair of affine structures coming from a special Lagrangian torus fibration. This identification of TB/Λ with $T^*\breve{B}/\breve{\Lambda}$, and $T^*B/\breve{\Lambda}$ with $T\breve{B}/\Lambda$ recovers, as promised, the mirror pair of Kähler manifolds X, $\breve{X}$.

EXAMPLE 3.2.9. The standard examples of affine manifolds without boundary or singularities are tori, which have natural flat co-ordinates. For example, taking the base manifold B to be S^1 and endowing $X = TB/\Lambda$ with the canonical complex structure described above yields an elliptic curve X.

EXAMPLE 3.2.10. Consider a polytope $P \subset \mathbb{R}^n$. The inclusion $P \to \mathbb{R}^n$ equips the interior B of P with the structure of an affine manifold. The non-compact symplectic manifold $T^\star B/\breve{\Lambda}$ admits a Hamiltonian action of $(S^1)^n$ for which the moment map is given by the projection to B. It is clear in such examples how to extend the construction of this torus bundle over B to the boundary strata of P: indeed this is nothing other than Delzant's construction of symplectic toric varieties from their moment polytopes [**29**].

REMARK 3.2.11. As the last example demonstates we shall often be interested in cases where B (or B_0) is a *manifold with corners*. A discussion of mirror symmetry for toric varieties from this perspective may be found in [**6**]. Auroux explains there that one may define complex co-ordinates on $\breve{X}$ by taking the areas of certain holomorphic cylinders in X, together with certain $U(1)$-holonomies. After adding compactifying divisors to X, these cylinders become discs, and so co-ordinates on the mirror manifold $\breve{X}$ are determined by computing the areas of certain holomorphic discs. In the toric setting (Example 3.2.10) this construction gives global co-ordinates on $\breve{X}$. In general, and certainly in our case (where singularities are present), computing areas of holomorphic discs will give only local co-ordinates on $\breve{X}$, with the transition functions between these co-ordinate patches reflecting *instanton corrections*. From this perspective, much of the rest of this chapter consists of a careful analysis of the instanton corrections in our setting: computing them explicitly where possible, and determining how they vary in certain simple families. We return to this point in the Conclusion on page 82.

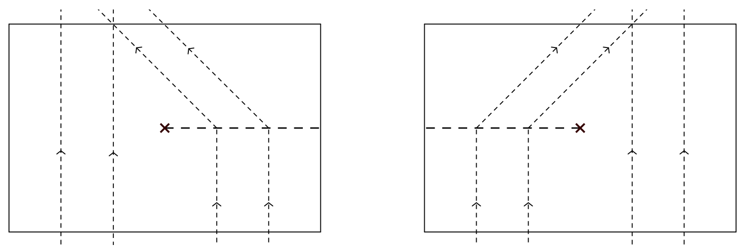
3.2.1. Focus-focus singularities. In the rest of this paper, we will primarily be concerned with affine manifolds that arise from polytopes, but rather than taking the polytope Q itself as the affine manifold, we shall instead *smooth the boundary*, exchanging the corners of Q for singularities in the interior of the polytope. The local model for this situation is as follows. Consider a two-dimensional affine manifold S_κ, where κ is a parameter, defined via a covering by two charts:

$$U_1 = \mathbb{R}^2 \setminus \left(\mathbb{R}_{\geq 0} \times \{0\}\right) \qquad\qquad U_2 = \mathbb{R}^2 \setminus \left(\mathbb{R}_{\leq 0} \times \{0\}\right)$$

with transition function ϕ from U_1 to U_2 given by:

$$(x, y) \mapsto \begin{cases} (x, y) & y > 0 \\ (x + \kappa y, y) & y < 0 \end{cases}$$

FIGURE 3.2.1. Straight lines in the two charts of a focus-focus singularity



The transition function is piecewise-linear: on the upper half-plane it is the identity transformation, and on the lower half-plane it is a horizontal shear with parameter κ. We will assume throughout that $\kappa \in \mathbb{Z}$; in this case, the affine manifold S_κ is integral. We will consider only affine manifolds with singularities that are locally modelled on some S_κ.

DEFINITION 3.2.12. A *singularity of type* κ in an affine manifold B is a point $p \in \Delta$ such that $p \notin \partial B$ and that there is a neighbourhood of p isomorphic as an affine manifold to a neighbourhood of $0 \in S_\kappa$.

REMARK 3.2.13. In the theory of Lagrangian fibrations such affine structures appear on the base of the fibration restricted to a neighbourhood of a nodal fiber. In this context κ is the number of nodes appearing the fiber and the vector $(1,0)$ in this standard model may be identified with the class of the vanishing cycle of this fibration.

DEFINITION 3.2.14. The monodromy around a singularity of type κ acts on $T_pB_0 \cong \mathbb{R}^2$ for $p \in B_0$. The monodromy operator fixes a one-dimensional subspace we refer to as its *monodromy invariant line*. In the local model S_κ this is the line $\mathbb{R} \times \{0\}$.

CONVENTION 3.2.15. Henceforth any affine manifold B that we consider will be two-dimensional and such that each $p \in \Delta$ is a singularity of type κ_p for some $\kappa_p \in \mathbb{Z}$. In particular, the singular locus Δ of B is disjoint from the boundary of B.

We will be primarily interested in one-parameter families of such affine structures, and in applying the Gross–Siebert algorithm 'fiberwise' to reconstruct a degenerating family.

REMARK 3.2.16. The Gross–Siebert algorithm for surfaces cannot be applied to certain 'illegal' configurations: one needs to insist that both monodromy-invariant lines and the rays introduced by scattering miss the singular locus. In practice one often guarantees this by ensuring that singularities have irrational co-ordinates. (In this context, monodromy-invariant lines and rays have rational slope.) But this approach generally precludes moving the singularities. As we shall see, smoothing the corners of a polygon is a particularly fortunate setting, where one can freely slide singularities along monodromy-invariant lines without risking illegal configurations.

3.2.2. Exchanging corners and focus-focus singularities. We shall now construct a local model for a degeneration. The most general definition of 'family of affine manifolds' we shall need consists of locally trivial families of affine structures together with finitely many copies of this local model.

Fix a rational, convex cone C in $\mathbb{R}^2$ and denote the primitive integral generators of its rays by v_1 and v_2. Fix a rational ray L contained in the interior of C, let ℓ be the primitive integer generator of L, and fix a $k \in \mathbb{Z}_{\geq 0}$ such that the (convex) rational cone generated by v_1 and $v_2 - k\ell$ either contains L or is itself a line in $\mathbb{R}^2$. We shall construct a topological manifold $\mathcal{B}_{C,L,k}$, together with a sheaf of affine functions on $\mathcal{B}_{C,L,k}$ and a map $\pi_k \colon \mathcal{B}_{C.L.k} \to \mathbb{R}_{\geq 0}$ of affine manifolds (where $\mathbb{R}_{\geq 0}$ has its canonical affine structure).

Definition 3.2.17. As a topological manifold $\mathcal{B}_{C,L,k}$ is equal to $C \times \mathbb{R}_{\geq 0}$. We give it an affine structure via an atlas with $k+1$ charts. We define each chart $U_i = (C \times \mathbb{R}_{\geq 0}) \setminus V_i$ where each V_i is a subset of $L \times \mathbb{R}_{\geq 0}$ as follows:

$$V_0 = \{(x\ell, t) : 0 \leq x \leq t < \infty\}$$
$$V_i = \{(x\ell, t) : 0 \leq x \leq it \text{ or } (i+1)t \leq x < \infty\} \qquad \text{for } i \in \{1, \cdots, k-1\}$$
$$V_k = \{(x\ell, t) : 0 < kt \leq x < \infty\}$$

with transition functions fixed by the requirement that, for $1 \leq i \leq k$ and for all $t > 0$, the charts U_{i-1}, U_i make the point $(it\ell, t)$ in the fiber $C \times \{t\}$ a type 1 singularity which monodromy invariant direction L.

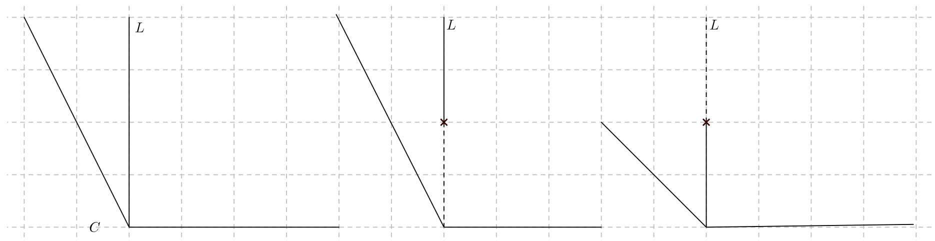
Note that this only makes the complement of $(0,0)$ an affine manifold, as the origin is in the closure of the singular locus but not contained in it. Despite this, the sheaf of affine functions is still defined in a neighbourhood of $(0,0)$.

Remark 3.2.18. We can interpret this family as 'exchanging corners for singularities' by studying the fibers of the projection to $\mathbb{R}_{\geq 0}$. For $t = 0$ the fiber is the cone C we wish to 'deform'. For all non-zero t the tangent wedge at the apex of the cone C is a different cone C' obtained from C by a PL transformation which 'flattens' it. In some cases (which we will see are precisely the 'T' or 'smoothable' cases) the boundary of the cone can be made entirely flat by this process and the tangent wedge ceases to be strictly convex. In Figure 3.2.2 we see an example (geometrically coming from an A_1 singularity) of a cone C, line L and generators v_1, v_2. In this example precisely one focus-focus point is introduced.

Remark 3.2.19. Later on we will restrict this family to a subset, replacing $C \times \mathbb{R}_{\geq 0}$ by $U \times [0, T)$ where U is a neighbourhood of the origin in C and T is sufficiently small that there are k singular points on the fiber $U \times \{T\}$.

Remark 3.2.20. There is an obvious generalization of this local model, which would allow the construction of more complicated degenerations. Rather than introduce a singularity of

FIGURE 3.2.2. An example of a (partial) corner smoothing for given cone C and line L



type 1 for each $1 \leq i \leq k$, we may consider a partition $\mathbf{k} = (k_1, \cdots, k_m)$ of k and construct a version $\mathcal{B}_{C,L,\mathbf{k}}$ of $\mathcal{B}_{C,L,k}$, in which the fiber over $t \in \mathbb{R}_{\geq 0}$ contains a singularity of type k_i at $(it\ell, t)$ for $1 \leq i \leq m$.

3.2.3. One-parameter families.

DEFINITION 3.2.21. We define a *one-parameter degeneration of affine structures* to be a topological manifold with corners $\mathcal{B}$ and a continuous map:

$$\pi \colon \mathcal{B} \to \mathbb{R}_{\geq 0}$$

such that:

- for some finite set S of points in the boundary of $\pi^{-1}(0)$, $\mathcal{B} \backslash S$ is an affine manifold with corners and π is a locally trivial map of affine manifolds; and
- for each $p \in S$ there is a neighbourhood U of p in $\mathcal{B}$ and a triple (C, L, k) such that U is isomorphic, as an affine manifold, to an open set of $\mathcal{B}_{C,L,k}$, via an isomorphism that identifies π with $\pi_{C,L,k}$.

REMARK 3.2.22. We will need to consider only one-parameter degenerations of affine structure such that a neighbourhood of the central fiber is locally modelled on $\mathcal{B}_{C,L,k}$ for various triples (C, L, k), possibly with $k = 0$.

It would be interesting to consider the generalization of this notion to families over arbitrary affine manifolds, and the associated moduli problems.

3.2.4. Polygons and Singularity Content. In this section we shall construct a one-parameter degeneration of affine structures from a given Fano polygon which partially smooths each vertex, in the sense we have described above. This is closely related to the notions of *singularity content*, *class T* and *class R* singularities which appear in [**3**].

A polygon P is *Fano* if it is integral, contains the origin and has primitive vertices. Fix such a polygon P and denote its polar polygon $Q := P^\circ$. In particular the origin is contained in the interior of Q. Fix a polyhedral decomposition $\mathscr{P}$ of Q by taking the spanning fan and restricting this fan to the polytope Q.

Fix a vertex $v \in \mathrm{Vert}(Q)$. The decomposition $\mathscr{P}$ induces a canonical choice of 1-cell L_v for each $v \in \mathrm{Vert}\,(Q)$: the 1-cell which is a cone of the spanning fan of Q. Consider the subset $U_v = \mathrm{Star}(v) \subset Q$; U_v is isomorphic to an open subset of a cone C_v with origin v and bounded by the rays containing each edge of Q incident to v. The 1-cell L_v becomes the restriction of a ray in this cone. To form a triple (C_v, L_v, k) as in §3.2.2 we still require the choice of a suitable integer k.

Definition 3.2.23. We shall refer to the maximal integer k such that (C, L, k) satisfy the conditions just before Definition 3.2.17 as the *singularity content* of the pair (C, L).

For each vertex $v \in Q$ denote by k_v the singularity content of (C_v, L_v), and choose a function $k\colon \mathrm{Vert}(Q) \to \mathbb{Z}_{\geq 0}$ such that $0 \leq k(v) \leq k_v$. We may now form the families $\mathcal{B}_{C_v,L_v,k(v)}$. Restrict each family to $U_v \times [0, T_v)$ where the fiber over T_v contains $k(v)$ singular points.

Definition 3.2.24. Let $\pi_{Q,k}\colon \mathcal{B}_{Q,k} \to [0, T)$ where $T = \min_v(T_v)$ be the following one-parameter degeneration of affine manifolds. As a topological manifold it is $Q \times [0, T)$, covered by the following charts:

- $U_v \times [0, T)$ as defined above for each vertex of Q and,
- $W \times [0, T)$ where W is a neighbourhood of the origin.

We may regard $U_v \times [0, T)$ as an affine manifold, with affine structure induced from $\mathcal{B}_{C_v,L_v,k(v)}$. We define the affine structure on $\mathcal{B}_{Q,k}$ by insisting that the transition functions between the $k(v)^{\mathrm{th}}$ chart of $U_v \times [0, T)$ and the $k(v')^{\mathrm{th}}$ chart of $U_{v'} \times [0, T)$ is the identity for vertices v and v', and the transition function between each of these charts and $W \times [0, T)$ is also the identity.

Notation 3.2.25. We will typically wish to smooth the corners as much as possible, so we use the notation $\pi_Q\colon \mathcal{B}_Q \to \mathbb{R}_{\geq 0}$ for the map $\pi_{Q,k}\colon \mathcal{B}_{Q,k} \to \mathbb{R}_{\geq 0}$ where k is the function sending each vertex to its singularity content.

We next show that our notion of singularity content (Definition 3.2.23) coincides with that of Akhtar–Kasprzyk [**3**]. We recall that given a Fano polygon $P \subset N_{\mathbb{R}}$ we may consider an edge e containing $v_1, v_2 \in \mathrm{Vert}(P)$. The edge defines an (inward-pointing, primitive) element of the dual lattice $w \in M$ such that $w(e)$ is a constant non-zero integer l. We may also consider the cone over the edge e, which we denote C_e. Let θ denote the lattice length of the line segment from v_1 to v_2. Writing $\theta = nl + r$ where $0 \leq r < l$, decomposes C_e into:

(1) A collection of n cones whose intersection with the affine hyperplane defined by $w(v) = l$ is a line segment of length l; and, if $r > 0$,
(2) A single cone of width $r < l$. This is the *residual* cone from [**3**].

If C_e contains no residual cone then we say that C_e is of *class T*. Akhtar–Kasprzyk call n the singularity content of C_e.

Consider an edge e of P with vertices v_1, v_2; this determines a vertex v_e of the polar polygon Q, and thus a cone C with origin at v_e, having rays dual to v_1 and v_2. The normal direction to e defines a ray in Q passing though v_e and the origin. Thus to the polygon P and edge e, we may associate a pair (C, L).

LEMMA 3.2.26. *The singularity content of* (C, L) *as in Definition 3.2.23 is equal to the singularity content of the cone over the edge* e *as defined in [**3**].*

PROOF. After a change of co-ordinates in N we may assume that the vertices v_1, v_2 of e are $(a_1, -h)$ and $(a_2, -h)$ respectively. The rational polygon Q then has a vertex $v_e = (0, -1/h)$ and edges which contain this vertex in directions $(-h, -a_1)$ and (h, a_2). This defines the cone C above. The ray L is vertical, and the singularity content of (C, L) is:

$$\max\{k \in \mathbb{Z}_{\geq 0} : a_2 - kh \geq a_1\}$$

This is the largest k such that $kh \leq a_2 - a_1$, and since $\theta = a_2 - a_1$ is the lattice length of the edge e, we see that the two definitions of singularity content coincide. □

DEFINITION 3.2.27. Let B be an affine manifold with singularities and corners, and $\mathscr{P}$ a polygonal decomposition of B. This pair is of *polygon type* if it is isomorphic to a fibre of a family $\pi_{Q,k} \colon \mathcal{B}_{Q,k} \to \mathbb{R}_{\geq 0}$.

3.3. From Affine Manifolds to Deformations: an Outline

We are now nearly in a position to apply the Gross–Siebert reconstruction algorithm to our base manifolds. Since we will require a slight generalization of the Gross–Siebert algorithm and since some of the details will be important later in the paper, we present the procedure in some detail. As a consequence sections 3.4 to 3.7 draw heavily on the paper [**59**] of Gross–Siebert and the book [**53**] by Gross.

As input data for this algorithm we require a two-dimensional affine manifold with singularities, plus some extra data attached to it. In section 3.4 we describe this extra data, introducing the notion of *log structure* and *open gluing data*, and explain how these data together determine the central fiber $X_0(B, \mathscr{P}, s)$ of a toric degeneration.

In section 3.5 we define the structure on the affine manifold with singularities plus log data, referred to simply as a *structure*, which encodes an nth-order deformation of $X_0(B, \mathscr{P}, s)$. Section 3.6 is then devoted to a description of the process ("scattering") by which an n-structure can be transformed into an $(n+1)$-structure; in other words, an nth-order deformation can be prolonged to an $(n+1)$st-order deformation. Finally we describe in section 3.7 how to pass from a structure to an nth-order deformation of the central fiber.

The rest of the chapter then applies this reconstruction algorithm to our original problem of smoothing cyclic quotient surface singularities. This is accomplished in a series of steps:

(1) In Section 3.8 we compute explicitly the local model at each boundary zero stratum.

(2) In Section 3.9 we return to the original problem: taking a polygon we show how the family of affine structures constructed in section 3.2 may be lifted order by order to give an algebraic family over $\operatorname{Spec}\mathbb{C}[\alpha][\![t]\!]$. Away from the central fiber, this is an application of the generalized Gross–Siebert algorithm; near the central fiber, this makes use of the local models computed in section 3.8. We further show that the local models at the vertices are compatible with the canonical cover construction, and thus that the family that we construct is $\mathbb{Q}$-Gorenstein.

(3) In section 3.10 we consider the special case in which a single singularity slides along its monodromy-invariant line from one corner into the opposite edge. Since there is no scattering diagram to consider, the tropical family here may be lifted to an algebraic family over $\mathbb{P}^1$; once again this algebraic family is $\mathbb{Q}$-Gorenstein.

3.4. Log Structures on the Central Fiber

In Section 3.2 we have considered the tropical analogue of smoothing the class-T singularities of a Fano toric surface. As explained, a version of the Gross–Siebert algorithm will allow us to reconstruct from this an algebraic family, the central fiber of which is itself the restriction to a formal neighbourhood of the central fiber of a degeneration of the Fano toric surface. The general fiber will be a different formal family with the same central fiber. The data appended to this central fiber that dicates which smoothing we take is a *log structure*. In this section we give a very functional description these log structures. However for a complete explanation of this notion, and its relevance to the Gross–Siebert algorithm, the reader is referred to [**58**, **59**]. For the rest of this section we fix a triple $(B, \mathscr{P}, s)$, where $\mathscr{P}$ is a polyhedral subdivison of B into convex, rational polyhedra. Here s is a choice of *open gluing data*, a concept we will also summarise in this section.

3.4.1. Construction of the central fiber. The method for constructing a scheme from the pair $(B, \mathscr{P})$ is straightforward. Each polygon in the decomposition $\mathscr{P}$ defines a toric variety via its normal fan, and the central fiber is constructed by gluing these along the strata they meet along in $\mathscr{P}$. Formally speaking, in order to define this gluing, we define a small category associated to a polyhedral decomposition:

Definition 3.4.1. Let $\mathscr{P}$ also denote the category which has:

Objects: The strata of the decompostion.

Morphisms: At most a single morphism between any two objects, where $e\colon \omega \to \tau$ exists iff $\omega \subseteq \tau$.

We next define a contravariant functor $V\colon \mathscr{P} \Rightarrow \text{AffSchemes}$. Its action on objects is as follows. Fix a vertex $v \in \mathscr{P}^0$. At v there is a fan $\Sigma_v \subseteq T_vB$ given by all the strata of $\mathscr{P}$ that meet v. Define K_ω to be the cone in Σ_v defined by the element $\omega \in \mathscr{P}$.

Definition 3.4.2 (of V on zero-dimensional objects). The co-ordinate ring of $V(v)$ is given by the Stanley–Reisner ring of the fan Σ_v: for lattice points m_1, $m_2 \in |\Sigma_v|$, we set

$$m_1.m_2 = \begin{cases} m_1.m_2 & \text{if } m_1,\, m_2 \in K_\omega \text{ for some } \omega \in \Sigma_v \\ 0 & \text{otherwise} \end{cases}$$

Given a stratum $\tau \in \mathscr{P}$ and a vertex v of τ, we define a fan around v:

$$\tau^{-1}\Sigma_v = \{K_e + \Lambda_{\tau,\mathbb{R}} : K_e \in \Sigma_v,\, e\colon v \to \sigma \text{ factoring though } \tau\}$$

recalling from [**59**] that $\Lambda_{\tau,\mathbb{R}}$ is the linear subspace generated by τ in T_vB. We remark, as in [**59**], that this subspace depends only on τ and not on the choice of vertex v. We can now define the image of a stratum τ under V:

Definition 3.4.3 (of V on positive-dimensional objects).

$$V(\tau) = \operatorname{Spec} \mathrm{k}\left[\tau^{-1}\Sigma_v\right]$$

where this k-algebra is interpreted as the Stanley–Reisner ring, as in Definition 3.4.2.

We now wish to define the functor V on morphisms. There is an obvious choice, namely sending a morphism $\tau \to \omega$ to the natural inclusion map $V(\tau) \to V(\omega)$ given by the fan. However one is free to compose this inclusion map with any choice of toric automorphism of $V(\tau)$. The choices of such automorphisms for every inclusion $\omega \hookrightarrow \tau$ form exactly the *Open gluing data* of [**59**], which we denote by s. This choice is not arbitrary, since V should be functorial: this constraint leads to the precise definition of open gluing data which we shall describe below. Once the definition of open gluing data is in place, and thus we have a well-defined functor V, we may then define the central fiber as the colimit:

$$\coprod_{\omega \in \mathscr{P}} V(\omega) \to X_0(B, \mathscr{P}, s) \tag{3.4.1}$$

3.4.1.1. *Open Gluing Data:* In [**59**] the authors explain that the toric automorphisms of an affine piece $V(\tau) = \operatorname{Spec}\left(\mathrm{k}\left[\tau^{-1}\Sigma_v\right]\right)$ for v a vertex of τ are in bijection with elements of a set $\mathrm{PM}(\tau)$ defined as follows.

Definition 3.4.4. Given $\tau \in \mathscr{P}$ and a vertex $v \in \tau$ we define $\mathrm{PM}(\tau)$ to be the set of maps $\mu\colon \Lambda_v \cap |\tau^{-1}\Sigma_v| \to k^\times$ such that:

- for any maximal cone σ of $\tau^{-1}\Sigma_v$, the restriction of μ to $\Lambda_v \cap \sigma$ is a homomorphism; and
- for any two maximal dimensional cones σ, σ', we have

$$\mu_\sigma|_{\Lambda_v\cap\sigma\cap\sigma'} = \mu_{\sigma'}|_{\Lambda_v\cap\sigma\cap\sigma'}.$$

As remarked in [**59**], whilst this description of $\mathrm{PM}(\tau)$ depends on $v \in \tau$, the set itself is independent of v.

REMARK 3.4.5. An elementary observation we shall use repeatedly in what follows is that the set of homomorphisms $\Lambda_v \cap \sigma \to k^\times$, where σ is a maximal dimensional cone, does not depend on the choice of maximal cone σ.

DEFINITION 3.4.6. A collection of *open gluing data* is a set

$$s = \{s_e \in \mathrm{PM}(\tau) \mid e\colon \omega \to \tau\}$$

such that if $e\colon \omega \to \tau$, $f\colon \tau \to \sigma$ then $s_f.s_e = s_{f\circ e}$ on the maximal cells where these are defined. We also insist that $s_{id} = 1$.

The conditions in Definition 3.4.6 are precisely those required to ensure that V is a functor.

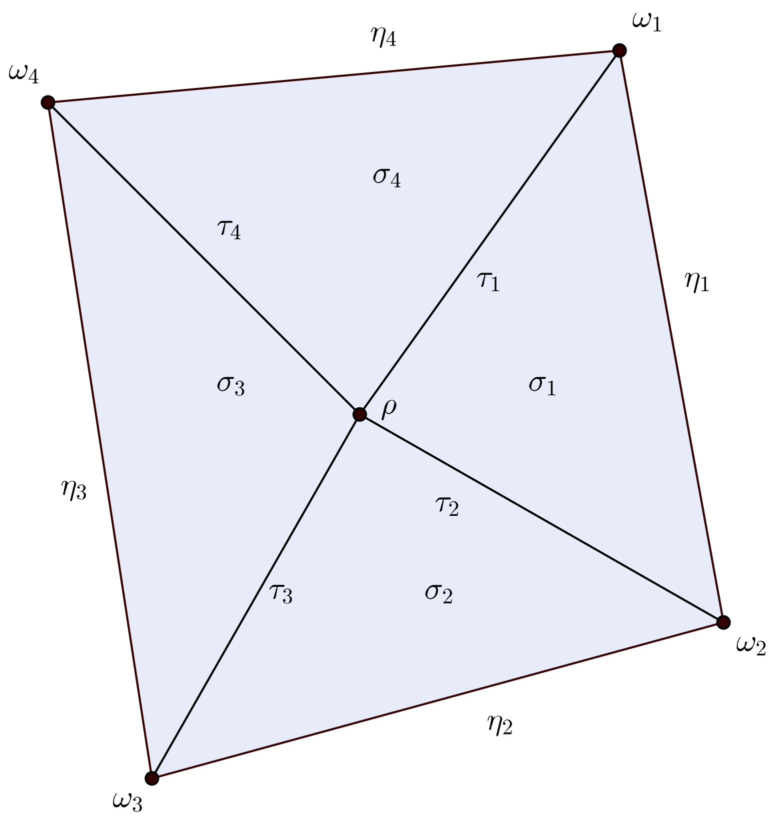
DEFINITION 3.4.7. Collections of open gluing data s_e, s'_e are *cohomologous* if there is a collection $\{t_\omega \in \mathrm{PM}(\omega) : \omega \in \mathscr{P}\}$ such that[2] $s'_e = t_\tau t_\omega^{-1} s_e$ whenever $e\colon \omega \to \tau$.

REMARK 3.4.8. In [**59**] it is proved that the schemes one obtains via (3.4.1) using cohomologous gluing data are isomorphic.

PROPOSITION 3.4.9. *Let $(B, \mathscr{P})$ be of polygon type. Then all choices of open gluing data are cohomologous.*

PROOF. Fix a polygon Q and label the various strata of $\mathscr{P}$:

[2]Here we use the fact that $t_\omega \in \mathrm{PM}(\omega)$ determines a unique element in $\mathrm{PM}(\tau)$, which we also denote by t_ω.



We need to show that, given any open gluing data s for $(B_Q, \mathscr{P})$, we can find a set $\{t_\omega \in \mathrm{PM}\,(\omega) : \omega \in \mathscr{P}\}$ such that $s_e = t_\tau t_\omega^{-1}$ for every $e\colon \omega \to \tau$. By Remark 3.4.5 we have that $\mathrm{PM}\,(\eta_j) \cong PM\,(\sigma_j)$ and $\mathrm{PM}\,(\omega_i) \cong PM\,(\tau_i)$ for all i and j. Open gluing data s are specified by the following five families of piecewise-multiplicative functions:

(1) $e_i^1\colon \rho \to \tau_i$
(2) $e_i^2\colon \tau_i \to \sigma_i$, $e_i^{2\prime}\colon \tau_i \to \sigma_{i-1}$
(3) $e_i^3\colon \omega_i \to \tau_i$
(4) $e_i^4\colon \omega \to \eta_i$, $e_i^{4\prime}\colon \omega \to \eta_{i-1}$
(5) $e_i^5\colon \eta_i \to \sigma_i$

We first define open gluing data s^1 cohomologous to s by setting $t_\tau = s_{e_i^1}^{-1}$. Thus $s^1_{e_i^1} = 1$. Next we observe that $s^1_{e_i^2} = s^1_{e'^2_{i+1}}$, since we have insisted that $s^1_{e_i^1} s^1_{e_i^2} = s^1_{e^1_{i+1}} s^1_{e'^2_{i+1}}$. Therefore we may define open gluing data s^2 cohomologous to s^1 by setting $t_{\sigma_i} = (s^1_{e_i^2})^{-1}$. By construction, s^2 associates the trivial element of PM to any morphism between any of ρ, τ_i and σ_j. We now define open gluing data s^3 cohomologous to s^2 using $t_{\omega_i} = (s^2_{e_i^3})^{-1}$ and $t_{\eta_i} = (s^2_{e_i^4})^{-1}$.

We claim that the open gluing data s^3 are trivial. First we check $s^3_{e_i^5}$. We have:

$$s^3_{e_i^5} = s^3_{e_i^4} s^3_{e_i^5} = s^3_{e_i^3} s^3_{e_i^2} = 1$$

where the first equality is the statement that $s^3_{e_i^4} = 1$ together with Remark 3.4.5. Finally we need to check that $s^3_{e_i'^4} = 1$. But $s^3_{e_{i+1}'^4} . s^3_{e_i^5} = s^3_{e_i^4} . s^3_{e_i^5}$, so this follows. Thus any open gluing data for $(B, \mathscr{P})$ are cohomologous to the trivial gluing data. □

Proposition 3.4.9 and Remark 3.4.8 together show that the scheme obtained from V by gluing (as in equation 3.4.1) is independent of the choice of open gluing data. Thus we will suppress the dependence on this choice in what follows, assuming that V is constructed using trivial gluing data.

In fact given an affine manifold $(B, \mathscr{P})$ of polygon type, obtained by smoothing the corners of a polygon Q, there is a well known family over $\mathbb{C}$ whose fiber over zero is $X_0(B, \mathscr{P}, s)$ and every other fiber is isomorphic to the toric variety defined by the normal fan of Q.

Definition 3.4.10. Fix an affine manifold B of polygon type and its polyhedral decomposition $\mathscr{P}$. Also choose a piecewise linear convex function ϕ whose maximal domains of linearity are precisely the maximal cells of the decomposition $\mathscr{P}$. Define a polyhedron $\tilde{Q}$ by setting

$$\tilde{Q} := \{(m, k) \in Q \times \mathbb{R} : k > \phi(m)\}.$$

The *Mumford degeneration* of $(B, \mathscr{P}, \phi)$ is the toric variety defined by the normal fan of $\tilde{Q}$.

Remark 3.4.11. We noramlly suppress the dependence on ϕ, since it is fixed by the data defining a *structure* in Section 3.5.

3.4.2. A Description Of The Log Structure. In this section we describe, following [**59**], how one may attach a space of log structures to a triple $(B, \mathscr{P}, s)$. We begin by describing a sheaf, of which log structures will be (certain) sections.

Definition 3.4.12. Let $\rho \in \mathscr{P}$ be a 1-cell and let V_ρ be the associated toric variety. Let k be the total number of singularities of the affine structure on ρ, counted with multiplicity[3]. Let v_1, v_2 be the vertices of ρ, and cover V_ρ with two charts $U_i = V(v_i) \cap V_\rho$. We shall define a sheaf $\mathcal{N}_\rho$ on V_ρ by setting $\mathcal{N}_\rho(U_i) = \mathcal{O}_{V_\rho}|_{U_i}$ and using the change of vertex formula

$$f_{\rho,v_1} = z^{k m^\rho_{v_1,v_2}} f_{\rho,v_2}$$

where $m^\rho_{v_1,v_2}$ is the primitive vector along ρ from v_1 to v_2.

This defines an invertible sheaf. If the vertices of ρ are integral then V_ρ is canonically isomorphic to $\mathbb{P}^1$ and the sheaf $\mathcal{N}_\rho$ is the line bundle $\mathcal{O}_{\mathbb{P}^1}(k)$. In particular the number of zeroes of a generic section of $\mathcal{N}_\rho$ is equal to the number of singular points of the affine manifold supported on this stratum, counted with multiplicity. When the vertices v_i are not integral

[3]This is the lattice length of what Gross–Siebert call the *monodromy polytope*, which here is a line segment.

the 1-strata are canonically identified with the weighted projective line $\mathbb{P}(a,b)$, where a and b are the indices of the respective vertices, and the sheaf $\mathcal{N}_\rho$ is the line bundle $\mathcal{O}\left(k\,\mathrm{lcm}(a,b)\right)$.

Remark 3.4.13. The orbifold structure here depends on the polarization of the central fiber. In any given example, one can repolarize the central fiber by scaling all the polygons until every vertex is integral; this induces a Veronese embedding on the 1-strata $\mathbb{P}(a,b)$ considered above. However this rescaling increases the number of interior integral points we need to consider, and in general leads to much more complicated embeddings.

Definition 3.4.14. The sheaf of pre-log structures $\mathcal{LS}^+_{pre,X}$ is defined to be $\oplus_\rho \mathcal{N}_\rho$ where $\mathcal{N}_\rho$ is the extension by zero of the sheaf in Definition 3.4.12.

Log structures will be sections of the sheaf $\mathcal{LS}^+_{pre,X}$ that satisfy a consistency condition that we now describe [**59**]. Given a vertex $v \in \mathscr{P}$ fix:

- A cyclic ordering of the 1-cells ρ_i containing v;
- Sections f_i of $\mathcal{N}_{\rho_i}$; and
- Dual vectors $\check{d}_{\rho_i}$ annihilating the tangent spaces of ρ_i, and chosen compatibly with the cyclic ordering of ρ_i.

The consistency condition that we require is:

$$\prod \check{d}_{\rho_i} \otimes_{\mathbb{Z}} f_i\big|_{V_v} = 0 \otimes 1$$

Remark 3.4.15. In [**59**] a further condition, *local rigidity*, is imposed on $X_0(B,\mathscr{P},s)$ which, roughly speaking, is that the sections f_i associated to the 1-strata by the log structure do not factorize. This is not a condition that we shall impose in our context.

Remark 3.4.16. Given a lattice polygon Q, we have constructed a family of affine manifolds $\mathcal{B}_{Q,k} \to \mathbb{R}_{\geq 0}$. One could also consider the affine manifold of polygon type $(B,\mathscr{P})$ constructed from Q, and place a log structure on the scheme $X_0(B,\mathscr{P},s)$. The choices involved in these two constructions are very closely related, as we now explain.

Definition 3.4.17. Given any one parameter degeneration of affine manifolds $\pi\colon \mathcal{B} \to \mathbb{R}_{\geq 0}$ observe that any fiber B of π gives the same variety $X_0(B,\mathscr{P},s)$. A one parameter family of log structures $s(x) \in \Gamma(\mathcal{LS}^+_{pre,X_0})$, over $\mathbb{C}$ is said to be *compatible with* $\mathcal{B}$ if for each interior 1-cell τ and for each $x \in \mathbb{C}$ the following two subsets of B coincide and have the same multiplicities:

(1) The image of the zero set of the section $s(x)$ under the moment map sending $V_\rho \to \rho$.
(2) The singular set $\Delta \subset B$, counted with multiplicity by singularity type.

Any one-parameter degeneration of affine manifolds $\pi\colon \mathcal{B} \to \mathbb{R}_{\geq 0}$ gives rise to a compatible one-parameter family of log structures.

3.5. Structures on Affine Manifolds

In this section we define a *structure* on $(B, \mathscr{P}, \phi)$. This is a purely combinatorial construction, which will encode the various functions used to reconstruct the formal deformation of the maximally degenerate variety $X_0(B, \mathscr{P}, s)$. This section is largely an exegesis of [**53**], Chapter 6.

3.5.1. Exponents and orders. Throughout this section we shall fix a triple $(B, \mathscr{P}, \phi)$ where:

(1) B is an affine manifold with singularities and corners.
(2) $\mathscr{P}$ is a polygonal decomposition of B into rational, convex polyhedra.
(3) ϕ is a multi-valued piecewise linear function which is linear when restricted to full-dimensional cells.

REMARK 3.5.1. The multi-valued nature of ϕ reflects the fact that B has singularities: ϕ may be defined as an affine function on the universal cover of $B \setminus \Delta$ but in general this will not take the same value on each point covering a given point $p \in B \setminus \Delta$. Picking a sheet of the covering around p is equivalent to making a choice of local representative for ϕ.

In view of this remark we shall define a sheaf twisted so as to ensure ϕ is a global section. Formally, we shall define a sheaf of abelian groups on B an extension by $\mathbb{Z}$ of Λ, the covariant lattice in the tangent space of B:

$$0 \to \mathbb{Z} \to \mathcal{P}_\phi \to \Lambda \to 0$$

To fix this sheaf we first choose a covering of B_0 by simply connected open sets U_i and a representative ϕ_i of ϕ for each U_i:

DEFINITION 3.5.2. The sheaf $\mathcal{P}$ is defined by taking $\mathcal{P}_\phi = \Lambda|_{U_i} \oplus \mathbb{Z}$ on restriction to each U_i. On the intersection $U_i \cap U_j$ we identify sections via

$$(r, m) \sim (r + d(\phi_j - \phi_i)(m), m)$$

noting that $\phi_j - \phi_i$ is a linear function and so has a well defined slope which we evaluate in the direction m.

DEFINITION 3.5.3. An *exponent* at $x \in B_0$ is an element of the stalk $\mathcal{P}_{\phi,x}$.

DEFINITION 3.5.4. There is a canonical projection $\mathcal{P}_{\phi,x} \to \Lambda_x$ for every $x \in B_0$. Given an exponent $m \in \mathcal{P}_{\phi,x}$ we denote the image of m under this projection by $\bar{m}$.

In the case where B has no singularities, the deformations of the central fiber described in this section arise from a toric construction, which we now sketch (see [**53**] for details). The input data for this construction are an affine manifold $B \subset \mathbb{R}^2$, a decomposition $\mathscr{P}$ of B into integral polygons and a convex function $\phi \colon B \to \mathbb{R}$ which is piecewise linear and linear on the elements of $\mathscr{P}$. The set $B' = \{(p, x) : x \geq \phi(p)\}$ is a polyhedron, with a well defined normal fan. The toric variety associated to this normal fan has a projection to $\mathbb{C}$ and the fiber over zero is equal to a reducible collection of toric varieties corresponding to the full-dimensional cells of $\mathscr{P}$.

Example 3.5.5. We consider a degeneration of $\mathbb{P}^1$: Let B be the union of the intervals $[-1,0]$, $[0,1]$ and consider:

$$\phi(x) = \begin{cases} 0 & x < 0 \\ x & x > 0 \end{cases}$$

The toric variety associated to B' is the blow up of $\mathbb{C}\times\mathbb{P}^1$ at $(0,\infty)$. The projection onto the first factor has general fiber $\mathbb{P}^1$ and central fiber equal to the union of 2 copies of $\mathbb{P}^1$ identified at a toric zero stratum.

Remark 3.5.6. Observe that in this construction each cell of $\mathscr{P}$ not contained in the boundary of B defines a cone via its tangent wedge in B' which is dual to a cone in the normal fan of B'. A chart of this degeneration is then given by taking the algebra over the monoid defined by the integral points of this tangent wedge.

We now localize this toric construction, so that it applies to $(B,\mathscr{P},\phi)$ such that B has singularities. In particular we shall define the analogue of the *monoid above the graph* from Remark 3.5.6. To state this definition we need two more locally defined objects:

(1) Σ_x: The fan in T_xB_0 induced by $\mathcal{P}$.
(2) $\phi_{i,x}$: the piecewise linear function induced by ϕ_i on T_xB_0. One may define this by defining its slope in each cell of Σ_x to be the slope of ϕ_i in the cell of $\mathscr{P}$ that cone corresponds to; see [**53**] for more details.

Definition 3.5.7. Fix an $x \in U_i$. We define a monoid $P_{\phi,x} \subseteq \mathcal{P}_{\phi,x}$ given by:

$$P_{\phi,x} = \{(r,m) : m \in |\Sigma_x|,\ r \geq \phi_{i,x}(m)\}$$

The fact that $P_{\phi,x}$ is independent of the chart used to define it is proven in [**53**], and a corollary of that calculation is the following observation.

Proposition 3.5.8. *The order of an exponent with respect to a maximal dimensional cell $\sigma \in \mathscr{P}$ given by the formula* $\mathrm{ord}_\sigma(p) = r - \phi_{i,\sigma}$ *is independent of the chart used to define it.*

In words this definition is simply: '*The order of m is its height above the hyperplane in $\mathcal{P}_{\phi,x}$ defined by σ*'. Thus we may extend the definition slightly:

Definition 3.5.9. For $\tau \in \mathscr{P}$ and $m \in |\Sigma|$, $\mathrm{ord}_\tau(m) = \max_{\tau\subseteq\sigma}\mathrm{ord}_\sigma(m)$ and $\mathrm{ord}(m) = \max_\sigma \mathrm{ord}_\sigma(m)$.

3.5.2. Slabs and rays on B. Structures on B consist of a collection of *slabs* and *rays*. We shall now define rays; these carry the instanton corrections analogous to gradient flow lines in [**79**]. We recall this definition from [**53**].

DEFINITION 3.5.10. A *naked ray* (Definition 6.16 of [**53**]) is an immersion $\mathfrak{d}\colon [0, L_{\mathfrak{d}}] \to B$ such that:

- whenever $\mathfrak{d}(x)$ is non-singular, $D\mathfrak{d}_x$ maps the integral tangent vectors to x to $\Lambda_{\mathfrak{d}(x)}$;
- the image of $\mathfrak{d}$ only intersects singular points in their monodromy invariant direction;
- if $L_{\mathfrak{d}}$ is finite then $\mathfrak{d}\left(L_{\mathfrak{d}}\right)$ is in ∂B.

A *ray* is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$ where $\mathfrak{d}$ is a naked ray, $f_{\mathfrak{d}} = 1 + c_m z^m$, and $m \in \Gamma\left(I_{\mathfrak{d}}, \mathfrak{d}^{-1}\mathcal{P}_{\phi}\right)$ is such that every germ m_x of m lies in $P_{\phi,\mathfrak{d}(x)}$

A crucial property of rays is that the order of an exponent *increases* as one moves from one cell of $\mathscr{P}$ to another; this follows from the strict convexity of the piecewise linear function ϕ:

LEMMA 3.5.11. *Consider a ray $(\mathfrak{d}, f_{\mathfrak{d}})$ and the section m giving the exponent of the ray function $f_{\mathfrak{d}}$. If $m_x \in P_{\phi,x}$ then for $x' > x, m'_x \in P_{\phi,x'}$.*

PROOF. This is an immediate consequence of Lemma 6.19 in [**53**]. □

REMARK 3.5.12. This Lemma implies that given an integer k, the set

$$\{x \in [0, L_{\mathfrak{d}}] : \mathrm{ord}_x\,(m) \leq k\}$$

is an interval of the form $\left[0, N_{\mathfrak{d}}^k\right]$; this defines the numbers $N_{\mathfrak{d}}^k$ for each pair $(\mathfrak{d}, k)$. In particular we can define the truncation of a ray at a given order:

DEFINITION 3.5.13. A k-truncated ray is a ray $(\mathfrak{d}, f_{\mathfrak{d}})$ restricted to the domain $[0, N_{\mathfrak{d}}^k]$.

We now encode the log structure in the structure on B. To do this we use a simplified version of the definition of a slab from [**59**]. We shall require the following preliminary observation:

LEMMA 3.5.14. *Given a codimension one cell ρ in $\mathscr{P}$ and a section $f_\rho \in \Gamma\left(V_\rho, \mathcal{O}\left(k\right)\right)$ defining the log structure along this stratum there is a canonical lift, which we also denote f_ρ, to a section of $\mathrm{k}\left[P_{\phi,v}\right]$ for any vertex $v \in \rho$.*

PROOF. The function $f_\rho|_{V(v)}$ is a polynomial function in z^m where m is the primitive generator of the tangent space to ρ. Therefore $f_\rho|_{V(v)}$ is canonically an element of the ring $\mathrm{k}\left[\Lambda_v\right]$. We take $f_{\rho,v}$ to be the canonical lift to $\mathcal{P}_{\phi,v}$, obtained from the observation that ϕ gives a section of the projection $\mathcal{P}_{\phi,v} \to \Lambda_v$. Notice that with respect to ρ the order of the slab function is always zero. □

DEFINITION 3.5.15. A *slab* consists of a codimension one cell ρ together with, for each non-singular point $x \in \rho$, a germ

$$f_{\rho,x} = \sum_{m \in P_x, \bar{m} \in \Lambda_\rho} c_m z^m \in \mathrm{k}\left[P_x\right]$$

such that the following two conditions hold:

(1) *Change of vertex formula*: Take x and x' and denote the corresponding connected components of $\rho \setminus \Delta$ by C_x and $C_{x'}$ respectively. Let k be the number of singularities (counted with multiplicity) between x and x', and define $m^{\rho}_{x,x'} \in \Lambda_x$ to be the k-fold dilate of the primitive generator of the ray from x to x'. Now we generalise the change of vertex formula of [**59**] to give the relation between the slab functions in different connected components:

$$f_{\rho,x'} = z^{m^{\rho}_{x,x'}} f_{\rho,x}$$

(2) *Agreement with log structure*: If $x \in C_v$ for some vertex $v \in \rho$, we have at v a function from the log structure: $f_\rho|_{V(v)}$. There is a canonical parallel transport map to the point x and we demand that, after parallel transport, we have $f_{\rho,x} = f_\rho|_{V(v)}$.

Remark 3.5.16. This definition of slab function relies on Proposition 3.4.9. Indeed the change of component formula in [**59**] is considerably more complicated and it is not clear what the correct general definition is in cases which are not locally rigid.

Remark 3.5.17. In [**59**] the authors ask only that the order zero part of the slab function agrees with the log structure; in [**53**] however *all* the corrections are carried by rays. Interpolating between these two, we shall regard slabs simply as placeholders for the log structure.

3.5.3. Defining a structure on $(B, \mathscr{P}, \phi)$.

Definition 3.5.18. A structure $\mathscr{S} = \mathscr{S}^s \cup \mathscr{S}^r$ is a finite collection $\mathscr{S}^s$ of slabs and a possibly infinite collection $\mathscr{S}^r$ of rays such that:

(1) The order of any exponent on any ray is strictly positive.
(2) The set
$$\mathscr{S}^r_k = \left\{ k\text{-truncated rays } (\mathfrak{d}, f_{\mathfrak{d}}) : N^k_{\mathfrak{d}} > 0 \right\}$$
is finite for each k.

Given a structure $\mathscr{S}$ and a non-negative integer k, we fix a polyhedral refinement $\mathscr{P}_k$ of $\mathscr{P}$ such that:

(1) The cells of $\mathscr{P}_k$ are rational convex polyhedra.
(2) For each $\mathfrak{d} \in \mathscr{S}^r_k$, the set $\mathfrak{d}\left(\left[0, N^k_{\mathfrak{d}}\right]\right)$ is a union of cells in $\mathscr{P}_k$.

We now define a category $\underline{\text{Glue}}(\mathscr{S}, k)$ and a functor to the category of commutative rings which will record each of the local pieces of the smoothing. This allows the problem of reconstructing the smoothing to be broken into two distinct problems: establishing functoriality, and then showing that the colimit of this functor produces a smoothing.

3.5.3.1. *The objects.* Let $(\omega, \tau, \mathfrak{u})$ be a triple such that:

(1) $\omega, \tau \in \mathscr{P}$ and a maximal cell $\mathfrak{u}$ of $\mathscr{P}_k$
(2) $\omega \subseteq \tau$
(3) $\omega \cap \mathfrak{u} \neq \varnothing$
(4) $\tau \subseteq \sigma_{\mathfrak{u}}$, where $\sigma_{\mathfrak{u}}$ is the maximal cell of $\mathscr{P}$ containing $\mathfrak{u}$

REMARK 3.5.19. Each of these will be used to define a small subscheme of the formally degenerating family by considering a certain thickening of the stratum corresponding to τ inside a formal smoothing of $\mathrm{Star}(\omega)$.

3.5.3.2. *The morphisms.* The space of morphisms between any two objects $(\omega, \tau, \mathfrak{u}), (\omega', \tau', \mathfrak{u}')$ has at most one element. It has one element precisely when $\omega \subseteq \omega'$, $\tau' \subseteq \tau$. We shall use the following basic observation about the morphisms of this category:

LEMMA 3.5.20. *Any morphism may be factored into morphisms of one of two types:*

(1) $\omega \subseteq \omega'$, $\tau' \subseteq \tau$, $\mathfrak{u} = \mathfrak{u}'$.
(2) $\omega = \omega'$, $\tau' = \tau$, $\mathfrak{u} \cap \mathfrak{u}'$ *is a one dimensional set containing* ω.

Note that this factorisation is generally non-unique.

3.5.4. The gluing functor. We now define the functor F_k from $\underline{\mathrm{Glue}}(\mathscr{S}, k)$ to $\underline{\mathrm{Rings}}$ from which we shall construct the kth-order formal degeneration. The definition of this functor is virtually identical to that of [**53**].

Having fixed an object $(\omega, \tau, \mathfrak{u})$ of $\underline{\mathrm{Glue}}(\mathscr{S}, k)$, we shall use the notation σ for the maximal cell in $\mathscr{P}$ containing $\mathfrak{u}$. We shall denote the ring $F_k(\omega, \tau, \mathfrak{u})$ by $R^k_{\omega,\tau,\mathfrak{u}}$; Spec $R^k_{\omega,\tau,\mathfrak{u}}$ is a thickening of the toric stratum corresponding to τ. We give the definition of these rings in three stages.

3.5.4.1. *Defining* $P_{\phi,\omega}$. Recall the monoid $P_{\phi,x}$ for $x \in \mathrm{Int}(\omega)$. If we pick a $y \in \sigma$ then since the interior of a cell in $\mathscr{P}_{max}$ is simply connected there is a well-defined inclusion $j\colon P_{\phi,x} \hookrightarrow \mathcal{P}_{\phi,y}$ via parallel transport.

DEFINITION 3.5.21. $P_{\phi,\omega} = j\left(P_{\phi,x}\right) \subseteq \mathcal{P}_{\phi,y}$.

3.5.4.2. *Defining the ideal* $I^k_{\omega,\tau,\sigma}$. The thickening of the stratum is defined by an ideal, $I^k_{\omega,\tau,\sigma} = \{m \in P_{\phi,\omega} : \mathrm{ord}_\tau(m) > k\}$. We set $R^k_{\omega\tau\sigma} = \mathrm{k}\left[P_{\phi,\omega}\right]/I^k_{\omega\tau\sigma}$.

3.5.4.3. *Localisation.* This is not yet a good enough definition of $F_k(\omega, \tau, \sigma)$ however. The change of vertex formula in the definition of slab demands that certain functions (which have zeroes on the toric 1-strata) should be invertible in these rings, therefore we need to localise with respect to these functions. This is broken into cases, depending on the strata ω, τ.

First assume that τ is an edge with non-trivial intersection with Δ. In this case we have a slab function attached to each smooth point of τ, and we form the localisation:

DEFINITION 3.5.22. $R^k_{\omega\tau\mathfrak{u}} = \left(R^k_{\omega\tau\sigma}\right)_{f_\tau}$

Precisely, we need to specify what f_τ means here. If $\omega = \tau$ it is irrelevant, the slab function is a polynomial in a single variable which is invertible in this ring. If ω is a vertex we simply take the germ of the slab function at this point.

In all other cases, namely $\tau \cap \Delta = \varnothing$, we define:

DEFINITION 3.5.23. $R^k_{\omega\tau\mathfrak{u}} = R^k_{\omega\tau\sigma}$

We are now able to define the functor F_k on objects:

$$F_k\left(\omega, \tau, \mathfrak{u}\right) = R^k_{\omega,\tau,\mathfrak{u}}$$

REMARK 3.5.24. We observe there are some canonical maps between various of these rings. If $\tau' \subseteq \tau$ and $\omega \subseteq \omega'$ there is a canonical inclusion $I^k_{\omega,\tau,\sigma} \hookrightarrow I^k_{\omega,\tau',\sigma}$ and thus a surjection $R^k_{\omega,\tau,\sigma} \to R^k_{\omega,\tau',\sigma}$. There is also an inclusion of monoids $P_{\phi,\omega,\sigma} \hookrightarrow P_{\phi,\omega',\sigma}$ and thus an injection $R^k_{\omega,\tau,\sigma} \hookrightarrow R^k_{\omega',\tau,\sigma}$. One may check that these maps survive the localisations at the slab functions.

Now we have defined the functor on objects we define the functor on morphisms. This is done case by case, recalling that any morphism may be factored into those of *change of strata* type and those of *change of chamber* type.

3.5.4.4. *Change of strata.* We specify a map:

$$R^k_{\omega,\tau,\sigma} \hookrightarrow R^k_{\omega',\tau',\sigma}$$

by composing the canonical maps we identified in the previous section, precisely, we define the change of strata map:

$$\psi_{(\omega,\tau),(\omega',\tau')} \colon R^k_{\omega,\tau,\mathfrak{u}} \to R^k_{\omega,\tau',\mathfrak{u}} \hookrightarrow R^k_{\omega',\tau',\mathfrak{u}}$$

to be the composition of the two maps above. See [**53**] for the verification that these are defined in the localised rings.

3.5.4.5. *Change of chamber maps.* Now we fix two chambers $\mathfrak{u}$, $\mathfrak{u}'$ with one dimensional intersection and such that $\omega \cap \mathfrak{u} \cap \mathfrak{u}' \neq \varnothing$. We also fix a point $y \in \mathrm{Int}\left(\mathfrak{u} \cap \mathfrak{u}'\right)$ such that the connected component of $B_0 \cap \mathfrak{u} \cap \mathfrak{u}'$ (recalling $B_0 := B \backslash \Delta$) containing y intersects ω. Note that either ω is a vertex, in which case there is a unique such component, or ω is an edge, in which case any connected component will do. We shall now define the change of chamber map $\theta_{\mathfrak{u},\mathfrak{u}'} \colon R^k_{\omega,\tau,\mathfrak{u}} \to R^k_{\omega,\tau,\mathfrak{u}'}$.

We consider two further cases, depending on whether or not $\sigma_{\mathfrak{u}} \cap \sigma_{\mathfrak{u}'} \cap \Delta = \varnothing$. If this is the case we define:

$$\theta_{\mathfrak{u},\mathfrak{u}',y}\left(z^m\right) = z^m \prod f^{\langle n,\bar{m}\rangle}_{(\mathfrak{d},x)}$$

Note that this is always an isomorphism – all the functions $f_{(\mathfrak{d},x)}$ are invertible. As rays propagate in the direction of $\bar{m}$ this is manifestly independent of the point y. If $\sigma_{\mathfrak{u}} \cap \sigma_{\mathfrak{u}'} \cap \Delta = \varnothing$, we shall define the map as follows:

$$\theta_{\mathfrak{u},\mathfrak{u}',y}\left(z^m\right) = z^m f^{\langle n,\bar{m}\rangle}_{\rho,y} \prod f^{\langle n,\bar{m}\rangle}_{(\mathfrak{d},x)}$$

REMARK 3.5.25. Notice that z^m in the left hand side is an element of $R^k_{\omega,\tau,\mathfrak{u}}$ whereas on the right it appears as an element of $R^k_{\omega,\tau,\mathfrak{u}'}$. The identification of these two rings is made via parallel transport along a 'short path' from $\mathfrak{u}$ to $\mathfrak{u}'$ which is contained in the union of these two chambers and which intersects the 1-cell between them only once.

Since $R^k_{\omega,\tau,\mathfrak{u}}$ is localised at the slab functions we see that all functions appearing in the product are invertible, and so this map is an automorphism. However, the above definition is not manifestly independent of y.

PROPOSITION 3.5.26. *$\theta_{\mathfrak{u},\mathfrak{u}',y}$ is independent of the choice of y.*

PROOF. Since this is proven in [**53**] we only provide a sketch of this proof. The key observation is that if we change from y to y' in a different component of $\mathfrak{u} \cap \mathfrak{u} \cap B_0$ we change the slab function by the transition function given in Definition 3.5.15. However we also change the identification of this stalk with $R^k_{\omega,\tau,\mathfrak{u}'}$ by parallel transport, which may be interpreted as precomposing this map with the isomorphism induced by a simple loop around the singular point. The factors in these two isomorphisms are the same, but occur with different signs, ensuring that the change of path does not alter the change of chamber map. □

3.5.4.6. *Functoriality.* We have now defined a map on objects and on 'elementary' morphisms; however we need to show both that this is well defined and that this is a functor. We first define a *joint* which will be used to formulate a necessary and sufficient condition for functoriality:

DEFINITION 3.5.27. A vertex of $\mathscr{P}_k$ not contained in the boundary of B is called a *joint*. The collection of joints of $\mathscr{P}_k$ is denoted Joints $(\mathscr{S}, k)$.

Indeed, fixing a $\mathfrak{j} \in$ Joints $(\mathscr{S}, k)$ and a cyclic ordering $\mathfrak{u}_1, \cdots, \mathfrak{u}_k$ of the chambers around this vertex one has a necessary condition for F_k to be a functor:

$$\theta_{\mathfrak{u}_1,\mathfrak{u}_2} \circ \cdots \circ \theta_{\mathfrak{u}_k,\mathfrak{u}_1} = \mathrm{Id} \tag{3.5.1}$$

The content of Theorem 6.28 of [**53**] is that it is sufficent to check this identity at every joint. Given what have said already, this is a purely formal exercise and the reader is referred to [**53**] for the proof of this result.

DEFINITION 3.5.28. Given a structure $\mathscr{S}$ and a joint $\mathfrak{j}$ we say $\mathscr{S}$ is *consistent at* $\mathfrak{j}$ *to order* k if and only if Equation 3.5.1 holds at $\mathfrak{j}$ to order k. $\mathscr{S}$ is called *compatible to order* k if it is consistent to order k at every joint.

By Theorem 6.28 of [**53**] compatibility of the structure $\mathscr{S}$ implies the existence of a well defined functor from the category $\underline{\text{Glue}}\,(\mathscr{S}, k)$ to $\underline{\text{Rings}}$.

3.6. Consistency and Scattering

We saw in the last section that in order for the gluing functor to be well defined we need to guarantee a consistency condition on the structure. In this section we shall describe an inductive algorithm for ensuring this is the case at each order. Theorem 6.28 of [**53**] has reduced this to a local computation at each joint. Indeed, fixing a joint $\mathfrak{j}$ we shall construct a *scattering diagram* $\mathfrak{D}_\mathfrak{j}$ which will encode this local data. We begin by outlining the necessary theory associated with scattering diagrams.

3.6.1. Scattering diagrams at joints. This section is based on Section 6.3.3 of [**53**] and on [**57**]. This section is also largely independent of the rest of the chapter; we can make these definitions independently of a structure $\mathscr{S}$ or an affine manifold B.

We shall fix the following data:

(1) A lattice $M \cong \mathbb{Z}^2$, and denote $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$.
(2) P a monoid, and a map $r\colon P \to M$. We shall denote $\mathfrak{m} := P \backslash P^{\times}$.

The scattering diagram itself will consist of a number of *rays* and *lines*:

Definition 3.6.1. A *ray* (resp. *line*) is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$. Here $\mathfrak{d} = m_0' - \mathbb{R}_{\geq 0} m_0$ for a ray (resp. $\mathfrak{d} = m_0' - \mathbb{R} m_0$ for a line). Viewing $\mathfrak{d}$ as a set gives the *support* of the ray (line). If $\mathfrak{d}$ is a ray we call m_0' the *initial point*. The function $f_{\mathfrak{d}}$ is an element of $\widehat{\mathrm{k}[P]}$, with the completion taken with respect to $\mathfrak{m}$, such that:

- $f_{\mathfrak{d}}$ is congruent to one modulo the maximal ideal, i.e. $f_{\mathfrak{d}} \in 1 \bmod \mathfrak{m}$
- $f_{\mathfrak{d}}$ may be written $f_{\mathfrak{d}} = 1 + \sum c_m z^m$ such that if $c_m \neq 0$, $r(m) = C m_0$ for a positive rational number C.

Definition 3.6.2. A *scattering diagram* $\mathfrak{D}$ over $\mathrm{k}[P]/I$ is a finite collection of rays and lines such that $f_{\mathfrak{d}} \in \mathrm{k}[P]$.

Given a ray or a line $(\mathfrak{d}, f_{\mathfrak{d}})$ we define an automorphism of $\mathrm{k}[P]/I$ as follows:

Fix a path γ that intersects $\mathfrak{d}$ transversely and a primitive element $n \in N$ annihilating the support of the ray such that the direction n is compatible with the orientation of the γ.

Given these choices, set $\theta_{\gamma,\mathfrak{d}}(z^m) = z^m f_{\mathfrak{d}}^{\langle n, r(m)\rangle}$. Composing these in sequence we can describe automorphisms arising from longer paths, or indeed loops, forming the *path ordered product* associated with these paths. Specifically, given a path γ we may define $\theta_{\gamma,\mathfrak{D}} = \theta_{\gamma,\mathfrak{d}_1} \cdots \theta_{\gamma,\mathfrak{d}_n}$ so long as γ intersects each of the $\mathfrak{d}_i$ transversely at time t_i, with $t_i > t_{i+1}$, and avoids the intersection points of any rays or lines.

Remark 3.6.3. One may equivalently define the wall crossing automorphism $\theta_{\gamma,\mathfrak{d}}$ by considering the element $f_{\mathfrak{d}} \partial_n$ of the Lie algebra of log derivations. The element $\theta_{\gamma,\mathfrak{d}}$ of $\mathrm{Aut}(\mathrm{k}[P]/I)$ is obtained by exponentiation from this Lie algebra. For more details the reader is referred to [**57**].

There is a natural notion of consistency for a scattering diagram:

Definition 3.6.4. A scattering diagram $\mathfrak{D}$ is *consistent* if and only if the path ordered product around any loop for which this product is defined is the identity in $\mathrm{Aut}(\mathrm{k}[P]/I)$.

One fundamental property of scattering diagrams is that one may add rays in an essentially unique fashion to achieve consistency. This is the content of the following result of Kontsevich–Soibelman:

Theorem 3.6.5. *Given a scattering diagram $\mathfrak{D}$, then there is a scattering diagram $S_I(\mathfrak{D})$ such that $S_I(\mathfrak{D}) \backslash \mathfrak{D}$ is entirely rays, and is consistent over the ring* $\mathrm{k}[P]/I$.

PROOF. The proof is a calculation in the Lie algebra of log derivations and the subalgebra which exponentiates to the tropical vertex group. This is discussed in much more detail in [**57**]. $\square$

We now have a framework in which we can introduce corrections to order k, inductively making a scattering diagram consistent. Recalling that we have fixed a joint $\mathfrak{j}$ in $\mathscr{S}$ on $(B, \mathscr{P}, \phi)$ we fix the data required to define a scattering diagram:

DEFINITION 3.6.6. Let the lattice be $M = \Lambda_{\mathfrak{j}}$, the monoid $P = P_{\phi,\sigma_{\mathfrak{j}},\sigma}$ and the map $r\colon P \to M$ be given by $m \mapsto \bar{m}$. Noting that in general we have a maximal ideal $\mathfrak{m} = P\backslash P^{\times}$ we fix an $\mathfrak{m}$-primary ideal, $I = I^k_{\sigma_j,\sigma_j,\sigma}$.

We construct the scattering diagram $\mathfrak{D}_{\mathfrak{j}}$ in two steps.

(1) If $\mathfrak{j} \subset \rho$ where ρ is a slab, that is $\rho \cap \Delta \neq \varnothing$, then we factorize $f_{\rho,x}$ for $x \in \rho$, writing $f_{\rho,x} = \prod_j 1 + c_{\rho,j} z^{l_j m_{\rho,x}}$. For each j we add the following line to the scattering diagram:
$$\left(\mathbb{R}m, 1 + c_{\rho,j} z^{l_j m_{\rho,x}}\right)$$
where m is the primitive vector in the direction of $T_x\rho$.

(2) For each ray $\mathfrak{d}$ in $\mathscr{S}_{k-1}$ such that there exists $x \in \left[0, N^k_{\mathfrak{d}}\right]$ with $\mathfrak{d}(x) \in \mathfrak{j}$ we add either a ray or a line. If $x = 0$ we add a ray:
$$\left(\mathbb{R}_{\geq 0}\mathfrak{d}'\left(x\right), 1 + c_{\mathfrak{d}} z^{m_{\mathfrak{d}},x}\right)$$
otherwise we add the line with the same function.

Section 6.3.3 of [**53**] establishes that if $\dim \sigma_j \in \{0, 2\}$ then in fact $\mathfrak{D}_j$ satisfies all the requirements of a scattering diagram and so one may apply the Kontsevich–Soibelman algorithm and obtain a consistent scattering diagram $S_I\left(\mathfrak{D}_j\right)$. The rays of $S_I\left(\mathfrak{D}_j\right)$ are then 'exponentiated' to give rays locally in the structure $\mathscr{S}$ which then propagate in B.

Of course we have not dealt with the case that $\dim \sigma_j = 1$. This is harder because the candidate scattering diagram does not satisfy the requirement that $f_{\mathfrak{d}} \in 1 \bmod \mathfrak{m}$ for those lines coming from the slabs. Indeed, those functions always have order zero in the interior of ρ. A solution would be to try and prove an analogue of the Kontsevich–Soibelman Lemma over the localised ring $(\mathrm{k}[P]/I)_{f_{\rho,x}}$. However, the approach taken in [**53**] is to work in an even larger ring, define a 'universal' scattering diagram and view the localised ring as a subring. Since we impose slightly weaker assumptions on the singular locus Δ than appear in [**53**] we require a slightly stronger result, which is the topic of the next section.

3.6.2. Localising scattering diagrams. This section details the required modest amendments to Proposition 6.47 of [**53**] needed in order to extend that result to 'non-simple' settings. Roughly, by replacing coefficients with formal variables one may embed the localised ring in a completion of the original ring with respect to a sequence of ideals I_e. Once one can show that the scattering diagrams $\mathrm{S}_{I_e}\left(\mathcal{D}\right)$ stabilize we may form the scattering diagram over this completed ring.

Before stating the proposition we require some results from [**53**] relating scattering diagrams and enumerative geometry. To state these we first consider a scattering digram of the following form:

$$\mathfrak{D} = \left\{ \mathbb{R}m_i, \left(\prod_{j=1}^{p_i} \prod_{k=1}^{l_{ij}} \left(1 + t_{ijk} z^{-jm_i}\right) \right) : 1 \leq i \leq p \right\} \tag{3.6.1}$$

Starting with this scattering diagram we shall study $S(\mathfrak{D})$, over the ring $\mathrm{k}[M][\![\{t_{ijk}\}]\!]$. Note that we can always reduce by an $\mathfrak{m}$-primary ideal I, to form $S_I(\mathfrak{D})$. We further assume that no two rays have the same support and fix a ray $(\mathfrak{d}, f_\mathfrak{d}) \in S(\mathfrak{D}) \setminus \mathfrak{D}$. Reducing mod I we can assume that $f_\mathfrak{d}$ is a polynomial. We now construct a toric variety corresponding to $\mathfrak{d}$:

Definition 3.6.7. Let $X_\mathfrak{d}$ be the non-singular toric surface associated to the complete fan $\Sigma_\mathfrak{d}$ which includes the rays: $\mathbb{R}_{\geq 0} m_i$ and $\mathfrak{d}$ for each m_i in the definition of the scattering diagram above. Let D_i denote the toric divisor corresponding to m_i and let D_{out} denote the toric divisor corresponding to $\mathfrak{d}$.

We also need some auxiliary combinatorial definitions to state an enumerative formula for $f_\mathfrak{d}$:

Definition 3.6.8. A graded partition G is a finite sequence $G = (P_1, \cdots, P_d)$ of ordered partitions $P_i = (p_{i1}, \cdots, p_{il_i})$, where $i \mid p_{ij}$ for each i and j. We call p_{ij} the *parts* of P_i and define $|P_i| = \sum_j p_{ij}$ and $|G| = \sum |P_i|$.

Now let $G = (G_1, \cdots G_p)$ be a tuple of graded partitions, where we denote by P_{ij} the jth piece of G_i and write $P_{ij} = \left(p_{ij1}, \cdots, p_{ijl_{ij}}\right)$.

As in [**53**] restrict to those G such that

$$-\sum |G_i| m_i = k_G m_\mathfrak{d} \tag{3.6.2}$$

for some $k_G \in \mathbb{Z}_{>0}$. Now fix the class $\beta \in H_2(X_\mathfrak{d}, \mathbb{Z})$ such that:

1. If $D \notin \{D_1, \cdots D_p, D_{\text{out}}\}$ then $\beta.D = 0$
2. $\beta.D_i = |G_i|$
3. $\beta.D_{\text{out}} = k_G$

If $D_{\text{out}} = D_i$ for some i replace the above prescription of $\beta.D_i$ with $\beta.D_i = |G_i| + k_G$. Next pick general points x_{ijk} on D_i and recall the notion of an orbifold blowup from [**57**]:

Definition 3.6.9. Let $p \in D$ be a point in a non-singular divisor in a surface S. There is a unique length j subscheme supported at p. Let $\mathcal{S}_j \to S_j \to S$ be the composition of the blowup map in this ideal sheaf and the coarse moduli map from the unique orbifold structure on the singular variety S_j.

Remark 3.6.10. The exceptional divisor E in the blown-up space has self intersection $[E]^2 = -1/j$

We now define a space by making the orbifold blow-ups designated by G.

DEFINITION 3.6.11. Let $\nu\colon X[G] \to X$ be the length j orbifold blow-up of X in each of the points x_{ijk}.

We shall use a Gromov-Witten invariant associated to the strict transform:

$$\beta_G = \nu^*(\beta) - \sum_{ijk} p_{ijk}\,[E_{ijk}]$$

Colloquially this is the virtual number of rational curves with tangency order k_G along D_{out} at exactly one point, and p_{ijk}/j branches tangent to D_i with order j at x_{ijk}. The precise definition is an integral over a moduli space of stable relative maps with orbifold target space $X^o_{\mathfrak{d}}$; see [**57**]. Here, conforming to the notation of [**57**], $X^o_{\mathfrak{d}}$ is the space obtained by removing the toric zero-strata from $X_{\mathfrak{d}}$. We call the result of the blow-up ν, $\widetilde{X}^o_{\mathfrak{d}}$.

Theorem 6.44 of [**53**] describes $\log(f_{\mathfrak{d}})$ in terms of these Gromov-Witten invariants:

THEOREM 3.6.12.

$$\log(f_{\mathfrak{d}}) = \sum_G k_G N_G t^G z^{-k_G m_{\mathfrak{d}}}$$

where $t^G = \prod t_{ijk}^{p_{ijk}/j}$ and the sum is over graded partitions G satisfying Equation 3.6.2.

We also recall Remarks 6.45 and 6.46 of [**53**]:

REMARK 3.6.13. The definition of relative stable maps includes the possibility of maps $f\colon C \to \widehat{X}^o_{\mathfrak{d}}$ to a reducible scheme, but $\widehat{X}^o_{\mathfrak{d}}$ comes with a map to $\widetilde{X}^o_{\mathfrak{d}}$ and thus fits into a diagram:

$$\begin{tikzcd}
C \arrow[r, "f"] \arrow[rr, bend left, "\tilde{f}"] \arrow[drr, "\bar{f}"'] & \widehat{X}^o_{\mathfrak{d}} \arrow[r] & \widetilde{X}^o_{\mathfrak{d}} \arrow[d, "\nu"] \\
 & & X^o_{\mathfrak{d}}
\end{tikzcd}$$

Results cited in [**53**] imply that $\tilde{f}(C) \cap \tilde{D}^o_i = \varnothing$. We can now make statement about the intersection properties of $\bar{f}_*[C]$. In particular as this represents β_G the intersection multiplicity at each of the points x_{ijk} must be exactly p_{ijk}. Futher there is a point $q \in D_{\text{out}}$ such that $\bar{f}_*[C] \cap \partial X_{\mathfrak{d}} = \{x_{ijk}\} \cup \{q\}$, and this point is constrained to lie on one of finitely many points of D_{out}. The full argument is in [**53**], but in short one can describe the restriction of $\bar{f}_*[C]$ to $\partial X_{\mathfrak{d}}$ in terms of q, but this is in the linear equivalence class given by $\beta|_{\partial X_{\mathfrak{d}}}$, so only those values of q which will land in this equivalence class are permitted.

We now relate general scattering diagrams to the apparently special type we described above.

Remark 3.6.14. A general scattering diagram consisting solely of lines is equivalent to one of the form:

$$\mathfrak{D} = \left\{ \left(\mathbb{R}\bar{m}_i, \prod_{j,k} \left(1 + c_{ijk} z^{-m_{ijk}}\right) \right) : 1 \leq i \leq p \right\}$$

such that $r\left(m_{ijk}\right)$ is proportional to $\bar{m}_i$ with index j. We now define a scattering diagram of the form considered in 3.6.1:

$$\mathfrak{D}' = \left\{ \left(\mathbb{R}\bar{m}_i, \prod_{j,k} \left(1 + t_{ijk} z^{-r\left(m_{ijk}\right)}\right) \right) : 1 \leq i \leq p \right\}$$

This is now a scattering diagram over $\mathrm{k}[M][\![\{t_{ijk}\}]\!]$. Thus we have an enumerative interpretation for the rays of $S(\mathfrak{D}')$. We shall refer to this as a 'universal scattering diagram'. Rather than defining $\mathfrak{D}'$ over all $\mathrm{k}[M][\![\{t_{ijk}\}]\!]$ we can consider the monoid $Q \subseteq M \oplus \mathbb{N}^l$ where the second factor corresponds to the t_{ijk} variables and $l = \sum_{i,j} l_{ij}$. There is a ring homomorphism $\phi\colon t_{ijk} z^{-r\left(m_{ijk}\right)} \mapsto c_{ijk} z^{-m_{ijk}}$ and we can define $\mathfrak{D}'$ over $\mathrm{k}[Q]/\phi^{-1}(I)$ for an $\mathfrak{m}$-primary ideal I. Following [**53**] we observe that there is a scattering diagram $\phi\left(S_{I'}\left(\mathfrak{D}'\right)\right)$ which is equivalent to $S_I(\mathfrak{D})$.

Remark 3.6.15. Given a joint $\mathfrak{j}$ supported on the interior of a 1-cell τ we may write down a collection of rays and lines as for a scattering diagram; we refer to this collection of rays and lines as $\mathfrak{D}_\mathfrak{j}$ and write:

$$\overline{\mathfrak{D}}_\mathfrak{j} = \{\mathfrak{d} \in \mathfrak{D}_\mathfrak{j} : \mathfrak{d} \text{ is a line}\}$$

By rewriting and factorising the functions attached to the slab and rays intersecting this joint we may assume $\overline{\mathfrak{D}}_\mathfrak{j}$ is of the form:

$$\overline{\mathfrak{D}}_\mathfrak{j} = \left\{ \mathbb{R}\bar{m}_i, \prod_{j,k} \left(1 + c_{ijk} z^{-m_{ijk}}\right) \right\}$$

Deviating from [**53**], there may be several factors (not just one) which are not in the maximal ideal $\mathfrak{m}$.

Definition 3.6.16. Notice that we have factorized the slab function at $\mathfrak{j}$; consequently we may define a set $\mathcal{J}$ of triples (i, j, k) such that:

$$f_{\tau,\mathfrak{j}} = \prod_{\mathcal{J}} \left(1 + c_{ijk} z^{-m_{ijk}}\right)$$

Recall that i here indexes the direction vectors of rays, and that any 'bad factor' (that is, any factor not of the form $1 + x$ with $x \in \mathfrak{m}$) is associated to the (one-dimensional) slab τ. Thus if $(i, j, k) \in \mathcal{J}$ then i must be one of at most two possibilities. If there are two distinct values of i denote them i_+, i_- and note that $\bar{m}_{i_-} = -\bar{m}_{i_+}$. Conversely, if all the elements of $\mathcal{J}$ have a unique value of i then we shall refer to this as i_+ and shall not define i_-.

We shall define an inverse system of ideals $I_e \subseteq \mathrm{k}[M] \otimes_k \mathrm{k}[\{t_{ijk}\}]$ such that $\overline{\mathfrak{D}}_\mathfrak{j}$ is a genuine scattering diagram with respect to each I_e and use the enumerative interpretation of these scattering diagrams to show these stabilise as $e \to \infty$. This will imply that we can define a scattering diagram over the completion with respect to the inverse system I_e; the required localisation is then a subring of this completion.

PROPOSITION 3.6.17. *Consider J a monomial ideal in the ring*

$$R = \mathrm{k}\left[t_{ijk} : (i,j,k) \notin \mathcal{J}\right]$$

with R/J artinian. For a non-negative integer e, let

$$I_e = \sum_{(i,j,k)\in\mathcal{J}} \left(t_{ijk}^e\right) + J$$

in $\mathrm{k}[M] \otimes_k \mathrm{k}\left[\{t_{ijk}\}\right]$. Now apply the Kontsevich–Soibelman algorithm to obtain $S_{I_e}(\mathfrak{D})$ and remove all $(\mathfrak{d}, f_{\mathfrak{d}})$ equal to 1 modulo I_e. The sequence of scattering diagrams $\mathfrak{D}_1, \mathfrak{D}_2, \cdots$ stabilizes.

PROOF. Take Γ to be the set of collections of graded partitions $G = (G_1, \cdots, G_p)$ such that:

$$\prod_{(i,j,k)\notin\mathcal{J}} t_{ijk}^{p_{ijk}/j} \notin J$$

and such that $p_{ijk} > 0$ for some $(i,j,k) \notin \mathcal{J}$. Now R/J Artinian implies that *having fixed* the values of $\{p_{ijk} : (i,j,k) \in \mathcal{J}\}$ there are finitely many choices of G, but these are themselves unconstrained. We proceed in two steps, following Proposition 6.47 in [**53**]. First we show that there are only a finite number with $N_G \neq 0$, then we bound the number of terms of any $\log f_{\mathfrak{d}}$ independently of e.

Suppose $G \in \Gamma$ and $N_G \neq 0$. Then there is a primitive integral vector $m_{\mathfrak{d}}$ such that:

$$-\sum_i |G_i| m_i = k_G m_{\mathfrak{d}}$$

and such a ray $(\mathfrak{d}, f_{\mathfrak{d}})$ must appear in the scattering diagram $\mathfrak{D}_e$ with support $\mathbb{R}_{\geq 0} m_{\mathfrak{d}}$. Let $\Sigma_{\mathfrak{d}}$ be as above; recall this fan is only determined up to arbitrary fan refinements so we may assume that both $\mathbb{R}_{\geq 0} m_{i_+}$ and $\mathbb{R}_{\leq 0} m_{i_+}$ appear in this fan, noting that the latter is equal to $\mathbb{R}_{\geq 0} m_{i_-}$ if i_- is defined. Hence there exists a toric morphism:

$$\pi \colon X_{\mathfrak{d}} \to \mathbb{P}^1$$

defined by these two rays. There are two toric sections of this morphism, which we shall refer to as D_+ and D_- corresponding to $\mathbb{R}_{\geq 0} m_{i_+}$ and $\mathbb{R}_{\leq 0} m_{i_+}$ respectively. Now $N_G \neq 0$ implies there is a map $\bar{f} \colon C \to X_{\mathfrak{d}}$ such that $\bar{f}_*(C)$ has intersection multiplicity p_{ijk} at x_{ijk} for $(i,j,k) \in \mathcal{J}$. Without loss of generality we assume that for any $t \neq 0, \infty$ the fiber $\pi^{-1}(t)$ contains at most one of the points x_{ijk}.

We wish to eliminate the possibility that the image of $\bar{f}$ contains $\pi^{-1}\left(\pi\left(x_{ijk}\right)\right)$ for any $(i,j,k) \in \mathcal{J}$. Observe that $\pi^{-1}\left(\pi\left(x_{ijk}\right)\right)$ meets $\partial X_{\mathfrak{d}}$ at a point other than any x_{ijk}; call this point q'. We also know that the divisor class $\sum p_{ijk} x_{ijk} + k_G q$ is of the class $\beta|_{\partial X_{\mathfrak{d}}}$ which is determined by G. Indeed, the set $\bar{f}(C) \cap \partial X_{\mathfrak{d}}$ is the collection x_{ijk} and one additional point q. If we assume that $\bar{f}(C)$ contains this fibre $\pi^{-1}\pi(x_{ijk})$, then we must have that $q = q'$. However we have assumed that there is at least one x_{ijk} such that $(i,j,k) \notin \mathcal{J}$ and $p_{ijk} > 0$, moving this point alone we obtain a contradiction.

As remarked, $\bar{f}_*(C)$ represents the class β and $\tilde{f}_*(C)$ represents β_G, the strict transform defined above. The total transform of $\pi^{-1}(\pi(x_{ijk}))$ contains the irreducible component E_{ijk}, and we know that $E_{ijk}.\beta_G = p_{ijk}$. Thus if F is the class of the fiber of π, $\beta.F \geq p_{ijk}$. Now assume $\pi^{-1}(0)$ does not contain D_{out}, indeed, swap it with $\pi^{-1}(\infty)$ if it does. The proper transform of $\pi^{-1}(0)$ is disjoint from $\tilde{f}(C)$ but $\beta.\pi^{-1}(0)$ is determined by the G_i for $i \neq i_+, i_-$. Thus p_{ijk} is bounded and this bound is independent of $\mathfrak{d}$, so there are a finite number of possiblities for $G \in \Gamma$.

The rest of the proof of Proposition 6.47 in [**53**] goes through as stated, expect that now we need to observe that

$$\left\{ \left(\mathbb{R}m_i, 1 + t_{ijk} z^{-m_i} \right) : (i,j,k) \in \mathcal{J} \right\}$$

contains no rays, meaning that the formula for any ray must have a coefficent t_{ijk} for some $(i,j,k) \notin \mathcal{J}$, and so the number of terms appearing in the formula for $\log(f_{\mathfrak{d}})$ is finite, and with bound determined by J, that is independent of e. One can now apply a factorization process and generate rays with functions $f_{\mathfrak{d}}$ all of the form $1 + cz^m$.

□

As remarked in [**53**] the purpose of this result to form $S(\mathfrak{D}) = \cup S_{I_e}(\mathfrak{D})$, which is a scattering diagram over the completion of $A = \mathbb{C}[M] \otimes \mathrm{k}[\{t_{ijk}\}]$ with respect to $\sum_{(i,j,k) \in \mathcal{J}} (t_{i,j,k})$; this completion contains the subring given by localising A at the various factors $1 + t_{ijk} z^{-m_{ijk}}$.

We can now generate rays in the structure $\mathscr{S}$ from the rays of this scattering diagram, yielding a compatible structure:

THEOREM 3.6.18. *$\mathscr{S}_k$ is compatible to order k.*

PROOF. The proof of Theorem 6.49 in [**53**] now goes through exactly, replacing Proposition 6.47 there with Proposition 3.6.17 above. □

3.7. Constructing the formal degeneration

We outline how the construction of the inverse system of rings in the last two sections allows one to construct a flat deformation by deforming each ring in turn. This section is a variation on Section 6.2.6 in [**53**].

3.7.1. Notation. We define an open set U^k_ω for each stratum ω, as follows. The sets U^k_ω together cover the kth-order smoothing, and U^k_ω defines a smoothing of the chart $V(\omega)$ on the central fiber defined in Section 3.4.

DEFINITION 3.7.1. Let

$$R^k_\omega := \varprojlim_{\omega \subseteq \tau} R^k_{\omega,\tau,\mathfrak{u}_\tau}$$

and set $U^k_\omega := \operatorname{Spec} R^k_\omega$.

Since the change of chamber maps are isomorphisms, a different choice of $\mathfrak{u}_\tau$ will yield an isomorphic inverse system – as proved in 6.2.6 of [**53**]. The main result of this section is:

PROPOSITION 3.7.2. *U^k_ω is a flat deformation of U^0_ω over $S_k := \operatorname{Spec} \mathrm{k}[t]/\left(t^{k+1}\right)$.*

We first compute the central fibre of this degeneration:

LEMMA 3.7.3. *U^0_ω is $\operatorname{Spec} \mathrm{k}\left[P_{\phi,x}\right]/(t)$ for $x \in \operatorname{Int}(\omega) \cap B_0$.*

PROOF. We give a brief outline of the proof from Lemma 6.30 of [**53**]:

(1) As all scattering diagrams are trivial we assume that chambers coincide with maximal cells of $\mathscr{P}$.
(2) There are no non-trivial change of chamber maps since the only non-zero elements of $R^0_{\omega,\tau,\sigma}$ for one-dimensional τ are parallel to τ.
(3) Thus the inverse system is just the one made up of all the canonical change of strata maps, and so we recover the toric picture as if there were no scattering.

□

The proof of flatness of U^k_ω over S_k is divided into three parts of increasing complexity, depending on the dimension of the stratum ω.

3.7.2. Codimension 0. For U^k_ω with ω two-dimensional we necessarily have that $\sigma_{\mathfrak{u}_\omega} = \omega$. Thus $P_{\phi,\omega,\sigma} = \Lambda_x \times \mathbb{N}$ and $U^k_\omega = U^0_k \times S_k$, i.e. a trivial deformation.

3.7.3. Codimension 1. For U^k_ω with ω one-dimensional we compute an explicit fiber product and show that this is flat. Following [**53**, **59**, **60**] we fix a one-dimensional ω and let $\sigma_\pm$ be the maximal cells containing ω. We assume that the piecewise linear function ϕ has slope zero on σ_- and slope $l\breve{d}_\omega$ on σ_+; here $\breve{d}_\omega$ is primitive.

There are three rings over which we shall compute the fiber product: $R_\pm = R^k_{\omega,\sigma_\pm,\mathfrak{u}_{\sigma_\pm}}$ and $R_\cap = R^k_{\omega\omega\mathfrak{u}_{\sigma_+}}$ - observe the choice of σ_+ made in defining $R_\cap$. We now define:

$$f_\omega := f_{\omega,x} \prod_{(\mathfrak{d},x)} f_{\mathfrak{d},x}$$

and regard this as lying in $\mathrm{k}[\Lambda_\omega][t]$. Lemma 6.33 of [**53**] then implies that:

LEMMA 3.7.4. *The fiber product $R_- \times_{R_\cap} R_+$ is isomorphic to the ring*

$$R_\cup = \mathrm{k}\left[\Lambda_\omega\right]\left[U,V,t\right]/\left(UV - f_\omega t^l, t^{k+1}\right)$$

PROOF. The reader is referred to the proof of Lemma 6.33 of [**53**] □

EXAMPLE 3.7.5. Consider the local models obtained by the above procedure when $\Delta \cap \rho$ is:

(1) one point with length 2 monodromy polytope;

(2) two distinct points, each with simple monodromy.

Applying Lemma 3.7.4 the two cases give the following rings:

(1) $\mathbb{C}[U,V,W,t]/\left(UV - t(W-a)^2, t^k\right)$
(2) $\mathbb{C}[U,V,W,t]/\left(UV - t(W-b)(W-c), t^k\right)$

where a,b,c are parameters.

We now consider the singularities of the generic fiber of each of these families. The first of these exhibits an ordinary double point at $(0,0,a,t) \in \mathbf{A}^3_{U,V,W} \times \{t\}$, while the second ring gives a smooth affine variety. We then see the connection between a family of affine varieties defined by varying the parameters b,c and sliding two singularites of an affine structure until they coalesce. This is precisely the behavour prohibited in [**53**, **59**] by demanding the affine manifold be *locally rigid*.

3.7.4. Codimension 2 strata. In [**53**, **59**] this is by far the most difficult step. However working with a more complicated singular locus than used in [**53**] does not change this argument and so details of the proof are not recalled here.

As usual, the rings corresponding to the local patch at the zero-cell ω are given by the inverse limit:

$$R^k_\omega = \varprojlim R^k_{\omega,\tau,\mathfrak{u}_\tau}$$

The inverse limit is over strata $\tau \supseteq \omega$, with a choice of chamber $\mathfrak{u}_\tau$ for each stratum. In [**53**] it is shown that the choice of this chamber does not change the isomorphism class of the inverse limit.

3.8. Local models at vertices

We wish to lift the operation of exchanging corners for singularities described in Section 3.2 to a deformation of the rings we have attached to these corners in Sections 3.5, 3.6. To define this deformation we will use an explicit description of the rings at the corners of B. In fact we give two descriptions; the first based on gluing the rings $R^k_{\omega,\tau,\mathfrak{u}}$, the second on the canonical cover construction for surface singularities. The equivalence of these formulations makes evident that we are constructing $\mathbb{Q}$-Gorenstein deformations.

3.8.1. Local description of the affine manifold. Fix a vertex ω of $\mathscr{P}$ contained in ∂B and a chart $U \subseteq B$ containing ω which intersects a minimal number of strata of $\mathscr{P}$. We shall assume for the rest of this section that:

(1) $\mathscr{P}$ divides U into two regions, described by intersecting U with a pair of 2-cells σ_1, σ_2 which meet along a 1-cell τ.
(2) we have fixed a structure $\mathscr{S}$ on B. Let $\mathscr{S}_\omega$ be the set of rays in $\mathscr{S}$ intersecting ω.
(3) If $\mathfrak{d} \in \mathscr{S}_\omega$ then $\mathfrak{d}|_U$ is supported on τ.

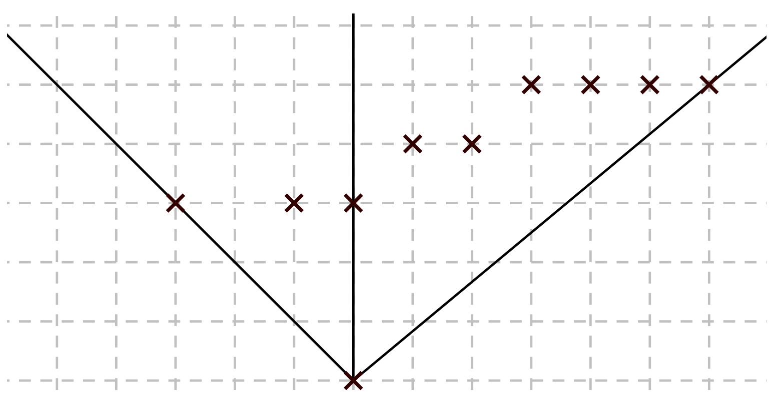
REMARK 3.8.1. These assumptions are automatically satisfied if B is of polygon type. Also, point 2 implies that there are two distinguished chambers independent of k whose boundary contains $\tau \cap U$. We refer to these as $\mathfrak{u}_1$ and $\mathfrak{u}_2$ respectively, where we have suppressed the dependence on k.

For ease of exposition we will assume without loss of generality that ϕ vanishes on the left-hand cone, i.e. on $\mathfrak{u}_1$.

NOTATION 3.8.2.

(1) Each σ_i for $i = 1, 2$ contains a 1-cell in ∂B intersecting ω. We denote these 1-cells τ_1, τ_2 respectively.
(2) Let n_0 be the unique primitive vector in $\Lambda^\star_\omega$ which annihilates the subspace defined by τ and evaluates postively on $\mathfrak{u}_1$.
(3) Denote by n_1, n_2 the unique primitive vectors in $\Lambda^\star_\omega$ annihilating τ_1, τ_2 respectively and evaluating non-negatively along τ.
(4) Let $f := f_\tau.\prod_{\mathfrak{d}} f_{\mathfrak{d}}$ where f_τ is the slab function on τ and the product is over rays $\mathfrak{d}$ supported on τ.

Now we have fixed this notation we describe the rings $R^k_{\omega,\rho,\mathfrak{u}_i}$ for different choices of ρ and i. Recalling that any such ring is a quotient of $k\,[P_{\omega,\phi}]$ we fix a generating set for the monoid $P_{\omega,\phi}$. After taking the projection $m \mapsto \bar{m}$ the generators are distributed in some fashion across the two subcones:



We will name the generators depending on the cone they project to. $\mathbb{C}\,[P_{\omega,\phi}]$ is generated as a $\mathbb{C}[t]$-module by three collections of monomials:

(1) x_i correspond to generators of the left-hand cone (not supported on τ). x_0 corresponds to a vector m_0 such that $\overline{m}_0 \in \tau_1$.
(2) y_j correspond to generators of the right-hand cone (not supported on τ). y_0 corresponds to a vector m_0 such that $\overline{m}_0 \in \tau_2$.
(3) w is the primitive generator of τ.

We recall the standard result in toric geometry that describes the corresponding ideal.

LEMMA 3.8.3. *If C is a cone in a lattice M with generating set $m_1, \cdots, m_s$ there is a natural short exact sequence:*

$$0 \to L \to \mathbb{Z}^s \to M \to 0$$

Writing $l \in L$ via the injective map into $\mathbb{Z}^s$ we can write $l = \sum l_i e_i$; now one may form the ideal $I = \left\langle \prod_{l_i > 0} x_i^{l_i} - \prod_{l_i < 0} x_i^{-l_i} \right\rangle$, and $k[x_1, \ldots, x_s]/I$ is the affine toric variety $\operatorname{Spec} k[C]$.

PROOF. See [**28**], chapter 1. □

The 2-cells σ_1, σ_2 define a pair of cones with their origin at the vertex ω. Let C_1, C_2 be the semigroups defined by the integral points of these cones respectively. Using Lemma 3.8.3 the relations between the generators specified for the monoid $P_{\omega,\phi}$ are generated by those of the form:

$$w^\gamma \prod x_i^{\alpha_i} \prod y_j^{\beta_j} - w^\delta \prod x_i^{\gamma_i} \prod y_j^{\delta_j}$$

Recall that in general we have:

$$R^k_{\omega,\sigma_1,\mathfrak{u}_1} = k\left[P_{\omega,\phi}\right]/I_{\omega,\sigma_1,\sigma_1}$$

Now we observe that the order of a monomial $M = t^\gamma \prod y_j^{\beta_j} w^\alpha$ in this monoid is given by:

$$\operatorname{ord}_\tau(M) = \sum \beta_j \phi_\omega\left(\bar{m}_j\right) + \gamma$$

This formula, together with the observation that over σ_1 ord_τ is just the t-degree fixes an explicit description of the ideal:

$$I_{\omega,\sigma_1,\sigma_1} = \langle M : \operatorname{ord}_\tau(M) > k \rangle$$

REMARK 3.8.4. We may view the ring $R^k_{\omega,\sigma_1,\mathfrak{u}_1}$ as a module over $S_k[w]$; letting $S_k[C_1]$, respectively $S_k[C_2]$ be the submodule of $k\left[P_{\omega,\phi}\right]$ generated by the x_i (respectively by the y_j) $R^k_{\omega,\sigma_1,\mathfrak{u}_1}$ may be expressed as a pushout:

$$\begin{tikzcd} S_k[w] \arrow[r] \arrow[d] & S_2 \arrow[d] \\ S_1 \arrow[r] & R^k_{\omega,\sigma_1,\mathfrak{u}_1} \end{tikzcd}$$

in which $S_1 = S_k[C_1]/(t^{k+1})$ and

$$S_2 = S_k[C_2]/\left\langle \prod t^\gamma y_j^{\beta_j} : \sum \beta_j \phi_\omega\left(\bar{m}_j\right) + \gamma > k \right\rangle$$

DEFINITION 3.8.5. For each cone C_i, $i = 1, 2$, let C_i° be the cone generated by $x_0, \cdots, x_N$, $y_0, \cdots, y_M$ respectively.

LEMMA 3.8.6. *The $S_k[w]$-module*

$$\widetilde{R}^k := S_k[C_1 \backslash \langle w \rangle] \oplus S_k[C_2 \backslash \langle w \rangle] \oplus S_k[w]$$

is a finitely generated $S_k[w]$-module and there is a surjective homomorphism $\widetilde{R}^k \to R^k_{\omega,\sigma_1,\mathfrak{u}_1}$.

Proof. Observe that the rings $S_k[C_i \backslash \langle w \rangle]$ are finitely generated $S_k[w]$-modules since there are canonical surjective homomorphisms: $S_k[w][C_i^\circ] \to S_k[C_i \backslash \langle w \rangle]$ for $i = 1, 2$. Each factor of $\widetilde{R}^k$ has a canonical map to a term of the push-out diagram above, together defining a map to $R^k_{\omega,\sigma_1,\mathrm{u}_1}$. Using this push-out and fixing an element of $R^k_{\omega,\sigma_1,\mathrm{u}_1}$ it may be expressed as a pair (u_1, u_2); in which u_i is a sum of monomials from σ_i for $i = 1, 2$. After removing terms involving only the variable w from each u_i we may express any element of $R^k_{\omega,\sigma_1,\mathrm{u}_1}$ as a triple of the form required. $\square$

We remark that analogous observations may be made about the rings $R^k_{\omega,\sigma_2,\mathrm{u}_2}$ and $R^k_{\omega,\tau,\mathrm{u}_1}$. Using this notation we now describe the co-ordinate ring of the affine patch containing the given vertex, that is the inverse limit of the following system.

$$
\begin{tikzcd}
 & & R^k_\Pi \arrow[dl, dotted] \arrow[dr, dotted] \arrow[dd, dotted] \arrow[ddll, dotted, bend right] \arrow[ddrr, dotted, bend left] \arrow[ddd, dotted, bend left] & & \\
 & R^k_{\omega,\sigma_1,\mathrm{u}_1} \arrow[dl] \arrow[dr] & & R^k_{\omega,\sigma_2,\mathrm{u}_2} \arrow[dl] \arrow[dr] & \\
R^k_{\omega,\tau_1,\mathrm{u}_1} \arrow[drr] & & R^k_{\omega,\tau,\mathrm{u}_1} \arrow[d] & & R^k_{\omega,\tau_2,\mathrm{u}_2} \arrow[dll] \\
 & & R^k_{\omega,\omega,\mathrm{u}_1} & &
\end{tikzcd}
$$

Remark 3.8.7. The inverse limit described above is manifestly isomorphic to the fiber product:

$$
\begin{tikzcd}
R^k_\Pi \arrow[r, dotted] \arrow[d, dotted] & R^k_{\omega,\sigma_1,\mathrm{u}_1} \arrow[d] \\
R^k_{\omega,\sigma_2,\mathrm{u}_2} \arrow[r] & R^k_{\omega,\tau,\mathrm{u}_1}
\end{tikzcd}
$$

If $u \in R^k_\Pi$, $u = (u_1, u_2)$ and the restrictions of u_i to $R^k_{\omega,\tau,\mathrm{u}_i}$ for $i = 1, 2$ respectively are related by the change of chamber map. Formally, we take the change of strata maps and compose the second with the change of chamber map:

$$
\begin{tikzcd}
R^k_{\omega,\sigma_1,\mathrm{u}_1} \arrow[d] & R^k_{\omega,\sigma_2,\mathrm{u}_2} \arrow[d] \\
R^k_{\omega,\tau,\mathrm{u}_1} & R^k_{\omega,\tau,\mathrm{u}_2} \arrow[l, dashed, "\theta_{\mathrm{u}_2,\mathrm{u}_1}"]
\end{tikzcd}
$$

Recall the following facts:

(1) Applying the change of chamber isomorphism $\theta_{\mathrm{u}_2,\mathrm{u}_1}$ to variables x_i, we have that: $\theta_{\mathrm{u}_2,\mathrm{u}_1}(x_i) = f^{\langle n_0, \bar{m} \rangle} x_i$.

(2) There is a similar formula for the $\theta_{\mathrm{u}_2,\mathrm{u}_1}(y_j)$ and w is always mapped to itself, as n_0 annihilates the tangent space to τ.
(3) The rings $R^k_{\omega,\tau,\mathrm{u}_i}$, $i = 1, 2$ have been localised at the slab function, ensuring that change of chamber map is an isomorphism.

We are now in a position to give an elementary description of the formal smoothing of the affine chart at a boundary vertex obtained from the Gross–Siebert reconstruction algorithm.

DEFINITION 3.8.8. $R^k_{\cup} = S_k[X_i, Y_j, W, t : 0 \le i \le N, 0 \le j \le M]/I_{\cup}$. To define $I_{\cup}$ consider each binomial relation

$$w^{\eta_1}\prod_{i,j} x_i^{\alpha_i} y_j^{\beta_j} = t^{\chi} w^{\eta_2} \prod_{k,l} x_k^{\gamma_k} y_l^{\delta_l}$$

in the usual monoid over ϕ on $C_1 \cup C_2$. We define an element of $I_{\cup}$ which may take one of two forms; if the monomials correspond to a lattice vector in C_1 consider the polynomial

$$f^{-\sum_l \delta_l \langle n_0, m_l\rangle} W^{\eta_1} \prod_{i,j} X_i^{\alpha_i} Y_j^{\beta_j} - f^{-\sum_j \beta_j \langle n_0, m_j\rangle} t^{\chi} W^{\eta_2} \prod_{k,l} X_k^{\gamma_k} Y_l^{\delta_l}$$

otherwise, if it is over C_2, consider the polynomial

$$f^{\sum_k \gamma_k \langle n_0, m_k\rangle} W^{\eta_1} \prod_{i,j} X_i^{\alpha_i} Y_j^{\beta_j} - f^{\sum_i \alpha_i \langle n_0, m_i\rangle} t^{\chi} W^{\eta_2} \prod_{k,l} X_k^{\gamma_k} Y_l^{\delta_l}$$

Here f is considered as an element of $S_k[W]$ (rather than $S_k[w]$). Divide out the given polynomial by as many factors of f as possible and append it to the generating set of $I_{\cup}$. For clarity we shall suppress the W^{η_i} in these relations from now on.

PROPOSITION 3.8.9. *There is a ring isomorphism* $\Phi \colon R^k_{\cup} \to R^k_{\Pi}$ *given on generators by:*

$$\begin{aligned} X_i &\mapsto \left(x_i, f^{\langle n_0, m_i\rangle} x_i\right) \\ Y_j &\mapsto \left(f^{\langle n_0, m_j\rangle} y_j, y_j\right) \\ W &\mapsto (w, w) \\ t &\mapsto (t, t) \end{aligned}$$

REMARK 3.8.10. Compare with the description around an interior 1-cell given in [**53**]. These rings are more complicated but the change of chamber map in the fiber product is essentially the same.

PROOF. To show this map is well-defined we consider the images under Φ of the generators of $I_\cup$. Indeed, we may simply compute Φ:

$$\begin{aligned}
&\Phi\left(f^{\sum_k \gamma_k\langle n_0,m_k\rangle}\prod_{i,j}X_i^{\alpha_i}Y_j^{\beta_j}\right)=\\
=&f^{\sum_k \gamma_k\langle n_0,m_k\rangle}\left(f^{-\sum\beta_j\langle n_0,m_j\rangle}\prod_{i,j}x_i^{\alpha_i}y_j^{\beta_j}, f^{\sum\alpha_i\langle n_0,m_i\rangle}\prod_{i,j}x_i^{\alpha_i}y_j^{\beta_j}\right)\\
=&f^{\sum_k \gamma_k\langle n_0,m_k\rangle+\sum_i\alpha_i\langle n_0,m_i\rangle}\left(f^{-\sum\beta_j\langle n_0,m_j\rangle-\sum\alpha_i\langle n_0,m_i\rangle}\prod_{i,j}x_i^{\alpha_i}y_j^{\beta_j},\prod_{i,j}x_i^{\alpha_i}y_j^{\beta_j}\right)\\
=&f^{\sum_k \gamma_k\langle n_0,m_k\rangle+\sum_i\alpha_i\langle n_0,m_i\rangle}\left(f^{-\sum\delta_l\langle n_0,m_l\rangle-\sum\gamma_k\langle n_0,m_k\rangle}\prod_{k,l}x_k^{\gamma_k}y_l^{\delta_l},\prod_{k,l}x_k^{\gamma_k}y_l^{\delta_l}\right)\\
=&f^{\sum_i\alpha_i\langle n_0,m_i\rangle}\left(f^{-\sum\delta_l\langle n_0,m_l\rangle}\prod_{k,l}x_k^{\gamma_k}y_l^{\delta_l}, f^{\sum_k\gamma_k\langle n_0,m_k\rangle}\prod_{k,l}x_k^{\gamma_k}y_l^{\delta_l}\right)\\
=&\Phi\left(f^{\sum_i\alpha_i\langle n_0,m_i\rangle}\prod_{k,l}X_k^{\gamma_k}Y_l^{\delta_l}\right)
\end{aligned}$$

To show Φ is surjective we use Lemma 3.8.6, which gives a generating set for the algebras $R^k_{\omega,\sigma_i,\mathfrak{u}_i}$ as $S_k[w]$ modules. Fix an element $(u_1,u_2)\in R^k_\Pi$, without loss of generality we assume that there are no terms in $u_i, i=1,2$ involving only w as any polynomial $g(w)$ may be accounted for by taking $\Phi(g(W))$. Now we (non-uniquely) write $u_1=\sum_k c_k\prod_i x_i^{\alpha_{i,k}}+h_1(y_j : 0\le j\le M)$ where the coefficents c_m lie in the ring $S_k[w]$. Similarly we write $u_2=\sum_l c_l\prod_j y_j^{\beta_{j,l}}+h_2(x_i : 0\le i\le N)$ using the same coefficent ring.

We claim that the pair (u_1,u_2) is in R^k_Π if and only if it is equal to:

$$\Phi\left(\sum_k c_k\prod_i x_i^{\alpha_{i,k}}+\sum_l c_l\prod_j y_j^{\beta_{j,l}}\right)$$

By the previous calculation this is certainly in the fiber product; furthermore this element agrees with all the x_i terms in f_1 and the y_j terms in f_2 by definition. All that remains is to check that this uniquely determines the h_1 and h_2. However the change of strata map is the identity on h_1 and h_2 and so we may express these in terms of previously determined quantities, for example:

$$h_1=\theta_{\mathfrak{u}_2,\mathfrak{u}_1}\psi_{(\omega,\sigma_2),(\omega,\tau)}\left(\sum_l c_l\prod_j y_j^{\beta_{j,l}}\right)$$

We next show that this map is injective. Assume we have a element $u\in R^k_\cup$ that is mapped to a pair (u_1,u_2) such that $u_1\in I^k_{\omega,\sigma_1,\mathfrak{u}_1}$ and $u_2\in I^k_{\omega,\sigma_2,\mathfrak{u}_2}$. Observe that we may rewrite any monomial $\prod_{i,j}x_i^{\alpha_i}y_j^{\beta_j}$, using the toric relations, in one of the following two forms:

(1) $\prod_{i,j}x_i^{\alpha_i}y_j^{\beta_j}-\prod_k x_k^{\gamma_k}$
(2) $\prod_{i,j}x_i^{\alpha_i}y_j^{\beta_j}-\prod_l y_l^{\delta_l}$

From Definition 3.8.8 we have a relations in $I_\cup$ of the form:

(1) $\prod_{i,j} X_i^{\alpha_i} Y_j^{\beta_j} - f^{\sum \alpha_i \langle n_0, m_i \rangle} \prod_k X_k^{\gamma_k}$
(2) $\prod_{i,j} X_i^{\alpha_i} Y_j^{\beta_j} - f^{-\sum \beta_j \langle n_0, m_j \rangle} \prod_l Y_l^{\delta_l}$

Thus we can assume that there are no terms involving both the X_i and the Y_j appearing in a representative of $R^k_\cup$, but by Proposition 3.8.9 Φ is the identity onto one of the two factors. Since the image onto this factor is in $I^k_{\omega,\sigma_i,\mathfrak{u}_i}$ for some i we may infer that the original element is in $I_\cup$. □

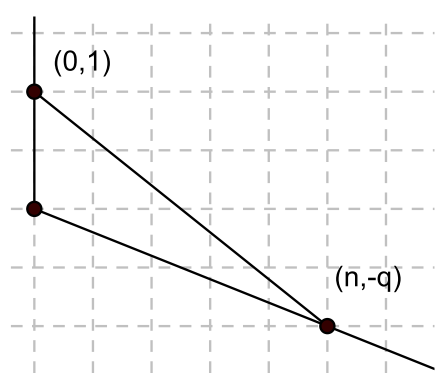
3.8.2. The canonical cover. We conclude this section by exhibiting a construction of the canonical cover for these rings; this will be used in the next section to construct a $\mathbb{Q}$-Gorenstein deformation.

Given a vertex $v \in B$ fix a chart of B containing v and let C denote the tangent cone at v. We shall assume for the remainder of this section that

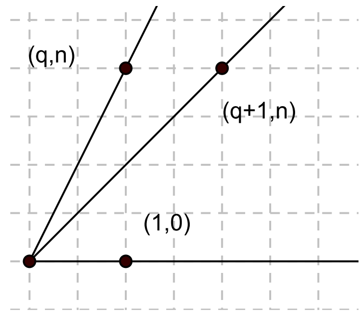
(1) $\mathscr{P}$ splits C into two cones C_i ,$i = 1, 2$, divided by a ray L.
(2) Denoting the primitive generators of C by v_1, v_2 respectively we have that $v_1 + v_2 \in L$.

Lemma 3.8.11. *Given a Fano polygon P fix a vertex v, its tangent wedge C and the ray L of the spanning fan of Q meeting v. The pair (C, L) satisfies the two conditions above.*

Proof. The first condition is obvious, the spanning fan introduces precisely one new ray intersecting v. For the second condition note that an edge of P may be put into the following standard form:



with the vertices of P at $(0, 1), (n, -q)$. Taking the dual cone:



We see the (rational) generators of this cone are $(1, 0), (q, n)$, the ray L defined by the normal to the edge of P is generated by $(q + 1, n)$ and satisfies the second condition. $\square$

We recall the canonical cover construction for the singularity $X = \frac{1}{n}(1, q)$, for which we use the following notation:

NOTATION 3.8.12.

(1) Define $p := q + 1$.
(2) Let $w := \mathrm{hcf}(n, p)$ and define a, r by requiring that $n = wr, p = wa$, so in particular $q = wa - 1$.
(3) Define m, w_0 by $w = mr + w_0$ with $0 \leq w_0 < r$.

REMARK 3.8.13. The singularity content of the singularity $X = \frac{1}{n}(1, q)$ is precisely m.

Having fixed this notation the canonical cover of X is:

CONSTRUCTION 3.8.14. Letting $X = \frac{1}{n}(1, q)$ there is an embedding $X \hookrightarrow \frac{1}{r}(1, q, a)$ which takes X onto the hypersurface $\{xy = z^w\}/\mu_r$. The $\mathbb{Q}$-Gorenstein deformations of X are determined by considering the space $\mathbb{C}^{m+1}$ of degree-m polynomials f_m and forming the family of hypersurfaces

$$\{xy = z^{w_0} f_m(z^r)\}/\mu_r$$

We shall show that our local model $R_\cup$ is always of this form and thus that the space of polynomials defined by the log-structure on this line segment may be identified with the parameter space of $\mathbb{Q}$-Gorenstein deformations.

In order to prove this relation, we compare the cones constructed in the proof of Lemma 3.8.11 to Construction 3.8.14.

CONSTRUCTION 3.8.15. Given $X = \frac{1}{n}(1, q)$, the fan of X is given by

$$\mathrm{Cone}((0, 1), (n, -q))$$

as in the proof of Lemma 3.8.11. This is isomorphic to the cone $\mathrm{Cone}((1, 0), (0, 1))$ in the lattice: $\mathbb{Z}^2 + \frac{1}{n}(1, q)$. Similarly $Y = \frac{1}{r}(1, q, a)$ is determined by $\mathrm{Cone}((1, 0, 0), (0, 1, 0), (0, 0, 1))$ in the lattice: $\mathbb{Z}^3 + \frac{1}{r}(1, q, a)$. Following Construction 3.8.14 we should consider the hypersurface

$\{xy = z^w\}$. This is the image of the embedding $X \hookrightarrow Y$. This embedding is induced by a map $\iota\colon \mathbb{Z}^2 + \frac{1}{n}(1,q) \to \mathbb{Z}^3 + \frac{1}{r}(1,q,a)$ between the respective lattices which may be expressed as the following matrix, which we also call ι.

$$\iota = \begin{pmatrix} w & 0 \\ 0 & w \\ 1 & 1 \end{pmatrix}$$

In particular $\iota(\frac{1}{n}(1,q)) = \frac{1}{n}(w, qw, 1+q) = \frac{1}{wr}(w, qw, wa) = \frac{1}{r}(1,q,a)$. We wish to compute the map between the dual lattices induced by ι. Observe that $(\mathbb{Z}^2 + \frac{1}{n}(1,q))^\vee$ is the sublattice

$$\left\{\alpha \in \mathbb{Z}^{2^\vee} : \alpha((1,q)) \in n\mathbb{Z}\right\}$$

of the dual lattice $\mathbb{Z}^{2\vee}$. There is an analogous expression for the lattice dual to $\mathbb{Z}^3 + \frac{1}{r}(1,q,a)$. From the matrix ι we may easily compute $\iota^\star$, in particular $\iota^\star(x^r) = x^n$, $\iota^\star(y^r) = y^n$ and $\iota^\star(z^r) = x^r y^r$.

Remark 3.8.16. Recall that the image of $\iota^\star$ is a sublattice of $\mathbb{Z}^{2\vee}$. The lattice elements corresponding to $x^n, x^r y^r, y^n$ are all primitive in this lattice, for example $x^r y^r$ is the generator of the cone previously called W.

Using these constructions we shall define a ring $R'^k_\cup$ and prove that it is isomorphic to $R^k_\cup$.

Definition 3.8.17. Given a zero stratum v of $\mathscr{P}$ contained in ∂B we may form the pair (C, L) as above. Note that C need not be strictly convex. In particular we may define the integers n, q, w for this cone.

$$R'^k_\cup = S_k[x,y,z]^{\mu_r}/\big(xy = t^l z^{w_0} f_\tau(z^r)\big)$$

where the μ_r action has weights $(1,q,a)$ and l is the slope of the piecewise linear function ϕ.

Proposition 3.8.18. *$R'^k_\cup$ is isomorphic to $R^k_\cup$.*

Proof. There is an obvious spanning set of $R'^k_\cup$ as an S_k-module; namely monomials with exponents in the sublattice of $\mathbb{Z}^{3\vee}$ dual to $(\mathbb{Z}^3 + \frac{1}{r}(1,q,a))$. Consider the submodule generated by the monomials $x^a z^b$ and $y^c z^d$; these give a basis for $R'^k_\cup$ as a S_k-module. Making the analogous statement for $R^k_\cup$ we observe that $R^k_\cup$ is generated as an S_k-module by monomials with exponents projecting to integral points in the cone C. There is an obvious identification of these two bases, which extends linearly to a map of S_k-modules; we now show this is an isomorphism of algebras. As a preliminary step we replace f_τ in the definition of $R'^k_\cup$ with $f = f_\tau \prod f_\mathfrak{d}$ where the product is over the rays $\mathfrak{d}$ of the scattering diagram supported on τ. Note each $f_\mathfrak{d}$ is invertible in $R'^k_\cup$, so an automorphism of $S_k[x,y,z]^{\mu_r}$ sending $xy \mapsto xy \prod f_\mathfrak{d}$ induces an isomorphism of $R'^k_\cup$ with $S_k[x,y,z]^{\mu_r}/\big(xy = t^l z^{w_0} f(z^r)\big)$.

Fix $U, V \in R^k_\cup$ and write $U = \bar{U} t^{l_1}$ and $V = \bar{V} t^{l_2}$ where $\bar{U} \in C_1$ and $\bar{V} \in C_2$. Now take the corresponding elements in $R'^k_\cup$: $\iota^\star(x^a z^b t^{l_1}), \iota^\star(y^c z^d t^{l_2})$. Suppose we have that UV projects to an element in C_1 and write $-\langle n_0, \bar{V}\rangle = \gamma$ so that $UV = \prod X_i^{a_i} W^b t^{l_1+l_2+\gamma l} f^\gamma$ where the X_i correspond to elements of the Hilbert basis of C_1. Writing

$$\iota^\star(x^a z^b t^{l_1}).\iota^\star(y^c z^d t^{l_2}) = \iota^\star(x^a y^c z^{b+d} t^{l_1+l_2})$$

UV in C_1 means that $c < a$ so using the relations in $R'^k_{\cup}$,

$$\iota^\star(x^a y^c z^{b+d} t^{l_1+l_2}) = \iota^\star\left(x^{a-c} z^{b+d+c.w_0} t^{l_1+l_2+cl} f(z^r)^c\right)$$

Our S_k-module isomorphism identifies

$$\iota^\star\left(x^{a-c} z^{b+d+c.w_0} t^{l_1+l_2+cl}\left(f(z^r)\right)^c\right)$$

with $\prod X_i^{a_i} W^b t^{l_1+l_2+cl} f^c$; thus we only need to show that $\gamma = c$. Recall we have identifed C with the quadrant in a sublattice of $\mathbb{Z}^{2\vee}$. Therefore we can compute $\langle n_0, (v_1, v_2)\rangle$ directly. The primitive generator of L in this sublattice of $\mathbb{Z}^{2\vee}$ is (r, r); the obvious element annihilating (r, r) is $(1, -1)$, but this has index w ($(1, 1) = wr\frac{1}{n}(1, q) - wa(0, 1)$), so in fact $\langle n_0, (v_1, v_2)\rangle = (v_1 - v_2)/w$. Now consider an element $\iota^\star y^c z^d = x^{wc+d} y^d$, evaluating $\gamma = \langle n_0, (v_1, v_2)\rangle$ for this lattice point we find that indeed $\gamma = c$. □

3.9. Smoothing quotient singularities of del Pezzo surfaces

Consider an affine manifold of polygon type, B_Q. In the previous sections we have:

(1) Defined the notion of a *one-parameter degeneration* of such affine manifolds
(2) Defined a family of log structures on the variety $X_0(B_Q, \mathscr{P}, s)$
(3) Outlined the Gross–Siebert algorithm for constructing a formal smoothing of this using the log-structure
(4) Explicitly computed the various rings and the family in the case of an isolated boundary singularity.

In this section we combine these to construct a flat family $\mathcal{X} \to \operatorname{Spec}\mathbb{C}[\alpha][\![t]\!]$ which will satisfy the conditions of Theorem 1.3.1, namely:

- Fixing a nonzero α the restriction of $\mathcal{X}$ over $\operatorname{Spec}\mathbb{C}[\![t]\!]$ is the flat formal family produced by the Gross–Siebert algorithm.
- Fixing $\alpha = 0$ the restriction of $\mathcal{X}$ over $\operatorname{Spec}\mathbb{C}[\![t]\!]$ is precisely the restriction of the Mumford degeneration of the pair $(Q, \mathscr{P})$.
- Fixing $t = 0$, the restriction of $\mathcal{X}$ is $X_0(Q, \mathscr{P}, s) \times \operatorname{Spec}\mathbb{C}[\alpha]$.
- For each boundary zero-stratum p of $X_0(Q, \mathscr{P}, s)$ there is neighbourhood U_p in $\mathcal{X}$ isomorphic to a family $\mathcal{Y} \to \operatorname{Spec}\mathbb{C}[\alpha]$ obtained by first taking a one-parameter $\mathbb{Q}$-Gorenstein smoothing of the singularity of X_Q at v, taking a simultaneous maximal degeneration of every fiber and restricting to a formal neighbourhood of the central fiber.

So far we have shown how the techniques developed in [**53, 59**] can be used to pass from an affine manifold with log structure data to a toric degeneration, but we have no explained how to produce (algebraic) families of degenerations (varying α). Indeed, that we can form this family is peculiar to our 'polygon type' setting since singularities must not meet rays transverse to the monodromy invariant line. The prinicipal obstacle to simply applying the Gross–Siebert algorithm to the family fiberwise is the jump in the log-structure at the central fiber; sections defining the log-structure are not permitted to vanish on any zero stratum. In fact we wish to choose log-structures from a different bundle at the central fiber, as the singular locus has changed. Therefore we have no *a priori* reason to suppose these glue to a family. However, we shall prove that our explicit construction at boundary zero-strata enables one to extend the obvious family over $\mathbb{C}^\star$ to one over $\mathbb{C}$.

REMARK 3.9.1. Methods in deformation theory shwo that the $\mathbb{Q}$-Gorenstein deformation families of a toric del Pezzo surface with cyclic quotient singularities are unobstructed, and thus our result is not unexpected. There is, of course, no clear connection between these families and the mutations of polygons or affine structures seen by these methods.

Recall we have a family of affine manifolds $\pi_Q \colon \mathcal{B}_Q \to \mathbb{R}$ defined by smoothing the corners, as described in Section 3.2. Fix a one parameter family of log-structures compatible with the family of affine manifolds in the sense of Definition 3.4.17.

REMARK 3.9.2. Consider the scattering diagram $\mathfrak{D}_\omega$ at the central vertex; this is equivalent to a scattering diagram of the following form:

$$\mathfrak{D} = \left\{ \left(\mathbb{R}\bar{m}_i, \prod_{j,k} \left(1 + c_{ijk} z^{-m_{ijk}}\right) \right) : 1 \leq i \leq p \right\}$$

Assuming $c_{ijk} \in \mathbb{C}[\alpha]$ the assumptions on a family of log structures imply that $c_{ijk} \in \alpha.\mathbb{C}[\alpha]$.

DEFINITION 3.9.3. For this section a *family of scattering diagrams* (with parameter α) is a scattering diagram defined via a map $r \colon P \to M$ and an $\mathfrak{m}$-primary ideal I, but now for $\mathfrak{d} \in \mathfrak{D}$, $f_\mathfrak{d} \in \mathbb{C}[\alpha][P]/I$. Further, write $\mathfrak{D}(\alpha)$ for the scattering diagram where all the functions have been evaluated at α.

LEMMA 3.9.4. *Given a family of scattering diagrams $\mathfrak{D}$ there is another one $S_I\big(\mathfrak{D}\big)$ such that:*

$$S_I\big(\mathfrak{D}\big)(\alpha) = S_I\big(\mathfrak{D}(\alpha)\big)$$

for all $\alpha \in \mathbb{C}$.

PROOF. We use the notion of a universal scattering diagram, indeed, writing:

$$\mathfrak{D} = \left\{ \left(\mathbb{R}\bar{m}_i, \prod_{j,k} \left(1 + c_{ijk} z^{-m_{ijk}}\right) \right) : 1 \leq i \leq p \right\}$$

we can form:

$$\mathfrak{D}' = \left\{ \left(\mathbb{R}\bar{m}_i, \prod_{j,k} \left(1 + t_{ijk} z^{-r\left(m_{ijk}\right)}\right) \right) : 1 \leq i \leq p \right\}$$

Where in the first scattering diagram is c_{ijk} is polynomial in α and the second scattering diagram is defined over the ring $\mathbb{C}[M][\![\{t_{ijk}\}]\!]$. In fact, following [**53**], this scattering diagram is defined over $\mathbb{C}[Q]$ where $Q \subseteq M \oplus \mathbb{N}^l$ is the monoid freely generated by pairs $(-r(m_{ijk}), e_{ijk})$, where e_{ijk} corresponds to t_{ijk}. Thus given an ideal I of P we obtain a scattering diagram $S_{I'}(\mathfrak{D}')$ by reduction modulo $I' = \phi^{-1}(I)$ where:

$$\phi\colon \mathbb{C}[Q] \to \mathbb{C}[\alpha][P]$$

via $t_{ijk} z^{-r\left(m_{ijk}\right)} \mapsto c_{ijk} z^{-m_{ijk}}$. Composing this with the evaluation map $\psi_\alpha\colon \mathbb{C}[\alpha][P] \to \mathbb{C}[P]$ we obtain a scattering diagram: $\psi_\alpha \circ \phi\left(S_{I'(\mathfrak{D}')}\right)$, which must be equivalent to $S_I(\mathfrak{D}(\alpha))$ by uniqueness. Thus we set $S_I(\mathfrak{D}) = \phi(S_{I'}(\mathfrak{D}'))$. □

PROPOSITION 3.9.5. *Varying α gives an algebraic family $\pi\colon \operatorname{Spec} \widetilde{R}^k_\omega \to \operatorname{Spec} \mathbb{C}[\alpha]$.*

PROOF. We construct $\mathbb{C}[\alpha]$-algebras $\widetilde{R}^k_\omega$ the fibers of which are the rings R^k_ω defined using the various log structures.

First let ω be a vertex contained in ∂B. From Section 3.8 we have a description of these rings via the isomorphism with the ring $R^k_\cup$. We denote by $\widetilde{R}^k_\cup$ the $\mathbb{C}[\alpha]$-algebra:

$$\mathbb{C}[\alpha]\left[X_i, Y_j, W\right]/I_\cup$$

Let ω be the central vertex of $\mathscr{P}$. The ring R^k_ω is a fiber product of rings of the form $R^k_{\omega\tau\mathfrak{u}}$ which is a quotient of the algebra $\mathbb{C}\left[P_{\omega,\phi}\right]$. We form the trivial algebra $\mathbb{C}[\alpha]\left[P_{\omega,\phi}\right]$ and so form the analogous rings $\tilde{R}^k_{\omega,\tau,\mathfrak{u}}$. Firstly setting

$$\tilde{R}^k_{\omega,\tau,\sigma_\mathfrak{u}} = R^k_{\omega,\tau,\sigma_\mathfrak{u}} \otimes_\mathbb{C} \mathbb{C}[\alpha]$$

and then defining:

$$\tilde{R}^k_{\omega,\tau,\mathfrak{u}} = \left(\tilde{R}^k_{\omega,\tau,\sigma_\mathfrak{u}}\right)_{f_\tau}$$

noting again that f_τ has non-trivial dependance on α. The change of chamber maps now give morphisms:

$$\theta_{\mathfrak{u},\mathfrak{u}'}\colon \tilde{R}^k_{\omega,\tau,\mathfrak{u}} \to \tilde{R}^k_{\omega,\tau,\mathfrak{u}'}$$

via the natural extension of the original definition:

$$\theta_{\mathfrak{u},\mathfrak{u}'}\left(z^m\right) = \left(\prod f_\mathfrak{d}\right)^{\langle n_0, m\rangle} z^m$$

These are isomorphisms of the rings $\tilde{R}^k_{\omega,\tau,\mathfrak{u}}$, giving $\tilde{R}^k_\omega$ the structure of a $\mathbb{C}[\alpha]$-algebra by taking the inverse limit of the rings $\tilde{R}^k_{\omega,\tau,\mathfrak{u}}$. Finally we need to check that varying α the functions on rays of the scattering diagram are polynomial in α, but this we know from Lemma 3.9.4. □

DEFINITION 3.9.6. We define the scheme $\mathcal{X}_Q \to \operatorname{Spec} \mathbb{C}[\alpha][\![t]\!]$ via the inverse limit over the system $\tilde{R}^k_\omega$, each of which is a $\mathbb{C}[\alpha][\![t]\!]$-algebra.

REMARK 3.9.7. In Theorem 1.3.1 we demand that $\mathcal{X}_Q$ is flat over $\operatorname{Spec} \mathbb{C}[\alpha][\![t]\!]$. Since flatness is local, we can consider $\mathbb{C}[\alpha][\![t]\!]$-algebras $\tilde{R}^k_\omega$ for each zero-dimensional stratum ω. We break these into two cases:

- If ω is a boundary zero-stratum flatness is an immediate consequence of Proposition 3.8.18 which gives an explicit description of this algebra.
- If ω is the central vertex we observe that by Lemma 3.9.4 the functions $f_{\mathfrak{d}}$ on each ray of the scattering diagram at order k is an element of $\mathbb{C}[\alpha,t]/(t^{k+1})$. We can now follow the proof of the case $\dim\omega = 0$ in Theorem 6.32 of [**53**] over the ring $\mathbb{C}[\alpha,t]/(t^{k+1})$.

We now prove that this satisfies the various conditions of Theorem 1.3.1, first identifying the restriction to $\alpha = 0$.

Proposition 3.9.8. *The restriction of $\mathcal{X}_Q \to \operatorname{Spec}\mathbb{C}[\alpha][\![t]\!]$ to $\alpha = 0$ is a thickening of the central fiber of the Mumford degeneration.*

Proof. Firstly we address the local model R^k_ω for ω the vertex of $\mathscr{P}$ in the interior of B. However the fiber $\alpha = 0$ is trivial, in the sense that all the slab functions are equal to 1, therefore the scattering diagram is trivial and there is a bijection between chambers and 2-cells of $\mathscr{P}$. Therefore the inverse limit simply reconstructed a local piece of the Mumford degeneration, as claimed.

Of greater interest are the local models at the vertices. As we remarked we cannot use the inverse limit, but rather we use the $R^k_\cup$ model constructed above. Using the notation from Section 3.8 we recall that the non-trivial relations were between generators projecting to different cones, for example:

$$\Big(\prod f_{\mathfrak{d}}\Big)^{\sum_k \gamma_k\langle n_0,m_k\rangle}\prod_{i,j}X_i^{\alpha_i}Y_j^{\beta_j} = \Big(\prod f_{\mathfrak{d}}\Big)^{\sum_i \alpha_i\langle n_0,m_i\rangle}\prod_{k,l}X_k^{\gamma_k}Y_l^{\delta_l}$$

Observe that $\prod f_{\mathfrak{d}} = f_\tau \prod_{\mathfrak{d}\text{ ray}} f_{\mathfrak{d}}$ where f_τ is the slab function associated to τ, and in particular that the our assumptions on the one-parameter family of log-structures imply that $f_\tau|_{\alpha=0} = w^{\deg f_\tau}$. Observe also that $\prod_{\mathfrak{d}\text{ ray}} f_{\mathfrak{d}}|_{\alpha=0} = 1$. This is a consequence of the fact that $S\big(\mathfrak{D}\big)(\alpha) = S\big(\mathfrak{D}(\alpha)\big)$: for the scattering diagram at the central vertex, setting $\alpha = 0$ the scattering diagram is trivial – every line has function $f_{\mathfrak{d}} = 1$. Therefore this is already consistent to all orders. The rays of this scattering diagram propagate until they intersect ∂B and indeed give all the rays in this structure. Combining these two observations we see that the fiber over zero has co-ordinate ring with relation:

$$\Big(w^l\Big)^{\sum_k \gamma_k\langle n_0,m_k\rangle}\prod_{i,j}X_i^{\alpha_i}Y_j^{\beta_j} = \Big(w^l\Big)^{\sum_i \alpha_i\langle n_0,m_i\rangle}\prod_{k,l}X_k^{\gamma_k}Y_l^{\delta_l}$$

Here $l = \deg(f_\tau)$, which is also the lattice length of the monodromy polytope of the discriminant locus on τ. Thus the local models near the boundary vertices, when α is set equal to zero, recover the local models for the Mumford degeneration. □

To conclude the proof of Theorem 1.3.1 we need to show that near the boundary vertices the family $\mathcal{X}_Q$ is induced by a $\mathbb{Q}$-Gorenstein smoothing of the singularities of Q.

PROPOSITION 3.9.9. *The family obtained in Proposition 3.9.5 in each of the charts containing a vertex of Q is isomorphic to a one parameter $\mathbb{Q}$-Gorenstein smoothing.*

PROOF. This is immediate from Proposition 3.8.18, as we may rewrite the families using the canonical cover. Indeed, by Proposition 3.8.18 deforming the log-structure simply deforms the equation in this cover, so in particular R'^k_{U} is defined for any fiber, not just away from the special fiber. □

We remark that for each k, $f = f_\tau \prod_{\mathfrak{d}} f_{\mathfrak{d}}$ is a polynomial in α, but as $k \to \infty$ the degree of this polynomial will, in general, tend to infinity. However there are local co-ordinates near boundary vertices with respect to which the family $\mathcal{X}_Q$ is algebraic to all orders.

3.10. Ilten families

We have studied Fano polygons P and smoothings of the associated toric varieties X_P. From the perspective of mirror symmetry [**4**,**21**] Fano polygons have a different interpretation – as Newton polygons of a Laurent polynomial W referred to as the mirror *superpotential*. Indeed, information pertaining to the enumerative geometry of a smoothing of X_P is encoded in the periods of W. However, there are potentially infinitely many Laurent polynomials (with different Newton polygons) that encode this enumerative information. These Laurent polynomials are related by certain birational transformations, referred to as *mutations* [**4**], or *symplectomorphisms of cluster type* [**72**]. Mutation of W defines an operation on the Newton polygon P of W and, by duality, an operation on $Q = P^\vee$. This dual action is the restriction of a piecewise linear transformation on the lattice M, where $Q \subset M_{\mathbb{R}}$. The following easy proposition gives a hint to *why* mutations are related to deformations.

PROPOSITION 3.10.1. *The piecewise linear transformation given by a mutation is precisely the transition function between the two charts defining the affine manifold obtained by exchanging a corner of Q for an interior singular point.*

One may then consider a family of affine manifolds in which the singularity is introduced, traverses its monodromy invariant line, and creates a corner in the opposing edge. This is made precise in the following way:

PROPOSITION 3.10.2. *Given a mutation between polygons $Q, Q' \subset M_{\mathbb{R}}$ there is family of affine manifolds $\pi\colon \mathcal{B} \to [0,1]$ for which:*
(1) $Q = \pi^{-1}(0)$, $Q' = \pi^{-1}(1)$.
(2) The generic fiber contains a single type-1 singularity.

This will be referred to as the *tropical Ilten family.*

PROOF. Take $\pi\colon \mathcal{B} \to [0,1]$ to be the trivial family with fiber Q. Construct a line segment l contained in the interior of Q as follows; The mutation is defined as a piecewise linear transformation on Q and Q', there is a distinguished line dividing M into two chambers;

intersecting this line with Q defines l. We shall refer to the two chambers contained in Q as Q_1, Q_2 and Q'_1, Q'_2 in Q'. Take a parameterization of l, writing now $l\colon [0,1] \to Q$.

We define the affine structure on the total space by covering it with two charts:

(1) Let $\mathcal{B}$ be the topological space $Q \times [0,1]$.
(2) Take $U_1 \subset \mathcal{B}$ to be

$$U_1 = \mathcal{B} \backslash \{(l(t), u) : u, t \in [0,1],\, t \leq u \text{ and } u \neq 0\}$$

(3) Similarly take $U_2 \subset \mathcal{B}$ to be

$$U_2 = \mathcal{B} \backslash \{(l(t), u) : u, t \in [0,1], t > u \text{ and } u \neq 1\}$$

(4) Take the transition function such that the fiber $\pi^{-1}(1)$ becomes Q' in the chart U_2 and in every $\pi^{-1}(x), x \in (0,1)$ exhibits a simple singularity in its interior.

Note that these two sets are not open, but the affine structure extends over the two corners. □

Observe that this family provides us both with an affine manifold B – a general fiber of π – and a polyhedral decomposition $\mathscr{P}$ of B, which subdivides B along l. We also require a family of log-structures compatible with the family of affine manifolds. The line segment l determines a one-dimensional projective toric stack $\mathbb{P}(a,b)$, with the log-structure a section of $\mathcal{O}(\mathrm{lcm}(a,b))$. The line segment l is the only interior 1-cell so there is no consistency condition to check. Sections of the bundle $\mathcal{O}(\mathrm{lcm}(a,b))$ are parameterized, up to scale, by $\mathbb{P}^1$ and we pick a family of sections such that the image of the zero set follows the singular locus of the affine structure. After choosing a piecewise linear ϕ on B we can apply the Gross–Siebert algorithm.

Applying the Gross–Siebert algorithm fiberwise, as in Theorem 1.3.1, and using the local models 3.8.8 to understand the central fiber as in Proposition 3.9.8, we obtain families $\pi_i\colon \mathcal{X}_i \to \operatorname{Spec} \mathbb{C}[\alpha, t]$ for $i = 1, 2$. We now describe these families; as there is no scattering these families are in fact *polynomial* in t. Relating the π_i to [**1, 64**], denote the Ilten family for Q, Q' as $\pi'\colon \mathcal{Y} \to \mathbb{P}^1$. The construction in [**1, 64**] also defines a family over the affine cone $\mathbb{C}^2$ of $\mathbb{P}^1$, we shall recover this family by gluing together the families π_i and contracting the resulting exceptional curve.

PROPOSITION 3.10.3. *There is a family $\pi\colon \mathcal{X} \to Bl_0(\mathbb{C}^2)$ from which we obtain each π_i as follows.*

(1) Cover the base with the standard toric charts U_1, U_2.
(2) Restricting $\pi|_{U_i}$ to a formal neighbourhood of the exceptional divisor recovers π_i.
(3) The family over the exceptional divisor is trivial, and after restricting to the strict transform of a line in $\mathbb{C}^2$ the family becomes a toric degeneration endowing the restriction of π to the exceptional divisor with a family of divisorial log-structures.

REMARK 3.10.4. It would be entirely legitimate at this point to embark on a description of this smoothing via the usual local model and inverse limit construction. For example these

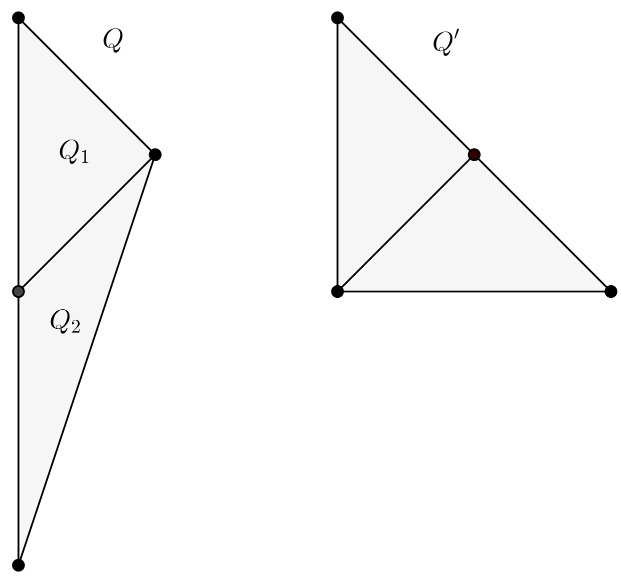


FIGURE 3.10.1. An example of Q, Q', Q_1 and Q_2

must contain the local model:

$$R_{\tau\tau\mathfrak{u}} \cong S_k[x, y, w^{\pm}]/(xy - (\alpha + w)t)$$

Indeed, all the families discussed in this section are compactifications of this affine local model. There are no non-trivial scattering diagrams around any joint of the structure so the family is obtained by taking a colimit over a finite system of algebras. However, we shall take a different approach, following [**60**], which projectivises this construction. This will greatly reduce the number of rings we need to keep track of and also produce an embedded family with the log structure encoded in the equations defining this family. We shall prove the equivalence with the original construction in Lemma 3.10.11.

Recall that the polygon $P^{\vee} = Q \subset M_{\mathbb{R}}$ defines a toric variety via $X_P = \mathrm{Proj}(\mathbb{C}[C(Q)])$ where $C(Q)$ is the semigroup defined by the integral points of the cone in $M_{\mathbb{R}} \oplus \mathbb{R}$ with height one slice equal to Q. As the vertices of Q are rational this graded ring need not be generated in degree one.

The prototypical example we shall refer to is the pair of polygons Q, Q' for $\mathbb{P}^2$ and $\mathbb{P}(1, 1, 4)$ respectively, they are shown below with the embedding from $\mathcal{O}(i)$, $i = 1, 2$ as shown below.

Take a generating set for $C(Q)$ and refer to a general element of the generating set as u_i. The generating set naturally subdivides into three disjoint sets:

(1) Any generators lying in the cone over Q_1 and outside Q_2 are denoted X_i.
(2) Any generators lying in the cone over Q_2 and outside Q_1 are denoted Y_j.
(3) Any generators lying over both Q_1 and Q_2 are denoted W_k. We observe that $(\mathbf{0}, 1) \in C(Q)$ is always in the generating set.

Indeed we write $C(Q_1), C(Q_2), C(Q_1 \cap Q_2)$ for the three sub-cones respectively. We shall insist that the union $\{X_i\} \cup \{W_k\}$ generates $C(Q_1)$, $\{Y_j\} \cup \{W_k\}$ generates $C(Q_2)$ and $\{W_k\}$ generate $C(Q_1 \cap Q_2)$. We denote the height of a generator u_i as $\kappa(u_i)$.

Remark 3.10.5. In the example above we can take a generating set with four elements, which we shall call $\{s_0, s_1, s_2, u\}$ with heights $1, 1, 1, 2$ respectively. Thus we see $\mathbb{P}^2$ embedded as $s_1 s_2 = u$ and $\mathbb{P}(1, 1, 4)$ embedded as $s_1 s_2 = s_0^2$ in $\mathbb{P}(1, 1, 1, 2)$.

Recalling that the affine manifold is equipped with a piecewise-linear function ϕ, we assume this has slope zero on Q_2 and slope k on Q_1, i.e. $\phi(X_i)$ is $k \langle n_0, \tilde{m}_i \rangle$ where n_0 is the primitive vector in N annihilating the tangent space to l, and $\tilde{m}_i$ is the rational point of Q defined by the exponent m_i of X_i. We shall assume k is chosen such that ϕ is integral on each generator. We can now write out the Proj of this algebra explicitly: we can construct an ambient weighted projective space $\mathbb{P}(\vec{a})$, where $\vec{a} \in \mathbb{Z}^N_{>0}$ and N is the size of the generating set, given by $\vec{a} = \sum_i \kappa(u_i) e_i$, the vector of heights.

The toric variety is then cut out in this space by the binomial equations given by the relations between these generators. We call the ideal generated I_Q. The toric degeneration corresponding to $\mathscr{P}$ is given by the following ideal, denoted $I_P(t)$:

Definition 3.10.6. For each binomial relation $M_1 - M_2 \in I_P$ such that $d = \mathrm{ord}_l(M_1) - \mathrm{ord}_l(M_2) \geq 0$ define a new binomial relation $M_1 - t^d M_2$. Take $I_P(t)$ to be the ideal generated by these new relations.

Remark 3.10.7. If $F \in I_P$ is an element of $\mathbb{C}[\{X_i\} \cup \{W_k\}]$, then $\mathrm{ord}_l(M_1) - \mathrm{ord}_l(M_2) = 0$ and the binomial relation remains unchanged in $I_P(t)$. The same is true of those relations in $\mathbb{C}[\{Y_j\} \cup \{W_k\}]$

Note this has recovered the Mumford degeneration for the pair $(Q, \mathscr{P})$. We have thus completed the first step, this family will be the family over the strict transform of a line through the origin in $\mathbb{C}^2$.

Remark 3.10.8. One can apply exactly the same procedure to Q' and obtain a toric degeneration of the second toric variety, the family over the fiber at ∞. In fact one may take exactly the same generating set, and get a different set of binomial relations. As in Section 3.9 we now describe a family 'interpolating' between them.

To construct such a family first consider that in the construction in Section 3.9 we used a variable that corresponded to a primitive vector along the monodromy invariant direction.

In this construction we find such a variable by looking at the part of $C(Q)^{\mathrm{gp}}$ generated by the exponents of the variables W_k. This is a rank 2 free abelian subgroup of $C(Q)^{\mathrm{gp}}$, that contains $(\mathbf{0}, 1)$. There is another canonical monomial $\mathcal{W}$, determined up to sign by requiring it to lie at height zero and lie in the monodromy invariant direction. In $\mathbb{C}\,[C(Q)^{\mathrm{gp}}]$ this has the form $\mathcal{W} = \frac{\prod_k W_k^{\alpha_k}}{\prod_l W_l^{\beta_l}}$. Note there may be many choices for the representation of $\mathcal{W}$ via the relations between the W_k.

Remark 3.10.9. In the example of $\mathbb{P}^2 \subset \mathbb{P}\,(1,1,1,2)$ we may take $\mathcal{W} = u/s_0^2$.

The interpolating family is then given by replacing elements in $I_Q(t)$ analogously to the procedure in Section 3.8:

Definition 3.10.10. The ideal $I_Q(t,\alpha)$ is the ideal generated by relations defined in Definition 3.8.8, where we replace C_i by Q_i and f by $(1+\alpha\mathcal{W})$.

In the example we have been considering, for $\mathbb{P}^2 \subset \mathbb{P}\,(1,1,1,2)$, we replace the relation $s_1s_2 = u$ with $s_1s_2 = ut(1+\alpha s_0^2/u)$ i.e. with $s_1s_2 = t(u+\alpha s_0^2)$. Observe that the fibers of this family are isomorphic to $\mathbb{P}^2$. The other family, that deforming $\mathbb{P}\,(1,1,4)$, is given by $s_1s_2 = t(s_0^2+\alpha u)$. This gives a smoothing of $\mathbb{P}\,(1,1,4)$ to $\mathbb{P}^2$.

To complete a proof of Proposition 3.10.3 we glue this pair of families in the obvious fashion. Define $\mathcal{X} \to \mathrm{Bl}_0(\mathbb{C}^2) =: E$ by taking $\mathcal{X} \hookrightarrow \mathbb{P}(\vec{a}) \times E$. Giving E homogenous co-ordinates, α, β of weight one and t the weight -1 co-ordinate, elements of $I_P(t,\alpha)$ may be homogenized to obtain: $M_1 = t^d(\beta+\alpha\mathcal{W})^d M_2$ homogenous of weight zero. These generate a homogeneous ideal, the equations of which define $\mathcal{X}$.

Given the family produced by Proposition 3.10.3 we can establish a family over $\mathbb{C}^2$ by contracting the exceptional curve, so that α and β become the coordinates on the plane and the new family is defined by equations $M_1 = (\beta+\alpha\mathcal{W})^d M_2$. Thus we have established Theorem 1.3.2.

In the running example the homogeneous equation is:

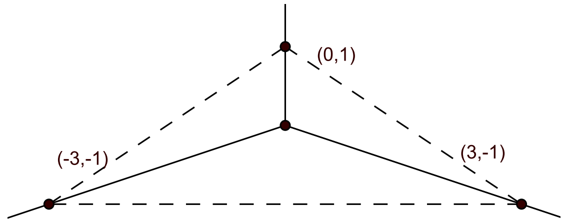
$$\{s_1s_2 = (\beta s_0^2 + \alpha u)\} \subset \mathbb{P}(1,1,1,2) \times \mathbb{P}^2_{(t:\alpha:\beta)}$$

Lemma 3.10.11. *Restricting to the ideal of* $\mathbb{C}[\alpha][\![t]\!]$ *generated by* $(\alpha-\alpha_0, t^{k+1})$ *for fixed* $\alpha_0 \neq 0$ *denote the restriction of* $\mathcal{X}$ *by* $\mathcal{X}_{\alpha_0,k}$*, this scheme is isomorphic to the scheme obtained in Sections 3.6, 3.7 from* $(B,\mathscr{P})$ *with log-structure fixed by the parameter* α.

Proof. Considering this $(B,\mathscr{P})$, there is no scattering, so we have $\mathscr{S}^r = \varnothing$, and the set of slabs $\mathscr{S}^s = \{l\}$. The category $\underline{\mathrm{Glue}}(\mathscr{S}, k)$ consists of objects $(\omega, \tau, \mathfrak{u})$ where:

(1) ω is an end-point of l, $\tau = l$ and $\mathfrak{u}$ is either of the two maximal cells of $\mathscr{P}$.
(2) In any other case the chamber is fixed by the choice of ω, τ. In particular τ is a boundary edge of B and contained in precisely one two-cell of $\mathscr{P}$.

Firstly R^k_ω is recovered by localizing $\mathcal{X}_{\alpha,k}$ with respect to the variable W_k corresponding to the vertex ω in $C(Q)$. This is immediate from the usual Proj construction and performing



this localisation we recover $R^k_{\cup}$ for this vertex, by construction. Indeed the same argument applies for any vertex of Q. The final check is that the gluing of these rings according to Section 3.7 coincides with that of Proj. □

Corollary 3.10.12. *The family given by Theorem 1.3.2 is* $\mathbb{Q}$*-Gorenstein.*

Proof. We can cover the family by neighbourhoods around each boundary vertex. By Lemma 3.10.11 each of these is equal to the local model described in Section 3.8 and is therefore $\mathbb{Q}$-Gorenstein. □

We remark the analogous families in both [**64**] and [**1**] are independently known to be $\mathbb{Q}$-Gorenstein, making this an expected outcome.

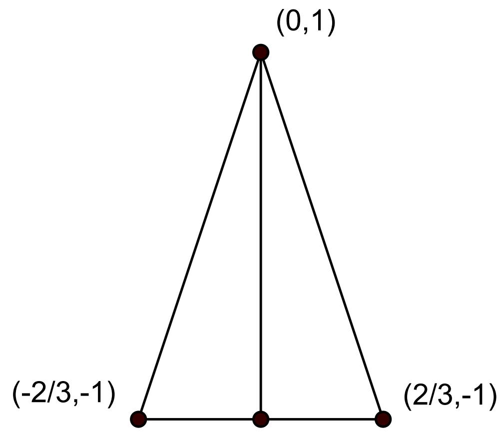
3.11. Examples

3.11.1. A single smoothing direction. Consider the hypersurface:

$$X_6 \subset \mathbb{P}(1,3,3,1)$$

This exhibits a toric degeneration in this ambient space to a toric variety with fan shown in Figure 3.11.1. The fan exhibits 2 residual singularities which persist after the smoothing and an A_5 singularity, $\frac{1}{6}(1,5)$ which is a T-singularity. Constructing the dual polygon one observes that the one-parameter family of affine manifolds obtained by smoothing all possible corners has a general fiber B with all six singularities ranged along a single edge. Therefore there is no scattering diagram to construct so one can construct a family (the multi-parameter analogue of the family appearing in Section 3.10) for which all the mutation equivalent toric varieties are special fibers.

To write down the family constructed in Section 3.10 for this polygon we consider the dual polygon $Q^\vee$ shown in Figure 3.11.1. Now form the monoid of integral points of the cone for which $Q^\vee$ is the height one slice. However, note that the polygon is that obtained from the polarisation $\mathcal{O}(2)$; using the more economical polarisation $\mathcal{O}(1)$ (embedding $Q^\vee$ at height 2) the associated relation is a binomial in $\mathbb{P}(1,1,3,3)$. Indeed the vertices of the polygon at height one are now $(0,1),(0,0),(-1/3,0),(1/3,0)$ after a translation, naming the corresponding variables X_0, X_1, Y, Z respectively gives: $YZ = X_1^6$. Applying the method of



Section 3.10, we find the Ilten family:

$$\{tYZ = (\alpha X_1^6 + \beta X_1^5 X_0)\}$$

Of course we can consider a general homogenous degree six polynomial in X_0, X_1 and so find a family over $\mathbb{P}^5$ which has 6 toric zero strata, each of which corresponds to a particular toric variety. There is redundancy in this description, since for example $YZ = X_0^6$ manifestly gives the same variety as $YZ = X_1^6$.

3.11.2. The cubic surface. In this example we place Example 4.4 of [**60**] in this context. The toric cubic surface $\{X_0X_1X_2 = X_3^3\} \subset \mathbb{P}^3$ exhibits $3 \times A_2$ singularities which may all be smoothed. However this situation is much more chaotic than the previous examples – the mutation graph is necessarily infinite and we cannot expect to capture all degenerations in a single algebraic family. However following [**60**] we may ask an easier question; rather than smoothing the corners completely we can simply introduce three type 1 singularities. This should produce a family of cubic surfaces which all exhibit at least ordinary double points. In [**60**] this scattering diagram is explicitly computed, in particular it is shown to be finite, producing a toric degeneration embedded in $\mathbb{P}^3$.

Having produced the scattering diagram one can construct a toric degeneration as explained above. The equation from [**60**] is:

$$\{XYZ = t((1+t)U^3 + (X+Y+Z)U^2)\} \subset \mathbb{P}^3 \times \mathbb{C}_t$$

To recover the family partially smoothing these A_2 singularities we simply repeat the derivation of this, but place general coefficents in the sections defining the log-structure. We know from Section 3.8 that this will give the correct family as these sections degenerate.

This calculation gives a family over $\mathbb{C}^3_{\alpha,\beta,\gamma}$:

$$\{XYZ = t((1+\alpha\beta\gamma t)U^3 + (\alpha X + \beta Y + \gamma Z)U^2)\}$$

For completeness we also compute an Ilten family for the cubic surface:
Subdividing using the x-axis, we have zero strata:

$$(1,0),(0,1),(-1,-1),(0,0),(-1/2,0)$$

Naming the corresponding variables X, Y, Z, U, W respectively we obtain the toric degeneration:

$$\{XYZ = tU^3, YZ = tW\} \subset \mathbb{P}(1,1,1,1,2)$$

Performing the construction of Section 3.10 we obtain the family:

$$\{XYZ = tU^2(\alpha U + \beta X), YZ = t(\alpha W + \beta U^2)\} \subset \mathbb{P}(1,1,1,1,2) \times \mathbb{C}^2_{\alpha,\beta}$$

3.12. Conclusion

An intuitive picture begins to emerge: If we fix a del Pezzo surface X which is a smoothing of a toric variety X_P we have various *mutation equivalent* toric varieties, namely those associated to the polygons obtained by mutating P. Rather than directly analysing the deformation theory of these varieties we studied the moduli space of log structures after taking a toric degeneration of X. This produced a 'tropical analogue' of the deformation theory, in which one mimics the $\mathbb{Q}$-Gorenstein deformations of X_P by introducing singularities into the affine manifold P. As well as recovering the entire theory of combinatorial mutations we have shown how to recover, order by order, an algebraic family with general fiber X via the Gross–Siebert algorithm.

Moving singularities defines a 'moduli problem' of its own, a topological orbifold (with isotropy due to automorphisms of the polygons) which carries an affine structure, first mentioned in [**79**]. There is also a stratification of this space: The zero strata being the polygons themselves, one strata the tropical Ilten families and so on. To relate this space to the study of $\mathbb{Q}$-Gorenstein degenerations one must understand how to lift these families to algebraic ones. From this perspective we have described this lift for the 1-skeleton of this space in this chapter.

CHAPTER 4

The Tropical Superpotential

4.1. A Tropical Superpotential

Given a smooth genus one curve in $\mathbb{P}^2$ we may form an affine manifold $B_{\mathbb{P}^2}$ with a *smooth boundary* as depicted in Figure 4.1.1. Recall we obtained this affine manifold in Chapter 3 by 'smoothing the corners' of the moment polytope Q for $(\mathbb{P}^2, D)$ where D is the toric boundary of $\mathbb{P}^2$. Indeed, we recall that the family

$$\{xy = t\} \subset \mathbb{C}^2_{x,y} \times \mathbb{C}_t$$

carries a family of special Lagrangian torus fibrations, and the family of affine structures on the base recovers the family of affine structures introduced in Chapter 3 smoothing the corner of the polytope. If we attempt to smooth all the vertices of Q simultaneously, we lose a simple geometric model, but, in its place we may apply the techniques of Gross–Siebert [**53**, **59**], as we explained in Chapter 3. This example is also discussed in [**6**].

As in the closed case, there is a notion of *Legendre duality* for affine structures with boundary, allowing us to consider the 'fan picture' or 'B-model' affine structure $B^{\vee}_{\mathbb{P}^2}$ for the pair $(\mathbb{P}^2, E)$. Building on work of Mikhalkin [**88**,**89**] and Nishinou–Siebert [**91**], it is expected that holomorphic curves in the total space of a special Lagrangian torus fibration (or more generally, a non-zero fiber of a toric degeneration) are in correspondence with *tropical curves* in this affine base.

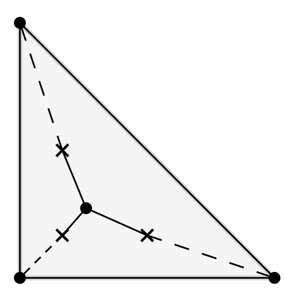


FIGURE 4.1.1. Intersection complex for a toric degeneration of $(\mathbb{P}^2, E)$

Recall from Chapter 2 that given a pair (X, D) for X a Fano manifold and $D \in |-K_X|$, the mirror superpotential is conjecturally defined by the $\mathfrak{m}_0$ obstruction, by definition a sum over Maslov index two discs,

$$\mathfrak{m}_0(L, \nabla) = \sum_\beta n_\beta(L) e^{-\int_\beta \omega} \mathrm{hol}_\nabla(\partial\beta)$$

Combining these two observations, it is expected that there is an enumerative interpretation of the superpotential mirror-dual to (X, D) in terms of counts of tropical *discs* in $B^\vee_{\mathbb{P}^2}$. This idea is pursued by Carl–Pumperla–Siebert in [**18**], where they prove many important technical results and provide several fundamental examples.

Focusing on the easiest case, $(\mathbb{P}^2, E)$, we extend their calculation of the tropical superpotential by computing the wall-and-chamber decomposition for the entire structure $\mathscr{S}$ on a larger domain of the affine manifold $U \subset B^\vee_{\mathbb{P}^2}$. In doing so we shall identify a canonical bijection between the chambers $\mathfrak{u}$ of $\mathscr{S}$ contained in U and the toric degenerations of $\mathbb{P}^2$. Moreover the Laurent polynomials defined by the count of tropical discs, or *broken lines*, are precisely the maximally mutable Laurent polynomials considered in Chapter 2. This is the first non-trivial example in which we can establish an interpretation of mirror-dual Laurent polynomials as disc counts.

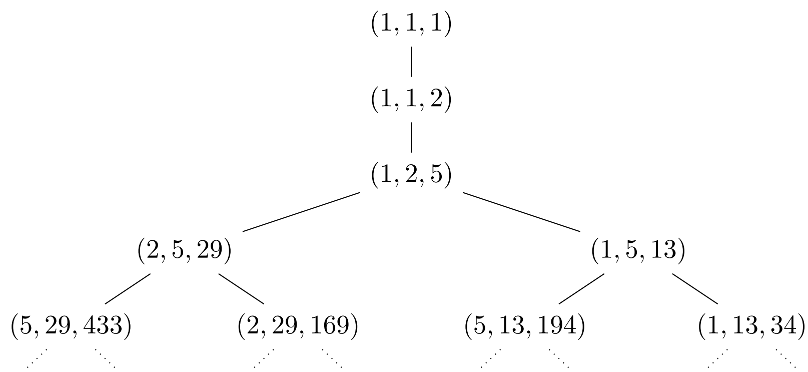
Recall we have the following classification result for toric degenerations of $\mathbb{P}^2$, see [**61**].

Theorem 4.1.1. *The set of toric varieties to which $\mathbb{P}^2$ admits a toric degeneration is in canonical bijection with the integral solutions of the* Markov equation $a^2 + b^2 + c^2 = 3abc$. *Consequently all toric degenerations of $\mathbb{P}^2$ are related by mutation.*

The Markov equation here also appears in many different areas of mathematics. Its solutions are completely described by the following lemma.

Lemma 4.1.2. *Given a solution (a, b, c) of $x^2 + y^2 + z^2 = 3xyz$ another solution is given by $\left(b, c, \frac{3bc-a^2}{a}\right)$. Given the initial solution $(1, 1, 1)$ this process generates all integral solutions to the Markov equation.*

Thus the solutions of the Markov equation may be encoded in a trivalent graph $\mathcal{G}$, in which each node is a triple (a, b, c) solving the Markov equation. Starting from the initial solution $(1, 1, 1)$ one may inductively define a *distance* $\mathrm{dist}(v), v \in \mathcal{V}(\mathcal{G})$, of a node from $(1, 1, 1)$ via successive mutations.



We shall show that the Markov equation can also be recovered from the wall-and-chamber decomposition given by the natural structure $\mathscr{S}$ on $B^\vee_{\mathbb{P}^2}$.

Having performed this detailed study of $\mathscr{S}$ we show, following [**18**], that the maximally mutable Laurent polynomials f with

$$\pi_f(t) = \sum_{m\geq 0} \frac{(3m)!}{(m!)^3} t^{3m}$$

that is, those mirror-dual to $\mathbb{P}^2$, can be expressed via counts of tropical discs (or *broken lines*) in this affine manifold.

In this chapter we prove Theorem 1.3.3 and Theorem 1.3.4. Theorem 1.3.3 is a computation of the chambers defined by all the scattering diagrams in $B^\vee_{\mathbb{P}^2}$. In particular we observe that $B^\vee_{\mathbb{P}^2}$ contains a region densely covered with rays and the complement of this region has a wall and chamber structure. The dual cell complex to these walls and chambers is a trivalent tree and each chamber is a triangle similar to the Fano triangle defined by the corresponding degeneration of the projective plane.

The second result, Theorem 1.3.4, is obtained by combining Theorem 1.3.3 with the results of [**18**]. These results determine the set of broken lines in each chamber and from this a candidate Laurent polynomial mirror. We check that these Laurent polynomials are precisely those whose period sequence is the quantum period sequence of $\mathbb{P}^2$.

4.2. A Normal Form for Fano Polygons

In this section we study the special class of polygons associated to the $\mathbb{Q}$-Gorenstein degenerations of $\mathbb{P}^2$ and the combinatorial mutations between them in considerable detail. In particular we present a *standard form* for a pair (P, v) of a Fano polygon P with singularity content $(3, \varnothing)$ and a vertex $v \in \mathcal{V}(P)$. Later, this will allow us to compare the polygons obtained by mutation with a standard scattering diagram.

The class of Fano polygons of $\mathbb{Q}$-Gorenstein toric degenerations of $\mathbb{P}^2$ is well-understood. Indeed, combining the results of [**61**] and [**2**] we have the following theorem.

THEOREM 4.2.1. *Given a Fano polygon P the following are equivalent:*

(1) P is the polygon of a degeneration of $\mathbb{P}^2$.

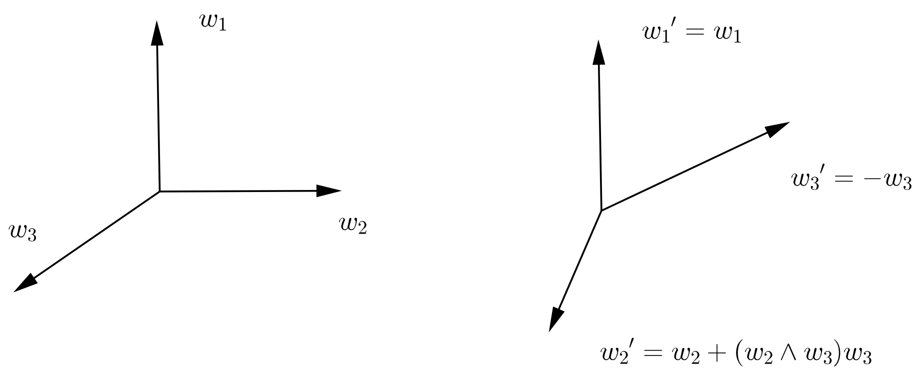


FIGURE 4.2.1. Normal vectors changing under a mutation

(2) The singularity content of P is $(3, \varnothing)$.
(3) P is mutation equivalent to the polygon $P_0 := \mathrm{Conv}\{(1,0),(0,1),(-1,-1)\}$.

REMARK 4.2.2. Theorem 4.2.1 implies that the polygon P of a toric surface X_P which admits a $\mathbb{Q}$-Gorenstein smoothing to $\mathbb{P}^2$ is a triangle. Performing any mutation out of P completely removes an edge and a new edge appears at the opposite vertex.

The effect of the M-side mutation on the triple of dual vectors defining the edges of P is similarly straightforward. As shown in Figure 4.2.1, the inward-pointing normal w_3 to the mutating edge changes sign, and the remaining pair undergo a piecewise linear map in which one is fixed and the other undergoes a shear with invariant direction spanned by w_3.

Thus:

LEMMA 4.2.3. *Given a Fano polygon P with singularity content* $(3, \varnothing)$ *a mutation of P is exactly (i.e. not only up to* $\mathrm{GL}(2,\mathbb{Z})$*) determined by a* mutating *edge E and* fixed *edge E'.*

Given a triangle P with singularity content $(3, \varnothing)$ and a vertex $v \in \mathcal{V}(P)$, we describe a 'normal form' for P making these edges orthogonal, at the expense of embedding P into a finer lattice.

DEFINITION 4.2.4. Given a pair (P, v), $v \in \mathcal{V}(P)$ let E_1, E_2 be the edges incident to v and let w_1, $w_2 \in M$ be their normal vectors respectively. Consider the map

$$\rho\colon \mathbb{Z}^2 \to M \qquad\qquad \rho\colon e_i \mapsto w_i, \text{ for } i = 1, 2$$

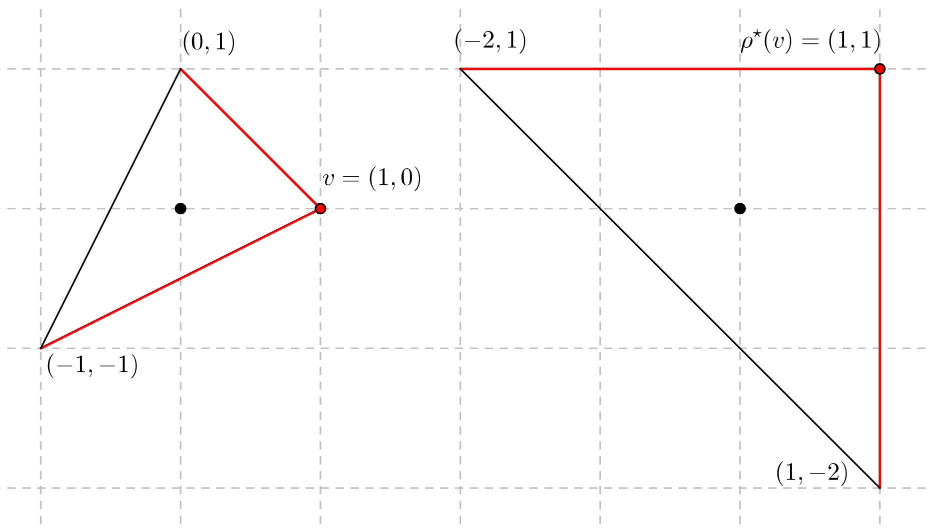


FIGURE 4.2.2. A triangle in normal form

where e_i are elements of the standard basis. The dual map $\rho^\star\colon N \hookrightarrow \mathbb{Z}^2$ embeds N into the lattice $\mathbb{Z}^{2\vee}$, consequently P is embeded in the new lattice $\mathbb{Z}^{2\vee}$ with fixed coordinate directions. We refer to the embedding $\rho^\star$ as the *normal form* for (P, v).

The prototypical example of a triangle in standard form is shown in Figure 4.2.2

REMARK 4.2.5. This construction is intimately related to the construction of a *cluster algebra* associated to P, [**70**] c.f. [**54**]. There one defines *seed data* by fixing the lattice $\mathbb{Z}^m$ with its standard basis, and define a skew-symmetric bilinear form on $\mathbb{Z}^m$. The mutation of seed data then involves making a choice of basis vector, for example e_k, and transforming this basis to:

$$e'_i = \begin{cases} -e_k & \text{if } i = k \\ e_i + \max{(e_k \wedge e_i, 0)}.e_k & \text{otherwise} \end{cases}$$

Given any Fano polygon P with singularity content $m_P = \sum_E m_E$, where m_E is the singularity content of $\mathrm{Cone}(E)$ and the sum is over the edges of P, we can define a map $\hat{\rho}\colon \mathbb{Z}^{m_P} \to M$ sending m_E basis vectors to the inward-pointing normal vector to the edge E. We can then form a skew-symmetric matrix $(u_1, u_2) := \hat{\rho}(u_1) \wedge \hat{\rho}(u_2)$. Restricting this definition to a pair of basis vectors we recover our previous definition of ρ. We return to this point in detail in Chapter 5.

NOTATION 4.2.6. We fix notation for the local index of various cones that appear in P and its mutations in E_1, E_2 respectively.

- Let ℓ_i be the local index of the cone over the edge E_i.
- Let ℓ'_ibe the local index of the cone over the edge E'_i. Where E'_i is the new edge formed by mutating the edge E_i.

- Let $L_i = \ell_i + \ell_i'$.

Putting P in normal form embeds the edges E_i of P into affine coordinate lines. This means we have a very simple description of the result of the pair of mutations in E_i for $i = 1, 2$. To make this precise we need to define what it means for a polygon to mutate *with respect to a sublattice*.

Definition 4.2.7. Given a sublattice $\rho^\star \colon N \hookrightarrow \mathbb{Z}^2$ and P a polygon in N, a vector $w \in \mathbb{Z}^{2\star}$ and a polygon $F = \rho^\star(F')$ for a polygon $F' \subset w^\perp$ of $\rho^\star(P)$ we define the *mutation with respect to N*

$$\mathrm{mut}_{(w,\rho^\star)}(\rho^\star(P), F) := \rho^\star(\mathrm{mut}_{\rho(w)}(P, F))$$

We prove two lemmas which will simplify our future calculations considerably.

Lemma 4.2.8. *Given a Fano polygon P with singularity content $(3, \varnothing)$ put P in normal form with respect to $v \in \mathcal{V}(P)$, incident to the edges E_1, E_2. The two corresponding mutations of $\rho^\star(P)$ with respect to N have factors*

$$F_1 = \mathrm{Conv}\{(0,0),(s,0)\} \qquad\qquad F_2 = \mathrm{Conv}\{(0,0),(0,s)\}$$

respectively, where s is the index $[\mathbb{Z}^2 : N]$.

Proof. The factors F_i are, by definition, line segments, with one vertex at the origin. Since F_i lies in $w_i^\perp$,

$$F_1 = \mathrm{Conv}\{(0,0),(k_1,0)\} \qquad\qquad F_2 = \mathrm{Conv}\{(0,0),(0,k_2)\}$$

for some k_i. To see that $k_i = s$ for $i = 1, 2$ we observe that

$$k_1 = \langle (1,0), \rho^\star(v_1)\rangle = \rho^\star(v_1)(1,0) = v_1(\rho(1,0)) = \langle w_2, v_1\rangle$$

where $v_i \in \mathcal{V}(P)$ is incident to E_i and not equal to v. Using the non-degenerate pairing the functional $\langle -, v_1\rangle \colon M \to \mathbb{Z}$ is equal to the functional $w_1 \wedge -$. But $w_1 \wedge w_2 = s$ by the definition of the map ρ. □

Lemma 4.2.9. *Given a Fano polygon P with singularity content $(3, \varnothing)$ fix a pair of edges E_1, E_2, then $L_1/\ell_2 = L_2/\ell_1 = s$.*

Proof. Since the cones over the distinguished edges E_1 and E_2 are primitive T-cones, the mutation in these edges completely removes the edge. Consequently the mutation of $\rho^\star(P)$ with respect to N also removes an entire edge. Thus, while the local index of the cones over $\rho^\star(E_i)$ remains unchanged, the width $L_1 = w = rs$, where the second equality follows from Lemma 4.2.8. However, for example, for the cone over E_1, $w = L_1$ and $r = \ell_2$. Applying the same consideration for E_2 completes the proof. □

4.3. Scattering Diagrams from Combinatorial Mutations

We now turn to a construction of the support of a scattering diagram in terms of combinatorial mutations. We shall build a collection of triangles using successive mutations and use the edges of these polygons to define a sequence of rays. In the next section we shall directly compare this with the rays of a certain simple scattering diagram. The collection of triangles we consider in this section fit together into an embedded simplicial complex which we refer to as a *diagram.*

DEFINITION 4.3.1. Given a triple (P, v, k) where P is a Fano polygon P with singularity content $(3, \varnothing)$, $v \in \mathcal{V}(P)$ and $k \in \mathbb{Z}_{\geq 0}$. Let E_1, E_2 denote the edges of P adjacent to v and let E denote the opposite edge to v, and allow this labelling to persist after mutation. We define a *diagram* $\mathrm{Diag}_k(P, v) := (\mathscr{P}_k(P, v), \iota_{P,v,k})$, where

- $\mathscr{P}_k(P, v)$ is a two-dimensional simplicial complex with $2k + 1$ maximal cells and;
- $\iota_{P,v,k} \colon |\mathscr{P}_k(P, v)| \to \mathbb{R}^2$ is a continuous map from a geometric realisation of $\mathscr{P}_k(P, v)$ to $\mathbb{R}^2$.

Denote the chambers of $\mathscr{P}_k(P, v)$ by $\mathfrak{u}_{k,0}$, $\mathfrak{u}^i_{k,j}$ for $i \in \{1, 2\}$ and $1 \leq j \leq k$ (suppressing the dependence on P and v). We fix $\mathscr{P}_k(P, v)$ by requiring that:

- The following intersections each consist of exactly one edge of the respective chambers,
  - $\tau_{i,j} := \mathfrak{u}^i_{k,j} \cap \mathfrak{u}^i_{k,j-1}$ for $1 \leq j \leq k$
  - $\tau_{i,0} := \mathfrak{u}^i_{k,1} \cap \mathfrak{u}_{k,0}$
- Every chamber $\mathfrak{u}^i_{k,j}$ contains the distiguished vertex v.

We fix the map $\iota_{P,v,k}$ by requiring that,

- $\iota_{P,v,k} \colon \mathfrak{u}_{k,0} \to P$ isomorphically.
- Up to scale and translation $\iota_{P,v,k}(\mathfrak{u}^i_{k,1})$ is equal to the mutation of P with mutating edge E_i and fixed edge E_{2-i}.
- For $1 \leq j \leq k$, up to scale and translation $\iota_{P,v,k}(\mathfrak{u}^i_{k,j})$ is equal to the mutation of the Fano polygon corresponding to $\mathfrak{u}^i_{k,j-1}$ with mutaing edge $E_{a_{i,j}}$ and fixed edge E. Here $a_{i,j}$ is defined by the requirement that $a_{i,j+1} = 2 - a_{i,j}$ and $a_{i,0} = i$.

In Figure 4.3.1 we give a schematic example of $\mathrm{Diag}_3(P, v)$ (for P in normal form with respect to v). The remainder of this section is devoted to showing that for any P, v and k, $\iota_{P,v,k}$ is an embedding.

To describe the slopes of the rays in $T_v\mathbb{R}^2$ induced by $\mathrm{Diag}_k(P, v)$ we use the description of M-side mutation to describe the normal vectors of the region $\mathfrak{u}^i_{k,j}$.

DEFINITION 4.3.2. Define $u^{i,1}_j$, $u^{i,2}_j \in M$ to be the normal vectors to the edges E_1, E_2 of $\mathfrak{u}^i_{k,j}$ (for any $k > j$) respectively.

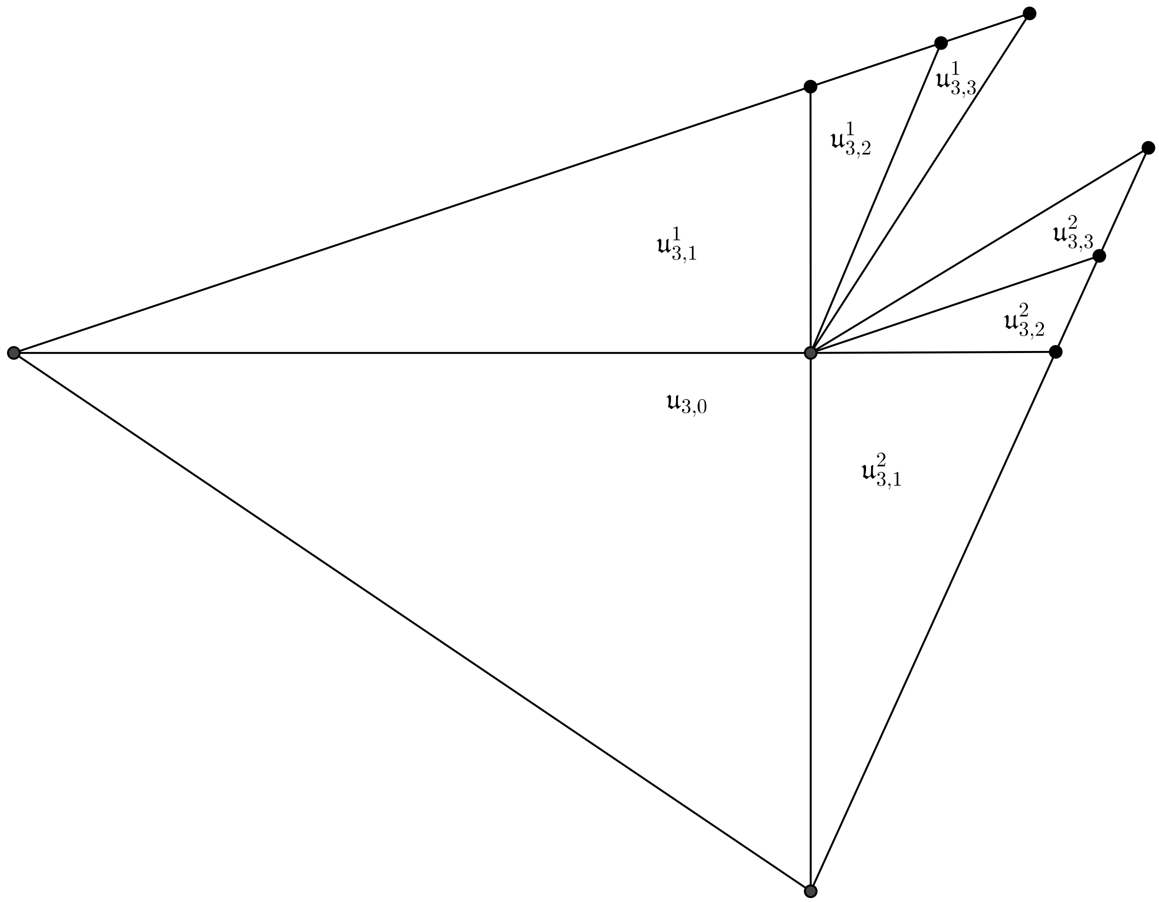


FIGURE 4.3.1. A schematic picture of $\mathrm{Diag}_3(P, v)$

The first few terms of $u_j^{1,1}$, $u_j^{1,2} \in \mathbb{Z}^2$ are:

$$\begin{tikzcd}
(-1,0) \arrow[dr] & (-i,1) \arrow[dr] & (1-i^2,i) \arrow[dr] & (2i-i^3,i^2-1) \arrow[dr] & \cdots \arrow[dr] & u_j^{1,1} \\
(0,1) \arrow[ur] & (1,0) \arrow[ur] & (i,-1) \arrow[ur] & (i^2-1,-i) \arrow[ur] & \cdots \arrow[ur] & u_j^{1,2}
\end{tikzcd}$$

There is an analogous sequence for the sequence $u_j^{2,l}$. From the piecewise linear map in M it is easy to determine that

$$u_k^{1,1} = (-a_{k+1}, a_k), u_k^{1,2} = (a_k, -a_{k-1})$$

where the a_k satisfy the recursive relationship:

$$a_{k+1} = sa_k - a_{k-1}$$

LEMMA 4.3.3. *These normal vectors define lines in $\mathbb{R}^2$ which converge to a line with slope $\frac{-s+\sqrt{s^2-4}}{2}$*

PROOF. Straightforward calculation. $\square$

REMARK 4.3.4. Making this calculation for the sequence $u_k^{2,l}$, the sequence of slopes of the induced lines converges to $\frac{-s-\sqrt{s^2-4}}{2}$.

We now put P in normal form, with respect to the edges incident to v. Combining this with the formula for the normal vectors we will compute the slopes of these rays in $T_v\mathbb{R}^2$ generated by the edges E_1 E_2 for the chambers $\mathfrak{u}_{k,j}^i$.

DEFINITION 4.3.5. Let v_j^i be the set of vectors generating a sequence of ray $\mathfrak{d}_j^i$ defined recursively as follows.

- $v_1^1 := (-1, 0)$, $v_2^1 := (0, 1)$
- $v_1^2 := (0, -1)$, $v_2^2 := (1, 0)$
- $v_{j-1}^i + v_{j+1}^i = sv_j^i$, recalling that $s = \det(\rho)$.

We record the following easy comparision result,

LEMMA 4.3.6. *The set of rays generated by $v_j^{i,l}$ in $T_v\mathbb{R}^2$ is equal to that induced on $T_v\mathbb{R}^2$ by the chambers $\mathfrak{u}_{k,j}^i$ for $i = 1$, 2, for all $j \geq 1$, $k \geq j$.*

PROOF. This follows immediately from our considerations of the normal vectors after taking the orthogonal directions using the volume form $(- \wedge -)$. $\square$

In particular these rays converge to a pair of asymptotes:

LEMMA 4.3.7. *The sequence of rays $\mathfrak{d}_j^i$ converge to the ray $\mathfrak{d}_\infty^i$ with slope*

$$m_\infty^i := \frac{s \pm \sqrt{s^2-4}}{2}$$

where $i = 1$ corresponds to the asymptote with positive sign.

PROOF. These are the orthogonal directions to the rays in $M_\mathbb{R}$ obtained in Lemma 4.3.3. $\square$

Next consider the union of the fixed edges E of $\mathfrak{u}_{j,k}^i$. This defines a pair of rays in $\mathbb{R}^2$ which we denote by $\mathfrak{e}_i$. In order to prove $\iota_{P,v,k}$ is an embedding we require a bound on the gradient of $\mathfrak{e}_i$. This bound makes use of the fact that P has singularity content $(3, \varnothing)$ and thus is of the form $\mathbb{P}(a^2, b^2, c^2)$. Our first step is to relate the triple of weights (a, b, c) for defined by P to the numbers ℓ_1, ℓ_2, s.

LEMMA 4.3.8. *Given a triangle mutation equivalent to $\mathbb{P}^2$ in standard form with respect to the vertex corresponding to a, let the other two vertices in clockwise order correspond to b and c then $\ell_1 = b$, $\ell_2 = c$ and $s = 3a$.*

PROOF. That $s = 3a$ is clear from the definition, since fixing the two edges adjacent to the distinguished vertex as E_1, E_2, $s = \det(\rho) = \rho(E_1) \wedge \rho(E_2)$. To prove $\ell_1 = b$ and $\ell_2 = c$ recall that ℓ_i is the height (or local index) of the edges E_1, E_2 respectively and observe that for a Fano polygon P such that $X_P = \mathbb{P}(a^2, b^2, c^2)$ such that $a^2 + b^2 + c^2 = 3abc$, the triple of local indices is given by (a, b, c). □

The main technical result of this section tells us that once the rays $\mathfrak{e}_i$ enter the region defined by the pair of asymptotes of the two sequences of rays v^i_∞ they never emerge again.

PROPOSITION 4.3.9. *For $i = 1, 2$ denote the slope of the ray $\mathfrak{e}_i$ by m_i, then:*

$$m^1_\infty \geq m_i \geq m^2_\infty$$

PROOF. First we need to compute the gradient of the rays $\mathfrak{e}_i$. Considering the initial chamber $\mathfrak{u}_{k,0}$ the edge e opposite v has normal vector (ℓ_1, ℓ_2). Consequently the normal directions to E in the chambers $\mathfrak{u}^i_{k,1}$ are $(\ell_1, \ell_2 - s\ell_1)$ and $(\ell_1 - s\ell_2, \ell_2)$ for $i = 1, 2$ and $k \geq 1$. Thus the orthogonal lines in N have slopes $\frac{\ell_1}{s\ell_1 - \ell_2}$ and $s - \frac{\ell_1}{\ell_2}$. As we can interchange b and c this situation is symmetric and we need only consider the second case, i.e. we only need to prove that

$$\frac{s - \sqrt{s^2 - 4}}{2} \leq s - \frac{\ell_1}{\ell_2} \leq \frac{s + \sqrt{s^2 - 4}}{2}$$

By Lemma 4.3.8 these are equivalent to the inequality

$$\sqrt{(3a)^2 - 4} \geq \left| 3a - \frac{2b}{c} \right|$$

But squaring both sides and rearranging we may reduce this inequality to a tautology:

$$\begin{aligned} (3a)^2 - 4 > (3a)^2 - 4.3\frac{ab}{c} + 4\frac{b^2}{c^2} &\Leftrightarrow \\ \frac{b}{c}\left(\frac{3ac - b}{c}\right) > 1 &\Leftrightarrow \\ 1 + \frac{a^2}{c^2} > 1 & \end{aligned}$$

□

COROLLARY 4.3.10. *The map $\iota_{P,v,k}$ is an embedding for every triple (P, v, k).*

In the proof of Theorem 1.3.3 we will make use of a gluing construction, whereby $\mathrm{Diag}(P, v, k)$ is created from iterated embeddings of $\mathrm{Diag}(P_i, v_i, 1)$ where P_i and v_i form a sequences of embedded polygons and vertices.

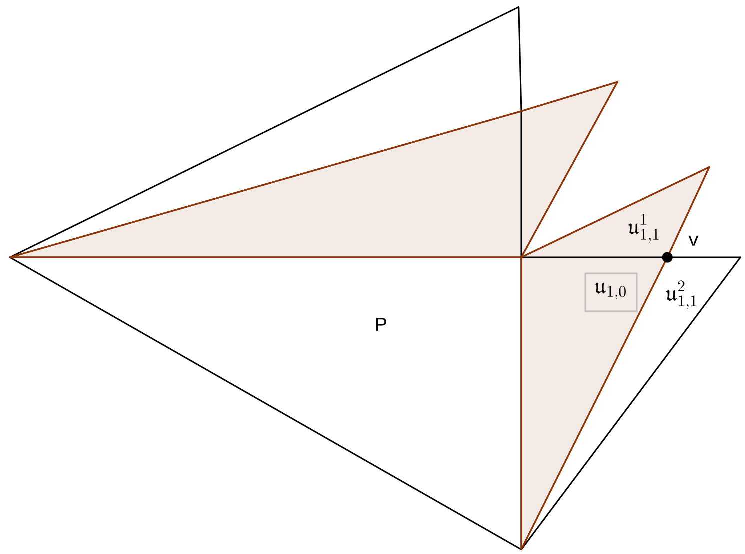


FIGURE 4.3.2. Gluing copies of $\mathrm{Diag}(P_i, v_i, 1)$ to form $\mathrm{Diag}(P, v, k)$

CONSTRUCTION 4.3.11. Given a diagram, $\mathrm{Diag}(P, v, k)$ define $P^1_k := \mathfrak{u}^1_{k,k}$ and $P^2_k := \mathfrak{u}^2_{k,k}$. Further, let v^1_k and v^2_k denote the vertices of P^1_k and P^2_k respectively such that $v^i_k \notin \mathcal{V}\left(\mathfrak{u}^i_{a,b}\right)$ for any $(a, b) \neq (k, k)$. There are Fano polygons $\bar{P}^i_k$ associated with P^i_k for $i = 1$, 2 such that the normal directions of $\bar{P}^i_k$ coincide with those of P^i_k. Forming the diagram $\mathrm{Diag}(\bar{P}^i_k, v^i_k, 1)$ and composing the embedding $\iota_{P^i_k, v^i_k}$ with a scale and translation, we can ensure that

$$\iota_{\bar{P}^i_k, v^i_k}(\bar{P}^i_k) = P^i_k$$

The simplical complex underlying $\mathrm{Diag}(P^i_k, v^i_k, 1)$ has two other chambers which extend the simplicial complex $\mathrm{Diag}(P, v, k)$. In fact, extending $\mathrm{Diag}(P, v, k)$ by the chambers $\mathfrak{u}^{2-i}_{1,1}$ of $\mathrm{Diag}(P^i_k, v^i_k, 1)$ we obtain the embedded complex $\mathrm{Diag}(P, v, k+1)$. An example of the process is shown in Figure 4.3.11

Eventually we shall want to mutate not just E_1 and E_2 but the third edge as well, to do this we shall need to recursively apply the construction we have just described to various triangular regions. However, before we describe this construction we will give a different interpretion of the set of rays we have defined, in a somewhat ad-hoc fashion, in this section. That is, we shall describe this set of rays as a subset of the rays of a *scattering diagram*.

4.4. Broken Lines in Affine Manifolds

Recall the definition of scattering diagrams from Chapter 3, following [**53**, **59**]. The only examples of scattering diagrams we shall use are the most basic examples, studied, for example in [**56**, **57**].

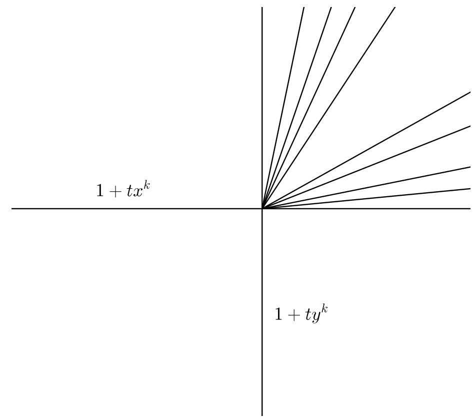


FIGURE 4.4.1. Rays of a scattering diagram $\mathfrak{D}_k(s)$.

$$\mathfrak{D}(k) = \{(\mathbb{R}(1,0), 1 + tx^k), (\mathbb{R}(0,1), 1 + ty^k)\}$$

The rays added by the scattering process admit a periodicity, resulting in a recursive formula identical to 4.3.6. As we have seen, the gradients of these rays converge to values which are, in general, different. In the region between these asymptotes there are non-zero functions supported on rays of any rational slope.

There is an obvious parallel to be drawn with the collection of rays we considered from the mutating family of triangles in Section 4.2. In particular outside the region defined by the two asymptotes the recursive formulae for generating the support of the rays are identical.

PROPOSITION 4.4.1. *Fix a triangle P and put it in standard form with respect to a vertex v. The collection of rays $\left\{\mathfrak{d}^i_j : i \in \{1,2\}, 1 \leq j\right\}$ formed in the construction described in Section 4.2 is equal to:*

$$\left\{(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}(s) : m_{\mathfrak{d}} > \frac{s + \sqrt{s^2 - 4}}{2} \text{ or } m_{\mathfrak{d}} < \frac{s - \sqrt{s^2 - 4}}{2}\right\}$$

where s is the determinant of the map ρ and $m_{\mathfrak{d}}$ is the slope of $\mathfrak{d}$.

PROOF. Both collections of rays are generated by the same recursive formula, and have the same initial configuration. □

REMARK 4.4.2. While the association of this set of rays with the triangles of the previous section seems mysterious, it is in fact tautological given the connection that both concepts have with cluster algebras. See Chapter 5 for details of this connection, and [**56**] for the connection to scattering diagrams.

As well as the notion of a scattering diagram we will utilize the notion of a *broken line* from [**18**, **52**]. These will provide an enumerative interpretation of the Laurent polynomials mirror to $\mathbb{P}^2$ as described in Theorem 4.1.1. The notion of broken line is very close to that of a *tropical disc*: broken lines can *bend* on the walls of a scattering diagram and one can canonically complete these bends so that the resulting object is a tropical curve *with stops* (following the terminology of [**90**]). For more details see Lemma 5.4 of [**18**].

The idea of calculating a superpotential tropically, utilising broken lines in the affine manifold, was first explored in [**18**]. In Section 4.6 we show that there is a domain U in the dual intersection complex $B^\vee$ for a toric degeneration of $(\mathbb{P}^2, E)$ such that the tropically defined superpotential is equal to the family of Laurent polynomials described in [**2**].

In an ideal setting tropical curves should be the 'spines' of images of holomorphic curves under a special Lagrangian torus fibration. Tropical discs are similar, but now the curve has boundary so there is a 'stop' where the tropical disc terminates. For a more detailed discussion of this point see [**18**, **52**, **90**]. There are many technical results in [**18**] showing that the tropical superpotential behaves well which we do not include here, but rather present a summary of those definitions and results required for the proof of Theorem 1.3.3. The results are presented for an affine manifold B satisfying the imposed by [**18**], any such conditions will be satisifed by the affine manifold $B^\vee_{\mathbb{P}^2}$.

Definition 4.4.3. A *broken line* is a proper continuous map

$$\beta\colon (-\infty, 0] \to B$$

with 'bends' at a sequence of points $-\infty = t_0 < t_1 < \cdots < t_r = 0$ such that $\beta|_{(t_j,t_{j+1})}$ is an affine map with image disjoint from the rays of $\mathscr{S}$.

Additionally a broken line carries a sequence of monomials $a_j z^{m_j}$ such that $\beta'(t) = \bar{m}_j$ which are naturally transported as sections of $\beta^{-1}P_\phi$. At a point $\beta(t_i) \in \tau$ for τ a one cell in $\mathscr{P}_k$ the monomial $a_j z^{m_j}$ defines a unique element in $R^k_{\tau,\tau,\mathfrak{u}}$ where $\beta(t_i - \epsilon) \in \mathfrak{u}$ for sufficently small $\epsilon > 0$. The wall-crossing formula $\theta_{\mathfrak{u},\mathfrak{u}'}$ defines a collection of monomials with order $\leq k$; these are the *results of transport* of $a_j z^{m_j}$.

For the broken line β we require that $\mathfrak{u}' \neq \mathfrak{u}$ and that the monomial attached to β is a result of transport. We also insist that $a_1 = 1$ and there is an unbounded 1-cell of $\mathscr{P}$ parallel to $\bar{m}_1$ for which m_1 has order zero.

This records all the important aspects of the definition, for a systematic treatment of broken lines the reader is referred to [**18**].

Given a general[1] point $p \in B$, denote the set of broken lines β with $\beta(0) = p$ by $\mathfrak{B}(p)$. For a given structure $\mathscr{S}_k$ on B, the ring $R^k_{\omega,\tau,\mathfrak{u}}$ and a general $p \in \mathfrak{u}$ we can produce the

[1]This is a generic condition, see Proposition 4.4 and Definition 4.6 of [**18**] for details.

superpotential at order k as an element of $R^k_{\omega,\tau,\mathfrak{u}}$, taking

$$W^k_{\omega,\tau,\mathfrak{u}}(p) = \sum_{\beta\in\mathfrak{B}(p)} a_\beta z^{m_\beta}$$

In [**18**] the authors obtain various results for $W^k_{\omega,\tau,\mathfrak{u}}(p)$, two of which we shall utilize in Section 4.6:

- The superpotential $W^k_{\omega,\tau,\mathfrak{u}}(p)$ is independent of the choice of $p \in \mathfrak{u}$ (Lemma 4.7 of [**18**]).
- The superpotentials are compatible with changing strata and chambers (Lemma 4.9 of [**18**]).

The content of the second point here is that, applying a change of chamber map to the superpotential, one obtains

$$\theta_{\mathfrak{u},\mathfrak{u}'}(W^k_{\omega,\tau,\mathfrak{u}}) = W^k_{\omega,\tau,\mathfrak{u}'}$$

where we have suppressed the dependence of $W^k_{\omega,\tau,\mathfrak{u}}(p)$ on p using the first point. This formula is intimately connected to the *mutations* we discussed in Chapter 2. To see this, we need to compare the rings $R^k_{\omega,\tau,\mathfrak{u}}$ and $\mathrm{k}[M]$ of which the respective superpotentials are elements. In Section 4.6 we shall find that the superpotential $W^k_{\omega,\tau,\mathfrak{u}}$ is, in the terminology of [**18**], *manifestly algebraic* in a domain U we shall specify. The main conseqence of this is that the limit

$$W = \lim_{\leftarrow} W^k$$

is a *finite* sum. Recall that $R^k_{\omega,\tau,\mathfrak{u}}$ is a localisation of

$$\mathrm{k}[P_{\phi,\omega}]/I^k_{\omega,\tau,\sigma_\mathfrak{u}}$$

So for sufficiently large k the obvious lift of $W^k_{\omega,\tau,\mathfrak{u}}$ to $\mathrm{k}[P_{\phi,\omega}]$ is independent of k. In fact, to make this precise, we will also show that $\mathscr{P}_k$ can be chosen so that $\mathfrak{u}$ is not further subdivided as $k \to \infty$. Taking the projection $\mathrm{k}[P_{\phi,\omega}] \to \mathrm{k}[M]$ induced by setting $t = 1$, we can represent W as a single Laurent polynomial. To summarise, we have

Lemma 4.4.4. *Let $\mathscr{S}$ be structure on an affine manifold B, and $U \subset B$ a domain such that $\mathscr{P}_k$ is eventually constant in U and such that for any ray $(\mathfrak{d}, f_\mathfrak{d})$ in U, the sequence of functions $f_\mathfrak{d}$ is eventually constant in k. In this setting we may define the ring*

$$\widehat{R}_{\omega,\tau,\mathfrak{u}} := \lim_{\leftarrow} R^k_{\omega,\tau,\mathfrak{u}}$$

which contains a subring of manifestly algebraic *elements $\overline{R}_{\omega,\tau,\mathfrak{u}}$, that is, the localisation of $\mathrm{k}[P_{\phi,\omega}]$ by the function $f_\mathfrak{d}$ appearing in the definition of $R^k_{\omega,\tau,\mathfrak{u}}$; see Chapter 3. This subring* *projects by taking $t = 1$:*

$$\begin{tikzcd} \overline{R}_{\omega,\tau,\mathfrak{u}} \arrow[r, "\iota"] \arrow[d, "(t=1)"] & \widehat{R}_{\omega,\tau,\mathfrak{u}} \\ R^{alg}_{\omega,\tau,\mathfrak{u}} & \end{tikzcd}$$

where $R^{alg}_{\omega,\tau,\mathfrak{u}} := \mathrm{k}[M]$, unless τ is one dimensional and intersects $\Delta \subset B$, in which case

$$R^{alg}_{\omega,\tau,\mathfrak{u}} := (\mathrm{k}[M])_{(1+c_m z^{\bar{m}})}$$

where $(1 + c_m z^m) = f_{\mathfrak{d}}$, the same localising function.

Remark 4.4.5. In Section 4.6 we shall see that for our example there is a domain U for which any general point p is contained in a chamber $\mathfrak{u}$ satisfying the hypotheses of Lemma 4.4.4. Thus the superpotential $W^k_{\omega,\tau,\mathfrak{u}}(p)$ may be identified with a Laurent polynomial.

Lemma 4.4.6. *The wall crossing formulae*

$$\theta^k_{\mathfrak{u},\mathfrak{u}'}(z^m) = z^m \prod f_{\mathfrak{d}}^{\langle n,\bar{m}\rangle}$$

define birational maps $\theta^k_{\mathfrak{u},\mathfrak{u}'} \colon \mathrm{k}(M) \to \mathrm{k}(M)$ for all sufficently large k. If there is only a single ray supported on $\mathfrak{d}$ and $f_{\mathfrak{d}} = 1 + c_m z^m$ for some exponent m then the birational map $\theta^k_{\mathfrak{u},\mathfrak{u}'}$ is an algebraic mutation*, in the sense of Chapter 2, with factor polynomial $(1 + c_m z^{\bar{m}})$.*

Thus the result of *crossing a wall* is that the function recorded at the base point, viewed simply as a Laurent polynomial, undergoes a birational change of variables which is precisely the mutation with factor given by the line segment in the direction of the wall. This will be an essential ingredient in the proof of Theorem 1.3.3, since it will allow us to compute the superpotential in every chamber from a calculation of broken lines in a single chamber.

4.5. The Affine Manifold $B^{\vee}_{\mathbb{P}^2}$

We now consider the affine structure on the dual intersection complex for a toric degeneration of $\mathbb{P}^2$. This is described in Example 2.4 of [**18**]. In [**18**], the authors consider the affine structure on the intersection complex and dual intersection complex of a so-called *distinguished* toric degeneration $(\mathfrak{X} \to T, \mathfrak{D})$. Given the pair $(\mathbb{P}^2, E)$ for a smooth genus one curve E, a distinguished toric degeneration will give an intersection complex as shown in Figure 4.1.1, as shown in the proof of Theorem 6.4 in [**18**].

Remark 4.5.1. While we only know how to construct such a degeneration formally, a close approximation to a globally defined toric degeneration is easily obtained. Indeed, consider the family

$$\begin{tikzcd} \{x_0x_1x_2 = t(y+g_3)\} \arrow[r] & \mathcal{X} \arrow[r] \arrow[d, "\pi"] & \mathbb{P}(1,1,1,3)\times\mathbb{C}_t \arrow[dl] \\ & \mathbb{C}_t & \end{tikzcd} \tag{4.5.1}$$

The variables x_0, x_1, x_2, y correspond to coordinates on the weighted projective space

$$\mathbb{P}(1,1,1,3) := \operatorname{Proj}\mathbb{C}[x_0, x_1, x_2, y]$$

with the grading determined by assigning the degrees 1,1,1,3 to the variables respectively, and $g_3 \in \mathbb{C}[x_0, x_1, x_2]$ is a general degree three polynomial. Observe that all of the fibres, except the fibre over $t = 0$, are isomorphic to $\mathbb{P}^2$ while the zero fibre of π is given by the equation $\{x_0x_1x_2 = 0\} \subset \mathbb{P}(1,1,1,3)$ and is made up of three copies of $\mathbb{P}(1,1,3)$. However this is *not* a suitable toric degeneration as it inherits the $\frac{1}{3}(1,1,1)$ singularity from the ambient space at $t = x_0 = x_1 = x_2 = 0$.

For a precise definition of the discrete Legendre duality between $B_{\mathbb{P}^2}$ and $B^\vee_{\mathbb{P}^2}$ see [**53**,**59**]. Rather than provide this definition here we will describe $B^\vee_{\mathbb{P}^2}$ as an affine manifold. The manifold $B^\vee_{\mathbb{P}^2}$ was described in [**18**], and is shown in Figure 4.5.1. The affine manifold $B_{\mathbb{P}^2}$ associated to the *intersection* complex is shown in Figure 4.1.1. The affine structure on $B^\vee_{\mathbb{P}^2}$ is such that the three 'outgoing' unbounded 1-cells of $\mathscr{P}$ are parallel to each other, the dual condition to the requirement that $B_{\mathbb{P}^2}$ have smooth (flat) boundary. Charts may be formed as usual, by cutting along the invariant lines of each focus-focus singularity. Note that each such charts will in general be disconnected, so we replace each chart with its connected component containing the unique closed 2-cell of $\mathscr{P}$.

Remark 4.5.2. Following the work of Gross–Hacking–Keel for log Calabi–Yau manifolds [**54**, **55**] one might attempt to consider the affine manifold obtained by regarding all the singularities of $B^\vee_{\mathbb{P}^2}$ as lying at the origin, which would be the 'U^{trop}' in those papers, for a log Calabi–Yau U. However in this case we do not have *maximal boundary*: the resulting affine manifold is a single ray and does not fit easily into that framework.

Following the philosophy of the Gross–Siebert programme [**59**], we endow the 1-cells τ of B supporting Δ with functions f_τ, or more precisely sections of $\mathcal{P}_\phi$ restricted to τ, defining a *log-structure* on a union of toric varieties. We shall make the standard choices of normalisation here so that f_τ is $(1 + z^m)$ where m is an exponent such that $\bar{m}$, a section of $\Lambda|_\tau$, is primitive and lies in the direction in τ *toward* the focus-focus singularity.

The data of $(B, \mathscr{P})$ together with the log-structure defines an initial structure, $\mathscr{S}_0$. Following the Gross–Siebert algorithm we shall consider various scattering diagrams in order to construct $\mathscr{S}_k$ for each k. In fact in Section 4.6 we shall compute a collection of regions using mutation of polygons and argue by our comparison result, Proposition 4.4.1, that this is the support of the union of scattering diagrams, away from a region densely filled with rays.

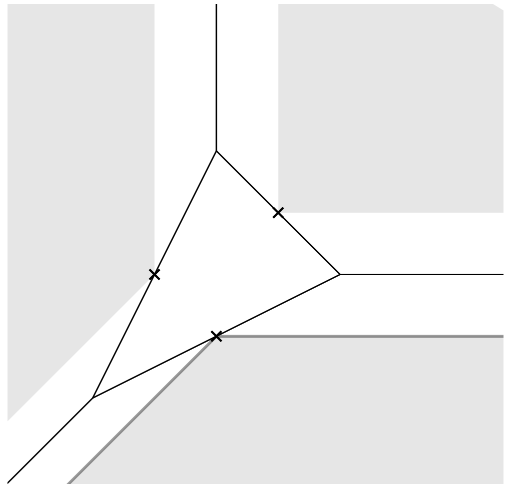


FIGURE 4.5.1. Affine structure on the dual intersection complex $B^{\vee}_{\mathbb{P}^2}$

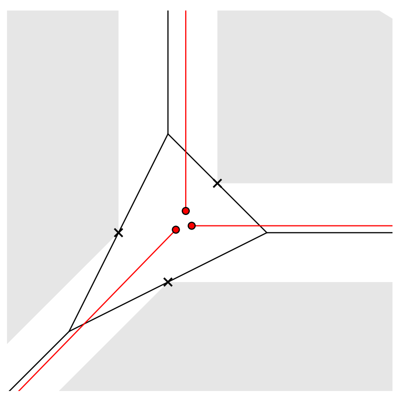


FIGURE 4.5.2. Broken lines in the central region

To compute the superpotential, we will use Corollary 6.8 of [**18**], which states that, for a base point in the interior of the bounded cell of $\mathscr{P}$, the superpotential for this structure is given by the usual Givental/Hori–Vafa superpotential:

$$W = x + y + \frac{1}{xy}$$

Each term here coming from a different broken line as shown in Figure 4.5.2. Using Lemma 4.9 of [**18**] this calculation determines the superpotential in every other chamber, using the wall-crossing formula $\theta_{\mathfrak{u},\mathfrak{u}'}$ to change chambers.

4.6. The Proof of Theorem 1.3

In this section we shall prove Theorem 1.3.3. To do this we will inductively build regions $U_n \supset U_{n-1}$, covered by the triangles corresponding to vertices of $\mathcal{G}$ with distance $\leq n$ from $(1,1,1)$, by considering new complexes $\mathrm{Diag}(P,v,k)$ at each stage. Defining $U := \lim_{n\geq 0} U_n$, we show that U is covered by (infinitely many) triangular regions and that the set of rays determined by the edges of these triangles is identical to the set of support of rays $\mathfrak{d}$ of the structure $\mathscr{S}$ intersected with U.

Proof of Theorem 1.3.3. *We only sketch this here as the notation quickly becomes dense, but we hope that the general methodology is clear.*

We shall define the domains U_k together with a polyhedral decomoposition by induction. Indeed, define $U_0 = P$ the bounded 2-cell in $B^{\vee}_{\mathbb{P}^2}$ and consider $\mathrm{Diag}(P,v,1)$ for each vertex $v \in \mathcal{V}(P)$. While the affine structure on $B^{\vee}_{\mathbb{P}^2}$ is not trivial we can compose $\iota_{P,v,k}$ with a chart on $B^{\vee}_{\mathbb{P}^2}$ which contains v. Thus we define U_1 as the union of chambers $\mathfrak{u}^i_{1,1}$ for each vertex $v \in \mathcal{V}(P)$ (this gluing is shown in Figure 1(a)). Observe that pairs of these chambers are identified by the transition functions on $B^{\vee}_{\mathbb{P}^2}$; there is only one additional chamber in U_1 for each edge of P. Also note that each $\mathfrak{u} = \mathfrak{u}^i_{1,1}$ has a distinguished vertex $v_{\mathfrak{u}}$ which is disjoint from U_0.

Given $k > 0$ we consider the set

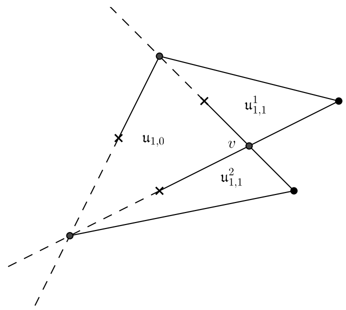
$$A_k = \{\mathfrak{u} \in \mathrm{Chambers}(U_k) \setminus \mathrm{Chambers}(U_{k-1})\}$$

where $\mathrm{Chambers}(U_k)$ is the set of maximal cells of the underlying complex of U_k. We form U_{k+1} as follows:

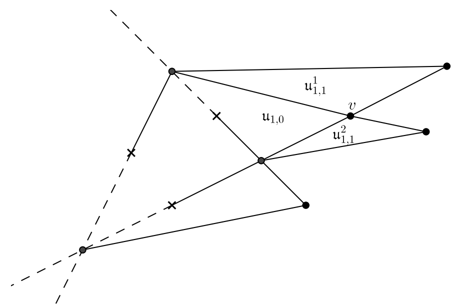
- By induction, there is a vertex $v_{\mathfrak{u}} \in \mathcal{V}(\mathfrak{u})$ for each $\mathfrak{u} \in A$ such that $v_{\mathfrak{u}}$ is not contained in U_{k-1}.
- For every $\mathfrak{u} \in A_k$ form the diagram $\mathrm{Diag}(\mathfrak{u}, v_{\mathfrak{u}}, 1)$.
- Embed $\mathrm{Diag}(\mathfrak{u}, v_{\mathfrak{u}}, 1)$ into $B^{\vee}_{\mathbb{P}^2}$ by composing $\iota_{\mathfrak{u},v_{\mathfrak{u}},1}$ with a chart of $B^{\vee}_{\mathbb{P}^2}$, thus identifying $\mathfrak{u}$ with the corresponding chamber in U_k.
- Identifying those chambers with the same image in $B^{\vee}_{\mathbb{P}^2}$ we form a collection U_{k+1} of chambers each of which has a unique vertex $v_{\mathfrak{u}}$ disjoint from U_k.

Examples of this gluing process are shown in Figure 4.6.1. Observe that, using Construction 4.3.11, given a pair $(\mathfrak{u}, v_{\mathfrak{u}})$, as $k \to \infty$, this process generates all of $\mathrm{Diag}(\mathfrak{u}, v_{\mathfrak{u}}, l)$ for *any* $l > 0$.

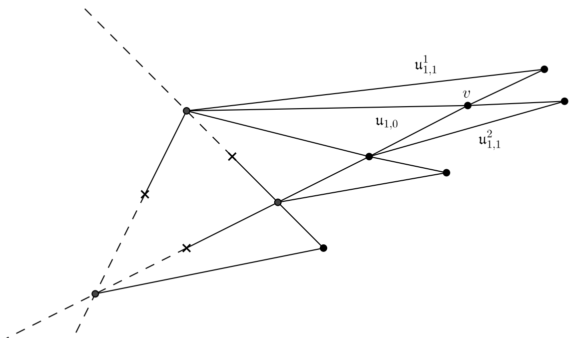
By the comparison result Proposition 4.4.1, every line segment appearing from this construction is a segment of a ray generating by a scattering diagram at a joint. This process also prolongs each of these segments of rays of the scattering diagram until they leave the domain U. The bound from Proposition 4.3.9 on their slope ensures they never enter U again. Combining these observations the proof is then reduced to a series of exercises.



(a) Gluing in a diagram



(b) Gluing a second $\mathrm{Diag}(P, v, 1)$



(c) Gluing a third $\mathrm{Diag}(P, v, 1)$

FIGURE 4.6.1. Building up $\mathrm{Chambers}(\mathscr{S}, k)$ using polygon mutation

(1) Each new edge produced by the chambers $\mathfrak{u}_{1,1}^{i}$ of $\mathrm{Diag}(P, v, k)$ is a segment of a ray of the scattering diagram induced by the segments appearing in U_k at the *other two* vertices of P.
(2) Every ray is prolonged until it enters the region between the two asymptotes in a unique diagram $\mathrm{Diag}(\mathfrak{u}, v, k)$.
(3) An initial segment of every ray in the structure $\mathscr{S}$ intersected with U is produced for some k.

Once we have concluded that the chambers in U_k are stable (that is, they undergo no further subdivision as k increases), we combine this with the discussion in Section 4.4 to conclude that, since the superpotential is manifestly algebraic in the central chamber P of U, and since the functions attached to each rays are of the form $1 + z^m$, the effect of the wall crossing formula is to produce another, algebraic superpotential – which we have observed is precisely the result of mutating the original superpotential.

$\square$

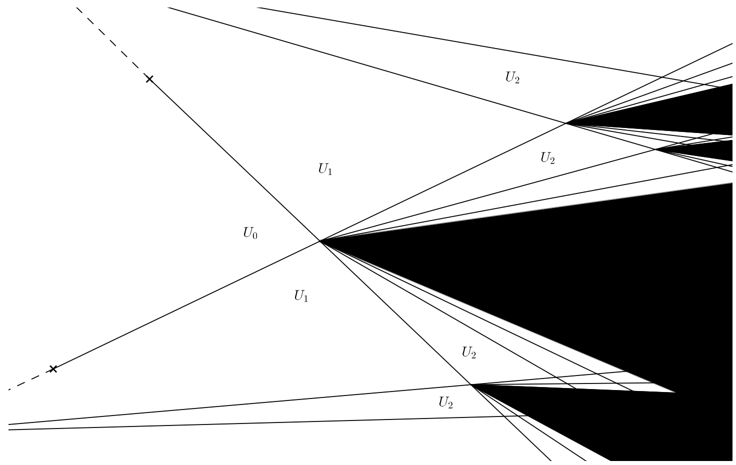


FIGURE 4.6.2. Part of U_2, defined by scattering diagrams in $\mathscr{S}$

CHAPTER 5

The Cluster Algebra Structure

5.1. Cluster Algebras and Mutations

In Chapter 2 we saw that, regarding Mirror Symmetry as the identification of a certain pair of variations of Hodge structure, there are potentially infinitely many Laurent polynomials mirror dual to a Fano variety. We saw in Chapter 2 a notion of *mutation* for Laurent polynomials, which preserves the period sequence, and it is natural to ask whether there is an algebraic structure controlling all of these mutations. Indeed, if we restrict to the surface case there is already a striking resemblence to the notion of a *cluster algebra*. Very roughly, a cluster algebra is an algebra generated by a recursively defined set of *cluster variables* associated to *seeds*. In particular, from one seed one can move to others via a process, also called mutation. Just as for Laurent polynomials in two variables, the number of possible mutations remains (in a suitable sense) constant, and performing the same mutation twice returns one to the original 'seed'.

In this chapter we will make this analogy precise, use it to produce a powerful mutation invariant of polygons and apply this to certain classification problems. Our first goal will be to reformulate the factors and weight vectors used to define polygon mutation as suitable *seed data*. The next goal will be to rigorously define a global object (a cluster variety) mirror to a Fano variety on which the various Laurent polynomials exist as the restriction of a *single* regular function to various torus charts. Recall that by rough analogy, the variety obtained by forgetting the superpotential is mirror–dual to the complement of a divisor $D \in |-K_X|$, so we expect to see the same cluster varieties that appear in the work of Gross–Hacking–Keel on log Calabi–Yau surfaces. This is indeed the case and we describe this connection in Section 5.6. We shall use these constructions to establish, or help establish, classification results. Perhaps most strikingly, using foundational results in cluster algebra theory (finite type and finite mutation type classification) we can classify those Fano polygons with finite mutation class, a result which would have been highly mysterious without this connection.

5.2. Background on Cluster Algebras

The definition of a cluster algebra is somewhat involved, so we devote this section to fixing the various conventions and notation, as well as recalling the basic definitions. We recall the definition of cluster algebra, and in order to address both geometric and combinatorial applications we shall adapt our treatment from the work of Fomin–Zelevinsky [**35**] and Gross–Hacking–Keel [**54**], which follows the treatment of Fock–Goncharov in [**33**]. We first fix the following data:

- N, a fixed rank n lattice with skew-symmetric form $\{\cdot,\cdot\}\colon N \times N \to \mathbb{Z}$.
- A saturated sublattice $N_{uf} \subseteq N$, the *unfrozen* sublattice.
- An index set I, $|I| = \mathrm{rk}(N)$ together with a subset $I_{uf} \subseteq I$ such that $|I_{uf}| = \mathrm{rk}(N_{uf})$. For later convenience we shall define $m := |I_{uf}|$.

REMARK 5.2.1. The requirement that the form is integral is not necessary, but is sufficently general for our applications and simplifies the exposition considerably.

REMARK 5.2.2. We use the notation standard in the cluster algebra literature, in particular N in this section is more closely related to the lattice M used in previous sections. This unfortunate exchange is directly related to the fact the cluster variety describes the the *mirror-dual* of the original log del Pezzo surface. We return to this point in Section 5.3.

DEFINITION 5.2.3. A *(labelled) seed* is a pair $\mathbf{s} = (\mathcal{E}, C)$, where:

- $\mathcal{E}$ is a basis of N indexed by I, such that $\mathcal{E}|_{I_{uf}}$ is a basis for N_{uf}.
- C is a transcendence basis of $\mathcal{F}$, the field of rational functions in n independent variables over $\mathbb{Q}(x_i | i \in I \backslash I_{uf})$, referred to as a *cluster*.

REMARK 5.2.4. The basis $\mathcal{E}$ is what the authors of [**33, 54**] refer to as *seed data*. Since we have fixed the lattice N and skew-symmetric form $\{\cdot,\cdot\}$ the variables x_i can be identified with coordinate functions on the *seed torus* T_N.

DEFINITION 5.2.5. Given a seed $\mathbf{s} = (\mathcal{E}, C)$ with $\mathcal{E} = \{e_1, \ldots, e_n\}$ and $C = \{x_1, \ldots, x_n\}$, the *jth mutation* of $(\mathcal{E}, C)$ is the seed $(\mathcal{E}', C')$, where $\mathcal{E}' = \{e'_1, \ldots, e'_n\}$ and $C' = \{x'_1, \ldots, x'_n\}$ are defined by:

$$e'_k = \begin{cases} -e_j, & \text{if } k = j \\ e_k + \max(b_{kj}, 0)e_j, & \text{otherwise} \end{cases}$$

where $b_{kl} = \{e_k, e_l\}$ and is often referred to as the *exchange matrix*,

$$x'_k = x_k \text{ if } k \neq j, \qquad \text{and} \qquad x_j x'_j = \prod_{\substack{k \text{ such that} \\ b_{jk}>0}} x_k^{b_{jk}} + \prod_{\substack{l \text{ such that} \\ b_{jl}<0}} x_l^{b_{lj}}. \tag{5.2.1}$$

DEFINITION 5.2.6. A *cluster algebra* is the subalgebra of $\mathcal{F}$ generated by the cluster variables appearing in the union of all clusters obtained by mutation from a given seed.

REMARK 5.2.7. This is really only a special case of the definition of a cluster algebra, a class referred to as the *skew-symmetric cluster algebras of geometric type*. In the general case the form $\{\cdot,\cdot\}$ need only be *skew-symmetrizable*. One consequence of the skew-symmetry of the form $\{\cdot,\cdot\}$ is the identification of each exchange matrix with a *quiver* Q. One may assign this quiver in the obvious way, namely assigning a vertex to each basis element of N, and b_{ij} arrows $v_i \to v_j$, with sign denoting the orientation of the arrows. Having divided the vertex set into *frozen* vertices and unfrozen ones one can replace the basis $\mathcal{E}$ with Q. There is a well-known notion of quiver mutation, going back to Bernstein–Gelfand–Ponomarev [**14**], Fomin–Zelevinsky [**35**], and others. Mutating a seed in a skew-symmetric cluster algebra induces a corresponding mutation of the associated quiver.

Definition 5.2.8. Given a quiver Q and a vertex v of Q, the *mutation of Q at v* is the quiver $\mathrm{mut}(Q, v)$ obtained from Q by:

(1) adding, for each subquiver $v_1 \to v \to v_2$, an arrow from v_1 to v_2;
(2) deleting a maximal set of disjoint two-cycles;
(3) reversing all arrows incident to v.

The resulting quiver is well-defined up to isomorphism, regardless of the choice of two-cycles in (2).

Since we shall refer to quivers frequently we shall make the following conventions

Definition 5.2.9. Given a quiver Q, we define

- Q_0 to be the set of vertices of Q.
- $\mathrm{Arr}(v_i, v_j)$ to be the set of arrows from $v_i \in Q_0$ to $v_j \in Q_0$.
- b_{ij} to be the cardinality of $\mathrm{Arr}(v_i, v_j)$, with sign indicating orientation.

We shall always assume Q has no vertex-loops or 2-cycles.

Given a seed $\mathbf{s}$ we shall also fix notation for the dual basis $\mathcal{E}^\star$ of $M := \mathrm{Hom}(N, \mathbb{Z})$ and for each $i \in I$, set $v_i := \{e_i, \cdot\} \in M$. We now define the $\mathcal{A}$ and $\mathcal{X}$ cluster *varieties* defined by Fock–Goncharov [**33**]. Toward this, observe to a seed $\mathbf{s}$ we can associate a pair of tori

$$\mathcal{X}_{\mathbf{s}} = T_M \qquad\qquad \mathcal{A}_{\mathbf{s}} = T_N$$

The dual pair of bases for the respective lattices define identifications of these tori with split tori.

$$\mathcal{X}_{\mathbf{s}} \to \mathbb{G}_m^n \qquad\qquad \mathcal{A}_{\mathbf{s}} \to \mathbb{G}_m^n$$

We also give birational maps

$$\mu_k^\star z^n = z^n(1 + z^{e_k})^{-\{n, e_k\}} \qquad\qquad \mu_k^\star z^m = z^m(1 + z^{v_k})^{\langle e_k, m\rangle}$$

Pulling these back along the identifications with the split torus given by the seed, the birational map $\mu_k \colon \mathcal{A}_{\mathbf{s}} \dashrightarrow \mathcal{A}_{\mu_k(\mathbf{s})}$ is given by the exchange relation 5.2.1. That is, this birational map is the coordinate-free expression of the exchange relation once we identify the standard coordinates on T_N with the cluster variables $x_i \in C$ (including the frozen variables $x_{n+1} \cdots, x_m$). We obtain schemes $\mathcal{X}$ and $\mathcal{A}$ by gluing the seed tori $\mathcal{A}_s$ and $\mathcal{X}_s$ along the birational maps defined by the mutations μ_k.

5.3. From Fano Polygons to Seeds

In this section we demonstrate how to associate a cluster algebra to a Fano polygon P. We then show that the mutations (combinatorial and algebraic) are compatible with the definitions of mutation for seed-data (and quivers) and the birational maps μ_k respectively. To avoid overloading notation the Fano polygon P lies in a lattice $\overline{M}$ for the rest of this section; this may be identified with the lattice N in previous chapters.

Definition 5.3.1. Given a Fano polygon $P \subset \overline{M}_{\mathbb{Q}}$ with singularity content $(n, \mathcal{B})$ and $m := |\mathcal{B}| + n$, we define:

- An index set I of size m, with a subset I_{uf} of size n and functions:

$$\phi_{uf} \colon I_{uf} \to \{\text{edges of } P\} \qquad\qquad \phi_f \colon I \backslash I_{uf} \to \mathcal{B}$$

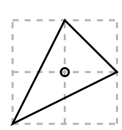
  Here the fiber $\phi_{uf}^{-1}(E)$ has m_E elements, where m_E is the singularity content of $\mathrm{Cone}(E)$, and ϕ_f is a bijection.
- A lattice map $\rho \colon \mathbb{Z}^m \to \overline{N}$ sending each basis element to the primitive, inward-pointing normal to the edge of P defined by the cone given by the specified functions ϕ_{uf} and ϕ.
- A form $\{e_i, e_j\} := \rho(e_i) \wedge \rho(e_j)$. Note that this is an integral skew-symmetric form.

Definition 5.3.2. Given a Fano polygon P by fixing the based lattice $N \cong \mathbb{Z}^m$ equipped with the skew-symmetric form from Definition 5.3.1 let E be the standard basis and C be the standard generating set. We define the cluster algebra $\mathcal{C}_P$ associated to P to be the cluster algebra generated by this seed. We define the *unfrozen* cluster algebra associated to P by forming the same seed $\mathbf{s} = (\mathcal{E}, C)$ and setting all frozen variables equal to 1.

We refer to the quiver obtained from the exchange matrix of $\{\cdot, \cdot\}$ in this basis as Q_P. Unless we specify otherwise the 'cluster algebra associated to P' shall refer to the *unfrozen* cluster algebra. The importance of the 'full' cluster algebra will not be explored in detail, but we do observe that we do not know of a counter-example to the following conjecture.

Conjecture 5.3.3. *The cluster algebras $\mathcal{C}_P$ for Fano polygons P together with a bijection between the set of frozen variables and $\mathcal{B}$ is a complete mutation invariant Fano polygons.*

Example 5.3.4. Consider the Fano polygon P for $\mathbb{P}^2$

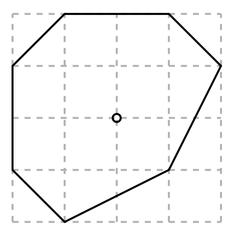


Computing the determinant of the inward-pointing normals we obtain the quiver Q_P

$$\begin{tikzcd} & \bullet \arrow[dr, "3"] & \\ \bullet \arrow[ur, "3"] & & \bullet \arrow[ll, "3"] \end{tikzcd}$$

The mutations of this quiver are well-known, and the triple $(3a, 3b, 3c)$ of non-zero entries of the exchange matrix satisfy the Markov equation $a^2 + b^2 + c^2 = 3abc$. Indeed, as the polygon P is mutated the corresponding toric surfaces are $\mathbb{P}(a^2, b^2, c^2)$ for the same triples (a, b, c). We see that in this case the mutations of the quivers exactly capture the mutations of the polygon. In the last section we saw a bijection between the seeds of this cluster algebra and chambers induced by scattering in a structure on an affine manifold.

Example 5.3.5. Consider the toric surface[1] $X_{5,5/3}$ associated with the Fano polygon shown below.



The unfrozen quiver of this surface is simply the A_2 quiver

$$\bullet \longrightarrow \bullet$$

This example is important, both in this section, because it is an example of a *finite-type* polygon, and in later chapters, since a smoothing of this surface is given by 5 Pfaffian equations, a fact closely connected to the A_2 quiver we see here.

To prove that mutations of polygons induce the mutations of the corresponding seed, we need only show that the associated quivers are related by the appropriate mutation.

Proposition 5.3.6 (Mutations of polygons induce mutations of quivers). *Let P be a Fano polygon, let v be a vertex of Q_P corresponding to a edge of P, and let P' be the corresponding mutation of P. We have $Q_{P'} = \mathrm{mut}(Q_P, v)$.*

Proof. Let E denote the edge of P corresponding to v, and let $w \in \overline{N}$ denote the primitive inner normal vector to E. Mutation with respect to w acts on $\overline{N}$ as a piecewise-linear transformation that is the identity in one half-space, and on the other half-space is a shear transformation $u \mapsto u + (w \wedge u)w$. Thus determinants between the pairs of normal vectors change as follows:

[1] Using the notation for these surfaces appearing in [**1**].

(1) The inner normal vector w to the mutating edge E becomes $-w$, so that all arrows into v change direction;
(2) For a pair of normal vectors in the same half-space (as defined by w), the determinant does not change;
(3) Consider edges with inner normal vectors in different half-spaces (as defined by w), let the corresponding vertices of Q_P be v_1 and v_2, and let the corresponding inner normal vectors in $\overline{N}$ be w_1 and w_2. Without loss of generality we may assume that $w_1 \wedge w > 0$ and $w_2 \wedge w < 0$, so that there are arrows $v_1 \to v \to v_2$ in Q_P. Under mutation, the primitive inner normal vectors change as $w_1 \mapsto w_1'$, $w_2 \mapsto w_2'$ where $w_1' = w_1$, $w_2' = w_2 + (w \wedge w_2)w$. Thus:
$$w_1' \wedge w_2' = w_1 \wedge w_2 + (w \wedge w_2)(w_1 \wedge w)$$
and so we add an arrow for each path $v_1 \to v \to v_2$. Cancelling two-cycles results in precisely the result of calculating the signed total number of arrows from v_1 to v_2.

Observing finally that if v_1, v_2 give normal vectors in the same half-space then there are no paths $v_1 \to v \to v_2$ or $v_2 \to v \to v_1$, we see that this description coincides with that of a quiver mutation. □

To compare the birational maps associated to the two notions of mutations fix a basis vector $e_k \in \mathcal{E}$ for $k \in I_{uf}$ and consider the following diagram:

(5.3.1)
$$\begin{tikzcd} \mathcal{A}_{\mathbf{s}} \arrow[r, dashed, "\mu_k"] \arrow[d, "\rho"] & \mathcal{A}_{\mu_k(\mathbf{s})} \arrow[d, "\rho"] \\ T_{\overline{N}} \arrow[r, dashed, "\mu_{(\rho(e_k),F)}"'] & T_{\overline{N}} \end{tikzcd}$$

where F is the factor canonically associated with the weight vector $w := \rho(e_k)$. In fact the definition of the cluster algebra associated to a polygon was discovered by insisting that this diagram commutes.

Proposition 5.3.7. *Diagram 5.3.1 commutes*

Proof. This is an exercise in writing out the definitions of the respective mutations, see [**70**, Section 3]. □

We now have a geometric reformulation of the notion of a *maximally-mutable Laurent polynomial* in the surface case as an element of the *upper cluster algebra* of $\mathcal{C}_P$.

Definition 5.3.8. The upper cluster algebra $\mathrm{up}(\mathcal{C})$ of a cluster algebra $\mathcal{C}$ is the intersection of the Laurent polynomial rings defined by its various clusters.

Proposition 5.3.9. *Given a Laurent polynomial f with support a Fano polygon P, f is maximally mutable if and only if $\rho^\star f \in \mathrm{up}(\mathcal{C}_P)$, where we identify $\overline{N}$ with the image of ρ.*

PROOF. This is a tautology: by definition f is maximally mutable if it remains Laurent under all possible mutations of P. $\square$

REMARK 5.3.10. The upper cluster algebra may be identified with the global functions on the $\mathcal{A}$ cluster variety of Fock–Goncharov [**33**], that is, $\Gamma(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$. As we shall see in Section 5.6 there is also a very close connection to the '$\mathcal{X}$-cluster algebra', $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

5.4. Finite Type Classifcation

In the last section we constructed a powerful invariant of Fano polygons up to mutation classes. As we shall see in the next section this invariant can be used to distinguish mutation classes that have the same singularity content and basket. As well as this, quiver mutation is a computationally very cheap way of checking if two polygons are likely to be mutation equivalent. In this section we use this to answer the question

Which mutation classes of Fano polygons are finite?

This will utilise general classification results for skew-symmetric cluster algebras of geometric type.

DEFINITION 5.4.1. A polygon P is said to be of *finite-type* if its mutation equivalence class is finite.

DEFINITION 5.4.2. Given an undirected graph G we say that a quiver Q is an *orientation* of G if it has the same set of vertices and for each edge of G there is precisely one arrow between the respective vertices. For a simply-laced Dynkin diagram D we say that Q is of *type* D if it is an orientation of the underlying graph of D.

THEOREM 5.4.3. *P is of finite type if and only if Q_P is mutation equivalent to one of the following types:*

- $(A_1)^n$*, which we refer to as type* I_n*.*
- A_2*, which we refer to as type* II*.*
- A_3*, which we refer to as type* III*.*
- D_4 *which we refer to as type* IV*.*

REMARK 5.4.4. The names of these classes are given in analogy with Kodaira's monodromy matrices. The connection with monodromy in affine manifolds and the presence of certain broken lines is explored in great detail in [**86**].

Before we attempt to prove Theorem 5.4.3 we make two simple, but important, observations. First, the cluster algebra $\mathcal{C}_P$ induces a sequence of surjections

$$\begin{tikzcd} \{\text{Clusters of } \mathcal{C}_P\} \arrow[d] \\ \{\text{Polygons mutation equivalent to } P\} \arrow[d] \\ \{\text{Quivers mutation equivalent to } Q_P\} \end{tikzcd} \tag{5.4.1}$$

Remark 5.4.5. In fact, as a consequence of Lemma 5.3.6 if we had defined the notion of a *mutation graph* in which the arrows were the various mutations in each context this would be a surjection of graphs. However the presence of automorphisms of P and Q_P makes these graphs awkward to define precisely. One can however define a graph for which the vertices are seeds and the arrows are cluster mutations, the *cluster exchange graph.*

Secondly we have a famous example of a cluster algebra: the cluster algebra generated by the seed defined by (x_1, x_2) and an A_2 quiver consisting of 5 clusters. Given the tower of surjections above, we have the following immediate corollary.

Corollary 5.4.6. *If a Fano polygon P has singularity content $(2, \mathcal{B})$ and the primitive inward-pointing normal vectors of the two edges corresponding to the unfrozen variables of $\mathcal{C}_P$ form a basis of the lattice M, then the mutation-equivalence class of P has at most five members.*

Proof. The quiver associated to P is precisely an orientation of the A_2 quiver. The cluster algebra $\mathcal{C}_P$ is well-known and its cluster exchange graph forms a pentagon. Note however that the *quiver* mutation graph is trivial, as the A_2 quiver mutates only to itself. Proposition 5.3.7 implies that the mutation class of P has at most five elements. (Proposition 5.3.6 does not give a non-trivial lower bound here, indeed the polygon considered in 5.3.5 has only a single polygon in its mutation class, up to $\mathrm{GL}(2, \mathbb{Z})$ equivalence.) $\square$

Definition 5.4.7. A cluster algebra $\mathcal{C}$ is said to be of *finite type* if it contains only finitely many seeds. $\mathcal{C}$ is said to be of *finite mutation type* if the mutation equivalence class of a quiver Q associated to a seed of $\mathcal{C}$ is finite.

Thus the sequence of surjections 5.4.1 give the obvious implications:

$$\mathcal{C}_P \text{ finite type} \Rightarrow P \text{ finite type} \Rightarrow \mathcal{C}_P \text{ finite mutation type}$$

We need one additional Lemma before we complete the proof of Theorem 5.4.3.

Lemma 5.4.8. *Given a Fano polygon P of finite type, Q_P does not contain a Kronecker subquiver*

$$Q_k := \ v_1 \xrightarrow{k} v_2$$

where $k > 1$ is the number of arrows from v_1 to v_2.

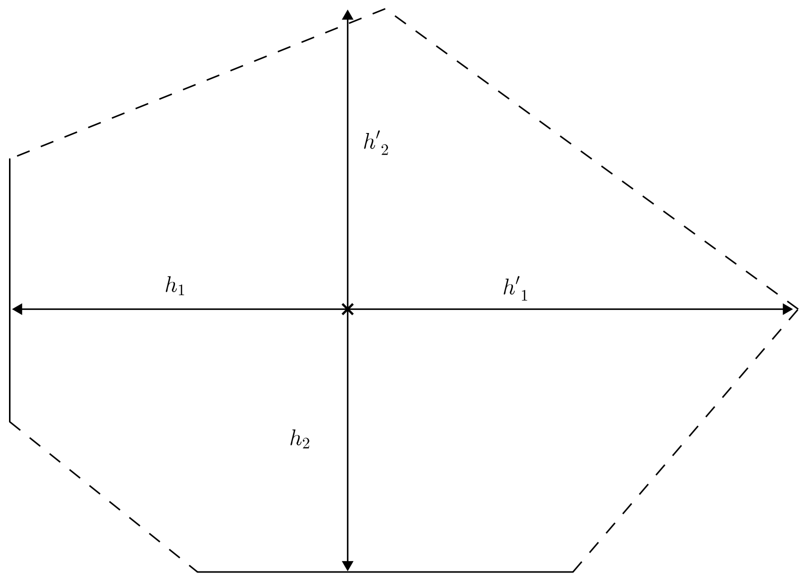


FIGURE 5.4.1. Schematic diagram of a polygon in standard form

PROOF. On the level of the cluster algebra, this is obvious, as restricting to a pair of cluster variables one can reduce to studying a 'rank 2' cluster algebra which is known not to be of finite type. Given the superpotential is a combination of cluster monomials this result is certainly expected, however we prove it directly from the combinatorics of P.

Assume there is such a subquiver of Q_P, with vertices v_1, v_2. To simplify notation, we shall put P in *normal form* with respect to the two edges E_1, E_2 of P corresponding to v_1 and v_2 of Q_P. Recall this is the image of P under the map $\rho^\star\colon \overline{M} \hookrightarrow M_{red}$, where ρ has been restricted to the sublattice N_{red} generated by e_1, $e_2 \in N$. The resulting polygon in $M_{red} := \mathrm{Hom}(N_{red}, \mathbb{Z})$ has the general form shown in Figure 5.4.1.

Consider the pair of local indices (h_1, h_2). As P undergoes various mutations this pair changes. For the two distinguished mutations out of P we have

$$(h_1, h'_2) \longleftarrow (h_1, h_2) \longrightarrow (h'_1, h_2)$$

From Figure 5.4.1 we observe that:

$$h'_1 \geq kh_2 - h_1 \qquad\qquad h'_2 \geq kh_1 - h_2$$

where $k = \rho(e_1) \wedge \rho(e_2)$ is the index of $\overline{M}$ in M_{red}. We consider two cases: first assume that $k \geq 3$, and assume without loss of generality that $h_2 \geq h_1$. Now $h'_1 \geq 3h_2 - h_1 \geq 2h_2 \geq 2h_1$. Thus in this case the values in the pair (h_1, h_2) grow exponentially with mutation.

Next consider the case $k = 2$. Now the inequalities above are simply:

$$h_1' \geq 2h_2 - h_1 \qquad\qquad h_2' \geq 2h_1 - h_2$$

and we are again free to assume that $h_2 \geq h_1$. Indeed if $h_2 \geq h_1$ then $h_1' \geq 2h_2 - h_1 \geq h_1$, and if $h_2 > h_1$, $h_1' > h_1$. But in this case $h_1' > h_2$: if not simply replace (h_1, h_2) with (h_1', h_2) to obtain a contradiction. So assuming $h_1 \neq h_2$ one can generate an infinite set of distinct values h_i. The only remaining case is if $h := h_1 = h_2 = h_1' = h_2'$. To eliminate this possibility observe that since the index k is two the edges e_1, e_2 must meet in a vertex with coordinates $(-h, -h)$. But the requirement that $\rho^\star$ doubles the lattice length of e_1 and e_2 fixes the sublattice given by the image of $\rho^\star$. The lattice vectors (h, h) are in this sublattice for all $h \in \mathbb{Z}$, and so by primitivity of the vertices in P, $h = 1$. These special cases can then eliminated individually. $\square$

We now prove Theorem 5.4.3, exploiting the known classification results for finite type and finite mutation type cluster algebras.

Proof of Theorem 5.4.3. In [**36**] the authors prove that a skew-symmetric cluster algebra without frozen variables has a seed with Q an orientation of products of simply-laced Dynkin diagrams. Recalling that in our construction the form $\{\cdot, \cdot\}$ is defined as the determinant of a pair of vectors in the plane, we obtain the following useful necessary condition for a quiver Q to be Q_P of a Fano polygon:

Lemma 5.4.9. *Given a Fano polygon P and vertices v_1, v_2, v_3 of Q_P such that there are no arrows $v_i \to v_{i+1}$ for $i = 1$, 2, there are no arrows between v_1 and v_3.*

In particular given a Fano polygon P such that Q_P is not connected, $Q_P = A_1^n$ for some n. Similarly if Q_P is of type A or D then it must be one of A_2, A_3 or D_4. We are now reduced to showing that there are is no Fano polygon P of finite type such that $\mathcal{C}_P$ is not of finite-type. However $\mathcal{C}_P$ is of finite mutation type, and there is a classification result here too, given in [**32**], which we now recall.

Theorem 5.4.10. *Given a quiver Q with finite mutation class, its adjacency matrix b_{ij} is the adjacency matrix of a triangulation of a bordered surface or is mutation equivalent to one of eleven exceptional types.*

In fact from Lemma 5.4.9 none of the eleven exceptional types can occur as Q_P for a Fano polygon P. To understand the class of quivers arising from triangulated surfaces we use another classification result, from [**34**].

Definition 5.4.11. A quiver Q is said to admit a *block decomposition* if it may be assembled from the 6 pieces (*blocks*) shown in Figure 5.4.2 by identifying the vertices of quivers shown with unfilled circles, the *outlets*. Having connected two vertices in such a way the vertex is no longer an outlet. Attaching outlets in the same block together is not permitted.

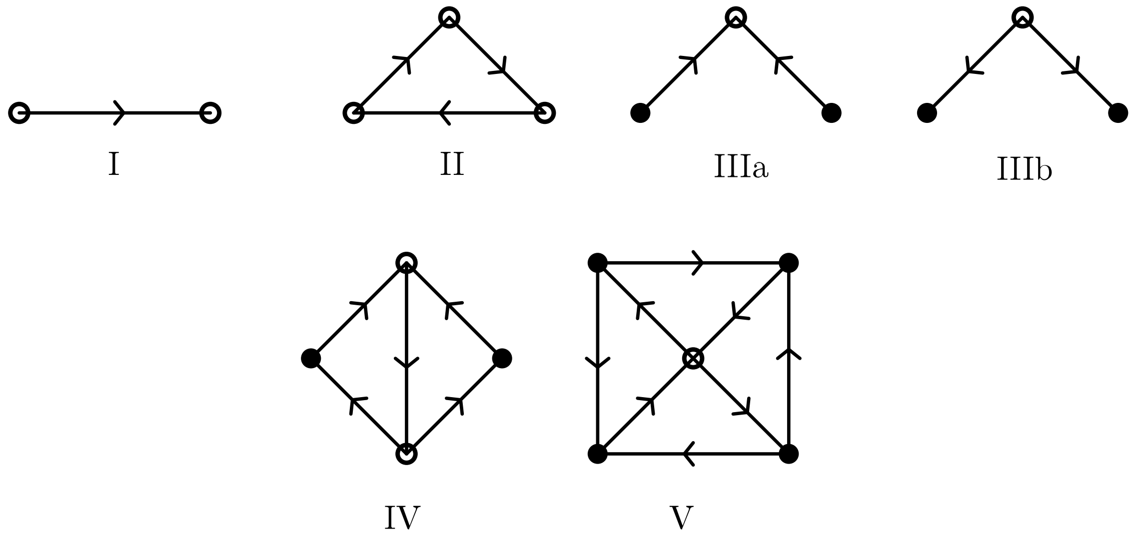


FIGURE 5.4.2. the blocks of a *block decomposition*

For a more detailed definition, see [**34**].

The following theorem, from [**34**], gives an explicit characterisation of those quivers coming from triangulations.

THEOREM 5.4.12. *A quiver Q given by the adjacency matrix of a triangulation of a surface is mutation equivalent to a quiver which admits a* block decomposition

To conclude the proof we claim that every quiver Q_P associated to a Fano polygon P which admits a box decomposition is either mutation equivalent to an orientiation of a simply laced Dynkin diagram or to a quiver which contains a subquiver Q_k for $k > 1$. This is a case-by-case analysis of the possible block decompositions of Q_P. For the rest of the proof we assume for condtradition that Q_P is the quiver associated to a Fano polygon P of finite-type which is not mutation equivalent to a simply laced Dynkin diagram.

Block V:

First observe that since only one vertex of the block V is an outlet the block is a subquiver of any quiver which contains V in its block decomposition. However this mutates to a quiver with a Q_2 subquiver as shown in Figure 5.4.3.

Therefore block V never appears in a decomposition of a quiver Q_P. For later use we shall fix the following intermediate quiver, V', as shown in Figure 5.4.4.

Blocks IIIa and IIIb:

If a type III block (a or b) is connected to a quiver Q' at a vertex v, then assuming Q' is an intermediate quiver in a block decomposition of Q_P, by Lemma 5.4.9 the only arrows

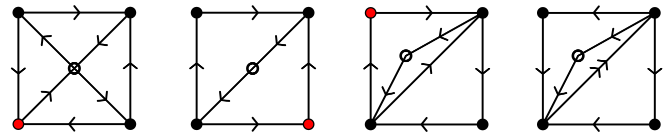


Figure 5.4.3. Mutations of block V

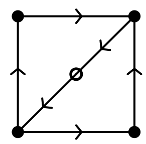


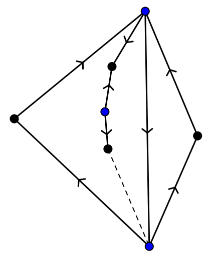
Figure 5.4.4. Quiver V'

in Q_P are incident to v are those in the type III block we attached to Q'. However Q_P is connected (as it has at least one arrow), giving only the A_3 and D_4 types.

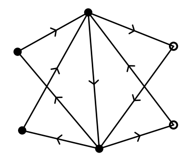
Block IV: Consider the case of a decomposition only using type IV blocks. First consider attaching two type IV blocks. If only one pair of outlets is attached we do not meet the condition of Lemma 5.4.9 for this to be Q_P for a polygon P. In fact it is easy to see that it is impossible to add additonal type IV blocks to meet this condition. If two pairs of outlets are attached there are two possible quivers depending on the relative orientations of the arrow between the outlets, one orientation produces a Q_2 subquiver automatically, the other produces a quiver contianing the quiver V' as a subquiver.

The only case which is not ruled out by Lemma 5.4.9 or automatically contains a Q_2 subquiver, is a pair of IV blocks glued to cancel the edge between the outlets. However this contains the quiver V' as a subquiver. So for a type IV block to appear in a decomposition of Q_P it must include a type I or II block.

Now consider decompositions using types I and IV. First we see that there must be exactly one type IV block, since if we use both outlets gluing type IV blocks the quiver is disconnected, and as before only connecting a single outlet will always result in a quiver violating the conditions of Lemma 5.4.9. Attaching a chain of type I quivers, we see the chain is at most two arrows long, or violates Lemma 5.4.9.



(a) Attaching I blocks to a IV block



(b) Attaching II blocks to a IV block

We now consider the possible cases. Attaching a type I block to cancel the arrow between the two outlets produces a quiver mutation equivalent to D_4, a simply laced Dynkin diagram. For chains of length two, if a 3-cycle is produced, a mutation in the vertex between the type I blocks produces the V' quiver. If not, the same mutation produces a Q_2 subquiver.

So a decomposition of Q_P with a type IV block must contain a type II block. Attaching a type II block along two outlets of the type IV block recovers the V' or Q_2 subquiver cases we have already seen. Attaching type II blocks to a single outlet each, attaching a single block must violate the conditions of Lemma 5.4.9. Attaching a second to meet this condition, we find the quiver:
This quiver mutates to one with a Q_2 subquiver. Attaching further type II blocks, we are forced to violate the conditions of Lemma 5.4.9. Attaching type I blocks between the remaining outlets, the only case satisfying Lemma 5.4.9 is obtained by attaching a single type I block to both outlets. This mutates in one step to a quiver with a Q_2 subquiver.

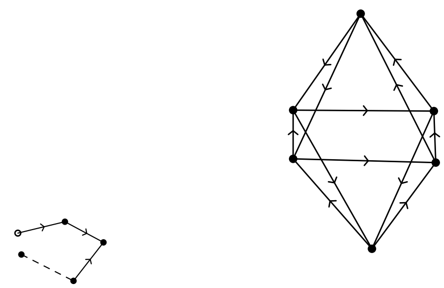
Blocks I and II:

From what we have shown above, the block decomposition of Q_P consists only of type I and type II blocks. For decompositions of only type I blocks it is easy to see that $(A_1)^{2n}$, A_2, A_3 and the 4-cycle, mutation equivalent to D_4, can be produced.

If connected, any such quiver is a path (with possibly changing orientations) which possibly closes up into a cycle. The only cases not violating Lemma 5.4.9 are those we have listed.

For decompositions of Q_P with type I and II blocks we order by the number of type II blocks. For a single type II block, we can attach a type I block to two outlets, we reduce to the case of a type III block, producing the A_3 and D_4 types. Attaching each type I block to a type II block in at most one outlet, the longest chain of type I blocks before returning to a vertex of the type II block is at most two, by Lemma 5.4.9. This again reduces to simple cases and only produces D_4.

For a pair of type II blocks, we reduce to the case the type II blocks are attached together, any case they are seperated by a type I block violates Lemma 5.4.9. If we attach along all three outlets we produce two easy cases. If we attach along a pair of outlets, we generate either a



(c) Chains of I blocks (d) Octahedron of II blocks

Q_2 subquiver or a 4-cycle. Considering the 4-cycle with two outlets (on non-adjacent corners) to meet the conditions of Lemma 5.4.9 a vertex adjacent to one outlet must be adjacent to the other. Further, if the resulting quiver features an arrow between the two outlets a mutation at one of the block nodes gives a Q_2 subquiver. Thus given a vertex v adjacent to each outlet, if this defines a path between the outlets mutating at this node and a black node in the four cycle produces a Q_2 subquiver. Otherwise mutating at both outlets produces a Q_2 subquiver.

Attaching at a single outlet, the four arrows incident to this vertex are now fixed, so any new vertex must touch each of the other 4 outlets, by Lemma 5.4.9. Thus there is at most one other vertex. However by the same Lemma there must be edges between the 4 remaining outlets and so, if we are only permitted to attach type I blocks there are no cases where this can be achieved.

Attaching more than two type II blocks together we can eliminate the case where two are connected to form a 4-cycle as above, and so each type II block meets every other in at most one outlet. Applying Lemma 5.4.9 repeatedly, the only possible quiver is can be represented as an octahedron (with some orientation),

Now, if any triangle is not a cycle we can mutate to form a Q_2 subquiver. However, possibly after a mutation, taking the 'top' of the octahedron we see a type V block subquiver, by the same reasoning as for the type V block case (although here the type V block is not part of a block decomposition) these cases can be eliminated. □

5.5. Classifying Mutation Classes of Polygons

In a different direction, one may consider the following, geometrically motivated, problem:

Given a permitted set of residual singularities, $\mathcal{B}$, classify all mutation classes of polygons with those residual singularities

In other words, find the polygons corresponding to $\mathbb{Q}$-Gorenstein toric degenerations of a log del Pezzo surface with residual basket $\mathcal{B}$. An algorithm to solve this problem was given in the joint work [**70**]. First one finds a finite collection of *minimal* polygons with the given collection of singularities, observing that every mutation class contains such a polygon, and then determine which of these lie in the same mutation equivalence class.

The cluster algebras associated to Fano polygons have been useful both to prove that polygons are *not* mutation equivalent and as an easy way of producing a candidate mutations between polygons, which are more subtle. These results also feed into the wider research programme: from here we can attempt to produce new examples of locally rigid del Pezzo surfaces, attempt to understand the moduli space of affine structures that appear and determine which surfaces admit deformations as complete intersections or other classical constructions.

DEFINITION 5.5.1. A Fano polygon is *minimal* if for every $P' := \mathrm{mut}_{w,F}(P)$ we have that $|\partial P \cap N| \leq |\partial Q \cap N|$.

The first problem considered in [**70**] is to identify minimal polygons with $\mathcal{B} = \varnothing$.

THEOREM 5.5.2. *There are 35 minimal polygons with $\mathcal{B} = \varnothing$, of which 16 are the well-known* reflexive polygons. *These define 10 mutation equivalence classes of Fano polygons.*

The singularity content n distinguishes every mutation equivalence class, except for the classes of the reflexive polygons P_1, P_2 corresponding to $\mathbb{F}_1$ and $\mathbb{P}^1 \times \mathbb{P}^1$ respectively.

EXAMPLE 5.5.3. The quivers for polygons P_1 and P_2 are:

$$\begin{tikzcd} \bullet \arrow[r, "2"] & \bullet \arrow[d, "2"] \\ \bullet \arrow[u, "2"] & \bullet \arrow[l, "2"] \end{tikzcd} \qquad\qquad \begin{tikzcd} \bullet \arrow[r, "2"] & \bullet \arrow[d, "1"] \\ \bullet \arrow[u, "3"] & \bullet \arrow[l, "2"] \end{tikzcd}$$

However if every number of arrows between two vertices is divisible by $k \in \mathbb{Z}$, this property persists under mutation, so these quivers are not mutation equivalent.

REMARK 5.5.4. A similar argument is used in the classification, up to mutation, of Fano polygons with singularity content $(n, k \times \frac{1}{3}(1,1))$. See [**70**, Example 3.19].

REMARK 5.5.5. From the perspective of toric degenerations of surfaces this is compelling. The content is that every toric degeneration of a (smooth) del Pezzo surface is related by a sequence of mutations. Thus as well as providing a classification of smoothable toric del Pezzo surfaces it also entails a strong statement on the boundary of the moduli stack of del Pezzo surfaces.

The finiteness result for Fano polygons with $\mathcal{B} = \varnothing$, extends to the case with $\mathcal{B} \neq \varnothing$ and the maximal local index of the cones in the residual basket, $m_{\mathcal{B}}$, is bounded. Indeed this is the central result of [**70**].

THEOREM 5.5.6. *Given a bound for the local index of the cones in the residual basket $\mathcal{B}$, there are only finitely many minimal Fano polygons, up to the action of* $\mathrm{GL}(2, \mathbb{Z})$*, satisfying this bound.*

The proof of this result is constructive, and so, having prescribed a collection of singularities one can apply the algorithm to find the minimal polygons. Applying this the methods used in the proof of Theorem 6.3 of [**70**] to Fano polygons with basket $\mathcal{B} = \{k \times \frac{1}{3}(1,1)\}$ we obtain the following result (Theorem 1.3 and Theorem 7.1 in [**70**]).

THEOREM 5.5.7. *Up to* $\mathrm{GL}(2,\mathbb{Z})$ *equivalence there are 64 minimal polygons with basket* $\mathcal{B} = \{k \times \frac{1}{3}(1,1)\}$. *These identify 26 mutation equivalence classes of Fano polygons with these singularities.*

The lists of the minimal polygons and representatives in the mutation equivalence classes can be found in [**70**].

REMARK 5.5.8. The vast majority of the polygons appear with the minimal local index $m_P = m_{\mathcal{B}}$, and it is known that there are only finitely many Fano polygons with bounded maximal local index. By comparison, in the case $\mathcal{B} = \varnothing$ the polygons with maximal local index 1 are precisely the reflexive polygons, which are 16 of the 35 minimal polygons with empty residual basket.

5.6. Relation to the construction of Gross–Hacking–Keel

In the final section of this chapter we discuss the geometry of the cluster variety that we have constructed in greater detail. In particular our general construction, with minor adjustments, matches the construction of a cluster algebra studied in Section 5 of [**54**]. Consequently the main results of [**54**], which concern a birational construction of the cluster variety may be applied to give a rich geometric description of the candidate Landau–Ginzburg model mirror-dual to a Fano variety. To avoid various technical details that appear from the slight differences in context this treatment will be slightly informal.

A maximally mutable Laurent polynomial f determines a surface $\mathcal{X}_f$, defined by gluing two-dimensional tori along the mutations supported by f. As we will now see the surface $\mathcal{X}_f$ is contained, as a fiber, in the $\mathcal{X}$-type cluster variety. We first require another notion from the theory of cluster algebras, that of a cluster algebra with *principal coefficents*. A detailed treatment of this notion is given in [**54**], and we briefly recall it here.

DEFINITION 5.6.1. Given the same fixed data, $N, \{\cdot,\cdot\}$ we define $\tilde{N} = N \oplus M$ with form $\{(n_1,m_1),(n_2,m_2)\} = \{n_1,n_2\} + \langle n_1,m_2\rangle - \langle n_2,m_1\rangle$. Given a basis $\mathcal{E}$ for N this defines a basis $\tilde{\mathcal{E}}$ for $\tilde{N}$, consisting of elements $(e,0),(0,e^\star)$ for $e \in \mathcal{E}$.

There is a natural map $p\colon \mathcal{A} \to \mathcal{X}$ defined[2] on seeds by the form $\{\cdot,\cdot\}$. The cluster variety $\mathcal{X}$ is itself a family, fibering over the torus $T_{K^\star}$ for $K = \ker p^\star$. $T_{K^\star}$ is invariant under

[2]This is just the map ρ from Definition 5.3.1

mutation, thus we have a commutative diagram:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "p"] \arrow[d] & \mathcal{X} \arrow[d] \\ \{1\} \arrow[r] & T_{K^\star} \end{tikzcd} \tag{5.6.1}$$

The surface $\mathcal{X}_f$ is the fiber of $\mathcal{X}$ over $1 \in T_{K^\star}$. The advantage of using principal coefficents is that the diagram 5.6.1 now extends to the commutative diagram (2.14) of [**54**]:

$$\begin{tikzcd} \mathcal{A}_{prin} \arrow[r, "\bar{p}"] \arrow[d, "\pi"] & \mathcal{X} \arrow[d] \\ T_M \arrow[r] & T_{K^\star} \end{tikzcd}$$

Restricting to the fiber over the identity of T_M recovers the diagram 5.6.1. As we saw above, a maximally mutable Laurent polynomial f supported on a Fano polygon P defines a section $p^\star f \in \Gamma(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$.

LEMMA 5.6.2. *If $\mathcal{B} \neq \varnothing$, then a maximally mutable Laurent polynomial f supported on P may be extended to a section $f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. If $\mathcal{B} = \varnothing$, then f extends over a codimension one subvariety of $\mathcal{X}$.*

PROOF. Informally, this involves forming a maximally mutable Laurent polynomial with additional parameters t_i which admit the same collection of mutations after changing the factor polynomials from $(1 + z^m)$ to $(1 + t_i z^m)$. Indeed, the t_i are the (frozen) variables $z^{e_i} \in k[N] \hookrightarrow k[\tilde{M}]$; fixing the t_i and consequently a point $t \in T_M$ defines an affine subtorus of $T_{\tilde{N}}$ which fibers over a translate of $T_{\overline{N}} \hookrightarrow T_M$ by t. That is, reparametrising the factors $F = (1 + t_i z^{v_k})$ defines a collection of birational maps on the affine subtorus

$$t \cdot T_{\overline{N}} \hookrightarrow T_M \cong \mathcal{X}_s$$

We now attempt to extend the section of $\mathcal{A}$ defined by a maximally mutable Laurent polynomial f over $\mathcal{A}_{prin}$ by solving the equations on the coefficents of the Laurent polynomial f which guarantee f admits all possible mutations, now using the factor polynomials $F = (1 + t_i z^{v_k})$. If $\mathcal{B} \neq \varnothing$ then there is can be achieved for any values of the varaibles t_i and a family of maximally mutable Laurent polynomials may be found over T_M[3]. If $\mathcal{B} = \varnothing$, there is a consistency condition $\prod_i t_i = 1$. □

EXAMPLE 5.6.3. Any polynomial

$$\lambda_1 x + \lambda_2 y + \frac{\lambda_3}{xy}.$$

[3]More precisely over $T_{K^\star}$: since for $t \in T_{\overline{N}}$ the translation does not move the subtorus and simply reparameterizes the same Laurent polynomial

is maximally mutable with factor polynomials $F_i = (1 + \frac{\lambda_i}{\lambda_{i+1}} z^i)$, where z^i is a monomial in x, y and 0, 1 and 2 are cyclically ordered. Setting $t_i = \lambda_i/\lambda_{i+1}$, we see that $t_1 \cdot t_2 \cdot t_3 = 1$.

Our goal now is to apply the results of Section 5 of [**54**] to replace the cluster variety $\mathcal{X}$ with a variety defined using an explicit birational construction.

CONSTRUCTION 5.6.4. From the collection of vectors w_i normal to the edges e_i in P we may form a toric surface $\bar{Y}$ by regarding the w_i as generators of rays of a fan in $\overline{N}$. Let $\bar{D}_i$ be the divisor in $\bar{Y}$ corresponding to w_i.

With minor adjustments (replacing $\overline{N}$ with the lattice generated by the vectors w_i and encoding the previous indices in numbers ν_i) Construction 5.3 of [**54**] will generate the seed data we have defined for the Fano polygon P. In particular we define a fan Σ by regarding the above rays as cones in a fan in M via an embedding defined using the wedge product on $\overline{N}$, and hence a toric variety $\mathrm{TV}(\Sigma)$.

Writing D_i for the toric divisor corresponding to w_i, the authors show that blowing up the subvarieties

$$Z_i = \Big(D_i \cap \overline{V}((1 + z^{e_i})^{\mathrm{ind}\, d_i v_i})\Big)$$

where $d_i := \nu_i / \gcd(\nu_i : 1 \leq i \leq n)$, one obtains a family $\mathcal{Y} \to T_{K^\star}$, let $\mathcal{D}$ denote the strict transform of the toric boundary.

In [**54**] the authors identify $\mathcal{Y}\backslash\mathcal{D}$ with the cluster variety $\mathcal{X}$, away from a codimension 2 subset. This provides a much simpler construction of the mirror Landau–Ginzburg model, supporting the same holomorphic functions.

REMARK 5.6.5. In Section 6 of [**54**] the authors study a pair of examples arising from Fano polygons, showing that $\Gamma(\mathcal{A}_{prin}, \mathcal{O}_{\mathcal{A}_{prin}})$ and $\Gamma(\mathcal{A}_t, \mathcal{O}_{\mathcal{A}_t})$ for very general t are non-Noetherian. From our perspective, both of these examples come from (dual) Fano polygons which are in fact reflexive – the polygons for the cubic surface and $\mathbb{P}^2$ respectively. Moreover, given any Fano polyon P with $\mathcal{B} = \varnothing$ one will be able to produce a cluster variety satisfying the same non-Noetherian property: the underlying variety of the candidate mirror-dual to a (smooth) del Pezzo surface.

CHAPTER 6

Complete Intersections and the Givental/Hori–Vafa Model

Over the previous three chapters we have developed an intricate theory to study log del Pezzo surfaces using Mirror Symmetry. Indeed, we encapsulated the notion of *mutation* from [**4**] into a theory of affine manifolds, before showing that the deformations one obtained by applying the Gross–Siebert algorithm to certain families exactly recovers the $\mathbb{Q}$-Gorenstein deformations of these toric surfaces. We then extended this theory in two directions, demonstrating that:

- In an example, the mutation equivalent Laurent polynomials can be identified with a tropical disc counts.
- The mutation classes can be related directly to the geometry of *cluster varieties* and the algebraic structure that controls them leads to classification results on the mutation classes of polyons.

Very little of this material applies directly outside of the surface case. In this chapter we begin to explore how the theory we have been developing extends to higher dimensions, restricting for the most part to complete intersections in toric varieties. The motivation for the techniques developed in this section is a natural question that arose in the programme of Coates–Corti–Galkin–Golyshev–Kasprzyk.

Given a Laurent polynomial f, conjecturally mirror-dual to a Fano variety X, how can one construct the variety X from f?

Since X should be provably mirror-dual to f it is logical to look first among the toric complete intersections. Indeed, this is a rare setting in which mirror-duality, on the level of the respective local systems, may actually be proved, see [**47**].

To begin to answer this question we first explore a technique, the *Przyjalkowski Method*, for obtaining a torus chart on the mirror-dual (Givental/Hori–Vafa) Landau–Ginzburg model for a Fano complete intersection X in a toric variety Y. We also consider how, via degenerations of the ambient space, this technique may be extended to complete intersections in homogenous spaces. In the final part of the dissertation we present a simple form of an inverse algorithm, *Laurent Inversion*, which in many cases allows one to reconstruct the Fano variety X directly from f.

There is also a fascinating connection between Laurent Inversion, the toric degenerations of the Gross–Siebert programme and the affine manifold techniques we have investigated for surfaces. This connection entails a dictionary between the singular locus of X_P, the toric variety corresponding to a Fano polytope P, and the weight matrix of the ambient variety

Y in which these singularites are expected to smooth. This dictionary goes via the singular locus of an affine manifold. Since, via the Przyjalkowski method, the weight matrix of Y is related to the coefficents of the mirror-dual Laurent polynomial f, this provides the first general connection between the maximal mutability of f and the existence of a smoothing of X_P for $P = \mathrm{Newt}(f)$.

6.1. The Przyjalkowski Method

In this section we explain, given data specifying a complete intersection in a toric variety, how to find a Laurent polynomial f that is mirror-dual to X. This is a slight generalization of a technique that we learned from V. Przyjalkowski [**96**].

Let Y be a toric Fano manifold and fix a collection $L_1, \ldots, L_c$ of nef line bundles on Y such that $-K_Y - \Lambda$ is nef, where $\Lambda = c_1(L_1) + \cdots + c_1(L_c)$. We define $X \subset Y$ to be a smooth complete intersection defined by a regular section of $\oplus_i L_i$ and assume that X is Fano (this is not automatic if any of the bundles is strictly nef). The data $(X; Y; L_1, \ldots, L_c)$ is the input for the construction of a mirror-dual Laurent polynomial f.

Recall that for any toric variety we have the following, dual pair of exact sequences,

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^N \xrightarrow{\ \rho\ } \mathbb{Z}^d \longrightarrow 0$$

$$0 \longleftarrow \mathbb{L}^\star \xleftarrow{\ D\ } (\mathbb{Z}^N)^\star \xleftarrow{\ \rho^\star\ } (\mathbb{Z}^d)^\star \longleftarrow 0$$

where the map ρ is defined by the N rays of a fan Σ for Y. Next define the elements $D_i \in \mathbb{L}^\star$, $1 \le i \le N$ to be the images of standard basis elements of $(\mathbb{Z}^N)^\star$. Also recall that $\mathbb{L}^\star \cong \mathrm{Pic}(Y)$, so that each line bundle L_m defines a class in $\mathbb{L}^\star$. Choose disjoint subsets E, S_1,...,S_c of $\{1, 2, \ldots, N\}$ such that:

- $\{D_j : j \in E\}$ is a basis for $\mathbb{L}^\star$;
- each D_i is a non-negative linear combination of $\{D_j : j \in E\}$;
- $\sum_{k \in S_m} D_k = L_m$ for each $m \in \{1, 2, \ldots, c\}$;

and distinguished elements $s_m \in S_m$, $1 \le m \le c$. Set $S_m^\circ = S_m \setminus \{s_m\}$.

Writing the map D in terms of the standard basis for $(\mathbb{Z}^N)^\star$ and the basis $\{D_j : j \in E\}$ for $\mathbb{L}^\star$ defines an $(N-d) \times N$ matrix (m_{ji}) of non-negative integers. Let $(x_1, \ldots, x_N)$ denote the standard co-ordinates on $(\mathbb{C}^\times)^N$, let $r = N - d$, and define $q_1, \ldots, q_r$ and $F_1, \ldots, F_c$ by:

$$q_j = \prod_{i=1}^{N} x_i^{m_{ji}} \qquad\qquad F_m = \sum_{k \in S_m} x_k$$

Givental [**47**] and Hori–Vafa [**63**] have shown that:

$$G_X = \int_\Gamma e^{tW} \frac{\bigwedge_{i=1}^N \frac{dx_i}{x_i}}{\bigwedge_{m=1}^c dF_m \wedge \bigwedge_{j=1}^r \frac{dq_j}{q_j}} \tag{6.1.1}$$

where $W = x_1 + \cdots + x_N$ and Γ is a certain cycle in the submanifold of $(\mathbb{C}^\times)^N$ defined by

$$q_1 = \cdots = q_r = 1 \qquad\qquad F_1 = \cdots = F_c = 1$$

Introducing new variables y_i for $i \in \bigcup_{m=1}^c S_m^\circ$, setting

$$x_i = \begin{cases} \frac{1}{1+\sum_{k\in S_m^\circ} y_k} & \text{if } i = s_m \\ \frac{y_i}{1+\sum_{k\in S_m^\circ} y_k} & \text{if } i \in S_m^\circ \end{cases}$$

and using the relations $q_1 = \cdots = q_r = 1$ to eliminate the variables x_j, $j \in E$, allows us to write $W - c$ as a Laurent polynomial f in the variables:

$$\{y_i : i \in \textstyle\bigcup_{m=1}^c S_m^\circ\} \qquad \text{and} \qquad \{x_i : i \notin E \text{ and } i \notin \textstyle\bigcup_{m=1}^c S_m^\circ\}$$

The mirror theorem (6.1.1) then implies that $\widehat{G}_X = \pi_f$, or in other words that f is mirror-dual to X.

The Laurent polynomial f produced by Przyjalkowski's method depends on our choices of E, S_1,…,S_c, and s_1,…,s_c, but up to mutation this is not the case:

THEOREM 6.1.1. *Let Y be a toric Fano manifold and let $L_1, \ldots, L_c$ be nef line bundles on Y such that $-K_Y - \Lambda$ is ample, where $\Lambda = c_1(L_1) + \cdots + c_1(L_c)$. Let $X \subset Y$ be a smooth complete intersection defined by a regular section of $\oplus_i L_i$. Let f and g be Laurent polynomial mirrors to X obtained by applying Przyjalkowski's method to $(X; Y; L_1, \ldots, L_c)$ as above, but with possibly-different choices for the subsets E, S_1,…,S_c and the elements s_1,…,s_c. Then we have equality $\pi_f = \pi_g$.*

PROOF. This follows directly from Theorem 2.24 of [**30**] in which the authors construct a birational map preserving the volume form. Thus the period integral of the pull-back Laurent polynomial is equal to the period integral of f, that is, $\pi_f = \pi_g$. ☐

EXAMPLE 6.1.2. Let Y be the projectivization of the vector bundle $\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 2}$ over $\mathbb{P}^2$. Choose a basis for the two-dimensional lattice $\mathbb{L}^\star$ such that the matrix (m_{ji}) of the map D is:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Consider the line bundle $L_1 \to Y$ defined by the element $(2,1) \in \mathbb{L}^\star$, and the Fano hypersurface $X \subset Y$ defined by a regular section of L_1. Applying Przyjalkowski's method to the triple $(X; Y; L_1)$ with $E = \{3,4\}$, $S_1 = \{1,2,5\}$, and $s_1 = 1$ yields the Laurent polynomial

$$f = \frac{(1+y_2+y_5)^2}{y_2 x_6 x_7} + \frac{1+y_2+y_5}{y_5 x_6 x_7} + x_6 + x_7$$

mirror-dual to X. Applying the method with $E = \{3,4\}$, $S_1 = \{1,6\}$, and $s_1 = 1$ yields:

$$g = x_2 + \frac{(1+y_6)^2}{x_2 y_6 x_7} + \frac{1+y_6}{x_5 y_6 x_7} + x_5 + x_7$$

We have that $f \overset{\varphi}{\dashrightarrow} g$ where the mutation $\varphi\colon (\mathbb{C}^\times)^4 \to (\mathbb{C}^\times)^4$ is given by:

$$(y_2, y_5, x_6, x_7) = \left(\frac{x_2}{x_5 y_6}, \frac{1}{y_6}, x_7, x_2 + x_5\right)$$

REMARK 6.1.3. Observe that, for a complete intersection of dimension n and codimension c, Przyjalkowski's method requires partitioning $n+c$ variables into c disjoint subsets. If $\frac{n+c}{c} < 2$ then at least one of the subsets must have size one and so the corresponding variable, x_j say, is eliminated from the Laurent polynomial via the equation $x_j = 1$. One could therefore have obtained the resulting Laurent polynomial from a complete intersection with smaller codimension: new Laurent polynomials are found only when $\frac{n+c}{c} \geq 2$, that is, when the codimension is at most the dimension. In particular, all possible mirrors to 4-dimensional Fano toric complete intersections given by the Przyjalkowski method occur for complete intersections in toric manifolds of dimension at most 8.

6.2. Finding Four-Dimensional Fano Toric Complete Intersections

In joint work [**20**] we find all four-dimensional Fano manifolds X of the form described in Section 6.1 such that the codimension c is at most 4 and $-K_Y - \Lambda$ is ample. In this case the Adjunction Formula gives that

$$-K_X = \big(-K_Y - \Lambda\big)\big|_X$$

so X is automatically Fano.

To place this work in context, four-dimensional Fano manifolds of higher Fano index have been classified [**38**–**41**, **66**–**68**, **73**, **75**, **102**, **104**]—there are 35 in total—but the most interesting case, where the Fano variety has index 1, is completely open. In [**20**] we find at least 738 examples, 717 of which have Fano index 1 and 527 of which are new.

We recall the method followed in [**20**]. Toric Fano manifolds are classified up to dimension 8 by Batyrev, Watanabe–Watanabe, Sato, Kreuzer–Nill, and Øbro [**10**,**11**,**80**,**92**,**98**,**108**]. For each toric Fano manifold Y of dimension $d = 4 + c$, in [**20**] we:

(1) compute the nef cone of Y;
(2) find all $\Lambda \in H^2(Y;\mathbb{Z})$ such that both Λ and $-K_X - \Lambda$ are nef;
(3) decompose Λ as the sum of c nef line bundles $L_1, \ldots, L_c$ in all possible ways.

Each such decomposition determines a 4-dimensional Fano manifold $X \subset Y$, defined as the zero locus of a regular section of the vector bundle $\oplus_i L_i$. To compute the nef cone in step (i), we recall the exact sequence

$$0 \longleftarrow \mathbb{L}^\star \xleftarrow{\;D\;} (\mathbb{Z}^N)^\star \xleftarrow{\;\rho^\star\;} (\mathbb{Z}^d)^\star \longleftarrow 0$$

from §6.1. There are canonical identifications $\mathbb{L}^\star \cong H^2(Y;\mathbb{Z}) \cong \mathrm{Pic}(Y)$, and the nef cone of Y is the intersection of cones

$$\mathrm{NC}(Y) = \bigcap_{\sigma \in \Sigma} \langle D_i : i \notin \sigma \rangle$$

where D_i is the image under D of the ith standard basis vector in $(\mathbb{Z}^N)^\star$. The classes Λ in step (ii) are the lattice points in the polyhedron $P = \mathrm{NC}(Y) \cap \big({-K_Y} - \mathrm{NC}(Y)\big)$. Since $\mathrm{NC}(Y)$ is a strictly convex cone, P is compact and the number of lattice points is finite. We implement step (iii) by first expressing Λ as a sum of Hilbert basis elements in $\mathrm{NC}(Y)$ in all possible ways:

$$\Lambda = b_1 + \cdots + b_r \qquad b_i \text{ an element of the Hilbert basis for } \mathrm{NC}(Y) \tag{6.2.1}$$

where some of the b_i may be repeated; this is a knapsack-style problem. We then, for each decomposition (6.2.1), partition the b_i into c subsets $S_1, \ldots, S_c$ in all possible ways, and define the line bundle L_i to be the sum of the classes in S_i.

We found 117173 distinct triples $(X; Y; L_1, \ldots, L_c)$, with a total of 17934 distinct ambient toric varieties Y. Note that the representation of a given Fano manifold X as a toric complete intersection is far from unique: for example, if X is a complete intersection in Y given by a section of $L_1 \oplus \cdots \oplus L_c$ then it is also a complete intersection in $Y \times \mathbb{P}^1$ given by a section of $\pi_1^\star L_1 \oplus \cdots \oplus \pi_1^\star L_c \oplus \pi_2^\star \mathcal{O}_{\mathbb{P}^1}(1)$. Thus we have found far fewer than 117173 distinct four-dimensional Fano manifolds. We show, in Section 3 and the electronic supplementary material in [**20**], by calculating quantum periods of the Fano manifolds X, that we find at least 738 non-isomorphic Fano manifolds. Since the quantum period is a very strong invariant—indeed no examples of distinct Fano manifolds $X \not\cong X'$ with the same quantum period $G_X = G_{X'}$ are known—we believe that we found precisely 738 non-isomorphic Fano manifolds. Eliminating the quantum periods found in [**25**], we see that at least 527 of our examples are new.

Remark 6.2.1. There exist Fano manifolds which do not occur as complete intersections in toric Fano manifolds. But in low dimensions, most Fano manifolds arise this way: 8 of the 10 del Pezzo surfaces, and at least 78 of the 105 smooth 3-dimensional Fano manifolds, are complete intersections in toric Fano manifolds [**22**].

Remark 6.2.2. It may be the case that any d-dimensional Fano manifold which occurs as a toric complete intersection in fact occurs as a toric complete intersection in codimension d; we know of no counterexamples. But even if this holds in dimension 4, our search will probably not find all 4-dimensional Fano manifolds which occur as toric complete intersections. This is because, if one of the line bundles L_i involved is strictly nef, then the Kähler cone for X can be strictly bigger than the Kähler cone for Y. In other words, it is possible for $-K_X$ to be ample on X even if $-K_Y - \Lambda$ is not ample on Y. For an explicit example of this in dimension 3, see [**22**, §55].

6.3. Examples

6.3.1. The Cubic 4-fold. Let X be the cubic 4-fold. This arises in our classification from the complete intersection data $(X;Y;L)$ with $Y = \mathbb{P}^5$ and $L = \mathcal{O}_{\mathbb{P}^5}(3)$. The Przyjalkowski method yields [**65**, §2.1] a Laurent polynomial:

$$f = \frac{(1+x+y)^3}{xyzw} + z + w$$

mirror-dual to X, and elementary calculation gives:

$$\pi_f(t) = \sum_{d=0}^{\infty} \frac{(3d)!(3d)!}{(d!)^6} t^{3d}$$

Indeed, $\widehat{G}_X = \pi_f$, and the corresponding regularized quantum differential operator is:

$$L_X = D^4 - 729t^3(D+1)^2(D+2)^2$$

The local log-monodromies for the local system of solutions $L_X g \equiv 0$ are:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{at } t = 0$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{at } t = \tfrac{1}{9}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{at the roots of } 81t^2 + 9t + 1 = 0$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{at } t = \infty$$

and the operator L_X is extremal.

6.3.2. A (3,3) Complete Intersection in $\mathbb{P}^6$. Let X be a complete intersection in $Y = \mathbb{P}^6$ of type $(3,3)$. This arises in our classification from the complete intersection data $(X;Y;L_1,L_2)$ with $L_1 = L_2 = \mathcal{O}_{\mathbb{P}^6}(3)$. The Przyjalkowski method yields a Laurent polynomial:

$$f = \frac{(1+x+y)^3(1+z+w)^3}{xyzw} - 36$$

mirror-dual to X, and [**22**, Corollary D.5] gives:

$$\widehat{G}_X = \pi_f(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(3l)!(3l)!(k+l)!}{k!(l!)^7}(-36)^k t^{k+l}$$

The corresponding regularized quantum differential operator L_X is:

$$\begin{aligned}
&(36t+1)^4(693t-1)D^4\\
&+18t(36t+1)^3(13860t+61)D^3\\
&+9t(36t+1)^2(3492720t^2+57672t+77)D^2\\
&+144t(36t+1)(11226600t^3+377622t^2+2754t+1)D\\
&+15552t^2(1796256t^3+98496t^2+1605t+7)
\end{aligned}$$

The local log-monodromies for the local system of solutions $L_X g \equiv 0$ are:

$$\begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} \quad \text{at } t=0$$

$$\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix} \quad \text{at } t=\tfrac{1}{693}$$

$$\begin{pmatrix} \frac{2}{3}&1&0&0\\ 0&\frac{2}{3}&0&0\\ 0&0&\frac{1}{3}&1\\ 0&0&0&\frac{1}{3} \end{pmatrix} \quad \text{at } t=-\tfrac{1}{36}$$

and so the operator L_X is extremal.

6.4. Complete Intersections in Grassmannians of Planes

Whilst the method defined in Section 6.1 is inherently toric, one may attempt to apply it to complete intersections in more general ambient spaces via a toric degeneration of the ambient space. In this section we apply this to obtain a Laurent polynomial mirror for a complete intersection in a Grassmannian of planes $\mathrm{Gr}(2, 2+k)$. This is achieved by first forming the Givental/Hori–Vafa mirror model for the flat degeneration of $\mathrm{Gr}(2, 2+k)$ to the toric variety $P(2, 2+k)$ described in [**13**]. A different algorithm for achieving this was given in the preprint [**94**], of which this section forms an appendix. We provide examples in this section that show that while the Laurent polynomials obtained by the respective methods are not the same, they are related by algebraic mutations.

Recalling the procedure for forming the Givental/Hori–Vafa mirror model, the equations $q_1 = \cdots = q_r = 1$ are imposing *binomial* equations determined by the toric variety Y and independent of the line bundles L_i. Applying these equations, one obtains the Laurent polynomial

$$W = \sum_{1 \le i \le R} z^{\rho_i}$$

formed by passing to the subtorus of $(\mathbb{C}^\star)^R$ defined by

$$\prod_{j=1}^{R} x_j^{m_{i,j}} = 1$$

i.e. on the subtorus $T_{(\mathbb{Z}^d)^\star} \hookrightarrow T_{(\mathbb{Z}^N)^\star}$. Observe that the polynomial W encodes the ray generators of $P(2, 2+k)$ in the exponents ρ_i. For a certain choice of coordinates $a_{i,j}$, explained in [**31**, **94**], the Laurent polynomial has the form

$$f_{\mathrm{Gr}(2,2+k)} = a_{1,1} + \sum_{j=2}^{k} \frac{a_{1,j}}{a_{1,j-1}} + \sum_{j=1}^{k} \frac{a_{2,j}}{a_{1,j}} + \sum_{j=2}^{k} \frac{a_{2,j}}{a_{2,j-1}} + \frac{1}{a_{2,k}}.$$

REMARK 6.4.1. Though we effectively apply the Przyjalkowski method to complete intersections in $P(2, 2+k)$, this is not $\mathbb{Q}$-factorial. However, we recall that replacing $\mathrm{Gr}(2, 2+k)$ with $P(2, 2+k)$ is justified in Section 4 of [**94**], following the results of [**13**].

In order to encode all possible combinations of complete intersection in $\mathrm{Gr}(2, 2+k)$ we consider the $k+2$ equations

$$f_1 = a_{1,1}, \qquad f_j = \frac{a_{1,j}}{a_{1,j-1}} + \frac{a_{2,j}}{a_{2,j-1}}, j \in [2, k], \qquad f_{k+1} = \frac{1}{a_{2,k}}.$$

Given a complete intersection of hypersurfaces of degree d_i, $i \in [1, l]$ we fix a partition

$$[1, k+1] = E_0 \sqcup E_1 \sqcup \ldots \sqcup E_l$$

with $|E_j| = d_j$ for $j > 0$, and we form the Givental/Hori–Vafa mirror to the complete intersection in $\mathrm{Gr}(2, 2+k)$ by restricting to the subvariety

$$\left\{ F_j = \sum_{r \in E_j} f_r = 1 \mid j = 1, \ldots, l \right\}.$$

In order to apply the Przyjalkowski method for finding a birational torus chart we apply a change of variables

$$x_{1,1} = a_{1,1}, \qquad x_{1,j} = \frac{a_{1,j}}{a_{1,j-1}}, j \in [2, k],$$

$$x_{2,j} = \frac{a_{2,j}}{a_{2,j-1}}, j \in [2, k], \qquad x_{2,k+1} = \frac{1}{a_{2,k}}.$$

With these changes of variables, we see that $2k$ of the $3k$ terms of $f_{\mathrm{Gr}(2,2+k)}$ are now simply variables $x_{i,j}$ and the remaining k terms are monomials

$$M_i = \frac{a_{2,i}}{a_{1,i}} = \big(\prod_{j \le i} x_{1,j} \prod_{j \ge i} x_{2,j} \big)^{-1}$$

The polynomials f_j have become the following:

$$f_1 = x_{1,1}, \qquad f_j = x_{1,j} + x_{2,j}, j \in [2,k], \qquad f_{k+1} = x_{2,k+1}.$$

We may now apply the general procedure given above, namely we define the E_m and E used there according to the terms appearing in the polynomials F_m and $\sum_{j=1}^{k} M_j$. Observe that all the requirements for finding a birational torus chart in the first section are met, so we may apply the change of coordinates above. This provides a completely explicit way of forming (several) Laurent polynomial mirrors for a given complete intersection in $\mathrm{Gr}(2,k+2)$. We now consider some examples of this method.

EXAMPLE 6.4.2. Consider a cubic intersected with the quadric $\mathrm{Gr}(2,4)$. The weight matrix D for $P(2,4)$ is the following:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

The Picard group is the sublattice generated by the column $(1,1)^t$. The Laurent polynomial mirror may be obtained either by applying the condition $D = 1$ above, or by changing $f_{\mathrm{Gr}(2,4)}$ to the $x_{i,j}$ variables, in either case there is a Laurent polynomial presentation given by the following:

$$f_{\mathrm{Gr}(2,4)} = x_{1,1} + x_{1,2} + \frac{1}{x_{1,1}x_{1,2}x_{2,3}} + \frac{1}{x_{1,1}x_{2,2}x_{2,3}} + x_{2,2} + x_{2,3}.$$

Where the first k and final k columns of D correspond to basis elements in $\mathcal{N}$ and to variables $x_{1,j}$, $j \in [1,l]$ and $x_{2,j}$, $j \in [2,l+1]$ respectively. The column $(3,3)^t$ may be obtained by adding the first, second, fifth, and sixth columns, giving the relation

$$x_{1,1} + x_{1,2} + x_{2,2} + x_{2,3} = 1.$$

Let $E_1 = \{1,2,5,6\}$, $E = \{3,4\}$, and $s_1 = 1$. Denoting the new variables $y_{i,j}$, consistent with the variables $x_{i,j}$ we have the following:

$$x_{1,1} = \frac{1}{1 + y_{1,2} + y_{2,2} + y_{2,3}}, \qquad x_{1,2} = \frac{y_{1,2}}{1 + y_{1,2} + y_{2,2} + y_{2,3}},$$
$$x_{2,2} = \frac{y_{2,2}}{1 + y_{1,2} + y_{2,2} + y_{2,3}}, \qquad x_{2,3} = \frac{y_{2,3}}{1 + y_{1,2} + y_{2,2} + y_{2,3}}.$$

The superpotential then becomes

$$\psi^* f_{\mathrm{Gr}(2,4)} = \frac{(1 + y_{1,2} + y_{2,2} + y_{2,3})^3}{y_{1,2}y_{2,3}} + \frac{(1 + y_{1,2} + y_{2,2} + y_{2,3})^3}{y_{2,2}y_{2,3}} =$$
$$= \frac{(y_{2,2} + y_{1,2})}{y_{1,2}y_{2,2}y_{2,3}}(1 + y_{1,2} + y_{2,2} + y_{2,3})^3.$$

We shall now show this is equivalent to the result of the algorithm given in [**94**] applied to this variety up to mutations ([**21**], [**72**]). Consider the birational map ϕ_1, defined by the following:

$$\phi_1^* y_{1,2} = y_{1,2}, \qquad \phi_1^* y_{2,2} = y_{2,2}, \qquad \phi_1^* y_{2,3} = (y_{1,2} + y_{2,2})y_{2,3}.$$

Computing $\phi_1^*\psi^* f_{\mathrm{Gr}(2,4)}$ we obtain

$$g_{\mathrm{Gr}(2,4)} = \phi_1^*\psi^* f_{\mathrm{Gr}(2,4)} = \frac{1}{y_{1,2}y_{2,2}y_{2,3}}(1 + y_{1,2} + y_{2,2} + (y_{1,2} + y_{2,2})y_{2,3})^3.$$

Now, setting

$$y_{1,2} = a_{2,1}, \qquad y_{2,2} = \frac{a_{2,1}^2}{a_{1,1}}, \qquad y_{2,3} = a_{1,2}$$

we have that

$$\begin{aligned} g_{\mathrm{Gr}(2,4)} &= \frac{a_{1,1}}{a_{2,1}^3 a_{1,2}}\left(1 + \left(a_{2,1} + \frac{a_{2,1}^2}{a_{1,1}}\right)(1 + a_{1,2})\right)^3 = \\ &= \frac{a_{1,1}}{a_{1,2}}\left(\frac{1}{a_{2,1}} + \left(1 + \frac{a_{2,1}}{a_{1,1}}\right)(1 + a_{1,2})\right)^3. \end{aligned}$$

Apply another birational change of coordinates ϕ_2, sending

$$a_{i,j} \mapsto a_{i,j}\left(1 + \frac{a_{2,1}}{a_{1,1}}\right)^{-1}$$

for each (i, j). We obtain

$$\begin{aligned} h_{\mathrm{Gr}(2,4)} = \phi_2^* g_{\mathrm{Gr}(2,4)} &= \frac{a_{1,1}}{a_{1,2}}\left(\left(1 + \frac{1}{a_{2,1}}\right)\left(1 + \frac{a_{2,1}}{a_{1,1}}\right) + a_{1,2}\right)^3 = \\ &= \frac{a_{1,1}}{a_{1,2}}\left(a_{1,2} + \frac{1 + a_{1,1}a_{2,1} + a_{1,1} + a_{2,1}}{a_{1,1}a_{2,1}}\right)^3. \end{aligned}$$

Which is the resulting polynomial in [**94**].

Example 6.4.3. A fourfold of index 2 given by 4 hyperplane sections of $\mathrm{Gr}(2, 6)$. The matrix D for $P(2, 6)$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The four bundles L_i are all equal to $(1, 1, 1, 1) \in \mathbb{L}^\vee$. We fix the nef-partition by taking the collections of basis elements D_i corresponding to columns $1, 12, \{2, 9\}, \{3, 10\}$ of the matrix D. Applying the notation employed in Example 6.4.2 we compute

$$\psi^* f_{\mathrm{Gr}(2,6)} = x_{1,4} + x_{2,4} + \frac{(1 + y_{2,2})(1 + y_{2,3})}{y_{2,2}y_{2,3}x_{2,4}}(1 + y_{2,2} + y_{2,2}y_{2,3}) + \frac{(1 + y_{2,2})(1 + y_{2,3})}{x_{1,4}}.$$

Noting that columns 4, 11 are in neither E nor any of the sets E_m , so the variables $x_{1,4}$ and $x_{2,4}$ persist. As in the previous example, this polynomial also agrees with the result of the algorithm described in [**94**] up to mutations, which in particular preserve the period sequence of the Laurent polynomial.

6.5. The Przyjalkowski Method Revisited

The goal for the remainder of this chapter is to describe an inverse to the Przyjalkowski method. In other words, we wish to produce a construction of a Fano variety X, expressed as a complete intersection, from a Laurent polynomial f, such that X provably corresponds to f under Mirror Symmetry.

We first treat the problem of finding torus charts on Landau–Ginzburg models somewhat more generally than in Section 6.1, since it will maximise the scope of our inversion technique. The problem of finding torus charts on Landau–Ginzburg models has been considered by many authors [**20**, **30**, **47**, **63**, **93**, **96**], and the construction below (in §6.9) generalises and unifies all these perspectives below. Consider first the ambient toric variety or toric stack Y. We consider the case where:

(6.5.1)

1. Y is a proper toric Deligne–Mumford stack;
2. the coarse moduli space of Y is projective;
3. the generic isotropy group of Y is trivial, that is, Y is a toric *orbifold*; and
4. at least one torus-fixed point in Y is smooth.

Conditions (i)–(iii) here are essential; condition (iv) is less important and will be removed in §6.9. In the original work by Borisov–Chen–Smith [**16**], toric Deligne–Mumford stacks are defined in terms of stacky fans. In our context, since the generic isotropy is trivial, giving a stacky fan that defines Y amounts to giving a triple $(N; \Sigma; \rho_1, \ldots, \rho_R)$ where N is a lattice, Σ is a rational simplicial fan in $N \otimes \mathbb{Q}$, and $\rho_1, \ldots, \rho_R$ are elements of N that generate the rays of Σ. It will be more convenient for our purposes, however, to represent Y as a GIT quotient $\left[\mathbb{C}^R /\!\!/_\omega (\mathbb{C}^\times)^r\right]$. Any such Y can be realised this way, as we now explain.

Definition 6.5.1. We say that $(K; \mathbb{L}; D_1, \ldots, D_R; \omega)$ are *GIT data* if $K \cong (\mathbb{C}^\times)^r$ is a connected torus of rank r; $\mathbb{L} = \mathrm{Hom}(\mathbb{C}^\times, K)$ is the lattice of subgroups of K; $D_1, \ldots, D_R \in \mathbb{L}^*$ are characters of K that span a strictly convex full-dimensional cone in $\mathbb{L}^* \otimes \mathbb{Q}$, and $\omega \in \mathbb{L}^* \otimes \mathbb{Q}$ lies in this cone.

GIT data $(K; \mathbb{L}; D_1, \ldots, D_R; \omega)$ determine a quotient stack $\left[V_\omega / K\right]$ with $V_\omega \subset \mathbb{C}^R$, as follows. The characters $D_1, \ldots, D_R$ define an action of K on $\mathbb{C}^R$. Write $[R] := \{1, 2, \ldots, R\}$. Say that a subset $I \subset [R]$ *covers* ω if and only if $\omega = \sum_{i \in I} a_i D_i$ for some strictly positive rational numbers a_i, set $\mathcal{A}_\omega = \{I \subset [R] \mid I \text{ covers } \omega\}$, and set

$$V_\omega = \bigcup_{I \in \mathcal{A}_\omega} (\mathbb{C}^\times)^I \times \mathbb{C}^{\bar{I}} \qquad \text{where} \qquad (\mathbb{C}^\times)^I \times \mathbb{C}^{\bar{I}} = \left\{(x_1, \ldots, x_R) \in \mathbb{C}^R \mid x_i \neq 0 \text{ if } i \in I\right\}.$$

The subset $V_\omega \subset \mathbb{C}^R$ is K-invariant, and $\left[V_\omega / K\right]$ is the GIT quotient (stack) given by the action of K on $\mathbb{C}^R$ and the stability condition ω. The convexity hypothesis in Definition 6.5.1 ensures that $\left[V_\omega / K\right]$ is proper.

Remark 6.5.2. The quotient $\left[V_\omega / K\right]$ here depends on ω only via the minimal cone σ of the secondary fan such that $\omega \in \sigma$. The *secondary fan* for GIT data $(K; \mathbb{L}; D_1, \ldots, D_R; \omega)$

is the fan defined by the wall-and-chamber decomposition of the cone in $\mathbb{L}^* \otimes \mathbb{Q}$ spanned by $D_1, \ldots, D_R$, where the walls are given by the cones spanned by $\{D_i \mid i \in I\}$ such that $I \subset [R]$ and $|I| = r - 1$.

Definition 6.5.3. *Orbifold* GIT data are those such that the quotient $\left[V_\omega/K\right]$ is a toric orbifold.

The quotient $\left[V_\omega/K\right]$ is a toric Deligne–Mumford stack if and only if ω lies in the strict interior of a maximal cone in the secondary fan. A toric orbifold Y satisfying the conditions (6.5.1) above arises as the quotient $\left[V_\omega/K\right]$ for GIT data $(K; \mathbb{L}; D_1, \ldots, D_R; \omega)$ as follows. Suppose that Y is defined, as discussed above, by the stacky fan data $(N; \Sigma; \rho_1, \ldots, \rho_R)$. There is an exact sequence

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^R \xrightarrow{\rho} N \longrightarrow 0 \tag{6.5.2}$$

where ρ maps the ith element of the standard basis for $\mathbb{Z}^R$ to ρ_i; this defines $\mathbb{L}$ and $K = \mathbb{L} \otimes \mathbb{C}^\times$. Dualizing gives

$$0 \longleftarrow \mathbb{L}^* \xleftarrow{D} (\mathbb{Z}^*)^R \longleftarrow M \longleftarrow 0 \tag{6.5.3}$$

where $M := \operatorname{Hom}(N, \mathbb{Z})$, and we set $D_i \in \mathbb{L}^*$ to be the image under D of the ith standard basis element for $(\mathbb{Z}^*)^R$. The stability condition ω is taken to lie in the strict interior of

$$C = \bigcap_{\text{maximal cones } \sigma \text{ of } \Sigma} C_\sigma$$

where C_σ is the cone in $\mathbb{L}^* \otimes \mathbb{Q}$ spanned by $\{D_i \mid i \in \sigma\}$; projectivity of the coarse moduli space of Y implies that C is a maximal cone of the secondary fan, and in particular that C has non-empty interior.

We can reverse this construction, defining a stacky fan $(N; \Sigma; \rho_1, \ldots, \rho_n)$ from GIT data $(K; \mathbb{L}; D_1, \ldots, D_R; \omega)$ such that $D_1, \ldots, D_R$ span $\mathbb{L}^*$, as follows. The lattice $\mathbb{L}$ and elements $D_1, \ldots, D_R \in \mathbb{L}^*$ define the exact sequence (6.5.3), and dualising gives (6.5.2). This defines the lattice N and $\rho_1, \ldots, \rho_R$. The fan Σ consists of the cones spanned by $\{\rho_i \mid i \in I\}$ where $I \subset [R]$ satisfies $[R] \setminus I \in \mathcal{A}_\omega$.

Remark 6.5.4. Once K, $\mathbb{L}$, and $D_1, \ldots, D_R$ have been fixed, choosing ω such that the GIT data $(K; \mathbb{L}; D_1, \ldots, D_R; \omega)$ define a toric Deligne–Mumford stack amounts to choosing a maximal cone in the secondary fan.

Under our hypotheses there is a canonical isomorphism between $\mathbb{L}^*$ and the Picard lattice $\operatorname{Pic}(Y)$. We will denote the line bundle on Y corresponding to a character $\chi \in \mathbb{L}^*$ also by χ.

Definition 6.5.5. Let $\Theta = (K; \mathbb{L}; D_1, \ldots, D_R; \omega)$ be orbifold GIT data, and let Y denote the corresponding toric orbifold. A *convex partition with basis* for Θ is a partition $B, S_1, \ldots, S_k, U$ of $[R]$ such that:

(1) $\{D_b \mid b \in B\}$ is a basis for $\mathbb{L}^*$;

(2) ω is a non-negative linear combination of $\{D_b \mid b \in B\}$;
(3) each S_i is non-empty;
(4) for each $i \in [k]$, the line bundle $L_i := \sum_{j \in S_i} D_j$ on Y is convex[1]; and
(5) for each $i \in [k]$, L_i is a non-negative linear combination of $\{D_b \mid b \in B\}$.

We allow $k = 0$, and we allow $U = \varnothing$.

REMARK 6.5.6. Since ω here is taken to lie in the strict interior of a maximal cone in the secondary fan, it is in fact a positive linear combination of $\{D_b \mid b \in B\}$. This positivity guarantees that the maximal cone spanned by $\{\rho_i \mid i \in [R] \setminus B\}$ defines a smooth torus-fixed point in Y.

REMARK 6.5.7. It would be more natural to replace the condition that L_i be convex here with the weaker condition that L_i be nef. But, since we currently lack a Mirror Theorem that applies to toric complete intersections beyond the convex case, we will require convexity. If the ambient space Y is a manifold, rather than an orbifold, then convexity and nef-ness coincide.

Given:

(6.5.4)
(1) orbifold GIT data $\Theta = (K; \mathbb{L}; D_1, \ldots, D_R; \omega)$;
(2) a convex partition with basis $B, S_1, \ldots, S_k, U$ for Θ; and
(3) a choice of elements $s_i \in S_i$ for each $i \in [k]$;

we define a Laurent polynomial f, as follows. Without loss of generality we may assume that $B = [r]$. Writing $D_1, \ldots, D_R$ in terms of the basis $\{D_b \mid b \in B\}$ for $\mathbb{L}^*$ yields an $r \times R$ matrix $\mathcal{M} = (m_{i,j})$ of the form

$$\mathcal{M} = \left(\begin{array}{c|ccc} & m_{1,r+1} & \cdots & m_{1,R} \\ I_r & \vdots & & \vdots \\ & m_{r,r+1} & \cdots & m_{r,R} \end{array} \right) \tag{6.5.5}$$

where I_r is an $r \times r$ identity matrix. Consider the function

$$W = x_1 + x_2 + \cdots + x_R - k$$

subject to the constraints

$$\prod_{j=1}^{R} x_j^{m_{i,j}} = 1 \qquad i \in [r] \tag{6.5.6}$$

and

$$\sum_{j \in S_i} x_j = 1 \qquad i \in [k] \tag{6.5.7}$$

[1]A line bundle L on a Deligne–Mumford stack Y is convex if and only if L is nef and is the pullback of a line bundle on the coarse moduli space $|Y|$ of Y along the structure map $Y \to |Y|$. See [**26**].

For each $i \in [k]$, introduce new variables y_j, where $j \in S_i \setminus \{s_i\}$, and set $y_{s_i} = 1$. Solve the constraints (6.5.7) by setting:

$$x_j = \frac{y_j}{\sum_{l \in S_i} y_l} \qquad\qquad j \in S_i$$

and express the variables x_b, $b \in B$, in terms of the y_js and remaining x_is using (6.5.6). The function W thus becomes a Laurent polynomial f in variables

$$\begin{aligned} &x_i, && \text{where } i \in U, \\ \text{and} \quad &y_j, && \text{where } j \in (S_1 \cup \cdots \cup S_k) \setminus \{s_1, \ldots, s_k\}. \end{aligned} \tag{6.5.8}$$

We call the x_i here the *uneliminated variables*.

Given data as in (6.5.4), let f be the Laurent polynomial just defined. Let Y denote the toric orbifold determined by Θ, let $L_1, \ldots, L_k$ denote the line bundles on Y from Definition 6.5.5, and let $X \subset Y$ be a complete intersection defined by a regular section of the vector bundle $\oplus_i L_i$. If X is Fano, then Mirror Theorems due to Givental [**47**], Hori–Vafa [**63**], and Coates–Corti–Iritani–Tseng [**23, 24**] imply that f corresponds to X under Mirror Symmetry (c.f. [**20**, §5]). We say that *f is a Laurent polynomial mirror* for X.

REMARK 6.5.8. If f is a Laurent polynomial mirror for X then the Picard–Fuchs local system for $f\colon (\mathbb{C}^\times)^n \to \mathbb{C}$ coincides, after translation of the base if necessary, with the Fourier–Laplace transform of the quantum local system for X; see [**21, 22**]. Thus we regard f and $g := f - c$, where c is a constant, as Laurent polynomial mirrors for the same manifold Y, since the Picard–Fuchs local systems for f and g differ only by a translation of the base (by c).

REMARK 6.5.9. If f and g are Laurent polynomials that differ by an invertible monomial change of variables then the Picard–Fuchs local systems for f and g coincide. Thus f is a Laurent polynomial mirror for X if and only if g is a Laurent polynomial mirror for X.

EXAMPLE 6.5.10. Let X be a smooth cubic surface. The ambient toric variety $Y = \mathbb{P}^3$ is a GIT quotient $\mathbb{C}^4 /\!\!/ \mathbb{C}^\times$ where $\mathbb{C}^\times$ acts on $\mathbb{C}^4$ with weights $(1, 1, 1, 1)$. Thus Y is given by GIT data $(K; \mathbb{L}; D_1, \ldots, D_4; \omega)$ with $K = \mathbb{C}^\times$, $\mathbb{L} = \mathbb{Z}$, $D_1 = D_2 = D_3 = D_4 = 1$, and $\omega = 1$. We consider the convex partition with basis B, S_1, $\varnothing$, where $B = \{1\}$ and $S_1 = \{2, 3, 4\}$, and take $s_1 = 4$. This yields

$$\mathcal{M} = \left(\begin{array}{c:ccc} 1 & 1 & 1 & 1 \end{array} \right)$$

and

$$W = x_1 + x_2 + x_3 + x_4 - 1$$

subject to

$$x_1 x_2 x_3 x_4 = 1 \qquad\qquad \text{and} \qquad\qquad x_2 + x_3 + x_4 = 1.$$

We set:

$$x_1 = \frac{1}{x_2 x_3 x_4} \qquad x_2 = \frac{x}{1 + x + y} \qquad x_3 = \frac{y}{1 + x + y} \qquad x_4 = \frac{1}{1 + x + y}$$

where, in the notation above, $x = y_2$ and $y = y_3$. Thus

$$f = \frac{(1+x+y)^3}{xy}$$

is a Laurent polynomial mirror to Y.

EXAMPLE 6.5.11. Let Y be the projective bundle $\mathbb{P}\big(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1)\big) \to \mathbb{P}^3$. This arises from the GIT data $(K; \mathbb{L}; D_1, \ldots, D_7; \omega)$ where $K = (\mathbb{C}^\times)^2$, $\mathbb{L} = \mathbb{Z}^2$,

$$D_1 = D_4 = D_6 = D_7 = (1,0) \qquad D_2 = D_3 = (0,1) \qquad D_5 = (-1,1)$$

and $\omega = (1,1)$. We consider the convex partition with basis B, S_1, S_2, U where $B = \{1,2\}$, $S_1 = \{3,4\}$, $S_2 = \{5,6\}$, $U = \{7\}$. This yields:

$$\mathcal{M} = \left(\begin{array}{cc|ccccc} 1 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right)$$

Choosing $s_1 = 3$ and $s_2 = 5$, we find that

$$f = \frac{(1+x)}{xyz} + (1+x)(1+y) + z$$

Here, in the notation above, $x = y_4$, $y = y_6$, and $z = x_7$.

6.6. Laurent Inversion

To invert the process described in §6.5, that is, to pass from a Laurent polynomial f to orbifold GIT data Θ, a convex partition with basis $B, S_1, \ldots, S_k, U$ for Θ, and elements $s_i \in S_i$, $i \in [k]$, would amount to expressing f in the form

$$f = f_1 + \cdots + f_r + \sum_{u \in U} x_u \tag{6.6.1}$$

where

$$f_a = \prod_{i=1}^{k} \prod_{j \in S_i} \left(\frac{\sum_{l \in S_i} y_l}{y_j} \right)^{m_{a,j}} \times \prod_{u \in U} x_u^{-m_{a,u}}.$$

In favourable circumstances, we can obtain from a decomposition (6.6.1) a smooth toric orbifold Y and convex line bundles $L_1, \ldots, L_k$ on Y such that the complete intersection $X \subset Y$ defined by a regular section of the vector bundle $\oplus_i L_i$ is Fano and corresponds to f under Mirror Symmetry. In general there are many such decompositions of f. Not every decomposition gives rise to a smooth toric orbifold Y, for example because not every decomposition gives rise to valid GIT data[2]. Even when the decomposition (6.6.1) gives orbifold GIT data $(K; \mathbb{L}; D_1, \ldots, D_R; \omega)$, and hence an ambient toric orbifold Y, it is not always possible to choose the stability condition ω such that Y has a smooth torus-fixed point, or such that the line bundles $L_1, \ldots, L_k$ are simultaneously convex, or such that X is Fano. In practice, however, this technique is surprisingly effective.

[2] The characters $D_1, \ldots, D_R$ of $K = (\mathbb{C}^\times)^r$ defined, via equation (6.5.5), by a decomposition (6.6.1) may not span a strictly convex full-dimensional cone.

DEFINITION 6.6.1. We refer to a decomposition (6.6.1) as a *scaffolding* for f, and to the Laurent polynomials f_a involved as *struts*.

ALGORITHM 6.6.2. We remark – and this is a key methodological point – that scaffoldings of f can be enumerated algorithmically. Let $A = \mathbb{Z}^s$ denote the lattice containing $\operatorname{Newt} f$. A partition $S'_1, \ldots, S'_k, U'$ of the standard basis for A, where we allow $k = 0$ and allow $U' = \varnothing$, defines a collection of standard simplices

$$\Delta(i) = \operatorname{Conv}\left(\{0\} \cup S'_i\right) \qquad\qquad i \in [k].$$

We call a polytope Δ a *strut* if it is a translation of a Minkowski sum of dilations of these standard simplices. A scaffolding (6.6.1) for f determines a collection of struts Δ_a and lattice points p_u, each contained in $P := \operatorname{Newt} f$, where $\Delta_a = \operatorname{Newt} f_a$ and p_u is the standard basis element corresponding to the uneliminated variable x_u. The struts Δ_a may overlap, and may overlap with the p_u. We refer to a collection $\{\Delta_a \mid a \in [r]\}$, $\{p_u \mid u \in U'\}$ of:

(1) struts $\{\Delta_a \mid a \in [r]\}$ with respect to some partition $S'_1, \ldots, S'_k, U'$; and
(2) standard basis elements $\{p_u \mid u \in U'\}$;

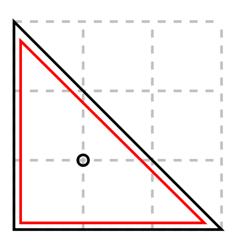
all of which are contained in a polytope P, as a *scaffolding* for P. One can check whether a scaffolding for $\operatorname{Newt} f$ arises from a scaffolding (6.6.1) for f by checking if the coefficients from the associated struts f_a and uneliminated variables x_u sum to give the coefficients of f. Since all coefficients of the struts f_a are positive, only finitely many scaffoldings for $\operatorname{Newt} f$ need to be checked. We are free to relax our notion of scaffolding, demanding that the left- and right-hand sides of (6.6.1) agree only up to a constant monomial – see Remark 6.5.8. This extra flexibility is often useful.

REMARK 6.6.3. It is more meaningful, in view of Remark 6.5.9, to allow scaffoldings of $\operatorname{Newt} f$ that are based on a partition $S'_1, \ldots, S'_k, U'$ of an arbitrary basis for A, rather than the standard basis. For fixed f, only finitely many such generalised scaffoldings need be checked.

EXAMPLE 6.6.4 (dP_3). Consider now the Laurent polynomial

$$f = \frac{(1 + x + y)^3}{xy}$$

from Example 6.5.10. A scaffolding for $\operatorname{Newt} f$ is given by a single standard 2-simplex, dilated by a factor of three:



Indeed f is equal to a single strut, with no uneliminated variables. From this we read off $r = 1$, $k = 1$, $B = \{1\}$, $S_1 = \{2, 3, 4\}$, $U = \varnothing$, and the exponents of the strut give:

$$\mathcal{M} = \left(\begin{array}{c:ccc} 1 & 1 & 1 & 1 \end{array} \right)$$

This gives GIT data $\Theta = (K; \mathbb{L}; D_1, \ldots, D_4; \omega)$ with $K = \mathbb{C}^\times$, $\mathbb{L} = \mathbb{Z}$, $D_1 = D_2 = D_3 = D_4 = 1$, and $\omega = 1$; note that the secondary fan here has a unique maximal cone. The corresponding toric variety is $Y = \mathbb{P}^3$. The line bundle $L_1 = \sum_{j \in S_1} D_j = \mathcal{O}(3)$ is nef. Thus $B, S_1, \varnothing$ is a convex partition with basis for Θ. That is, by scaffolding f we obtain the cubic hypersurface as in Example 6.5.10.

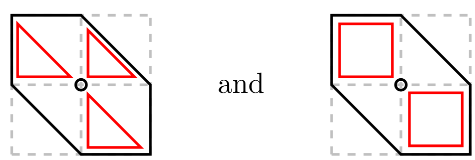
Example 6.6.5 (dP_6). The projective plane blown up in three points, dP_6, is toric, but it has two famous models as a complete intersection:

(1) as a hypersurface of type $(1, 1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$;
(2) as the intersection of two bilinear equations in $\mathbb{P}^2 \times \mathbb{P}^2$.

Let us see how these arise from Laurent inversion. The Laurent polynomial mirror to dP_6 that we shall use is:

$$f = x + y + \frac{1}{x} + \frac{1}{y} + \frac{x}{y} + \frac{y}{x}.$$

We may scaffold $\mathrm{Newt}(f)$ in two different ways: using three triangles, and using a pair of squares:



These choices correspond, respectively, to the scaffoldings

$$f = (1+x+y) + \frac{(1+x+y)}{x} + \frac{(1+x+y)}{y} - 3 \qquad \text{and} \qquad f = \frac{(1+x)(1+y)}{x} + \frac{(1+x)(1+y)}{y} - 2.$$

As discussed, we ignore the constant terms.

From the first scaffolding we read off $r = 3$, $k = 1$, $B = \{1, 2, 3\}$, $S_1 = \{4, 5, 6\}$, $U = \varnothing$, and the exponents of the struts give:

$$\mathcal{M} = \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right)$$

This gives GIT data $\Theta = (K; \mathbb{L}; D_1, \ldots, D_6; \omega)$ with $K = (\mathbb{C}^\times)^3$, $\mathbb{L} = \mathbb{Z}^3$, $D_1 = D_4 = (1, 0, 0)$, $D_2 = D_5 = (0, 1, 0)$, $D_3 = D_6 = (0, 0, 1)$, and $\omega = (1, 1, 1)$; the secondary fan here again has a unique maximal cone. The corresponding toric variety is $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. The line bundle $L_1 = \sum_{j \in S_1} D_j$ is $\mathcal{O}(1, 1, 1)$, so we see that f is a Laurent polynomial mirror to a hypersurface of type $(1, 1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

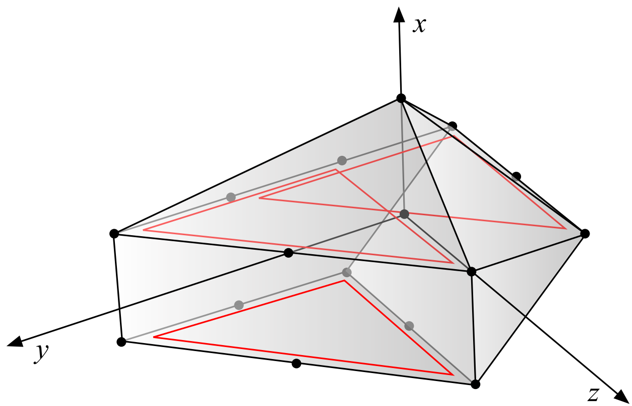


FIGURE 6.6.1. A scaffolding for Newt f in Example 6.6.6.

From the second scaffolding we read off $r = 2$, $k = 2$, $B = \{1, 2\}$, $S_1 = \{3, 4\}$, $S_2 = \{5, 6\}$, $U = \varnothing$, and the exponents of the struts give:

$$\mathcal{M} = \left(\begin{array}{cc:cccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

This gives GIT data $\Theta = (K; \mathbb{L}; D_1, \ldots, D_6; \omega)$ with $K = (\mathbb{C}^\times)^2$, $\mathbb{L} = \mathbb{Z}^2$, $D_1 = D_4 = D_5 = (1, 0)$, $D_2 = D_3 = D_6 = (0, 1)$, and $\omega = (1, 1)$; once again the secondary fan has a unique maximal cone. The corresponding toric variety Y is $\mathbb{P}^2 \times \mathbb{P}^2$. The line bundles $L_1 = D_3 + D_4$ and $L_2 = D_5 + D_6$ are both equal to $\mathcal{O}(1, 1)$, so we see that f is a Laurent polynomial mirror to the complete intersection of two hypersurfaces defined by bilinear equations in $\mathbb{P}^2 \times \mathbb{P}^2$.

EXAMPLE 6.6.6. Consider the rigid maximally-mutable Laurent polynomial

$$f = x + \frac{y^2}{z} + 2y + \frac{3y}{z} + z + \frac{3}{z} + \frac{z}{y} + \frac{2}{y} + \frac{1}{yz} + \frac{y^2}{xz} + \frac{2y}{x} + \frac{2y}{xz} + \frac{z}{x} + \frac{2}{x} + \frac{1}{xz}.$$

The Newton polytope of f can be scaffolded as in Figure 6.6.1, and there is a corresponding scaffolding of f:

$$f = x + \frac{(1 + y + z)^2}{xz} + \frac{(1 + y + z)^2}{z} + \frac{(1 + y + z)^2}{yz}$$

From this we read off $r = 3$, $k = 1$, $B = \{1, 2, 3\}$, $U = \{4\}$, $S_1 = \{5, 6, 7\}$, and the exponents of the struts give:

$$\mathcal{M} = \left(\begin{array}{ccc:cccc} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right)$$

This gives GIT data $\Theta = (K; \mathbb{L}; D_1, \ldots, D_6; \omega)$ with $K = (\mathbb{C}^\times)^3$, $\mathbb{L} = \mathbb{Z}^3$, $D_1 = D_4 = (1, 0, 0)$, $D_2 = (0, 1, 0)$, $D_3 = D_6 = (0, 0, 1)$, $D_4 = (1, 1, 0)$, and $D_7 = (1, 1, 1)$. The secondary fan is as shown in Figure 6.6.2. Choosing $\omega = (3, 2, 1)$ yields a weak Fano toric manifold Y such that the line bundle $L_1 = \sum_{j \in S_1} D_j$ is convex. Let X denote the hypersurface in Y defined by a regular section of L_1. The class $-K_Y - L_1$ is nef but not ample on Y, but it becomes ample

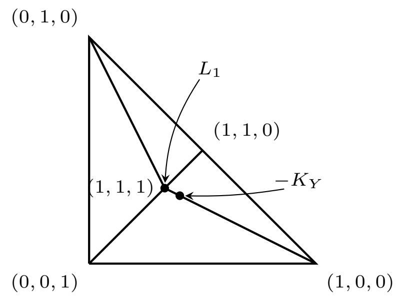


FIGURE 6.6.2. The secondary fan for Example 6.6.6, sliced by the plane $x + y + z = 1$.

on restriction to X; thus X is Fano (cf. [**22**, §57]). We see that f is a Laurent polynomial mirror to X. This example shows that our Laurent inversion technique applies in cases where the ambient space Y is not Fano. In fact Y need not even be weak Fano.

6.7. A New Four-Dimensional Fano Manifold

Consider

$$f = \frac{(1+x)^2}{xyw} + \frac{x}{z} + y + z + w$$

This is a rigid maximally-mutable Laurent polynomial in four variables. It is presented in scaffolded form, and we read off $r = 2$, $k = 1$, $B = \{1, 2\}$, $S_1 = \{3, 4\}$, $U = \{5, 6, 7\}$. The exponents of the struts give:

$$\mathcal{M} = \left(\begin{array}{cc:ccccc} 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 & 1 & 0 \end{array} \right)$$

This yields GIT data $\Theta = (K; \mathbb{L}; D_1, \ldots, D_6; \omega)$ with $K = (\mathbb{C}^\times)^2$, $\mathbb{L} = \mathbb{Z}^2$, $D_1 = D_5 = D_7 = (1, 0)$, $D_2 = D_6 = (0, 1)$, $D_3 = (1, 1)$, and $D_4 = (1, -1)$. We choose the stability condition $\omega = (5, 2)$, thus obtaining a Fano toric orbifold Y such that the line bundle $L_1 = D_3 + D_4$ on Y is convex. Let X denote the four-dimensional Fano manifold defined inside Y by a regular section of L_1.

The Fano manifold X is new. To see this, we can compute the regularised quantum period $\widehat{G}_X$ of X. Since f is a Laurent polynomial mirror to X, the regularised quantum period $\widehat{G}_X$ coincides with the classical period of f:

$$\pi_f(t) = \sum_{d=0}^{\infty} c_d t^d \qquad \text{where} \qquad c_d = \operatorname{coeff}_1\left(f^d\right).$$

This is explained in detail in [**21**, **22**]. In the case at hand,

$$\widehat{G}_X = \pi_f(t) = 1 + 12t^3 + 120t^5 + 540t^6 + 20160t^8 + 33600t^9 + \cdots$$

and we see that $\widehat{G}_X$ is not contained in the list of regularised quantum periods of known four-dimensional Fano manifolds [**20**, **25**]. Thus X is new. We did not find X in our systematic search for four-dimensional Fano toric complete intersections [**20**], because there we considered only ambient spaces that are Fano toric *manifolds* whereas the ambient space Y here has non-trivial orbifold structure. This is striking because the degree $K_X^4 = 433$ of X is not that low – compare with Figure 5 in [**20**]. In dimensions 2 and 3 only Fano manifolds of low degree fail to occur as complete intersections in toric manifolds. The space Y can be obtained as the unique non-trivial flip of the projective bundle $\mathbb{P}\big(\mathcal{O}(-1) \oplus \mathcal{O}^{\oplus 3} \oplus \mathcal{O}(1)\big)$ over $\mathbb{P}^1$. As was pointed out to us by Casagrande, the other extremal contraction of Y, which is small, exhibits X as the blow-up of $\mathbb{P}^4$ in a plane conic. This suggests that restricting to smooth ambient spaces when searching for Fano toric complete intersections may omit many Fano manifolds with simple classical constructions.

6.8. From Laurent Inversion to Toric Degenerations

Suppose now that we have a scaffolding (6.6.1) for the Laurent polynomial f, and that this gives rise to:

(1) orbifold GIT data $\Theta = (K; \mathbb{L}; D_1, \ldots, D_R; \omega)$;
(2) a convex partition with basis $B, S_1, \ldots, S_k, U$ for Θ; and
(3) a choice of elements $s_i \in S_i$ for each $i \in [k]$.

We now explain how to pass from this data to a toric degeneration of the complete intersection $X \subset Y$ defined by a regular section of the vector bundle $\oplus_i L_i$. This degeneration was discovered independently by Doran–Harder [**30**]; see §6.9 for an alternative view on their construction. In favourable circumstances, as we will explain, the central fiber of this toric degeneration is the Fano toric variety X_f defined by the spanning fan of Newt f. The existence of such a degeneration is predicted by Mirror Symmetry.

By assumption we have, as in §6.5, an $r \times R$ matrix $\mathcal{M} = (m_{i,j})$ of the form:

$$\mathcal{M} = \left(\begin{array}{c|ccc} & m_{1,r+1} & \cdots & m_{1,R} \\ I_r & \vdots & & \vdots \\ & m_{r,r+1} & \cdots & m_{r,R} \end{array} \right)$$

such that $l_{b,i} := \sum_{j \in S_i} m_{b,j}$ is non-negative for all $b \in [r]$ and $i \in [k]$. The exact sequence (6.5.2) becomes

$$0 \longrightarrow \mathbb{Z}^r \xrightarrow{\mathcal{M}^T} \mathbb{Z}^R \xrightarrow{\rho} N \longrightarrow 0$$

and, writing $\rho_i \in N$ for the image under ρ of the ith standard basis vector in $\mathbb{Z}^R$, we find that $\{\rho_i \mid r < i \leq R\}$ is a distinguished basis for N and that

$$\rho_i = -\sum_{j=r+1}^{R} m_{i,j} \rho_j \qquad \text{for all } i \in [r].$$

Let $M = \mathrm{Hom}(N, \mathbb{Z})$ and define $u_j \in M$, $j \in [k]$, by

$$u_j(\rho_i) = \begin{cases} 0 & \text{if } r < i \leq R \text{ and } i \notin S_j; \\ 1 & \text{if } r < i \leq R \text{ and } i \in S_j. \end{cases}$$

Let $N' := N \cap H_{u_1} \cap \ldots \cap H_{u_k}$ be the sublattice of N given by restricting to the intersection of the hyperplanes $H_{u_i} := \{v \in N \mid u_i(v) = 0\}$. Let Σ' denote the fan defined by intersecting Σ with $N'_{\mathbb{Q}}$, and let X' be the toric variety defined by Σ'.

Proposition 6.8.1. *There is a flat degeneration $\mathbb{X} \to \mathbb{A}^1$ with general fiber $\mathbb{X}_t$ isomorphic to X and special fiber $\mathbb{X}_0$ isomorphic to X'.*

Proof. Recall that X is cut out of the toric variety Y by regular sections s_i of the line bundles L_i, $i \in [k]$. By deforming s_i to the binomial section s'_i of L_i given by

$$s_i = \prod_{a \in [r]} x_a^{l_{a,i}} - \prod_{j \in S_i} x_j$$

we can construct a flat degeneration with general fiber X and special fiber a toric variety X'. Since $u_i(\rho_a) = -l_{a,i}$, we see that the fan Σ' defining X' is the intersection of the fan Σ defining Y with $H_{u_1} \cap \cdots \cap H_{u_k}$, as claimed. □

Our choice of elements $s_i \in S_i$, $i \in [k]$, gives rise to a distinguished basis for N', consisting of

$$\begin{aligned} & \rho_i, && \text{where } i \in U, \\ \text{and} \quad & \rho_i - \rho_{s_j}, && \text{where } i \in S_j \setminus \{s_j\} \text{ for some } j \in [k]. \end{aligned} \tag{6.8.1}$$

Comparing (6.5.8) with (6.8.1), we see that this choice of basis also specifies an isomorphism between N' and the lattice A that contains $\mathrm{Newt}\, f$. Thus it makes sense to ask whether the fan Σ' coincides with the spanning fan of $\mathrm{Newt}\, f$; in this case we will say that Σ' *is the spanning fan.* If Σ' is the spanning fan then the above construction gives a degeneration from X to the (singular) toric variety X_f, as predicted by Mirror Symmetry.

Remark 6.8.2. In any given example it is easy to check whether Σ' is the spanning fan. This is often the case – it holds, for example, for all of the examples in this paper – but it is certainly not the case in general. It would be interesting to find a geometrically meaningful condition that guarantees that Σ' is the spanning fan. This problem is challenging because, at this level of generality, we do not have much control over what the fan Σ looks like. It is easy to see that each ray of Σ' passes through some vertex of a strut in the scaffolding of $\mathrm{Newt}\, f$, and that the cone $C'_a \subset N'_{\mathbb{Q}}$ over the strut $\Delta_a = \mathrm{Newt}\, f_a$ is given by the intersection with $N'_{\mathbb{Q}}$ of the cone $C_a \subset N$ spanned by $\{\rho_a\} \cup \{\rho_i \mid i \in S_1 \cup \cdots \cup S_k\}$. But typically only some of the C_a lie in Σ (indeed typically the cones C'_a overlap with each other) and in general it is hard to say more. Doran–Harder [**30**] give sufficient conditions for Σ' to be a refinement of the spanning fan, but for applications to Mirror Symmetry this is not enough.

6.9. Torus Charts on Landau–Ginzburg Models

Suppose, as before, that we have:

(6.9.1)

(1) orbifold GIT data $\Theta = (K; \mathbb{L}; D_1, \ldots, D_R; \omega)$;
(2) a convex partition with basis $B, S_1, \ldots, S_k, U$ for Θ; and
(3) a choice of elements $s_i \in S_i$ for each $i \in [k]$.

Let Y be the corresponding toric orbifold, and $X \subset Y$ the complete intersection defined by a regular section of the vector bundle $\oplus_i L_i$. Givental [**47**] and Hori–Vafa [**63**] have defined a Landau–Ginzburg model that corresponds to X under Mirror Symmetry. In this section we explain how to write down a torus chart on the Givental/Hori–Vafa mirror model on which the superpotential restricts to a Laurent polynomial. This gives an alternative perspective on Doran–Harder's notion of *amenable collection subordinate to a nef partition* [**30**, §§2.2–2.3].

Definition 6.9.1. Suppose that we have fixed orbifold GIT data Θ defining Y, as in (6.9.1-i). The Landau–Ginzburg model mirror to Y is the family of tori equipped with a superpotential:

$$\begin{tikzcd} (\mathbb{C}^\times)^R \arrow[r, "W"] \arrow[d, "D"'] & \mathbb{C} \\ T_{\mathbb{L}^*} \end{tikzcd}$$

where $W = \sum_{j=1}^R x_j$; $x_1, \ldots, x_R$ are the standard co-ordinates on $(\mathbb{C}^\times)^R$; D is the map from (6.5.3); and $T_{\mathbb{L}^*}$ is the torus $\mathbb{L}^* \otimes \mathbb{C}^\times$.

In our context, rather than considering the whole family over $T_{\mathbb{L}^*}$, we restrict to the fiber over 1. Extending the diagram defining the Landau–Ginzburg model to include this fiber we have:

$$\begin{tikzcd} T_M \arrow[r, "\rho^\vee"] & (\mathbb{C}^\times)^R \arrow[r, "W"] \arrow[d, "D"'] & \mathbb{C} \\ & T_{\mathbb{L}^*} \end{tikzcd}$$

where $T_M = M \otimes \mathbb{C}^\times$ and $\rho^\vee$ is the dual to the fan map ρ from (6.5.2).

Definition 6.9.2. Suppose that we have fixed orbifold GIT data and a nef partition with basis, as in (6.9.1). The Landau–Ginzburg model mirror to X is the restriction of the mirror model for Y to a subvariety $X^\vee$, defined by the following commutative diagram:

$$\begin{tikzcd}
 & \mathbb{C} & \\
X^\vee \arrow[ur] \arrow[r, "j"] & T_M \arrow[r, "\rho^\vee"] \arrow[d] & (\mathbb{C}^\times)^R \arrow[ul, "W"'] \arrow[dl, "\Phi"] \arrow[d, "D"] \\
 & \mathbb{C}^l & T_{\mathbb{L}^*}
\end{tikzcd}$$

where $\Phi := \left(\sum_{i\in S_1} x_i, \ldots, \sum_{i\in S_k} x_i\right)$ and j is the inclusion of the fiber over 1. The Landau–Ginzburg model mirror to X is the map

$$\left(\rho^\vee \circ j\right)^* W \colon X^\vee \to \mathbb{C}$$

We now present a general technique for finding torus charts on $X^\vee$ on which the restriction of the superpotential $\left(\rho^\vee \circ j\right)^* W$ is a Laurent polynomial. To do this we will construct a birational map μ such that the pullback $\chi := (\rho^\vee \circ \mu)^*\Phi$ of Φ becomes regular, as in the following diagram.

$$\begin{tikzcd}
 & X^\vee \arrow[r, "j"] & T_M \arrow[d, "(\rho^\vee)^*\Phi"] \\
T_{\mathrm{Ker}(\chi)} \arrow[ur, dashed] \arrow[r, "\theta:=\ker(\chi)"'] & T_M \arrow[ur, dashed, "\mu"] \arrow[r, "\chi"'] & \mathbb{C}^l
\end{tikzcd}$$

Remark 6.9.3. Via the bijection between monomials in the variables x_i, $1 \le i \le R$, with their exponents in $\mathbb{Z}^R$ we identify the monomials $(\rho^\vee)^*(x_i)$ with their exponents $\rho_i \in N$. In this notation:

$$(\rho^\vee)^*\Phi = \Big(\sum_{i\in S_1} x^{\rho_i}, \cdots, \sum_{i\in S_k} x^{\rho_i}\Big)$$

Recall that the vectors ρ_i generate the rays of the fan Σ that defines Y.

We construct our birational map μ from the data in (6.9.1) together with a choice of lattice vectors $w_i \in M$ such that:

(6.9.2)

(1) $\langle w_i, \rho_j\rangle = -1$ for all $j \in S_i$ and all i;
(2) $\langle w_i, \rho_j\rangle = 0$ for all $j \in S_l$ such that $l < i$ and all i;
(3) $\langle w_i, \rho_j\rangle \ge 0$ for all $j \in S_l$ such that $l > i$ and all i.

This is exactly Doran–Harder's notion of an *amenable collection subordinate to a nef partition.*

Definition 6.9.4. A *weight vector* $w \in M$ and a *factor* $F \in \mathbb{C}[w^\perp]$ together determine a birational transformation $\theta \colon T_M \dashrightarrow T_M$ called an *algebraic mutation.* This is given by the automorphism $x^\gamma \mapsto x^\gamma F^{\langle \gamma, w\rangle}$ of the field of fractions $\mathbb{C}(N)$ of $\mathbb{C}[N]$.

We define the birational map μ as the composition of a sequence of algebraic mutations $\mu_1, \ldots, \mu_k$, where the mutation μ_i has weight vector w_i and factor given by

$$F_i := \frac{\big(\mu_1 \circ \cdots \circ \mu_{i-1}\big)^* \Big(\sum_{j \in S_i} x^{\rho_j} \Big)}{x^{\rho_{s_i}}}$$

The conditions (6.9.2) guarantee that F_i is a Laurent polynomial, that $F_i \in \mathbb{C}[w_i^\perp]$, and that $\big(\mu_1 \circ \cdots \circ \mu_i\big)^* W$ is a Laurent polynomial for all $i \in [k]$.

We can always take the weight vectors w_i in (6.9.2) to be equal to the $-u_i$ from §6.8, but many other choices are possible. We get a toric degeneration in this more general context, too (cf. [**30**]):

LEMMA 6.9.5. *The lattice vector $w_i \in M$ defines a binomial section of the line bundle $L_i \in \operatorname{Pic}(Y)$.*

PROOF. The lattice M is the character lattice of the torus T_N, and so w_i defines a rational function on Y. The image $\rho^\vee(w_i) \in (\mathbb{Z}^*)^R$ defines a pair of effective torus invariant divisors by taking the positive entries and minus the negative entries of this vector, written in the standard basis. The only negative entries are those in S_i, which are equal to minus one. Both monomials have the same image under D, and so they are both in the linear system defined by L_i. □

6.10. From Laurent Inversion to the SYZ conjecture

In this section we very briefly sketch, and provide a simple example of, an interpretation of Laurent Inversion in terms of the toric degenerations of the Gross–Siebert programme and affine/tropical geometry. We will return to this in detail elsewhere.

This interpretation is achieved in a number of stages:

(1) First interpret the input data for the Przyjalkowski method (or output data from Laurent Inversion) as a smoothing of the variety $\prod_{j \in S_1} x_j = 0, \cdots, \prod_{j \in S_k} x_j = 0$. Thus, given a complete intersection to which we may apply the Przyjalkowski method, there is an embedded degeneration of the complete intersection to a (compact) 'vertex' variety.
(2) Use the degeneration to this vertex to form a tropical version of the complete intersection. As the equations defining the complete intersection change, the singular locus of its tropicalisation varies along substrata of the intersection complex.
(3) The singularities which appear imply the existence of certain elements in $H^0(X, L_i)$. The existence of these elements impose certain relations on the weight matrix.
(4) In the Przyjalkowski method the weight matrix D encodes the exponents appearing in the mirror-dual Laurent polynomial. Remarkably, we see an equivalence between the condition that the Laurent polynomial f obtained via our construction is maximally mutable and the condition that the toric variety defined by $\operatorname{Newt}(f)$ admits an embedded smoothing.

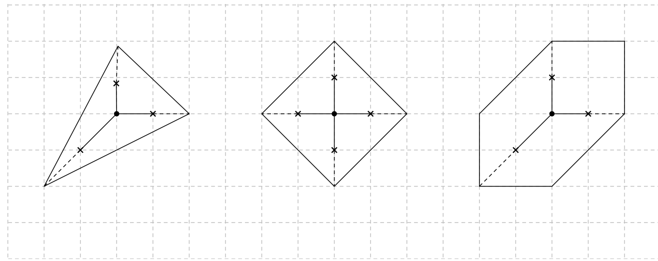


FIGURE 6.10.1. Affine manifolds corresponding to scaffoldings of dP_3, dP_4, and dP_6

EXAMPLE 6.10.1. Recall Example 6.6.5, in which dP_6 is endowed with a scaffolding by triangles. This gave rise to the toric variety $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with the line bundle $L = \mathcal{O}(1,1,1)$, that is, the weight matrix for the ambient toric variety is

$$\mathcal{M} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

The degeneration referred to above is obtained by deforming the section

$$s_0 = x_0x_1x_2 \in H^0(Y, L)$$

Since each cone in the spanning fano over the reflexive polygon P considered in Example 6.6.5 is smooth, we expect each slab in the affine manifold to contain a single focus-focus singularity. Indeed, taking a generic element $s \in H^0(Y, L)$ and a one parameter family $s_0 + ts$ the singularities of the embedded family are contained in the lines $x_i = x_j = 0$ for $1 \leq i, j \leq 3$ and $i \neq j$. In particular the family is singular when $s|_{x_i=x_j=0}$ vanishes. This restriction is a linear equation in one variable, thus the family is singular over three reduced points in the base, one on each line.

There is only a short list of possible affine manifolds (after forgetting the boundary and the monodromies, which depend on $H^0(X; \bigoplus_i L_i)$); the type of affine manifold which occurs is determined by the shapes of the struts used in the scaffolding. For surfaces, the 'maximal' (no uneliminated variables) cases are:

- X is a hypersurface in Y, and L is the sum of three divisors $D_i = D(e_i)$; and
- X is codimension 2 and L_1, L_2 are sums of disjoint pairs of divisors D_i.

The affine manifolds corresponding to these two possibilities are shown in Figure 6.10.2. The prototypical examples here are the cubic surface, for which each singularity has monodromy polytope equal to an interval of length three, and the intersection of two quadrics in $\mathbb{P}^4$, for which each singularity has monodromy polytope equal to an interval of length two. These examples, together with the dP_6 example where the boundary divisor is given by three curves with zero self-intersection, are shown in Figure 6.10.1.

For the threefold case there are three maximal cases:

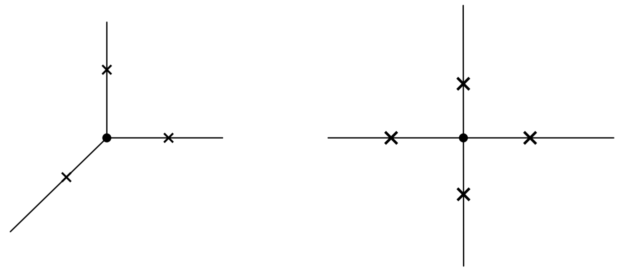


FIGURE 6.10.2. Affine manifolds for triangle and square scaffoldings

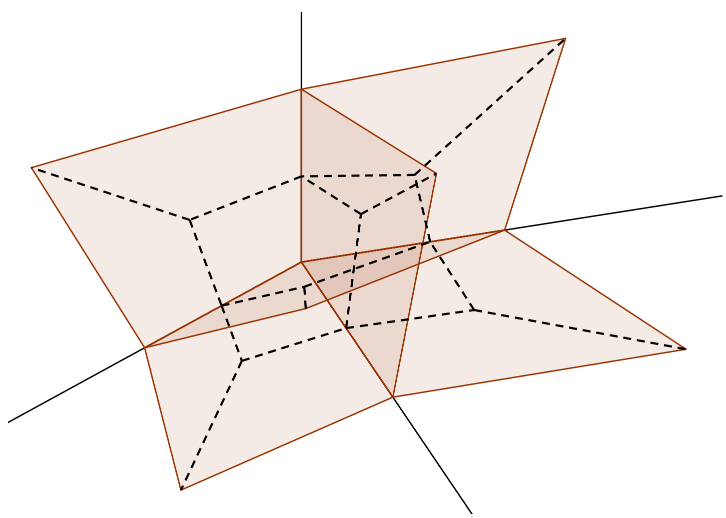


FIGURE 6.10.3. Affine manifold corresponding to tetrahedral scaffoldings

- X is a hypersurface in Y, and L is the sum of four divisors $D_i = D(e_i)$; or
- X is codimension 2 in Y, and L_1, L_2 is the sum of two and three divisors $D_i = D(e_i)$ respectively; or
- X is codimension 3 and L_1, L_2, L_3 are sums of disjoint pairs of divisors D_i.

The affine manifolds obtained from the first and third of these are shown in Figures 6.10.3, 6.10.4.

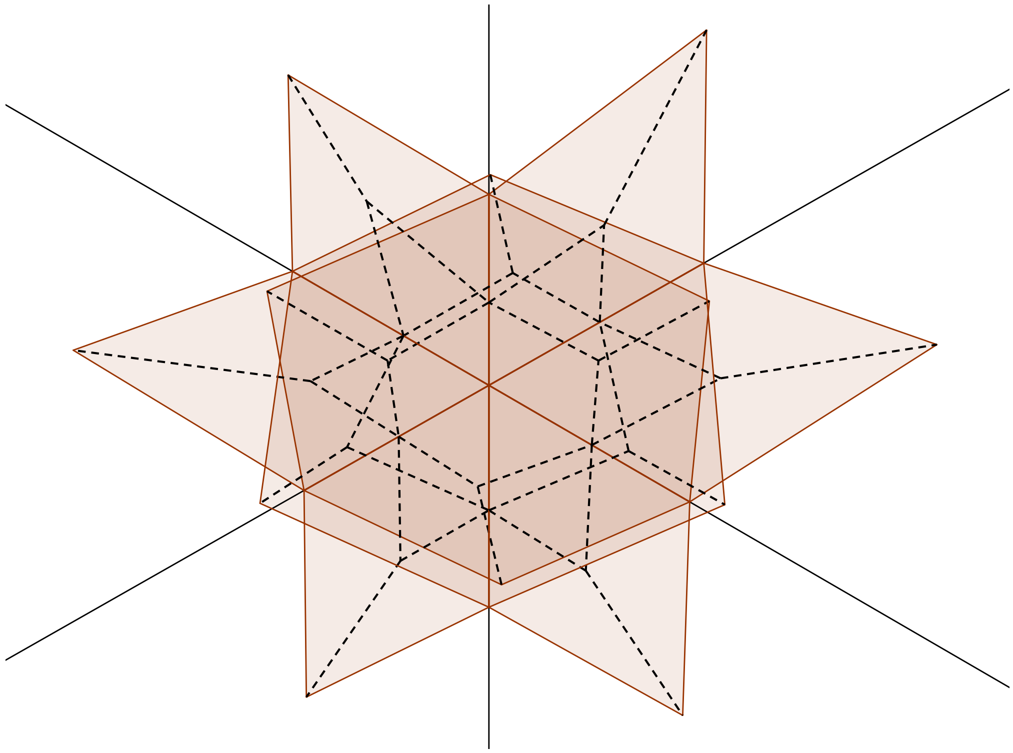


FIGURE 6.10.4. Affine manifold corresponding to cuboid scaffoldings

CHAPTER 7

From Commuting Mutations to Complete Intersections

In the previous chapter we introduced a technique, *Laurent Inversion*, which, starting from a Laurent Polynomial f, constructs a Fano Variety which for which f is a mirror in the sense of Givental/Hori–Vafa. The principal difficulty with this method is that while we are able to recover a collection of rays of a fan in general, recovering the correct fan to produce a smoothing of the toric variety $X_{\mathrm{Newt}(f)}$ requires a careful analysis in each case. In this chapter we consider a Fano polytope P together with a collection of *commuting* mutations Ξ and construct a 'minimal' Laurent polynomial f such that $\mathrm{Newt}(f) = P$ and f admits this collection of mutations. We then run Laurent Inversion on f *and* construct a specific stability condition. Thus we produce a toric variety $Y_{P,\Xi}$ and line bundles L_i such that each toric variety $X_{\mu(P)}$ for $\mu \in \Xi$ is the vanishing locus of a section of $\oplus_i L_i$.

THEOREM 7.0.2. *Given a Fano polytope P and a collection of commuting mutations Ξ there is a toric variety $Y_{P,\Xi}$ and line bundles L_i such that for each $\mu \in \Xi$ there is a section $s_\mu \in H^0(\oplus_i L_i)$ such that $s_\mu \cap Y_{P,\Xi} = X_{\mu(P)}$.*

REMARK 7.0.3. While we do not make a general analysis of a general section in this chapter the effect of *simultaneously* producing the deformations corresponding to a collection of mutations is that we expect to often be able to smooth every singularity, and thus we expect this construction, applied to reflexive polytopes, to yield many new Fano manifolds.

7.1. Overview

This chapter will make extensive use of the notion of *combinatorial mutation* in arbitrary dimensions, which directly generalises the definition for polygons detailed in Section 2.1. We recall the definition of mutation below, but for more details see [**4**].

Throughout this section $N \cong \mathbb{Z}^n$ is a lattice, $M = \mathrm{Hom}(N, \mathbb{Z})$ its dual lattice, $P \subset N$ is be an n-dimensional Fano polytope and $Q := P^\circ$ its polar (dual) polytope. Given a k-dimensional stratum s of ∂P we denote by $s^\star$ its dual $(n - k - 1)$-dimensional stratum of ∂Q. Recall that given a Fano polygon P we denote its *spanning* (or *face*) fan by Σ_P and the toric variety corresponding to this fan by X_P. Note that this is not an entirely standard convention, which would more usually define Σ_P as the *normal* fan to P.

A *mutation* of P is defined using two additional pieces of data:

- A *weight vector* $w \in M$.
- A *factor* polytope, $F \subset w^\perp \subset N$.

This data cannot be chosen arbitrarily, for a mutation to exist additional criteria must be satisfied, see [**4**] for details. It is possible to directly write out the polytope $\mu_{w,F}(P)$ which immediately generalises Definition 2.2.2, however it is easier to write the dual operation, which we recall in Definition 7.1.1 and refer to [**4**] (or Definition 2.2.2) for the direct 'N-side' definition.

DEFINITION 7.1.1 (M-side mutation). If a mutation $\mu_{w,F}$ exists, it takes P to a Fano polytope $\mu_{w,F}(P)$, which is uniquely determined by its polar polytope. The piecewise linear map $\mu_{w,F} \colon M \to M$ defined by

$$\mu_{w,F} \colon u \mapsto u - \min_{v \in \mathrm{verts}(F)} \langle v, u \rangle$$

sends Q to $\mu_{w,F}(P)^{\circ}$, and will be the definition of mutation used in this chapter. We will assume throughout that $0 \in \mathrm{verts}\, F$ so there is a chamber on which the piecewise linear map $\mu_{w,F}$ is the identity.

The definition for polytopes was inspired by an operation on Laurent polynomials which we refer to as *algebraic mutation*. This is a birational map from the torus T_M to itself. An algebraic mutation is determined by a weight vector $w \in M$ and a polynomial F^{alg} such that $\mathrm{Newt}(F^{alg}) \subset w^{\perp}$. We recall the precise definition which we gave in Section 2.1.

DEFINITION 7.1.2. Given a Laurent polynomial $f \in \mathbb{C}[T_N]$ say f mutates to $\theta^{\star}_{w,F^{alg}}(f)$ if the latter is also a Laurent polynomial on T_M. The birational map $\theta_{w,F^{alg}}$ is defined to by setting

$$\theta^{\star}_{w,F^{alg}}(z^n) = z^n \left(F^{alg}\right)^{\langle w,n \rangle}$$

In particular if f mutates to $\theta^{\star}_{w,F^{alg}}(f)$, $P = \mathrm{Newt}(f)$ mutates to

$$\mu_{w,F}(P) = \mathrm{Newt}(\theta^{\star}_{w,F^{alg}}(f))$$

where $F = \mathrm{Newt}(F^{alg})$. Every combinatorial mutation can be expressed in terms of the Newton polyhedra of an algebraic mutation.

DEFINITION 7.1.3. Given a collection Ξ of mutations, denote the set of factors $\mathrm{Factors}(\Xi)$ and the set of weight vectors $\mathrm{Weights}(\Xi)$. Given a factor $F \in \mathrm{Factors}(\Xi)$ define $\mathrm{Weights}(\Xi, F)$ to be those weights of mutations in Ξ with factor F.

7.2. Commuting Mutations

In this section we classify collections of commuting mutations. We must first make precise the sense in which we insist that two mutations *commute*.

DEFINITION 7.2.1. Given two mutations $\mu_{w,F}$, $\mu_{w',F'}$ we say that these *commute* if the (integral) piecewise linear transformations induced on M commute.

REMARK 7.2.2. It may be that $\mu_{w,F}$, $\mu_{w',F'}$ *do not* commute but that still $\mu_{w',F'} \circ \mu_{w,F}(P)$ is $\mathrm{GL}(n, \mathbb{Z})$ equivalent to $\mu_{w,F} \circ \mu_{w',F'}(P)$ due to automorphisms of P.

If Ξ is a collection of commuting mutations and $\mu \in \Xi$, define $\mu(\Xi)$ to be the set of mutations

$$\mu(\Xi) := \{\eta \in \Xi : F_\eta \neq F_\mu\} \cup \{(w, F) : w = w_\eta - w_\mu, F = F_\eta \text{ for all } \eta \in \Xi \text{ such that } F_\mu = F_\eta\}$$

Definition 7.2.3. A collection of commuting mutations Ξ is called *a collection of commuting mutations of P* if for all $\mu \in \Xi$ and all $\eta \in \mu(\Xi)$, η defines a mutation of $\mu(P)$.

Thus a collection Ξ of commuting mutations of P can be followed under every mutation $\mu \in \Xi$. In the next section we construct a Laurent polynomial which admits all of the mutations in Ξ. Before doing so we prove a result classifying commuting sets of mutations.

Proposition 7.2.4. *Let P be a Fano polygon and Ξ be a collection of mutations of P. The mutations in Ξ pairwise commute if and only if for all factors F_i and weight vectors w_j we have $F_i \subset w_j^\perp$.*

Proof. One direction is easy, writing out the definition of the piecewise linear map in M we see that if $F_i \subset w_j^\perp$ for all i, j then the mutations must commute. Assume that there is some pair of mutations $\mu_i \in \Xi$ for $i = 1, 2$ such that F_2 is not contained in $w_1^\perp$. Dually we have that

$$w_1 \notin F_2^\perp := \bigcap_{v \in \text{verts}(F_2)} v^\perp \subset M$$

The mutation μ_2 subdivides M into chambers, and assuming the origin of N is a vertex of F_2, there is a chamber on which μ_2 is the identity. However this chamber is not preserved by translation in $\pm w_1$, since the smallest subspace it contains is $F^\perp$. We claim there is an element $u \in M$ contained in the chamber fixed by μ_2 that is moved out of this chamber by μ_1. Having shown this, we see that

$$\mu_1(\mu_2(u)) = \mu_1(u)$$

But also,

$$\mu_2(\mu_1(u)) \neq \mu_1(u)$$

Observe that w_1 is not in the kernel of the projection $M \to M/F_2^\perp$ and the identity chamber C_{μ_2} of μ_2 defines a strictly convex cone in $M/F_2^\perp$. Thus there is a wall of C_{μ_2} such that for any point y on this wall $y + \epsilon w_1 \notin C_{\mu_2}$ for all $\epsilon > 0$ and possibly replacing w_1 with $-w_1$. Translating the factor F_1 appropriately the mutation μ_1 acts non-trivially on a given point u lying in this wall, thus we can take any such element u. $\square$

7.3. Constructing $f_{P,\Xi}$

Following the procedure in Section 6.6 we construct a scaffolding of P by constructing a Laurent polynomial $f_{P,\Xi}$ which admits each mutation $\mu \in \Xi$, lifting the combinatorial mutation μ to an algebraic mutation with the same weight vector and factor $F^{alg} := \sum_{v \in \text{verts}(F)} z^v$.

Before we define $f_{P,\Xi}$ we define some useful auxillary polynomials.

Definition 7.3.1. Given a function $\mathbf{a} \in \mathbb{Z}_{\geq 0}^{\mathrm{Factors}(\Xi)}$, define the *strut polynomial*

$$\mathbb{S}_{\mathbf{a}} := \prod_{F \in \mathrm{Factors}(\Xi)} \left(F^{alg}\right)^{\mathbf{a}(F)}$$

We will refer to the entries $\mathbf{a}(F)$ as *dilation factors*, noting that the Newton polytope of $\mathbb{S}_{\mathbf{a}}$ is the Minkowski sum of each $F \in \mathrm{Factors}(\Xi)$ dilated by a factor of $\mathbf{a}(F)$.

Definition 7.3.2. Define the polynomial

$$f_{P,\varnothing} := \sum_{v \in \mathrm{verts}(P)} z^v$$

We define the *maximum weight vector* for each vertex of P; this determines the dilation factors of the strut that appears at each vertex.

Definition 7.3.3. Fix a $v \in \mathrm{verts}(P)$ and polytope $F \in F_\Xi$ we define the vector $w_v \in \mathbb{Z}_{\geq 0}^{\mathrm{Factors}(\Xi)}$ by insisting that

$$w_v(F) := \max_{w \in \mathrm{Weights}(\Xi, F) \cup \{0\}} -\langle w, v\rangle$$

Observe this number is always non-negative since $0 \in \mathrm{verts}(F)$.

Given a vertex v of P and factor F there is a vertex $v_F(v)$ of F such that

$$\{v + w_v(F)x : x \in (\mathrm{conv}(F) - v_F(v))\} \subset P$$

That is, there is a vertex $v_F(v)$ of F such that scaling F by $w_v(F)$ and translating it so that $v_F(v)$ lies at v the transformed factor lies inside P (since P admits a mutation with this factor and weight vector putting v as the height $w_v(F)$).

Remark 7.3.4. The vertex $v_F(v)$ is *not* necessarily unique, but is unique if $w_v(F) > 0$. If $w_v(F) = 0$, $v_F(v)$ is chosen arbitrarily from $\mathrm{verts}(F)$.

We can then form subsets $\mathrm{verts}(P)_u \subset \mathrm{verts}(P)$ for any given $u \in \mathrm{verts}(F)$,

$$\mathrm{verts}(P)_u := \{v \in \mathrm{verts}\, P : v_F(v) \text{ can be taken to be } u\}$$

Ranging over the different $F \in \mathrm{Factors}(\Xi)$ we can refine this collection of subsets.

Definition 7.3.5. Given a function $U\colon \mathrm{Factors}(\Xi) \to N$ such that $U(F) \in \mathrm{verts}(F)$ define

$$\mathrm{verts}(P)_U := \{v \in \mathrm{verts}\, P : v_F(v) = U(F)\, (\forall F \in \mathrm{Factors}(\Xi))\}$$

We call such a function U a *vertex picking function* and call $v \in \mathrm{verts}(P)_U$ a *U-vertex* of P if $v \in \mathrm{verts}(P)_U$.

Example 7.3.6. Let $F_1 = \{0, (1, 0, 0)\}$ and $F_2 = \{0, (0, 1, 0)\}$ where corresponding to $\mathrm{Newt}(1 + x)$ and $\mathrm{Newt}(1 + y)$ in $\mathbb{C}[x, y, z]$ respectively. Assume that

$$\mathrm{Factors}(\Xi) = \{F_1, F_2\}$$

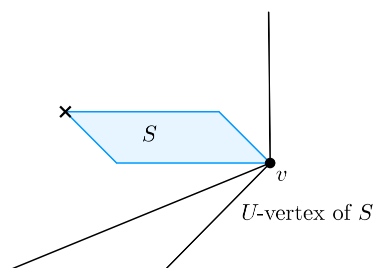


FIGURE 7.3.1. A vertex $v \in \mathrm{verts}(P)_U$

Then a strut polynomial has the general form

$$\mathbb{S}_{\mathbf{a}} = x^a y^b z^c (1+x)^d (1+x)^e$$

The polyhedra F_1, F_2 and a general strut $\mathrm{Newt}(\mathbb{S}_{\mathbf{a}})$ is shown in Figure 7.3.2. Let U be the vertex picking function which sends $F_1 \mapsto (1,0,0)$ and $F_2 \mapsto (0,1,0)$. Figure 7.3 shows an example of a U-vertex v of a strut.

REMARK 7.3.7. Neither $\mathrm{verts}(P)_u$ nor $\mathrm{verts}(P)_U$ need give partitions of $\mathrm{verts}(P)$, since factors can have height $w_v(F) = 0$.

DEFINITION 7.3.8. We now define a preliminary version of $f_{P,\Xi}$ by writing

$$f^{pre}_{P,\Xi} := \sum_{\substack{U\colon \mathrm{Factors}(\Xi)\to N\\ U(F)\in \mathrm{verts}(F)}} \sum_{v\in \mathrm{verts}(P)_U} \left(z^{v-\sum_{F\in \mathrm{Factors}\,\Xi} w_v(F)U(F)} \cdot \mathbb{S}_{w_v} \right)$$

This polynomial will, by construction, admit each mutation in Ξ and we claim that $\mathrm{Newt}(f^{pre}_{P,\Xi}) = P$.

LEMMA 7.3.9. *The Newton polytope of $f^{pre}_{P,\Xi}$ is P.*

PROOF. Given a vertex $v \in P$ we have replaced the term z^v in $f_{P,\varnothing}$ with the polynomial

$$S^{alg}_v = z^{v-\sum_{F\in \mathrm{Factors}\,\Xi} w_v(F)U(F)} \cdot \mathbb{S}_{w_v}$$

Where $v \in \mathrm{verts}(P)_U$, Figure 7.3 shows the Newton polytope of this polynomial and we require that it is contained within P. However since Ξ is a collection of *commuting* mutations we can mutate each factor F_i in turn using the weight vector w_F which attains $w_v(F)$. Since $U(F)$ was chosen such that a copy of $w_v(F)F$ lies within P and meets v we can argue inductively that the Minkowski sum $\sum_F w_v(F)F$, translated so its U-vertex meets v lies within P. Specifically, induct on the number of terms in the Minkowski sum, apply a mutation with factor F and weight vector w_F and use that the mutations commute. □

Whilst being close to what we need, $f^{pre}_{P,\Xi}$ generally contains repeated terms which we shall trim. In particular we have a mapping from monomial terms in $f_{P,\varnothing}$ to terms in $f^{pre}_{P,\Xi}$ by sending:

$$\phi\colon z^v \mapsto \left(z^{v-\sum_{F\in\text{Factors}\,\Xi} w_v(F)U(F)}\cdot \mathbb{S}_{w_v}\right)$$

This function need not be injective, so we will trim the repeated terms. Before we do this we recall the notion of a *strut*. Recall from Section REF that we can think of the Newton polytope of $f^{pre}_{P,\Xi}$ as being 'tiled' by polyhedra, called *struts*.

Definition 7.3.10. For any $v \in \text{verts}(P)$ the *strut* S_v is the polyhedral set

$$S_v := y + \text{Newt}(\mathbb{S}_{w_v})$$

Where $y \in N$ is uniquely determined by requiring that $S_v \subset P$ and v is a vertex of S_v, that is, the point

$$y = v - \sum_{F\in\text{Factors}\,\Xi} w_v(F)U(F)$$

Define the set $\text{Struts}(P) = \{S_v : v \in \text{verts}(P)\}$. Observe that there is a surjection, which we also call $\phi\colon \text{verts}(P) \to \text{Struts}(P)$

We can extend the notion of a U-vertex to all the vertices of a strut S_v.

Definition 7.3.11. Given a vertex picking function U we refer the element of N corresponding to taking the monomial $z^{U(F)}$ in each term of the sum $\mathbb{S}_{w_v} = \prod_{F\in\text{Factors}(\Xi)}\left(F^{alg}\right)^{w_v(F)}$ as the *U-vertex* of S_v. Clearly if $v' \in \text{verts}(S_v)\cap\text{verts}(P)$ then the U-vertex of S_v is a U-vertex of P, that is, $v' \in \text{verts}(P)_U$.

Definition 7.3.12. Given a strut S_v we define $\text{Root}(S_v) = v - \sum_{F\in\text{Factors}\,\Xi} w_v(F)U(F)$. This defines an injection $\text{Root}\colon \text{Struts}(P) \to N$.

We collect several elementary observations about struts into the following Lemma.

Lemma 7.3.13. *Given a strut $S \in \text{Struts}(P,\Xi)$ we have,*

- *The lattice point $\text{Root}(S) \in N$ is independent of the choice of vertex v of the strut S_v.*
- *If $0 \in \text{verts}(F)$ for each $F \in \text{Factors}(\Xi)$ then $\text{Root}(S) \in P$.*
- *The values $w_v(F)$ are constant over vertices $v \in \text{verts}(S)$.*

Remark 7.3.14. Using the third point, given a strut $S \in \text{Struts}(P,\Xi)$ we will write w_S in place of w_v for any $v \in \text{verts}(S)$.

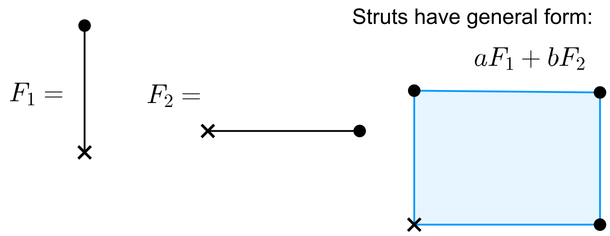


FIGURE 7.3.2. Two factors and general strut shape

EXAMPLE 7.3.15. Returning to Example 7.3.6 the strut S_v is exactly the polytope

$$\mathrm{Newt}(x^a y^b z^c (1+x)^d (1+y)^e)$$

and the point $(a, b, c) \in N$ is $\mathrm{Root}(S_v)$. An example of a scaffolding of a polytope P with these factors is shown in Figure 7.3.3 in which the roots of each strut are marked with a $\times$ symbol. From this example we see that one strut may meet many vertices of P and by construction meets at least one.

Utilizing the terminology of struts we define the polynomial $f_{P,\Xi}$.

DEFINITION 7.3.16.

$$f_{P,\Xi} := \sum_{S \in \mathrm{Struts}(P)} z^{\mathrm{Root}(S)} \cdot \mathbb{S}_{w_S}$$

Recall from Section 6.6 that we can reconstruct the rays of the Givental/Hori–Vafa mirror to a toric variety $Y_{P,\Xi}$, a candidate ambient space for a complete intersection model of $X_{P,\Xi}$. Schematically, the steps in this procedure are as follows.

- Introduce k variables where k is the codimension of the complete intersection we are attempting to construct.
- Apply a mutation with its weight vector determined by these new variables.

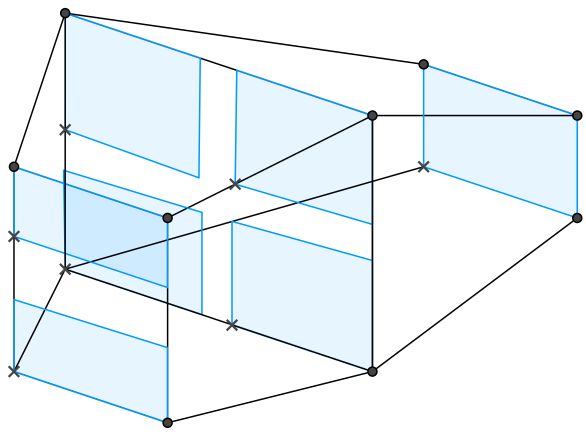


FIGURE 7.3.3. An example of a scaffolding

- The monomials of the resulting polynomial define rays in a lattice $\tilde{N} = N \oplus \langle e_1, \cdots e_k \rangle$.

Applying this to $f_{P,\Xi}$ we first add new variables X_F for each $F \in \text{Factors}(\Xi)$.

DEFINITION 7.3.17. Define the lattice $\tilde{N} := N \oplus \langle e_F : F \in \text{Factors}(\Xi) \rangle$. The Laurent polynomial Givental/Hori–Vafa mirror to the ambient toric variety $Y_{P,\Xi}$ will have exponents in this lattice. Denote the dual lattice $\tilde{M}$ and the dual vectors to $e_F \in \tilde{N}$ by $e_F^\star$.

Introducing new variables we write down an intermediate Laurent polynomial,

$$f^{(1)}_{P,\Xi} := \sum_{F \in \text{Factors}(\Xi)} X_F + \sum_{S \in \text{Struts}(P)} z^{\text{Root}(P)} \cdot \mathbb{S}_{w_S} \cdot \prod_{F \in \text{Factors}(\Xi)} X_F^{-w_S(F)}$$

Successively apply mutations with factor F and weight vector $e_F^\star$.

DEFINITION 7.3.18. Define the polynomial

$$\tilde{f}_{P,\Xi} := \sum_{F \in \text{Factors}(\Xi)} X_F F^{alg} + \sum_{S \in \text{Struts}(P)} z^{\text{Root}(P)} \cdot \prod_{F \in \text{Factors}(\Xi)} X_F^{-w_S(F)}$$

The exponents of these monomials lie in the lattice $\tilde{N}$ and define the following set of rays.

DEFINITION 7.3.19.

$$\begin{aligned}\text{Rays}(P,\Xi) = &\{\langle e_F + v \rangle : \forall F \in \text{Factors}(\Xi), v \in \text{verts}(F)\} \cup \\ &\left\{ \left\langle \text{Root}(S) - \sum_{F \in \text{Factors}(\Xi)} w_S(F) e_F \right\rangle : S \in \text{Struts}(P) \right\}\end{aligned}$$

For brevity, we will denote these rays as:

- $\rho_{F,v} := \langle e_F + v \rangle$ for $F \in \mathrm{Factors}(\Xi), v \in \mathrm{verts}(F)$
- $\rho_S := \langle \mathrm{Root}(S) - \sum_{F \in \mathrm{Factors}\,\Xi} w_S(F) e_F \rangle$ for $S \in \mathrm{Struts}(P)$, observing that the function U is determined by v.

Let $g_{F,v}$ denote the primitive integral generator of the ray $\rho_{F,v}$ and g_S denote the primitive integral generator of the ray ρ_S.

7.4. The polytope $\tilde{Q}_{P,\Xi}$

Now that we have the Laurent polynomial $\tilde{f}_{P,\Xi}$ we can write down the rays $\mathrm{Rays}(P, \Xi)$. We will define a polytope $\tilde{Q}_{P,\Xi} \subset \tilde{M}$ using these rays and define $Y_{P,\Xi}$ as the toric variety assoicated to the normal fan of $\tilde{Q}_{P,\Xi}$.

Definition 7.4.1. The polytope $\tilde{Q}_{P,\Xi} \subset \tilde{M}$ defined using inequalities defined by evaluation in the primitive integral generators of elements of $\mathrm{Rays}(P, \Xi)$.

$$\tilde{Q}_{P,\Xi} := \bigcap_{\substack{F \in \mathrm{Factors}(F), \\ v \in \mathrm{verts}(F)}} \{u \in \tilde{M} : \langle u, g_{F,v} \rangle \geq 0\} \cap \bigcap_{S \in \mathrm{Struts}(P,\Xi)} \{u \in \tilde{M} : \langle u, g_S \rangle \geq -1\}$$

The rest of this section is devoted to understanding certain important strata of this polytope. To begin with we observe that each $F \in \mathrm{Factors}(\Xi)$ determines a polyhedral decomposition of $Q := P^\circ$.

Definition 7.4.2. Given an element $F \in \mathrm{Factors}(\Xi)$ we define a polyhedral decomposition of Q by fixing a vertex $v \in \mathrm{verts}(F)$ and defining a chamber

$$C_v := \{x \in Q_{\mathbb{R}} : \min_{v \in \mathrm{verts}(F)} \langle v, x \rangle \text{ is attained on } v\}$$

These are closed polyhedral subsets, and define a polyhedral decomposition $\mathcal{P}(F)$. Given a vertex picking function U we define

$$C_U := \{x \in Q_{\mathbb{R}} : \min_{U(F) \in \mathrm{verts}(F)} \langle U(F), x \rangle \text{ is attained on } U(F) \text{ for all } F\}$$

The C_U are chambers of a polyhedral decomposition.

Later we will need to understand the relationship between the scaffolding of P induced by the polynomial $f_{P,\Xi}$ and the decompositions of Q induced by the factors F. We collect the observations we will need in the following Lemma.

Lemma 7.4.3. *Given a vertex picking function U, the sets* $\mathrm{verts}(P)_U$ *are dual to the chambers C_U in the following sense.*

(1) Given a vertex $v \in \mathrm{verts}(P)$, $v \in \mathrm{verts}(P)_U$ if and only if $v^\star \cap C_U \neq \varnothing$.
(2) Given a vertex $v' \in \mathrm{verts}(Q)$, $v' \in C_U$ if and only if $v'^\star$ contains a U-vertex.

Proof. Choose a vertex $v \in \mathrm{verts}(P)_U$ and let $v'^\star$ be a facet of P containing v with $v' \in \mathrm{verts}(Q)$. Since $S_v \subset P$, evaluating v' on points in S_v it attains its minimum of -1

precisely along $v'^{\star}$. That is, ranging over vertex picking functions, the minimum is attained at v, a U-vertex, thus $v' \in C_U$ and $v'^{\star} \subset v^{\star}$ since duality is inclusion reversing. This proves the forward implication of the first point and the opposite implication of the second.

For the other directions, if $v^{\star} \cap C_U \neq \varnothing$ we pick a rational point $r \in M_{\mathbb{Q}}$ in the intersection. Given a factor $F \in \mathrm{Factors}(\Xi)$ we know that, ranging over the vertices of F, r achieves its minimum on $U(F)$, since $r \in C_U$. Thus, ranging over the vertices of S_v, r is minimal on the U-vertex. But $r \in v^{\star}$ so this occurs precisely at v. This proves the other implication of the first point. Choosing a facet $v^{\star}$ such that $v' \in v^{\star}$ then letting $r = v'$ proves the forward implication of the second point. $\square$

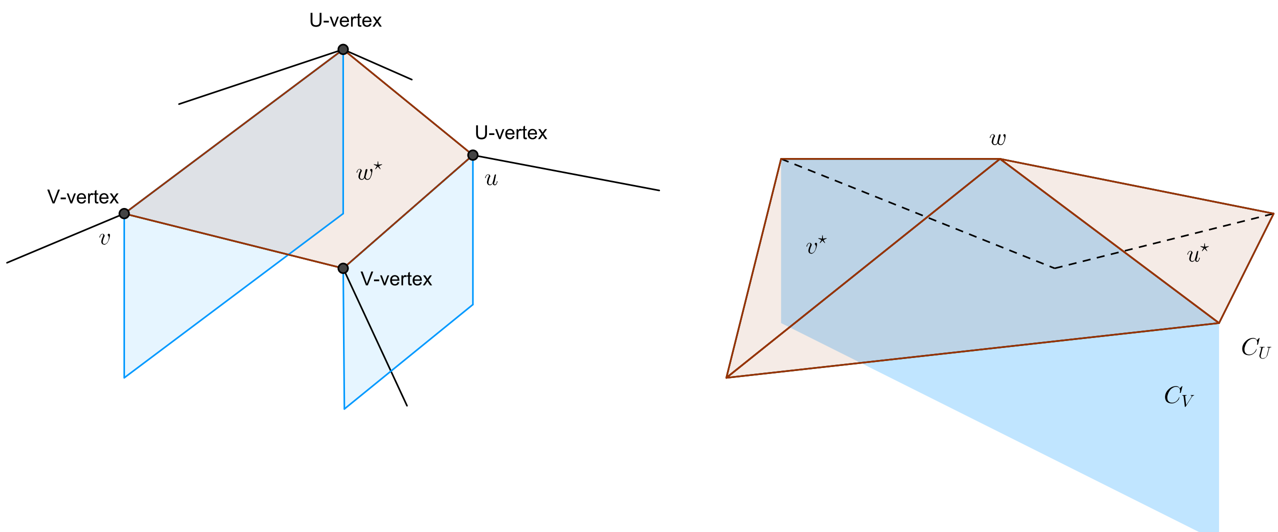


FIGURE 7.4.1. An example demonstrating Lemma 7.4.3

The function $\min_{v \in \mathrm{verts}(F)}$ defines a piecewise linear function on Q linear on the chambers of $\mathcal{P}(F)$, there is an important connection between the graph of this function and the polytope $\tilde{Q}_{P,\Xi}$ which we now explore.

DEFINITION 7.4.4. Considering the graphs of the functions $\min_{v \in \mathrm{verts}(F)}$ simultaneously we define a piecewise linear function $\iota \colon M_{\mathbb{R}} \to \tilde{M}_{\mathbb{R}}$.

$$\iota \colon x \mapsto x - \sum_{F \in \mathrm{Factors}(\Xi)} \min_{v \in \mathrm{verts}(F)} (\langle x, v \rangle) e_F^{\star}$$

Observe that ι is a linear function on each polyhedral subset $C_U \subset Q_{\mathbb{R}}$. The main result of this section is that each C_U is a stratum of the boundary of $\tilde{Q}_{P,\Xi}$.

PROPOSITION 7.4.5. *The image $\iota(Q)$ is equal to a collection of strata of $\tilde{Q}_{P,\Xi}$.*

PROOF. We prove this in two stages, first forgetting the boundary of Q and identifying the piecewise linear subspace of $\tilde{M}$ defined by extending the functions ι to all of M and using the inequalities $\langle g_{F,v}, u\rangle \geq 0$ to recover this subspace. We then show that applying the inequalities $\langle g_S, u\rangle \geq -1$ reproduces the image of ∂Q on this (piecewise linear) subspace of $\tilde{M}$.

Consider the polyhedral subspace defined by the inequalities $\{u \in \tilde{M} : \langle g_{F,v}, u\rangle \geq 0\}$. A stratum of the boundary of this subspace is uniquely determined by a collection of pairs (v, F) such that $\langle g_{F,v}, u\rangle = 0$. Given a vertex picking function U, there is a boundary stratum R_U given by choosing precisely the pairs $(F, U(F))$. Observe that the image $\iota(C_U)$ has the same dimension as R_U, and we claim that $\iota(C_U)$ is annihilated by each vector $e_F + U(F)$, making $\iota(C_U)$ a maximal dimensional polyhedral subset of R_U. However, by definition

$$y \in \iota(C_U) \Leftrightarrow y = y' - \sum_{F\in\mathrm{Factors}(\Xi)} \langle U(F), y'\rangle e_F, \qquad \text{for some } y' \in M$$

Evaulating $\rho_{(F,v)} = (e_F + U(F))$ at y we find

$$(e_F + U(F))(y' - \sum_{F\in\mathrm{Factors}(\Xi)} \langle U(F), y'\rangle e_F) = \langle U(F), y'\rangle - \langle U(F), y'\rangle = 0$$

So the the piecewise linear subspace into which $M_{\mathbb{R}}$ maps under ι is equal to the union of the R_U. We still need to check that the inequalities ρ_v restricted to the union $R := \cup_U R_U$ defines the image $\iota(Q)$. We first show that $\iota(Q) \subseteq \tilde{Q}_{P,\Xi} \cap R$.

Choose a vertex $v \in \mathrm{verts}(P)$ such that the facet $v^\star$ has non-empty intersection with C_U. The subspace R_U is precisely the subspace on which $e_F + U(F) = 0$ for each $F \in \mathrm{Factors}(\Xi)$, so the inequality $\rho_{F,v} = v - \sum_{F\in\mathrm{Factors}(\Xi)} w(F)(e_F + U(F)) \geq -1$ restricts to the inequality $v \geq -1$, an inequality defining Q. Observe that this argument only applies on those subsets R_U of R for which $v \in \mathrm{verts}(P)_U$ (using Lemma 7.4.3).

To show the reverse inclusion $\tilde{Q}_{P,\Xi} \cap R \subseteq \iota(Q)$ we still need to show that the inequality defined by ρ_S does not interfere with other sets R_U, as illustrated in Figure 7.4.2. However taking any $g_S = \mathrm{Root}(S) - \sum_{F\in\mathrm{Factors}(\Xi)} w_S(F)e_F$ and $\tilde{y}$ any point in the image of ι,

$$\tilde{y} = y - \sum_{F\in\mathrm{Factors}(\Xi)} a_F(y)e_F^\star$$

where $a_F(y) = \min_{u\in\mathrm{verts}(F)} \langle u, y\rangle$. Evaluating g_S at y we find,

$$g_S(\tilde{y}) = \mathrm{Root}(S)(y) + \sum_{F\in\mathrm{Factors}(\Xi)} w_v(F)a_F$$

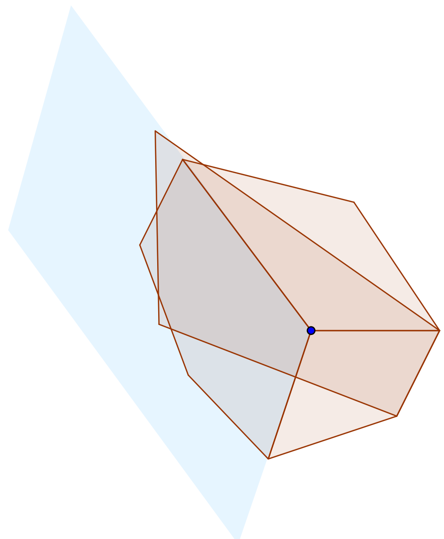


FIGURE 7.4.2. Impossible behaviour of the facets of $\tilde{Q}_{P,\Xi}$ by Proposition 7.4.5

Expanding out $\mathrm{Root}(S) = v - \sum_F w_S(F)U(F)$ for *any* $v \in \mathrm{verts}(S)$ where v is a U-vertex of S we have

$$\rho_S(\tilde{y}) = v(y) + \sum_{F \in \mathrm{Factors}(\Xi)} w_S(F)(a_F - U(F)(y))$$

Fix a vertex picking function U by requiring that $U(F)$ is the vertex of $\mathrm{verts}(F)$ on which $\langle y, - \rangle$ attains its minimum and let v be the U-vertex of S. Now, U is a vertex picking function such that $a_F(y) = U(F)(y)$, thus the evaluation of g_S at y reduces to

$$g_S(\tilde{y}) = v(y) \geq -1$$

This also shows that equality is attained precisely when

$$v \in \mathrm{verts}(P) \cap \mathrm{verts}(S) \text{ and } y \in v^\star \subset \partial Q$$

Thus $\iota(Q) \subseteq \tilde{Q}_{P,\Xi} \cap R$, but from our local calculation we know that $\tilde{Q}_{P,\Xi} \cap R \subseteq \iota(Q)$. $\square$

7.5. Recovering toric varieties $X_{\mu(P)}$

In this section we study the toric variety defined by $\tilde{Q}_{P,\Xi}$ in detail, and prove the main result of this chapter, Theorem 7.0.2.

DEFINITION 7.5.1. Define the fan $\Sigma_{P,\Xi}$ to be the normal fan of the polytope $\tilde{Q}_{P,\Xi}$.

LEMMA 7.5.2. *The set of rays* $\Sigma_{P,\Xi}(1)$ *is equal to the set* $\mathrm{Rays}(P,\Xi)$*, that is, there are no redundant inequalities defining* $\tilde{Q}_{P,\Xi}$*.*

PROOF. Recall that the set $\mathrm{Rays}(P,\Xi)$ naturally divides into two pieces

$$\mathrm{Rays}(P,\Xi) := \{\rho_{F,v} : F \in \mathrm{Factors}(\Xi), v \in \mathrm{verts}(F)\} \cup \{\rho_S : S \in \mathrm{Struts}(P,\Xi)\}$$

and recall that these are defined by primitive integral generators which we refer to as $g_{F,v}$ and g_S respectively. Choose a factor $F \in \mathrm{Factors}(\Xi)$, a vertex $v \in \mathrm{verts}(F)$ and an $\epsilon > 0$ such that for any $x \in B_\epsilon(0)$, $\langle g_v, x\rangle > -1$. We will find a $y \in \tilde{M}_\mathbb{R}$ such that $g_{F',v'}(y) = (e'_F + v')(y) > 0$ if $(F',v') \neq (F,v)$ and $g_{F,v}(y) = \langle (e_F + v), y\rangle = 0$. First let

$$y' := \sum_{F' \in \mathrm{Factors}(\Xi)\setminus\{F\}} \frac{\epsilon}{K} e_{F'}$$

For $K > |\,\mathrm{Factors}(\Xi)|$, now $g_{F',v'}(y') > 0$ for all $F' \neq F$, but $g_{F,v'}(y') = 0$ for all $v' \in \mathrm{verts}(F)$. Therefore choose an element $u \in \iota(\mathrm{relint}(C_U))$. Consider points $y_\eta := \eta u + y'$ for $\eta > 0$. For sufficently small η, $y_\eta \in B_\epsilon(0)$, and the inequality $g_{F',v'}(y_\eta) > 0$ holds for all $\eta > 0$. Further,

$$g_{F,v'}(y_\eta) = \langle e_F, y_\eta\rangle + \langle v', y_\eta\rangle = \eta\langle e_F, u\rangle\eta\langle v', u\rangle$$

But $u = u' - \sum_{F \in \mathrm{Factors}(\Xi)} \min_{v \in \mathrm{verts}(F)}(\langle u', v\rangle)e_F$ for some $u' \in \mathrm{relint}(C_U)$, so

$$g_{F,v'}(y_\eta) = -\min_{v \in \mathrm{verts}(F)}(\langle u', v\rangle) + \langle u', v'\rangle > 0$$

Since as $u' \in \mathrm{relint}(C_U)$ $\min_{v \in \mathrm{verts}(F)}(\langle u', v\rangle)$ is achieved uniquely on v.

For rays ρ_S we adopt a similar approach. First choose a vertex $v \in \mathrm{verts}(P)$ such that $S_v = S$. Choose a point u' in the relative interior of the facet $v^\star$ of Q and consider $u = \iota(u')$. By the the proof of Proposition 7.4.5 we can choose a small ball around u so that $g_{S'}(z) > -1$ for all $S' \neq S_v \in \mathrm{Struts}(P,\Xi)$. As before we choose an element z of the subspace $\{x \in \tilde{M} : \langle g_S, x\rangle = -1\}$ such that $g_{F,v}(z) > 0$ for all pairs F, v. Considering $u + \eta z$ for small $\eta > 0$ we find the required point. Figure 7.5.1 illustrates the idea of each of the two parts of this proof. □

We are interested in cones of $\Sigma_{\Xi,P}$ which have a non-trivial intersection with N, that is, cones which will induce non-trivial cones in the fan $\Sigma_{\Xi,P} \cap N_\mathbb{Q}$. We now classify precisely these cones by comparing the scaffolding of P with the set $\mathrm{Rays}(P,\Xi)$. Indeed, let E be a stratum of ∂P and let $E^\star$ denote the dual stratum of Q. E is uniquely determined by the following information assoicated to the scaffolding of P:

- The set $S(E) \subset \mathrm{Struts}(P,\Xi)$ of struts such that $S \in S(E)$ if and only if $\mathrm{verts}(S) \cap \mathrm{verts}(E) \neq \varnothing$.
- The set

$U(E) = \cap_{v \in S(E)}$ vertex picking U such that $\mathrm{verts}\, E$ contains the U-vertex of S_v

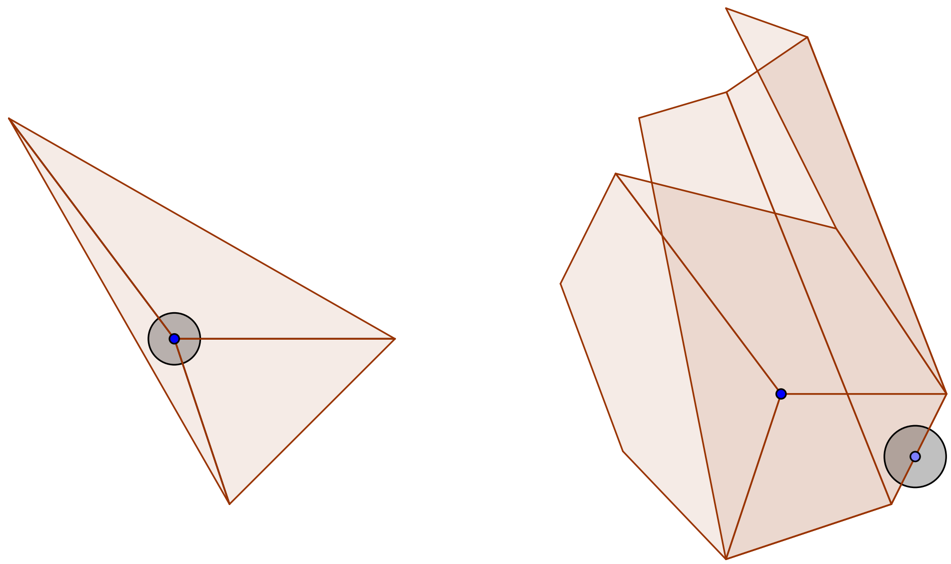


FIGURE 7.5.1. Illustrating the proof of Lemma 7.5.2

Observe that each $S_v \in S(E)$ may be identified with a ray $\rho_{S_v} \in \mathrm{Rays}(P, \Xi) = \Sigma_{P,\Xi}(1)$, in fact $S(E)$ is in bijection with the set of rays $\{\rho_{S_v} : v \in \mathrm{verts}(E)\}$. Also observe that each $U \in U(E)$ may be identified with a cone:

$$\mathrm{Cone}(U) := \langle \rho_{F,v} : v = U(F) \quad \forall F \in \mathrm{Factors}(\Xi) \rangle$$

DEFINITION 7.5.3. Define the cone

$$\widetilde{\mathrm{Cone}(E)} := \langle \mathrm{Cone}(U), \rho_S : \forall S \in S(E), U \in U(E) \rangle$$

PROPOSITION 7.5.4. *Restricting* $\widetilde{\mathrm{Cone}(E)}$ *to* N *recovers* $\mathrm{Cone}(E)$*, that is,* $\widetilde{\mathrm{Cone}(E)} \cap N = \mathrm{Cone}(E)$*.*

PROOF. We show that $\mathrm{Cone}(E) \subseteq \widetilde{\mathrm{Cone}(E)} \cap N$ by showing every ray generator of $\mathrm{Cone}(E)$ appears in $\widetilde{\mathrm{Cone}(E)} \cap N$. By definition, the ray generators of $\mathrm{Cone}(E)$ are precisely the vertices of E. Writing,

$$g_{S_v} = v - \sum_F w_v(F)(e_F + U(F))$$

for U such that $v \in \mathrm{verts}(P)_U$ we see that

$$g_{S_v} + \sum_{F \in \mathrm{Factors}\,\Xi} w_v(F) g_{F,U(F)} = v \in N \oplus \langle e_F : F \in \mathrm{Factors}(F) \rangle$$

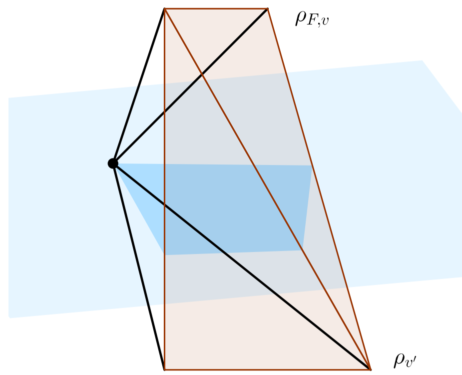


FIGURE 7.5.2. Calculating $\widetilde{\mathrm{Cone}(E)} \cap N$

So we can recover $v \in N$ as a sum of g_{S_v} and $g_{F,U(F)}$ such that v is the U-vertex of S_v. Thus $\mathrm{Cone}(E) \subseteq \widetilde{\mathrm{Cone}(E)} \cap N$.

Now suppose that $\mathrm{Cone}(E)$ is strictly contained in $\widetilde{\mathrm{Cone}(E)} \cap N$. Thus there is a ray of $\widetilde{\mathrm{Cone}(E)} \cap N$ not contained in $\mathrm{Cone}(E)$ and hence a cone of $\widetilde{\mathrm{Cone}(E)}$ whose intersection with N is positive dimensional but meets $\mathrm{Cone}(E)$ only at the origin. Now take any cone C generated by rays of $\widetilde{\mathrm{Cone}(E)}$. For C to meet N in a positive dimensional cone it must contain:

- A ray ρ_S for some $S \in \mathrm{Struts}(P, \Xi)$.
- A ray $\rho_{F,v}$ for any F and some $v \in \mathrm{verts}(F)$.

But making choices of S and $v \in \mathrm{verts}(F)$ for every F the subcone of C generated by these rays meets N along the ray over the U-vertex of the strut S (arguing as in the first part). But the ray over any vertex of any $S \in \mathrm{Struts}(P, \Xi)$ is contained in $\mathrm{Cone}(E)$, which gives a contradiction.

□

The central result of this section is that the square shown in Diagram 7.5.1 is well-defined and commutes. Using this it will follow that X_P is a toric subvariety of $Y_{P,\Xi}$, corresponding to the inclusion $N \hookrightarrow \tilde{N}$. This in turn is a significant step in proving all toric varieties $X_{\mu P}$ are subvarieties of $Y_{P,\Xi}$.

$$
\begin{tikzcd}
\Sigma_{P,\Xi}^{max} \arrow[r] \arrow[d, "-\cap N_{\mathbb{R}}"] & \mathrm{verts}(\tilde{Q}_{P,\Xi}) \\
\Sigma_{P}^{max} \arrow[r] & \mathrm{verts}(Q) \arrow[u, "\iota"']
\end{tikzcd}
\tag{7.5.1}
$$

The horizontal arrows are the usual bijections between k-strata of a polytope and the codimension k cones of its normal fan, and Σ_P is the spanning fan of the polytope P. The right-hand vertical arrow is the inclusion by the piecewise linear map ι and the left-hand vertical map is intersection with the subspace $N_{\mathbb{R}}$.

Lemma 7.5.5. *Diagram 7.5.1 is well defined and commutes.*

Proof. Fix a vertex $\tilde{v} = \iota(v) \in \mathrm{verts}(\tilde{Q}_{P,\Xi})$. The vertex $\iota(v)$ lies in C_U for a collection of vertex picking functions U. By Lemma 7.4.3 for each such U there is a vertex v' of $v^\star \subset P$ which is the U-vertex of its strut S_v. Thus $\iota(v)$ is intersection of facets of $\tilde{Q}_{P,\Xi}$ defined by the following rays of $\Sigma_{P,\Xi}$:

- $\rho_{S_{v'}}$ for $v' \in \mathrm{verts}\, v^\star$.
- $\rho_{F,U(F)}$ for U such that $v \in C_u$.

But by Proposition 7.5.4, if C the cone generated by these rays $C \cap N = \mathrm{Cone}(v^\star)$, thus the square commutes as required. □

Proposition 7.5.6. *The inclusion of lattices $N \hookrightarrow \tilde{N}$ induces an inclusion of toric varieties $X_P \hookrightarrow Y_{P,\Xi}$.*

Proof. Consider the fan $\Sigma'_P := \Sigma_{P,\Xi} \cap N_{\mathbb{Q}}$. To show that $\Sigma'_P = \Sigma_P$ we consider the maximal cones of Σ'_P. A subset of maximal cones is determined by Lemma 7.5.5: fixing a vertex of Q Lemma 7.5.5 produces a maximal cone in Σ'_P equal to a maximal cone of Σ_P. But since Σ_P is a complete fan its maximal cones cover all of N and thus Σ'_P can contain no other maximal cones. □

We conclude the proof of Theorem 7.0.2 by proving that, given any $\mu \in \Xi$, $\Sigma_{\mu(P),\mu(\Xi)}$ differs from $\Sigma_{P,\Xi}$ by a linear transformation which identifies the respective fans $\tilde{\Sigma}_{P,\Xi}$ in each case.

Lemma 7.5.7. *Fix an element $\mu \in \Xi$. Then there is a linear equivalence $T_\mu \colon \tilde{N} \to \tilde{N}_\mu$ such that $\Sigma_{\mu(P),\mu(\Xi)} = T_\mu\left(\Sigma_{P,\Xi}\right)$.*

Proof. Given a mutation $\mu \in \Xi$ with weight vector w_μ and factor F_μ we will first determine a sublattice $N_\mu \subset \tilde{N}$ which we will use in the definition of $\tilde{N}_\mu$ and T_μ. Recalling that N can be described as the subspace $\cap_{F \in \mathrm{Factors}\,\Xi} e_F^\perp$ define,

$$N_\mu := \bigcap_{\substack{F \in \mathrm{Factors}\,\Xi \\ F \neq F_\mu}} e_F^\perp \cap (e_{F_\mu} + w_\mu)^\perp$$

and define $\tilde{N}_\mu := N_\mu \oplus \langle e_F : F \in \mathrm{Factors}(\Xi)\rangle$. Given an element $n + \sum_{F\in\mathrm{Factors}(\Xi)} a_F e_F \in \tilde{N}$ We define the map $T_\mu \colon \tilde{N} \to \tilde{N}_\mu$ as follows,

$$T_\mu \colon n + \sum_{F\in\mathrm{Factors}(\Xi)} a_F e_F \mapsto n + w_\mu(n) e_{F_\mu} + \sum_{F\in\mathrm{Factors}(\Xi)} a_F e_F$$

which sends $e_F \mapsto e_F$ for each factor F. Similarly, by Proposition 7.2.4, w_μ annihilates every vertex of every factor, so

$$T_\mu \colon g_{F,v} \mapsto g_{F,v} = \langle e_F + v\rangle$$

Finally, applying T_μ to a ray ρ_S for $S \in \mathrm{Struts}(P,\Xi)$ we calculate that

$$T_\mu \colon g_S \mapsto g_S + w_\mu(v) e_{F_\mu}$$

where v is some vertex of S. We now show an equality,

$$\{T_\mu(\rho) : \rho \in \mathrm{Rays}(P,\Xi)\} = \{\rho \in \mathrm{Rays}(\mu(P),\mu(\Xi))\}$$

To find $\mathrm{Rays}(\mu(P),\mu(\Xi))$ first form the following Laurent polynomial from $(\mu(P),\mu(\Xi))$,

$$f_{\mu(P),\mu(\Xi)} := \sum_{S\in\mathrm{Struts}(\mu(P),\mu(\Xi))} z^{\mathrm{Root}(S)} \mathbb{S}_{w_S^\mu}$$

where w_S^μ is the maximal weight vector ranging over $\mathrm{Weights}(\mu(\Xi))$ for any vertex of S. We claim that $f_{\mu(P),\mu(\Xi)} = \theta^\star_{w,F}(f_{P,\Xi})$, where w, F are the weight vector and factor polynomial of $\mu \in \Xi$ respectively. Indeed, mutating $f_{P,\Xi}$ we see that

$$\theta^\star_{w,F}(f_{P,\Xi}) := \sum_{S\in\mathrm{Struts}(P,\Xi)} z^{\mathrm{Root}(\mu(S))} \mathbb{S}_{w_S^\mu}$$

where $\mu(S)$ is the polytope obtained applying the mutation to S. We will show that there is an identification $\mathrm{Struts}(P,\Xi) \to \mathrm{Struts}(\mu(P),\mu(\Xi))$ which preserves the roots of the struts.

The polytope $\mu(S) = y + \mathrm{Newt}(\mathbb{S}_{\mathbf{a}})$ for some $y, \mathbf{a}$ and so has the general form of a strut and it is easy to check it has the correct dilation factors (given by the maximal weight vector for y defined over weights in $\mu(\Xi)$) and is contained in $\mu(P)$. Thus, if $\mu(S)$ contains a vertex v' of $\mu(P)$ then $\mu(S)$ appears as a strut in the scaffolding of $\mu(P)$, namely as the strut $S_{v'} \subset \mu(P)$. By construction the root vertex is unchanged by the mutation.

If $\mu(S)$ does not meet a vertex of $\mu(P)$ we can delete terms corresponding to $\mu(S)$ in the Laurent polynomial $\theta^\star_{w,F}(f_{P,\Xi})$ and not change the polytope $\mu(P)$. Thus we can form a polynomial g which mutates by $\theta^{-1}_{w,F}$ to a Laurent polynomial with Newton polytope P but without any of terms from the strut S. But this is a contradiction since S meets at least one vertex of P disjoint from all other struts. Thus there must be a natural inclusion of $\mathrm{Struts}(P,\Xi)$ into $\mathrm{Struts}(\mu(P),\mu(\Xi))$ arguing similarly with μ^{-1} we have the other inclusion.

Using $f_{\mu(P),\mu(\Xi)}$ we can form the ray set via $\tilde{f}_{\mu(P),\mu(\Xi)}$. However, note that the rays $\rho_{F,v}$ are identical in both cases, and the struts $S \in \mathrm{Struts}(\mu(P),\mu(\Xi))$ give ray generators $\mathrm{Root}(S) + \sum_F w_v(F) e_F$ which differs from the corresponding generator for a strut $S' \in \mathrm{Struts}(P,\Xi)$ by $w_\mu(v) e_F$ for any $v \in \mathrm{verts}(S')$, thus the rays differ by the linear map T_μ.

We now find $\tilde{Q}_{P,\Xi}$ and apply $T^{\star}_{\mu}$ to obtain $\tilde{Q}_{\mu(P),\mu(\Xi)}$. Note that we find the fan of $X_{\mu(P),\mu(\Xi)}$ by intersecting $\tilde{\Sigma}_{P,\Xi}$ with the sublattice $N_{\mu} \subset \tilde{N}$.

□

Thus we conclude the proof of Theorem 7.0.2.

PROOF OF THEOREM 7.0.2. For each $\mu \in \Xi$ we have the subspace N_{μ} defined in Lemma 7.5.7. For each $F \in \mathrm{Factors}(\Xi)$ write

$$L_F = \mathcal{O}\left(\prod_{v \in \mathrm{verts}(F)} X_{F,v}\right)$$

where $X_{F,v}$ is the Cox co-ordinate corrsponding to the ray $\rho_{F,v}$. Note that this defines a complete intersection if the L_i are Cartier, which we do not investigate here. Note that $N \hookrightarrow \tilde{N}$ corresponds to a complete intersection found by taking one binomial in each L_i since

$$N := \bigcap_{F \in \mathrm{Factors}\,\Xi} e^{\star}_{F}$$

and $e^{\star}_{F}(g_{F,v}) = 1$ for all $v \in \mathrm{verts}(F)$ and $e^{\star}_{F}(g_S) \leq 0$ for all $S \in \mathrm{Struts}(P,\Xi)$. In fact, ranging over $\mu \in \Xi$ and considering the sublattice N_{μ} we see that the monomial in Cox co-ordinates defined by those rays on which $e^{\star}_{F}$ evaulates positively is always identically $\prod_{v \in \mathrm{verts}(F)} X_{F,v}$. Therefore each toric subvariety defined by $N_{\mu} \hookrightarrow N$ is defined by a series of binomials in $\bigoplus_F L_F$. □

Bibliography

[1] M. Akhtar, T. Coates, A. Corti, L. Heuberger, A. Kasprzyk, A. Oneto, A. Petracci, T. Prince, and K. Tveiten. Mirror symmetry and the classification of orbifold del pezzo surfaces. *preprint arXiv:1501.05334*, 2015.

[2] M. Akhtar and A. Kasprzyk. Mutations of fake weighted projective planes. *preprint math.AG/1302.1152*, 2013.

[3] M. Akhtar and A. Kasprzyk. Singularity content. *preprint: math.AG/1401.5458*, 2014.

[4] Mohammad Akhtar, Tom Coates, Sergey Galkin, and Alexander M. Kasprzyk. Minkowski polynomials and mutations. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 8:Paper 094, 17, 2012.

[5] V. A. Alekseev and V. V. Nikulin. Classification of del Pezzo surfaces with log-terminal singularities of index ≤ 2 and involutions on $K3$ surfaces. *Dokl. Akad. Nauk SSSR*, 306(3):525–528, 1989.

[6] Denis Auroux. Mirror symmetry and T-duality in the complement of an anticanonical divisor. *J. Gökova Geom. Topol. GGT*, 1:51–91, 2007.

[7] Denis Auroux. Special Lagrangian fibrations, wall-crossing, and mirror symmetry. In *Surveys in differential geometry. Vol. XIII. Geometry, analysis, and algebraic geometry: forty years of the Journal of Differential Geometry*, volume 13 of *Surv. Differ. Geom.*, pages 1–47. Int. Press, Somerville, MA, 2009.

[8] Denis Auroux, Ludmil Katzarkov, and Dmitri Orlov. Mirror symmetry for del Pezzo surfaces: vanishing cycles and coherent sheaves. *Invent. Math.*, 166(3):537–582, 2006.

[9] Denis Auroux, Ludmil Katzarkov, and Dmitri Orlov. Mirror symmetry for weighted projective planes and their noncommutative deformations. *Ann. of Math. (2)*, 167(3):867–943, 2008.

[10] V. V. Batyrev. Toric Fano threefolds. *Izv. Akad. Nauk SSSR Ser. Mat.*, 45(4):704–717, 927, 1981.

[11] V. V. Batyrev. On the classification of toric Fano 4-folds. *J. Math. Sci. (New York)*, 94(1):1021–1050, 1999. Algebraic geometry, 9.

[12] Victor V. Batyrev. Quantum cohomology rings of toric manifolds. *Astérisque*, (218):9–34, 1993. Journées de Géométrie Algébrique d'Orsay (Orsay, 1992).

[13] Victor V. Batyrev, Ionuţ Ciocan-Fontanine, Bumsig Kim, and Duco van Straten. Conifold transitions and mirror symmetry for Calabi-Yau complete intersections in Grassmannians. *Nuclear Phys. B*, 514(3):640–666, 1998.

[14] I. N. Bernšteĭn, I. M. Gel′fand, and V. A. Ponomarev. Coxeter functors, and Gabriel's theorem. *Uspehi Mat. Nauk*, 28(2(170)):19–33, 1973.

[15] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.*, 23(2):405–468, 2010.

[16] Lev A. Borisov, Linda Chen, and Gregory G. Smith. The orbifold Chow ring of toric Deligne-Mumford stacks. *J. Amer. Math. Soc.*, 18(1):193–215 (electronic), 2005.

[17] Philip Candelas, Xenia C. de la Ossa, Paul S. Green, and Linda Parkes. A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. *Nuclear Phys. B*, 359(1):21–74, 1991.

[18] M. Carl, M. Pumperla, and B. Siebert. A tropical view on landau ginzburg models. *unpublished (preliminary version)*, 2011.

[19] Cheol-Hyun Cho and Yong-Geun Oh. Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds. *Asian J. Math.*, 10(4):773–814, 2006.

[20] T. Coates, A. Kasprzyk, and T. Prince. Four-dimensional Fano toric complete intersections. *Proc. A.*, 471(2175):20140704, 14, 2015.

[21] Tom Coates, Alessio Corti, Sergey Galkin, Vasily Golyshev, and Alexander M. Kasprzyk. Mirror symmetry and Fano manifolds. In *European Congress of Mathematics Kraków, 2–7 July, 2012*, pages 285–300, 2014.

[22] Tom Coates, Alessio Corti, Sergey Galkin, and Alexander M. Kasprzyk. Quantum periods for 3-dimensional Fano manifolds. `arXiv:1303.3288 [math.AG]`, 2013.

[23] Tom Coates, Alessio Corti, Hiroshi Iritani, and Hsian-Hua Tseng. A mirror theorem for toric stacks. `arXiv:1310.4163 [math.AG]`, 2013.

[24] Tom Coates, Alessio Corti, Hiroshi Iritani, and Hsian-Hua Tseng. Some applications of the mirror theorems for toric stacks. `arXiv:1401.2611 [math.AG]`, 2014.

[25] Tom Coates, Sergey Galkin, Alexander M. Kasprzyk, and Andrew Strangeway. Quantum periods for certain four-dimensional Fano manifolds. `arXiv:1406.4891 [math.AG]`, 2014.

[26] Tom Coates, Amin Gholampour, Hiroshi Iritani, Yunfeng Jiang, Paul Johnson, and Cristina Manolache. The quantum Lefschetz hyperplane principle can fail for positive orbifold hypersurfaces. *Math. Res. Lett.*, 19(5):997–1005, 2012.

[27] Tom Coates and Alexander Givental. Quantum Riemann-Roch, Lefschetz and Serre. *Ann. of Math. (2)*, 165(1):15–53, 2007.

[28] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[29] Thomas Delzant. Hamiltoniens périodiques et images convexes de l'application moment. *Bull. Soc. Math. France*, 116(3):315–339, 1988.

[30] Charles F. Doran and Andrew Harder. Toric degenerations and the Laurent polynomials related to Givental's Landau–Ginzburg models. `arXiv:1502.02079 [math.AG]`, 2015.

[31] Tohru Eguchi, Kentaro Hori, and Chuan-Sheng Xiong. Gravitational quantum cohomology. *Internat. J. Modern Phys. A*, 12(9):1743–1782, 1997.

[32] Anna Felikson, Michael Shapiro, and Pavel Tumarkin. Skew-symmetric cluster algebras of finite mutation type. *J. Eur. Math. Soc. (JEMS)*, 14(4):1135–1180, 2012.

[33] V. V. Fock and A. B. Goncharov. Cluster ensembles, quantization and the dilogarithm. II. The intertwiner. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I*, volume 269 of *Progr. Math.*, pages 655–673. Birkhäuser Boston, Inc., Boston, MA, 2009.

[34] Sergey Fomin, Michael Shapiro, and Dylan Thurston. Cluster algebras and triangulated surfaces. I. Cluster complexes. *Acta Math.*, 201(1):83–146, 2008.

[35] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529 (electronic), 2002.

[36] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. II. Finite type classification. *Invent. Math.*, 154(1):63–121, 2003.

[37] K. Fujita and K. Yasutake. Classification of log del Pezzo surfaces of index three. `arXiv:1401.1283 [math.CO]`, 2014.

[38] Takao Fujita. On the structure of polarized manifolds with total deficiency one. I. *J. Math. Soc. Japan*, 32(4):709–725, 1980.

[39] Takao Fujita. On the structure of polarized manifolds with total deficiency one. II. *J. Math. Soc. Japan*, 33(3):415–434, 1981.

[40] Takao Fujita. On the structure of polarized manifolds with total deficiency one. III. *J. Math. Soc. Japan*, 36(1):75–89, 1984.

[41] Takao Fujita. *Classification theories of polarized varieties*, volume 155 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1990.

[42] Kenji Fukaya. Multivalued Morse theory, asymptotic analysis and mirror symmetry. In *Graphs and patterns in mathematics and theoretical physics*, volume 73 of *Proc. Sympos. Pure Math.*, pages 205–278. Amer. Math. Soc., Providence, RI, 2005.

[43] Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. *Lagrangian intersection Floer theory: anomaly and obstruction. Part I*, volume 46 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.

[44] S. Galkin and A. Usnich. Mutations of potentials. *IPMU preprint*, 2011.

[45] Alexander Givental. Homological geometry and mirror symmetry. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 472–480. Birkhäuser, Basel, 1995.

[46] Alexander Givental. Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture. In *Topics in singularity theory*, volume 180 of *Amer. Math. Soc. Transl. Ser. 2*, pages 103–115. Amer. Math. Soc., Providence, RI, 1997.

[47] Alexander Givental. A mirror theorem for toric complete intersections. In *Topological field theory, primitive forms and related topics (Kyoto, 1996)*, volume 160 of *Progr. Math.*, pages 141–175. Birkhäuser Boston, Boston, MA, 1998.

[48] Alexander B. Givental. Equivariant Gromov-Witten invariants. *Internat. Math. Res. Notices*, (13):613–663, 1996.

[49] Vasily Golyshev. Spectra and strains. `arXiv:0801.0432 [math.AG]`, 2008.

[50] Vasily V. Golyshev. Classification problems and mirror duality. In *Surveys in geometry and number theory: reports on contemporary Russian mathematics*, volume 338 of *London Math. Soc. Lecture Note Ser.*, pages 88–121. Cambridge Univ. Press, Cambridge, 2007.

[51] Mark Gross. The Strominger-Yau-Zaslow conjecture: from torus fibrations to degenerations. In *Algebraic geometry—Seattle 2005. Part 1*, volume 80 of *Proc. Sympos. Pure Math.*, pages 149–192. Amer. Math. Soc., Providence, RI, 2009.

[52] Mark Gross. Mirror symmetry for $\mathbb{P}^2$ and tropical geometry. *Adv. Math.*, 224(1):169–245, 2010.

[53] Mark Gross. *Tropical geometry and mirror symmetry*, volume 114 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2011.

[54] Mark Gross, Paul Hacking, and Sean Keel. Birational geometry of cluster algebras. *Algebr. Geom.*, 2(2):137–175, 2015.

[55] Mark Gross, Paul Hacking, and Sean Keel. Mirror Symmetry for log Calabi-Yau surfaces I. *Publications mathematiques de l'IHES*, 2015.

[56] Mark Gross, Paul Hacking, Sean Keel, and Maxim Kontsevich. Canonical bases for cluster algebras. `arXiv:1411.1394 [math.AG]`, 2014.

[57] Mark Gross, Rahul Pandharipande, and Bernd Siebert. The tropical vertex. *Duke Math. J.*, 153(2):297–362, 2010.

[58] Mark Gross and Bernd Siebert. Mirror symmetry via logarithmic degeneration data. I. *J. Differential Geom.*, 72(2):169–338, 2006.

[59] Mark Gross and Bernd Siebert. From real affine geometry to complex geometry. *Ann. of Math. (2)*, 174(3):1301–1428, 2011.

[60] Mark Gross and Bernd Siebert. An invitation to toric degenerations. In *Surveys in differential geometry. Volume XVI. Geometry of special holonomy and related topics*, volume 16 of *Surv. Differ. Geom.*, pages 43–78. Int. Press, Somerville, MA, 2011.

[61] Paul Hacking and Yuri Prokhorov. Smoothable del Pezzo surfaces with quotient singularities. *Compos. Math.*, 146(1):169–192, 2010.

[62] K. Hori and C. Vafa. Mirror symmetry. `arXiv:hep-th/0002222`, 2000.

[63] Kentaro Hori, Sheldon Katz, Albrecht Klemm, Rahul Pandharipande, Richard Thomas, Cumrun Vafa, Ravi Vakil, and Eric Zaslow. *Mirror symmetry*, volume 1 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2003. With a preface by Vafa.

[64] Nathan Owen Ilten. Mutations of Laurent polynomials and flat families with toric fibers. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 8:Paper 047, 7, 2012.

[65] Nathan Owen Ilten, Jacob Lewis, and Victor Przyjalkowski. Toric degenerations of Fano threefolds giving weak Landau-Ginzburg models. *J. Algebra*, 374:104–121, 2013.

[66] V. A. Iskovskih. Fano threefolds. I. *Izv. Akad. Nauk SSSR Ser. Mat.*, 41(3):516–562, 717, 1977.

[67] V. A. Iskovskih. Anticanonical models of three-dimensional algebraic varieties. In *Current problems in mathematics, Vol. 12 (Russian)*, pages 59–157, 239 (loose errata). VINITI, Moscow, 1979.

[68] V. A. Iskovskikh and Yu. G. Prokhorov. Fano varieties. In *Algebraic geometry, V*, volume 47 of *Encyclopaedia Math. Sci.*, pages 1–247. Springer, Berlin, 1999.

[69] Dominic Joyce. Singularities of special Lagrangian fibrations and the SYZ conjecture. *Comm. Anal. Geom.*, 11(5):859–907, 2003.

[70] A. Kasprzyk, B. Nill, and T. Prince. Minimality and mutation-equivalence of polygons. `arXiv:1501.05335 [math.AG]`, 2015.

[71] Alexander M. Kasprzyk and Ketil Tveiten. Maximally mutable laurent polynomials. In preparation, 2015.

[72] Ludmil Katzarkov and Victor Przyjalkowski. Landau-Ginzburg models—old and new. In *Proceedings of the Gökova Geometry-Topology Conference 2011*, pages 97–124. Int. Press, Somerville, MA, 2012.

[73] Shoshichi Kobayashi and Takushiro Ochiai. Characterizations of complex projective spaces and hyperquadrics. *J. Math. Kyoto Univ.*, 13:31–47, 1973.

[74] J. Kollár and N. I. Shepherd-Barron. Threefolds and deformations of surface singularities. *Invent. Math.*, 91(2):299–338, 1988.

[75] Yános Kollár. Higher-dimensional Fano varieties of large index. *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, (3):31–34, 80–81, 1981.

[76] M. Kontsevich. Lectures at ens. 1998.

[77] Maxim Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 120–139. Birkhäuser, Basel, 1995.

[78] Maxim Kontsevich and Yan Soibelman. Homological mirror symmetry and torus fibrations. In *Symplectic geometry and mirror symmetry (Seoul, 2000)*, pages 203–263. World Sci. Publ., River Edge, NJ, 2001.

[79] Maxim Kontsevich and Yan Soibelman. Affine structures and non-Archimedean analytic spaces. In *The unity of mathematics*, volume 244 of *Progr. Math.*, pages 321–385. Birkhäuser Boston, Boston, MA, 2006.

[80] Maximilian Kreuzer and Benjamin Nill. Classification of toric Fano 5-folds. *Adv. Geom.*, 9(1):85–97, 2009.

[81] Naichung Conan Leung and Margaret Symington. Almost toric symplectic four-manifolds. *J. Symplectic Geom.*, 8(2):143–187, 2010.

[82] Bong H. Lian, Kefeng Liu, and Shing-Tung Yau. Mirror principle. I [MR1621573 (99e:14062)]. In *Surveys in differential geometry: differential geometry inspired by string theory*, volume 5 of *Surv. Differ. Geom.*, pages 405–454. Int. Press, Boston, MA, 1999.

[83] Bong H. Lian, Kefeng Liu, and Shing-Tung Yau. Mirror principle. II [MR1701925 (2001a:14057)]. In *Surveys in differential geometry: differential geometry inspired by string theory*, volume 5 of *Surv. Differ. Geom.*, pages 455–509. Int. Press, Boston, MA, 1999.

[84] Bong H. Lian, Kefeng Liu, and Shing-Tung Yau. Mirror principle. III. *Asian J. Math.*, 3(4):771–800, 1999.

[85] Bong H. Lian, Kefeng Liu, and Shing-Tung Yau. Mirror principle. IV. In *Surveys in differential geometry*, Surv. Differ. Geom., VII, pages 475–496. Int. Press, Somerville, MA, 2000.

[86] Travis Mandel. Classification of rank 2 cluster varieties. `arXiv:1407.6241 [math.AG]`, 2014.

[87] Robert C. McLean. Deformations of calibrated submanifolds. *Comm. Anal. Geom.*, 6(4):705–747, 1998.

[88] Grigory Mikhalkin. Counting curves via lattice paths in polygons. *C. R. Math. Acad. Sci. Paris*, 336(8):629–634, 2003.

[89] Grigory Mikhalkin. Enumerative tropical algebraic geometry in $\mathbb{R}^2$. *J. Amer. Math. Soc.*, 18(2):313–377, 2005.

[90] Takeo Nishinou. Disk counting on toric varieties via tropical curves. *Amer. J. Math.*, 134(6):1423–1472, 2012.

[91] Takeo Nishinou and Bernd Siebert. Toric degenerations of toric varieties and tropical curves. *Duke Math. J.*, 135(1):1–51, 2006.

[92] Mikkel Øbro. An algorithm for the classification of smooth Fano polytopes. `arXiv:0704.0049 [math.CO]`, 2007.

[93] V. V. Przhiyalkovskiĭ. Weak Landau-Ginzburg models of smooth Fano threefolds. *Izv. Ross. Akad. Nauk Ser. Mat.*, 77(4):135–160, 2013.

[94] V. Przyjalkowski and C. Shramov. Laurent phenomenon for Landau–Ginzburg models of complete intersections in grassmannians of planes. `arXiv:1409.3729 [math.AG]`, 2015.

[95] Victor Przyjalkowski. On Landau-Ginzburg models for Fano varieties. *Commun. Number Theory Phys.*, 1(4):713–728, 2007.

[96] Victor Przyjalkowski. Hori-Vafa mirror models for complete intersections in weighted projective spaces and weak Landau-Ginzburg models. *Cent. Eur. J. Math.*, 9(5):972–977, 2011.

[97] Miles Reid. Update on 3-folds. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 513–524. Higher Ed. Press, Beijing, 2002.

[98] Hiroshi Sato. Toward the classification of higher-dimensional toric Fano varieties. *Tohoku Math. J. (2)*, 52(3):383–413, 2000.

[99] Paul Seidel. More about vanishing cycles and mutation. In *Symplectic geometry and mirror symmetry (Seoul, 2000)*, pages 429–465. World Sci. Publ., River Edge, NJ, 2001.

[100] Paul Seidel. Vanishing cycles and mutation. In *European Congress of Mathematics, Vol. II (Barcelona, 2000)*, volume 202 of *Progr. Math.*, pages 65–85. Birkhäuser, Basel, 2001.

[101] Paul Seidel. *Fukaya categories and Picard-Lefschetz theory*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[102] Maria Ezia Serpico. Fano varieties of dimensions $n \geq 4$ and of index $r \geq n-1$. *Rend. Sem. Mat. Univ. Padova*, 62:295–308, 1980.

[103] Nick Sheridan. On the Fukaya category of a Fano hypersurface in projective space. `arXiv:1306.4143 [math.AG]`, 2014.

[104] V. V. Shokurov. A nonvanishing theorem. *Izv. Akad. Nauk SSSR Ser. Mat.*, 49(3):635–651, 1985.

[105] Andrew Strominger, Shing-Tung Yau, and Eric Zaslow. Mirror symmetry is T-duality. *Nuclear Phys. B*, 479(1-2):243–259, 1996.

[106] Margaret Symington. Four dimensions from two in symplectic topology. In *Topology and geometry of manifolds (Athens, GA, 2001)*, volume 71 of *Proc. Sympos. Pure Math.*, pages 153–208. Amer. Math. Soc., Providence, RI, 2003.

[107] Jonathan M. Wahl. Elliptic deformations of minimally elliptic singularities. *Math. Ann.*, 253(3):241–262, 1980.

[108] Keiichi Watanabe and Masayuki Watanabe. The classification of Fano 3-folds with torus embeddings. *Tokyo J. Math.*, 5(1):37–48, 1982.