Black Holes in $D = 4$ Higher-Derivative Gravity

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ABSTRACT

Extensions of Einstein gravity with higher-order derivative terms are natural generalizations of Einstein's theory of gravity. They may arise in string theory and other effective theories, as well as being of interest in their own right. In this paper we study static black-hole solutions in the example of Einstein gravity with additional quadratic curvature terms in four dimensions. A Lichnerowicz-type theorem simplifies the analysis by establishing that they must have vanishing Ricci scalar curvature. By numerical methods we then demonstrate the existence of further black-hole solutions over and above the Schwarzschild solution. We discuss some of their thermodynamic properties, and show that they obey the first law of thermodynamics.

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2015 marks the centennial of Einstein's General Theory of Relativity. In Chinese tradition, it is customary for one to celebrate the 9'th rather than the 10'th anniversary. It is thus appropriate to talk about black holes in this International Conference of Gravitation and Cosmology/the 4'th Galileo-xu Gaungqi Meeting at Beijing in May 2015. The first black hole solution in Einstein's gravity was constructed by Karl Schwarzschild 99 years ago [1]. Since then, there have been tremendous progresses in constructing exact solutions of black holes in diverse dimensions, within the framework of Einstein gravity with or without a cosmological constant. These include the four-dimensional asymptotically flat rotating Kerr black hole [2], its higher dimensional generalizations [3], asymptotically (A)dS rotating black holes in four [4], five [5] and general dimensions [6, 7]. However, there was little progress in the construction of black holes involving higher-derivative terms in four dimensions.

The well-known problem of the non-renormalisability of Einstein gravity has given rise to many attempts to view it as an effective low-energy theory that will receive higher-order corrections that become important as the energy scale increases (see, for example, [8]). In string theory, the Einstein-Hilbert action becomes just the first term in an infinite series of gravitational corrections built from powers of the curvature tensor and its derivatives. In other approaches, the possibility is envisaged that just a finite number of additional terms might be added. For example, as was shown in [9], if one adds all possible quadratic curvature invariants to the usual Einstein-Hilbert action one obtains a renormalisable theory, albeit at the price of introducing ghost-like modes in the theory.

Black holes can be viewed as the most fundamental objects in a theory of gravity, and they provide powerful probes for studying some of the more subtle global aspects of the theory. For this reason, it is of considerable interest to investigate the structure of black-hole solutions in theories of gravity with higher-order curvature terms. In this paper, we report on some investigations of the static, spherically-symmetric black-hole solutions in Einstein-Hilbert gravity with added quadratic curvature terms. Since the Gauss-Bonnet integrand is purely topological in four dimensions, the most general possibilities for additional quadratic curvature terms can be parameterised by adding the square of the Ricci tensor and the square of the Ricci scalar, with arbitrary coefficients. Equivalently, in view of the topological nature of the Gauss-Bonnet combination, we can parameterise the most general action with quadratic curvature in the form

$$I = \int d^4x\sqrt{-g}\left(\gamma R - \alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \beta R^2\right) , \tag{0.1}$$

where α, β and γ are constants and $C_{\mu\nu\rho\sigma}$ is the Weyl tensor. We shall work in units where we set $\gamma=1$, and the equations of motion following from (0.1) are then

$$
\begin{aligned}
& R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} - 4\alpha B_{\mu\nu} + 2\beta R (R_{\mu\nu} - \tfrac{1}{4} R g_{\mu\nu}) \\
& + 2\beta (g_{\mu\nu} \Box R - \nabla_\mu \nabla_\nu R) = 0\,,
\end{aligned} \tag{0.2}
$$

where $B_{\mu\nu} = (\nabla^\rho \nabla^\sigma + \frac{1}{2} R^{\rho\sigma}) C_{\mu\rho\nu\sigma}$ is the Bach tensor, which is tracefree.

In general, the theory describes a system with a massive spin-2 mode with mass-squared $m_2^2 = 1/(2\alpha)$ and a massive spin-0 mode with mass-squared $m_0^2 = 1/(6\beta)$, in addition to the massless spin-2 graviton. These massive modes will be associated with rising and falling Yukawa type behaviour in the metric modes near infinity [10], of the form $\frac{1}{r} e^{\pm m_2 r}$ and $\frac{1}{r} e^{\pm m_0 r}$. In particular, one can expect that if generic initial data is set at some small distance, the rising exponentials will eventually dominate, leading to singular asymptotic behaviour. In seeking black-hole solutions, the question then arises as to whether the rising exponentials can be avoided for appropriately finely-tuned initial data.

It can easily be seen that any solution of pure Einstein gravity will also be a solution of (0.2), and so in particular the usual Schwarzschild black hole continues to be a solution in the higher-order theory. The question we wish to address, then, is whether there exist any other static black hole solutions, over and above the Schwarzschild solution.

Static, spherically-symmetric black-hole solutions have been investigated by Nelson [11], using generalisations of the Lichnerowicz and Israel theorems for Einstein gravity. Since we will arrive at somewhat different conclusions, we shall briefly summarise the key elements in Nelson's discussion, although derived in a different notation. We consider static metrics of the form

$$
ds_4^2 = -\lambda^2\, dt^2 + d\bar{s}_3^2\,, \qquad d\bar{s}_3^2 = h_{ij}\, dx^i dx^j\,, \tag{0.3}
$$

where λ and h_{ij} are functions only of the three spatial coordinates x^i. We shall shortly perform a Kaluza-Klein type reduction on the time coordinate. First take the trace of the field equations (0.2), obtaining $\beta\,(\Box - m_0^2) R = 0$. Then multiply by λR and integrate over the spatial domain from a putative horizon out to infinity. Expressed in terms of the covariant derivative D_i with respect to the spatial 3-metric h_{ij}, this gives

$$
\int \sqrt{h}\, d^3x \Big[D^i (R D_i R) - \lambda (D_i R)^2 - m_0^2 \lambda R^2 \Big] = 0\,. \tag{0.4}
$$

Since λ vanishes on the horizon, it follows that if $D_i R$ goes to zero sufficiently rapidly at spatial infinity the total derivative (i.e. surface term) gives no contribution, and the non-positivity of the remaining terms then implies $R=0$. In other words, as shown in [11], any static black-hole solution of (0.1) must have vanishing Ricci scalar: $R=0$. This leads to a great simplification, and it means that one can, without loss of generality, study the case of pure Einstein-Weyl gravity (i.e. (0.1) with $\beta = 0$), since obviously the term quadratic in R makes no contribution to the field equations for a configuration with $R = 0$. Furthermore, the trace of the field equations (0.2) for Einstein-Weyl gravity immediately implies $R = 0$. In fact, the two differential equations for h and f are both now of only second order in derivatives.

The second stage in Nelson's discussion then involved looking at the remaining content of (0.2), i.e. the non-trace part. According to [11], this led to another integral identity that then implied, under certain assumptions, that $R_{\mu\nu} = 0$. If this were correct, then the conclusion would be that the usual Schwarzschild solution was the only static black hole solution of the theory described by (0.1). However, we find that there are sign errors in the expression given in [11]. Setting $R = 0$, as already argued above, multiplying (0.2) by $\lambda R^{\mu\nu}$, and then integrating over the spatial region outside the horizon gives

$$\begin{aligned}
&\int \sqrt{h} d^3x \Big[D^i W_i - \tfrac{1}{4}\lambda (D_i \bar{R} - 4 D^j R_{ij})^2 + 4\lambda (D^j R_{ij})^2 \\
&-4\lambda (D_{[i} R_{j]k})^2 + \lambda (D_i R_{jk})^2 - \tfrac{1}{4}\lambda \bar{R}^2 (m_2^2 + \bar{R}) \\
&-\lambda (m_2^2 R^{ij} R_{ij} - 2 R^{ij} R_{jk} R^k{}_i) \Big] = 0\,,
\end{aligned} \tag{0.5}$$

where $W_i = \lambda R^{jk} D_i R_{jk} + \frac{1}{4}\lambda \bar{R} D_i \bar{R} - 2\lambda R^{jk} D_j R_{ik} - \lambda \bar{R} D_j R_i{}^j$, and $\bar{R}$ is the Ricci scalar of the spatial metric h_{ij}. Although the surface term will give zero, the mix of positive and negative signs in the bulk terms precludes one from obtaining any kind of vanishing theorem for the Ricci tensor of the four-dimensional metric. This raises the intriguing possibility that there might in fact exist static, spherically symmetric black-hole solutions over and above the Schwarzschild solution.

The equations of motion following from (0.1) are too complicated to be able to solve explicitly, even for the case of the static, spherically-symmetric ansatz

$$ds^2 = -h(r) dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{0.6}$$

In our work, we have therefore carried out a numerical investigation of the solutions. To do this, we begin by supposing that there exists a black-hole horizon at some radius $r = r_0 > 0$, at which the metric functions h and f vanish, and we then obtain near-horizon Taylor expansions for $h(r)$ and $f(r)$, of the form

$$\begin{aligned}
h(r) &= c \Big[(r - r_0) + h_2 (r - r_0)^2 + h_3 (r - r_0)^3 + \cdots \Big]\,, \\
f(r) &= f_1 (r - r_0) + f_2 (r - r_0)^2 + f_3 (r - r_0)^3 + \cdots
\end{aligned} \tag{0.7}$$

Substituting into the equations of motion (0.2), with β set to zero for the reasons discussed above, the coefficients h_i and f_i for $i \geq 2$ can be solved for in terms of the two non-trivial free parameters r_0 and f_1. There is also a "trivial" parameter, corresponding to the freedom to rescale the time coordinate, which we have accordingly written in the form of an overall scaling of $h(r)$. Thus we have

$$h_2 = \frac{1-2f_1\, r_0}{f_1\, r_0^2} + \frac{1-f_1\, r_0}{8\alpha f_1^2\, r_0}\,, \quad f_2 = \frac{1-2f_1\, r_0}{r_0^2} - \frac{3(1-f_1\, r_0)}{8\alpha f_1\, r_0}\,,$$

and so on. (We used Taylor expansions to $\mathcal{O}((r-r_0)^9)$ in our numerical integrations.) The Schwarzschild solution corresponds to $f_1 = 1/r_0$, and so it is convenient to parameterise f_1 as

$$f_1 = \frac{1+\delta}{r_0}\,, \tag{0.8}$$

with non-vanishing δ characterising the extent to which the near-horizon solution deviates from Schwarzschild.

We use the expansions to set initial data at a radius r_i just outside the horizon, and then use numerical routines in Mathematica to integrate the equations out to large radius. Generically, one finds that for a given choice of the parameters r_0 and δ the solution rapidly becomes singular as one integrates outwards from $r = r_i$, as expected in view of our earlier observations about the rising Yukawa terms in the asymptotic form for the metric. If we fix a particular value for r_0, we can then use the shooting method to try to home in on a special value of δ for which the outward integration can proceed without encountering a singularity. Of course in practice, because of accuracy limitations in the integrations, the solution will always eventually become singular at large enough r. The signal for a good black-hole solution is that it should be possible to see $f(r)$ approaching very close to 1 as r increases, with $h(r)$ approaching a constant also, and that by stepping up the accuracy and precision goals in the calculations one can extend at will the maximum upper limit $r = r_f$ for which the smooth behaviour can be achieved. In practice, by running the routines with accuracy and precision goals of order 20 decimal places, we have been able to obtain very clean and trustworthy solutions out to at least 60 times the horizon radius.

Our findings are that there exists a range of values for the horizon radius, bounded below by a certain multiple of the length $\sqrt{\alpha}$, for which we can obtain precisely one static black-hole solution in addition to the Schwarzschild solution. In order to make the statement of our results in the most concise possible way, it is convenient, without loss of mathematical generality, to make a specific choice for the value of α in (0.1). We shall take

$$\alpha = \frac{1}{2}\,. \tag{0.9}$$

We then find that for each choice of $r_0 > r_0^{\rm min}$, where

$$r_0^{\rm min} \approx 0.876\,, \qquad (0.10)$$

we can find a non-Schwarzschild static black hole. For each such r_0, there is a corresponding value $\delta = \delta^*$ of the "non-Schwarzschild parameter" that yields the non-singular black-hole solution. As r_0 is taken closer and closer to the value $r_0^{\rm min}$, the required value δ^* becomes smaller and smaller, tending to zero at $r_0 = r_0^{\rm min}$. Thus the Schwarzschild and the non-Schwarzschild black holes "coalesce" as $r_0 = r_0^{\rm min}$ is approached.

As r_0 is increased above $r_0^{\rm min}$, the Schwarzschild and non-Schwarzschild black-hole solutions separate more from one another (and in particular the required value of δ increases). The mass of the Schwarzschild black hole is simply $\frac{1}{2}r_0$, and thus it increases linearly as r_0 increases. By constrast, the mass of the non-Schwarzschild black hole decreases as r_0 increases, until at $r_0 = r_0^{\rm m=0}$ it becomes massless, where

$$r_0^{\rm m=0} \approx 1.143\,. \qquad (0.11)$$

(The definition of mass in higher-derivative theories was discussed in detail in [12, 13]. For asymptotically-flat black holes it is just $\frac{1}{2}$ the coefficient of $1/r$ in g_{tt} (assuming t is normalised canonically at infinity).) Interestingly, if r_0 is increased beyond $r_0^{\rm m=0}$, one can still obtain a non-Schwarzschild black-hole solution for an appropriate choice of δ, but now the mass is actually negative. In other words there is still a regular horizon, and the metric is asymptotically flat at large distances, but the metric function f now rises above 1 as r increases from r_i, before sinking down to 1 again in the asymptotic region. Figure 1 shows the masses and Hawking temperatures of the Schwarzschild and non-Schwarzschild black holes as a function of r_0. We can then compare how the masses are related to the temperature of these two black holes. The maximum possible mass for the non-Schwarzschild black hole, attained when $r_0 = r_0^{\rm min}$, is given by $M^{\rm max} = \frac{1}{2}r_0^{\rm min} \approx 0.438$.

The plots of the metric functions f and h for the examples of a positive-mass black hole $r_0 = 1$ and a negative-mass black hole $r_0 = 2$ are shown in Figure 2.

Having established the existence of the non-Schwarzschild black holes, it is instructive to study some of their thermodynamic properties, and to compare these with the properties of the Schwarzschild black holes. In order to do this, we have collected the numerical results for a sequence of black-hole solutions with r_0 in the range $r_0^{\rm min} \approx 0.876 < r_0 < 1.5$, and then fitted the data to appropriate polynomials. Because we are working with a higher-derivative theory, the entropy is not simply given by one quarter of the area of the event horizon, and

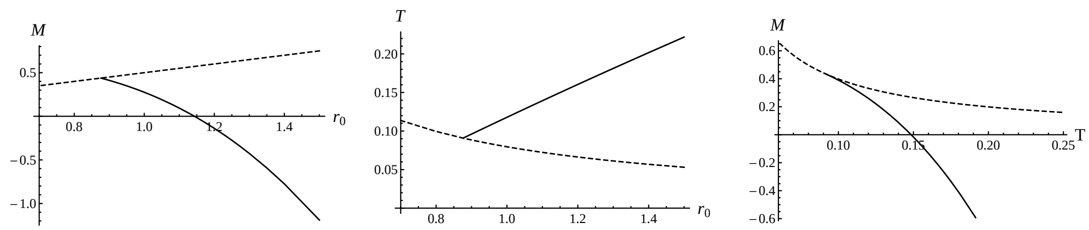

Figure 1: *The masses (left plot) and temperatures (middle plot) of the Schwarzschild (dashed line) and non-Schwarzschild (solid line) black holes as a function of the horizon radius r_0. The right plot shows the masses of the two black holes as a function of the temperature.*

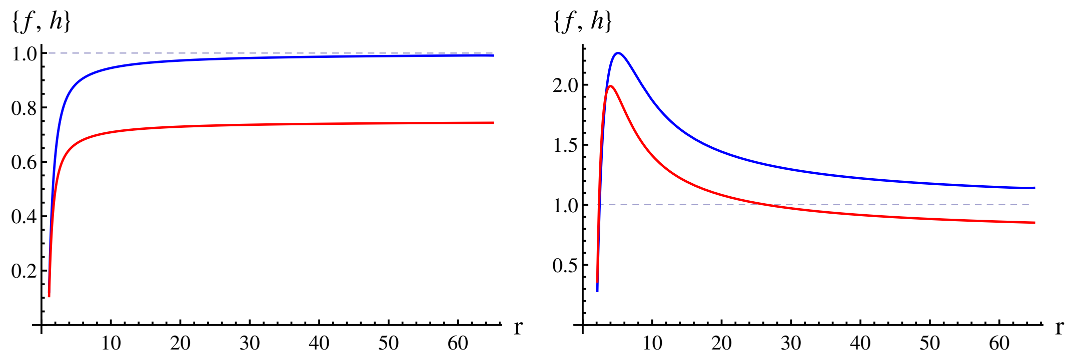

Figure 2: *The non-Schwarzschild black hole for $r_0 = 1$ (left plot) and $r_0 = 2$ (right plot). In each plot the upper curve is $f(r)$ and the lower curve is $h(r)$. For clarity we have chosen a rescaling of h so that it approaches $\frac{3}{4}$, rather than 1, to avoid an asymptotic overlap of the curves.*

instead we need to use the formula derived by Wald [14, 15]. This has been evaluated for the ansatz (0.6) in quadratic curvature gravities in [16], and applied to our case with $\beta = 0$ and $\gamma = 1$ in (0.1) this gives $S = \pi r_0^2 + 4\pi\alpha(1 - f_1\, r_0) = \pi r_0^2 - 4\pi\alpha\delta^*$. (There is a freedom to add a constant multiple of the Gauss-Bonnet invariant to the Lagrangian, which shifts the entropy by a (parameter-independent) constant without affecting the equations of motion. We have used this to ensure the entropy of the Schwarzschild black hole vanishes when the mass vanishes.) We then find that the mass and the temperature of these non-Schwarzschild black holes, as a function of the entropy, take the form

$$
\begin{aligned}
M &\approx 0.168 + 0.131\, S - 0.00749\, S^2 - 0.000139\, S^3 + \cdots\,, \\
T &\approx 0.131 - 0.0151\, S - 0.000428\, S^2 + \cdots\,. \qquad (0.12)
\end{aligned}
$$

It can be seen that $\partial M/\partial S \approx 0.131 - 0.0150\, S - 0.000417\, S^2$, which is very close to the expression for the temperature. Thus the non-Schwarzschild black holes are seen to obey the first law $dM = TdS$ to quite a high precision. Note that the expressions for M and T as a function of S for the Schwarzschild black holes are very different in form, with $M = (S/4\pi)^{1/2}$ and $T = \frac{1}{4}(\pi S)^{-1/2}$.

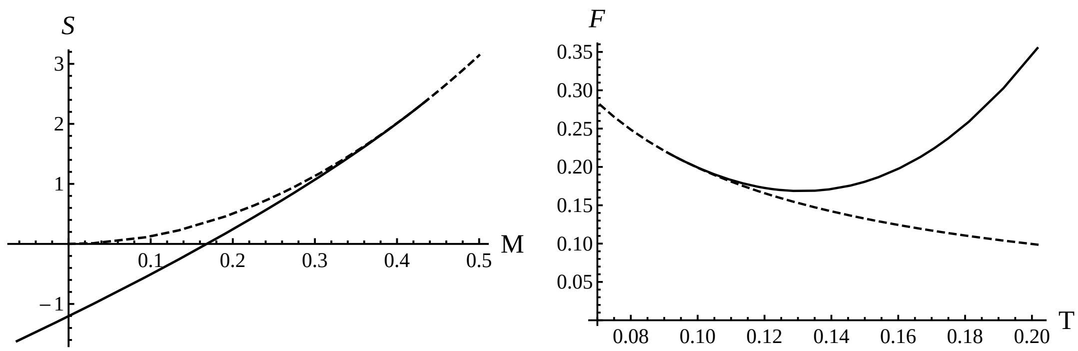

Figure 3: *The first plot shows the entropy as a function of mass, and the second shows the free energy $F = M - TS$ as a function of T, for the Schwarzschild (dashed line) and non-Schwarzschild (solid line) black holes.*

It is interesting to note that the entropy of the non-Schwarzschild black hole of a given mass is always less than the entropy of the Schwarzschild black hole of the same mass. The two entropies approach each other asymptotically as r_0 approaches $r_0^{\rm min}$. This can be seen in the left-hand plot in Figure 3. It is also of interest to look at the free energy $F = M - TS$ as a function of temperature. This is shown in the right-hand plot in Figure 3. It can be seen that the free energy is always larger for the non-Schwarzschild black hole at a given temperature, with the two curves again meeting at the lower limit when $r_0 = r_0^{\rm min}$.

In this paper, we have used black holes to probe some of the consequences of interpreting the action (0.1) as complete classical action in its own right. We have seen that there exists a second branch of static, spherically symmetric black holes, over and above the Schwarzschild solutions. These are not Ricci flat, although they do have vanishing Ricci scalar. Restoring the factors of α and γ that we fixed in our numerical simulations, the second branch of black holes have can masses, which can become negative, bounded approximately by $M \le 0.438\sqrt{2\alpha\gamma}$. Thus in a regime where α is small, which one might hope would correspond to a small correction to Einstein gravity, the second branch of black holes will be tiny, and will actually have very large curvature near the horizon, thus tending to invalidate the requirement that the curvature-squared should be small. In particular, the fact that their mass can be negative, violating the usual positive-mass theorem of standard Einstein gravity, is a reflection of the fact that the ghost-like pathologies of the quadratically-corrected action are becoming dominant in this regime. Thus it could be viewed as a satisfactory outcome of our investigation that the only indications of the existence of black holes with potentially pathological properties in the quadratic-curvature theories occur in a regime where yet higher-order corrections, as in string theory, are going to be important also. It would be interesting to obtain analytical proofs of the existence of the numerical solutions we have found. Although this could be challenging, it might perhaps be easier to obtain restricted no hair theorems that confirmed the apparent absence of non-Schwarzschild black holes outside the parameter range where we have found them.

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