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Van Duong DINH

Strichartz estimates and the nonlinear Schrödinger-type equations

JURY

Valeria BANICA	Professeur d'Université	Examinateur
Jean Marc BOUCLET	Professeur d'Université	Directeur de thèse
Rémi CARLES	Directeur de Researche	Rapporteur
Stefan LE COZ	Maître de Conférences	Examinateur
Mihai MARIŞ	Professeur d'Université	Examinateur
Nikolay TZVETKOV	Professeur d'Université	Rapporteur

École doctorale et spécialité :
MITT : Domaine Mathématiques : Mathématiques appliquées
Unité de Recherche :
Institut de Mathématiques de Toulouse (UMR 5219)
Directeur de Thèse :
Jean Marc BOUCLET
Rapporteurs :
Rémi CARLES et **Nikolay TZVETKOV**

To my wife, Uyen Cong.

Contents

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Publications

Published papers:

1. V. D. Dinh, Strichartz estimates for the fractional Schrödinger and wave equations on compact manifolds without boundary, **J. Differential Equations 263 (2017)**, No. 12, 8804–8837.
2. V. D. Dinh, On the focusing mass-critical nonlinear fourth-order Schrödinger equation below the energy space, **Dyn. Partial Differ. Equ. 14 (2017)**, No. 3, 295–320.
3. V. D. Dinh, Global well-posedness for a L^2-critical nonlinear higher-order Schrödinger equation, **J. Math. Anal. Appl. 458 (2018)**, No. 1, 174–192.
4. V. D. Dinh, On the Cauchy problem for the nonlinear semirelativistic equation in Sobolev spaces, **Discrete Contin. Dyn. Syst. 38 (2018)**, No. 3, 1127–1143.
5. V. D. Dinh, On well-posedness, regularity and ill-posedness for the nonlinear fourth-order Schrödinger equation, **Bull. Belg. Math. Soc. Simon Stevin 25 (2018)**, No. 3, 415–437.
6. V. D. Dinh, Global existence and scattering for a class of nonlinear fourth-order Schrödinger equation below the energy space, **Nonlinear Anal. 172 (2018)**, 115–140.
7. V. D. Dinh, Blowup of H^1 solutions for a class of the focusing inhomogeneous nonlinear Schrodinger equation, **Nonlinear Anal. 174 (2018)**, 169–188.
8. V. D. Dinh, On blowup solutions to the focusing mass-critical nonlinear fractional Schrodinger equation, **Commun. Pure Appl. Anal. 18 (2019)**, No.2, 689–708.
9. V. D. Dinh, Global existence and blowup for a class of the focusing nonlinear Schrödinger equation with inverse-square potential, **J. Math. Anal. Appl. 468 (2018)**, No. 1, 270–303.
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12. V. D. Dinh, Global Strichartz estimates for the fractional Schrodinger equations on asymptotically Euclidean manifolds, **J. Funct. Anal. 275 (2018)**, No. 8, 1943–2014.
13. V. D. Dinh, Well-posedness of nonlinear fractional Schrodinger and wave equations in Sobolev spaces, **Int. J. Appl. Math. 31 (2018)**, No. 4, 483–525.
14. A. Bensouilah, V. D. Dinh and S. Zhu, On stability and instability of standing waves for the nonlinear Schrodinger equation with inverse-square potential, **J. Math. Phys. 59 (2018)**, 101505.
15. V. D. Dinh, Energy scattering for a class of the defocusing inhomogeneous nonlinear Schrödinger equation, **J. Evol. Equ**, to appear, 2019.

Preprints:

1. V. D. Dinh, Scattering theory in a weighted L^2 space for a class of the defocusing inhomogeneous nonlinear Schrödinger equation, preprint arXiv:1710.01392.
2. V. D. Dinh, On instability of radial standing waves for the nonlinear Schrödinger equation with inverse-square potential, preprint, arXiv:1806.01068.
3. V. D. Dinh, On instability of standing waves for the mass-supercritical fractional nonlinear Schrödinger equation, preprint, arXiv:1806.08935.

Unpublished papers:

1. V. D. Dinh, Global existence for the defocusing mass-critical nonlinear fourth-order Schrödinger equation below the energy space, arXiv:1706.06517.

Publications related to this thesis

The work presented in this thesis is based on the following articles:

1. V. D. Dinh, Well-posedness of nonlinear fractional Schrödinger and wave equations in Sobolev spaces, **Int. J. Appl. Math. 31 (2018)**, No. 4, 483–525.
2. V. D. Dinh, Strichartz estimates for the fractional Schrödinger and wave equations on compact manifolds without boundary, **J. Differential Equations 263 (2017)**, No. 12, 8804–8837.
3. V. D. Dinh, On the focusing mass-critical nonlinear fourth-order Schrödinger equation below the energy space, **Dyn. Partial Differ. Equ. 14 (2017)**, No. 3, 295–320.
4. V. D. Dinh, Global well-posedness for a L^2-critical nonlinear higher-order Schrödinger equation, **J. Math. Anal. Appl. 458 (2018)**, No. 1, 174–192.
5. V. D. Dinh, On the Cauchy problem for the nonlinear semirelativistic equation in Sobolev spaces, **Discrete Contin. Dyn. Syst. 38 (2018)**, No. 3, 1127–1143.
6. V. D. Dinh, On well-posedness, regularity and ill-posedness for the nonlinear fourth-order Schrödinger equation, **Bull. Belg. Math. Soc. Simon Stevin 25 (2018)**, No. 3, 415–437.
7. V. D. Dinh, Global Strichartz estimates for the fractional Schrödinger equations on asymptotically Euclidean manifolds, **J. Funct. Anal. 275 (2018)**, No. 8, 1943–2014.

Résumé

Cette thèse est consacrée à l'étude des aspects linéaires et non-linéaires des équations de type Schrödinger

$$i\partial_t u + |\nabla|^\sigma u = F, \quad |\nabla| = \sqrt{-\Delta}, \quad \sigma \in (0,\infty).$$

Quand $\sigma = 2$, il s'agit de l'équation de Schrödinger bien connue dans de nombreux contextes physiques tels que la mécanique quantique, l'optique non-linéaire, la théorie des champs quantiques et la théorie de Hartree-Fock. Quand $\sigma \in (0,2)\backslash\{1\}$, c'est l'équation Schrödinger fractionnaire, qui a été découverte par Laskin (voir par exemple [Las00] et [Las02]) en lien avec l'extension de l'intégrale de Feynman, des chemins quantiques de type brownien à ceux de Lévy. Cette équation apparaît également dans des modèles de vagues (voir par exemple [IP14] et [Ngu16]). Quand $\sigma = 1$, c'est l'équation des demi-ondes qui apparaît dans des modèles de vagues (voir [IP14]) et dans l'effondrement gravitationnel (voir [ES07], [FL07]). Quand $\sigma = 4$, c'est l'équation Schrödinger du quatrième ordre ou biharmonique introduite par Karpman [Kar96] et par Karpman-Shagalov [KS00] pour prendre en compte le rôle de la dispersion du quatrième ordre dans la propagation d'un faisceau laser intense dans un milieu massif avec non-linéarité de Kerr.

Cette thèse est divisée en deux parties. La première partie étudie les estimations de Strichartz pour des équations de type Schrödinger sur des variétés comprenant l'espace plat euclidien, les variétés compactes sans bord et les variétés asymptotiquement euclidiennes. Ces estimations de Strichartz sont utiles pour l'étude de l'équations dispersives non-linéaire à régularité basse. La seconde partie concerne l'étude des aspects non-linéaires tels que les caractères localement puis globalement bien posés sous l'espace d'énergie, ainsi que l'explosion de solutions peu régulières pour des équations non-linéaires de type Schrödinger.

Dans le Chapitre 1, nous discutons des estimations de Strichartz pour les équations de type Schrödinger avec $\sigma \in (0,\infty)$ sur l'espace euclidien $\mathbb{R}^d$.

Dans le Chapitre 2, nous prouvons des estimations de Strichartz pour les équations de type Schrödinger avec $\sigma \in (0,\infty)\backslash\{1\}$ sur $\mathbb{R}^d$ équipé d'une métrique lisse bornée g.

Au Chapitre 3, nous utilisons les estimations de Strichartz prouvées au Chapitre 2 pour montrer les estimations de Strichartz pour les équations de type Schrödinger avec $\sigma \in (0,\infty)\backslash\{1\}$ sur les variétés compactes sans bord.

Au Chapitre 4, nous montrons des estimations de Strichartz globales pour les équations de type Schrödinger avec $\sigma \in (0,\infty)\backslash\{1\}$ sur les variétés asymptotiquement euclidiennes sous la condition de non-capture.

Dans le Chapitre 5, nous utilisons les estimations de Strichartz données au Chapitre 1 (entre autres) pour étudier le caractère localement bien posé des équations non-linéaires de type Schrödinger avec la non-linéarité de type puissance et $\sigma \in (0,\infty)$ posées sur $\mathbb{R}^d$.

Dans le Chapitre 6, nous étudions le le caractère globalement bien posé de l'équation de Schrödinger non-linéaire du quatrième ordre $\sigma = 4$ défocalisante et L^2 critique, en considérant séparément deux cas $d = 4$ et $d \geq 5$ qui correspondent respectivement à la non-linéarité algébrique et non-algébrique.

Dans le Chapitre 7, nous étudions l'explosion des solutions peu régulières de l'équation de Schrödinger non-linéaire du quatrième ordre focalisante L^2 critique. Comme au Chapitre 6, nous considérons aussi séparément deux cas $d = 4$ et $d \geq 5$.

Mots-clés: Equations non-linéaires de type Schrödinger; Estimations de Strichartz; problème localement bien posé; prolème globalement bien posé; explosion; méthode-I; Estimations bilinéaires de Strichartz; Inégalités d'interaction de Morawetz; Variétés compactes sans bord ; Variétés asymptotiquement euclidiennes.

MSC2010: 35A01, 35A17, 35B44, 35B45, 35E15, 35G20, 35G25, 35Q55.

Abstract

This dissertation is devoted to the study of linear and nonlinear aspects of the Schrödinger-type equations

$$i\partial_t u + |\nabla|^\sigma u = F, \quad |\nabla| = \sqrt{-\Delta}, \quad \sigma \in (0, \infty).$$

When $\sigma = 2$, it is the well-known Schrödinger equation arising in many physical contexts such as quantum mechanics, nonlinear optics, quantum field theory and Hartree-Fock theory. When $\sigma \in (0,2)\backslash\{1\}$, it is the fractional Schrödinger equation, which was discovered by Laskin (see e.g. [Las00] and [Las02]) owing to the extension of the Feynman path integral, from the Brownian-like to Lévy-like quantum mechanical paths. This equation also appears in the water waves model (see e.g. [IP14] and [Ngu16]). When $\sigma = 1$, it is the half-wave equation which arises in water waves model (see [IP14]) and in gravitational collapse (see [ES07], [FL07]). When $\sigma = 4$, it is the fourth-order or biharmonic Schrödinger equation introduced by Karpman [Kar96] and by Karpman-Shagalov [KS00] taking into account the role of small fourth-order dispersion term in the propagation of intense laser beam in a bulk medium with Kerr nonlinearity.

This thesis is divided into two parts. The first part studies Strichartz estimates for Schrödinger-type equations on manifolds including the flat Euclidean space, compact manifolds without boundary and asymptotically Euclidean manifolds. These Strichartz estimates are known to be useful in the study of nonlinear dispersive equation at low regularity. The second part concerns the study of nonlinear aspects such as local well-posedness, global well-posedness below the energy space and blowup of rough solutions for nonlinear Schrödinger-type equations.

In Chapter 1, we discuss Strichartz estimates for Schrödinger-type equations with $\sigma \in (0,\infty)$ on the Euclidean space $\mathbb{R}^d$.

In Chapter 2, we derive Strichartz estimates for Schrödinger-type equations with $\sigma \in (0,\infty)\backslash\{1\}$ on $\mathbb{R}^d$ equipped with a smooth bounded metric g.

In Chapter 3, we make use of Strichartz estimates proved in Chapter 2 to show Strichartz estimates for Schrödinger-type equations with $\sigma \in (0,\infty)\backslash\{1\}$ on compact manifolds without boundary.

In Chapter 4, we prove global in time Strichartz estimates for Schrödinger-type equations with $\sigma \in (0,\infty)\backslash\{1\}$ on asymptotically Euclidean manifolds under the non-trapping condition.

In Chapter 5, we use Strichartz estimates given in Chapter 1 (among other things) to study the local well-posedness of the power-type nonlinear Schrödinger-type equations with $\sigma \in (0,\infty)$ posed on $\mathbb{R}^d$.

In Chapter 6, we study the global well-posedness for the defocusing mass-critical nonlinear fourth-order Schrödinger equation $\sigma = 4$ below the energy space. We will consider separately two cases $d = 4$ and $d \geq 5$ which respectively correspond to the algebraic and non-algebraic nonlinearity.

In Chapter 7, we study the blowup of rough solutions to the focusing mass-critical nonlinear fourth-order Schrödinger equation. As in Chapter 6, we also consider separately two cases $d = 4$ and $d \geq 5$.

Keywords: Nonlinear Schrödinger-type equations; Strichartz estimates; local well-posedness; global well-posedness; blowup; I-method; bilinear Strichartz estimates; Interaction Morawetz inequalities; compact manifolds without boundary; asymptotically Euclidean manifolds.

MSC2010: 35A01, 35A17, 35B44, 35B45, 35E15, 35G20, 35G25, 35Q55.

Introduction

This thesis is devoted to the study of Schrödinger-type equations such as the fractional Schrödinger (including the well-known Schrödinger equation and the fourth-order Schrödinger equation) and the half-wave equations.

The first part of this thesis is devoted to Strichartz estimates for Schrödinger-type equations on manifolds including the flat Euclidean space, compact manifolds without boundary and asymptotically Euclidean manifolds. These Strichartz estimates are known to be useful in the study of nonlinear dispersive equation at low regularity. Let us first discuss Strichartz estimates for Schrödinger-type equations on the flat Euclidean space $\mathbb{R}^d$. Consider

$$i\partial_t u + |\nabla|^\sigma u = 0, \quad u(0) = \psi, \quad |\nabla| = \sqrt{-\Delta}, \quad \sigma \in (0, \infty).$$

In the case $\sigma = 2$, it is well-known that one can compute explicitly the solution to this equation, that is

$$e^{-it\Delta}\psi(x) = \frac{e^{\pm i\frac{\pi d}{4}}}{|4\pi t|^{\frac{d}{2}}} \int e^{-i\frac{|x-y|^2}{4t}} \psi(y) dy, \quad \pm := \text{ sign of } t.$$

This implies the dispersive estimate

$$\|e^{-it\Delta}\psi\|_{L^\infty} \lesssim |t|^{-d/2}, \quad t \in \mathbb{R}.$$

Using this dispersive estimate and the isometry $\|e^{-it\Delta}\psi\|_{L^2} = \|\psi\|_{L^2}$, the so-called $TT^\star$-criterion (see [KT98]) shows Strichartz estimates

$$\|e^{-it\Delta}\psi\|_{L^p(\mathbb{R}, L^q)} \lesssim \|\psi\|_{L^2},$$

for any sharp Schrödinger admissible pair (p, q), i.e.

$$(p, q) \in [2, \infty]^2, \quad (p, q, d) \neq (2, \infty, 2), \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2}. \tag{0.0.1}$$

When $\sigma \neq 2$, since we do not have the explicit formula of the solution $e^{it|\nabla|^\sigma}\psi(x)$, the above method does not work. Fortunately, since the equation enjoys a scaling invariance in frequency space, we are able to use the scaling technique to derive Strichartz estimates. More precisely, we decompose the solution in dyadic pieces, namely

$$e^{it|\nabla|^\sigma}\psi \sim \sum_{N \in 2^{\mathbb{Z}}} e^{it|\nabla|^\sigma} P_N \psi,$$

where P_N is a Fourier multiplier by $\chi_N(\xi) = \chi(N^{-1}\xi)$ with $\chi \in C_0^\infty(\mathbb{R}^d)$ and $\operatorname{supp}(\chi) \subset \{\xi \in \mathbb{R}^d : 1/2 \leq |\xi| \leq 2\}$. By a change of variables, we find that

$$[e^{it|\nabla|^\sigma} P_N \psi](t, x) = [e^{it|\nabla|^\sigma} P_1 \psi_N](N^\sigma t, Nx),$$

where $\psi_N(x) := \psi(N^{-1}x)$. This implies

$$\|e^{it|\nabla|^\sigma} P_N \psi\|_{L^p(\mathbb{R}, L^q)} = N^{-\frac{d}{q} - \frac{\sigma}{p}} \|e^{it|\nabla|^\sigma} P_1 \psi_N\|_{L^p(\mathbb{R}, L^q)},$$
$$\|P_1 \psi_N\|_{L^2} = N^{\frac{d}{2}} \|P_N \psi\|_{L^2}.$$

The problem is then reduced to show

$$\|e^{it|\nabla|^\sigma} P_1 \psi\|_{L^p(\mathbb{R}, L^q)} \lesssim \|P_1 \psi\|_{L^2}.$$

By the $TT^\star$-criterion, it suffices to prove the following energy and dispersive estimates

$$\|e^{it|\nabla|^\sigma} P_1\|_{L^2\to L^2} \lesssim 1,$$
$$\|e^{it|\nabla|^\sigma} P_1\|_{L^1\to L^\infty} \lesssim (1+|t|)^{-\upsilon}.$$

By the stationary phase theorem, we learn that

$$\upsilon = \begin{cases} \frac{d}{2} & \text{for } d\geq 1, \sigma\neq 1, \\ \frac{d-1}{2} & \text{for } d \geq 2, \sigma = 1. \end{cases} \tag{0.0.2}$$

This shows that

$$\|e^{it|\nabla|^\sigma} P_N\psi\|_{L^p(\mathbb{R},L^q)} \lesssim N^{\gamma_{p,q}}\|P_N\psi\|_{L^2}, \tag{0.0.3}$$

where

$$\gamma_{p,q} := \frac{d}{2}-\frac{d}{q}-\frac{\sigma}{p},$$

and (p,q) satisfies for $d\geq 1$ and $\sigma\neq 1$,

$$(p,q)\in[2,\infty]^2, \quad (p,q,d)\neq(2,\infty,2), \quad \frac{2}{p}+\frac{d}{q}\leq\frac{d}{2}, \tag{0.0.4}$$

and for $d\geq 2$ and $\sigma = 1$,

$$(p,q)\in[2,\infty]^2, \quad (p,q,d)\neq(2,\infty,3), \quad \frac{2}{p}+\frac{d-1}{q}\leq\frac{d-1}{2}. \tag{0.0.5}$$

We combine the Littlewood-Paley theorem, the Minkowski inequality and (0.0.3) to obtain Strichartz estimates

$$\|e^{it|\nabla|^\sigma}\psi\|_{L^p(\mathbb{R},L^q)} \lesssim \Big(\sum_{N\in 2^{\mathbb{Z}}} N^{2\gamma_{p,q}}\|P_N\psi\|^2_{L^2}\Big)^{1/2} \sim \|\psi\|_{\dot{H}^{\gamma_{p,q}}},$$

where (p,q) satisfies either (0.0.4) or (0.0.5) with $q\in[2,\infty)$. We refer to Chapter 1 for more general Strichartz estimates and its variants.

In Chapter 2, we extend Strichartz estimates studied in Chapter 1 by considering the same equations with the Laplacian operator of variable coefficients. More precisely, we consider

$$i\partial_t u + |\nabla_g|^\sigma u = 0, \quad u(0)=\psi, \quad |\nabla_g| = \sqrt{-\Delta_g}, \quad \sigma\in(0,\infty)\backslash\{1\},$$

on $\mathbb{R}^d$ equipped with a smooth bounded metric g. Let $g(x) = (g_{jk}(x))^d_{j,k=1}$ and denote $g^{-1}(x) = (g^{jk}(x))^d_{j,k=1}$. The Laplace-Beltrami operator associated to g reads

$$\Delta_g = \sum_{j,k=1}^d |g(x)|^{-1}\partial_j(g^{jk}(x)|g(x)|\partial_k),$$

where $|g(x)| := \sqrt{\det g(x)}$. We make the following assumptions:

- (Ellipticity) There exists $C>0$ such that for all $x,\xi\in\mathbb{R}^d$,

$$C^{-1}|\xi|^2 \leq p(x,\xi) := \sum_{j,k=1}^d g^{jk}(x)\xi_j\xi_k \leq C|\xi|^2. \tag{0.0.6}$$

- (Boundedness) For all $\alpha \in \mathbb{N}^d$, there exists $C_\alpha > 0$ such that for all $x \in \mathbb{R}^d$,

$$|\partial^\alpha g^{jk}(x)| \leq C_\alpha, \quad j,k \in \{1,...,d\}. \tag{0.0.7}$$

To study Strichartz estimates, we decompose the solution into dyadic pieces as follows

$$e^{it|\nabla_g|^\sigma}\psi \sim e^{it|\nabla_g|^\sigma}\varphi_0(-\Delta_g)\psi + \sum_{h^{-2}\in 2^{\mathbb{N}}} e^{it|\nabla_g|^\sigma}\varphi(-h^2\Delta_g)\psi,$$

where $\varphi_0 \in C_0^\infty(\mathbb{R})$ and $\varphi \in C_0^\infty(\mathbb{R}\backslash\{0\})$. Here $f(-\Delta_g)$ is the functional calculus is defined by spectral theorem. Since we are interested in local in time Strichartz estimates, we do not need to decompose the solution at low frequencies, i.e. terms of the form $e^{it|\nabla_g|^\sigma}\varphi(-\epsilon^{-2}\Delta_g), \epsilon^{-2} \in 2^{\mathbb{N}}$. Indeed, the low frequency part can be bounded easily using the Sobolev embedding. In the context of variable coefficients, there is no scaling technique as on $\mathbb{R}^d$. We thus need to estimate separately each localized piece. The main goal is to establish dispersive estimates for semi-classical operators $e^{ith^{-1}(h|\nabla_g|)^\sigma}\varphi(-h^2\Delta_g)$ on some small time interval independent of h, namely

$$\|e^{ith^{-1}(h|\nabla_g|)^\sigma}\varphi(-h^2\Delta_g)\|_{L^1\to L^\infty} \lesssim h^{-d}(1+|t|h^{-1})^{-\frac{d}{2}}, \quad t \in [-t_0,t_0], \tag{0.0.8}$$

for some $t_0 > 0$. Here the implicit constant does not depend on the parameter $h \in (0,1]$. With this dispersive estimate, the semi-classical version of $TT^\star$-criterion implies Strichartz estimates for each semi-classical terms $e^{ith^{-1}(h|\nabla_g|)^\sigma}\varphi(-h^2\Delta_g)\psi$. Rescaling in time and summing over all dyadic pieces, we derive Strichartz estimates for the solution. To study the dispersive estimate (0.0.8), we first use the semi-classical expansion of $\varphi(-h^2\Delta_g)$, namely

$$\varphi(-h^2\Delta_g) = \sum_{j=0}^{N-1} h^j Op_h(a_j) + h^N R_N(h),$$

where

$$Op_h(a_j)\psi(x) = (2\pi h)^{-d}\iint e^{ih^{-1}(x-y)\cdot\xi}a_j(x,\xi)\psi(y)dyd\xi, \tag{0.0.9}$$

for some $a_j \in S(-\infty)$ with $\text{supp}(a_j) \subset p^{-1}(\text{supp}(\varphi))$ and $R_N(h)$ satisfying for all $m \geq 0$,

$$\|R_N(h)\|_{H^{-m}\to H^m} \lesssim h^{-2m}.$$

By the Sobolev embedding, the remainder term is bounded by

$$\|e^{ith^{-1}(h|\nabla_g|)^\sigma}h^N R_N(h)\|_{L^1\to L^\infty} \lesssim h^N\|e^{ith^{-1}(h|\nabla_g|)^\sigma}R_N(h)\|_{H^{-m}\to H^m} \lesssim h^{N-2m}.$$

Taking N sufficiently large, we obtain dispersive estimate for the remainder term. Therefore, the study of (0.0.8) is reduced to the study of dispersive estimate for $e^{ith^{-1}(h|\nabla_g|)^\sigma}Op_h(a)$ with $a \in S(-\infty)$ and $\text{supp}(a) \subset p^{-1}(\text{supp}(\varphi))$. To do so, we make use of the WKB method to construct an approximation to $w(t) = e^{ith^{-1}(h|\nabla_g|)^\sigma}Op_h(a)\psi$ of the form

$$w(t) = J_N(t)\psi + R_N(t)\psi, \quad t \in [-t_0,t_0],$$

for some $t_0 > 0$, where

$$J_N(t) := \sum_{j=0}^{N-1} h^j J_h(S(t),a_j(t)), \quad J_N(0) = Op_h(a),$$

with

$$J_h(S(t),a_j(t))\psi(x) = (2\pi h)^{-d}\iint e^{ih^{-1}(S(t,x,\xi)-y\cdot\xi)}a_j(t,x,\xi)\psi(y)dyd\xi,$$

and the remainder term satisfies a "nice" estimate, for instance, $R_N(t) = O_{L^2\to L^2}(h^{N-1})$ uniformly with respect to $t \in [-t_0, t_0]$. We observe from the fundamental theorem of calculus that

$$\begin{aligned} e^{-ith^{-1}(h|\nabla_g|)^\sigma} J_N(t)\psi &= J_N(0)\psi + \int_0^t \frac{d}{ds}\Big(e^{-ish^{-1}(h|\nabla_g|)^\sigma} J_N(s)\Big)\psi ds \\ &= Op_h(a)\psi + ih^{-1}\int_0^t e^{-ish^{-1}(h|\nabla_g|)^\sigma}(hD_s - (h|\nabla_g|)^\sigma)J_N(s)\psi ds, \end{aligned}$$

where $D_s = i^{-1}\partial_s$. This implies

$$w(t) = e^{ith^{-1}(h|\nabla_g|)^\sigma} Op_h(a)\psi = J_N(t)\psi - ih^{-1}\int_0^t e^{i(t-s)h^{-1}(h|\nabla_g|)^\sigma}(hD_s - (h|\nabla_g|)^\sigma)J_N(s)\psi ds.$$

We want the last term to have a small contribution. To do this, we need to study the action of $hD_s - (h|\nabla_g|)^\sigma$ on $J_N(s)$. The first action of hD_s on $J_N(s)$ is easy to compute, and we have

$$hD_s \circ J_N(s) = \sum_{l=0}^N h^l J_h(S(s), b_l(s)),$$

where

$$\begin{aligned} b_0(s) &= \partial_s S(s) a_0(s), \\ b_l(s) &= \partial_s S(s) a_l(s) + D_s a_{l-1}(s), \quad l = 1, ..., N-1, \\ b_N(s) &= D_s a_{N-1}(s). \end{aligned}$$

The second action of $(h|\nabla_g|)^\sigma$ on $J_N(s)$ is complicated. In the case $\sigma = 2$, we have an explicit form of $-h^2\Delta_g$, that is,

$$-h^2\Delta_g = Op_h(p) + hOp_h(p_1), \tag{0.0.10}$$

where p is as in (0.0.6) and $p_1(x,\xi) = \sum_{l=1}^d n_l(x)\xi_l$. A direct computation shows

$$\begin{aligned} Op_h(p) \circ J_h(S(s), q) &= J_h\Big(S(s), p(x, \nabla_x S(s))q + ih\nabla_\xi p(x, \nabla_x S(s)) \cdot \nabla_x q + ihOp(p)S(s)q \\ &\qquad + h^2 Op(p)q\Big), \\ hOp_h(p_1) \circ J_h(S(s), q) &= J_h\Big(S(s), ihOp(p_1)S(s)q + h^2 Op(p_1)q\Big). \end{aligned}$$

Here we use the notation $Op(a) = Op_1(a)$, i.e. $h = 1$ in (0.0.9). This shows

$$-h^2\Delta_g \circ J_h(S(s), q) = J_h\Big(S(s), E(s)q + ihT(s)q - h^2\Delta_g q\Big),$$

where

$$\begin{aligned} E(s)q &= p(x, \nabla_x S(s))q, \\ T(s)q &= -\nabla_\xi p(x, \nabla_x S(s)) \cdot \nabla_x q - \Delta_g S(s)q. \end{aligned}$$

Hence the action of $-h^2\Delta_g$ on $J_N(s)$ can be computed explicitly as

$$-h^2\Delta_g \circ J_N(s) = \sum_{l=0}^{N+1} h^l J_h(S(s), c_l(s)),$$

where

$$\begin{aligned}
c_0(s) &= E(s)a_0(s),\\
c_1(s) &= E(s)a_1(s) + iT(s)a_0(s),\\
c_l(s) &= E(s)a_l(s) + iT(s)a_{l-1}(s) - \Delta_g a_{l-2}(s), \quad l = 2, \cdots, N-1,\\
c_N(s) &= iT(s)a_{N-1}(s) - \Delta_g a_{N-2}(s),\\
c_{N+1}(s) &= -\Delta_g a_{N-1}(s).
\end{aligned}$$

We thus get

$$(hD_s + h^2\Delta_g)J_N(s) = \sum_{k=0}^{N+1} h^k J_h(S(s), d_k(s)),$$

with

$$\begin{aligned}
d_0(s) &= (\partial_s S(s) - E(s))a_0(s),\\
d_1(s) &= (\partial_s S(s) - E(s))a_1(s) + (D_s - iT(s))a_0(s),\\
d_k(s) &= (\partial_s S(s) - E(s))a_k(s) + (D_s - iT(s))a_{k-1}(s) + \Delta_g a_{k-2}(s), \quad k = 2, \cdots, N-1,\\
d_N(s) &= (D_s - iT(s))a_{N-1}(s) + \Delta_g a_{N-2}(s),\\
d_{N+1}(s) &= \Delta_g a_{N-1}(s).
\end{aligned}$$

Therefore, in order to make $(hD_s + h^2\Delta_g)J_N(s)$ to have a small contribution, we need to study the "Eikonal" or Hamilton-Jacobi equation

$$\partial_s S(s) - p(x, \nabla_x S(s)) = 0, \quad S(0) = x \cdot \xi,$$

and transport equations

$$\begin{aligned}
&D_s a_0(s) - iT(s)a_0(s) = 0,\\
&D_s a_k(s) - iT(s)a_k(s) = -\Delta_g a_{k-1}(s), \quad k = 1, \cdots, N-1,
\end{aligned}$$

with initial data

$$a_0(0) = a(x, \xi), \quad a_k(0) = 0, \quad k = 1, \cdots, N-1.$$

When $\sigma \ne 2$, we do not have an explicit formula for $(h|\nabla_g|)^\sigma$, thus the above calculation does not hold. However, we can overcome this difficulty by means of pseudo-differential calculus as follows. Thanks to the support of φ, we replace $e^{ith^{-1}(h|\nabla_g|)^\sigma}Op_h(a)$ by $e^{ith^{-1}\omega(-h^2\Delta_g)}Op_h(a)$ with $\omega(\lambda) = \tilde{\varphi}(\lambda)\sqrt{\lambda}^\sigma$, where $\tilde{\varphi} \in C_0^\infty(\mathbb{R}\backslash\{0\})$ satisfying $\tilde{\varphi} = 1$ on the support of φ. The interest of this replacement is that we can write $\omega(-h^2\Delta_g)$ in terms of semi-classical pseudo-differential operators, namely

$$\omega(-h^2\Delta_g) = \sum_{k=0}^{N-1} h^k Op_h(q_k) + h^N R_N(h), \tag{0.0.11}$$

where $q_k \in S(-\infty)$ satisfies $q_0(x,\xi) = \omega \circ p(x,\xi)$ and $\text{supp}(q_k) \subset p^{-1}(\text{supp}(\omega))$ and $R_N(h)$ is bounded in L^2 uniformly in $h \in (0,1]$. As above if we set $w(t) = e^{ith^{-1}\omega(-h^2\Delta_g)}Op_h(a)\psi$, then we have

$$w(t) = J_N(t)\psi - ih^{-1}\int_0^t e^{i(t-s)h^{-1}\omega(-h^2\Delta_g)}(hD_s - \omega(-h^2\Delta_g))J_N(s)\psi ds.$$

We need to study the action of $\omega(-h^2\Delta_g)$ on $J_N(s)$. To do so, we use the action of pseudo-differential operators on Fourier integral operators, namely

$$Op_h(b) \circ J_h(S,c) = \sum_{j=1}^{N-1} h^j J_h(S,(b \triangleleft c)_j) + h^N J_h(S, r_N(h)),$$

where $(b \triangleleft c)_j$ is a universal linear combination of

$$\partial_\xi^\beta b(x, \nabla_x S(x,\xi)) \partial_x^{\beta-\alpha} c(x,\xi) \partial_x^{\alpha_1} S(x,\xi) \cdots \partial_x^{\alpha_k} S(x,\xi),$$

with $\alpha \leq \beta, \alpha_1 + \cdots + \alpha_k = \alpha$ and $|\alpha_l| \geq 2$ for all $l = 1, \cdots, k$ and $|\beta| = j$. In particular,

$$\begin{aligned} (b \triangleleft c)_0(x,\xi) &= b(x, \nabla_x S(x,\xi)) c(x,\xi), \\ i(b \triangleleft c)_1(x,\xi) &= \nabla_\xi b(x, \nabla_x S(x,\xi)) \cdot \nabla_x c(x,\xi) + \frac{1}{2} \mathrm{tr}(\nabla_\xi^2 b(x, \nabla_x S(x,\xi)) \cdot \nabla_x^2 S(x,\xi)) c(x,\xi). \end{aligned}$$

This combined with (0.0.11) yield

$$\begin{aligned} \omega(-h^2\Delta_g) \circ J_N(t) &= \sum_{k=0}^{N-1} h^k Op_h(q_k) \circ \sum_{j=0}^{N-1} h^j J_h(S(t), a_j(t)) + h^N R_N(h) J_N(t) \\ &= \sum_{k+j+l=0}^{N} h^{k+j+l} J_h(S(t), (q_k \triangleleft a_j(t))_l) + h^{N+1} J_h(S(t), r_{N+1}(h,t)) + h^N R_N(h) J_N(t). \end{aligned}$$

This implies that

$$(hD_t - \omega(-h^2\Delta_g)) J_N(t) = \sum_{r=0}^{N} h^r J_h(S(t), c_r(t)) - h^N R_N(h) J_N(t) - h^{N+1} J_h(S(t), r_{N+1}(h,t)),$$

where

$$\begin{aligned} c_0(t) &= (\partial_t S(t) - q_0(x, \nabla_x S(t))) a_0(t), \\ c_r(t) &= (\partial_t S(t) - q_0(x, \nabla_x S(t))) a_r(t) + D_t a_{r-1}(t) - (q_0 \triangleleft a_{r-1}(t))_1 - (q_1 \triangleleft a_{r-1}(t))_0 \\ &\quad - \sum_{\substack{k+j+l=r \\ j \leq r-2}} (q_k \triangleleft a_j(t))_l, \quad r = 1, \cdots, N-1, \\ c_N(t) &= D_t a_{N-1}(t) - (q_0 \triangleleft a_{N-1}(t))_1 - (q_1 \triangleleft a_{N-1}(t))_0 - \sum_{\substack{k+j+l=N \\ j \leq N-2}} (q_k \triangleleft a_j(t))_l. \end{aligned}$$

The system of equations $c_r(t) = 0$ for $r = 0, \cdots, N$ leads to the following "eikonal" or Hamilton-Jacobi equation

$$\partial_t S(t) - q_0(x, \nabla_x S(t)) = 0, \quad S(0) = x \cdot \xi,$$

and transport equations

$$\begin{aligned} &D_t a_0(t) - (q_0 \triangleleft a_0(t))_1 - (q_1 \triangleleft a_0(t))_0 = 0, \\ &D_t a_r(t) - (q_0 \triangleleft a_r(t))_1 - (q_1 \triangleleft a_r(t))_0 = \sum_{\substack{k+j+l=r+1 \\ j \leq r-1}} (q_k \triangleleft a_j(t))_l, \quad r = 1, \cdots, N-1, \end{aligned}$$

with initial data

$$a_0(0) = a, \quad a_r(0) = 0, \quad r = 1, \cdots, N-1.$$

After solving the Hamilton-Jacobi equation and these transport equations on the time interval

$[-t_0, t_0]$ for some $t_0 > 0$, we show the L^2-boundedness of the remainder term

$$\|R_N(t)\|_{L^2 \to L^2} \lesssim h^{N-1},$$

for all $t \in [-t_0, t_0]$ and all $h \in (0, 1]$. We also have dispersive estimates for the main term

$$\|J_N(t)\|_{L^1 \to L^\infty} \lesssim h^{-d}(1 + |t|h^{-1})^{-\frac{d}{2}},$$

for all $t \in [-t_0, t_0]$ and all $h \in (0, 1]$. We thus obtain dispersive estimates for semi-classical Schrödinger-type operators

$$\|e^{ith^{-1}(h|\nabla_g|)^\sigma} \varphi(-h^2\Delta_g)\psi\|_{L^\infty} \lesssim h^{-d}(1 + |t|h^{-1})^{-\frac{d}{2}} \|\psi\|_{L^1},$$

for all $t \in [-t_0, t_0]$ and all $h \in (0, 1]$. These estimates together with energy estimates and the $TT^\star$-criterion yield

$$\|e^{ith^{-1}(h|\nabla_g|)^\sigma} \varphi(-h^2\Delta_g)\psi\|_{L^p([-t_0,t_0],L^q)} \lesssim h^{-\left(\frac{d}{2}-\frac{d}{q}-\frac{1}{p}\right)} \|\psi\|_{L^2}.$$

By scaling in time, we obtain

$$\|e^{it|\nabla_g|^\sigma} \varphi(-h^2\Delta_g)\psi\|_{L^p(h^{\sigma-1}[-t_0,t_0],L^q)} \lesssim h^{-\gamma_{p,q}} \|\psi\|_{L^2}.$$

In the case $\sigma \in (0, 1)$, we obviously bound estimates on a finite time interval I by estimates on intervals of size $h^{\sigma-1}$ and obtain the following local in time Strichartz estimates

$$\|e^{it|\nabla_g|^\sigma} \varphi(-h^2\Delta_g)\psi\|_{L^p(I,L^q)} \lesssim h^{-\gamma_{p,q}} \|\psi\|_{L^2}.$$

In the case $\sigma \in (1, \infty)$, we cumulate $O(h^{1-\sigma})$ estimates on intervals of size $h^{\sigma-1}$ to get estimates on a finite interval I and obtain

$$\|e^{it|\nabla_g|^\sigma} \varphi(-h^2\Delta_g)\psi\|_{L^p(I,L^q)} \lesssim h^{-\gamma_{p,q}-\frac{\sigma-1}{p}} \|\psi\|_{L^2}.$$

Moreover, we can replace the norm $\|\psi\|_{L^2}$ in the right hand side of above Strichartz estimates by $\|\varphi(-h^2\Delta_g)\psi\|_{L^2}$. By the Littlewood-Paley decomposition and the almost orthogonality, we obtain Strichartz estimates for Schrödinger-type equations

$$\sigma \in (1, \infty), \quad \|e^{it|\nabla_g|^\sigma}\psi\|_{L^p(I,L^q)} \lesssim \|\psi\|_{H^{\gamma_{p,q}+\frac{\sigma-1}{p}}},$$

and

$$\sigma \in (0, 1), \quad \|e^{it|\nabla_g|^\sigma}\psi\|_{L^p(I,L^q)} \lesssim \|\psi\|_{H^{\gamma_{p,q}}}.$$

We see that in the case $\sigma \in (1, \infty)$, there is a loss of $\frac{\sigma-1}{p}$ derivatives compared to those on $\mathbb{R}^d$.

In Chapter 3, we use Strichartz estimates obtained in Chapter 2 to show Strichartz estimates for Schrödinger-type equations on compact manifolds without boundary. More precisely, we consider

$$i\partial_t u + |\nabla_g|^\sigma u = 0, \quad u(0) = \psi, \quad |\nabla_g| = \sqrt{-\Delta_g}, \quad \sigma \in (0, \infty)\backslash\{1\},$$

on compact manifolds without boundary (M, g), where Δ_g is the Laplace-Beltrami operator on (M, g). In the case $\sigma = 2$, Burq-Gérard-Tzvetkov established in [BGT04] Strichartz estimates with a loss of $1/p$ derivatives, i.e.

$$\|e^{-it\Delta_g}\psi\|_{L^p(I,L^q(M))} \lesssim \|\psi\|_{H^{1/p}(M)},$$

where (p, q) is sharp Schrödinger admissible with $q < \infty$ (see (0.0.1)). In the case $\sigma \neq 2$, we use the Littlewood-Paley decomposition (see e.g. [BGT04, Corollary 2.3]), that is for $q \in [2,\infty)$,

$$\|v\|_{L^q(M)} \lesssim \|v\|_{L^2(M)} + \Big(\sum_{h^{-2}\in 2^{\mathbb{N}}} \|\varphi(-h^2\Delta_g)v\|^2_{L^q(M)}\Big)^{1/2}$$

and the Minkowski inequality to have for any finite time interval I,

$$\|v\|_{L^p(I,L^q(M))} \lesssim \|v\|_{L^p(I,L^2(M))} + \Big(\sum_{h^{-2}\in 2^{\mathbb{N}}} \|\varphi(-h^2\Delta_g)v\|^2_{L^p(I,L^q(M))}\Big)^{1/2}.$$

Applying this estimate together with the L^2 isometry of the Schrödinger-type operator, we have

$$\|e^{it|\nabla_g|^\sigma}\psi\|_{L^p(I,L^q(M))} \lesssim \|\psi\|_{L^2(M)} + \Big(\sum_{h^{-2}\in 2^{\mathbb{N}}} \|e^{it|\nabla_g|^\sigma}\varphi(-h^2\Delta_g)\psi\|^2_{L^p(I,L^q(M))}\Big)^{1/2}.$$

The problem is then reduced to showing local in time Strichartz estimates for the localized Schrödinger-type operator $e^{it|\nabla_g|^\sigma}\varphi(-h^2\Delta_g)\psi$, hence for $e^{ith^{-1}(h|\nabla_g|)^\sigma}\varphi(-h^2\Delta_g)\psi$, namely

$$\|e^{ith^{-1}(h|\nabla_g|)^\sigma}\varphi(-h^2\Delta_g)\psi\|_{L^p([-t_0,t_0],L^q(M))} \lesssim \|\psi\|_{L^2(M)},$$

for some $t_0 > 0$ independent of $h \in (0,1]$. To do so, it suffices to show dispersive estimates

$$\|e^{ith^{-1}(h|\nabla_g|)^\sigma}\varphi(-h^2\Delta_g)\psi\|_{L^\infty(M)} \lesssim h^{-d}(1+|t|h^{-1})^{-\frac{d}{2}}\|\psi\|_{L^1(M)}, \tag{0.0.12}$$

for all $t \in [-t_0,t_0]$ and all $h \in (0,1]$. Thanks to the localization φ, we can replace $(h|\nabla_g|)^\sigma$ by $\omega(-h^2\Delta_g)$ where $\omega(\lambda) = \tilde{\varphi}(\lambda)\sqrt{\lambda}^\sigma$ with $\tilde{\varphi} \in C_0^\infty(\mathbb{R}\backslash\{0\})$ such that $\tilde{\varphi} = 1$ on $\text{supp}(\varphi)$. The partition of unity allows us to consider only on a local coordinates $(U_\kappa, V_\kappa, \kappa)_\kappa$, i.e. $\sum_k e^{ith^{-1}\omega(-h^2\Delta_g)}$ $\varphi(-h^2\Delta_g)\phi_\kappa$, where $\phi_\kappa \in C_0^\infty(U_\kappa)$ and $1 = \sum_\kappa \phi_\kappa$. By the functional calculus, we can express $\varphi(-h^2\Delta_g)\phi_\kappa$ in terms of semi-classical pseudo-differential operators, namely

$$\varphi(-h^2\Delta_g)\phi_\kappa = \sum_{j=0}^{N-1} h^j \tilde{\phi}_\kappa Op_h^\kappa(a_{\kappa,j})\phi_\kappa + h^N R_{\kappa,N}(h),$$

where $\tilde{\phi}_\kappa \in C_0^\infty(U_\kappa)$ satisfies $\tilde{\phi}_\kappa = 1$ on $\text{supp}(\phi_\kappa)$, the operators

$$Op_h^\kappa(a_{\kappa,j}) = \kappa^* Op_h(a_{\kappa,j})\kappa_*,$$

with $a_{\kappa,j} \in S(-\infty)$ and $\text{supp}(a_{\kappa,j}) \subset \text{supp}(\varphi \circ p_\kappa)$, and for any $m \geq 0$,

$$\|R_{\kappa,N}(h)\|_{H^{-m}(M)\to H^m(M)} \lesssim h^{-2m}.$$

Here p_κ is the pricipal symbol of $-\Delta_g$ in $(U_\kappa, V_\kappa, \kappa)$ and κ_*, κ^* are the pullback and pushforward operators respectively. Thus, it suffices to show dispersive estimates for $e^{ith^{-1}\omega(-h^2\Delta_g)}\tilde{\phi}_\kappa Op_h^\kappa(a_\kappa)\phi_\kappa$ with $a_\kappa \in S(-\infty)$ and $\text{supp}(a_\kappa) \subset \text{supp}(\varphi \circ p_\kappa)$. If we set $w(t) = e^{ith^{-1}\omega(-h^2\Delta_g)}\tilde{\phi}_\kappa Op_h^\kappa(a_\kappa)\phi_\kappa\psi$, then w solves the semi-classical evolution equation

$$(hD_t - \omega(-h^2\Delta_g))w(t) = 0, \quad w(0) = \tilde{\phi}_\kappa Op_h^\kappa(a_\kappa)\phi_\kappa\psi.$$

The WKB method allows us to construct an approximation of the solution in a finite time interval independent of $h \in (0,1]$. To do so, we first find an operator, denoted by P, globally defined on $\mathbb{R}^d$ of the form

$$P = \sum_{j,k=1}^d g^{jk}(x)\partial_j\partial_k + \sum_{l=1}^d n_l(x)\partial_l,$$

which coincides with $-\Delta_g$ on a large relatively compact subset V_0 of V_κ. For instance, we can take $P = -\chi\Delta_g - (1-\chi)\Delta$, where $\chi \in C_0^\infty(V_\kappa)$ takes values in $[0,1]$ satisfying $\chi = 1$ on V_0. Here Δ is the free Laplacian operator on $\mathbb{R}^d$. The principal symbol of P is

$$p(x,\xi) = \sum_{j,k=1}^d g^{jk}(x)\xi_j\xi_k, \quad g^{jk}(x) = \chi(x)g_\kappa^{jk}(x) + (1-\chi(x))\delta_{jk},$$

where $\sum_{j,k=1}^d g_\kappa^{jk}(x)\xi_j\xi_k$ is the principal symbol of $-\Delta_g$ in $(U_\kappa, V_\kappa, \kappa)$. It is easy to see that $(g^{jk}(x))_{j,k=1}^d$ satisfies (0.0.6 and (0.0.7) and n_l are bounded on $\mathbb{R}^d$ together with all of theirs derivatives. We next write for some $\vartheta_\kappa \in C_0^\infty(U_\kappa)$ satisfying $\vartheta_\kappa = 1$ on $\mathrm{supp}(\tilde{\phi}_\kappa)$,

$$\omega(-h^2\Delta_g)\vartheta_\kappa = \sum_{l=0}^{N-1} h^l \tilde{\vartheta}_\kappa Op_h^\kappa(b_{\kappa,l})\vartheta_\kappa + h^N \mathfrak{R}_{\kappa,N}(h).$$

We thus can apply the WKB approximation given in Chapter 2 to find $t_0 > 0$, a function $S_\kappa \in C^\infty([-t_0,t_0]\times\mathbb{R}^{2d})$ and a sequence $a_{\kappa,j}(t) \in S(-\infty)$ satisfying $\mathrm{supp}(a_{\kappa,j}(t)) \subset p^{-1}(J)$ for some small neighborhood J of $\mathrm{supp}(\varphi)$ not containing the origin uniformly in $t \in [-t_0,t_0]$ such that

$$\left(hD_t - \sum_{l=0}^{N-1} h^l Op_h(b_{\kappa,l})\right) J_{\kappa,N}(t) = R_{\kappa,N}(t), \tag{0.0.13}$$

where

$$J_{\kappa,N}(t) := \sum_{j=0}^{N-1} h^j J_h(S_\kappa(t), a_{\kappa,j}(t)), \quad J_{\kappa,N}(0) = Op_h(a_\kappa),$$

satisfies for all $t \in [-t_0,t_0]$ and all $(x,\xi) \in p^{-1}(J)$,

$$|\partial_x^\alpha \partial_\xi^\beta (S_\kappa(t,x,\xi) - x\cdot\xi)| \le C_{\alpha\beta}|t|, \quad |\alpha+\beta| \ge 1,$$
$$\left|\partial_x^\alpha \partial_\xi^\beta \left(S_\kappa(t,x,\xi) - x\cdot\xi + t\sqrt{p(x,\xi)}^\sigma\right)\right| \le C_{\alpha\beta}|t|^2,$$

and for all $h \in (0,1]$,

$$\|J_{\kappa,N}(t)\|_{L^1\to L^\infty} \lesssim h^{-d}(1+|t|h^{-1})^{-\frac{d}{2}}, \tag{0.0.14}$$
$$R_{\kappa,N}(t) = O_{L^2\to L^2}(h^{N-1}). \tag{0.0.15}$$

Now let us set

$$J_N^\kappa(t) := \kappa^* J_{\kappa,N}(t)\kappa_*, \quad R_N^\kappa(t) := \kappa^* R_{\kappa,N}(t)\kappa_*.$$

By the fundamental theorem of calculus, we have

$$\begin{aligned} w(t) &= e^{ith^{-1}\omega(-h^2\Delta_g)}\tilde{\phi}_\kappa Op_h^\kappa(a_\kappa)\phi_\kappa\psi \\ &= \tilde{\phi}_\kappa J_N^\kappa(t)\phi_\kappa\psi - ih^{-1}\int_0^t e^{i(t-s)\omega(-h^2\Delta_g)}(hD_s - \omega(-h^2\Delta_g))\tilde{\phi}_\kappa J_N^\kappa(s)\phi_\kappa\psi ds. \end{aligned}$$

By (0.0.13), we write

$$(hD_s - \omega(-h^2\Delta_g))\tilde{\phi}_\kappa J_N^\kappa(s)\phi_\kappa = \tilde{\phi}_\kappa hD_s J_N^\kappa(s)\phi_\kappa - \tilde{\vartheta}_\kappa Op_h^\kappa(b_\kappa(h))\tilde{\phi}_\kappa J_N^\kappa(s)\phi_\kappa - h^N\mathfrak{R}_{\kappa,N}(h)\tilde{\phi}_\kappa J_N^\kappa(s)\phi_\kappa,$$

where $b_\kappa(h) = \sum_{l=0}^{N-1} h^l b_{\kappa,l}$. Note that up to a smoothing $O_{L^2\to L^2}(h^\infty)$ operator, the operator $J_h(S(t),a(t))\chi$ can be replaced by $\tilde{\chi}J_h(S(t),a(t))\chi$ for any $\tilde{\chi} \in C_0^\infty$ satisfying $\tilde{\chi} = 1$ on $\mathrm{supp}(\chi)$.

Thus,

$$\begin{aligned}(hD_s - \omega(-h^2\Delta_g))\tilde{\phi}_\kappa J_N^\kappa(s)\phi_\kappa &= \tilde{\vartheta}_\kappa \kappa^*(hD_s - Op_h(b_\kappa(h)))J_N(s)\kappa_*\phi_\kappa - R_\kappa(s) \\ &\quad - h^N \mathfrak{R}_{\kappa,N}(h)\tilde{\phi}_\kappa J_N^\kappa(s)\phi_\kappa \\ &= \tilde{\vartheta}_\kappa R_N^\kappa(s)\phi_\kappa - R_\kappa(s) - h^N \mathfrak{R}_{\kappa,N}(h)\tilde{\phi}_\kappa J_N^\kappa(s)\phi_\kappa,\end{aligned}$$

where $R_\kappa(s) = O_{L^2(M)\to L^2(M)}(h^\infty)$. Here we also use the L^2-boundedness of pseudo-differential operators with symbol in $S(-\infty)$. We thus get

$$w(t) = \tilde{\phi}_\kappa J_N^\kappa(t)\phi_\kappa\psi + \mathcal{R}_N^\kappa(t)\psi,$$

where

$$\mathcal{R}_N^\kappa(t)\psi = ih^{-1}\int_0^t e^{i(t-s)h^{-1}\omega(-h^2\Delta_g)}\left(\tilde{\vartheta}_\kappa R_N^\kappa(s)\phi_\kappa - R_\kappa(s) - h^N\mathfrak{R}_{\kappa,N}(h)\tilde{\phi}_\kappa J_N^\kappa(s)\phi_\kappa\right)\psi ds.$$

Thanks to the dispersive estimate (0.0.14) and the L^2-boundedness (0.0.15), we obtain for all $t \in [-t_0, t_0]$ and all $h \in (0,1]$,

$$\|e^{ith^{-1}\omega(-h^2\Delta_g)}\varphi(-h^2\Delta_g)\phi_\kappa\psi\|_{L^\infty(M)} \lesssim h^{-d}(1+|t|h^{-1})^{-\frac{d}{2}}\|\psi\|_{L^1(M)}.$$

These dispersive esimates combined with the partition of unity show (0.0.12).

In Chapter 4, we study global in time Strichartz estimates for Schrödinger-type equations [1]

$$i\partial_t u - |\nabla_g|^\sigma u = 0, \quad u(0) = \psi, \quad |\nabla_g| = \sqrt{-\Delta_g}, \quad \sigma \in (0,\infty)\backslash\{1\},$$

on asymptotically Euclidean manifolds, i.e. $\mathbb{R}^d$ equipped with a smooth long range pertubation metric g. More precisely the metric g satisfies the following assumptions:

- (Ellipticity) There exists $C > 0$ such that for all $x, \xi \in \mathbb{R}^d$,

$$C^{-1}|\xi|^2 \leq p(x,\xi) := \sum_{j,k=1}^d g^{jk}(x)\xi_j\xi_k \leq C|\xi|^2. \tag{0.0.16}$$

- (Long range pertubation) There exists $\rho > 0$ such that for all $\alpha \in \mathbb{N}^d$, there exists $C_\alpha > 0$ such that for all $x \in \mathbb{R}^d$,

$$|\partial^\alpha(g^{jk}(x) - \delta_{jk})| \leq C_\alpha \langle x\rangle^{-\rho-|\alpha|}. \tag{0.0.17}$$

In some situation, we assume that the geodesic flow associated to g is non-trapping. It means that the Hamiltonian flow $(X(t),\Xi(t)) := (X(t,x,\xi),\Xi(t,x,\xi))$ associated to the principal symbol p of $g^{-1}(x) = (g^{jk}(x))_{j,k=1}^d$, i.e.

$$\left\{\begin{array}{ll}\dot{X}(t) &= \nabla_\xi p(X(t),\Xi(t)), \\ \dot{\Xi}(t) &= -\nabla_x p(X(t),\Xi(t)),\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{ll}\dot{X}(0) &= x, \\ \dot{\Xi}(0) &= \xi,\end{array}\right.$$

satisfies for all $(x,\xi) \in T^\star\mathbb{R}^d$ with $\xi \neq 0$,

$$|X(t)| \to \infty \text{ as } |t| \to \infty.$$

Remark that by the conservation of energy and (0.0.16), all geodesics starting from (x,ξ) are defined globally in time. We also assume that there exists $M > 0$ large enough such that for all

[1] This is different from the previous chapters with the minus sign in front of $|\nabla_g|^\sigma$. It is technical due to the construction of the Isozaki-Kitada parametrix.

$\chi \in C_0^\infty(\mathbb{R}^d)$,

$$\|\chi(-\Delta_g - \lambda \pm i0)^{-1}\chi\|_{L^2 \to L^2} \lesssim \lambda^M, \quad \lambda \geq 1. \tag{0.0.18}$$

Note that this assumption holds in certain trapping situations (see e.g. [Dat09], [NZ09] or [BGH10]), for instance,

$$\|\chi(-\Delta_g - \lambda \pm i0)^{-1}\chi\|_{L^2 \to L^2} \lesssim \lambda^{-\frac{1}{2}} \log \lambda, \quad \lambda \geq 1,$$

as well as in non-trapping condition (see [Rob92] or [Vod04])

$$\|\chi(-\Delta_g - \lambda \pm i0)^{-1}\chi\|_{L^2 \to L^2} \lesssim \lambda^{-\frac{1}{2}}, \quad \lambda \geq 1.$$

In order to study global in time Strichartz estimates for Schrödinger-type equations on asymptotically Euclidean manifolds, we need to split the solution into low and high frequency pieces, namely

$$e^{-it|\nabla_g|^\sigma}\psi = f_0(-\Delta_g)e^{-it|\nabla_g|^\sigma}\psi + (1-f_0)(-\Delta_g)e^{-it|\nabla_g|^\sigma}\psi =: u_{\text{low}}(t) + u_{\text{high}}(t),$$

where $f_0 \in C_0^\infty(\mathbb{R})$ satisfies $f_0 = 1$ on $[-1, 1]$.

Let us consider the high frequency term. For a given $\chi \in C_0^\infty(\mathbb{R}^d)$, we write $u_{\text{high}} = \chi u_{\text{high}} + (1-\chi)u_{\text{high}}$. In our consideration, we have the following version of Littlewood-Paley decomposition: for any $q \in [2, \infty)$, $N \geq 1$ and $\chi \in C_0^\infty(\mathbb{R}^d)$,

$$\|(1-\chi)(1-f_0)(-\Delta_g)v\|_{L^q} \lesssim \Big(\sum_{h^{-2} \in 2^{\mathbb{N}}} \|(1-\chi)f(-h^2\Delta_g)v\|^2_{L^q} + h^N \| \langle x \rangle^{-N} f(-h^2\Delta_g)v\|^2_{L^2} \Big)^{1/2},$$

where $f(\lambda) = f_0(\lambda) - f_0(2\lambda)$. The same estimate holds true for χ in place of $1-\chi$. By the Minkowski inequality,

$$\begin{aligned}\|(1-\chi)u_{\text{high}}\|_{L^p(\mathbb{R}, L^q)} \lesssim \Big(&\sum_{h^{-2} \in 2^{\mathbb{N}}} \|(1-\chi)f(-h^2\Delta_g)e^{-it|\nabla_g|^\sigma}\psi\|^2_{L^p(\mathbb{R}, L^q)} \\ &+ h^N \| \langle x \rangle^{-N} f(-h^2\Delta_g)e^{-it|\nabla_g|^\sigma}\psi\|^2_{L^p(\mathbb{R}, L^2)} \Big)^{1/2}.\end{aligned}$$

The same estimate holds for $\|\chi u_{\text{high}}\|_{L^p(\mathbb{R}, L^q)}$ with χ instead of $1-\chi$. To estimate the weighted term $\langle x \rangle^{-N} f(-h^2\Delta_g)e^{-it|\nabla_g|^\sigma}\psi$, we use the L^2 integrability which is available on $(\mathbb{R}^d, g)$ under the assumption (0.0.18), namely

$$\| \langle x \rangle^{-1} f(-h^2\Delta_g)e^{-ith^{-1}(h|\nabla_g|)^\sigma}\psi\|_{L^2(\mathbb{R}, L^2)} \lesssim h^{\frac{1-N_0}{2}} \|\psi\|_{L^2},$$

for some $N_0 > 0$. Interpolating between $L^2(\mathbb{R})$ and $L^\infty(\mathbb{R})$, we have

$$\| \langle x \rangle^{-1} f(-h^2\Delta_g)e^{-ith^{-1}(h|\nabla_g|)^\sigma}\psi\|_{L^p(\mathbb{R}, L^2)} \lesssim h^{\frac{1-N_0}{p}} \|\psi\|_{L^2},$$

or

$$\| \langle x \rangle^{-1} f(-h^2\Delta_g)e^{-i|\nabla_g|^\sigma}\psi\|_{L^p(\mathbb{R}, L^2)} \lesssim h^{\frac{\sigma-N_0}{p}} \|\psi\|_{L^2}.$$

Thus,

$$h^{\frac{N}{2}} \| \langle x \rangle^{-N} f(-h^2\Delta_g)e^{-it|\nabla_g|^\sigma}\psi\|_{L^p(\mathbb{R}, L^2)} \lesssim h^{\frac{N}{2} + \frac{\sigma-N_0}{p}} \|\psi\|_{L^2}.$$

Moreover, we can replace the norm $\|\psi\|_{L^2}$ by $\|f(-h^2\Delta_g)\psi\|_{L^2}$. Therefore, by taking N large enough, we see that this weighted term is bounded by $h^{-\gamma_{p,q}}\|f(-h^2\Delta_g)\psi\|_{L^2}$. By the almost orthogonality, Strichartz estimates for the high frequency piece are reduced to showing

$$\|\chi e^{-it|\nabla_g|^\sigma} f(-h^2\Delta_g)\psi\|_{L^p(\mathbb{R},L^q)} \lesssim h^{-\gamma_{p,q}}\|f(-h^2\Delta_g)\psi\|_{L^2}, \tag{0.0.19}$$

$$\|(1-\chi) e^{-it|\nabla_g|^\sigma} f(-h^2\Delta_g)\psi\|_{L^p(\mathbb{R},L^q)} \lesssim h^{-\gamma_{p,q}}\|f(-h^2\Delta_g)\psi\|_{L^2}. \tag{0.0.20}$$

Here the almost orthogonality means formally that $\mathrm{supp}[f(2^{-k}\cdot)] \cap \mathrm{supp}[f(2^{-l}\cdot)] = \emptyset$ for $k,l \in \mathbb{N}$ and $|k-l| \geq 2$. This allows us to show, for instance,

$$\Bigg(\sum_{\substack{h^{-2}=2^{-k}\\ k\in\mathbb{N}}} h^{-2\gamma_{p,q}}\|f(-h^2\Delta_g)\psi\|^2_{L^2}\Bigg)^{1/2} \lesssim \|\psi\|_{\dot{H}^{\gamma_{p,q}}}.$$

For the low frequency term, we use the following Littlewood-Paley decomposition: for any $\chi \in C_0^\infty(\mathbb{R}^d)$ satisfying $\chi(x) = 1$ for $|x| \leq 1$,

$$\|f_0(-\Delta_g)v\|_{L^q} \lesssim \Big(\sum_{\epsilon^{-2}\in 2^{\mathbb{N}}} \|(1-\chi)(\epsilon x) f(-\epsilon^{-2}\Delta_g)v\|^2_{L^q} + \epsilon^{2\left(\frac{d}{2}-\frac{d}{q}\right)}\| \langle \epsilon x\rangle^{-1} f(-\epsilon^{-2}\Delta_g)v\|^2_{L^2}\Big)^{1/2}$$

to bound

$$\begin{aligned}\|u_{\mathrm{low}}\|_{L^p(\mathbb{R},L^q)} \lesssim \Big(\sum_{\epsilon^{-2}\in 2^{\mathbb{N}}} &\|(1-\chi)(\epsilon x) f(-\epsilon^{-2}\Delta_g)e^{-it|\nabla_g|^\sigma}\psi\|^2_{L^p(\mathbb{R},L^q)} \\ &+ \epsilon^{2\left(\frac{d}{2}-\frac{d}{q}\right)}\| \langle \epsilon x\rangle^{-1} f(-\epsilon^{-2}\Delta_g)e^{-it|\nabla_g|^\sigma}\psi\|^2_{L^p(\mathbb{R},L^2)}\Big)^{1/2}.\end{aligned}$$

Using the L^p-integrability, which follows from the low frequency resolvent estimates,

$$\| \langle \epsilon x\rangle^{-1} f(-\epsilon^{-2}\Delta_g)e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma}\psi\|_{L^p(\mathbb{R},L^2)} \lesssim \epsilon^{-\frac{1}{p}}\|\psi\|_{L^2},$$

we estimate

$$\epsilon^{\frac{d}{2}-\frac{d}{q}}\| \langle \epsilon x\rangle^{-1} f(-\epsilon^{-2}\Delta_g)e^{-it|\nabla_g|^\sigma}\psi\|_{L^p(\mathbb{R},L^2)} \lesssim \epsilon^{\gamma_{p,q}}\|f(-\epsilon^{-2}\Delta_g)\psi\|_{L^2}.$$

Therefore, by an almost orthogonality argument, it suffices to show

$$\|(1-\chi)(\epsilon x) f(-\epsilon^{-2}\Delta_g)e^{-it|\nabla_g|^\sigma}\psi\|_{L^p(\mathbb{R},L^q)} \lesssim \epsilon^{\gamma_{p,q}}\|f(-\epsilon^{-2}\Delta_g)\psi\|_{L^2}. \tag{0.0.21}$$

To show (0.0.19), we assume that the geodesic flow associated to g is non-trapping. It is crucial in our argument. We make use of the local in time Strichartz estimates for the localized Schrödinger-type operator, namely

$$\|e^{-ith^{-1}(h|\nabla_g|)^\sigma}\varphi(-h^2\Delta_g)v\|_{L^p(\mathbb{R},L^q)} \lesssim h^{-\kappa_{p,q}}\|v\|_{L^2},$$

with $\kappa_{p,q} = \frac{d}{2} - \frac{d}{q} - \frac{1}{p}$ as well as the inhomogenous Strichartz estimates

$$\Big\| \int_0^t e^{-i(t-s)h^{-1}(h|\nabla_g|)^\sigma}\varphi^2(-h^2\Delta_g)G(s)ds\Big\|_{L^p(\mathbb{R},L^q)} \lesssim h^{-\kappa_{p,q}}\|G\|_{L^1(I,L^2)}.$$

These Strichartz estimates are proved in Chapter 2. Note that the long range assumption (0.0.17) implies that the metric g satisfies (0.0.7). With these localized Strichartz estimates and the sharp L^2-integrability

$$\| \langle x\rangle^{-1} f(-h^2\Delta_g)e^{-ith^{-1}(h|\nabla_g|)^\sigma} f(-h^2\Delta_g)\psi\|_{L^2(\mathbb{R},L^2)} \lesssim \|\psi\|_{L^2},$$

we prove (0.0.19). Note that the non-trapping condition is needed to have the above sharp L^2-

integrability.

The proofs of (0.0.20) and (0.0.21) are based on the Isozaki-Kitada parametrix and local energy decay estimates, namely for $k \geq 0$,

$$\begin{aligned} \| \langle x \rangle^{-1-k} e^{-ith^{-1}(h|\nabla_g|)^\sigma} f(-h^2\Delta_g) \langle x \rangle^{-1-k} \|_{L^2 \to L^2} &\lesssim h^{-N_k} \langle th^{-1} \rangle^{-k}, \\ \| \langle \epsilon x \rangle^{-1-k} e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma} f(-\epsilon^{-2}\Delta_g) \langle \epsilon x \rangle^{-1-k} \|_{L^2 \to L^2} &\lesssim \langle \epsilon t \rangle^{-k}. \end{aligned}$$

We refer the reader to Chapter 4 for more details.

The second part of this thesis concerns nonlinear aspects of the nonlinear Schrödinger-type equations such as local well-posedness, global well-posedness, global existence and blowup for low regularity initial data. In Chapter 5, we study the local well-posedness in Sobolev spaces for nonlinear Schrödinger-type equations. More precisely, we consider

$$i\partial_t u + |\nabla|^\sigma u = \pm |u|^{\nu-1} u, \quad u(0) = \psi, \quad \sigma \in (0,\infty), \quad \nu > 1. \tag{NLST}$$

This equation enjoys formally the conservation of mass and energy

$$\begin{aligned} M(u(t)) &= \int |u(t,x)|^2 dx = M(\psi), \\ E(u(t)) &= \frac{1}{2} \int ||\nabla|^{\sigma/2} u(t,x)|^2 dx \mp \frac{1}{\nu+1} \int |u(t,x)|^{\nu+1} dx = E(\psi). \end{aligned}$$

The equation (NLST) also has the scaling invariance

$$u_\lambda(t,x) = \lambda^{-\frac{\sigma}{\nu-1}} u(\lambda^\sigma t, \lambda^{-1} x), \quad \lambda > 0.$$

By a direct computation, we have

$$\|u_\lambda(0)\|_{\dot{H}^\gamma} = \lambda^{\frac{d}{2} - \frac{\sigma}{\nu-1} - \gamma} \|\psi\|_{\dot{H}^\gamma}.$$

From this, we define the critical regularity exponent by

$$\gamma_{\mathrm{c}} := \frac{d}{2} - \frac{\sigma}{\nu-1}.$$

In Chapter 5, we are interested in the well-posedness result for (NLST) when $\gamma \geq \gamma_{\mathrm{c}}$. Since we are working in Sobolev spaces of fractional order γ and γ_{c}, we need the nonlinearity $F(z) = \pm |z|^{\nu-1} z$ to have enough regularity. When ν is an odd integer, the nonlinearity is smooth. When $\nu > 1$ is not an odd integer, we need the following assumption

$$\lceil \gamma \rceil, \lceil \gamma_{\mathrm{c}} \rceil \leq \nu, \tag{0.0.22}$$

where $\lceil \gamma \rceil$ is the smallest integer greater than or equal to γ, similarly for $\lceil \gamma_{\mathrm{c}} \rceil$.

In order to study the local well-posedness of (NLST) in Sobolev spaces, we need two important tools: Strichartz estimates and nonlinear estimates. Strichartz estimates for Schrödinger-type equations are shown in Chapter 1. Note that in the case $\sigma \in (0,2)$, admissible conditions (0.0.4) and (0.0.5) yield $\gamma_{p,q} > 0$ for any (p,q) except $(\infty, 2)$. Thus, in this case, there is a loss of derivatives in the sense that if we use Strichartz estimates at H^γ-level, then we need the initial data to belong to $H^{\gamma+\gamma_{p,q}}$. This loss of derivatives leads to a weak local well-posedness result for $\sigma \in (0,2)$ compared to the one for $\sigma \in [2,\infty)$. Therefore, we will consider three cases $\sigma \in (0,2)\backslash\{1\}$, $\sigma = 1$ and $\sigma \in [2,\infty)$ respectively. We also need the Kato fractional derivative estimates, namely for $\gamma \geq 0$ and $1 < r, p < \infty, 1 < q \leq \infty$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{\nu-1}{q}$:

- if $\nu > 1$ is an odd integer or $\lceil \gamma \rceil \leq \nu$ otherwise, then there exists $C = C(d,\nu,\gamma,r,p,q) > 0$

such that for all $u \in \mathscr{S}$,

$$\begin{aligned} \||\nabla|^\gamma(|u|^{\nu-1}u)\|_{L^r} &\leq C\|u\|^{\nu-1}_{L^q}\||\nabla|^\gamma u\|_{L^p}, \\ \|\langle\nabla\rangle^\gamma(|u|^{\nu-1}u)\|_{L^r} &\leq C\|u\|^{\nu-1}_{L^q}\|\langle\nabla\rangle^\gamma u\|_{L^p}. \end{aligned}$$

- if $\nu > 1$ is an odd integer or $\lceil\gamma\rceil \leq \nu-1$ otherwise, then there exists $C = C(d, \nu, \gamma, r, p, q) > 0$ such that for all $u \in \mathscr{S}$,

$$\begin{aligned} \||\nabla|^\gamma(|u|^{\nu-1}u - |v|^{\nu-1}v)\|_{L^r} \leq C\Big(&(\|u\|^{\nu-1}_{L^q} + \|v\|^{\nu-1}_{L^q})\||\nabla|^\gamma(u-v)\|_{L^p} \\ &+(\|u\|^{\nu-2}_{L^q} + \|v\|^{\nu-2}_{L^q})(\||\nabla|^\gamma u\|_{L^p} + \||\nabla|^\gamma v\|_{L^p})\|u-v\|_{L^q}\Big), \\ \|\langle\nabla\rangle^\gamma(|u|^{\nu-1}u - |v|^{\nu-1}v)\|_{L^r} \leq C\Big(&(\|u\|^{\nu-1}_{L^q} + \|v\|^{\nu-1}_{L^q})\|\langle\nabla\rangle^\gamma(u-v)\|_{L^p} \\ &+(\|u\|^{\nu-2}_{L^q} + \|v\|^{\nu-2}_{L^q})(\|\langle\nabla\rangle^\gamma u\|_{L^p} + \|\langle\nabla\rangle^\gamma v\|_{L^p})\|u-v\|_{L^q}\Big). \end{aligned}$$

The proof of the local well-posedness is based on Strichartz estimates and the standard contraction mapping argument. By Duhamel's formula, it suffices to show the functional

$$\Phi(u) := e^{it|\nabla|^\sigma} \mp i\int_0^t e^{i(t-s)|\nabla|^\sigma}|u(s)|^{\nu-1}u(s)ds$$

is a contraction on a suitable Banach space (X, d).

Let us consider $\sigma \in (0, 2)$. In the subcritical case, i.e. $\gamma > \gamma_c$, we choose X as

$$X := \Big\{u \in L^\infty(I, H^\gamma) \cap L^p(I, H^{\gamma-\gamma_{p,q}}_q) \ : \ \|u\|_{L^\infty(I,H^\gamma)} + \|u\|_{L^p(I,H^{\gamma-\gamma_{p,q}}_q)} \leq M\Big\},$$

and the distance

$$d(u, v) := \|u - v\|_{L^\infty(I,L^2)} + \|u - v\|_{L^p(I,H^{-\gamma_{p,q}}_q)},$$

where $I = [0, T]$ with $M, T > 0$ to be determined momentarily. Here (p, q) is an admissible pair satisfying either (0.0.4) or (0.0.5) to be chosen shortly, and H^γ_q is the generalized Sobolev space (see Chapter 1 for the notation). Due to the loss of derivatives, we have to use Strichartz estimate for the special pair $(\infty, 2)$ to get

$$\begin{aligned} \|\Phi(u)\|_{L^\infty(I,H^\gamma)} + \|\Phi(u)\|_{L^p(I,H^{\gamma-\gamma_{p,q}}_q)} &\lesssim \|\psi\|_{H^\gamma} + \||u|^{\nu-1}u\|_{L^1(I,H^\gamma)}, \\ \|\Phi(u) - \Phi(v)\|_{L^\infty(I,L^2)} + \|\Phi(u) - \Phi(v)\|_{L^p(I,H^{-\gamma_{p,q}}_q)} &\lesssim \||u|^{\nu-1}u - |v|^{\nu-1}v\|_{L^1(I,L^2)}. \end{aligned}$$

The nonlinear term can be bounded by

$$\||u|^{\nu-1}u\|_{L^1(I,H^\gamma)} \lesssim |I|^{1-\frac{\nu-1}{p}}\|u\|^{\nu-1}_{L^p(I,L^\infty)}\|u\|_{L^\infty(I,H^\gamma)},$$

similarly for $|u|^{\nu-1}u - |v|^{\nu-1}v$. In order to close the contraction ball, we need to choose (p, q) such that $\nu - 1 < p$ and $L^p(I, H^{\gamma-\gamma_{p,q}}_q) \subset L^p(I, L^\infty)$ or $H^{\gamma-\gamma_{p,q}}_q \subset L^\infty$. By the Sobolev embedding, we choose: for $\sigma \in (0, 2)\backslash\{1\}$, (p, q) satisfies (0.0.4) and

$$p > \begin{cases} \max(\nu - 1, 4) & \text{if } d = 1, \\ \max(\nu - 1, 2) & \text{if } d \geq 2, \end{cases}$$

and for $\sigma = 1$, (p, q) satisfies (0.0.5) and

$$p > \begin{cases} \max(\nu - 1, 4) & \text{if } d = 2, \\ \max(\nu - 1, 2) & \text{if } d \geq 3. \end{cases}$$

In the critical case $\gamma = \gamma_c$, the Sobolev embedding does not help. To overcome the loss of derivatives, we consider

$$X := \Big\{ u \in L^\infty(I, H^{\gamma_c}) \cap L^p(I, B_q^{\gamma_c - \gamma_{p,q}}) \ : \ \|u\|_{L^\infty(I,\dot{H}^{\gamma_c})} \leq M, \|u\|_{L^p(I,\dot{B}_q^{\gamma_c-\gamma_{p,q}})} \leq N \Big\},$$

and

$$d(u,v) := \|u - v\|_{L^\infty(I,L^2)} + \|u-v\|_{L^p(I,\dot{B}_q^{-\gamma_{p,q}})},$$

where $I = [0,T]$ with $M, N, T > 0$ to be determined. Here B_q^γ and $\dot{B}_q^\gamma$ are generalized inhomogeneous and homogeneous Besov spaces respectively (see again Chapter 1 for the notation). As in the subcritical case, by using Strichartz estimate for the pair $(\infty, 2)$ and the Hölder inequality, it suffices to bound $\|u\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)}$. To do so, we use the argument of Hong-Sire [HS15] (see also [CKSTT5]) to have: for $\sigma \in (0,2)\backslash\{1\}$,

$$\|u\|^{\nu-1}_{L^{\nu-1}(\mathbb{R},L^\infty)} \lesssim \begin{cases} \|u\|^4_{L^4(\mathbb{R},\dot{B}_\infty^{\gamma_c-\gamma_{4,\infty}})} \|u\|^{\nu-5}_{L^\infty(\mathbb{R},\dot{B}_2^{\gamma_c})} & \text{when } d=1, \\ \|u\|^p_{L^p(\mathbb{R},\dot{B}_{p^\star}^{\gamma_c-\gamma_{p,p^\star}})} \|u\|^{\nu-1-p}_{L^\infty(\mathbb{R},\dot{B}_2^{\gamma_c})} & \text{where } \nu-1>p>2 \text{ when } d=2, \\ \|u\|^2_{L^2(\mathbb{R},\dot{B}_{2^\star}^{\gamma_c-\gamma_{2,2^\star}})} \|u\|^{\nu-3}_{L^\infty(\mathbb{R},\dot{B}_2^{\gamma_c})} & \text{when } d\geq 3, \end{cases}$$

where $p^\star = 2p/(p-2)$ and $2^\star = 2d/(d-2)$, and for $\sigma = 1$,

$$\|u\|^{\nu-1}_{L^{\nu-1}(\mathbb{R},L^\infty)} \lesssim \begin{cases} \|u\|^4_{L^4(\mathbb{R},\dot{B}_\infty^{\gamma_c-\gamma_{4,\infty}})} \|u\|^{\nu-5}_{L^\infty(\mathbb{R},\dot{B}_2^{\gamma_c})} & \text{when } d=2, \\ \|u\|^p_{L^p(\mathbb{R},\dot{B}_{p^\star}^{\gamma_c-\gamma_{p,p^\star}})} \|u\|^{\nu-1-p}_{L^\infty(\mathbb{R},\dot{B}_2^{\gamma_c})} & \text{where } 2<p<\nu-1 \text{ when } d=3, \\ \|u\|^2_{L^2(\mathbb{R},\dot{B}_{2^\star}^{\gamma_c-\gamma_{2,2^\star}})} \|u\|^{\nu-3}_{L^\infty(\mathbb{R},\dot{B}_2^{\gamma_c})} & \text{when } d\geq 4, \end{cases}$$

where $p^\star = 2p/(p-2)$ and $2^\star = 2(d-1)/(d-3)$. We thus choose for $\sigma \in (0,2)\backslash\{1\}$,

$$(p,q) = \begin{cases} (4,\infty) & \text{if } d=1, \\ (p,p^\star) & \text{if } d=2, \\ (2,2^\star) & \text{if } d\geq 3, \end{cases}$$

and for $\sigma = 1$,

$$(p,q) = \begin{cases} (4,\infty) & \text{if } d=2, \\ (p,p^\star) & \text{if } d=3, \\ (2,2^\star) & \text{if } d\geq 4. \end{cases}$$

In the case $\sigma \in [2,\infty)$, thanks to Strichartz estimates without loss of derivatives, we show Φ is a contraction on (X,d) with

$$X := \Big\{ u \in L^p(I, H_q^\gamma) \ : \ \|u\|_{L^p(I,\dot{H}_q^\gamma)} \leq M \Big\}, \quad d(u,v) := \|u-v\|_{L^p(I,L^q)},$$

where

$$p = \frac{2\sigma(\nu+1)}{(\nu-1)(d-2\gamma)}, \quad q = \frac{d(\nu+1)}{d+(\nu-1)\gamma}.$$

By Strichartz estimates, we bound

$$\|\Phi(u)\|_{L^p(I,\dot{H}_q^\gamma)} \lesssim \|\psi\|_{\dot{H}^\gamma} + \||u|^{\nu-1}u\|_{L^{p'}(I,\dot{H}^\gamma_{q'})},$$
$$\|\Phi(u)-\Phi(v)\|_{L^p(I,L^q)} \lesssim \||u|^{\nu-1}u - |v|^{\nu-1}v\|_{L^{p'}(I,L^{q'})}.$$

The Hölder inequality then implies

$$\||u|^{\nu-1}u\|_{L^{p'}(I,\dot{H}^\gamma_{q'})} \lesssim |I|^{1-\frac{(\nu-1)(d-2\gamma)}{2\sigma}}\|u\|^\nu_{L^p(I,\dot{H}^\gamma_q)},$$
$$\||u|^{\nu-1}u-|v|^{\nu-1}v\|_{L^{p'}(I,L^{q'})} \lesssim |I|^{1-\frac{(\nu-1)(d-2\gamma)}{2\sigma}}\Big(\|u\|^{\nu-1}_{L^p(I,\dot{H}^\gamma_q)}+\|v\|^{\nu-1}_{L^p(I,\dot{H}^\gamma_q)}\Big)\|u-v\|_{L^p(I,L^q)}.$$

With these estimates at hand, we can easily show that Φ is a contraction on (X,d).

In Chapter 6, we consider the defocusing nonlinear fourth-order Schrödinger equation

$$i\partial_t u+\Delta^2 u=-|u|^{\frac{8}{d}}u, \quad u(0)=\psi. \tag{dNL4S}$$

By the local theory given in Chapter 5, (dNL4S) is locally well-posed in H^γ for $\gamma>0$ satisfying, in the case $d\neq 1,2,4$,

$$\lceil\gamma\rceil \leq 1+\frac{8}{d}. \tag{0.0.23}$$

This conditon ensures the nonlinearity to have enough regularity. The conservation of mass and energy together with the persistence of regularity yield the global well-posedness for (dNL4S) in H^γ with $\gamma\geq 2$ satisfying for $d\neq 1,2,4$, (0.0.23). The main goal of Chapter 6 is to prove the global well-posedness for (dNL4S) in low regularity spaces $H^\gamma(\mathbb{R}^d)$ with $d\geq 4$ and $0<\gamma<2$. Since we are working with low regularity data, the conservation of energy does not hold. In order to overcome this difficulty, we make use of the I-method introduced by [CKSTT1] and the interaction Morawetz inequality (which is available for $d\geq 5$). We thus consider separately two cases $d=4$ and $d\geq 5$.

In the case $d=4$, we use the I-method in Bourgain spaces, which is an adaptation of the one given in [CKSTT1] to prove the low regularity global well-posedness of the defocusing cubic nonlinear Schrödinger equation on $\mathbb{R}^2$. The idea of the I-method is to replace the conserved energy $E(u)$, which is not available when $\gamma<2$, by an "almost conserved" quantity $E(I_N u)$ with $N\gg 1$. Here I_N is a smoothing operator which behaves like the identity for low frequencies $|\xi|\leq N$ and like a fractional integral operator of order $2-\gamma$ for high frequencies $|\xi|\geq 2N$. Since $I_N u$ is not a solution to the equation, we may expect an energy increment. The key idea is to show that on the time interval of local existence, the increment of the modified energy $E(I_N u)$ decays with respect to a large parameter N. This allows to control $E(I_N u)$ on time interval where the local solution exists, and we can iterate this estimate to obtain a global in time control of the solution by means of the bootstrap argument. In the case $d=4$, the nonlinearity is algebraic. It allows to write explicitly the commutator between the I-operator and the nonlinearity by means of the Fourier transform, and then control it by multi-linear analysis. We will show in Chapter 6 that (dNL4S) is globally well-posed in $H^\gamma(\mathbb{R}^4)$ for any $\frac{60}{53}<\gamma<2$.

In the case $d\geq 5$, we use the I-method combined with the interaction Morawetz inequality. In this consideration, the nonlinearity is no longer algebraic. Thus we cannot apply the Fourier transform technique to estimate the increment of the modified energy. Fortunately, thanks to Strichartz estimates with a "gain" of derivatives, namely

$$\|\Delta u\|_{L^p(\mathbb{R},L^q)} \lesssim \|\Delta\psi\|_{L^2}+\|\nabla(|u|^{\frac{8}{d}}u)\|_{L^2(\mathbb{R},L^{\frac{2d}{d+2}})},$$

we are able to apply the technique given in [VZ09] to control the commutator. Due to the presence of the biharmonic operator Δ^2, we need the nonlinearity to have enough regularity. This leads to a restriction on dimensions $d=5,6$ and 7. The interaction Morawetz inequality for the nonlinear fourth-order Schrödinger equation was first introduced in [Pau1] for $d\geq 7$, and was extended for $d\geq 5$ in [MWZ15]. As a byproduct of Strichartz estimates and the I-method, we show global well-posedness for (dNL4S) in $H^\gamma(\mathbb{R}^d)$ for any $\gamma(d)<\gamma<2$, where $\gamma(5)=\frac{8}{5},\gamma(6)=\frac{5}{3}$ and $\gamma(7)=\frac{13}{7}$. However, this result is not new since one has a better result due to Pausader-Shao in [PS10]. In [PS10], the authors proved the global well-posedness for (dNL4S) with initial data in

$L^2(\mathbb{R}^d), d \geq 5$. They also proved that the solution satisfies the uniform bound

$$\|u\|_{L^{2+\frac{8}{d}}(\mathbb{R}, L^{2+\frac{8}{d}})} \leq C(\|\psi\|_{L^2}).$$

With this uniform bound, the persistence of regularity shows the global well-posedness for (dNL4S) in $H^\gamma(\mathbb{R}^d)$ for any $0 < \gamma < 2$ satisfying (0.0.23).

In the last chapter, we consider the focusing nonlinear fourth-order Schrödinger equation

$$i\partial_t u + \Delta^2 u = |u|^{\frac{8}{d}} u, \quad u(0) = \psi. \tag{fNL4S}$$

The main goal of this chapter is to study dynamical properties such as L^2-concentration, limiting profile with minimal mass, ... for low regularity blowup solutions. The study of blowup solutions is closely related to the notion of ground states of (fNL4S) which are solutions to the elliptic equation

$$\Delta^2 Q + Q - |Q|^{\frac{8}{d}} Q = 0. \tag{0.0.24}$$

This elliptic equation is obtained by considering solitary solutions (standing waves) of (fNL4S) of the form $u(t,x) = e^{-it}Q(x)$. The existence of solutions to (0.0.24) was proved in [ZYZ10], but the uniqueness still remains open. In the case $\|\psi\|_{L^2} < \|Q\|_{L^2}$, using the sharp Gagliardo-Nirenberg inequality

$$\|u\|^{2+\frac{8}{d}}_{L^{2+\frac{8}{d}}} \leq C(d)\|u\|^{\frac{8}{d}}_{L^2}\|\Delta u\|^2_{L^2}, \quad C(d) = \frac{1+\frac{4}{d}}{\|Q\|^{\frac{8}{d}}_{L^2}},$$

together with the conservation of energy, it is easy to see that (fNL4S) is globally well-posed in H^2. Moreover, Fibich-Ilan-Papanicolaou in [FIP02] provided some numerical observations showing that the H^2 solution to (fNL4S) may blow up if the initial data satisfies $\|\psi\|_{L^2} \geq \|Q\|_{L^2}$. Recently, Boulenger-Lenzmann in [BL17] showed the existence of radial blowup solutions to (fNL4S). More precisely, the authors proved that for any negative radial initial data ψ in H^2, the corresponding solution $u(t)$ either blows up in finite time or blows up infinite time and satisfies

$$\|u(t)\|_{\dot{H}^2} \geq Ct^2, \quad \forall t \geq t_0,$$

with some constant $C = C(\psi) > 0$ and $t_0 = t_0(\psi) > 0$. Baruch-Fibich-Mandelbaum in [BFM10] proved some dynamical properties of radially symmetric blowup solutions such as blowup rate, L^2-concentration. Later, Zhu-Yang-Zhang in [ZYZ10] removed the radially symmetric assumption and established the profile decomposition, the existence of ground states for the elliptic equation (0.0.24) and the following concentration compactness lemma for (fNL4S): for any bounded sequence $(v_n)_{n\geq 1}$ of H^2 functions satisfying

$$\limsup_{n\to\infty} \|\Delta v_n\|_{L^2} \leq M \quad \text{and} \quad \limsup_{n\to\infty} \|v_n\|_{L^{2+\frac{8}{d}}} \geq m,$$

there exists a sequence $(x_n)_{n\geq 1}$ of $\mathbb{R}^d$ such that up to a subsequence

$$v_n(\cdot + x_n) \rightharpoonup V \text{ weakly in } H^2 \text{ as } n \to \infty,$$

with $\|V\|^{\frac{8}{d}}_{L^2} \geq \frac{\|Q\|^{\frac{8}{d}}_{L^2} m^{2+\frac{8}{d}}}{(1+\frac{4}{d})M^2}$, where Q is the solution to the elliptic equation (0.0.24). Consequently, the authors in [ZYZ11] used the I-method and the compactness lemma to establish the limiting profile and the L^2-concentration for (fNL4S) with initial data in $H^\gamma(\mathbb{R}^4)$, $\frac{9+\sqrt{721}}{20} < \gamma < 2$. In Chapter 7, we aim to lower the required regularity of [ZYZ11] in the fourth dimensional case and to extend the result of [ZYZ11] to higher dimensions $d \geq 5$.

In the case $d = 4$, we make use of the I-method which is essentially established in Chapter 6. This allows us to show dynamical properties of blowup solutions in $H^\gamma(\mathbb{R}^4)$ with $\frac{67+\sqrt{40489}}{150} < \gamma <$

2. This is an improvement of the result of [ZYZ11].

In the case $d \geq 5$, we also make use of the I-method used in Chapter 6. As mentioned above, due to the high-order term $\Delta^2 u$, we need the nonlinearity to have at least two orders of derivatives in order to successfully establish the energy increment. We thus restrict ourself in spatial space of dimensions $d = 5, 6$ and 7. With the help of the I-method, we are able to study dynamical properties of blowup solutions in $H^\gamma(\mathbb{R}^d)$ with $d = 5, 6, 7$ and $\frac{56-3d+\sqrt{137d^2+1712d+3136}}{2(2d+32)} < \gamma < 2$.

Notations:

Throughout this thesis, we will use the following notations. The various constant will be denoted by C. The constants with subscripts $C_1, C_2, ...$ will be used when we need to compare them to one another. The notation $A \lesssim B$ means that there exists a universal constant $C > 0$ such that $A \leq CB$. The notation $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. We also use the Japanese bracket $\langle a \rangle := \sqrt{1+|a|^2} \sim 1 + |a|$ and $a\pm := a \pm \epsilon$ with some universal constant $0 < \epsilon \ll 1$. For Banach spaces X and Y, the notation $\|\cdot\|_{X \to Y}$ denotes the operator norm from X to Y. The one $T = O_{X \to Y}(A)$ means that $\|T\|_{X \to Y} \lesssim A$.

Part I

Strichartz estimates for Schrödinger-type equations on manifolds

CHAPTER 1

Strichartz estimates for Schrödinger-type equations on the flat Euclidean space

Contents

In this chapter, we derive Strichartz estimates for the inhomogeneous linear Schrödinger-type equations

$$\begin{cases} i\partial_t u(t,x) + |\nabla|^\sigma u(t,x) &= F(t,x), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ u(0,x) &= \psi(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{LST}$$

where $\sigma \in (0,\infty)$ and $|\nabla|^\sigma$ is the Fourier multiplier by $|\xi|^\sigma$. To do so, let us first recall the abstract $TT^\star$-criterion due to Keel-Tao [KT98].

Theorem 1.0.1 ($TT^\star$-criterion)**.** *Let (X, dx) be a measure space and H a Hilbert space. Suppose for each time $t \in \mathbb{R}$, an operator $T(t): H \to L^2(X)$ which obeys:*

1. *For all $t \in \mathbb{R}$ and all $f \in H$,*

$$\|T(t)f\|_{L^2(X)} \lesssim \|f\|_H. \tag{1.0.1}$$

2. *There exists $\delta > 0$ so that one of the following decay estimates holds for all $g \in L^1(X)$,*

$$\|T(t)T(s)^\star g\|_{L^\infty(X)} \lesssim |t-s|^{-\delta}\|g\|_{L^1(X)}, \quad \forall t \ne s, \tag{1.0.2}$$

$$\|T(t)T(s)^\star g\|_{L^\infty(X)} \lesssim (1+|t-s|)^{-\delta}\|g\|_{L^1(X)}, \quad \forall t, s \in \mathbb{R}. \tag{1.0.3}$$

If $T(t)$ obeys (1.0.1) *and* (1.0.2)*, then the estimates*

$$\|T(t)f\|_{L^p(\mathbb{R},L^q(X))} \lesssim \|f\|_H, \tag{1.0.4}$$

$$\Big\|\int T(s)^\star F(s)ds\Big\|_H \lesssim \|F\|_{L^{p'}(\mathbb{R},L^{q'}(X))}, \tag{1.0.5}$$

$$\Big\|\int_{s<t} T(t)T(s)^\star F(s)ds\Big\|_{L^p(\mathbb{R},L^q(X))} \lesssim \|F\|_{L^{a'}(\mathbb{R},L^{b'}(X))}, \tag{1.0.6}$$

hold for all sharp δ-admissible pairs (p,q) and (a,b), i.e.

$$(p,q) \in [2,\infty]^2, \quad (p,q,\delta) \ne (2,\infty,1), \quad \frac{1}{p} + \frac{\delta}{q} = \frac{\delta}{2}.$$

Furthermore, if $T(t)$ obeys (1.0.1) *and* (1.0.3)*, then* (1.0.4), (1.0.5) *and* (1.0.6) *hold for all δ-*

admissible pairs (p,q) *and* (a,b)*, i.e.*

$$(p,q) \in [2,\infty]^2, \quad (p,q,\delta) \neq (2,\infty,1), \quad \frac{1}{p} + \frac{\delta}{q} \leq \frac{\delta}{2}.$$

Here (p,p') *is a conjugate pair, and similarly for* $(q,q'), (a,a')$ *and* (b,b').

To our knowledge, there are two ways to derive Strichartz estimates for (LST). One way is to use directly dispersive and energy estimates as for the linear Schrödinger equation, i.e. $\sigma = 2$, More precisely, one can compute the Schwartz kernel of the Schrödinger group $e^{-it\Delta}$

$$e^{-it\Delta}\psi(x) = \frac{e^{\pm i\frac{\pi d}{4}}}{|4\pi t|^{\frac{d}{2}}} \int e^{-i\frac{|x-y|^2}{4t}} \psi(y)dy, \quad \pm := \text{ sign of } t.$$

From this, we obtain the following dispersive estimate

$$\|e^{-it\Delta}\psi\|_{L^\infty} \lesssim |t|^{-\frac{d}{2}} \|\psi\|_{L^1}.$$

In the case $\sigma \neq 2$, it is not clear that one can compute the Schwartz kernel of $e^{it|\nabla|^\sigma}$, and thus dispersive estimates for $e^{it|\nabla|^\sigma}$ are not obtained directly. Another way is to decompose the solution in dyadic pieces and use the scaling technique to reduce to estimates at frequency one. Since (LST) enjoys a scaling invariance in the frequency space, it allows us to use the scaling technique.

Before entering some details, let us introduce some standard notations (see [GV85, Appendix], [Tri83, Chapter 5] or [BL76, Chapter 6]). Let [1] $\chi_0 \in C_0^\infty(\mathbb{R}^d)$ be such that $\chi_0(\xi) = 1$ for $|\xi| \leq 1$ and $\text{supp}(\chi_0) \subset \{\xi \in \mathbb{R}^d, |\xi| \leq 2\}$. We set $\chi(\xi) := \chi_0(\xi) - \chi_0(2\xi)$. It is easy to see that $\chi \in C_0^\infty(\mathbb{R}^d)$ and $\text{supp}(\chi) \subset \{\xi \in \mathbb{R}^d, 1/2 \leq |\xi| \leq 2\}$. We denote the Littlewood-Paley projections by $P_0 := \chi_0(D), P_N := \chi(N^{-1}D)$ with $N = 2^k, k \in \mathbb{Z}$ where $\chi_0(D), \chi(N^{-1}D)$ are Fourier multipliers by $\chi_0(\xi)$ and $\chi(N^{-1}\xi)$ respectively. Given $\gamma \in \mathbb{R}$ and $1 \leq q \leq \infty$, the generalized inhomogeneous Sobolev H^γ_q and Besov B^γ_q spaces are defined respectively as closures of the Schwartz space $\mathscr{S}$ under the norms

$$\|u\|_{H^\gamma_q} := \| \langle\nabla\rangle^\gamma u\|_{L^q}, \quad \langle\nabla\rangle := \sqrt{1-\Delta},$$
$$\|u\|_{B^\gamma_q} := \|P_0 u\|_{L^q} + \Big(\sum_{N\in 2^{\mathbb{N}}} N^{2\gamma}\|P_N u\|^2_{L^q}\Big)^{1/2},$$

where Δ is the free Laplace operator on $\mathbb{R}^d$. Now, let $\mathscr{S}_0$ be a subspace of $\mathscr{S}$ consisting of functions ϕ satisfying $D^\alpha\hat{\phi}(0) = 0$ for all $\alpha \in \mathbb{N}^d$, where $\hat{\cdot}$ is the Fourier transform on $\mathscr{S}$. The generalized homogeneous Sobolev and Besov spaces are defined respectively as closures of $\mathscr{S}_0$ under the norms

$$\|u\|_{\dot{H}^\gamma_q} := \||\nabla|^\gamma u\|_{L^q},$$
$$\|u\|_{\dot{B}^\gamma_q} := \Big(\sum_{N\in 2^{\mathbb{Z}}} N^{2\gamma}\|P_N u\|^2_{L^q}\Big)^{1/2}.$$

We again refer the reader to [GV85, Appendix], [Tri83, Chapter 5] or [BL76, Chapter 6] for various properties of these function spaces. It is easy to see that the spaces B^γ_q and $\dot{B}^\gamma_q$ do not depend on the choice of χ_0. Note that $H^\gamma_q, B^\gamma_q, \dot{H}^\gamma_q$ and $\dot{B}^\gamma_q$ are Banach spaces with the norms $\|u\|_{H^\gamma_q}, \|u\|_{B^\gamma_q}, \|u\|_{\dot{H}^\gamma_q}$ and $\|u\|_{\dot{B}^\gamma_q}$ respectively. In the sequel, we shall use $H^\gamma := H^\gamma_2$, $\dot{H}^\gamma := \dot{H}^\gamma_2$. By the Littlewood-Paley theorem, we see that if $2 \leq q < \infty$, then $\dot{B}^\gamma_q \subset \dot{H}^\gamma_q$ with the reverse inclusion for $1 < q \leq 2$. In particular, $\dot{B}^\gamma_2 = \dot{H}^\gamma$ and $\dot{B}^0_2 = \dot{H}^0_2 = L^2$. Moreover, if $\gamma > 0$, then $H^\gamma_q = L^q \cap \dot{H}^\gamma_q$ and $B^\gamma_q = L^q \cap \dot{B}^\gamma_q$.

[1] Note that one can choose χ_0 to be radially symmetric, and then so is χ.

Throughout this thesis, we denote for $(p,q) \in [1,\infty]^2$,

$$\gamma_{p,q} = \frac{d}{2} - \frac{d}{q} - \frac{\sigma}{p}. \tag{1.0.7}$$

1.1 Strichartz estimates for Schrödinger-type equations on the flat Euclidean space

Let $\sigma \in (0,\infty)\backslash\{1\}$ and consider the inhomogeneous linear Schrödinger-type equations on $\mathbb{R}^d, d \geq 1$,

$$\begin{cases} i\partial_t u(t,x) + |\nabla|^\sigma u(t,x) &= F(t,x), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ u(0,x) &= \psi(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{LST}$$

where $|\nabla|^\sigma$ is the Fourier multiplier by $|\xi|^\sigma$ with $\Delta = \sum_{j=1}^d \partial_j^2$ the free Laplace operator on $\mathbb{R}^d$. The Duhamel formula (see e.g. [Tao06, Proposition 1.35]) shows that the (LST) is essentially equivalent to the integral equation

$$u(t) = e^{it|\nabla|^\sigma}\psi - i\int_0^t e^{i(t-s)|\nabla|^\sigma} F(s)ds. \tag{1.1.1}$$

The purpose of this section is to derive Strichartz estimates for the (LST). To do so, we introduce the following admissible condition.

Definition 1.1.1. A pair (p,q) is said to be **Schrödinger admissible** if

$$(p,q) \in [2,\infty]^2, \quad (p,q,d) \neq (2,\infty,2), \quad \frac{2}{p} + \frac{d}{q} \leq \frac{d}{2}. \tag{1.1.2}$$

Theorem 1.1.2. *Let $d \geq 1, \sigma \in (0,\infty)\backslash\{1\}, \gamma \in \mathbb{R}$. If u is a solution to the* (LST) *for some data ψ, F, then for all (p,q) and (a,b) Schrödinger admissible pairs,*

$$\|u\|_{L^p(\mathbb{R},\dot{B}^\gamma_q)} \lesssim \|\psi\|_{\dot{H}^{\gamma+\gamma_{p,q}}} + \|F\|_{L^{a'}(\mathbb{R},\dot{B}^{\gamma+\gamma_{p,q}-\gamma_{a',b'}-\sigma}_{b'})}, \tag{1.1.3}$$

where $\gamma_{p,q}$ and $\gamma_{a',b'}$ are as in (1.0.7). *In particular,*

$$\|u\|_{L^p(\mathbb{R},\dot{B}^{\gamma-\gamma_{p,q}}_q)} \lesssim \|\psi\|_{\dot{H}^\gamma} + \|F\|_{L^1(\mathbb{R},\dot{H}^\gamma)}, \tag{1.1.4}$$

and

$$\|u\|_{L^\infty(\mathbb{R},\dot{B}^{\gamma_{p,q}}_2)} + \|u\|_{L^p(\mathbb{R},\dot{B}^0_q)} \lesssim \|\psi\|_{\dot{H}^{\gamma_{p,q}}} + \|F\|_{L^{a'}(\mathbb{R},\dot{B}^0_{b'})}, \tag{1.1.5}$$

provided that

$$\gamma_{p,q} = \gamma_{a',b'} + \sigma. \tag{1.1.6}$$

Here (a,a') and (b,b') are conjugate pairs.

Proof. We first note that the Minkowski inequality with $p,q \geq 2$ gives

$$\|u\|_{L^p(\mathbb{R},\dot{B}^\gamma_q)} \lesssim \Big(\sum_{N\in 2^{\mathbb{Z}}} N^{2\gamma}\|P_N u\|^2_{L^p(\mathbb{R},L^q)}\Big)^{1/2}. \tag{1.1.7}$$

Therefore, the theorem is proved if we establish

$$\|e^{it|\nabla|^\sigma} P_1\psi\|_{L^p(\mathbb{R},L^q)} \lesssim \|P_1\psi\|_{L^2}, \tag{1.1.8}$$

$$\Big\| \int_0^t e^{i(t-s)|\nabla|^\sigma} P_1F(s)ds \Big\|_{L^p(\mathbb{R},L^q)} \lesssim \|P_1F\|_{L^{a'}(\mathbb{R},L^{b'})}, \tag{1.1.9}$$

for all (p,q), (a,b) Schrödinger admissible pairs. Indeed, by change of variables, we see that

$$\|e^{it|\nabla|^\sigma} P_N\psi\|_{L^p(\mathbb{R},L^q)} = N^{-(d/q+\sigma/p)}\|e^{it|\nabla|^\sigma} P_1\psi_N\|_{L^p(\mathbb{R},L^q)},$$
$$\|P_1\psi_N\|_{L^2} = N^{d/2}\|P_N\psi\|_{L^2},$$

where $\psi_N(x)=\psi(N^{-1}x)$. To see this, we write

$$\begin{aligned} e^{it|\nabla|^\sigma} P_N\psi(t,x) &= (2\pi)^{-d}\int e^{ix\xi}e^{it|\xi|^\sigma}\chi(N^{-1}\xi)\hat{\psi}(\xi)d\xi \\ &= (2\pi)^{-d}\int e^{iNx\xi}e^{itN^\sigma|\xi|^\sigma}\chi(\xi)\widehat{\psi_N}(\xi)d\xi = e^{it|\nabla|^\sigma}P_1\psi_N(N^\sigma t, Nx), \end{aligned}$$

where $\psi_N(x):=\psi(N^{-1}x)$. The estimate (1.1.8) implies that

$$\|e^{it|\nabla|^\sigma} P_N\psi\|_{L^p(\mathbb{R},L^q)} \lesssim N^{\gamma_{p,q}}\|P_N\psi\|_{L^2}, \tag{1.1.10}$$

for all $N\in 2^{\mathbb{Z}}$. Similarly,

$$\Big\| \int_0^t e^{i(t-s)|\nabla|^\sigma} P_NF(s)ds \Big\|_{L^p(\mathbb{R},L^q)} = N^{-(d/q+\sigma/p+\sigma)}\Big\| \int_0^t e^{i(t-s)|\nabla|^\sigma} P_1F_N(s)ds \Big\|_{L^p(\mathbb{R},L^q)},$$

where $F_N(t,x)=F(N^{-\sigma}t,N^{-1}x)$. We also have from (1.1.9) and the fact

$$\|P_1F_N\|_{L^{a'}(\mathbb{R},L^{b'})} = N^{(d/b'+\sigma/a')}\|P_NF\|_{L^{a'}(\mathbb{R},L^{b'})}$$

that

$$\Big\| \int_0^t e^{i(t-s)|\nabla|^\sigma} P_NF(s)ds \Big\|_{L^p(\mathbb{R},L^q)} \lesssim N^{\gamma_{p,q}-\gamma_{a',b'}-\sigma}\|P_NF\|_{L^{a'}(\mathbb{R},L^{b'})}, \tag{1.1.11}$$

for all $N\in 2^{\mathbb{Z}}$. We see from (1.1.10) and (1.1.11) that

$$N^\gamma\|P_Nu\|_{L^p(\mathbb{R},L^q)} \lesssim N^{\gamma+\gamma_{p,q}}\|P_N\psi\|_{L^2} + N^{\gamma+\gamma_{p,q}-\gamma_{a',b'}-\sigma}\|P_NF\|_{L^{a'}(\mathbb{R},L^{b'})}.$$

By taking the $\ell^2(2^{\mathbb{Z}})$ norm both sides and using (1.1.7), we get (1.1.3). The estimate (1.1.4) follows from (1.1.3) by taking $\gamma=\gamma-\gamma_{p,q}$ and $(a,b)=(\infty,2)$. The estimate (1.1.5) follows again from (1.1.3) by taking $(p,q)=(\infty,2)$ with $\gamma=\gamma_{p,q}$ and $\gamma=0$. Let us prove (1.1.8) and (1.1.9). By the TT^*-criterion given in Theorem 1.0.1, we need to show

$$\|T(t)\|_{L^2\to L^2} \lesssim 1, \tag{1.1.12}$$

$$\|T(t)\|_{L^1\to L^\infty} \lesssim (1+|t|)^{-d/2}, \tag{1.1.13}$$

for all $t\in\mathbb{R}$ where $T(t):=e^{it|\nabla|^\sigma}P_1$. The energy estimate (1.1.12) is obvious by using the Plancherel theorem. It remains to prove the dispersive estimate (1.1.13). To do this, we first write the kernel of $T(t)$ as

$$K(t,x,y) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i((x-y)\cdot\xi - t|\xi|^\sigma)}\chi(\xi)d\xi.$$

The estimate (1.1.13) is then in turn equivalent to

$$|K(t,x,y)| \lesssim (1+|t|)^{-d/2}, \tag{1.1.14}$$

for all $t \in \mathbb{R}$ and all $x, y \in \mathbb{R}^d$. We only prove (1.1.14) for $t \geq 0$, the case $t < 0$ is similar. Thanks to the compact support of χ, we have $|K(t,x,y)| \lesssim 1, \forall t \in \mathbb{R}, x, y \in \mathbb{R}^d$. In the case $0 \leq t \leq C$ for some constant $C > 0$ large enough, we have

$$|K(t,x,y)| \lesssim 1 \lesssim (1+t)^{-d/2}, \quad \forall x, y \in \mathbb{R}^d.$$

In the case $t \geq C$, we rewrite

$$K(t,x,y) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{it\Phi(t,x,y,\xi)} \chi(\xi) d\xi,$$

where $\Phi(t,x,y,\xi) = (x-y)\cdot \xi/t - |\xi|^\sigma$.

If $|x-y|/t \geq C_1$ for some constant $C_1 > 0$ large enough, then using that $\sigma \neq 1$ and $1/2 \leq |\xi| \leq 2$, we have

$$|\nabla_\xi \Phi(t,x,y,\xi)| \geq |x-y|/t - \sigma|\xi|^{\sigma-1} \geq C > 0.$$

The non stationary phase theorem implies that for all $N \geq 0$,

$$|K(t,x,y)| \lesssim t^{-N} \lesssim (1+t)^{-d/2},$$

for all $t \geq C$ and all $x, y \in \mathbb{R}^d$ satisfying $|x-y|/t \geq C_1$ provided that N is taken larger than $d/2$. A similar result holds with $|x-y|/t \leq C_2$ for some constant $C_2 > 0$ small enough.

We can now assume that

$$C_2 \leq |x-y|/t \leq C_1. \tag{1.1.15}$$

We have

$$\nabla^2_\xi \Phi(t,x,y,\xi) = -\sigma|\xi|^{\sigma-2}\Big(I_{\mathbb{R}^d} + (\sigma-2)\frac{\xi\cdot\xi^T}{|\xi|^2}\Big).$$

This implies that

$$|\det \nabla^2_\xi \Phi| = \sigma^d|\sigma-1||\xi|^{(\sigma-2)d} \geq C > 0.$$

Thus, the map $\xi \mapsto \nabla_\xi \Phi$ from a neighborhood of $\{\xi \in \mathbb{R}^d, 1/2 \leq |\xi| \leq 2\}$ to its range is a local diffeomorphism. The stationary phase theorem then implies that

$$|K(t,x,y)| \lesssim t^{-d/2} \lesssim (1+t)^{-d/2},$$

for all $t \geq C$ and all $x, y \in \mathbb{R}^d$ satisfying (1.1.15). This completes the proof. □

We next give some applications of Strichartz estimates given in Theorem 1.1.2.

Corollary 1.1.3. *Let $d \geq 1, \sigma \in (0,\infty)\backslash\{1\}, \gamma \in \mathbb{R}$. If u is a solution to the* (LST) *for some data ψ, F, then for all (p,q) and (a,b) Schrödinger admissible with $q < \infty$ and $b < \infty$ satisfying* (1.1.6),

$$\|u\|_{L^p(\mathbb{R},\dot{H}^{\gamma-\gamma_{p,q}}_q)} \lesssim \|\psi\|_{\dot{H}^\gamma} + \|F\|_{L^1(\mathbb{R},\dot{H}^\gamma)}, \tag{1.1.16}$$

$$\|u\|_{L^\infty(\mathbb{R},\dot{H}^{\gamma_{p,q}})} + \|u\|_{L^p(\mathbb{R},L^q)} \lesssim \|\psi\|_{\dot{H}^{\gamma_{p,q}}} + \|F\|_{L^{a'}(\mathbb{R},L^{b'})}. \tag{1.1.17}$$

Corollary 1.1.4. *Let $d \geq 1$, $\sigma \in (0,\infty)\backslash\{1\}$, $\gamma \geq 0$ and I a bounded interval. If u is a solution to the* (LST) *for some data φ, F, then for all (p,q) Schrödinger admissible satisfying $q < \infty$,*

$$\|u\|_{L^p(I,H^{\gamma-\gamma_{p,q}}_q)} \lesssim \|\psi\|_{H^\gamma} + \|F\|_{L^1(I,H^\gamma)}. \tag{1.1.18}$$

Proof. We first note that when $\gamma_{p,q} \geq 0$ (or at least $\sigma \in (0,2]\backslash\{1\}$), we can obtain (1.1.18) for any $\gamma \in \mathbb{R}$ and $I = \mathbb{R}$. To see this, we write $\|u\|_{L^p(\mathbb{R},H^{\gamma-\gamma_{p,q}}_q)} = \|\langle\nabla\rangle^{\gamma-\gamma_{p,q}} u\|_{L^p(\mathbb{R},L^q)}$ and use (1.1.16) with $\gamma = \gamma_{p,q}$ to obtain

$$\|u\|_{L^p(\mathbb{R},H_q^{\gamma-\gamma_{p,q}})} \lesssim \| \langle\nabla\rangle^{\gamma-\gamma_{p,q}} \psi\|_{\dot{H}^{\gamma_{p,q}}} + \| \langle\nabla\rangle^{\gamma-\gamma_{p,q}} F\|_{L^1(\mathbb{R},\dot{H}^{\gamma_{p,q}})}.$$

This gives the claim since $\|v\|_{\dot{H}^{\gamma_{p,q}}} \leq \|v\|_{H^{\gamma_{p,q}}}$ using that $\gamma_{p,q} \geq 0$. It remains to treat the case $\gamma_{p,q} < 0$. By the Minkowski inequality and the unitarity of $e^{it|\nabla|^\sigma}$ in L^2, the estimate (1.1.18) is proved if we can show for $\gamma \geq 0$, $I \subset \mathbb{R}$ a bounded interval and all (p,q) Schrödinger admissible with $q < \infty$ that

$$\|e^{it|\nabla|^\sigma}\psi\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \lesssim \|\psi\|_{H^\gamma}. \tag{1.1.19}$$

Indeed, if we have (1.1.19), then

$$\begin{aligned}\Big\| \int_0^t e^{i(t-s)|\nabla|^\sigma} F(s)ds\Big\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} &\leq \int_I \|\mathbb{1}_{[0,t]}(s)e^{i(t-s)|\nabla|^\sigma}F(s)\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})}ds \\ &\leq \int_I \|e^{i(t-s)|\nabla|^\sigma}F(s)\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})}ds \\ &\lesssim \int_I \|F(s)\|_{H^\gamma}ds = \|F\|_{L^1(I,H^\gamma)}.\end{aligned}$$

We now prove (1.1.19). To do so, we write

$$\langle\nabla\rangle^{\gamma-\gamma_{p,q}} e^{it|\nabla|^\sigma}\psi = \omega(D)\langle\nabla\rangle^{\gamma-\gamma_{p,q}} e^{it|\nabla|^\sigma}\psi + (1-\omega)(D)\langle\nabla\rangle^{\gamma-\gamma_{p,q}} e^{it|\nabla|^\sigma}\psi,$$

for some $\omega \in C_0^\infty(\mathbb{R}^d)$ valued in $[0,1]$ and equal to 1 near the origin. Here $\omega(D)$ is the Fourier multiplier by $\omega(\xi)$. For the first term, the Sobolev embedding implies

$$\|\omega(D)\langle\nabla\rangle^{\gamma-\gamma_{p,q}} e^{it\nabla^\sigma}\psi\|_{L^q} \lesssim \|\omega(D)\langle\nabla\rangle^{\gamma-\gamma_{p,q}} e^{it\nabla^\sigma}\psi\|_{H^\delta},$$

for some $\delta > d/2 - d/q$. Thanks to the support of ω and the unitary property of $e^{it\nabla^\sigma}$ in L^2, we get

$$\|\omega(D)\langle\nabla\rangle^{\gamma-\gamma_{p,q}} e^{it|\nabla|^\sigma}\psi\|_{L^p(I,L^q)} \lesssim \|\psi\|_{L^2} \lesssim \|\psi\|_{H^\gamma}.$$

Here the boundedness of I is crucial to have the first estimate. For the second term, using (1.1.17), we obtain

$$\|(1-\omega)(D)\langle\nabla\rangle^{\gamma-\gamma_{p,q}} e^{it|\nabla|^\sigma}\psi\|_{L^p(I,L^q)} \lesssim \|(1-\omega)(D)\langle\nabla\rangle^{\gamma-\gamma_{p,q}}\psi\|_{\dot{H}^{\gamma_{p,q}}} \lesssim \|\psi\|_{H^\gamma}.$$

Combining the two terms, we have (1.1.19). This completes the proof. □

Another application of Strichartz estimates for the (LST) is the following Strichartz estimates for the following inhomogeneous linear wave-type equations,

$$\left\{\begin{array}{rcl} \partial_t^2 v(t,x) + (-\Delta)^\sigma v(t,x) &=& G(t,x), \quad (t,x)\in\mathbb{R}\times\mathbb{R}^d, \\ v(0,x) = \psi(x), && \partial_t v(0,x) = \phi(x), \quad x\in\mathbb{R}^d. \end{array}\right. \tag{LWT}$$

Corollary 1.1.5. *Let $d \geq 1, \sigma \in (0,\infty)\backslash\{1\}, \gamma \in \mathbb{R}$. If v is a solution to the* (LWT) *for some data ψ, ϕ, G, then for all (p,q) and (a,b) Schrödinger admissible pairs,*

$$\|[v]\|_{L^p(\mathbb{R},\dot{B}_q^\gamma)} \lesssim \|[v](0)\|_{\dot{H}^{\gamma+\gamma_{p,q}}} + \|G\|_{L^{a'}(\mathbb{R},\dot{B}_{b'}^{\gamma+\gamma_{p,q}-\gamma_{a',b'}-2\sigma})}, \tag{1.1.20}$$

where

$$\begin{aligned}\|[v]\|_{L^p(\mathbb{R},\dot{B}_q^\gamma)} &:= \|v\|_{L^p(\mathbb{R},\dot{B}_q^\gamma)} + \|\partial_t v\|_{L^p(\mathbb{R},\dot{B}_q^{\gamma-\sigma})}, \\ \|[v](0)\|_{\dot{H}^{\gamma+\gamma_{p,q}}} &:= \|\psi\|_{\dot{H}^{\gamma+\gamma_{p,q}}} + \|\phi\|_{\dot{H}^{\gamma+\gamma_{p,q}-\sigma}}.\end{aligned}$$

In particular,

$$\|[v]\|_{L^p(\mathbb{R},\dot{B}_q^{\gamma-\gamma_{p,q}})} \lesssim \|[v](0)\|_{\dot{H}^\gamma} + \|G\|_{L^1(\mathbb{R},\dot{H}^{\gamma-\sigma})}, \tag{1.1.21}$$

and

$$\|[v]\|_{L^\infty(\mathbb{R},\dot{B}_2^{\gamma_{p,q}})} + \|[v]\|_{L^p(\mathbb{R},\dot{B}_q^0)} \lesssim \|[v](0)\|_{\dot{H}^{\gamma_{p,q}}} + \|G\|_{L^{a'}(\mathbb{R},\dot{B}^0_{b'})}, \tag{1.1.22}$$

provided that

$$\gamma_{p,q} = \gamma_{a',b'} + 2\sigma. \tag{1.1.23}$$

Proof. By Duhamel's formula, the solution to (LWT) is given by

$$v(t) = \cos(t|\nabla|^\sigma)\psi + \frac{\sin(t|\nabla|^\sigma)}{|\nabla|^\sigma}\phi + \int_0^t \frac{\sin((t-s)|\nabla|^\sigma)}{|\nabla|^\sigma} G(s)ds.$$

The desired estimates follow easily from Theorem 1.1.2 and the fact that

$$\cos(t|\nabla|^\sigma) = \frac{e^{it|\nabla|^\sigma} + e^{-it|\nabla|^\sigma}}{2}, \quad \sin(t|\nabla|^\sigma) = \frac{e^{it|\nabla|^\sigma} - e^{-it|\nabla|^\sigma}}{2i}.$$

□

As in Corollary 1.1.3, we have the following usual Strichartz estimates for fractional wave equations.

Corollary 1.1.6. *Let $d \geq 1, \sigma \in (0,\infty)\backslash\{1\}, \gamma \in \mathbb{R}$. If v is a solution to the* (LWT) *for some data ψ, ϕ, G, then for all (p,q) and (a,b) Schrödinger admissible satisfying $q<\infty, b<\infty$ and* (1.1.23),

$$\|v\|_{L^p(\mathbb{R},\dot{H}_q^{\gamma-\gamma_{p,q}})} \lesssim \|[v](0)\|_{\dot{H}^\gamma} + \|G\|_{L^1(\mathbb{R},\dot{H}^{\gamma-\sigma})}, \tag{1.1.24}$$

$$\|[v]\|_{L^\infty(\mathbb{R},\dot{H}^{\gamma_{p,q}})} + \|v\|_{L^p(\mathbb{R},L^q)} \lesssim \|[v](0)\|_{\dot{H}^{\gamma_{p,q}}} + \|G\|_{L^{a'}(\mathbb{R},L^{b'})}. \tag{1.1.25}$$

The following result, which is similar to Corollary 1.1.4, gives the local Strichartz estimates for the fractional wave equation.

Corollary 1.1.7. *Let $d \geq 1$, $\sigma \in (0,\infty)\backslash\{1\}$, $\gamma \geq 0$ and $I \subset \mathbb{R}$ a bounded interval. If v is a solution to the inhomogeneous linear wave-type equation for some data ψ, ϕ, G, then for all (p,q) Schrödinger admissible satisfying $q < \infty$,*

$$\|v\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \lesssim \|[v](0)\|_{H^\gamma} + \|G\|_{L^1(I,H^{\gamma-\sigma})}. \tag{1.1.26}$$

Proof. The proof is similar to the one of Corollary 1.1.4. Thanks to the Minkowski inequality, it suffices to prove for all $\gamma \geq 0$, all $I \subset \mathbb{R}$ bounded interval and all (p,q) Schrödinger admissible pair with $q < \infty$,

$$\|\cos(t|\nabla|^\sigma)\psi\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \lesssim \|\psi\|_{H^\gamma}, \tag{1.1.27}$$

$$\Big\|\frac{\sin(t|\nabla|^\sigma)}{|\nabla|^\sigma}\phi\Big\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \lesssim \|\phi\|_{H^{\gamma-\sigma}}. \tag{1.1.28}$$

The estimate (1.1.27) follows from the ones of $e^{\pm it|\nabla|^\sigma}$. We will give the proof of (1.1.28). To do this, we write

$$\langle\nabla\rangle^{\gamma-\gamma_{p,q}} \frac{\sin(t|\nabla|^\sigma)}{|\nabla|^\sigma} = \omega(D)\langle\nabla\rangle^{\gamma-\gamma_{p,q}} \frac{\sin(t|\nabla|^\sigma)}{|\nabla|^\sigma} + (1-\omega)(D)\langle\nabla\rangle^{\gamma-\gamma_{p,q}} \frac{\sin(t|\nabla|^\sigma)}{|\nabla|^\sigma},$$

for some ω as in the proof of Corollary 1.1.4. For the first term, the Sobolev embedding and the fact $\left\|\frac{\sin(t|\nabla|^\sigma)}{|\nabla|^\sigma}\right\|_{L^2\to L^2} \leq |t|$ imply

$$\left\|\omega(D)\langle\nabla\rangle^{\gamma-\gamma_{p,q}}\frac{\sin(t|\nabla|^\sigma)}{|\nabla|^\sigma}\phi\right\|_{L^q} \lesssim |t|\|\omega(D)\langle\nabla\rangle^{\gamma+\delta-\gamma_{p,q}}\phi\|_{L^2},$$

for some $\delta > d/2 - d/q$. This gives

$$\left\|\omega(D)\langle\nabla\rangle^{\gamma-\gamma_{p,q}}\frac{\sin(t|\nabla|^\sigma)}{|\nabla|^\sigma}\phi\right\|_{L^p(I,L^q)} \lesssim \|\phi\|_{H^{\gamma-\sigma}}.$$

Here we use that $\|\omega(D)\langle\nabla\rangle^{\delta+\sigma-\gamma_{p,q}}\|_{L^2\to L^2} \lesssim 1$. For the second term, we apply (1.1.19) with the fact $\sin(t|\nabla|^\sigma) = (e^{it|\nabla|^\sigma} - e^{-it|\nabla|^\sigma})/2i$ and get

$$\left\|(1-\omega)(D)\langle\nabla\rangle^{\gamma-\gamma_{p,q}}\frac{\sin(t|\nabla|^\sigma)}{|\nabla|^\sigma}\phi\right\|_{L^p(I,L^q)} \lesssim \|(1-\omega)(D)|\nabla|^{-\sigma}\phi\|_{H^\gamma} \lesssim \|\phi\|_{H^{\gamma-\sigma}}.$$

Here we also use that $\|(1-\omega)(D)\langle\nabla\rangle^\sigma|\nabla|^{-\sigma}\|_{L^2\to L^2} \lesssim 1$ by functional calculus. Combining two terms, we have (1.1.28). The proof is complete. □

1.2 Strichartz estimates for the half-wave equation on the flat Euclidean space

Let us now consider the inhomogeneous linear half-wave equation, namely

$$\left\{\begin{array}{rcl} i\partial_t u(t,x) + |\nabla|u(t,x) &=& F(t,x), \quad (t,x)\in\mathbb{R}\times\mathbb{R}^d, \\ u(0,x) &=& \psi(x), \quad x\in\mathbb{R}^d. \end{array}\right. \tag{LHW}$$

As for the (LST), the solution of (LHW) is given in terms of the Duhamel formula as

$$u(t,x) = e^{it|\nabla|}\psi - i\int_0^t e^{i(t-s)|\nabla|}F(s)ds. \tag{1.2.1}$$

In order to state Strichartz estimates for the (LHW), we introduce some notations.
Definition 1.2.1. A pair (p,q) is said to be **wave admissible** if

$$(p,q)\in[2,\infty]^2, \quad (p,q,d)\neq(2,\infty,3), \quad \frac{2}{p}+\frac{d-1}{q}\leq\frac{d-1}{2}.$$

Theorem 1.2.2 ([BCD11], [KT98], [KTV14])**.** *Let $d\geq 2, \gamma\in\mathbb{R}$ and u be a solution to the* (LHW)*, for some data ψ, F. Then for all (p,q) and (a,b) wave admissible pairs,*

$$\|u\|_{L^p(\mathbb{R},\dot{B}^\gamma_q)} \lesssim \|\psi\|_{\dot{H}^{\gamma+\gamma_{p,q}}} + \|F\|_{L^{a'}(\mathbb{R},\dot{B}^{\gamma+\gamma_{p,q}-\gamma_{a',b'}-1}_{q'})}, \tag{1.2.2}$$

where $\gamma_{p,q}$ and $\gamma_{a',b'}$ are as in (1.0.7) *with $\sigma=1$. In particular,*

$$\|u\|_{L^p(\mathbb{R},\dot{B}^{\gamma-\gamma_{p,q}}_q)} \lesssim \|\psi\|_{\dot{H}^\gamma} + \|F\|_{L^1(\mathbb{R},\dot{H}^\gamma)}, \tag{1.2.3}$$

and

$$\|u\|_{L^\infty(\mathbb{R},\dot{B}^{\gamma_{p,q}}_2)} + \|u\|_{L^p(\mathbb{R},\dot{B}^0_q)} \lesssim \|\psi\|_{\dot{H}^{\gamma_{p,q}}} + \|F\|_{L^{a'}(\mathbb{R},\dot{B}^0_{b'})}, \tag{1.2.4}$$

provided that

$$\gamma_{p,q} = \gamma_{a',b'} + 1. \tag{1.2.5}$$

Here (a,a') and (b,b') are conjugate pairs.

The proof of this result is based on the following spherical Fourier transform (see e.g. [Wol03]).

Lemma 1.2.3 (Spherical Fourier Transform). *Let σ be the hyper-surface measure of the sphere $\mathbb{S}^{d-1}$. Then, the spherical Fourier transform*

$$\hat{\sigma}(\xi) = \int_{\mathbb{S}^{d-1}} e^{-i\xi\cdot\theta} d\sigma(\theta)$$

satisfies

$$|\hat{\sigma}(|\xi|)| \lesssim |\xi|^{-\frac{d-1}{2}},$$

for $\xi \in \mathbb{R}^d$ with $|\xi|$ large.

Proof. Let us recall the fact that for any A an invertible linear maps from $\mathbb{R}^d$ to itself, we have

$$\widehat{f\circ A} = \frac{1}{|\det A|}\hat{f}\circ (A^T)^{-1},$$

where A^T is the transpose matrix of A. In particular, if A is an orthogonal transformation, i.e. $AA^t = I_{\mathbb{R}^d}$, then $\widehat{f\circ A} = \hat{f}\circ A$. From this and the facts that σ is invariant by orthogonal transformations and orthogonal transformations act transitively on $\mathbb{S}^{d-1}$, we have $\hat{\sigma}$ is radial. Moreover, $\hat{\sigma}$ is smooth. It then suffices to prove for $\xi = |\xi|e_d$, where $e_d = (0,...,0,1)$. We first choose an atlas on $\mathbb{S}^{d-1}$ as follows: $(U_j^\pm, B(0,1), \kappa_j^\pm)_{j=1}^d$ where $U_j^\pm = \left\{(x_1,...,x_j,...,x_d)\in\mathbb{S}^{d-1}, \pm x_j > 0\right\}$, and $B(0,1)$ is the open unit ball in $\mathbb{R}^{d-1}$ and

$$\begin{array}{rcl}\kappa_j^\pm : U_j^\pm \subset \mathbb{S}^{d-1} & \rightarrow & B(0,1)\in\mathbb{R}^{d-1}\\ (x_1,...,x_j,...,x_d) & \mapsto & (x_1,...,x_{j-1},x_{j+1},...,x_d).\end{array}$$

Let $(\phi_j^\pm)_{j=1}^d$ be a partition of unity associated to $(U_j^\pm)_{j=1}^d$. We can write

$$\hat{\sigma}(\xi) = \sum_{j=1}^d \left(\int_{U_j^+} e^{-i|\xi|e_d\cdot\theta}\phi_j^+(\theta)d\sigma(\theta) + \int_{U_j^-} e^{-i|\xi|e_d\cdot\theta}\phi_j^-(\theta)d\sigma(\theta)\right).$$

We separate this quantity into two terms. The first term is for the sum over $j=1$ to $d-1$ and second term for $j=d$. For the first term, we treat for $j=1$ only, the other ones are treated similarly. By writing

$$\begin{array}{rcl}\kappa_1^{\pm -1} : B(0,1)\in\mathbb{R}^{d-1} & \rightarrow & U_1^\pm \subset\mathbb{S}^{d-1}\\ z=(z_1,...,z_{d-1}) & \mapsto & \left(\pm\sqrt{1-|z|^2}, z\right),\end{array}$$

we have $\int_{U_1^+} e^{-i|\xi|e_d\cdot\theta}\phi_1^+(\theta)d\sigma(\theta) + \int_{U_1^-} e^{-i|\xi|e_d\cdot\theta}\phi_1^-(\theta)d\sigma(\theta)$ equals to

$$\int_{B(0,1)} e^{-i|\xi|z_{d-1}}\phi_1^+(\sqrt{1-|z|^2},z)\frac{dz}{\sqrt{1-|z|^2}} + \int_{B(0,1)} e^{-i|\xi|z_{d-1}}\phi_1^-(-\sqrt{1-|z|^2},z)\frac{dz}{\sqrt{1-|z|^2}}.$$

We see that in above integrals, the phases are non stationary, thus the first term can be bounded by $|\xi|^{-N}$ for all $N\geq 0$. For the second term, we process as above and it equals to

$$\int_{B(0,1)} e^{-i|\xi|\sqrt{1-|z|^2}}\phi_d^+(z,\sqrt{1-|z|^2})\frac{dz}{\sqrt{1-|z|^2}} + \int_{B(0,1)} e^{i|\xi|\sqrt{1-|z|^2}}\phi_d^-(z,-\sqrt{1-|z|^2})\frac{dz}{\sqrt{1-|z|^2}}.$$

The phase function $\sqrt{1-|z|^2}$ has only one critical point at zero and we have

$$\partial^2_{z_j z_k}\sqrt{1-|z|^2} = \begin{cases} -\frac{1}{\sqrt{1-|z|^2}} - \frac{z_j^2}{\sqrt{1-|z|^2}^3} & \text{when } j = k, \\ -\frac{z_j z_k}{\sqrt{1-|z|^2}^3} & \text{when } j \neq k, \end{cases}$$

for all $j, k = 1, ..., d-1$. This implies the Hessian of $\sqrt{1-|z|^2}$ at zero is $-I_{\mathbb{R}^{d-1}}$, so is invertible. We can apply the stationary phase theorem and the second term can be bounded by $|\xi|^{-\frac{d-1}{2}}$. Combining the two terms, we have the result. □

Proof of Theorem 1.2.2. As in the proof of Theorem 1.1.2, using

$$\begin{aligned} \|e^{it|\nabla|}P_N\psi\|_{L^p(\mathbb{R},L^q)} &= N^{-(d/q+1/p)}\|e^{it|\nabla|}P_1\psi_N\|_{L^p(\mathbb{R},L^q)}, \\ \|P_1\psi_N\|_{L^2} &= N^{d/2}\|P_N\psi\|_{L^2}, \\ \Big\| \int_0^t e^{i(t-s)|\nabla|}P_N F(s)ds \Big\|_{L^p(\mathbb{R},L^q)} &= N^{-(d/q+1/p+1)}\Big\| \int_0^t e^{i(t-s)|\nabla|}P_1 F_N(s)ds \Big\|_{L^p(\mathbb{R},L^q)}, \\ \|P_1F_N\|_{L^{a'}(\mathbb{R},L^{b'})} &= N^{(d/b'+1/a')}\|P_N F\|_{L^{a'}(\mathbb{R},L^{b'})}, \end{aligned}$$

where $\psi_N(x) = \psi(N^{-1}x)$ and $F_N(t,x) = F(N^{-1}t, N^{-1}x)$, the theorem is proved if we have

$$\|e^{it|\nabla|}P_1\psi\|_{L^p(\mathbb{R},L^q)} \lesssim \|P_1\psi\|_{L^2}, \tag{1.2.6}$$

$$\Big\| \int_0^t e^{i(t-s)|\nabla|}P_1 F(s)ds \Big\|_{L^p(\mathbb{R},L^q)} \lesssim \|P_1 F\|_{L^{a'}(\mathbb{R},L^{b'})}, \tag{1.2.7}$$

for all (p,q), (a,b) wave admissible pairs. By the $TT^\star$-criterion, it suffices to prove

$$\|T(t)\|_{L^2\to L^2} \lesssim 1, \tag{1.2.8}$$

$$\|T(t)\|_{L^1\to L^\infty} \lesssim (1+|t|)^{-(d-1)/2}, \tag{1.2.9}$$

for all $t\in\mathbb{R}$ where $T(t) = e^{it|\nabla|}P_1$. The energy estimate (1.2.8) again follows from the Plancherel theorem. We need to prove (1.2.9). To do so, we write the integral kernel of $T(t)$ as [2]

$$K(t,x,y) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i((x-y)\cdot\xi + t|\xi|)}\chi(|\xi|)d\xi.$$

Since χ is compactly supported, we have for all $t\in\mathbb{R}$,

$$|K(t,x,y)| \leq C,$$

for some constant $C>0$. It implies the required estimate if $|t|$ is small. Indeed, if $|t| \leq C'$ for some fixed $C'>0$ large, then $1+|t| \leq (1+C')$ or $(1+|t|)^{-\frac{d-1}{2}} \geq (1+C')^{-\frac{d-1}{2}} \gtrsim C$. Thus, we can assume that $|t| \geq C'$. The phase function $\Phi(t,x,y,\xi) = (x-y)\cdot\xi + t|\xi|$ satisfies

$$\nabla_\xi\Phi(t,x,y,\xi) = (x-y) + t\frac{\xi}{|\xi|}.$$

We remark that $\nabla_\xi\Phi = 0$ only if $|x-y| = |t|$ and the critical points of Φ lie on a line, hence are not isolated. So, the stationary phase theorem can not be applied directly. To overcome this difficulty, we use the polar coordinates, i.e. $\xi = r\theta$ with $r\in(0,+\infty)$ and $\theta\in\mathbb{S}^{d-1}$. The kernel reads

$$K(t,x,y) = (2\pi)^{-d}\int_0^{+\infty}\int_{\mathbb{S}^{d-1}} e^{ir((x-y)\cdot\theta + t)}\chi(r)r^{d-1}drd\sigma(\theta).$$

[2] Here χ is radially symmetric, i.e. there exists a function which is still denoted by χ so that $\chi(\xi) = \chi(|\xi|)$.

If $|t| \geq 2|x-y|$, then the phase is non stationary. By integration by parts with respect to r together with the fact

$$\partial_r \left(e^{ir((x-y)\cdot\theta+t)}\right) = i((x-y)\cdot\theta+t)e^{ir((x-y)\cdot\theta+t)}.$$

We have for all $N \geq 0$,

$$\begin{aligned}\left|\int_0^{+\infty} e^{ir((x-y)\cdot\theta+t)}\chi(r)r^{d-1}dr\right| &= \left|\frac{1}{(i((x-y)\cdot\theta+t))^N}\int_0^{+\infty} e^{ir((x-y)\cdot\theta+t)}(-\partial_r)^N(\chi(r)r^{d-1})dr\right| \\ &\lesssim |(x-y)\cdot\theta+t|^{-N} \leq 2^N|t|^{-N} \lesssim (1+|t|)^{-N}.\end{aligned}$$

If $|t| \leq 2|x-y|$, we can write the kernel

$$K(t,x,y) = (2\pi)^{-d}\int_0^{+\infty} e^{irt}\hat{\sigma}(r(y-x))\chi(r)r^{d-1}dr.$$

Using Lemma 1.2.3 we see that

$$|K(t,x,y)| \leq (2\pi)^{-d}\int_0^{+\infty} |r(y-x)|^{-\frac{d-1}{2}}\chi(r)r^{d-1}dr,$$

Since χ is compactly supported, we have

$$|K(t,x,y)| \lesssim |x-y|^{-\frac{d-1}{2}} \lesssim |t|^{-\frac{d-1}{2}} \lesssim (1+|t|)^{-\frac{d-1}{2}}.$$

Combine two cases, we have $|K(t,x,y)| \lesssim (1+|t|)^{-\frac{d-1}{2}}$ and this proves (1.2.9). The proof is complete. □

Corollary 1.2.4. *Let $d \geq 2$ and $\gamma \in \mathbb{R}$. If u is a solution to the* (LHW) *for some data ψ, F, then for all (p,q) wave admissible satisfying $q < \infty$,*

$$\|u\|_{L^p(\mathbb{R},H_q^{\gamma-\gamma_{p,q}})} \lesssim \|\psi\|_{H^\gamma} + \|F\|_{L^1(\mathbb{R},H^\gamma)}. \tag{1.2.10}$$

Proof. We first remark that (1.2.3) together with the Littlewood-Paley theorem yield for any (p,q) wave admissible satisfying $q < \infty$,

$$\|u\|_{L^p(\mathbb{R},\dot{H}_q^{\gamma-\gamma_{p,q}})} \lesssim \|u_0\|_{\dot{H}^\gamma} + \|F\|_{L^1(\mathbb{R},\dot{H}^\gamma)}. \tag{1.2.11}$$

We next write $\|u\|_{L^p(\mathbb{R},H_q^{\gamma-\gamma_{p,q}})} = \|\langle\nabla\rangle^{\gamma-\gamma_{p,q}} u\|_{L^p(\mathbb{R},L^q)}$ and apply (1.2.11) with $\gamma = \gamma_{p,q}$ to get

$$\|u\|_{L^p(\mathbb{R},H_q^{\gamma-\gamma_{p,q}})} \lesssim \|\langle\nabla\rangle^{\gamma-\gamma_{p,q}} u_0\|_{\dot{H}^{\gamma_{p,q}}} + \|\langle\nabla\rangle^{\gamma-\gamma_{p,q}} F\|_{L^1(\mathbb{R},\dot{H}^{\gamma_{p,q}})}.$$

The estimate (1.2.10) then follows by using the fact that $\gamma_{p,q} > 0$ for all (p,q) is wave admissible satisfying $q < \infty$. □

Another consequence of Theorem 1.2.2 is the following Strichartz estimates for the following inhomogeneous linear wave equation,

$$\left\{\begin{array}{rcl}\partial_t^2 v(t,x) - \Delta v(t,x) &=& G(t,x), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ v(0,x) = \psi(x), && \partial_t v(0,x) = \phi(x), \quad x \in \mathbb{R}^d.\end{array}\right. \tag{LWE}$$

Corollary 1.2.5. *Let $d \geq 2, \gamma \in \mathbb{R}$. If v is a solution to the* (LWE) *for some data ψ, ϕ, G, then for all (p,q) and (a,b) wave admissible pairs,*

$$\|[v]\|_{L^p(\mathbb{R},\dot{B}_q^\gamma)} \lesssim \|[v](0)\|_{\dot{H}^{\gamma+\gamma_{p,q}}} + \|G\|_{L^{a'}(\mathbb{R},\dot{B}_{b'}^{\gamma+\gamma_{p,q}-\gamma_{a',b'}-2})}, \tag{1.2.12}$$

where

$$\|[v]\|_{L^p(\mathbb{R},\dot{B}^\gamma_q)} := \|v\|_{L^p(\mathbb{R},\dot{B}^\gamma_q)} + \|\partial_t v\|_{L^p(\mathbb{R},\dot{B}^{\gamma-1}_q)},$$
$$\|[v](0)\|_{\dot{H}^{\gamma+\gamma_{p,q}}} := \|\psi\|_{\dot{H}^{\gamma+\gamma_{p,q}}} + \|\phi\|_{\dot{H}^{\gamma+\gamma_{p,q}-1}}.$$

In particular,

$$\|[v]\|_{L^p(\mathbb{R},\dot{B}^{\gamma-\gamma_{p,q}}_q)} \lesssim \|[v](0)\|_{\dot{H}^\gamma} + \|G\|_{L^1(\mathbb{R},\dot{H}^{\gamma-1})}, \tag{1.2.13}$$

and

$$\|[v]\|_{L^\infty(\mathbb{R},\dot{B}^{\gamma_{p,q}}_2)} + \|[v]\|_{L^p(\mathbb{R},\dot{B}^0_q)} \lesssim \|[v](0)\|_{\dot{H}^{\gamma_{p,q}}} + \|G\|_{L^{a'}(\mathbb{R},\dot{B}^0_{b'})}, \tag{1.2.14}$$

provided that

$$\gamma_{p,q} = \gamma_{a',b'} + 2. \tag{1.2.15}$$

Proof. By Duhamel's formula, the solution to (LWE) is given by

$$v(t) = \cos(t|\nabla|)\psi + \frac{\sin(t|\nabla|)}{|\nabla|}\phi + \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|}G(s)ds.$$

The desired estimates follow easily from Theorem 1.2.2 and the fact that

$$\cos(t|\nabla|) = \frac{e^{it|\nabla|}+e^{-it|\nabla|}}{2}, \quad \sin(t|\nabla|) = \frac{e^{it|\nabla|}-e^{-it|\nabla|}}{2i}.$$

□

As in Corollary 1.2.4, we have the following usual Strichartz estimates for the inhomogeneous linear wave equation.

Corollary 1.2.6. *Let $d \geq 1, \gamma \in \mathbb{R}$. If v is a solution to the* (LWE) *for some data ψ, ϕ, G, then for all (p,q) and (a,b) wave admissible satisfying $q<\infty, b<\infty$ and* (1.2.15)*,*

$$\|v\|_{L^p(\mathbb{R},\dot{H}^{\gamma-\gamma_{p,q}}_q)} \lesssim \|[v](0)\|_{\dot{H}^\gamma} + \|G\|_{L^1(\mathbb{R},\dot{H}^{\gamma-1})}, \tag{1.2.16}$$
$$\|[v]\|_{L^\infty(\mathbb{R},\dot{H}^{\gamma_{p,q}})} + \|v\|_{L^p(\mathbb{R},L^q)} \lesssim \|[v](0)\|_{\dot{H}^{\gamma_{p,q}}} + \|G\|_{L^{a'}(\mathbb{R},L^{b'})}. \tag{1.2.17}$$

The following result, which is similar to Corollary 1.1.4, gives local Strichartz estimates for the inhomogeneous linear wave equation.

Corollary 1.2.7. *Let $d \geq 1$, $\gamma \geq 0$ and $I \subset \mathbb{R}$ a bounded interval. If v is a solution to the* (LWE) *for some data ψ, ϕ, G, then for all (p,q) wave admissible satisfying $q<\infty$,*

$$\|v\|_{L^p(I,H^{\gamma-\gamma_{p,q}}_q)} \lesssim \|[v](0)\|_{H^\gamma} + \|G\|_{L^1(I,H^{\gamma-1})}. \tag{1.2.18}$$

Proof. The proof is similar to the one of Corollary 1.1.7. Thanks to the Minkowski inequality, it suffices to prove for all $\gamma \geq 0$, all $I \subset \mathbb{R}$ bounded interval and all (p,q) wave admissible pair with $q<\infty$,

$$\|\cos(t|\nabla|)\psi\|_{L^p(I,H^{\gamma-\gamma_{p,q}}_q)} \lesssim \|\psi\|_{H^\gamma}, \tag{1.2.19}$$
$$\Big\|\frac{\sin(t|\nabla|)}{|\nabla|}\phi\Big\|_{L^p(I,H^{\gamma-\gamma_{p,q}}_q)} \lesssim \|\phi\|_{H^\gamma}. \tag{1.2.20}$$

The estimate (1.2.19) follows from the ones of $e^{\pm it|\nabla|}$. We will give the proof of (1.2.20). To do this, we write

$$\langle \nabla \rangle^{\gamma-\gamma_{p,q}} \frac{\sin(t|\nabla|)}{|\nabla|} = \omega(D) \langle \nabla \rangle^{\gamma-\gamma_{p,q}} \frac{\sin(t|\nabla|)}{|\nabla|} + (1-\omega)(D) \langle \nabla \rangle^{\gamma-\gamma_{p,q}} \frac{\sin(t|\nabla|)}{|\nabla|},$$

for some ω as in the proof of Corollary 1.1.4. For the first term, the Sobolev embedding and the fact $\left\| \frac{\sin(t|\nabla|)}{|\nabla|} \right\|_{L^2 \to L^2} \leq |t|$ imply

$$\left\| \omega(D) \langle \nabla \rangle^{\gamma-\gamma_{p,q}} \frac{\sin(t|\nabla|)}{|\nabla|} \phi \right\|_{L^q} \lesssim |t| \| \omega(D) \langle \nabla \rangle^{\gamma+\delta-\gamma_{p,q}} \phi \|_{L^2},$$

for some $\delta > d/2 - d/q$. This gives

$$\left\| \omega(D) \langle \nabla \rangle^{\gamma-\gamma_{p,q}} \frac{\sin(t|\nabla|)}{|\nabla|} \phi \right\|_{L^p(I, L^q)} \lesssim \|\phi\|_{H^{\gamma-1}}.$$

Here we use that $\|\omega(D) \langle \nabla \rangle^{\delta+1-\gamma_{p,q}} \|_{L^2 \to L^2} \lesssim 1$. For the second term, we apply (1.1.19) with $\sin(t|\nabla|) = (e^{it|\nabla|} - e^{-it|\nabla|})/2i$ to get

$$\left\| (1-\omega)(D) \langle \nabla \rangle^{\gamma-\gamma_{p,q}} \frac{\sin(t|\nabla|)}{|\nabla|} \phi \right\|_{L^p(I, L^q)} \lesssim \|(1-\omega)(D) |\nabla|^{-1} \phi\|_{H^\gamma} \lesssim \|\phi\|_{H^{\gamma-1}}.$$

Here we also use that $\|(1-\omega)(D) \langle \nabla \rangle |\nabla|^{-1}\|_{L^2 \to L^2} \lesssim 1$ by functional calculus. Combining two terms, we have (1.2.20). The proof is complete. □

Chapter 2

Strichartz estimates for the linear Schrödinger-type equations on bounded metric Euclidean spaces

Contents

This chapter deals with Strichartz estimates for the homogeneous linear Schrödinger-type equations on $\mathbb{R}^d$ equipped with a smooth bounded metric g, namely

$$\left\{ \begin{array}{rcl} i\partial_t u + |\nabla_g|^\sigma u &=& 0, \\ u(0) &=& \psi, \end{array} \right. \tag{2.0.1}$$

where $\sigma \in (0,\infty)\backslash\{1\}$ and $|\nabla_g| = \sqrt{-\Delta_g}$ with Δ_g the Laplace-Beltrami operator associated to the metric g.

Let $g(x) = (g_{jk}(x))_{j,k=1}^d$ be a metric on $\mathbb{R}^d$, and denote $G(x) = (g^{jk}(x))_{j,k=1}^d := g^{-1}(x)$. The Laplace-Beltrami operator associated to g reads

$$\Delta_g = \sum_{j,k=1}^d |g(x)|^{-1}\partial_j \left(g^{jk}(x)|g(x)|\partial_k\right),$$

where $|g(x)| := \sqrt{\det g(x)}$. Denote $P := -\Delta_g$ the self-adjoint realization of $-\Delta_g$. Recall that the principal symbol of P is

$$p(x,\xi) = \xi^t G(x)\xi = \sum_{j,k=1}^d g^{jk}(x)\xi_j\xi_k.$$

In this chapter, we assume that g satisfies the following assumptions.

1. There exists $C > 0$ such that for all $x, \xi \in \mathbb{R}^d$,

$$C^{-1}|\xi|^2 \leq \sum_{j,k=1}^d g^{jk}(x)\xi_j\xi_k \leq C|\xi|^2. \tag{2.0.2}$$

2. For all $\alpha \in \mathbb{N}^d$, there exists $C_\alpha > 0$ such that for all $x \in \mathbb{R}^d$,

$$|\partial^\alpha g^{jk}(x)| \leq C_\alpha, \quad j,k \in \{1,...,d\}. \tag{2.0.3}$$

We first note that the elliptic assumption (2.0.2) implies that $|g(x)|$ is bounded from below and above by positive constants. This shows that the space $L^q(\mathbb{R}^d, d\text{vol}_g), 1 \leq q \leq \infty$ where $d\text{vol}_g = |g(x)|dx$ and the usual Lebesgue space $L^q(\mathbb{R}^d)$ coincide. Thus in this chapter, the notation $L^q(\mathbb{R}^d)$

stands for either $L^q(\mathbb{R}^d, d\mathrm{vol}_g)$ or the usual Lebesgue space $L^q(\mathbb{R}^d)$. We will denote the space $L^q(\mathbb{R}^d)$ by L^q for short.

Let us first recall local (in time) Strichartz estimates for Schrödinger-type operators on $\mathbb{R}^d$ given in Corollary 1.1.4. For $\sigma \in (0,\infty)\backslash\{1\}$ and $I \subset \mathbb{R}$ a bounded interval, one has

$$\|e^{it|\nabla|^\sigma}\psi\|_{L^p(I,L^q)} \leq C\|\psi\|_{H^{\gamma_{p,q}}}, \tag{2.0.4}$$

where $|\nabla| = \sqrt{-\Delta}$, (p,q) is Schrödinger admissible with $q<\infty$ and $\gamma_{p,q}$ is as in (1.0.7).

It is well-known that under the assumptions (2.0.2) and (2.0.3), Strichartz estimates (2.0.4) may fail at least for the Schrödinger equation (see [BGT04, Appendix]) and in this case (i.e. $\sigma = 2$) one has a loss of $1/p$ derivatives, that is the right hand side of (2.0.4) is replaced by $\|\psi\|_{H^{\gamma_{p,q}+1/p}}$. Note that in [BGT04], the authors consider the sharp Schrödinger admissible condition with $q<\infty$ (see (0.0.1)). In this Chapter, we extend the result of Burq-Gérard-Tzvetkov to a more general setting, i.e. $\sigma \in (0,\infty)\backslash\{1\}$ and obtain Strichartz estimates with a "loss" of $(\sigma-1)/p$ derivatives when $\sigma \in (1,\infty)$ and without "loss" of derivatives when $\sigma \in (0,1)$. Throughout this chapter, the "loss" compares to (2.0.4).

Theorem 2.0.1. *Consider $\mathbb{R}^d, d \geq 1$ equipped with a smooth metric g satisfying* (2.0.2), (2.0.3) *and let $I \subset \mathbb{R}$ a bounded interval. If $\sigma \in (1,\infty)$, then for all (p,q) Schrödinger admissible with $q<\infty$, there exists $C>0$ such that for all $\psi \in H^{\gamma_{p,q}+(\sigma-1)/p}$,*

$$\|e^{it|\nabla_g|^\sigma}\psi\|_{L^p(I,L^q)} \leq C\|\psi\|_{H^{\gamma_{p,q}+(\sigma-1)/p}}, \tag{2.0.5}$$

where $\|u\|_{H^\gamma} := \|\langle\nabla_g\rangle^\gamma u\|_{L^2}$. If $\sigma \in (0,1)$, then (2.0.5) *holds with $\gamma_{p,q}+(\sigma-1)/p$ is replaced by $\gamma_{p,q}$.*

The proof of (2.0.5) is based on the WKB approximation which is similar to [BGT04]. Since we are working on manifolds, a good way is to decompose the semi-classical Schrödinger-type operator, namely $e^{ith^{-1}(h|\nabla_g|)^\sigma}$, at localized frequency, i.e. $e^{ith^{-1}(h|\nabla_g|)^\sigma}\varphi(h^2P)$ for some $\varphi \in C_0^\infty(\mathbb{R}\backslash\{0\})$. The main difficulty is that in general we do not have the exact form of the semi-classical fractional Laplace-Beltrami operator in order to use the usual construction in [BGT04]. To overcome this difficulty we write $e^{ith^{-1}(h|\nabla_g|)^\sigma}\varphi(h^2P)$ as $e^{ith^{-1}\omega(h^2P)}\varphi(h^2P)$ where $\omega(\lambda) = \tilde{\varphi}(\lambda)\sqrt{\lambda}^\sigma$ for some $\tilde{\varphi} \in C^\infty(\mathbb{R}\backslash\{0\})$ satisfying $\tilde{\varphi} = 1$ near $\mathrm{supp}(\varphi)$. We then approximate $\omega(h^2P)$ in terms of pseudo-differential operators and use the action of pseudo-differential operators on Fourier integral operators in order to construct an approximation for $e^{ith^{-1}\omega(h^2P)}\varphi(h^2P)$. This approximation gives dispersive estimates for $e^{ith^{-1}(h|\nabla_g|)^\sigma}\varphi(h^2P)$ on some small time interval independent of h. After scaling in time, we obtain Strichartz estimates without "loss" of derivatives over time intervals of size $h^{\sigma-1}$. When $\sigma \in (1,\infty)$, we can cumulate estimates over intervals of size $h^{\sigma-1}$ and get local in time Strichartz estimates with $(\sigma-1)/p$ loss of derivatives. In the case $\sigma \in (0,1)$, we can bound the estimates over time intervals of size 1 by the ones of size $h^{\sigma-1}$ and obtain the same Strichartz estimates as on $\mathbb{R}^d$. It is not a surprise that we recover the same Strichartz estimates as in the free case for $\sigma \in (0,1)$ since $e^{it|\nabla_g|^\sigma}$ has micro-locally the finite propagation speed property which is similar to $\sigma = 1$ for the half-wave equation. Intuitively, if we consider the free Hamiltonian $H(x,\xi) = |\xi|^\sigma$, then the spatial component of geodesic flow reads $x(t) = x(0) + t\sigma\xi|\xi|^{\sigma-2}$. After a time t, the distance $d(x(t),x(0)) \sim t|\xi|^{\sigma-1} \lesssim t$ if $\sigma - 1 \leq 0$ and $|\xi| \geq 1$. By decomposing the solution to $i\partial_t u + |\nabla|^\sigma u = 0$ as $u = \sum_{k\geq 0} u_k$ where $u_k = \varphi(2^{-k}D)u$ is localized near $|\xi| \sim 2^k \geq 1$, we see that after a time t, all components u_k have traveled at a distance t from the data $u_k(0)$.

Corollary 2.0.2. *Consider $\mathbb{R}^d, d \geq 1$ equipped with a smooth metric g satisfying* (2.0.2), (2.0.3) *and let $I \subset \mathbb{R}$ a bounded interval. Let u be a solution to the inhomogeneous linear Schrödinger-type equation on $(\mathbb{R}^d, g)$,*

$$\left\{ \begin{array}{rcl} i\partial_t u(t,x) + |\nabla_g|^\sigma u(t,x) &=& F(t,x), \quad (t,x) \in I \times \mathbb{R}^d, \\ u(0,x) &=& \psi(x), \quad x \in \mathbb{R}^d, \end{array} \right.$$

for some data ψ, F. If $\sigma \in (1,\infty)$, then for all (p,q) Schrödinger admissible with $q<\infty$, there *exists $C > 0$ such that*

$$\|u\|_{L^p(I,L^q)} \le C\Big(\|\psi\|_{H^{\gamma_{p,q}+(\sigma-1)/p}} + \|F\|_{L^1(I,H^{\gamma_{p,q}+(\sigma-1)/p})}\Big).$$

If $\sigma \in (0,1)$, then the above inequality holds with $\gamma_{p,q}$ in place of $\gamma_{p,q}+(\sigma-1)/p$.

Remark 2.0.3. By the same technique as in the proof of Theorem 2.0.1 up to some minor modifications, one can obtain easily local Strichartz estimates for the homogeneous linear half-wave equation on $(\mathbb{R}^d, g)$ which is similar to those (local in time) on $\mathbb{R}^d$ (see Corollary 1.2.4).

As a consequence of Theorem 2.0.1, we have the following Strichartz estimates for inhomogeneous linear wave-type equations posed on $(\mathbb{R}^d, g)$. Let us consider the following inhomogeneous linear wave-type equations posed on $(\mathbb{R}^d, g)$,

$$\begin{cases} \partial_t^2 v(t,x) + (-\Delta_g)^\sigma v(t,x) &= G(t,x), \quad (t,x) \in I \times \mathbb{R}^d, \\ v(0,x) = \psi(x), & \partial_t v(0,x) = \phi(x), \quad x \in \mathbb{R}^d. \end{cases} \tag{2.0.6}$$

We refer to [CH03] or [Her14] for the introduction of wave-type equations which arise in physics. Comparing with local Strichartz estimates for the inhomogeneous linear wave-type equations given in Section 1.1, we obtain estimates with a loss of derivatives $(\sigma-1)/p$ when $\sigma \in (1,\infty)$ and with no loss when $\sigma \in (0,1)$. More precisely, we have the following result.

Corollary 2.0.4. *Consider $\mathbb{R}^d, d \ge 1$ equipped with a smooth bounded metric g satisfying* (2.0.2), (2.0.3) *and let $I \subset \mathbb{R}$ a bounded interval. Let v be a solution to the inhomogeneous linear wave-type equation* (2.0.6). *If $\sigma \in (1,\infty)$, then for all (p,q) Schrödinger admissible with $q < \infty$, there exists $C > 0$ such that for all $(\psi, \phi) \in H^{\gamma_{p,q}+(\sigma-1)/p} \times H^{\gamma_{p,q}+(\sigma-1)/p-\sigma}$,*

$$\|v\|_{L^p(I,L^q)} \le C\Big(\|[v](0)\|_{H^{\gamma_{p,q}+(\sigma-1)/p}} + \|G\|_{L^1(I,H^{\gamma_{p,q}+(\sigma-1)/p-\sigma})}\Big), \tag{2.0.7}$$

where

$$\|[v](0)\|_{H^{\gamma_{p,q}+(\sigma-1)/p}} := \|\psi\|_{H^{\gamma_{p,q}+(\sigma-1)/p}} + \|\phi\|_{H^{\gamma_{p,q}+(\sigma-1)/p-\sigma}}.$$

If $\sigma \in (0,1)$, then (2.0.7) *holds with $\gamma_{p,q}+(\sigma-1)/p$ is replaced by $\gamma_{p,q}$.*

2.1 Reduction of problem

In this subsection, we give a reduction of Theorem 2.0.1 due to the Littlewood-Paley decomposition. To do so, we first recall some useful facts on pseudo-differential calculus. For $m \in \mathbb{R}$, we consider the symbol class $S(m)$ the space of smooth functions a on $\mathbb{R}^{2d}$ satisfying

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta} \langle \xi \rangle^{m-|\beta|},$$

for all $x, \xi \in \mathbb{R}^d$. We also need $S(-\infty) := \cap_{m\in\mathbb{R}} S(m)$. We define the semi-classical pseudo-differential operator with a symbol $a \in S(m)$ by

$$Op_h(a)u(x) := (2\pi h)^{-d} \iint_{\mathbb{R}^{2d}} e^{ih^{-1}(x-y)\cdot\xi} a(x,\xi) u(y) dy d\xi,$$

where $u \in \mathscr{S}$ is a Schwartz function. The following result gives the $L^q \to L^r$-bound for pseudo-differential operators (see e.g. [BT07, Proposition 2.4]).

Proposition 2.1.1. *Let $m > d$ and a be a continuous function on $\mathbb{R}^{2d}$ smooth with respect to the second variable satisfying for all $\beta \in \mathbb{N}^d$, there exists $C_\beta > 0$ such that for all $x, \xi \in \mathbb{R}^d$,*

$$|\partial_\xi^\beta a(x,\xi)| \le C_\beta \langle \xi \rangle^{-m}.$$

Then for all $1 \le q \le r \le \infty$, there exists $C > 0$ such that for all $h \in (0,1]$,

$$\|Op_h(a)\|_{L^q \to L^r} \le C h^{-\left(\frac{d}{q}-\frac{d}{r}\right)}.$$

For a given $f \in C_0^\infty(\mathbb{R})$, we can approximate $f(h^2P)$ in term of pseudo-differential operators, where P is the Laplace-Beltrami operator. We have the following result (see e.g [BT07, Proposition 2.5] or [BGT04, Proposition 2.1]).

Proposition 2.1.2. *Consider $\mathbb{R}^d$ equipped with a smooth metric g satisfying* (2.0.2) *and* (2.0.3). *Then for a given $f \in C_0^\infty(\mathbb{R})$, there exist a sequence of symbols $q_j \in S(-\infty)$ satisfying $q_0 = f \circ p$ and $\operatorname{supp}(q_j) \subset \operatorname{supp}(f \circ p)$ such that for all $N \geq 1$,*

$$f(h^2P) = \sum_{j=0}^{N-1} h^j Op_h(q_j) + h^N R_N(h),$$

and for all $m \geq 0$ and all $1 \leq q \leq r \leq \infty$, there exists $C > 0$ such that for all $h \in (0,1]$,

$$\|R_N(h)\|_{L^q \to L^r} \leq Ch^{-\left(\frac{d}{q}-\frac{d}{r}\right)}.$$
$$\|R_N(h)\|_{H^{-m} \to H^m} \leq Ch^{-2m}.$$

A direct consequence of Proposition 2.1.1 and Proposition 2.1.2 is the following $L^q \to L^r$-bound for $f(h^2P)$.

Proposition 2.1.3. *Let $f \in C_0^\infty(\mathbb{R})$. Then for all $1 \leq q \leq r \leq \infty$, there exists $C > 0$ such that for all $h \in (0,1]$,*

$$\|f(h^2P)\|_{L^q \to L^r} \leq Ch^{-\left(\frac{d}{q}-\frac{d}{r}\right)}.$$

Next, we need the following version of the Littlewood-Paley decomposition (see e.g. [BGT04, Corollary 2.3] or [BT07, Proposition 2.10]).

Proposition 2.1.4. *There exist $\varphi_0 \in C_0^\infty(\mathbb{R})$ and $\varphi \in C_0^\infty(\mathbb{R}\backslash\{0\})$ such that*

$$\varphi_0(P) + \sum_{h^{-1}:\text{dya}} \varphi(h^2P) = \text{Id},$$

where h^{-1} : dya means $h^{-1} = 2^k, k \in \mathbb{N}\backslash\{0\}$. Moreover, for all $q \in [2,\infty)$, there exists $C > 0$ such that for all $u \in \mathscr{S}$,

$$\|u\|_{L^q} \leq C\Big(\sum_{h^{-1}:\text{dya}} \|\varphi(h^2P)u\|^2_{L^q}\Big)^{1/2} + C\|u\|_{L^2}.$$

We end this subsection with the following reduction.

Proposition 2.1.5. *Consider $\mathbb{R}^d, d \geq 1$ equipped with a smooth metric g satisfying* (2.0.2), (2.0.3). *Let $\sigma \in (0,\infty)\backslash\{1\}$ and $\varphi \in C_0^\infty(\mathbb{R}\backslash\{0\})$. If there exist $t_0 > 0$ small enough and $C > 0$ such that for all $\psi \in L^1$ and all $h \in (0,1]$,*

$$\|e^{ith^{-1}(h|\nabla_g|)^\sigma}\varphi(h^2P)\psi\|_{L^\infty} \leq Ch^{-d}(1+|t|h^{-1})^{-d/2}\|\psi\|_{L^1}, \tag{2.1.1}$$

for all $t \in [-t_0, t_0]$, then Theorem 2.0.1 *holds true.*

The proof of Proposition 2.1.5 is based on the following semi-classical version of $TT^\star$-criterion (see [KT98], [Zwo12, Theorem 10.7] or [Zha15, Proposition 4.1]).

Theorem 2.1.6. *Let $(X, \mathcal{M}, \mu)$ be a σ-finite measured space, and $T : \mathbb{R} \to \mathcal{B}(L^2(X,\mathcal{M},\mu))$ be a weakly measurable map satisfying, for some constants $C, \gamma, \delta > 0$,*

$$\|T(t)\|_{L^2(X)\to L^2(X)} \leq C, \quad t \in \mathbb{R}, \tag{2.1.2}$$
$$\|T(t)T(s)^\star\|_{L^1(X)\to L^\infty(X)} \leq Ch^{-\delta}(1+|t-s|h^{-1})^{-\tau}, \quad t,s \in \mathbb{R}. \tag{2.1.3}$$

Then for all pair (p,q) satisfying

$$p \in [2,\infty], \quad q \in [1,\infty], \quad (p,q,\delta) \neq (2,\infty,1), \quad \frac{1}{p} \leq \tau\Big(\frac{1}{2}-\frac{1}{q}\Big),$$

one has

$$\|T(t)u\|_{L^p(\mathbb{R},L^q(X))} \leq Ch^{-\kappa}\|u\|_{L^2(X)},$$

where $\kappa = \delta(1/2-1/q)-1/p$.

Proof of Proposition 2.1.5. Using the energy estimates and dispersive estimates (2.1.1), we can apply Theorem 2.1.6 for $T(t) = \mathbb{1}_{[-t_0,t_0]}(t)e^{ith^{-1}(h|\nabla_g|)^\sigma}\varphi(h^2P), \delta = d, \tau = d/2$ and get

$$\|e^{ith^{-1}(h|\nabla_g|)^\sigma}\varphi(h^2P)\psi\|_{L^p([-t_0,t_0],L^q)} \leq Ch^{-(d/2-d/q-1/p)}\|\psi\|_{L^2}.$$

By scaling in time, we have

$$\begin{aligned}\|e^{it|\nabla_g|^\sigma}\varphi(h^2P)\psi\|_{L^p(h^{\sigma-1}[-t_0,t_0],L^q)} &= h^{(\sigma-1)/p}\|e^{ith^{-1}(h|\nabla_g|)^\sigma}\varphi(h^2P)\psi\|_{L^p([-t_0,t_0],L^q)} \\ &\leq Ch^{-\gamma_{p,q}}\|\psi\|_{L^2}. \end{aligned} \tag{2.1.4}$$

Using the group property and the unitary property of Schrödinger operator $e^{it|\nabla_g|^\sigma}$, we have similar estimates as in (2.1.4) for all intervals of size $2h^{\sigma-1}$. Indeed, for any interval I_h of size $2h^{\sigma-1}$, we can write $I_h = [c-h^{\sigma-1}t_0, c+h^{\sigma-1}t_0]$ for some $c \in \mathbb{R}$ and

$$\begin{aligned}\|e^{it|\nabla_g|^\sigma}\varphi(h^2P)\psi\|_{L^p(I_h,L^q)} &= \|e^{it|\nabla_g|^\sigma}\varphi(h^2P)e^{ic|\nabla_g|^\sigma}\psi\|_{L^p(h^{\sigma-1}[-t_0,t_0],L^q)} \\ &\leq Ch^{-\gamma_{p,q}}\|e^{ic|\nabla_g|^\sigma}\psi\|_{L^2} = Ch^{-\gamma_{p,q}}\|\psi\|_{L^2}.\end{aligned}$$

In the case $\sigma \in (1,\infty)$, we use a trick given in [BGT04], i.e. cumulating $O(h^{1-\sigma})$ estimates on intervals of length $2h^{\sigma-1}$ to get estimates on any finite interval I. Precisely, by writing I as a union of N intervals I_h of length $2h^{\sigma-1}$ with $N \lesssim h^{1-\sigma}$, we have

$$\begin{aligned}\|e^{it|\nabla_g|^\sigma}\varphi(h^2P)\psi\|_{L^p(I,L^q)} &\leq \Big(\sum_{I_h}\int_{I_h}\|e^{it|\nabla_g|^\sigma}\varphi(h^2P)\psi\|^p_{L^q}dt\Big)^{1/p} \\ &\leq CN^{1/p}h^{-\gamma_{p,q}}\|\psi\|_{L^2} \leq Ch^{-\gamma_{p,q}-(\sigma-1)/p}\|\psi\|_{L^2}.\end{aligned} \tag{2.1.5}$$

In the case $\sigma \in (0,1)$, we can obviously bound estimates over time intervals of size 1 by the ones of size $h^{\sigma-1}$ and obtain

$$\|e^{it|\nabla_g|^\sigma}\varphi(h^2P)\psi\|_{L^p(I,L^q)} \leq Ch^{-\gamma_{p,q}}\|\psi\|_{L^2}. \tag{2.1.6}$$

Moreover, we can replace the norm $\|\psi\|_{L^2}$ in the right hand side of (2.1.5) and (2.1.6) by $\|\varphi(h^2P)\psi\|_{L^2}$. Indeed, by choosing $\tilde{\varphi} \in C_0^\infty(\mathbb{R}\backslash\{0\})$ satisfying $\tilde{\varphi} = 1$ near supp(φ), we can write

$$e^{ith^{-1}(h|\nabla_g|)^\sigma}\varphi(h^2P)\psi = e^{ith^{-1}(h|\nabla_g|)^\sigma}\tilde{\varphi}(h^2P)\varphi(h^2P)\psi$$

and apply (2.1.5) and (2.1.6) with $\tilde{\varphi}$ in place of φ. Now, by using the Littlewood-Paley decomposition given in Proposition 2.1.4 and the Minkowski inequality, we have for all (p,q) Schrödinger admissible with $q<\infty$,

$$\|u\|_{L^p(I,L^q)} \leq C\Big(\sum_{h^{-1}:\text{dya}}\|\varphi(h^2P)u\|^2_{L^p(I,L^q)}\Big)^{1/2} + C\|u\|_{L^p(I,L^2)}. \tag{2.1.7}$$

We now apply (2.1.7) for $u = e^{it|\nabla_g|^\sigma}\psi$ together with (2.1.5) and get for $\sigma \in (1,\infty)$,

$$\|e^{it|\nabla_g|^\sigma}\psi\|_{L^p(I,L^q)} \leq C\Big(\sum_{h^{-1}:\text{dya}}h^{-2(\gamma_{p,q}+(\sigma-1)/p)}\|\varphi(h^2P)\psi\|^2_{L^2}\Big)^{1/2} + C\|\psi\|_{L^2}.$$

Here the boundedness of I is crucial to have a bound on the second term in the right hand side of

(2.1.7). The almost orthogonality and the fact that $\gamma_{p,q}+(\sigma-1)/p \geq 1/p$ imply for $\sigma\in(1,\infty)$,

$$\|e^{it|\nabla_g|^\sigma}\psi\|_{L^p(I,L^q)} \leq C\|\psi\|_{H^{\gamma_{p,q}+(\sigma-1)/p}}.$$

Similar results hold for $\sigma\in(0,1)$ with $\gamma_{p,q}$ in place of $\gamma_{p,q}+(\sigma-1)/p$ by using (2.1.6) instead of (2.1.5). This completes the proof. □

2.2 The WKB approximation

This subsection is devoted to the proof of dispersive estimates (2.1.1). To do so, we will use the so called WKB approximation (see [BGT04], [BT07], [Kap90] or [Rob87]), i.e. to approximate $e^{ith^{-1}(h|\nabla_g|)^\sigma}\varphi(h^2P)$ in terms of Fourier integral operators. The following result is the main goal of this subsection. To simplify the presentation, we denote $U_h(t) := e^{ith^{-1}(h|\nabla_g|)^\sigma}$.

Theorem 2.2.1. *Let $\sigma\in(0,\infty)\backslash\{1\}, \varphi\in C^\infty_0(\mathbb{R}\backslash\{0\})$, J a small neighborhood of* supp(φ) *not containing the origin, $a\in S(-\infty)$ with* supp$(a)\subset p^{-1}($supp$(\varphi))$*. Then there exist $t_0>0$ small enough, $S\in C^\infty([-t_0,t_0]\times\mathbb{R}^{2d})$ and a sequence of functions $a_j(t,\cdot,\cdot)\in S(-\infty)$ satisfying* supp$(a_j(t,\cdot,\cdot))\subset p^{-1}(J)$ *uniformly with respect to $t\in[-t_0,t_0]$ such that for all $N\geq 1$,*

$$U_h(t)Op_h(a)\psi = J_N(t)\psi + R_N(t)\psi,$$

where

$$\begin{aligned}J_N(t)\psi(x) &= \sum_{j=0}^{N-1} h^j J_h(S(t),a_j(t))\psi(x)\\ &= \sum_{j=0}^{N-1} h^j \left[(2\pi h)^{-d}\iint_{\mathbb{R}^{2d}} e^{ih^{-1}(S(t,x,\xi)-y\cdot\xi)}a_j(t,x,\xi)\psi(y)dyd\xi\right],\end{aligned}$$

$J_N(0)=Op_h(a)$ and the remainder $R_N(t)$ satisfies for all $t\in[-t_0,t_0]$ and all $h\in(0,1]$,

$$\|R_N(t)\|_{L^2\to L^2} \leq Ch^{N-1}. \tag{2.2.1}$$

Moreover, there exists a constant $C>0$ such that for all $t\in[-t_0,t_0]$ and all $h\in(0,1]$,

$$\|J_N(t)\|_{L^1\to L^\infty} \leq Ch^{-d}(1+|t|h^{-1})^{-d/2}. \tag{2.2.2}$$

Remark 2.2.2. Before entering to the proof of Theorem 2.2.1, let us show that Theorem 2.2.1 implies (2.1.1). We first note that the study of dispersive estimates for $U_h(t)\varphi(h^2P)$ is reduced to the one of $U_h(t)Op_h(a)$ with $a\in S(-\infty)$ satisfying supp$(a)\subset p^{-1}($supp$(\varphi))$. Indeed, by using the parametrix of $\varphi(h^2P)$ given in Proposition 2.1.2, we have for all $N\geq 1$,

$$\varphi(h^2P) = \sum_{j=0}^{N-1} h^j Op_h(\tilde{q}_j) + h^N\tilde{R}_N(h),$$

for some $\tilde{q}_j\in S(-\infty)$ satisfying supp$(\tilde{q}_j)\subset p^{-1}($supp$(\varphi))$ and the remainder satisfies for all $m\geq 0$,

$$\|\tilde{R}_N(h)\|_{H^{-m}\to H^m} \leq Ch^{-2m}.$$

Since $U_h(t)$ is bounded in H^m, the Sobolev embedding with $m>d/2$ implies

$$\|U_h(t)\tilde{R}_N(h)\|_{L^1\to L^\infty} \leq \|U_h(t)\tilde{R}_N(h)\|_{H^{-m}\to H^m} \leq Ch^{-2m}.$$

By choosing N large enough, the remainder term is bounded in $L^1\to L^\infty$ independent of t,h. We next show that Theorem 2.2.1 gives dispersive estimates for $U_h(t)Op_h(a)$, i.e.

$$\|U_h(t)Op_h(a)\|_{L^1\to L^\infty} \leq Ch^{-d}(1+|t|h^{-1})^{-d/2}, \tag{2.2.3}$$

for all $h \in (0,1]$ and all $t \in [-t_0, t_0]$. Indeed, by choosing $\tilde{\tilde{\varphi}} \in C_0^\infty(\mathbb{R}\backslash\{0\})$ which satisfies $\tilde{\tilde{\varphi}} = 1$ near $\mathrm{supp}(\varphi)$, we can write

$$\begin{aligned} U_h(t)Op_h(a) = \tilde{\tilde{\varphi}}(h^2P)U_h(t)Op_h(a)\tilde{\tilde{\varphi}}(h^2P) + (1-\tilde{\tilde{\varphi}})(h^2P)U_h(t)Op_h(a)\tilde{\tilde{\varphi}}(h^2P) \\ + U_h(t)Op_h(a)(1-\tilde{\tilde{\varphi}})(h^2P). \end{aligned} \tag{2.2.4}$$

Using Theorem 2.2.1, the first term is written as

$$\tilde{\tilde{\varphi}}(h^2P)U_h(t)Op_h(a)\tilde{\tilde{\varphi}}(h^2P) = \tilde{\tilde{\varphi}}(h^2P)J_N(t)\tilde{\tilde{\varphi}}(h^2P) + \tilde{\tilde{\varphi}}(h^2P)R_N(t)\tilde{\tilde{\varphi}}(h^2P).$$

We learn from Proposition 2.1.2 and (2.2.2) that the first term in the right hand side is of size $O_{L^1\to L^\infty}(h^{-d}(1+|t|h^{-1})^{-d/2})$ and the second one is of size $O_{L^1\to L^\infty}(h^{N-1-d})$. For the second and the third term of (2.2.4), we compose to the left and the right hand side with $(P+1)^m$ for $m \geq 0$ and use the parametrix of $(1-\tilde{\tilde{\varphi}})(h^2P)$. By composing pseudo-differential operators with disjoint supports, we obtain terms of size $O_{L^2\to L^2}(h^\infty)$. The Sobolev embedding with $m > d/2$ implies that the second and the third terms are of size $O_{L^1\to L^\infty}(h^\infty)$. By choosing N large enough, we have (2.2.3).

Proof of Theorem 2.2.1. Let us explain the strategy of the proof. As mentioned in the introduction, the main difficulty is that we do not have the exact form of the semi-classical fractional Laplace-Beltrami operator, namely $(h|\nabla_g|)^\sigma$, in order to use the usual construction of [BGT04]. Fortunately, thanks to the support of the symbol a, we can replace $U_h(t)$ by $e^{ith^{-1}\omega(h^2P)}$ for some smooth, compactly supported function ω. The interest of this replacement is that one can approximate $\omega(h^2P)$ in terms of pseudo-differential operators. We next use the action of pseudo-differential operators on Fourier integral operators and collect the powers of the semi-classical parameter h to yield the Hamilton-Jacobi equation for the phase and a system of transport equations for the amplitudes. After solving these equations, we control the remainder terms and prove dispersive estimates for the main terms. The proof of this theorem is done in several steps.

Step 1: Construction of the phase and amplitudes Due to the support of a, we can replace $(h|\nabla_g|)^\sigma$ by $\omega(h^2P)$ where $\omega(\lambda) = \tilde{\varphi}(\lambda)\sqrt{\lambda}^\sigma$ with $\tilde{\varphi} \in C_0^\infty(\mathbb{R}\backslash\{0\})$ and $\tilde{\varphi} = 1$ on J. The interest of this replacement is that we can use Proposition 2.1.2 to write

$$\omega(h^2P) = \sum_{k=0}^{N-1} h^k Op_h(q_k) + h^N R_N(h), \tag{2.2.5}$$

where $q_k \in S(-\infty)$ satisfy $q_0(x,\xi) = \omega \circ p(x,\xi)$, $\mathrm{supp}(q_k) \subset p^{-1}(\mathrm{supp}(\omega))$ and $R_N(h)$ is bounded in L^2 uniformly in $h \in (0,1]$. Next, using the fact

$$\frac{d}{dt}\Big(e^{-ith^{-1}\omega(h^2P)}J_N(t)\Big) = ih^{-1}e^{-ith^{-1}\omega(h^2P)}(hD_t - \omega(h^2P))J_N(t),$$

and $J_N(0) = Op_h(a)$, the fundamental theorem of calculus gives

$$e^{ith^{-1}\omega(h^2P)}Op_h(a)\psi = J_N(t)\psi - ih^{-1}\int_0^t e^{i(t-s)h^{-1}\omega(h^2P)}\left(hD_s - \omega(h^2P)\right)J_N(s)\psi ds.$$

We want the last term to have a small contribution. To do this, we need to consider the action of $hD_t - \omega(h^2P)$ on $J_N(t)$. We first compute the action of hD_t on $J_N(t)$ and have

$$hD_t \circ J_N(t) = \sum_{l=0}^{N} h^l J_h(S(t), b_l(t)),$$

where

$$\begin{aligned} b_0(t,x,\xi) &= \partial_t S(t,x,\xi) a_0(t,x,\xi), \\ b_l(t,x,\xi) &= \partial_t S(t,x,\xi) a_l(t,x,\xi) + D_t a_{l-1}(t,x,\xi), \quad l = 1,...,N-1, \\ b_N(t,x,\xi) &= D_t a_{N-1}(t,x,\xi). \end{aligned}$$

In order to study the action of $\omega(h^2P)$ on $J_N(t)$, we need the following action of a pseudo-differential operator on a Fourier integral operator (see e.g. [Rob87, Théorème IV.19], [RS06, Theorem 2.5] or [Bouc00, Appendix]).

Proposition 2.2.3. *Let $b \in S(-\infty)$ and $c \in S(-\infty)$ and $S \in C^\infty(\mathbb{R}^{2d})$ satisfy for all $\alpha, \beta \in \mathbb{N}^d, |\alpha+\beta| \geq 1$, there exists $C_{\alpha\beta} > 0$,*

$$|\partial_x^\alpha \partial_\xi^\beta (S(x,\xi) - x\cdot\xi)| \leq C_{\alpha\beta}, \quad \forall x,\xi \in \mathbb{R}^d. \tag{2.2.6}$$

Then

$$Op_h(b) \circ J_h(S,c) = \sum_{j=0}^{N-1} h^j J_h(S, (b \triangleleft c)_j) + h^N J_h(S, r_N(h)),$$

where $(b\triangleleft c)_j$ is a universal linear combination of

$$\partial_\xi^\beta b(x, \nabla_x S(x,\xi)) \partial_x^{\beta-\alpha} c(x,\xi) \partial_x^{\alpha_1} S(x,\xi) \cdots \partial_x^{\alpha_k} S(x,\xi),$$

with $\alpha \leq \beta, \alpha_1 + \cdots + \alpha_k = \alpha$ and $|\alpha_l| \geq 2$ for all $l = 1,...,k$ and $|\beta| = j$. The maps $(b,c) \mapsto (b\triangleleft c)_j$ and $(b,c) \mapsto r_N(h)$ are continuous from $S(-\infty) \times S(-\infty)$ to $S(-\infty)$. In particular, we have

$$\begin{aligned} (b\triangleleft c)_0(x,\xi) &= b(x,\nabla_x S(x,\xi)) c(x,\xi), \\ i(b\triangleleft c)_1(x,\xi) &= \nabla_\xi b(x, \nabla_x S(x,\xi)) \cdot \nabla_x c(x,\xi) + \frac{1}{2}\mathrm{tr}\left(\nabla_\xi^2 b(x,\nabla_x S(x,\xi)) \cdot \nabla_x^2 S(x,\xi)\right) c(x,\xi). \end{aligned}$$

Using (2.2.5), we can apply [1] Proposition 2.2.3 and obtain

$$\begin{aligned} \omega(h^2P) \circ J_N(t) &= \sum_{k=0}^{N-1} h^k Op_h(q_k) \circ \sum_{j=0}^{N-1} h^j J_h(S(t), a_j(t)) + h^N R_N(h) J_N(t) \\ &= \sum_{k+j+l=0}^{N} h^{k+j+l} J_h(S(t), (q_k \triangleleft a_j(t))_l) + h^{N+1} J_h(S(t), r_{N+1}(h,t)) + h^N R_N(h) J_N(t). \end{aligned}$$

This implies that

$$(hD_t - \omega(h^2P)) J_N(t) = \sum_{r=0}^{N} h^r J_h(S(t), c_r(t)) - h^N R_N(h) J_N(t) - h^{N+1} J_h(S(t), r_{N+1}(h,t)),$$

where

$$\begin{aligned} c_0(t) &= \partial_t S(t) a_0(t) - q_0(x, \nabla_x S(t)) a_0(t), \\ c_r(t) &= \partial_t S(t) a_r(t) - q_0(x,\nabla_x S(t)) a_r(t) + D_t a_{r-1}(t) - (q_0 \triangleleft a_{r-1}(t))_1 - (q_1 \triangleleft a_{r-1}(t))_0 \\ &\quad - \sum_{\substack{k+j+l=r \\ j \leq r-2}} (q_k \triangleleft a_j(t))_l, \quad r = 1,...,N-1, \end{aligned}$$

[1] We will see later that the phase satisfies requirements of Proposition 2.2.3.

and

$$c_N(t) = D_t a_{N-1}(t) - (q_0 \triangleleft a_{N-1}(t))_1 - (q_1 \triangleleft a_{N-1}(t))_0 - \sum_{\substack{k+j+l=N \\ j \le N-2}} (q_k \triangleleft a_j(t))_l.$$

The system of equations $c_r(t) = 0$ for $r = 0, ..., N$ leads to the following Hamilton-Jacobi equation

$$\partial_t S(t) - q_0(x, \nabla_x S(t)) = 0, \tag{2.2.7}$$

with $S(0) = x \cdot \xi$, and transport equations

$$D_t a_0(t) - (q_0 \triangleleft a_0(t))_1 - (q_1 \triangleleft a_0(t))_0 = 0, \tag{2.2.8}$$

$$D_t a_r(t) - (q_0 \triangleleft a_r(t))_1 - (q_1 \triangleleft a_r(t))_0 = \sum_{\substack{k+j+l=r+1 \\ j \le r-1}} (q_k \triangleleft a_j(t))_l, \tag{2.2.9}$$

for $r = 1, ..., N-1$ with initial data

$$a_0(0) = a, \quad a_r(0) = 0, \quad r = 1, ..., N-1. \tag{2.2.10}$$

The standard Hamilton-Jacobi equation gives the following result (see e.g. [Rob87, Théorème IV.14] or Appendix $A.1$).

Proposition 2.2.4. *There exist $t_0 > 0$ small enough and a unique solution $S \in C^\infty([-t_0, t_0] \times \mathbb{R}^{2d})$ to the Hamilton-Jacobi equation*

$$\left\{ \begin{array}{rcl} \partial_t S(t,x,\xi) - q_0(x, \nabla_x S(t,x,\xi)) & = & 0, \\ S(0,x,\xi) & = & x \cdot \xi. \end{array} \right. \tag{2.2.11}$$

Moreover, for all $\alpha, \beta \in \mathbb{N}^d$, there exists $C_{\alpha\beta} > 0$ such that for all $t \in [-t_0, t_0]$ and all $x, \xi \in \mathbb{R}^d$,

$$|\partial_x^\alpha \partial_\xi^\beta \left(S(t,x,\xi) - x \cdot \xi\right)| \le C_{\alpha\beta}|t|, \quad |\alpha + \beta| \ge 1, \tag{2.2.12}$$

$$|\partial_x^\alpha \partial_\xi^\beta (S(t,x,\xi) - x \cdot \xi - t q_0(x,\xi))| \le C_{\alpha\beta}|t|^2. \tag{2.2.13}$$

Note that the phase given in Proposition 2.2.4 satisfies requirements of Proposition 2.2.3. It remains to solve the transport equations (2.2.8), (2.2.9). To do so, we rewrite these equations as

$$\begin{aligned} \partial_t a_0(t,x,\xi) - V(t,x,\xi) \cdot \nabla_x a_0(t,x,\xi) - f(t,x,\xi) a_0(t,x,\xi) &= 0, \\ \partial_t a_r(t,x,\xi) - V(t,x,\xi) \cdot \nabla_x a_r(t,x,\xi) - f(t,x,\xi) a_r(t,x,\xi) &= g_r(t,x,\xi), \end{aligned}$$

for $r = 1, ..., N-1$ where

$$\begin{aligned} V(t,x,\xi) &= (\partial_\xi q_0)(x, \nabla_x S(t,x,\xi)), \\ f(t,x,\xi) &= \frac{1}{2} \mathrm{tr} \left[\nabla_\xi^2 q_0(x, \nabla_x S(t,x,\xi)) \cdot \nabla_x^2 S(t,x,\xi) \right] + i q_1(x, \nabla_x S(t,x,\xi)), \\ g_r(t,x,\xi) &= i \sum_{\substack{k+j+l=r+1 \\ j \le r-1}} (q_k \triangleleft a_j(t))_l. \end{aligned}$$

We now construct $a_r(t,x,\xi), r = 0, ..., N-1$ by using the method of characteristics as follows. Let $Z(t,s,x,\xi)$ be the flow associated to $V(t,x,\xi)$, i.e.

$$\partial_t Z(t,s,x,\xi) = -V(t, Z(t,s,x,\xi), \xi), \quad Z(s,s,x,\xi) = x.$$

By the fact that $q_0 \in S(-\infty)$ and (2.2.12) and using the same trick as in Lemma A.1.1, we have

$$|\partial_x^\alpha \partial_\xi^\beta (Z(t,s,x,\xi) - x)| \leq C_{\alpha\beta}|t-s|, \tag{2.2.14}$$

for all $|t|, |s| \leq t_0$. Now, we can define iteratively

$$\begin{aligned} a_0(t,x,\xi) &= a(Z(0,t,x,\xi),\xi) \exp\left(\int_0^t f(s, Z(s,t,x,\xi),\xi)ds\right), \\ a_r(t,x,\xi) &= \int_0^t g_r(s, Z(s,t,x,\xi),\xi) \exp\left(\int_\tau^t f(\tau, Z(\tau,t,x,\xi),\xi)d\tau\right) ds, \end{aligned}$$

for $r = 1, ..., N-1$. These functions are respectively solutions to (2.2.8) and (2.2.9) with initial data (2.2.10) respectively. Since $\mathrm{supp}(a) \subset p^{-1}(\mathrm{supp}(\varphi))$, we see that for $t_0 > 0$ small enough, $(Z(t,s,p^{-1}(\mathrm{supp}(\varphi))),\xi) \in p^{-1}(J)$ for all $|t|,|s| \leq t_0$. By extending $a_r(t,x,\xi)$ on $\mathbb{R}^{2d}$ by $a_r(t,x,\xi) = 0$ for $(x,\xi) \notin p^{-1}(J)$, the functions a_r are still smooth in $(x,\xi) \in \mathbb{R}^{2d}$. Using the fact that $a, q_k \in S(-\infty)$, (2.2.13) and (2.2.14), we have for $t_0 > 0$ small enough, $a_r(t,\cdot,\cdot)$ is a bounded set of $S(-\infty)$ and $\mathrm{supp}(a_r(t,\cdot,\cdot)) \in p^{-1}(J)$ uniformly with respect to $t \in [-t_0, t_0]$.

Step 2: L^2-boundedness of remainder We will use the so called Kuranishi trick (see e.g. [Rob87], [Miz13]). We first have

$$R_N(t) = ih^{N-1}\int_0^t e^{i(t-s)h^{-1}\omega(h^2P)}\Big(R_N(h)J_N(s) + hJ_h(S(s), r_{N+1}(h,s))\Big)ds.$$

Using that $e^{i(t-s)h^{-1}\omega(h^2P)}$ is unitary in L^2 and Proposition 2.1.2 that $R_N(h)$ is bounded in $L^2 \to L^2$ uniformly in $h \in (0,1]$, the estimate (2.2.1) follows from the L^2-boundedness of $J_h(S(t), a(t))$ uniformly with respect to $h \in (0,1]$ and $t \in [-t_0,t_0]$ where $(a(t))_{t\in[-t_0,t_0]}$ is bounded in $S(-\infty)$. For $t \in [-t_0, t_0]$, we define a map on $\mathbb{R}^{3d}$ by

$$\Lambda(t,x,y,\xi) := \int_0^1 \nabla_x S(t, y + s(x-y), \xi)ds.$$

Using (2.2.12), there exists $t_0 > 0$ small enough so that for all $t \in [-t_0, t_0]$,

$$\|\nabla_x\nabla_\xi S(t,x,\xi) - I_{\mathbb{R}^d}\| \ll 1, \quad \forall x, \xi \in \mathbb{R}^d.$$

This implies that

$$\|\nabla_\xi \Lambda(t,x,y,\xi) - I_{\mathbb{R}^d}| \leq \int_0^1 \|\nabla_\xi\nabla_x S(t, y+s(x-y),\xi) - I_{\mathbb{R}^d}\|ds \ll 1, \quad \forall t \in [-t_0,t_0].$$

Thus for all $t \in [-t_0, t_0]$ and all $x, y \in \mathbb{R}^d$, the map $\xi \mapsto \Lambda(t,x,y,\xi)$ is a diffeomorphism from $\mathbb{R}^d$ onto itself. If we denote $\xi \mapsto \Lambda^{-1}(t,x,y,\xi)$ the inverse map, then $\Lambda^{-1}(t,x,y,\xi)$ satisfies (see [Bouc00]) that: for all $\alpha, \alpha', \beta \in \mathbb{N}^d$, there exists $C_{\alpha\alpha'\beta} > 0$ such that

$$|\partial_x^\alpha \partial_y^{\alpha'} \partial_\xi^\beta (\Lambda^{-1}(t,x,y,\xi) - \xi)| \leq C_{\alpha\alpha'\beta}|t|, \tag{2.2.15}$$

for all $t \in [-t_0, t_0]$. Now, by change of variable $\xi \mapsto \Lambda^{-1}(t,x,y,\xi)$, the action $J_h(S(t),a(t)) \circ J_h(S(t),a(t))^\star$ becomes (see [Rob87]) a semi-classical pseudo-differential operator with the amplitude

$$a(t,x,\Lambda^{-1}(t,x,y,\xi))\overline{a(t,y,\Lambda^{-1}(t,x,y,\xi))}|\det \partial_\xi \Lambda^{-1}(t,x,y,\xi)|.$$

Using the fact that $(a(t))_{t\in[-t_0,t_0]}$ is bounded in $S(-\infty)$ and (2.2.15), this amplitude and its derivatives are bounded. By the Calderón-Vaillancourt theorem, we have the result.

Step 3: Dispersive estimates We prove the result for a general term, namely $J_h(S(t), a(t))$ with $(a(t))_{t\in[-t_0,t_0]}$ a bounded family in $S(-\infty)$ satisfying $\mathrm{supp}(a(t,\cdot,\cdot)) \in p^{-1}(J)$ for some small neighborhood J of $\mathrm{supp}(\varphi)$ not containing the origin uniformly with respect to $t \in [-t_0, t_0]$. The kernel of $J_h(S(t), a(t))$ reads

$$K_h(t,x,y) = (2\pi h)^{-d} \int_{\mathbb{R}^d} e^{ih^{-1}(S(t,x,\xi)-y\cdot\xi)} a(t,x,\xi) d\xi.$$

It suffices to show for all $t \in [-t_0, t_0]$ and all $h \in (0,1]$, $|K_h(t,x,y)| \leq Ch^{-d}(1+|t|h^{-1})^{-d/2}$, for all $x, y \in \mathbb{R}^d$. We only consider the case $t \geq 0$, for $t \leq 0$ it is similar. Since the amplitude is compactly supported in ξ and $a(t,x,\xi)$ is bounded uniformly in $t \in [-t_0, t_0]$ and $x, y \in \mathbb{R}^d$, we have $|K_h(t,x,y)| \leq Ch^{-d}$. If $0 \leq t \leq h$ hence $1 + th^{-1} \leq 2$, then

$$|K_h(t,x,y)| \leq Ch^{-d} \leq Ch^{-d}(1+th^{-1})^{-d/2}.$$

We now can assume that $h \leq t \leq t_0$ and write the phase function as $(S(t,x,\xi) - y\cdot\xi)/t$ with the parameter $\lambda := th^{-1} \geq 1$. By the choice of $\tilde{\varphi}$ (see Step 1 for $\tilde{\varphi}$), we see that on the support of the amplitude, i.e. on $p^{-1}(J)$, $q_0(x,\xi) = \sqrt{p(x,\xi)}^{\sigma}$. Thus we apply (2.2.11) to write

$$S(t,x,\xi) = x\cdot\xi + t\sqrt{p(x,\xi)}^{\sigma} + t^2 \int_0^1 (1-\theta)\partial_t^2 S(\theta t, x, \xi) d\theta.$$

Next, using that $p(x,\xi) = \xi^t G(x)\xi = |\eta|^2$ with $\eta = \sqrt{G(x)}\xi$ or $\xi = \sqrt{g(x)}\eta$ where $g(x) = (g_{jk}(x))_{j,k=1}^d$ and $G(x) = (g(x))^{-1} = (g^{jk}(x))_{j,k=1}^d$, the kernel can be written as

$$K_h(t,x,y) = (2\pi h)^{-d} \int_{\mathbb{R}^d} e^{i\lambda\Phi(t,x,y,\eta)} a(t,x,\sqrt{g(x)}\eta)|g(x)| d\eta,$$

where

$$\Phi(t,x,y,\eta) = \frac{\sqrt{g(x)}(x-y)\cdot\eta}{t} + |\eta|^{\sigma} + t\int_0^1 (1-\theta)\partial_t^2 S(\theta t, x, \sqrt{g(x)}\eta) d\theta.$$

Recall that $|g(x)| := \sqrt{\det g(x)}$. By (2.0.2), $\|\sqrt{G(x)}\|$ and $\|\sqrt{g(x)}\|$ are bounded from below and above uniformly in $x \in \mathbb{R}^d$. This implies that η still belongs to a compact set of $\mathbb{R}^d$ away from zero. We denote this compact support by $\mathcal{K}$. The gradient of the phase is

$$\nabla_\eta \Phi(t,x,y,\eta) = \frac{\sqrt{g(x)}(x-y)}{t} + \sigma\eta|\eta|^{\sigma-2} + t\Big(\int_0^1 (1-\theta)(\nabla_\xi \partial_t^2 S)(\theta t, x, \sqrt{g(x)}\eta) d\theta\Big)\sqrt{g(x)}.$$

Let us consider the case $|\sqrt{g(x)}(x-y)/t| \geq C$ for some constant C large enough. Thanks to the Hamilton-Jacobi equation (2.2.11) (see also $(A.1.9)$, $(A.1.2)$ and Lemma $A.1.2$) and the fact $\sigma \in (0,\infty)\backslash\{1\}$, we have for t_0 small enough,

$$|\nabla_\eta \Phi| \geq |\sqrt{g(x)}(x-y)/t| - \sigma|\eta|^{\sigma-1} - O(t) \geq C_1.$$

Hence we can apply the non stationary theorem, i.e. by integrating by parts with respect to η together with the fact that for all $\beta \in \mathbb{N}^d$ satisfying $|\beta| \geq 2$, $|\partial_\eta^\beta \Phi(t,x,y,\eta)| \leq C_\beta$, we have for all $N \geq 1$,

$$|K_h(t,x,y)| \leq Ch^{-d}\lambda^{-N} \leq Ch^{-d}(1+th^{-1})^{-d/2},$$

provided N is taken greater than $d/2$.

Thus we can assume that $|\sqrt{g(x)}(x-y)/t| \leq C$. In this case, we write

$$\nabla_\eta^2 \Phi(t,x,y,\eta) = \sigma|\eta|^{\sigma-2}\Big(I_{\mathbb{R}^d} + (\sigma-2)\frac{\eta\cdot\eta^T}{|\eta|^2}\Big) + O(t).$$

Using that

$$\Big| \det \sigma |\eta|^{\sigma-2} \Big(I_{\mathbb{R}^d} + (\sigma-2) \frac{\eta \cdot \eta^T}{|\eta|^2} \Big) \Big| = \sigma^d |\sigma - 1| |\eta|^{(\sigma-2)d} \geq C.$$

Therefore, for $t_0 > 0$ small enough, the map $\eta \mapsto \nabla_\eta \Phi(t,x,y,\eta)$ from a neighborhood of $\mathcal{K}$ to its range is a local diffeomorphism. Moreover, for all $\beta \in \mathbb{N}^d$ satisfying $|\beta| \geq 1$, we have $|\partial_\eta^\beta \Phi(t,x,y,\eta)| \leq C_\beta$. The stationary phase theorem then implies that for all $t \in [h,t_0]$ and all $x, y \in \mathbb{R}^d$ satisfying $|\sqrt{g(x)}(x-y)/t| \leq C$,

$$|K_h(t,x,y)| \leq Ch^{-d} \lambda^{-d/2} \leq Ch^{-d}(1 + th^{-1})^{-d/2}.$$

This completes the proof. □

Chapter 3

Strichartz estimates for Schrödinger-type equations on compact manifolds without boundary

Contents

In this chapter, we establish Strichartz estimates for Schrödinger-type equations posed on a compact Riemannian manifold without boundary (M, g), namely

$$\begin{cases} i\partial_t u(t,x) + |\nabla_g|^\sigma u(t,x) &= F(t,x), \quad (t,x) \in I \times M, \\ u(0,x) &= \psi(x), \quad x \in M, \end{cases} \tag{3.0.1}$$

where $\sigma \in (0,\infty)\backslash\{1\}$ and $|\nabla_g| = \sqrt{-\Delta_g}$ with Δ_g the Laplace-Beltrami operator on (M, g).

Before stating our main result, let us recall known results related to the problem. Burq-Gérard-Tzvetkov established in [BGT04] Strichartz estimates with a loss of $1/p$ derivatives for the homogeneous linear Schrödinger equation (i.e. $\sigma = 2$), namely

$$\|e^{-it\Delta_g}\psi\|_{L^p(I,L^q(M))} \leq C\|\psi\|_{H^{1/p}(M)}, \tag{3.0.2}$$

where (p, q) is a sharp Schrödinger admissible pair and $q < \infty$ (see (0.0.1)). When M is the flat torus $\mathbb{T}^d$, Bourgain showed in [Bou1], [Bou2] some estimates related to (3.0.2) by means of the Fourier series for the Schrödinger equation. A direct consequence of these estimates is

$$\|e^{-it\Delta_g}\psi\|_{L^4(\mathbb{T}\times\mathbb{T}^d)} \leq C\|\psi\|_{H^\gamma(T^d)}, \quad \gamma > \frac{d}{4} - \frac{1}{2}. \tag{3.0.3}$$

When $M = \mathbb{T}$ and $\sigma \in (1, 2)$, the authors in [DET16] established estimates related to (3.0.3), namely

$$\|e^{it|\nabla_g|^\sigma}\psi\|_{L^4(\mathbb{T}\times\mathbb{T})} \leq C\|\psi\|_{H^\gamma(\mathbb{T})}, \quad \gamma > \frac{2-\sigma}{8}. \tag{3.0.4}$$

The main purpose of this chapter is to extend the result of Burq-Gérard-Tzvetkov to the homogeneous linear Schrödinger-type equation (3.0.1). Precisely, we have the following result.

Theorem 3.0.1. *Consider (M, g) a smooth compact boundaryless Riemannian manifold of dimension $d \geq 1$ and let $I \subset \mathbb{R}$ a bounded interval. If $\sigma \in (1,\infty)$, then for all (p, q) Schrödinger* *admissible with $q < \infty$, there exists $C > 0$ such that for all $\psi \in H^{\gamma_{p,q}+(\sigma-1)/p}(M)$,*

$$\|e^{it|\nabla_g|^\sigma}\psi\|_{L^p(I,L^q(M))} \leq C\|\psi\|_{H^{\gamma_{p,q}+(\sigma-1)/p}(M)}. \tag{3.0.5}$$

Moreover, if u is a (weak) solution to (3.0.1)*, then*

$$\|u\|_{L^p(I,L^q(M))} \leq C\Big(\|\psi\|_{H^{\gamma_{p,q}+(\sigma-1)/p}(M)} + \|F\|_{L^1(I,H^{\gamma_{p,q}+(\sigma-1)/p}(M))}\Big). \tag{3.0.6}$$

If $\sigma \in (0,1)$, then (3.0.5) *and* (3.0.6) *hold with $\gamma_{p,q}$ in place of $\gamma_{p,q} + (\sigma-1)/p$.*

The proof of Theorem 3.0.1 is based on Strichartz estimates on $\mathbb{R}^d$ equipped with a smooth bounded metric g given in the previous chapter.

Remark 3.0.2.

1. Note that the exponents $\gamma_{p,q} + (\sigma-1)/p = d/2 - d/q - 1/p$ in the right hand side of (3.0.5) and $\gamma_{p,q} = d/2 - d/q - \sigma/p$ in the case of $\sigma \in (0,1)$ correspond to the gain of $1/p$ and σ/p derivatives respectively compared with the Sobolev embedding.
2. Using the same argument as in [BGT04], we see that the endpoint homogeneous Strichartz estimate (3.0.5) is sharp on $\mathbb{S}^d, d \geq 3$. Indeed, let ψ be a zonal spherical harmonic associated to eigenvalue $\lambda = k(d+k-1)$. One has (see e.g. [Sog86]) that for $\lambda \gg 1$,

$$\|\psi\|_{L^q(\mathbb{S}^d)} \sim \sqrt{\lambda}^{s(q)}, \quad s(q) = \frac{d-1}{2} - \frac{d}{q} \quad \text{if } \frac{2(d+1)}{d-1} \leq q \leq \infty.$$

   Moreover, the above estimates are sharp. Therefore,

$$\|e^{it|\nabla_g|^\sigma}\psi\|_{L^2(I,L^{2^\star}(\mathbb{S}^d))} = \|e^{it\sqrt{\lambda}^\sigma}\psi\|_{L^2(I,L^{2^\star}(\mathbb{S}^d))} \sim \sqrt{\lambda}^{s(2^\star)},$$

   where $2^\star = 2d/(d-2)$ and $s(2^\star) = 1/2$. This gives the optimality of (3.0.5) since $\gamma_{22^\star} + (\sigma - 1)/2 = 1/2 = s(2^\star)$.
3. By the same technique used in the proof of Theorem 3.0.1, we can prove with minor modifications Strichartz estimates for the homogeneous linear half-wave equation on (M, g) which is similar to the one given in Corollary 1.2.4).

As an application of Theorem 3.0.1, we obtain Strichartz estimates for inhomogeneous linear wave-type equations posed on (M, g). Let us consider the following inhomogeneous linear wave-type equations posed on (M, g),

$$\begin{cases} \partial_t^2 v(t,x) + (-\Delta_g)^\sigma v(t,x) &= G(t,x), \quad (t,x) \in I \times M, \\ v(0,x) = \psi(x), & \partial_t v(0,x) = \phi(x), \quad x \in M. \end{cases} \tag{3.0.7}$$

Corollary 3.0.3. *Consider (M, g) a smooth compact boundaryless Riemannian manifold of dimension $d \geq 1$. Let $I \subset \mathbb{R}$ be a bounded interval and v a (weak) solution to* (3.0.7)*. If $\sigma \in (1,\infty)$, then for all (p,q) Schrödinger admissible with $q < \infty$, there exists $C > 0$ such that for all $(\psi,\phi) \in H^{\gamma_{p,q}+(\sigma-1)/p}(M) \times H^{\gamma_{p,q}+(\sigma-1)/p-\sigma}(M)$,*

$$||v||_{L^p(I,L^q(M))} \leq C\Big(||[v](0)||_{H^{\gamma_{p,q}+(\sigma-1)/p}(M)} + ||G||_{L^1(I,H^{\gamma_{p,q}+(\sigma-1)/p-\sigma}(M))}\Big), \tag{3.0.8}$$

where

$$||[v](0)||_{H^{\gamma_{p,q}+(\sigma-1)/p}(M)} := ||\psi||_{H^{\gamma_{p,q}+(\sigma-1)/p}(M)} + ||\phi||_{H^{\gamma_{p,q}+(\sigma-1)/p-\sigma}(M)}.$$

If $\sigma \in (0,1)$, then (3.0.8) *holds with $\gamma_{p,q} + (\sigma-1)/p$ is replaced by $\gamma_{p,q}$.*

3.1 Notations

Coordinate charts and partition of unity Let M be a smooth compact Riemannian manifold without boundary. A coordinate chart $(U_\kappa, V_\kappa, \kappa)$ on M comprises an homeomorphism κ between an open subset U_κ of M and an open subset V_κ of $\mathbb{R}^d$. Given $\phi \in C_0^\infty(U_\kappa)$(resp. $\chi \in C_0^\infty(V_\kappa)$), we define the pushforward of ϕ (resp. pullback of χ) by $\kappa_*\phi := \phi \circ \kappa^{-1}$ (resp. $\kappa^*\chi := \chi \circ \kappa$). For a given finite cover of M, namely $M = \cup_{\kappa \in \mathcal{F}} U_\kappa$ with $\#\mathcal{F} < \infty$, there exist $\phi_\kappa \in C_0^\infty(U_\kappa), \kappa \in \mathcal{F}$ such that $1 = \sum_\kappa \phi_\kappa(m)$ for all $m \in M$.

Laplace-Beltrami operator For any coordinate chart $(U_\kappa, V_\kappa, \kappa)$, there exists a symmetric positive definite matrix $g_\kappa(x) := (g_{jk}^\kappa(x))_{j,k=1}^d$ with smooth and real valued coefficients on V_κ such that the Laplace-Beltrami operator $P = -\Delta_g$ reads in $(U_\kappa, V_\kappa, \kappa)$ as

$$P_\kappa := -\kappa_* \Delta_g \kappa^* = -\sum_{j,k=1}^d |g_\kappa(x)|^{-1} \partial_j \Big(|g_\kappa(x)| g_\kappa^{jk}(x) \partial_k \Big),$$

where $|g_\kappa(x)| = \sqrt{\det g_\kappa(x)}$ and $(g_\kappa^{jk}(x))_{j,k=1}^d := (g_\kappa(x))^{-1}$. The principal symbol of P_κ is

$$p_\kappa(x,\xi) = \sum_{j,k=1}^d g_\kappa^{jk}(x) \xi_j \xi_k.$$

3.2 Functional calculus

In this subsection, we recall well-known facts on pseudo-differential calculus on manifolds (see e.g. [BGT04]). For a given $a \in S(m)$, we define the operator

$$Op_h^\kappa(a) := \kappa^* Op_h(a) \kappa_*. \tag{3.2.1}$$

If nothing is specified about $a \in S(m)$, then the operator $Op_h^\kappa(a)$ maps $C_0^\infty(U_\kappa)$ to $C^\infty(U_\kappa)$. In the case $\mathrm{supp}(a) \subset V_\kappa \times \mathbb{R}^d$, we have that $Op_h^\kappa(a)$ maps $C_0^\infty(U_\kappa)$ to $C_0^\infty(U_\kappa)$ hence to $C^\infty(M)$. We have the following result.

Proposition 3.2.1. *Let $\phi_\kappa \in C_0^\infty(U_\kappa)$ be an element of a partition of unity on M and $\tilde{\phi}_\kappa, \tilde{\tilde{\phi}}_\kappa \in C_0^\infty(U_\kappa)$ be such that $\tilde{\phi}_\kappa = 1$ near $\mathrm{supp}(\phi_\kappa)$ and $\tilde{\tilde{\phi}}_\kappa = 1$ near $\mathrm{supp}(\tilde{\phi}_\kappa)$. Then for all $N \geq 1$, all $z \in [0, +\infty)$and all $h \in (0,1]$,*

$$(h^2 P - z)^{-1} \phi_\kappa = \sum_{j=0}^{N-1} h^j \tilde{\phi}_\kappa Op_h^\kappa(q_{\kappa,j}(z)) \phi_\kappa + h^N R_N(z,h),$$

where $q_{\kappa,j}(z) \in S(-2-j)$ is a linear combination of $a_k(p_\kappa - z)^{-1-k}$ for some symbol $a_k \in S(2k-j)$ independent of z and

$$R_N(z,h) = -(h^2 P - z)^{-1} \tilde{\tilde{\phi}}_\kappa Op_h^\kappa(r_{\kappa,N}(z,h)) \phi_\kappa,$$

where $r_{\kappa,N}(z,h) \in S(-N)$ with seminorms growing polynomially in $1/\mathrm{dist}(z, \mathbb{R}^+)$ uniformly in $h \in (0,1]$ as long as z belongs to a bounded set of $\mathbb{C} \backslash [0, +\infty)$.

Proof. Let us set $\chi_\kappa := \kappa_* \phi_\kappa$, similarly for $\tilde{\chi}_\kappa$ and $\tilde{\tilde{\chi}}_\kappa$. We get $\chi_\kappa, \tilde{\chi}_\kappa, \tilde{\tilde{\chi}}_\kappa \in C_0^\infty(V_\kappa)$ and $\tilde{\chi}_\kappa = 1$ near $\mathrm{supp}(\chi_\kappa)$ and $\tilde{\tilde{\chi}}_\kappa = 1$ near $\mathrm{supp}(\tilde{\chi}_\kappa)$. We first find an operator, still denoted by P, globally defined on $\mathbb{R}^d$ of the form

$$P = -\sum_{j,k=1}^d g^{jk}(x) \partial_j \partial_k + \sum_{l=1}^d b_l(x) \partial_l, \tag{3.2.2}$$

which coincides with P_κ on a large relatively compact subset V_0 of V_κ. By "large", we mean that $\mathrm{supp}(\tilde{\tilde{\chi}}_\kappa) \subset V_0$. For instance, we can take $P = \upsilon P_\kappa - (1-\upsilon)\Delta$ where $\upsilon \in C_0^\infty(V_\kappa)$ with values in

$[0,1]$ satisfying $\upsilon = 1$ on V_0. The principal symbol of P is

$$p(x,\xi) = \sum_{j,k=1}^{d} g^{jk}(x)\xi_j\xi_k, \quad \text{where } g^{jk}(x) = \upsilon(x) g^{jk}_\kappa(x) + (1-\upsilon(x))\delta_{jk}. \tag{3.2.3}$$

It is easy to see that $g(x) = (g^{jk}(x))$ satisfies (2.0.2) and (2.0.3) and b_l is bounded in $\mathbb{R}^d$ together with all of theirs derivatives. Using the standard elliptic parametrix for $(h^2P - z)^{-1}$ (see e.g [Rob87]), we have

$$(h^2P - z)Op_h(q_\kappa(z,h)) = I + h^N Op_h(\tilde{r}_{\kappa,N}(z,h)), \tag{3.2.4}$$

where $q_\kappa(z,h) = \sum_{j=0}^{N-1} h^j q_{\kappa,j}(z)$ with $q_{\kappa,j}(z) \in S(-2-j)$ and $\tilde{r}_{\kappa,N}(z,h) \in S(-N)$ with seminorms growing polynomially in $\langle z\rangle / \mathrm{dist}(z, \mathbb{R}^+)$ uniformly in $h \in (0,1]$. On the other hand, we can write

$$\begin{aligned}(h^2P_\kappa - z)\tilde{\chi}_\kappa Op_h(q_\kappa(z,h))\chi_\kappa \\ = \tilde{\chi}_\kappa(h^2P_\kappa - z)Op_h(q_\kappa(z,h))\chi_\kappa + [h^2P_\kappa, \tilde{\chi}_\kappa]Op_h(q_\kappa(z,h))\chi_\kappa.\end{aligned} \tag{3.2.5}$$

Here $[h^2P_\kappa, \tilde{\chi}_\kappa]$ and χ_κ have coefficients with disjoint supports. Thanks to (3.2.4) and the composition of pseudo-differential operators with disjoint supports, we have

$$(h^2P_\kappa - z)\tilde{\chi}_\kappa Op_h(q_\kappa(z,h))\chi_\kappa = \chi_\kappa + h^N \tilde{\tilde{\chi}}_\kappa Op_h(r_{\kappa,N}(z,h))\chi_\kappa,$$

with $r_{\kappa,N}(z,h)$ satisfying the required property. We then compose to the right and the left of above equality with κ^* and κ_* respectively and get

$$(h^2P - z)\tilde{\phi}_\kappa Op^\kappa_h(q_\kappa(z,h))\phi_\kappa = \phi_\kappa + h^N \tilde{\tilde{\phi}}_\kappa Op^\kappa_h(r_{\kappa,N}(z,h))\phi_\kappa.$$

This gives the result and the proof is complete. □

Next, we give an application of the parametrix given in Proposition 3.2.1 and have the following result (see [BGT04, Proposition 2.1] or [BT07, Proposition 2.5]).

Proposition 3.2.2. *Let $\phi_\kappa, \tilde{\phi}_\kappa, \tilde{\tilde{\phi}}_\kappa$ be as in* Proposition 3.2.1 *and $f \in C^\infty_0(\mathbb{R})$. Then for all $N \geq 1$ and all $h \in (0,1]$,*

$$f(h^2P)\phi_\kappa = \sum_{j=0}^{N-1} h^j \tilde{\phi}_\kappa Op^\kappa_h(a_{\kappa,j})\phi_\kappa + h^N R_{\kappa,N}(h), \tag{3.2.6}$$

where $a_{\kappa,j} \in S(-\infty)$ with $\mathrm{supp}(a_{\kappa,j}) \subset \mathrm{supp}(f \circ p_\kappa)$ *for $j = 0, ..., N-1$. Moreover, for all $m \geq 0$, there exists $C > 0$ such that for all $h \in (0,1]$,*

$$\|R_N(h)\|_{H^{-m}(M)\to H^m(M)} \leq Ch^{-2m}. \tag{3.2.7}$$

Proof. The proof is essentially given in [BGT04, Proposition 2.1]. For the reader's convenience, we recall some details. By using Proposition 3.2.1 and the Helffer-Sjöstrand formula (see [DS99]), namely

$$f(h^2P) = -\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)(h^2P - z)^{-1} dL(z),$$

where $\widetilde{f}$ is an almost analytic extension of f, the Cauchy formula implies (3.2.6) with

$$R_{\kappa,N}(h) = \frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{f}(z)(h^2P - z)^{-1}\tilde{\tilde{\phi}}_\kappa Op^\kappa_h(r_{\kappa,N}(z,h))\phi_\kappa dL(z).$$

It remains to prove (3.2.7). This leads to study the action on $L^2(\mathbb{R}^d)$ of the operator

$$\int_{\mathbb{C}} \overline{\partial} \widetilde{f}(z)(P_\kappa+1)^{m/2}(h^2P_\kappa - z)^{-1}\tilde{\tilde{\chi}}_\kappa Op_h(r_{\kappa,N}(z,h))\chi_\kappa(P_\kappa+1)^{m/2}dL(z).$$

Using a trick as in (3.2.5), we can find a globally defined operator P which coincides with P_κ on the support of $\tilde{\tilde{\chi}}_\kappa$. We see that $\|(h^2P-z)^{-1}\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leq C|\text{Im } z|^{-1}$ and

$$(P+1)^{m/2}Op_h(r_{\kappa,N}(z,h))\chi_\kappa(P+1)^{m/2} = h^{-2m}Op_h(\tilde{r}_{\kappa,N}(z,h)),$$

where $\tilde{r}_{\kappa,N}(z,h) \in S(-N+2m)$ with seminorms growing polynomially in $1/\text{dist}(z,\mathbb{R}^+)$ uniformly in $h \in (0,1]$ which are harmless since $\widetilde{f}$ is compactly supported and $\overline{\partial}\widetilde{f}(z) = O(|\text{Im } z|^\infty)$. By choosing N such that $N-2m > d$, the result then follows from the $\mathcal{L}(L^2(\mathbb{R}^d))$ bound of pseudo-differential operator given in Proposition 2.1.1. □

A direct consequence of Proposition 2.1.2 using partition of unity and Proposition 2.1.1 is the following result (see [BGT04, Corollary 2.2] or [BT07, Proposition 2.9]).

Corollary 3.2.3. *Let $f \in C_0^\infty(\mathbb{R})$. Then for all $1 \leq q \leq r \leq \infty$, there exists $C>0$ such that for all $h \in (0,1]$,*

$$\|f(h^2P)\|_{L^q(M)\to L^r(M)} \leq Ch^{-\left(\frac{d}{q}-\frac{d}{r}\right)}.$$

The next proposition gives the Littlewood-Paley decomposition on compact manifolds without boundary (see [BGT04, Corollary 2.3]) which is similar to Proposition 2.1.4.

Proposition 3.2.4. *There exist $\varphi_0 \in C_0^\infty(\mathbb{R})$ and $\varphi \in C_0^\infty(\mathbb{R}\backslash\{0\})$ such that for all $q \in [2,\infty)$, there exists $C>0$,*

$$\|u\|_{L^q(M)} \leq C\Big(\sum_{h^{-1}:\text{dya}} \|\varphi(h^2P)u\|^2_{L^q(M)}\Big)^{1/2} + C\|u\|_{L^2(M)},$$

for all $u \in C_0^\infty(M)$.

3.3 Reduction of problem

In this subsection, we first show how to get Corollary 3.0.3 from Theorem 3.0.1 and then give a reduction of Theorem 3.0.1.

Proof of Corollary 3.0.3 Since we are working on compact manifolds without boundary, it is well-known that there exists an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of $L^2(M) := L^2(M, d\text{vol}_g)$ of C^∞ functions on M such that

$$|\nabla_g|^\sigma e_j = \lambda_j^\sigma e_j,$$

with $0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lim_{j\to\infty}\lambda_j = +\infty$. For any f a piecewise continuous function, the functional $f(|\nabla_g|)$ is defined as

$$f(|\nabla_g|)u := \sum_{j\in\mathbb{N}} f(\lambda_j)u_je_j,$$

where

$$u_j := \langle e_j, u\rangle_{L^2(M)} = \int_M \overline{e_j(x)}u(x)d\text{vol}_g(x).$$

If we set $j_0 := \dim(\ker|\nabla_g|^\sigma)$, then $\lambda_0 = \lambda_1 = \cdots = \lambda_{j_0-1} = 0$ and $\lambda_j \geq \lambda_{j_0} > 0$ for $j \geq j_0$. Here the number j_0 stands for the number of connected components of M and the corresponding eigenfunctions $(e_j)_{j=0}^{j_0-1}$ are constant functions. We now define the projection on $\ker(|\nabla_g|^\sigma)$ by

$$\Pi_0 u := \sum_{j<j_0} u_je_j.$$

By the Duhamel formula, the equation (3.0.7) can be written as

$$v(t)=\cos(t|\nabla_g|^\sigma)\psi+\frac{\sin(t|\nabla_g|^\sigma)}{|\nabla_g|^\sigma}\phi+\int_0^t\frac{\sin((t-s)|\nabla_g|^\sigma)}{|\nabla_g|^\sigma}G(s)ds.$$

We remark that the only problem may happen on $\ker(|\nabla_g|^\sigma)$ of $\frac{\sin(t|\nabla_g|^\sigma)}{|\nabla_g|^\sigma}$. But it is not the case because

$$\Pi_0\frac{\sin(t|\nabla_g|^\sigma)}{|\nabla_g|^\sigma}\phi=\sum_{j<j_0}\frac{\sin(t\lambda_j^\sigma)}{\lambda_j^\sigma}v_{1,j}e_j=\sum_{j<j_0}t\frac{\sin(t\lambda_j^\sigma)}{t\lambda_j^\sigma}v_{1,j}e_j=t\sum_{j<j_0}v_{1,j}e_j=t\Pi_0\phi.$$

Since $\ker(|\nabla_g|^\sigma)$ is generated by constant functions, local in time Strichartz estimates of $\Pi_0 v$, namely $\|\Pi_0 v\|_{L^p(I,L^q(M))}$ with I a bounded interval, can be controlled by any Sobolev norms of data. Therefore, we only need to study local in time Strichartz estimates of v away from $\ker(|\nabla_g|^\sigma)$. Using the fact that

$$\cos(t|\nabla_g|^\sigma)=\frac{e^{it|\nabla_g|^\sigma}+e^{-it|\nabla_g|^\sigma}}{2},\quad \sin(t|\nabla_g|^\sigma)=\frac{e^{it|\nabla_g|^\sigma}-e^{-it|\nabla_g|^\sigma}}{2i},$$

Strichartz estimates (3.0.8) follow directly from the ones of $e^{\pm it|\nabla_g|^\sigma}$ as in (3.0.6). This gives Corollary 3.0.3. □

We now prove Theorem 3.0.1. To do so, we have the following reduction.

Proposition 3.3.1. *Consider (M,g) a smooth compact Riemannian manifold of dimension $d\geq 1$. Let $\sigma\in(0,\infty)\backslash\{1\}$ and $\varphi\in C_0^\infty(\mathbb{R}\backslash\{0\})$. If there exists $t_0>0$ small enough and $C>0$ such that for all $\psi\in L^1(M)$ and all $h\in(0,1]$,*

$$\|e^{ith^{-1}(h|\nabla_g|)^\sigma}\varphi(h^2P)\psi\|_{L^\infty(M)}\leq Ch^{-d}(1+|t|h^{-1})^{-d/2}\|\psi\|_{L^1(M)}, \tag{3.3.1}$$

for all $t\in[-t_0,t_0]$, then Theorem 3.0.1 *holds true.*

Proof. The proof of homogeneous Strichartz estimates follows similarly to the one given in Proposition 2.1.5. We only give the proof of (3.0.6), i.e. $\sigma\in(1,\infty)$, the one for $\sigma\in(0,1)$ is completely similar. The homogeneous part follows from (3.0.5). It remains to prove

$$\Big\|\int_0^t e^{i(t-s)|\nabla_g|^\sigma}F(s)ds\Big\|_{L^p(I,L^q(M))}\leq C\|F\|_{L^1(I,H^{\gamma_{p,q}+(\sigma-1)/p}(M))}. \tag{3.3.2}$$

The estimate (3.3.2) follows easily from (3.0.5) and the Minkowski inequality (see [BGT04], Corollary 2.10). Indeed, the left hand side reads

$$\begin{aligned}\Big\|\int_I \mathbb{1}_{[0,t]}(s)e^{i(t-s)|\nabla_g|^\sigma}F(s)ds\Big\|_{L^p(I,L^q(M))}&\leq\int_I\|\mathbb{1}_{[0,t]}(s)e^{i(t-s)|\nabla_g|^\sigma}F(s)\|_{L^p(I,L^q(M))}ds\\&\leq\int_I\|e^{i(t-s)|\nabla_g|^\sigma}F(s)\|_{L^p(I,L^q(M))}ds\\&\leq C\int_I\|F(s)\|_{H^{\gamma_{p,q}+(\sigma-1)/p}(M)}ds.\end{aligned}$$

This gives (3.3.2) and the proof of Proposition 3.3.1 is complete. □

3.4 Dispersive estimates

This subsection is devoted to prove the dispersive estimates (3.3.1). Again thanks to the localization φ, we can replace $(h|\nabla_g|)^\sigma$ by $\omega(h^2P)$ where $\omega(\lambda)=\tilde{\varphi}(\lambda)\sqrt{\lambda}^\sigma$ with $\tilde{\varphi}\in C_0^\infty(\mathbb{R}\backslash\{0\})$ such that $\tilde{\varphi}=1$ near $\operatorname{supp}(\varphi)$. The partition of unity allows us to consider only on a local coordinates, namely $\sum_\kappa e^{ith^{-1}\omega(h^2P)}\varphi(h^2P)\phi_\kappa$. By using the same argument as in Remark 2.2.2 and Proposition 3.2.2, the study of $e^{ith^{-1}\omega(h^2P)}\varphi(h^2P)\phi_\kappa$ is reduced to the one of $e^{ith^{-1}\omega(h^2P)}\tilde{\phi}_\kappa Op_h^\kappa(a_\kappa)\phi_\kappa$ with $a_\kappa \in S(-\infty)$ and $\text{supp}(a_\kappa) \subset \text{supp}(\varphi \circ p_\kappa)$. Let us set

$$u(t) = e^{ith^{-1}\omega(h^2P)}\tilde{\phi}_\kappa Op_h^\kappa(a_\kappa)\phi_\kappa\psi.$$

We see that u solves the following semi-classical evolution equation

$$\begin{cases} (hD_t - \omega(h^2P))u(t) &= 0, \\ u_{|t=0} &= \tilde{\phi}_\kappa Op_h^\kappa(a_\kappa)\phi_\kappa\psi. \end{cases} \tag{3.4.1}$$

The WKB method allows us to construct an approximation of the solution to (3.4.1) in finite time independent of h. To do so, we first choose $\vartheta_\kappa, \tilde{\vartheta}_\kappa, \tilde{\tilde{\vartheta}}_\kappa \in C_0^\infty(U_\kappa)$ such that $\vartheta_\kappa = 1$ near $\text{supp}(\tilde{\tilde{\phi}}_\kappa)$ (see Proposition 3.2.1 for $\tilde{\tilde{\phi}}_\kappa$), $\tilde{\vartheta}_\kappa = 1$ near $\text{supp}(\vartheta_\kappa)$ and $\tilde{\tilde{\vartheta}}_\kappa = 1$ near $\text{supp}(\tilde{\vartheta}_\kappa)$. Proposition 3.2.2 then implies

$$\omega(h^2P)\vartheta_\kappa = \tilde{\vartheta}_\kappa Op_h^\kappa(b_\kappa(h))\vartheta_\kappa + h^N\mathfrak{R}_{\kappa,N}(h), \tag{3.4.2}$$

where $b_\kappa(h) = \sum_{l=1}^{N-1} h^l b_{\kappa,l}$ with $b_{\kappa,l} \in S(-\infty)$ and $\mathfrak{R}_{\kappa,N}(h) = O_{L^2(M)\to L^2(M)}(1)$. We apply the construction of the WKB approximation given in Subsection 2.2 and find $t_0 > 0$ small enough, a function $S_\kappa \in C^\infty([-t_0,t_0]\times\mathbb{R}^{2d})$ and a sequence $a_{\kappa,j}(t,\cdot,\cdot) \in S(-\infty)$ satisfying $\text{supp}(a_{\kappa,j}(t,\cdot,\cdot)) \subset p^{-1}(J)$ (see (3.2.3) for the definition of p) for some small neighborhood J of $\text{supp}(\varphi)$ not containing the origin uniformly in $t \in [-t_0,t_0]$ such that

$$(hD_t - Op_h(b_\kappa(h)))J_{\kappa,N}(t) = R_{\kappa,N}(t), \tag{3.4.3}$$

where

$$J_{\kappa,N}(t) := \sum_{j=0}^{N-1} h^j J_h(S_\kappa(t), a_{\kappa,j}(t)), \quad J_{\kappa,N}(0) = Op_h(a_\kappa),$$

satisfies for all $t \in [-t_0,t_0]$ and all $(x,\xi) \in p^{-1}(J)$,

$$|\partial_x^\alpha\partial_\xi^\beta(S_\kappa(t,x,\xi) - x\cdot\xi)| \leq C_{\alpha\beta}|t|, \quad |\alpha+\beta| \geq 1, \tag{3.4.4}$$

$$\Big|\partial_x^\alpha\partial_\xi^\beta(S_\kappa(t,x,\xi) - x\cdot\xi + t\sqrt{p(x,\xi)}^\sigma)\Big| \leq C_{\alpha\beta}|t|^2, \tag{3.4.5}$$

and for all $h \in (0,1]$,

$$\|J_{\kappa,N}(t)\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \leq Ch^{-d}(1+|t|h^{-1})^{-d/2}, \tag{3.4.6}$$

$$R_{\kappa,N}(t) = O_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}(h^{N-1}). \tag{3.4.7}$$

Next, we need the following micro-local finite propagation speed.

Lemma 3.4.1. *Let $\sigma \in (0,\infty)\backslash\{1\}$, $\chi, \tilde{\chi} \in C_0^\infty(\mathbb{R}^d)$ such that $\tilde{\chi} = 1$ near $\text{supp}(\chi)$, $a(t) \in S(-\infty)$ with $\text{supp}(a(t,\cdot,\cdot)) \subset p^{-1}(J)$ uniformly in $t \in [-t_0,t_0]$ and $S \in C^\infty([-t_0,t_0]\times\mathbb{R}^{2d})$ satisfy (3.4.5) for all $t \in [-t_0,t_0]$ and all $(x,\xi) \in p^{-1}(J)$. Then for $t_0 > 0$ small enough,*

$$J_h(S(t),a(t))\chi = \tilde{\chi}J_h(S(t),a(t))\chi + \tilde{R}(t),$$

where $\tilde{R}(t) = O_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}(h^\infty)$.

Proof. The kernel of $J_h(S(t),a(t))\chi - \tilde{\chi}J_h(S(t),a(t))\chi$ is given by

$$K_h(t,x,y) = (2\pi h)^{-d}\int_{\mathbb{R}^d} e^{ih^{-1}(S(t,x,\xi)-y\cdot\xi)}(1-\tilde{\chi})(x)a(t,x,\xi)\chi(y)d\xi.$$

Using (3.4.5), we can write for $t_0 > 0$ small enough, $t \in [-t_0, t_0]$ and $(x,\xi) \in p^{-1}(J)$,

$$S(t,x,\xi) - y\cdot\xi = (x-y)\cdot\xi - t\sqrt{p(x,\xi)}^{\sigma} + O(t^2).$$

By change of variables $\eta = \sqrt{G(x)}\xi$ or $\xi = \sqrt{g(x)}\eta$, we have

$$K_h(t,x,y) = (2\pi h)^{-d}\int_{\mathbb{R}^d} e^{ih^{-1}\Phi(t,x,y,\eta)}(1-\tilde{\chi})(x)a(t,x,\sqrt{g(x)}\eta)\chi(y)\sqrt{\det g(x)}dx,$$

where $\Phi(t,x,y,\xi) = \sqrt{g(x)}(x-y)\cdot\eta - t|\eta|^{\sigma} + O(t^2)$. Thanks to the support of χ and $\tilde{\chi}$, we see that $|x-y| \geq C$. This gives for $t_0 > 0$ small enough that

$$|\nabla_\eta \Phi(t,x,y,\eta)| = |\sqrt{g(x)}(x-y) - t\sigma\eta|\eta|^{\sigma-2} + O(t^2)| \geq C(1+|x-y|).$$

Here we also use the fact that $\|\sqrt{g(x)}\|$ is bounded from below and above (see (3.2.3)). Using the fact that for all $\beta \in \mathbb{N}^d$ satisfying $|\beta| \geq 2$,

$$|\partial_\eta^\beta \Phi(t,x,y,\eta)| \leq C_\beta,$$

the non stationary phase theorem implies for all $N \geq 1$, all $t \in [-t_0, t_0]$ and all $x, y \in \mathbb{R}^d$,

$$|K_h(t,x,y)| \leq Ch^{N-d}(1+|x-y|)^{-N}.$$

The Schur's Lemma gives $\tilde{R}(t) = O_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)}(h^\infty)$. This ends the proof. □

Proof of dispersive estimates (3.3.1) With the same spirit as in (3.2.1), let us set $J_N^\kappa(t) = \kappa^* J_{\kappa,N}(t)\kappa_*$, $R_N^\kappa(t) = \kappa^* R_{\kappa,N}(t)\kappa_*$ where $J_{\kappa,N}(t)$ and $R_{\kappa,N}(t)$ given in (3.4.3). The Duhamel formula gives

$$\begin{aligned} u(t) &= e^{ith^{-1}\omega(h^2P)}\tilde{\phi}_\kappa Op_h^\kappa(a_\kappa)\phi_\kappa\psi \\ &= \tilde{\phi}_\kappa J_N^\kappa(t)\phi_\kappa\psi - ih^{-1}\int_0^t e^{i(t-s)h^{-1}\omega(h^2P)}(hD_s - \omega(h^2P))\tilde{\phi}_\kappa J_N^\kappa(s)\phi_\kappa\psi ds. \end{aligned}$$

We aslo have from (3.4.2) that

$$\begin{aligned} &(hD_s - \omega(h^2P))\tilde{\phi}_\kappa J_N^\kappa(s)\phi_\kappa \\ &\qquad = \tilde{\phi}_\kappa hD_s J_N^\kappa(s)\phi_\kappa - \tilde{\vartheta}_\kappa Op_h^\kappa(b_\kappa(h))\tilde{\phi}_\kappa J_N^\kappa(s)\phi_\kappa - h^N\mathfrak{R}_{\kappa,N}(h)\tilde{\phi}_\kappa J_N^\kappa(s)\phi_\kappa. \end{aligned}$$

The micro-local finite propagation speed given in Lemma 3.4.1 and (3.4.3) imply

$$\begin{aligned} &(hD_s - \omega(h^2P))\tilde{\phi}_\kappa J_N^\kappa(s)\phi_\kappa \\ &\qquad = \tilde{\vartheta}_\kappa\kappa^*(hD_s - Op_h(b_\kappa(h)))J_N(s)\kappa_*\phi_\kappa - \tilde{R}_\kappa(s) - h^N\mathfrak{R}_{\kappa,N}(h)\tilde{\phi}_\kappa J_N^\kappa(s)\phi_\kappa \\ &\qquad = \tilde{\vartheta}_\kappa R_N^\kappa(s)\phi_\kappa - \tilde{R}_\kappa(s) - h^N\mathfrak{R}_{\kappa,N}(h)\tilde{\phi}_\kappa J_N^\kappa(s)\phi_\kappa, \end{aligned}$$

where $\tilde{R}_\kappa(s) = O_{L^2(M)\to L^2(M)}(h^\infty)$. Here we also use the L^2-boundedness of pseudo-differential operators with symbols in $S(-\infty)$. We then get

$$u(t) = \tilde{\phi}_\kappa J_N^\kappa(t)\phi_\kappa\psi + \mathcal{R}_N^\kappa(t)\psi,$$

where

$$\mathcal{R}_N^\kappa(t)\psi = -ih^{-1}\int_0^t e^{i(t-s)h^{-1}\omega(h^2P)}(\tilde{\vartheta}_\kappa R_N^\kappa(s)\phi_\kappa - \tilde{R}_\kappa(s) - h^N\mathfrak{R}_{\kappa,N}(h)\tilde{\phi}_\kappa J_N^\kappa(s)\phi_\kappa)\psi ds.$$

By the same process as in Remark 2.2.2 using (3.4.6) and the fact $\mathcal{R}^{\kappa}_N(t) = O_{L^2(M)\to L^2(M)}(h^{N-1})$ for all $t \in [-t_0, t_0]$, we obtain

$$\|e^{ith^{-1}\omega(h^2P)}\varphi(h^2P)\phi_\kappa\psi\|_{L^\infty(M)} \leq Ch^{-d}(1+|t|h^{-1})^{-d/2}\|\psi\|_{L^1(M)},$$

for all $t \in [-t_0, t_0]$. The dispersive estimates (3.3.1) then follow from the above estimates and partition of unity. This completes the proof. □

Chapter 4

Global-in-time Strichartz estimates for Schrödinger-type equations on asymptotically Euclidean manifolds

Contents

In this chapter, we consider the time dependent Schrödinger-type equation on a Riemannian manifold (M,g), namely

$$\left\{ \begin{array}{rcl} i\partial_t u(t,x) - |\nabla_g|^\sigma u(t,x) &=& 0, \quad (t,x) \in \mathbb{R}\times M, \\ u(0,x) &=& \psi(x), \quad x\in M, \end{array} \right. \tag{4.0.1}$$

where $\sigma \in (0,\infty)\backslash\{1\}, |\nabla_g| = \sqrt{-\Delta_g}$ with Δ_g the Laplace-Beltrami operator associated to the metric g. Note that in (4.0.1) we consider the minus sign in front of $|\nabla_g|^\sigma$ which is different from the previous chapters. This irrelevant change is just for convenience to fit the usual construction of the Isozaki-Kitada parametrix.

When $M=\mathbb{R}^d$ and $g=\mathrm{Id}$, i.e. the flat Euclidean space, the solution to (4.0.1) enjoys global in time Strichartz estimates (see Corollary 1.1.4),

$$\|u\|_{L^p(\mathbb{R},L^q(\mathbb{R}^d))} \lesssim \|\psi\|_{\dot{H}^{\gamma_{p,q}}(\mathbb{R}^d)},$$

where (p,q) satisfies the Schrödinger admissible condition with $q<\infty$ and $\gamma_{p,q}$ is as in (1.0.7).

When $M=\mathbb{R}^d$ and g a smooth bounded metric satisfying (2.0.2) and (2.0.3) or (M,g) a smooth compact Riemannian manifold without boundary, we also have Strichartz estimates but only local in time (see Theorem 2.0.1 and Theorem 3.0.1),

$$\|u\|_{L^p([0,1],L^q(M))} \lesssim \|\psi\|_{H^\gamma(M)}.$$

In the case $\sigma \in (0,1)$, we have the same (local in time) Strichartz estimates as in $(\mathbb{R}^d, \mathrm{Id})$, i.e. $\gamma = \gamma_{p,q}$. In the case $\sigma \in (1,\infty)$, there is a "loss" of $(\sigma-1)/p$ derivatives compared to the one on $(\mathbb{R}^d, \mathrm{Id})$, i.e. $\gamma = \gamma_{p,q} + (\sigma-1)/p$.

When M is a non-compact Riemannian manifold, global in time Strichartz estimates for the Schrödinger equation (i.e. $\sigma = 2$) have been studied intensively over the last decade. In [BT08], Bouclet-Tzvetkov established global in time Strichartz estimates on asymptotically Euclidean manifold, i.e. $\mathbb{R}^d$ equipped with a long range perturbation metric g (see (4.0.3)) with a low frequency cutoff under non-trapping condition. The first breakthrough on this topic was done by Tataru in [Tat08] where he considered long range and globally small perturbations of the Euclidean metric with C^2 and time dependent coefficients. In this setting, no trapping could occur. Later, Marzuola-Metcalfe-Tataru in [MMT08] improved the results by considering more general perturbations in a compact set, including some weak trapping. Afterwards, Hassell-Zhang in [HZ16] extended those results for general geometric framework of asymptotically conic manifolds and including very short range potentials with non-trapping condition. Subsequencely, Bouclet-Mizutani in [BM16] established global in time Strichartz estimates for a more general class of asymptotically conic manifolds including all usual smooth long range perturbations of the Euclidean metric with hyperbolic trapping condition. After that, Zhang-Zheng [ZZ17] extended the result of Hassell-Zhang [HZ16] and proved global in time Strichartz estimates for Schrödinger operators with potentials on assymptotically conic manifoldswith non-trapping condition. They also extended Bouclet-Mizutani's result [BM16] by considering Schrödinger operators with short range potentials on asymptotically conic manifolds with hyperbobic trapping condition. Recently, Zhang-Zheng [ZZ18] established global in time Strichartz estimates for Schrödinger operators on metric cone.

In order to prove Strichartz estimates on curved backgrounds, one uses the Littlewood-Paley decomposition to localize the solution in frequency. One then uses microlocal techniques to derive dispersive estimates and obtain Strichartz estimates for each spectrally localized components. By summing over all frequency pieces, one gets Strichartz estimates for the solution. For local in time Strichartz estimates, this usual scheme works very well. However, for global in time Strichartz estimates, one has to face a difficulty arising at low frequency. Due to the uncertainty principle, one can only use microlocal techniques for data supported outside compact sets at low frequency. Therefore, one has to use another technique for data supported inside compact sets. Note also that on $\mathbb{R}^d$, one can use the scaling technique to reduce the analysis at low frequency to the study at frequency one, but this technique does not work on manifolds in general.

The goal of this chapter is to study global in time Strichartz estimates for the Schrödinger-type equation on asymptotically Euclidean manifolds. In the case of Schrödinger equation, it can be seen as a completion of those in [BT08] of spatial dimensions greater than or equal to 3. In order to achieve this goal, we will use the techniques of [BM16] combined with the analysis of [BT08]. Note that since we consider a larger range of admissible condition compared to the sharp Schrödinger admissible condition (see (0.0.1)) of [BM16], we have to be more careful in order to apply the techniques of [BM16].

Before giving the main results, let us introduce some notations. Let $g(x) = (g_{jk}(x))_{j,k=1}^d$ be a metric on $\mathbb{R}^d, d \geq 2$, and denote $G(x) = (g^{jk}(x))_{j,k=1}^d := g^{-1}(x)$. We consider the Laplace-Beltrami operator associated to g, i.e.

$$\Delta_g = \sum_{j,k=1}^d |g(x)|^{-1}\partial_{x_j}\left(g^{jk}(x)|g(x)|\partial_{x_k}\right),$$

where $|g(x)| := \sqrt{\det g(x)}$. Throughout the chapter, we assume that g satisfies the following assumptions.

1. There exists $C>0$ such that for all $x,\xi \in \mathbb{R}^d$,

$$C^{-1}|\xi|^2 \leq \sum_{j,k=1}^{d} g^{jk}(x)\xi_j\xi_k \leq C|\xi|^2. \tag{4.0.2}$$

2. There exists $\rho>0$ such that for all $\alpha \in \mathbb{N}^d$, there exists $C_\alpha>0$ such that for all $x \in \mathbb{R}^d$,

$$\left|\partial^\alpha \left(g^{jk}(x)-\delta_{jk}\right)\right| \leq C_\alpha \langle x\rangle^{-\rho-|\alpha|}. \tag{4.0.3}$$

3. The geodesic flow associated to g is non-trapping. It means that the Hamiltonian flow $(X(t),\Xi(t)) := (X(t,x,\xi),\Xi(t,x,\xi))$ associated to p, i.e.

$$\left\{\begin{array}{ll} \dot{X}(t) & = \nabla_\xi p(X(t),\Xi(t)), \\ \dot{\Xi}(t) & = -\nabla_x p(X(t),\Xi(t)), \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{ll} \dot{X}(0) & = x, \\ \dot{\Xi}(0) & = \xi, \end{array}\right.$$

satisfies: for all $(x,\xi)\in T^\star\mathbb{R}^d$ with $\xi \ne 0$,

$$|X(t)| \to +\infty \text{ as } t \to \pm\infty, \tag{4.0.4}$$

where p is the principal symbol of $-\Delta_g$ (see (4.0.5) below). Remark that by the conservation of energy and (4.0.2), all the geodesics starting from (x,ξ) are defined globally in time, i.e. $(X(t),\Xi(t))$ exists for all $t\in\mathbb{R}$.

The elliptic assumption (4.0.2) implies that $|g(x)|$ is bounded from below and above by positive constants. Thus for $1\leq q\leq\infty$, the spaces $L^q(\mathbb{R}^d, d_gx)$ where $d_gx = |g(x)|dx$ and $L^q(\mathbb{R}^d)$ coincide and have equivalent norms. In the sequel, we will use the same notation $L^q(\mathbb{R}^d)$ or L^q for short. It is well-known that $-\Delta_g$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ under the assumptions (4.0.2) and (4.0.3). We denote the unique self-adjoint extension on L^2 of $-\Delta_g$ by P. Note that the principal symbol of P is

$$p(x,\xi) = \xi^t G(x)\xi = \sum_{j,k=1}^{d} g^{jk}(x)\xi_j\xi_k. \tag{4.0.5}$$

Now let $\gamma\in\mathbb{R}$ and $q\in[1,\infty]$. The inhomogeneous Sobolev space $W_g^{\gamma,q}$ associated to the metric g is defined as a closure of the Schwartz space $\mathscr{S}$ under the norm

$$\|u\|_{W_g^{\gamma,q}} := \|\langle\nabla_g\rangle^\gamma u\|_{L^q}, \quad \langle\nabla_g\rangle = \sqrt{1-\Delta_g}.$$

It is very useful to recall that for all $\gamma\in\mathbb{R}$ and $q\in(1,\infty)$, there exists $C>1$ such that

$$C^{-1}\|\langle\nabla\rangle^\gamma u\|_{L^q} \leq \|u\|_{W_g^{\gamma,q}} \leq C\|\langle\nabla\rangle^\gamma u\|_{L^q}, \tag{4.0.6}$$

with $\langle\nabla\rangle = \sqrt{1-\Delta}$ where Δ is the free Laplace operator on $\mathbb{R}^d$. This fact follows from the L^q-boundedness of zero order pseudo-differential operators (see e.g [Sog86, Theorem 3.1.6]). The estimates (4.0.6) allow us to use the Sobolev embedding as on $\mathbb{R}^d$. For the homogeneous Sobolev space associated to g, one should be careful since the Schwartz space is not a good candidate due to the singularity at 0 of $\lambda\mapsto|\lambda|^\gamma$. Recall that (see [GV85, Appendix], [Tri83, chapter 5] and [BL76, Chapter 6]) on $\mathbb{R}^d$, the homogeneous Sobolev space $\dot{W}^{\gamma,q}$ is the closure of $\mathscr{L}$ under the norm

$$\|u\|_{\dot{W}^{\gamma,q}} := \||\nabla|^\gamma u\|_{L^q},$$

where

$$\mathscr{L} := \left\{u\in\mathscr{S} \mid D^\alpha\hat{u}(0)=0, \ \forall\alpha\in\mathbb{N}^d\right\}.$$

Here $\hat{\cdot}$ is the spatial Fourier transform. Since there is no Fourier transform on manifolds, we need to use the spectral theory instead. We denote

$$\mathscr{L}_g := \{\varphi(P)u \mid u \in \mathscr{S}, \varphi \in C_0^\infty((0,\infty))\}. \tag{4.0.7}$$

We define the homogeneous Sobolev space $\dot{W}_g^{\gamma,q}$ associated to g as the closure of $\mathscr{L}_g$ under the norm

$$\|u\|_{\dot{W}_g^{\gamma,q}} := \||\nabla_g|^\gamma u\|_{L^q}.$$

When $q=2$, we use $H^\gamma, \dot{H}^\gamma, H_g^\gamma$ and $\dot{H}_g^\gamma$ instead of $W^{\gamma,2}, \dot{W}^{\gamma,2}, W_g^{\gamma,2}$ and $\dot{H}_g^\gamma$ respectively. Thanks to the equivalence (4.0.6), we will only use the usual notation H^γ in the sequel. It is important to note (see [Bouc11, Proposition 2.3] or [SW10, Lemma 2.4]) that for $d \geq 2$,

$$\|u\|^2_{\dot{H}^1_g} = (|\nabla_g|u, |\nabla_g|u) = (u, Pu) \simeq \|\nabla u\|^2_{L^2} = \|u\|^2_{\dot{H}^1}. \tag{4.0.8}$$

By Stone's theorem, the solution to (4.0.1) is given by $u(t) = e^{-it|\nabla_g|^\sigma}\psi$. Let $f_0 \in C_0^\infty(\mathbb{R})$ be such that $f_0 = 1$ on $[-1,1]$. We split

$$u(t) = u_{\mathrm{low}}(t) + u_{\mathrm{high}}(t),$$

where

$$u_{\mathrm{low}}(t) := f_0(P)e^{-it|\nabla_g|^\sigma}\psi, \quad u_{\mathrm{high}}(t) = (1-f_0)(P)e^{-it|\nabla_g|^\sigma}\psi. \tag{4.0.9}$$

We see that $u_{\mathrm{low}}(t)$ and $u_{\mathrm{high}}(t)$ correspond to the low and high frequencies respectively. By the Littlewood-Paley decomposition which is very similar to the one given in [BM16, Subsection 4.2] (see Subsection 4.2.1), we split the high frequency term into two parts: inside and outside a compact set. Our first result concerns the global in time Strichartz estimates for the high frequency term inside a compact set.

Theorem 4.0.1. *Consider $\mathbb{R}^d, d \geq 2$ equipped with a smooth metric g satisfying* (4.0.2), (4.0.3) *and assume that the geodesic flow associated to g is non-trapping. Then for all $\chi \in C_0^\infty(\mathbb{R}^d)$ and all (p,q) Schrödinger admissible with $q<\infty$, there exists $C>0$ such that for all $\psi \in \mathscr{L}_g$,*

$$\|\chi u_{\mathrm{high}}\|_{L^p(\mathbb{R}, L^q)} \leq C\|\psi\|_{\dot{H}_g^{\gamma_{p,q}}}. \tag{4.0.10}$$

The proof of Theorem 4.0.1 is based on local in time Strichartz estimates and global L^2 integrability estimates of the Schrödinger-type operator. This strategy was first used in [ST02] for the Schrödinger equation. We will make use of dispersive estimates given in Chapter 2 to get Strichartz estimates with a high frequency spectral cutoff on a small time interval. Thanks to global L^2 integrability estimates, we can upgrade these local in time Strichartz estimates in to global in time Strichartz estimates. This strategy depends heavily on the non-trapping condition. We believe that one can improve this result to allow some weak trapping condition such as hyperbolic trapping in [BGH10]. We hope to come back to this interesting question in a future work.

Our next result is the following global in time Strichartz estimates for the high frequency term outside a compact set.

Theorem 4.0.2. *Consider $\mathbb{R}^d, d \geq 2$ equipped with a smooth metric g satisfying* (4.0.2), (4.0.3) *and assume that there exists $M>0$ large enough such that for all $\chi \in C_0^\infty(\mathbb{R}^d)$,*

$$\|\chi(P-\lambda \pm i0)^{-1}\chi\|_{L^2 \to L^2} \lesssim_\chi \lambda^M, \quad \lambda \geq 1. \tag{4.0.11}$$

Then there exists $R>0$ large enough such that for all (p,q) Schrödinger admissible with $q<\infty$, there exists $C>0$ such that for all $\psi \in \mathscr{L}_g$,

$$\|\mathbb{1}_{\{|x|>R\}} u_{\mathrm{high}}\|_{L^p(\mathbb{R}, L^q)} \leq C\|\psi\|_{\dot{H}_g^{\gamma_{p,q}}}. \tag{4.0.12}$$

The assumption (4.0.11) is known to hold in certain trapping situations (see e.g. [Dat09],

[NZ09] or [BGH10]) as well as in the non-trapping case (see [Rob92] or [Vod04]). We remark that under the trapping condition of [Dat09], [NZ09] or [BGH10], we have

$$\|\chi(P-\lambda \pm i0)^{-1}\chi\|_{L^2\to L^2} \lesssim_\chi \lambda^{-1/2}\log\lambda, \quad \lambda \geq 1,$$

and under non-trapping condition, we have (see e.g. [Bur02], [Rob92]) that

$$\|\chi(P-\lambda \pm i0)^{-1}\chi\|_{L^2\to L^2} \lesssim_\chi \lambda^{-1/2}, \quad \lambda \geq 1.$$

The proof of Theorem 4.0.2 relies on the so called Isozaki-Kitada parametrix (see [BT08]) and resolvent estimates given in [BM16] using (4.0.11). Recall that the Isozaki-Kitada parametrix was first introduced on $\mathbb{R}^d$ to study the scattering theory of Schrödinger operators with long range potentials [IK85]. It was then modified and successfully used to show the Strichartz estimates for Schrödinger equation outside a compact set in many papers (see e.g. [BT07], [BT08], [Bouc11], [Miz13], [Miz12] or [BM16]).

The low frequency term in (4.0.9) enjoys the following global in time Strichartz estimates.

Theorem 4.0.3. *Consider $\mathbb{R}^d, d \geq 3$ equipped with a smooth metric g satisfying* (4.0.2), (4.0.3). *Then for all (p,q) Schrödinger admissible with $q<\infty$, there exists $C>0$ such that for all $\psi \in \mathscr{L}_g$,*

$$\|u_{\rm low}\|_{L^p(\mathbb{R},L^q)} \leq C\|\psi\|_{\dot{H}^{\gamma_{p,q}}_g}. \tag{4.0.13}$$

As mentioned earlier, since we consider a larger range of admissible condition than the one studied in [BM16], we can not apply directly the low frequency Littlewood-Paley decomposition given in [BM16]. We thus need a "refined" version of Littlewood-Paley decomposition. To do so, we will take advantage of heat kernel estimates (see Subsection 4.2.1). As a result, we split the low frequency term into two parts: one supported outside a compact set and another one localized in a weak sense, i.e. by means of a spatial decaying weight. The term with a spatial decaying weight is treated easily by using global L^p integrability estimates of the Schrödinger-type operator at low frequency. Note that this type of global L^p integrability estimate relies on the low frequency resolvent estimates of [BR15] which is available for spatial dimensions $d \geq 3$. We expect that global in time Strichartz estimates for the Schrödinger-type equation at low frequency may hold in dimension $d=2$ as well. However, we do not know how to prove it at the moment. For the term outside a compact set, we make use of microlocal techniques and a low frequency version of the Isozaki-Kitada parametrix. We refer the reader to Section 4.4 for more details.

Combining Theorem 4.0.1, Theorem 4.0.2 and Theorem 4.0.3, we have the following result.

Theorem 4.0.4. *Consider $\mathbb{R}^d, d \geq 3$ equipped with a smooth metric g satisfying* (4.0.2), (4.0.3) *and assume that the geodesic flow associated to g is non-trapping. Let u be a weak solution to* (4.0.1). *Then for all (p,q) Schrödinger admissible with $q<\infty$, there exists $C>0$ such that for all $\psi \in \mathscr{L}_g$,*

$$\|u\|_{L^p(\mathbb{R},L^q)} \leq C\|\psi\|_{\dot{H}^{\gamma_{p,q}}_g}. \tag{4.0.14}$$

Remark 4.0.5. Global in time Strichartz estimates for the homogeneous linear half-wave equation $\sigma=1$ on asymptotically Euclidean manifolds $d \geq 3$ under non-trapping condition were established by Sogge-Wang [SW10] by applying the result of Metcalfe-Tataru [MT12]. The method presented in this chapter can be applied with a suitable modification to show Strichartz estimates for the half-wave equation on asymptotically Euclidean manifolds under non-trapping condition, and thus provides an alternative proof for global in time Strichartz estimates in the case $\sigma=1$.

Using the homogeneous Strichartz estimate (4.0.14) and the Christ-Kiselev Lemma, we get the following inhomogeneous Strichartz estimates.

Proposition 4.0.6. *Consider $\mathbb{R}^d, d \geq 3$ equipped with a smooth metric g satisfying* (4.0.2), (4.0.3) *and assume that the geodesic flow associated to g is non-trapping. Let $\sigma \in (0,\infty)\backslash\{1\}$ and u be a* *weak solution to the Cauchy problem*

$$\begin{cases} i\partial_t u(t,x) - |\nabla_g|^\sigma u(t,x) &= F(t,x), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ u(0,x) &= \psi(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{4.0.15}$$

with data $\psi \in \mathscr{L}_g$ and $F \in C(\mathbb{R}, \mathscr{L}_g)$. Then for all (p,q) and (a,b) Schrödinger admissible with $q<\infty$ and $b<\infty$, there exists $C>0$ such that

$$\|u\|_{L^p(\mathbb{R},L^q)} + \|u\|_{L^\infty(\mathbb{R},\dot{H}^{\gamma_{p,q}}_g)} \leq C\Big(\|\psi\|_{\dot{H}^{\gamma_{p,q}}_g} + \|F\|_{L^{a'}(\mathbb{R},L^{b'})}\Big), \tag{4.0.16}$$

provided that $(p,a) \neq (2,2)$ and

$$\gamma_{p,q} = \gamma_{a',b'} + \sigma. \tag{4.0.17}$$

Remark 4.0.7.

1. The homogeneous Strichartz estimates (4.0.14) and the Minkowski inequality imply

$$\|u\|_{L^p(\mathbb{R},L^q)} \leq C\Big(\|\psi\|_{\dot{H}^{\gamma_{p,q}}_g} + \|F\|_{L^1(\mathbb{R},\dot{H}^{\gamma_{p,q}}_g)}\Big). \tag{4.0.18}$$

2. When $\sigma \in (0,2)\backslash\{1\}$, we always have $\gamma_{p,q} > 0$ for any Schrödinger admissible pair (p,q) except $(p,q) = (\infty,2)$. Thus, condition (4.0.17) implies that $(p,a) \neq (2,2)$, and (4.0.16) includes the endpoint case. When $\sigma \geq 2$, the estimates (4.0.16) do not include the endpoint estimate.
3. In the case $\sigma \in (0,2]\backslash\{1\}$, we can replace the homogeneous Sobolev norms in (4.0.16) and (4.0.18) by the inhomogeneous ones.

Proposition 4.0.8. *Consider $\mathbb{R}^d, d \geq 3$ equipped with a smooth metric g satisfying* (4.0.2), (4.0.3) *and assume that the geodesic flow associated to g is non-trapping. Let $\sigma \in (0,\infty)\backslash\{1\}$ and v be a weak solution to the Cauchy problem*

$$\begin{cases} \partial_t^2 v(t,x) + (-\Delta_g)^\sigma v(t,x) &= F(t,x), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ v(0,x) = v_0(x), & \partial_t v(0,x) = v_1(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{4.0.19}$$

with data $v_0, v_1 \in \mathscr{L}_g$ and $F \in C(\mathbb{R}, \mathscr{L}_g)$. Then for all (p,q) and (a,b) Schrödinger admissible with $q<\infty$ and $b<\infty$, there exists $C>0$ such that

$$\|v\|_{L^p(\mathbb{R},L^q)} + \|[v]\|_{L^\infty(\mathbb{R},\dot{H}^{\gamma_{p,q}}_g)} \leq C\left(\|[v](0)\|_{\dot{H}^{\gamma_{p,q}}_g} + \|F\|_{L^{a'}(\mathbb{R},L^{b'})}\right), \tag{4.0.20}$$

where $[v](t) := (v(t), \partial_t v(t))$ and

$$\|[v]\|_{L^\infty(\mathbb{R},\dot{H}^{\gamma_{p,q}}_g)} := \|v\|_{L^\infty(\mathbb{R},\dot{H}^{\gamma_{p,q}}_g)} + \|\partial_t v\|_{L^\infty(\mathbb{R},\dot{H}^{\gamma_{p,q}-\sigma}_g)}$$

provided that $(p,a) \neq (2,2)$ and

$$\gamma_{p,q} = \gamma_{a',b'} + 2\sigma. \tag{4.0.21}$$

Remark 4.0.9. As in Remark 4.0.7, we have

$$\|v\|_{L^p(\mathbb{R},L^q)} \leq C\Big(\|[v](0)\|_{\dot{H}^{\gamma_{p,q}}_g} + \|F\|_{L^1(\mathbb{R},\dot{H}^{\gamma_{p,q}-\sigma}_g)}\Big). \tag{4.0.22}$$

4.1 Functional calculus and propagation estimates

In this section, we recall some well-known results on pseudo-differential operators and prove some propagation estimates related to our problem.

4.1.1 Pseudo-differential operators.

Let $\mu, m \in \mathbb{R}$. We consider the symbol class $S(\mu, m)$ the space of smooth functions a on $\mathbb{R}^{2d}$ satisfying

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \le C_{\alpha\beta} \langle x\rangle^{\mu-|\alpha|} \langle \xi\rangle^{m-|\beta|}.$$

In practice, we mainly use $S(\mu, -\infty) := \cap_{m\in\mathbb{R}} S(\mu, m)$.

For $a \in S(\mu, m)$ and $h \in (0,1]$, we consider the **semi-classical pseudo-differential operator** $Op^h(a)$ which is defined by

$$Op^h(a)u(x) = (2\pi h)^{-d} \iint_{\mathbb{R}^{2d}} e^{ih^{-1}(x-y)\cdot\xi} a(x,\xi) u(y) dy d\xi. \tag{4.1.1}$$

By the long range assumption (4.0.3), we see that $h^2 P = Op^h(p) + hOp^h(p_1)$ with $p \in S(0,2)$ given in (4.0.5) and $p_1 \in S(-\rho-1, 1) \subset S(-1,1)$. We recall that for $a \in S(\mu_1, m_1)$ and $b \in S(\mu_2, m_2)$, the composition $Op^h(a)Op^h(b)$ is given by

$$Op^h(a)Op^h(b) = \sum_{j=0}^{N-1} h^j Op^h((a\# b)_j) + h^N Op^h(r_N^\#(h)), \tag{4.1.2}$$

where $(a\#b)_j = \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha a D_x^\alpha b \in S(\mu_1+\mu_2-j, m_1+m_2-j)$ and $(r_N^\#(h))_{h\in(0,1]}$ is a bounded family in $S(\mu_1+\mu_2-N, m_1+m_2-N)$. The adjoint with respect to the Lebesgue measure $Op^h(a)^\star$ is given by

$$Op^h(a)^\star = \sum_{j=0}^{N-1} h^j Op^h(a_j^\star) + h^N Op^h(r_N^\star(h)), \tag{4.1.3}$$

where $a_j^\star = \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha \overline{a} \in S(\mu_1 - j, m_1 - j)$ and $(r_N^\star(h))_{h\in(0,1]}$ is a bounded family in $S(\mu_1 - N, m_1 - N)$.

We next recall the definition of rescaled pseudo-differential operator which is essentially given in [BM16]. This type of operator is very useful for the analysis at low frequency. Let $a \in S(\mu, m)$ and $\epsilon \in (0,1]$. The **rescaled pseudo-differential operator** $Op_\epsilon(a)$ is defined by

$$Op_\epsilon(a)u(x) = (2\pi)^{-d} \iint_{\mathbb{R}^{2d}} e^{i(x-y)\cdot\xi} a(\epsilon x, \epsilon^{-1}\xi) u(y) dy d\xi.$$

Setting $D_\epsilon u(x) := \epsilon^{d/2} u(\epsilon x)$. It is easy to see that D_ϵ is a unitary map on L^2 and

$$Op_\epsilon(a) = D_\epsilon Op(a) D_\epsilon^{-1}, \tag{4.1.4}$$

where $D_\epsilon^{-1} u(x) = \epsilon^{-d/2} u(\epsilon^{-1} x)$ and $Op(a) := Op^1(a)$, i.e. $h = 1$ in (4.1.1). Thanks to (4.1.2), (4.1.3) and (4.1.4), the composition $Op_\epsilon(a)Op_\epsilon(b)$ and the adjoint with respect to the Lebesgue measure $Op_\epsilon(a)^\star$ with $a \in S(\mu_1, m_1)$ and $b \in S(\mu_2, m_2)$ are given by

$$Op_\epsilon(a)Op_\epsilon(b) = \sum_{j=0}^{N-1} Op_\epsilon((a\#b)_j) + Op_\epsilon(r_N^\#), \quad Op_\epsilon(a)^\star = \sum_{j=0}^{N-1} Op_\epsilon(a_j^\star) + Op_\epsilon(r_N^\star).$$

4.1.2 Functional calculus.

In this subsection, we will recall the approximations for $\phi(h^2 P)$ and $\zeta(\epsilon x)\phi(\epsilon^{-2} P)$ in terms of semi-classical and rescaled pseudo-differential operators respectively, where $\phi \in C_0^\infty(\mathbb{R})$ and $\zeta \in C^\infty(\mathbb{R}^d)$ is supported outside $B(0,1)$ and equal to 1 near infinity. Here $B(0,1)$ is the open unit ball in $\mathbb{R}^d$.

We firstly recall the following $L^q \to L^r$-bound of pseudo-differential operators (see e.g. [BT07, Proposition 2.4]).

Proposition 4.1.1. *Let $m > d$ and a be a continuous function on $\mathbb{R}^{2d}$ smooth with respect to the second variable satisfying for all $\beta \in \mathbb{N}^d$, there exists $C_\beta > 0$ such that for all $x, \xi \in \mathbb{R}^d$,*

$$|\partial_\xi^\beta a(x,\xi)| \leq C_\beta \langle \xi \rangle^{-m}.$$

Then for $1 \leq q \leq r \leq \infty$, there exists $C > 0$ such that for all $h \in (0,1]$,

$$\|Op^h(a)\|_{L^q \to L^r} \leq Ch^{-(d/q - d/r)}.$$

The following proposition gives an approximation of $\phi(h^2 P)$ in terms of semi-classical pseudo-differential operators (see e.g. [BT07] or [Rob87]).

Proposition 4.1.2. *Consider $\mathbb{R}^d$ equipped with a smooth metric g satisfying* (4.0.2) *and* (4.0.3). *Then for a given $\phi \in C_0^\infty(\mathbb{R})$, there exist a sequence of symbols $q_j \in S(-j, -\infty)$ satisfying $q_0 = \phi \circ p$ and $\operatorname{supp}(q_j) \subset \operatorname{supp}(\phi \circ p)$ such that for all $N \geq 1$,*

$$\phi(h^2 P) = \sum_{j=0}^{N-1} h^j Op^h(q_j) + h^N R_N(h),$$

and for $m \geq 0$ and $1 \leq q \leq r \leq \infty$, there exists $C > 0$ such that for all $h \in (0,1]$,

$$\|R_N(h) \langle x \rangle^N \|_{L^q \to L^r} \leq Ch^{-(d/q - d/r)},$$
$$\|R_N(h) \langle x \rangle^N \|_{H^{-m} \to H^m} \leq Ch^{-2m}.$$

Combining Proposition 4.1.1 and Proposition 4.1.2, one has the following result (see e.g. [BT07, Proposition 2.9]).

Proposition 4.1.3. *Consider $\mathbb{R}^d$ equipped with a smooth metric g satisfying* (4.0.2) *and* (4.0.3). *Let $\phi \in C_0^\infty(\mathbb{R})$. Then for $1 \leq q \leq r \leq \infty$, there exists $C > 0$ such that for all $h \in (0,1]$,*

$$\|\phi(h^2 P)\|_{L^q \to L^r} \leq Ch^{-(d/q - d/r)}.$$

It is also known (see [BM16]) that the rescaled pseudo-differential operator is very useful to approximate the low frequency localization of P, i.e. operators of the form $\phi(\epsilon^{-2} P)$. By the uncertainty principle, one can only expect to get such approximation whenever $|x|$ is large, typically $|x| \gtrsim \epsilon^{-1}$.

Remark 4.1.4. Let $\mu \leq 0, m \in \mathbb{R}$ and $a \in S(\mu, m)$. If we set

$$a_\epsilon(x,\xi) := \epsilon^\mu a(\epsilon^{-1} x, \xi),$$

then for all $\alpha, \beta \in \mathbb{N}^d$, there exists $C_{\alpha\beta} > 0$ such that for all $|x| \geq 1, \xi \in \mathbb{R}^d$,

$$|\partial_x^\alpha \partial_\xi^\beta a_\epsilon(x,\xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m - |\beta|}, \quad \forall \epsilon \in (0,1].$$

We next rewrite $\epsilon^{-2} P$ as $D_\epsilon (D_\epsilon^{-1} (\epsilon^{-2} P) D_\epsilon) D_\epsilon^{-1}$. A direct computation gives

$$D_\epsilon^{-1} (\epsilon^{-2} P) D_\epsilon = Op(p_\epsilon) + Op(p_{\epsilon,1}) =: P_\epsilon,$$

where $p_\epsilon(x,\xi) = p(\epsilon^{-1} x, \xi)$ and $p_{\epsilon,1}(x,\xi) = \epsilon^{-1} p_1(\epsilon^{-1} x, \xi)$. We thus obtain

$$\epsilon^{-2} P = Op_\epsilon(p_\epsilon) + Op_\epsilon(p_{\epsilon,1}). \tag{4.1.5}$$

Using the fact that $p \in S(0,2), p_1 \in S(-1,1)$, Remark 4.1.4 allows us to construct the parametrix for the resolvent $\zeta(\epsilon x)(\epsilon^{-2} P - z)^{-k}$ with $\zeta \in C^\infty(\mathbb{R}^d)$ supported outside $B(0,1)$ and equal to 1 near infinity. Indeed, by writing $\zeta(\epsilon x)(\epsilon^{-2}P - z)^{-k} = D_\epsilon \left[\zeta(x)(P_\epsilon - z)^{-k}\right] D_\epsilon^{-1}$, we can apply the standard elliptic parametrix for $\zeta(x)(P_\epsilon - z)^{-k}$ and we have (see e.g. [BT07] or [BM16]) the following result.

Proposition 4.1.5. *Let $\zeta, \tilde{\zeta}, \tilde{\tilde{\zeta}} \in C^\infty(\mathbb{R}^d)$ be supported outside $B(0,1)$ and equal to 1 near infinity such that $\tilde{\zeta} = 1$ near $\mathrm{supp}(\zeta)$ and $\tilde{\tilde{\zeta}} = 1$ near $\mathrm{supp}(\tilde{\zeta})$. Then for all $k, N \geq 1$ integers and $z \in \mathbb{C}\backslash[0,+\infty)$, we have for $\epsilon \in (0,1]$,*

$$\zeta(\epsilon x)(\epsilon^{-2}P - z)^{-k} = \sum_{j=0}^{N-1} \zeta(\epsilon x) Op_\epsilon(b_{\epsilon,j}(z))\tilde{\zeta}(\epsilon x) + R_N(z,\epsilon),$$

where $(b_{\epsilon,j}(z))_{\epsilon\in(0,1]}$ is a bounded family in $S(-j,-2k-j)$ which is a linear combination of $d_{\epsilon,l}(p_\epsilon - z)^{-k-l}$ with $(d_{\epsilon,l})_{\epsilon\in(0,1]}$ a bounded family in $S(-j, 2l-j)$ and

$$R_N(z,\epsilon) = \zeta(\epsilon x) Op_\epsilon(r_N(z,\epsilon))\tilde{\tilde{\zeta}}(\epsilon x)(\epsilon^{-2}P - z)^{-k}$$

where $r_N(z,\epsilon) \in S(-N,-N)$ has seminorms growing polynomially in $1/\mathrm{dist}(z,\mathbb{R}^+)$ uniformly in $\epsilon \in (0,1]$ as long as z belongs to a bounded set of $\mathbb{C}\backslash[0,+\infty)$.

A first application of Proposition 4.1.5 is the following result.

Proposition 4.1.6. *Using the notations given in* Proposition 4.1.5*, let $k > d/2$ and $2 \leq q \leq \infty$. Then there exists $C > 0$ such that for all $\epsilon \in (0,1]$,*

$$\|\zeta(\epsilon x)(\epsilon^{-2}P + 1)^{-k}\|_{L^2\to L^q} \leq C\epsilon^{d/2-d/q}. \tag{4.1.6}$$

Proof. We apply Proposition 4.1.5 with $N > d$, we see that

$$\begin{aligned}\zeta(\epsilon x)(\epsilon^{-2}P + 1)^{-k} &= \sum_{j=0}^{N-1} \zeta(\epsilon x) Op_\epsilon(b_{\epsilon,j}(-1))\tilde{\zeta}(\epsilon x) + \zeta(\epsilon x) Op_\epsilon(r_N(-1,\epsilon))\tilde{\tilde{\zeta}}(\epsilon x)(\epsilon^{-2}P+1)^{-k}, \\ &= \sum_{j=0}^{N-1} D_\epsilon \left\{\zeta(x) Op(b_{\epsilon,j}(-1))\tilde{\zeta}(x) + \zeta(x) Op(r_N(-1,\epsilon))\tilde{\tilde{\zeta}}(x)(P_\epsilon + 1)^{-k}\right\} D_\epsilon^{-1},\end{aligned}$$

where $(b_{\epsilon,j}(-1))_{\epsilon\in(0,1]}, (r_N(-1,\epsilon))_{\epsilon\in(0,1]}$ are bounded in $S(-j,-2k-j)$ and $S(-N,-N)$ respectively. The result then follows from Proposition 4.1.1 with $h = 1$ and that

$$\|D_\epsilon\|_{L^q\to L^q} = \epsilon^{d/2-d/q}, \quad \|D_\epsilon^{-1}\|_{L^2\to L^2} = 1.$$

We also use that $\|(P_\epsilon + 1)^{-k}\|_{L^2\to L^2} \leq 1$ for the remainder term. ∎

Another application of Proposition 4.1.5 is the following approximation of $\zeta(\epsilon x)\phi(\epsilon^{-2}P)$ in terms of rescaled pseudo-differential operators.

Proposition 4.1.7. *Consider $\mathbb{R}^d$ equipped with a smooth metric g satisfying* (4.0.2) *and* (4.0.3)*. Let $\phi \in C_0^\infty(\mathbb{R})$ and $\zeta, \tilde{\zeta}, \tilde{\tilde{\zeta}}$ be as in* Proposition 4.1.5*. Then there exists a sequence of bounded families of symbols $(q_{\epsilon,j})_{\epsilon\in(0,1]} \in S(-j,-\infty)$ with $q_{\epsilon,0} = \phi \circ p_\epsilon$ and $\mathrm{supp}(q_{\epsilon,j}) \subset \mathrm{supp}(\phi\circ p_\epsilon)$ such that for all $N \geq 1$,*

$$\zeta(\epsilon x)\phi(\epsilon^{-2}P) = \sum_{j=0}^{N-1} \zeta(\epsilon x) Op_\epsilon(q_{\epsilon,j})\tilde{\zeta}(\epsilon x) + R_N(\epsilon). \tag{4.1.7}$$

Moreover, for any $m \geq 0$, there exists $C > 0$ such that for all $\epsilon \in (0,1]$,

$$\|(\epsilon^{-2}P + 1)^m R_N(\epsilon)\langle \epsilon x\rangle^N\|_{L^2\to L^2} \leq C. \tag{4.1.8}$$

Proof. By using Proposition 4.1.5 with $k = 1$ and the Helffer-Sjöstrand formula (see [DS99]) namely

$$\phi(\epsilon^{-2}P) = -\frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{\phi}(z)(\epsilon^{-2}P - z)^{-1}dL(z),$$

where $\widetilde{\phi}$ is an almost analytic extension of ϕ, the Cauchy formula gives (4.1.7) with

$$R_N(\epsilon) = \frac{1}{\pi}\int_{\mathbb{C}} \overline{\partial}\widetilde{\phi}(z)\zeta(\epsilon x)Op_\epsilon(r_N(z,\epsilon))\tilde{\tilde{\zeta}}(\epsilon x)(\epsilon^{-2}P - z)^{-1}dL(z). \tag{4.1.9}$$

Here $(r_N(z,\epsilon))_{\epsilon\in(0,1]}$ is bounded in $S(-N,-N)$ and has semi-norms growing polynomially in $|\mathrm{Im}\, z|^{-1}$ which is harmless since $\overline{\partial}\widetilde{\phi}(z) = O(|\mathrm{Im}\, z|^\infty)$. The left hand side of (4.1.8) is bounded by

$$\frac{1}{\pi}\int_{\mathbb{C}} |\overline{\partial}\widetilde{\phi}(z)|\|(\epsilon^{-2}P+1)^m\zeta(\epsilon x)Op_\epsilon(r_N(z,\epsilon))\tilde{\tilde{\zeta}}(\epsilon x)(\epsilon^{-2}P - z)^{-1}\langle\epsilon x\rangle^N\|_{L^2\to L^2}dL(z).$$

By choosing $\zeta_1 \in C^\infty(\mathbb{R}^d)$ supported outside $B(0,1)$ such that $\zeta_1 = 1$ near $\mathrm{supp}(\tilde{\tilde{\zeta}})$, we can write

$$(\epsilon^{-2}P - z)^{-1} = (\epsilon^{-2}P - z)^{-1}(1-\zeta_1)(\epsilon x) + (\epsilon^{-2}P - z)^{-1}\zeta_1(\epsilon x).$$

We note that $(1-\zeta_1)(\epsilon x)\langle\epsilon x\rangle^N$ is of size $O_{L^2\to L^2}(1)$ due to the compact support in ϵx, and $(\epsilon^{-2}P+1)(\epsilon^{-2}P-z)^{-1}$ is of size $O_{L^2\to L^2}(|\mathrm{Im}\, z|^{-1})$ by functional calculus. Moreover, using (4.1.5) and the same process as in the proof of Proposition 4.1.6, there exists $\tau(m)\in\mathbb{N}$ such that

$$\|(\epsilon^{-2}P+1)^m\zeta(\epsilon x)Op_\epsilon(r_N(z,\epsilon))\tilde{\tilde{\zeta}}(\epsilon x)(\epsilon^{-2}P+1)^{-1}\|_{L^2\to L^2} \le C|\mathrm{Im}\, z|^{-\tau(m)}.$$

This shows that

$$\|(\epsilon^{-2}P+1)^m R_N(\epsilon)(1-\zeta_1)(\epsilon x)\langle\epsilon x\rangle^N\|_{L^2\to L^2} \le C. \tag{4.1.10}$$

For the term $(\epsilon^{-2}P+1)^m R_N(\epsilon)\zeta_1(\epsilon x)\langle\epsilon x\rangle^N$, using Proposition 4.1.5 (by taking the adjoint), we see that

$$(\epsilon^{-2}P - z)^{-1}\zeta_1(\epsilon x) = \sum_{j=0}^{N'-1}\tilde{\zeta}_1(\epsilon x)Op_\epsilon(\tilde{b}_{\epsilon,j}(z))\zeta_1(\epsilon x) + \tilde{R}_{N'}(z,\epsilon),$$

with $(\tilde{b}_{\epsilon,j}(z))_{\epsilon\in(0,1]}$ a bounded family in $S(-j,-2-j)$ and

$$\tilde{R}_{N'}(z,\epsilon) = (\epsilon^{-2}P - z)^{-1}\tilde{\tilde{\zeta}}_1(\epsilon x)Op_\epsilon(\tilde{r}_{N'}(z,\epsilon))\zeta_1(\epsilon x),$$

where $\tilde{r}_{N'}(z,\epsilon) \in S(-N',-N')$ has seminorms growing polynomially in $|\mathrm{Im}\, z|^{-1}$ uniformly in $\epsilon\in(0,1]$. By the same argument as above, we obtain

$$\|(\epsilon^{-2}P+1)^m R_N(\epsilon)\zeta_1(\epsilon x)\langle\epsilon x\rangle^N\|_{L^2\to L^2} \le C. \tag{4.1.11}$$

Combining (4.1.10) and (4.1.11), we prove (4.1.8). □

As a consequence of Proposition 4.1.7, we have the following result.

Corollary 4.1.8. *Let $\phi\in C_0^\infty(\mathbb{R})$. Then for $2\le q\le r\le\infty$, there exists $C>0$ such that for all $\epsilon\in(0,1]$,*

$$\|\zeta(\epsilon x)\phi(\epsilon^{-2}P)\|_{L^q\to L^r} \le C\epsilon^{d/q-d/r}. \tag{4.1.12}$$

Proof. By (4.1.7) and (4.1.8) (see also (4.1.9)), we can write for any $N\ge 1$ and any $m\ge 0$,

$$\zeta(\epsilon x)\phi(\epsilon^{-2}P) = \sum_{j=0}^{N-1}\zeta(\epsilon x)Op_\epsilon(q_{\epsilon,j})\tilde{\zeta}(\epsilon x) + R_N(\epsilon),$$

where

$$R_N(\epsilon) = \tilde{\zeta}(\epsilon x)(\epsilon^{-2}P+1)^{-m}B_\epsilon \langle \epsilon x\rangle^{-N}$$

with $B_\epsilon = O_{L^2\to L^2}(1)$ uniformly in $\epsilon \in (0,1]$. The main terms can be estimated by using Proposition 4.1.1 (see also the proof of Proposition 4.1.6). It remains to treat the remainder term. We firstly note that $\langle \epsilon x\rangle^{-N} = O_{L^q\to L^2}(\epsilon^{d/q-d/2})$ provided $N > \frac{d(q-2)}{2q}$. Using this bound together with $B_\epsilon = O_{L^2\to L^2}(1)$ and (4.1.6), we see that

$$\begin{aligned}\|R_N(\epsilon)\|_{L^q\to L^r} &\lesssim \|\tilde{\zeta}(\epsilon x)(\epsilon^{-2}P+1)^{-m}\|_{L^2\to L^r}\|B_\epsilon\|_{L^2\to L^2}\| \langle \epsilon x\rangle^{-N}\|_{L^q\to L^2} \\ &\lesssim \epsilon^{d/2-d/r}\epsilon^{d/q-d/2} \lesssim \epsilon^{d/q-d/r}.\end{aligned}$$

This proves (4.1.12). □

Another consequence of Proposition 4.1.7 is the following estimate.

Corollary 4.1.9. *Let $\phi \in C_0^\infty(\mathbb{R})$. For $m \geq 0$, there exists $C>0$ such that for all $\epsilon \in (0,1]$,*

$$\| \langle \epsilon x\rangle^{-m} \phi(\epsilon^{-2}P) \langle \epsilon x\rangle^{m} \|_{L^2\to L^2} \leq C. \tag{4.1.13}$$

Proof. By choosing $\zeta \in C^\infty(\mathbb{R}^d)$ supported outside $B(0,1)$ and equal to 1 near infinity, we can write $\langle \epsilon x\rangle^{-m} \phi(\epsilon^{-2}P) \langle \epsilon x\rangle^{m}$ as

$$\langle \epsilon x\rangle^{-m} \phi(\epsilon^{-2}P)\zeta(\epsilon x) \langle \epsilon x\rangle^{m} + \langle \epsilon x\rangle^{-m} \phi(\epsilon^{-2}P)(1-\zeta)(\epsilon x) \langle \epsilon x\rangle^{m}.$$

The $L^2 \to L^2$-boundedness of the first term follows from the parametrix of $\phi(\epsilon^{-2}P)\zeta(\epsilon x)$ which is obtained by taking the adjoint of (4.1.7). The second term follows from the fact that $(1-\zeta)(\epsilon x)\langle \epsilon x\rangle^m$ is bounded on L^2 since $1-\zeta$ vanishes outside a compact set. □

4.1.3 Propagation estimates.

In this subsection, we recall some results on resolvent estimates and prove some propagation estimates both at high and low frequencies. Let us start with the following result.

Proposition 4.1.10. 1. *Consider $\mathbb{R}^d, d \geq 2$ equipped with a smooth metric g satisfying (4.0.2), (4.0.3) and suppose that the assumption (4.0.11) holds. Then for $k \geq 0$, there exists non-decreasing $N_k \in \mathbb{N}$ such that for λ belonging to a relatively compact interval of $(0,+\infty)$, there exists $C>0$ such that for all $h \in (0,1]$,*

$$\| \langle x\rangle^{-1-k} (h^2P-\lambda \mp i0)^{-1-k} \langle x\rangle^{-1-k} \|_{L^2\to L^2} \leq Ch^{-N_k}. \tag{4.1.14}$$

2. *Consider $\mathbb{R}^d, d \geq 3$ equipped with a smooth metric g satisfying (4.0.2), (4.0.3). Then for $k \geq 0$ and λ belonging to a relatively compact interval of $(0,+\infty)$, there exists $C>0$ such that for all $\epsilon \in (0,1]$,*

$$\| \langle \epsilon x\rangle^{-1-k} (\epsilon^{-2}P-\lambda \mp i0)^{-1-k} \langle \epsilon x\rangle^{-1-k} \|_{L^2\to L^2} \leq C. \tag{4.1.15}$$

The high frequency resolvent estimates (4.1.14) are given in [BM16, Proposition 7.5] and the low frequency resolvent estimates (4.1.15) are given in [BR15, Theorem 1.2]. Note that under the non-trapping condition, the estimates (4.1.14) hold with $N_k = k+1$ (see e.g. [Rob94, Theorem 2.8]). We next use the resolvent estimates given in Proposition 4.1.10 to have the following resolvent estimates for the Schrödinger-type operator.

Proposition 4.1.11. *Let $\sigma \in (0,\infty)$.*

1. *Consider $\mathbb{R}^d, d \geq 2$ equipped with a smooth metric g satisfying (4.0.2), (4.0.3) and suppose that the assumption (4.0.11) holds. Then for $k \geq 0$, there exists non-decreasing $N_k \in \mathbb{N}$ such that for λ belonging to a relatively compact interval of $(0,+\infty)$, there exists $C>0$ such that for all $h \in (0,1]$,*

$$\| \langle x\rangle^{-1-k} ((h|\nabla_g|)^\sigma - \mu \mp i0)^{-1-k} \langle x\rangle^{-1-k} \|_{L^2\to L^2} \leq Ch^{-N_k}. \tag{4.1.16}$$

2. *Consider* $\mathbb{R}^d, d \geq 3$ *equipped with a smooth metric* g *satisfying* (4.0.2), (4.0.3). *Then for* $k \geq 0$ *and* λ *belonging to a relatively compact interval of* $(0, +\infty)$*, there exists* $C > 0$ *such that for all* $\epsilon \in (0, 1]$*,*

$$\| \langle \epsilon x \rangle^{-1-k} ((\epsilon^{-1}|\nabla_g|)^\sigma - \mu \mp i0)^{-1-k} \langle \epsilon x \rangle^{-1-k} \|_{L^2 \to L^2} \leq C. \tag{4.1.17}$$

Proof. We only give the proof for (4.1.16), the one for (4.1.17) is similar using (4.1.13). We firstly note that the estimates (4.1.16) are equivalent to

$$\| \langle x \rangle^{-1-k} ((h|\nabla_g|)^\sigma - \mu \mp i0)^{-1-k} \phi(h^2 P) \langle x \rangle^{-1-k} \|_{L^2 \to L^2} \leq C h^{-N_k},$$

where $\phi \in C_0^\infty((0, +\infty))$ satisfying $\phi = 1$ near I. Note that here $|\nabla_g| = \sqrt{P}$. Next, we write $\mu = \lambda^{\sigma/2}$ with λ lying in a relatively compact interval of $(0, +\infty)$. By functional calculus, we write

$$(h|\nabla_g|)^\sigma - \mu \mp i0 = (h^2 P - \lambda \mp i0) Q(h^2 P, \mu),$$

where $Q(\cdot, \mu)$ is smooth and non vanishing on the support of ϕ. This implies for all $k \geq 0$,

$$((h|\nabla_g|)^\sigma - \mu \mp i0)^{-1-k} \phi(h^2 P) = (h^2 P - \lambda \mp i0)^{-1-k} \tilde{Q}(h^2 P, \mu),$$

where $\tilde{Q}(h^2 P, \mu) = \phi(h^2 P) Q^{-1-k}(h^2 P, \mu)$. This allows us to approximate $\tilde{Q}(h^2 P, \mu)$ by pseudo-differential operators by means of Proposition 4.1.2. Thus, we have that $\langle x \rangle^{1+k} \tilde{Q}(h^2 P, \mu) \langle x \rangle^{-1-k}$ is of size $O_{L^2 \to L^2}(1)$ uniformly in $\mu \in I \Subset (0, +\infty)$ and $h \in (0, 1]$. Therefore, (4.1.16) follows from (4.1.14). The proof is complete. □

We now give an application of resolvent estimates given in Proposition 4.1.11 when $k = 0$ and obtain the following global L^2 integrability estimates for the Schrödinger-type operators both at high and low frequencies.

Proposition 4.1.12. *Let* $\sigma \in (0, \infty)$ *and* $f \in C_0^\infty((0, +\infty))$.

1. *Consider* $\mathbb{R}^d, d \geq 2$ *equipped with a smooth metric* g *satisfying* (4.0.2), (4.0.3) *and suppose that the assumption* (4.0.11) *holds. Then there exists* $C > 0$ *such that for all* $\psi \in L^2$ *and all* $h \in (0, 1]$*,*

$$\| \langle x \rangle^{-1} f(h^2 P) e^{-ith^{-1}(h|\nabla_g|)^\sigma} \psi \|_{L^2(\mathbb{R}, L^2)} \leq C h^{(1-N_0)/2} \|\psi\|_{L^2}. \tag{4.1.18}$$

2. *Consider* $\mathbb{R}^d, d \geq 3$ *equipped with a smooth metric* g *satisfying* (4.0.2), (4.0.3). *Then there exists* $C > 0$ *such that for all* $\psi \in L^2$ *and all* $\epsilon \in (0, 1]$*,*

$$\| \langle \epsilon x \rangle^{-1} f(\epsilon^{-2} P) e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma} \psi \|_{L^2(\mathbb{R}, L^2)} \leq C \epsilon^{-1/2} \|\psi\|_{L^2}. \tag{4.1.19}$$

Remark 4.1.13.
1. By interpolating between $L^2(\mathbb{R})$ and $L^\infty(\mathbb{R})$, we get the following L^p integrability estimates

$$\| \langle x \rangle^{-1} f(h^2 P) e^{-ith^{-1}(h|\nabla_g|)^\sigma} \psi \|_{L^p(\mathbb{R}, L^2)} \leq C h^{(1-N_0)/p} \|\psi\|_{L^2}. \tag{4.1.20}$$

$$\| \langle \epsilon x \rangle^{-1} f(\epsilon^{-2} P) e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma} \psi \|_{L^p(\mathbb{R}, L^2)} \leq C \epsilon^{-1/p} \|\psi\|_{L^2}. \tag{4.1.21}$$

2. Thanks to the fact that P is non-negative, these estimates are still true for $f \in C_0^\infty(\mathbb{R} \backslash \{0\})$. Moreover, we can replace $\|\psi\|_{L^2}$ in the right hand side of (4.1.18) and (4.1.20) (resp. (4.1.19) and (4.1.21)) by $\|f(h^2 P)\psi\|_{L^2}$ (resp. $\|f(\epsilon^{-2} P)\psi\|_{L^2}$). Indeed, we choose $\tilde{f} \in C_0^\infty(\mathbb{R} \backslash \{0\})$ such that $\tilde{f} = 1$ near supp(f) and write $f(h^2 P) = \tilde{f}(h^2 P) f(h^2 P)$. We apply (4.1.18) and (4.1.20) with $\tilde{f}$ instead of f. Similarly for the low frequency case.

Proof of Proposition 4.1.12. We again only consider the high frequency case, the low frequency one is completely similar. By the limiting absorption principle (see [ReS78, Theorem XIII.25]), we see that $\| \langle x\rangle^{-1} f(h^2P)e^{-it(h|\nabla_g|)^\sigma}\psi\|^2_{L^2(\mathbb{R},L^2)}$ is bounded by

$$2\pi \sup_{\substack{\mu\in\mathbb{R}\\ \epsilon>0}} \| \langle x\rangle^{-1} f(h^2P)((h|\nabla_g|)^\sigma - \mu - i\epsilon)^{-1} f(h^2P)\langle x\rangle^{-1} \|_{L^2\to L^2}\|\psi\|^2_{L^2}.$$

By functional calculus and the holomorphy of the resolvent, it suffices to bound $\| \langle x\rangle^{-1} f(h^2P)$ $((h|\nabla_g|)^\sigma - \mu - i0)^{-1} f(h^2P)\langle x\rangle^{-1} \|_{L^2\to L^2}$, uniformly with respect to $\mu\in\mathbb{R}$. As a function of $h|\nabla_g|$, the operator $f(h^2P)((h|\nabla_g|)^\sigma-\mu-i0)^{-1}f(h^2P)$ reads $f(\lambda^2)(\lambda^\sigma-\mu-i0)^{-1}f(\lambda^2)$. Assume that $\mathrm{supp}(f)\subset \left[1/c^2,c^2\right]$ for some $c>1$, so $\lambda\in[1/c,c]$.

In the case $\mu\geq 2c^\sigma$ or $\mu\leq 1/2c^\sigma$, we have that $\mu-\lambda^\sigma\geq c^\sigma$ or $\lambda^\sigma-\mu\geq 1/2c^\sigma$. The functional calculus gives

$$\|f(h^2P)((h|\nabla_g|)^\sigma-\mu-i0)^{-1}f(h^2P)\|_{L^2\to L^2}\leq 2c^\sigma\|f\|^2_{L^\infty(\mathbb{R})}.$$

Thus we can assume that $\mu\in[1/2c^\sigma,2c^\sigma]$. Using (4.1.16) with $k=0$, we have

$$\| \langle x\rangle^{-1} ((h|\nabla_g|)^\sigma-\mu\mp i0)^{-1}\langle x\rangle^{-1}\|_{L^2\to L^2}\leq Ch^{-N_0}.$$

On the other hand, $\langle x\rangle^{-1} f(h^2P)\langle x\rangle$ is bounded on L^2 by pseudo-differential calculus. This implies

$$\| \langle x\rangle^{-1} f(h^2P)e^{-it(h|\nabla_g|)^\sigma}\psi\|_{L^2(\mathbb{R},L^2)}\leq Ch^{-N_0/2}\|\psi\|_{L^2}.$$

By scaling in time, this gives the result. □

Another application of the resolvent estimates given in Proposition 4.1.11 is the following local energy decay for the Schrödinger-type operators both at high and low frequencies.

Proposition 4.1.14. *Let $\sigma\in(0,\infty)$ and $f\in C_0^\infty(\mathbb{R}\backslash\{0\})$.*

1. *Consider $\mathbb{R}^d, d\geq 2$ equipped with a smooth metric g satisfying* (4.0.2), (4.0.3) *and suppose that the assumption* (4.0.11) *holds. Then for $k\geq 0$, there exist $C>0$ and non-decreasing $N_k\in\mathbb{N}$ such that for all $t\in\mathbb{R}$ and all $h\in(0,1]$,*

$$\| \langle x\rangle^{-1-k} e^{-ith^{-1}(h|\nabla_g|)^\sigma} f(h^2P)\langle x\rangle^{-1-k}\|_{L^2\to L^2}\leq Ch^{-N_k}\left\langle th^{-1}\right\rangle^{-k}. \tag{4.1.22}$$

2. *Consider $\mathbb{R}^d, d\geq 3$ equipped with a smooth metric g satisfying* (4.0.2), (4.0.3)*. Then for $k\geq 0$, there exists $C>0$ such that for all $t\in\mathbb{R}$ and all $\epsilon\in(0,1]$,*

$$\| \langle \epsilon x\rangle^{-1-k} e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma} f(\epsilon^{-2}P)\langle \epsilon x\rangle^{-1-k}\|_{L^2\to L^2}\leq C\left\langle \epsilon t\right\rangle^{-k}. \tag{4.1.23}$$

Proof. As above, we only give the proof for the high frequency case. Using the Stone formula, the operator $e^{-it(h|\nabla_g|)^\sigma}f(h^2P)$ reads

$$\frac{1}{2i\pi}\int_{\mathbb{R}} e^{-it\mu}f(\mu^{2/\sigma})(((h|\nabla_g|)^\sigma-\mu-i0)^{-1}-((h|\nabla_g|)^\sigma-\mu+i0)^{-1})d\mu.$$

We use the same trick as in [BT08]. By multiplying to above equality with $(it)^k$ and using integration by parts in the weighted spaces $\langle x\rangle^{-1-k} L^2$, we see that $(it)^k e^{-it(h|\nabla_g|)^\sigma}f(h^2P)$ is a linear combination with $l+n=k$ of terms of the form

$$\int_{\mathbb{R}} e^{-it\mu}\partial^l_\mu(f(\mu^{2/\sigma}))(((h|\nabla_g|)^\sigma-\mu-i0)^{-1-n}-((h|\nabla_g|)^\sigma-\mu+i0)^{-1-n})d\mu.$$

The compact support of f implies that μ is bounded from above and below. The resolvent estimates (4.1.16) then imply

$$\| \langle x\rangle^{-1-k} e^{-it(h|\nabla_g|)^\sigma} f(h^2P)\langle x\rangle^{-1-k}\|_{L^2\to L^2}\leq Ch^{-N_k}\left\langle t\right\rangle^{-k}.$$

Here we use that N_m is non-decreasing with respect to m. By scaling in time, we have (4.1.22). The proof is complete. $\square$

4.2 Reduction of the problem

4.2.1 The Littlewood-Paley theorems

In this subsection, we recall some Littlewood-Paley type estimates which are essentially given in [BM16]. Let us introduce $f(\lambda) = f_0(\lambda) - f_0(2\lambda)$, where f_0 given as in (4.0.9). We have $f \in C_0^\infty(\mathbb{R}\backslash\{0\})$ and

$$\sum_{k=1}^\infty f(2^{-k}\lambda) = (1-f_0)(\lambda), \quad \sum_{k=0}^\infty f(2^k\lambda) = \mathbb{1}_{\mathbb{R}\backslash\{0\}}(\lambda) f_0(\lambda), \quad \lambda \in \mathbb{R}.$$

The Spectral Theorem implies that

$$(1-f_0)(P) = \sum_{k=1}^\infty f(2^{-k}P), \quad f_0(P) = \sum_{k=0}^\infty f(2^k P). \tag{4.2.1}$$

Here we use the fact that 0 is not an eigenvalue of P in the second sum.

Theorem 4.2.1. *1. Let $N \geq 1$ and $\chi \in C_0^\infty(\mathbb{R}^d)$. Then for $q \in [2,\infty)$, there exists $C>0$ such that*

$$\|(1-\chi)(1-f_0)(P)v\|_{L^q} \leq C\Big(\sum_{h^2=2^{-k}} \|(1-\chi)f(h^2P)v\|^2_{L^q} + h^N \| \langle x\rangle^{-N} f(h^2P)v\|^2_{L^2}\Big)^{1/2}, \tag{4.2.2}$$

for all $v \in \mathscr{S}(\mathbb{R}^d)$, where $k \in \mathbb{N}\backslash\{0\}$. The same estimates hold for χ in place of $1-\chi$.

2. Let $\chi \in C_0^\infty(\mathbb{R}^d)$ be such that $\chi(x) = 1$ for $|x| \leq 1$. Then for $q \in (2,\infty)$, there exists $C>0$ such that for all $v \in L^2$,

$$\|f_0(P)v\|_{L^q} \leq C\Big(\sum_{\epsilon^{-2}=2^k} \|(1-\chi)(\epsilon x) f(\epsilon^{-2}P)v\|^2_{L^q} + \|\epsilon^{d/2-d/q}\langle \epsilon x\rangle^{-1} f(\epsilon^{-2}P)v\|^2_{L^2}\Big)^{1/2}. \tag{4.2.3}$$

Here we use in the sum that $k \in \mathbb{N}$.

Note that the Littlewood-Paley theorem at low frequency is slightly different from the one in [BM16, Theorem 4.1]. In [BM16], Bouclet-Mizutani considered the sharp Schrödinger admissible condition (see (0.0.1)). This allows to interpolate between the trivial Strichartz estimate for $(\infty,2)$ and the endpoint Strichartz estimate for the endpoint pair $(2,2^\star)$. The proof of the low frequency Littlewood-Paley theorem given in [BM16] makes use of the homogeneous Sobolev embedding

$$\|v\|_{L^{2^\star}} \leq C\||\nabla_g|v\|_{L^2}, \quad 2^\star = \frac{2d}{d-2}. \tag{4.2.4}$$

Since we consider a larger range of admissible condition (1.1.2), we can not apply this interpolation technique. To overcome this difficulty, we will take the advantage of heat kernel estimates. Our estimate (4.2.3) is robust and can be applied for another types of dispersive equations such as the wave or Klein-Gordon equations.

Let $K(t,x,y)$ be the kernel of the heat operator $e^{-tP}, t>0$, i.e.

$$e^{-tP}u(x) = \int_{\mathbb{R}^d} K(t,x,y)u(y)dy.$$

We recall some properties (see e.g. [Cha84], [Gri99]) of the heat kernel on arbitrary Riemannian manifold.

Lemma 4.2.2. *Let (M,g) be an arbitrary Riemannian manifold. Then the heat kernel K satisfies the following properties:*

(i) K is a strictly positive C^∞ function on $(0,\infty)\times M\times M$.
(ii) K is symmetric in the space components.
(iii) (Maximum principle)

$$\int_M K(t,x,y)d_g(y)\leq 1.$$

(iv) (Semi-group property)

$$\int_M K(s,x,y)K(t,y,z)d_g(y)=K(s+t,x,z).$$

In order to obtain the heat kernel estimate, we will make use of the Nash inequality (see e.g. [SC02, Theorem 3.2.1]), namely

$$\|u\|_{L^2}\leq C\|u\|_{L^1}^{\frac{2}{d+2}}\|\nabla u\|_{L^2}^{\frac{d}{d+2}}. \tag{4.2.5}$$

Note that the Nash inequality on $\mathbb{R}^d$ is valid for any $d\geq 1$. Thanks to (4.0.8), we have for $d\geq 2$,

$$\|u\|_{L^2}\leq C\|u\|_{L^1}^{\frac{2}{d+2}}\||\nabla_g|u\|_{L^2}^{\frac{d}{d+2}}. \tag{4.2.6}$$

Using (4.2.6), we have the following upper bound for the heat kernel.

Theorem 4.2.3. *There exists $C>0$ such that for all $x,y\in\mathbb{R}^d$ and all $t>0$ such that*

$$K(t,x,x)\leq Ct^{-d/2}, \tag{4.2.7}$$

$$K(t,x,y)\leq Ct^{-d/2}\exp\Big(-\frac{|x-y|^2}{Ct}\Big). \tag{4.2.8}$$

In particular,

$$\|e^{-tP}\|_{L^1\to L^\infty}\leq Ct^{-d/2},\quad t>0. \tag{4.2.9}$$

Proof. The proof is similar to the one given in [Gri99, Theorem 6.1] where the author shows how to get (4.2.7) from the homogeneous Sobolev embedding (4.2.4). For the reader's convenience, we give a sketch of the proof. Fix $x\in\mathbb{R}^d$ and denote $v(t,y)=K(t,y,x)$ and

$$J(t):=\|v(t)\|_{L^2}^2.$$

Using the fact that $\partial_t v(t,y)=-Pv(t,y)$, we have

$$J'(t)=2\left\langle v(t),\partial_t v(t)\right\rangle=-2\left\langle v(t),Pv(t)\right\rangle=-2\||\nabla_g|v(t)\|_{L^2}^2.$$

This implies that $J(t)$ is non-increasing. On the other hand, the maximum principle (see also [Gri99]) shows that

$$\|v(t)\|_{L^1}=\int_{\mathbb{R}^d}K(t,x,y)dy\leq 1.$$

This together with (4.2.6) yield

$$\|v(t)\|_{L^2}^2\leq C\|v(t)\|_{L^1}^{\frac{4}{d+2}}\||\nabla_g|v(t)\|_{L^2}^{\frac{2d}{d+2}}\leq C\||\nabla_g|v(t)\|_{L^2}^{\frac{2d}{d+2}}.$$

We thus get

$$J'(t)\leq -C\|v(t)\|_{L^2}^{\frac{2(d+2)}{d}}=-CJ(t)^{\frac{d+2}{d}}.$$

This implies that

$$J(t) \leq \left(\frac{2C}{d}t + \frac{1}{[J(0)]^{\frac{2}{d}}}\right)^{-\frac{d}{2}}$$

which together with the non-increasing property of $J(t)$ yield

$$J(t) \leq Ct^{-d/2}.$$

The estimate (4.2.7) then follows by the symmetric property of $K(t,x,y)$, i.e. $J(t) = K(2t,x,x)$. Using (4.2.7), the off-diagonal argument (see also [Gri99]) implies the following upper bound for the heat kernel

$$K(t,x,y) \leq Ct^{-d/2}\exp\Big(-\frac{d^2(x,y)}{Ct}\Big), \quad \forall x,y \in \mathbb{R}^d, t>0,$$

where $d(x,y)$ is the geodesic distance from x to y. Thanks to the elliptic condition (4.0.2) of the metric g, it is easy to see that

$$d(x,y) \sim |x-y|.$$

This shows (4.2.8) and the proof is complete. □

We now give some applications of the upper bound (4.2.8). A first application is the following homogeneous Sobolev embedding.

Lemma 4.2.4. *Let $q \in (2,\infty)$ and $\alpha = \frac{d}{2} - \frac{d}{q}$. Then the operator $|\nabla_g|^{-\alpha}$ maps L^2 to L^q. In particular, there exists $C>0$ such that*

$$\|u\|_{L^q} \leq C\||\nabla_g|^{\alpha}u\|_{L^2}. \tag{4.2.10}$$

Proof. We firstly recall the following version of Hardy-Littlewood-Sobolev theorem.

Theorem 4.2.5 ([HL28, Sob63])**.** *Let $1<p<q<\infty$, $\gamma = d + \frac{d}{q} - \frac{d}{p}$ and $K_\gamma(x) := |x|^{-\gamma}$. Then the convolution operator $T_\gamma := f * K_\gamma$ maps L^p to L^q. In particular, there exists $C>0$ such that*

$$\|T_\gamma u\|_{L^q} \leq C\|u\|_{L^p}.$$

Now let $\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt, \mathrm{Re}\,(z)>0$ be the Gamma function. The spectral theory with the fact $|\nabla_g| = \sqrt{P}$ gives

$$|\nabla_g|^{-\alpha} = P^{-\alpha/2} = \frac{1}{\Gamma(\alpha/2)}\int_0^\infty e^{-tP}t^{\alpha/2-1}dt.$$

Let $[|\nabla_g|^{-\alpha}](x,y)$ be the kernel of $|\nabla_g|^{-\alpha}$. By (4.2.8),

$$|[|\nabla_g|^{-\alpha}](x,y)| \leq \frac{C}{\Gamma(\alpha/2)}\int_0^\infty t^{-d/2}e^{-\frac{|x-y|^2}{Ct}}t^{\alpha/2-1}dt.$$

A change of variable shows

$$|[|\nabla_g|^{-\alpha}](x,y)| \leq \frac{C}{\Gamma(\alpha/2)}|x-y|^{-(d-\alpha)}\int_0^\infty t^{d/2-\alpha/2-1}e^{-t}dt = \frac{C\Gamma(d/2-\alpha/2)}{\Gamma(\alpha/2)}|x-y|^{-(d-\alpha)}.$$

The result follows by applying Theorem 4.2.5 with $\gamma = d-\alpha$ and $p=2$. □

Another application of the heat kernel upper bound (4.2.8) is the following $L^q - L^r$-bound of the heat operator.

Lemma 4.2.6. *Let $1 \leq q \leq r \leq \infty$. The heat operator $e^{-tP}, t>0$ maps L^q to L^r. In particular, there exists $C>0$ such that for all $t>0$,*

$$\|e^{-tP}\|_{L^q\to L^r} \leq Ct^{-\frac{d}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}.$$

Proof. By the symmetric and maximal principle properties of the heat kernel, the Schur's Test yields

$$\|e^{-tP}\|_{L^q\to L^q} \leq C, \quad t>0. \tag{4.2.11}$$

Interpolating between (4.2.9) and (4.2.11), we have the result. □

Corollary 4.2.7. *Let $f\in C_0^\infty(\mathbb{R}\backslash\{0\})$ and $q\in[2,\infty]$. Then there exists $C>0$ such that for all $\epsilon\in(0,1]$,*

$$\|f(\epsilon^{-2}P)\|_{L^2\to L^q} \leq C\epsilon^{d/2-d/q}.$$

Proof. By writing

$$f(\epsilon^{-2}P) = e^{-\epsilon^{-2}P}(e^{\epsilon^{-2}P}f(\epsilon^{-2}P)),$$

and using Lemma 4.2.6 with $t=\epsilon^{-2}$, we get

$$\|f(\epsilon^{-2}P)\|_{L^2\to L^q} \leq \|e^{-\epsilon^{-2}P}\|_{L^2\to L^q}\|e^{\epsilon^{-2}P}f(\epsilon^{-2}P)\|_{L^2\to L^2} \leq C\epsilon^{d/2-d/q}.$$

Here, using the compactly supported property of f and spectral theorem, we have $e^{\epsilon^{-2}P}f(\epsilon^{-2}P)$ is of size $O_{L^2\to L^2}(1)$. This gives the result. □

We now are able to prove Theorem 4.2.1. We only give the proof for the low frequency case. The high frequency one is essentially given in [BM16, Theorem 4.6].

Proof of Theorem 4.2.1. By the second term of (4.2.1), we have

$$\|f_0(P)v\|_{L^q} = \sup_{\|w\|_{L^{q'}}=1}|(w,f_0(P)v)| = \sup_{\|w\|_{L^{q'}}=1}\Big|\lim_{M\to\infty}\sum_{k=0}^{M}(w,f(\epsilon^{-2}P)v)\Big|, \tag{4.2.12}$$

where $\epsilon^{-2}=2^k$ and $(\cdot,\cdot)$ is the inner product on L^2. By choosing $\tilde f\in C_0^\infty(\mathbb{R}\backslash\{0\})$ satisfying $\tilde f=1$ near $\mathrm{supp}(f)$, we use Proposition 4.1.7 to write $(1-\chi)(\epsilon x)\tilde f(\epsilon^{-2}P)=Q(\epsilon)+R(\epsilon)$, where

$$Q(\epsilon)=(1-\chi)(\epsilon x)Op_\epsilon(\tilde f\circ p_\epsilon)\zeta(\epsilon x), \quad R(\epsilon)=\zeta(\epsilon x)(\epsilon^{-2}P+1)^{-m}B(\epsilon)\langle\epsilon x\rangle^{-1},$$

with $\zeta\in C^\infty(\mathbb{R}^d)$ supported outside $B(0,1)$ and equal to 1 near $\mathrm{supp}(1-\chi)$ and $B(\epsilon)=O_{L^2\to L^2}(1)$ uniformly in $\epsilon\in(0,1]$. We next write

$$f(\epsilon^{-2}P)=Q(\epsilon)(1-\chi)(\epsilon x)f(\epsilon^{-2}P)+A(\epsilon)\epsilon^\alpha\langle\epsilon x\rangle^{-1}f(\epsilon^{-2}P),$$

with $\alpha=d/2-d/q$ and

$$A(\epsilon)=\epsilon^{-\alpha}\Big((1-\chi)(\epsilon x)\tilde f(\epsilon^{-2}P)\chi(\epsilon x)+R(\epsilon)(1-\chi)(\epsilon x)+\chi(\epsilon x)\tilde f(\epsilon^{-2}P)\Big)\langle\epsilon x\rangle.$$

We now bound

$$\begin{aligned}\Big|\sum_{k=0}^{M}(w,f(\epsilon^{-2}P)v)\Big| &\lesssim \Big|\sum_{k=0}^{M}\big(w,Q(\epsilon)(1-\chi)(\epsilon x)f(\epsilon^{-2}P)v\big)\Big| + \Big|\sum_{k=0}^{M}(w,A(\epsilon)\epsilon^\alpha\langle\epsilon x\rangle^{-1}f(\epsilon^{-2}P)v)\Big| \\ &\lesssim \Big|\sum_{k=0}^{M}\big(Q^\star(\epsilon)w,(1-\chi)(\epsilon x)f(\epsilon^{-2}P)v\big)\Big| + \|w\|_{L^{q'}}\Big\|\sum_{k=0}^{M}A(\epsilon)\epsilon^\alpha\langle\epsilon x\rangle^{-1}f(\epsilon^{-2}P)v\Big\|_{L^q} \\ &=: \mathrm{(I)}+\mathrm{(II)}.\end{aligned} \tag{4.2.13}$$

We use the Cauchy-Schwarz inequality in k and the Hölder inequality in space to have

$$\mathrm{(I)}\lesssim\|\widetilde S_M w\|_{L^{q'}}\|S_M v\|_{L^q},$$

where

$$\widetilde{S}_M w := \Big(\sum_{k=0}^M |Q^\star(\epsilon) w|^2\Big)^{1/2}, \quad S_M v := \Big(\sum_{k=0}^M |(1-\chi)(\epsilon x) f(\epsilon^{-2}P) v|^2\Big)^{1/2}.$$

We now make use of the following estimate (see [BM16, Proposition 4.3]).

Proposition 4.2.8. *For $r \in (1,2]$, there exists $C>0$ such that for all $M \geq 0$ and all $w \in \mathscr{S}(\mathbb{R}^d)$,*

$$\|\widetilde{S}_M w\|_{L^r} \leq C \|w\|_{L^r}.$$

We thus get

$$\text{(I)} \lesssim \|S_M v\|_{L^q} \|w\|_{L^{q'}} \lesssim \Big(\sum_{k=0}^M \|(1-\chi)(\epsilon x) f(\epsilon^{-2}P) v\|^2_{L^q}\Big)^{1/2} \|w\|_{L^{q'}}. \tag{4.2.14}$$

For the second term in (4.2.13), we use the homogeneous Sobolev embedding (4.2.10) to have

$$\Big\| \sum_{k=0}^M A(\epsilon)\epsilon^\alpha \langle \epsilon x\rangle^{-1} f(\epsilon^{-2}P) v\Big\|_{L^q} \lesssim \Big\| \sum_{k=0}^M |\nabla_g|^\alpha A(\epsilon) \epsilon^\alpha \langle \epsilon x\rangle^{-1} f(\epsilon^{-2}P) v \Big\|_{L^2}.$$

We next write

$$|\nabla_g|^\alpha A(\epsilon) = (\epsilon^{-2}P)^{\alpha/2} (\epsilon^{-2}P+1)^{-\alpha} D(\epsilon), \tag{4.2.15}$$

with $D(\epsilon) = O_{L^2\to L^2}(1)$ uniformly in $\epsilon \in (0,1]$. It is easy to have (4.2.15) from the first two terms in $A(\epsilon)$ by using Proposition 4.1.7. The less obvious contribution in (4.2.15) is the uniform L^2 boundedness of $(\epsilon^{-2}P+1)^\alpha \chi(\epsilon x) \tilde{f}(\epsilon^{-2}P)\langle \epsilon x\rangle$. By the functional calculus, it is enough to show for N large enough the uniform L^2 boundedness of $(\epsilon^{-2}P+1)^N \chi(\epsilon x)\tilde{f}(\epsilon^{-2}P)\langle \epsilon x\rangle$. To see it, we write

$$(\epsilon^{-2}P+1)^N \chi(\epsilon x)\tilde{f}(\epsilon^{-2}P)\langle \epsilon x\rangle = \chi(\epsilon x)(\epsilon^{-2}P+1)^N \tilde{f}(\epsilon^{-2}P)\langle \epsilon x\rangle + [(\epsilon^{-2}P+1)^N, \chi(\epsilon x)]\tilde{f}(\epsilon^{-2}P)\langle \epsilon x\rangle,$$

where $[\cdot,\cdot]$ is the commutator. The L^2 boundedness of $\chi(\epsilon x)(\epsilon^{-2}P+1)^N \tilde{f}(\epsilon^{-2}P)\langle \epsilon x\rangle$ follows as in (4.1.13). On the other hand, note that the commutator $[(\epsilon^{-2}P+1)^N, \chi(\epsilon x)]$ can be written as a sum of rescaled pseudo-differential operators vanishing outside the support of $\zeta(\epsilon x)$ for some $\zeta \in C^\infty(\mathbb{R}^d)$ supported outside $B(0,1)$ and equal to 1 near infinity. This allows to use Proposition 4.1.7, and the L^2 boundedness of $[(\epsilon^{-2}P+1)^N, \chi(\epsilon x)]\tilde{f}(\epsilon^{-2}P)\langle \epsilon x\rangle$ follows. We next need to recall the following well-known discrete Schur estimate.

Lemma 4.2.9. *Let $\theta > 0$ and $(T_l)_l$ be a sequence of linear operators on a Hilbert space $\mathcal{H}$. If $\|T_l^\star T_k\|_{\mathcal{H}\to\mathcal{H}} \lesssim 2^{-\theta|k-l|}$, then there exits $C>0$ such that for all sequence $(v_k)_k$ of $\mathcal{H}$,*

$$\|\sum T_k v_k\|_{\mathcal{H}} \leq C\Big(\sum \|v_k\|^2_{\mathcal{H}}\Big)^{1/2}.$$

Now let $T_k = (\epsilon_k^{-2}P)^{\alpha/2}(\epsilon_k^{-2}P+1)^{-\alpha}D(\epsilon_k)$ with $\epsilon_k^{-2} = 2^k$. We see that

$$T_l^\star T_k = 2^{\frac{\alpha(l+k)}{2}} D^\star(\epsilon_l)(2^l P+1)^{-\alpha} P^\alpha (2^k P+1)^{-\alpha} D(\epsilon_k).$$

Note that $l+k = -|k-l|+2k$ for $k \geq l$ and $l+k = -|k-l|+2l$ for $l \geq k$. Thus for $k \geq l$,

$$\|T_l^\star T_k\|_{L^2\to L^2} = 2^{-\frac{\alpha|k-l|}{2}} \|D^\star(\epsilon_l)(2^lP+1)^{-\alpha}(2^kP)^\alpha(2^kP+1)^{-\alpha}D(\epsilon_k)\|_{L^2\to L^2} \lesssim 2^{-\frac{\alpha|k-l|}{2}}.$$

Similarly for $l \geq k$. Therefore, we can apply Lemma 4.2.9 for $T_k = (\epsilon_k^{-2}P)^{\alpha/2}(\epsilon_k^{-2}P+1)^{-\alpha}D(\epsilon_k)$

with $\epsilon_k^{-2} = 2^k, \mathcal{H} = L^2$ and $\theta = \alpha/2$ to get

$$\sup_M \Big\| \sum_{k=0}^{M} |\nabla_g|^\alpha A(\epsilon)\epsilon^\alpha \langle \epsilon x\rangle^{-1} f(\epsilon^{-2}P)v \Big\|_{L^2} \lesssim \Big(\sum_{k\geq 0} \|\epsilon^\alpha \langle \epsilon x\rangle^{-1} f(\epsilon^{-2}P)v\|^2_{L^2}\Big)^{1/2}. \tag{4.2.16}$$

Collecting (4.2.12), (4.2.13), (4.2.14) and (4.2.16), we prove (4.2.3). The proof of Theorem 4.2.1 is now complete. ☐

4.2.2 Reduction of the high frequency problem

Let us now consider the high frequency case. For a given $\chi \in C_0^\infty(\mathbb{R}^d)$, we write $u_{\text{high}} = \chi u_{\text{high}} + (1-\chi)u_{\text{high}}$. Using (4.2.2) and Minkowski inequality with $p, q \geq 2$, we have

$$\begin{aligned}\|(1-\chi)u_{\text{high}}\|_{L^p(\mathbb{R},L^q)} \leq C\Big(\sum_{h^2=2^{-k}} &\|(1-\chi)f(h^2P)e^{-it|\nabla_g|^\sigma}\psi\|^2_{L^p(\mathbb{R},L^q)} \\ &+ h^N\| \langle x\rangle^{-N} f(h^2P)e^{-it|\nabla_g|^\sigma}\psi\|^2_{L^p(\mathbb{R},L^2)}\Big)^{1/2}.\end{aligned} \tag{4.2.17}$$

The same estimate holds for $\|\chi u_{\text{high}}\|_{L^p(\mathbb{R},L^q)}$ with χ in place of $1-\chi$. We can apply the Item 2 of Remark 4.1.13 with scaling in time for the second term in the right hand side of the above quantity to get

$$h^{N/2}\| \langle x\rangle^{-N} f(h^2P)e^{-it|\nabla_g|^\sigma}\psi\|_{L^p(\mathbb{R},L^2)} \leq Ch^{N/2+(\sigma-N_0)/p}\|f(h^2P)\psi\|_{L^2}. \tag{4.2.18}$$

By taking N large enough, this term is bounded by $h^{-\gamma_{p,q}}\|f(h^2P)\psi\|_{L^2}$. Thus we have the following reduction.

Proposition 4.2.10. *1. Consider $\mathbb{R}^d, d \geq 2$ equipped with a smooth metric g satisfying* (4.0.2), (4.0.3) *and suppose that the geodesic flow associated to g is non-trapping. If for all $\chi \in C_0^\infty(\mathbb{R}^d)$ and all (p,q) Schrödinger admissible with $q < \infty$, there exists $C > 0$ such that for all $\psi \in \mathscr{L}_g$ and all $h \in (0,1]$,*

$$\|\chi e^{-it|\nabla_g|^\sigma} f(h^2P)\psi\|_{L^p(\mathbb{R},L^q)} \leq Ch^{-\gamma_{p,q}}\|f(h^2P)\psi\|_{L^2}, \tag{4.2.19}$$

then

$$\|\chi u_{\text{high}}\|_{L^p(\mathbb{R},L^q)} \leq C\|\psi\|_{\dot{H}_g^{\gamma_{p,q}}}, \tag{4.2.20}$$

i.e. Theorem 4.0.1 *holds true.*

2. Consider $\mathbb{R}^d, d \geq 2$ equipped with a smooth metric g satisfying (4.0.2), (4.0.3) *and suppose that* (4.0.11) *is satisfied. If there exists $R > 0$ large enough such that for all (p,q) Schrödinger admissible with $q < \infty$ and all $\chi \in C_0^\infty(\mathbb{R}^d)$ satisfying $\chi = 1$ for $|x| < R$, there exists $C > 0$ such that for all $\psi \in \mathscr{L}_g$ and all $h \in (0,1]$,*

$$\|(1-\chi)e^{-it|\nabla_g|^\sigma} f(h^2P)\psi\|_{L^p(\mathbb{R},L^q)} \leq Ch^{-\gamma_{p,q}}\|f(h^2P)\psi\|_{L^2}, \tag{4.2.21}$$

then

$$\|(1-\chi)u_{\text{high}}\|_{L^p(\mathbb{R},L^q)} \leq C\|\psi\|_{\dot{H}_g^{\gamma_{p,q}}}, \tag{4.2.22}$$

i.e. Theorem 4.0.2 *holds true.*

Moreover, combining (4.2.20) *and* (4.2.22)*, we have*

$$\|u_{\text{high}}\|_{L^p(\mathbb{R},L^q)} \leq C\|\psi\|_{\dot{H}_g^{\gamma_{p,q}}}.$$

Proof. We only consider the case $1-\chi$, for χ it is similar. By using (4.2.18) and (4.2.21), we see that (4.2.17) implies

$$\|(1-\chi)u_{\text{high}}\|_{L^p(\mathbb{R},L^q)} \leq C\Big(\sum_{h^2=2^{-k}} h^{-2\gamma_{p,q}}\|f(h^2P)\psi\|^2_{L^2}\Big)^{1/2} \leq C\|\psi\|_{\dot{H}^{\gamma_{p,q}}_g}.$$

Here we use the almost orthogonality and the support property of f to obtain the last inequality. This proves (4.2.22). □

4.2.3 Reduction of the low frequency problem

Let us consider the low frequency case. We only treat the case $q\in(2,\infty)$ since the Strichartz estimate for $(p,q)=(\infty,2)$ is trivial. We apply the Littlewood-Paley estimates (4.2.3) and Minkowski inequality with $p\geq 2$ to have

$$\begin{aligned}\|u_{\text{low}}\|_{L^p(\mathbb{R},L^q)} \leq C\Big(\sum_{\epsilon^{-2}=2^k} &\|(1-\chi)(\epsilon x)f(\epsilon^{-2}P)e^{-it|\nabla_g|^\sigma}\psi\|^2_{L^p(\mathbb{R},L^q)}\\ &+\|\epsilon^{d/2-d/q}\langle\epsilon x\rangle^{-1}f(\epsilon^{-2}P)e^{-it|\nabla_g|^\sigma}\psi\|^2_{L^p(\mathbb{R},L^2)}\Big)^{1/2}.\end{aligned}$$

We use global L^p integrability estimates (4.1.21) with rescaling in time to bound the second term in the right hand side as

$$\|\epsilon^{d/2-d/q}\langle\epsilon x\rangle^{-1}f(\epsilon^{-2}P)e^{-it|\nabla_g|^\sigma}\psi\|_{L^p(\mathbb{R},L^2)} \leq C\epsilon^{\gamma_{p,q}}\|f(\epsilon^{-2}P)\psi\|_{L^2}. \tag{4.2.23}$$

Here we recall that $\gamma_{p,q}=d/2-d/q-\sigma/p$. This leads to the following reduction.

Proposition 4.2.11. *Consider $\mathbb{R}^d, d\geq 3$ equipped with a smooth metric g satisfying (4.0.2), (4.0.3). If for all $\chi\in C_0^\infty(\mathbb{R}^d)$ satisfying $\chi(x)=1$ for $|x|\leq 1$ and all (p,q) Schrödinger admissible with $q<\infty$, there exists $C>0$ such that for all $\psi\in\mathscr{L}_g$ and all $\epsilon\in(0,1]$,*

$$\|(1-\chi)(\epsilon x)f(\epsilon^{-2}P)e^{-it|\nabla_g|^\sigma}\psi\|_{L^p(\mathbb{R},L^q)} \leq C\epsilon^{\gamma_{p,q}}\|f(\epsilon^{-2}P)\psi\|_{L^2}, \tag{4.2.24}$$

then

$$\|u_{\text{low}}\|_{L^p(\mathbb{R},L^q)} \leq C\|\psi\|_{\dot{H}^{\gamma_{p,q}}_g}.$$

Proof. Indeed, if the estimates (4.2.24) hold true, then the Littlewood-Paley estimates (4.2.3) and (4.2.23) give

$$\|u_{\text{low}}\|_{L^p(\mathbb{R},L^q)} \leq C\Big(\sum_{\epsilon^{-2}=2^k}\epsilon^{2\gamma_{p,q}}\|f(\epsilon^{-2}P)\psi\|^2_{L^2}\Big)^{1/2}.$$

Note that

$$\epsilon^{\gamma_{p,q}}\|f(\epsilon^{-2}P)\psi\|_{L^2} \leq \epsilon^{\gamma_{p,q}}\|\tilde{f}(\epsilon^{-2}P)|\nabla_g|^{-\gamma_{p,q}}\|_{L^2\to L^2}\|f(\epsilon^{-2}P)|\nabla_g|^{\gamma_{p,q}}\psi\|_{L^2},$$

where $\tilde{f}\in C_0^\infty(\mathbb{R}\backslash\{0\})$ satisfies $\tilde{f}=1$ near $\operatorname{supp}(f)$. By functional calculus, the first factor in the right hand side is bounded by

$$\epsilon^{\gamma_{p,q}}\sup_{\lambda\in\mathbb{R}}\left|\frac{\tilde{f}(\epsilon^{-2}\lambda^2)}{\lambda^{\gamma_{p,q}}}\right| \leq \epsilon^{\gamma_{p,q}}\frac{\|\tilde{f}\|_{L^\infty(\mathbb{R})}}{(\epsilon/c)^{\gamma_{p,q}}} \leq c^{\gamma_{p,q}}\|\tilde{f}\|_{L^\infty(\mathbb{R})}.$$

Here $\epsilon^{-2}\lambda^2\in\operatorname{supp}(\tilde{f})$ hence $|\lambda|\in[\epsilon/c,\epsilon c]$ for some constant $c>1$. Thus we have

$$\|u_{\text{low}}\|_{L^p(\mathbb{R},L^q)} \leq C\Big(\sum_{\epsilon^{-2}=2^k}\|f(\epsilon^{-2}P)|\nabla_g|^{\gamma_{p,q}}\psi\|^2_{L^2}\Big)^{1/2} \leq C\|\psi\|_{\dot{H}^{\gamma_{p,q}}_g},$$

the last inequality follows from the almost orthogonality. This completes the proof. □

4.3 Strichartz estimates inside compact sets

In this section, we will give the proof of (4.2.19). Our main tools are the local in time Strichartz estimates given in Chapter 2 and the L^2 integrability estimate at high frequency given in Proposition 4.1.12.

4.3.1 The WKB approximations

Let us start with the following result which is given in Theorem 2.2.1.

Theorem 4.3.1. *Let $\sigma \in (0,\infty)\backslash\{1\}$ and q be a smooth function on $\mathbb{R}^{2d}$ compactly support in ξ away from zero and satisfying for all $\alpha, \beta \in \mathbb{N}^d$, there exists $C_{\alpha\beta} > 0$ such that for all $x, \xi \in \mathbb{R}^d$,*

$$|\partial_x^\alpha \partial_\xi^\beta q(x,\xi)| \leq C_{\alpha\beta}.$$

Then there exist $t_0 > 0$ small enough, a function $S \in C^\infty([-t_0,t_0]\times\mathbb{R}^{2d})$ and a sequence of smooth functions $a_j(t,x,\xi)$ compactly supported in ξ away from zero uniformly in $t \in [-t_0,t_0]$ such that for all $N \geq 1$,

$$e^{-ith^{-1}(h|\nabla_g|)^\sigma} Op^h(q)\psi = J_N(t)\psi + R_N(t)\psi,$$

where

$$J_N(t)\psi(x) = (2\pi h)^{-d} \iint_{\mathbb{R}^{2d}} e^{ih^{-1}(S(t,x,\xi)-y\cdot\xi)} \sum_{j=0}^{N-1} h^j a_j(t,x,\xi)\psi(y)dyd\xi,$$

$J_N(0) = Op^h(q)$ and the remainder $R_N(t)$ satisfies for all $t \in [-t_0,t_0]$ and all $h \in (0,1]$,

$$\|R_N(t)\|_{L^2\to L^2} \leq Ch^{N-1}.$$

Moreover, there exists a constant $C > 0$ such that for all $t \in [-t_0,t_0]$ and all $h \in (0,1]$,

$$\|J_N(t)\|_{L^1\to L^\infty} \leq Ch^{-d}(1+|t|h^{-1})^{-d/2}.$$

In Chapter 2, we consider the smooth bounded metric, i.e. for all $\alpha \in \mathbb{N}^d$, there exists $C_\alpha > 0$ such that for all $x \in \mathbb{R}^d$,

$$|\partial^\alpha g^{jk}(x)| \leq C_\alpha, \quad j,k \in \{1,...d\}.$$

It is obvious to see that the above condition is always satisfied under the assumption (4.0.3). This theorem and the parametrix given in Proposition 4.1.2 give the following dispersive estimates for the Schrödinger-type equations (see Remark 2.2.2).

Proposition 4.3.2. *Let $\sigma \in (0,\infty)\backslash\{1\}$ and $\varphi \in C_0^\infty(\mathbb{R}\backslash\{0\})$. Then there exists $t_0 > 0$ small enough and $C > 0$ such that for all $\psi \in L^1(\mathbb{R}^d)$ and all $h \in (0,1]$,*

$$\|e^{-ith^{-1}(h|\nabla_g|)^\sigma}\varphi(h^2P)\psi\|_{L^\infty} \leq Ch^{-d}(1+|t|h^{-1})^{-d/2}\|\psi\|_{L^1}, \tag{4.3.1}$$

for all $t \in [-t_0,t_0]$.

Next, we recall the following version of $TT^\star$-criterion of Keel and Tao (see [Zha15], [KT98] or [Zwo12]).

Proposition 4.3.3. *Let $I \subseteq \mathbb{R}$ be an interval and $(T(t))_{t\in I}$ a family of linear operators satisfying for some constant $C > 0$ and $\delta, \tau, h > 0$,*

$$\|T(t)\|_{L^2\to L^2} \leq C, \tag{4.3.2}$$

$$\|T(t)T(s)^\star\|_{L^1\to L^\infty} \leq Ch^{-\delta}(1+|t-s|h^{-1})^{-\tau}, \tag{4.3.3}$$

for all $t, s \in I$. Then for all (p,q) satisfying

$$p \in [2,\infty], \quad q \in [1,\infty], \quad (p,q,\tau) \ne (2,\infty,1), \quad \frac{1}{p} \le \tau\left(\frac{1}{2}-\frac{1}{q}\right),$$

we have

$$\|Tv\|_{L^p(I,L^q)} \le Ch^{-\kappa}\|v\|_{L^2},$$

where $\kappa = \delta(1/2-1/q) - 1/p$.

Proposition 4.3.3 together with energy estimate and dispersive estimate (4.3.1) give the following result.

Corollary 4.3.4. *Let $\sigma \in (0,\infty)\backslash\{1\}, \varphi \in C_0^\infty(\mathbb{R}\backslash\{0\})$ and t_0 be as in* Theorem 4.3.1. *Denote $I = [-t_0, t_0]$. Then for all (p,q) Schrödinger admissible with $q < \infty$, there exists $C > 0$ such that*

$$\|\varphi(h^2P)e^{-ith^{-1}(h|\nabla_g|)^\sigma}v\|_{L^p(I,L^q)} \le Ch^{-\kappa_{p,q}}\|v\|_{L^2}, \tag{4.3.4}$$

where $\kappa_{p,q} = d/2 - d/q - 1/p$. Moreover,

$$\Big\|\int_0^t \varphi^2(h^2P)e^{-i(t-s)h^{-1}(h|\nabla_g|)^\sigma}G(s)ds\Big\|_{L^p(I,L^q)} \le Ch^{-\kappa_{p,q}}\|G\|_{L^1(I,L^2)}. \tag{4.3.5}$$

Proof. The homogeneous estimates (4.3.4) follow directly from Proposition 4.3.2 and Proposition 4.3.3 with $T(t) = \varphi(h^2P)e^{-ith^{-1}(h|\nabla_g|)^\sigma}$. It remains to prove the inhomogeneous estimates (4.3.5). Let us set

$$U_h(t) := h^{\kappa_{p,q}}\varphi(h^2P)e^{-ith^{-1}(h|\nabla_g|)^\sigma}.$$

Using the homogeneous Strichartz estimates (4.3.4), we see that $U_h(t)$ is a bounded operator from L^2 to $L^p(I,L^q)$. Similarly, we have $U_h(s) = \varphi(h^2P)e^{-ish^{-1}(h|\nabla_g|)^\sigma}$ is a bounded operator from L^2 to $L^\infty(I,L^2)$. Here we use the fact that $(\infty,2)$ is Schrödinger-tye admissible with $\kappa_{\infty,2} = 0$. Thus the adjoint $U_h(s)^\star$, namely

$$U_h(s)^\star : G \in L^1(I,L^2) \mapsto \int_I \varphi(h^2P)e^{ish^{-1}(h|\nabla_g|)^\sigma}G(s)ds \in L^2$$

is also a bounded operator. This implies $U_h(t)U_h(s)^\star$ is a bounded operator from $L^1(I,L^2)$ to $L^p(I,L^q)$. In particular, we have

$$\Big\|\int_I h^{\kappa_{p,q}}\varphi^2(h^2P)e^{-i(t-s)h^{-1}(h|\nabla_g|)^\sigma}G(s)ds\Big\|_{L^p(I,L^q)} \le C\|G\|_{L^1(I,L^2)}.$$

The Christ-Kiselev Lemma (see Lemma 4.5.1) implies that for all (p,q) Schrödinger admissible with $q < \infty$,

$$\Big\|\int_0^t \varphi^2(h^2P)e^{-i(t-s)h^{-1}(h|\nabla_g|)^\sigma}G(s)ds\Big\|_{L^p(I,L^q)} \le Ch^{-\kappa_{p,q}}\|G\|_{L^1(I,L^2)}.$$

This completes the proof. □

4.3.2 From local Strichartz estimates to global Strichartz estimates

We now show how to upgrade the local in time Strichartz estimates given in Corollary 4.3.4 to the global in time ones (4.2.19). We emphasize that the non-trapping assumption is supposed here.

Let us set $v(t) = \langle x\rangle^{-1} f(h^2P)e^{-ith^{-1}(h|\nabla_g|)^\sigma}\psi$. By choosing $f_1 \in C_0^\infty(\mathbb{R}\backslash\{0\})$ with $f_1 = 1$ near $\mathrm{supp}(f)$, we see that the study of $\|v\|_{L^p(\mathbb{R},L^q)}$ is reduced to the one of $\|f_1(h^2P)v\|_{L^p(\mathbb{R},L^q)}$. Indeed, we can write

$$v(t) = f_1(h^2P)v(t) + (1-f_1)(h^2P)v(t),$$

where the term $(1-f_1)(h^2P)v(t)$ can be written as

$$((1-f_1)(h^2P)\langle x\rangle^{-1}\tilde{f}_1(h^2P)\langle x\rangle)\langle x\rangle^{-1}f(h^2P)e^{-ith^{-1}(h|\nabla_g|)^\sigma}\psi,$$

with $\tilde{f}_1\in C_0^\infty(\mathbb{R}\backslash\{0\})$ such that $f_1=1$ near $\mathrm{supp}(\tilde{f}_1)$ and $\tilde{f}_1=1$ near $\mathrm{supp}(f)$. By pseudo-differential calculus, we have

$$(1-f_1)(h^2P)\langle x\rangle^{-1}\tilde{f}_1(h^2P)\langle x\rangle=O_{L^2\to L^q}(h^\infty),$$

for all $q\geq 2$. This implies that there exists $C>0$ such that for all $N\geq 1$,

$$\begin{aligned}\|v-f_1(h^2P)v\|_{L^p(\mathbb{R},L^q)}&\leq Ch^N\|\langle x\rangle^{-1}f(h^2P)e^{-ith^{-1}(h|\nabla_g|)^\sigma}\psi\|_{L^p(\mathbb{R},L^2)}\\&\leq Ch^N\|f(h^2P)\psi\|_{L^2}\leq Ch^{-\kappa_{p,q}}\|f(h^2P)\psi\|_{L^2}\end{aligned}\tag{4.3.6}$$

provided that N is taken large enough. Here we use (4.1.20) with $N_0=1$ due to the non-trapping condition.

We next write

$$v(t)=\langle x\rangle^{-1}f(h^2P)e^{-ith^{-1}\omega(h^2P)}\psi,$$

where $\omega(\lambda)=\tilde{f}(\lambda)\sqrt{\lambda}^\sigma$ with $\tilde{f}\in C_0^\infty(\mathbb{R}\backslash\{0\})$ and $\tilde{f}=1$ near $\mathrm{supp}(f)$. Now, let $t_0>0$ be as in Corollary 4.3.4. We next choose $\theta\in C_0^\infty(\mathbb{R},[0,1])$ satisfying $\theta=1$ near 0 and $\mathrm{supp}(\theta)\subset(-1,1)$ such that $\sum_{k\in\mathbb{Z}}\theta(t-k)=1$, for all $t\in\mathbb{R}$. We then write $v(t)=\sum_{k\in\mathbb{Z}}v_k(t)$, where $v_k(t)=\theta((t-t_k)/t_0)v(t)$ with $t_k=t_0k$. By the Duhamel formula, we have

$$v_k(t)=e^{-ith^{-1}\omega(h^2P)}v_k(0)+ih^{-1}\int_0^t e^{-i(t-s)h^{-1}\omega(h^2P)}(hD_s+\omega(h^2P))v_k(s)ds.$$

For $k\neq 0$, we compute the action of $hD_s+\omega(h^2P)$ on $v_k(s)$ and get

$$\begin{aligned}(hD_s+\omega(h^2P))v_k(s)&=h(it_0)^{-1}\theta'((s-t_k)/t_0)v(s)\\&\quad+\theta((s-t_k)/t_0)\left[\omega(h^2P),\langle x\rangle^{-1}\right]f(h^2P)e^{-ish^{-1}\omega(h^2P)}\psi=:v_k^1(s)+v_k^2(s).\end{aligned}$$

Due to the support property of θ, we have $v_k(0)=0$. Now, we have for $k\neq 0$,

$$f_1(h^2P)v_k(t)=ih^{-1}\int_0^t e^{-i(t-s)h^{-1}\omega(h^2P)}f_1(h^2P)(v_k^1(s)+v_k^2(s))ds.$$

We remark that both t,s belong to $I_k=(t_k-t_0,t_k+t_0)$. Up to a translation in time $t\mapsto t-t_k$ and the same for s, we can apply the inhomogeneous Strichartz estimates given in Corollary 4.3.4 with $\varphi^2=f_1$ and obtain

$$\begin{aligned}\|f_1(h^2P)v_k\|_{L^p(\mathbb{R},L^q)}&=\|f_1(h^2P)v_k\|_{L^p(I_k,L^q)}\\&\leq Ch^{-\kappa_{p,q}-1}\left(\|v_k^1\|_{L^1(I_k,L^2)}+\|v_k^2\|_{L^1(I_k,L^2)}\right).\end{aligned}$$

Here $\kappa_{p,q}$ is given in Corollary 4.3.4. We have

$$\begin{aligned}\|v_k^1\|_{L^1(I_k,L^2)}&=\|h(it_0)^{-1}\theta'((s-t_k)/t_0)\langle x\rangle^{-1}f(h^2P)e^{-ish^{-1}(h|\nabla_g|)^\sigma}\psi\|_{L^1(I_k,L^2)}\\&\leq\|h(it_0)^{-1}\theta'((s-t_k)/t_0)\|_{L^2(I_k)}\|\langle x\rangle^{-1}f(h^2P)e^{-ish^{-1}(h|\nabla_g|)^\sigma}\psi\|_{L^2(I_k,L^2)}\\&\leq Ch\|\langle x\rangle^{-1}f(h^2P)e^{-ish^{-1}(h|\nabla_g|)^\sigma}\psi\|_{L^2(I_k,L^2)},\end{aligned}$$

where we use Cauchy Schwarz inequality to go from the first to the second line. Similarly

$$\begin{aligned}\|v_k^2\|_{L^1(I_k,L^2)} &\leq \|[\omega(h^2P), \langle x\rangle^{-1}]f(h^2P)e^{-ith^{-1}(h|\nabla_g|)^\sigma}\psi\|_{L^2(I_k,L^2)}\\ &\leq Ch\| \langle x\rangle^{-1} f(h^2P)e^{-ith^{-1}(h|\nabla_g|)^\sigma}\psi\|_{L^2(I_k,L^2)},\end{aligned}$$

where we use the fact that $[\omega(h^2P), \langle x\rangle^{-1}]\tilde{f}_1(h^2P)\langle x\rangle$ is of size $O_{L^2\to L^2}(h)$ by pseudo-differential calculus. This implies that for $k \neq 0$,

$$\|f_1(h^2P)v_k\|_{L^p(I_k,L^q)} \leq Ch^{-\kappa_{p,q}}\| \langle x\rangle^{-1} f(h^2P)e^{-ith^{-1}(h|\nabla_g|)^\sigma}\psi\|_{L^2(I_k,L^2)}.$$

For $k = 0$, we have

$$\|f_1(h^2P)v_0\|_{L^p(\mathbb{R},L^q)} \leq C\|f(h^2P)e^{-ith^{-1}(h|\nabla_g|)^\sigma}\psi\|_{L^p(I,L^q)} \leq Ch^{-\kappa_{p,q}}\|f(h^2P)\psi\|_{L^2}.$$

Here the first inequality follows from the facts that $\theta(t/t_0)$ and $f_1(h^2P)\langle x\rangle^{-1}$ are bounded from$L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ and $L^q \to L^q$ respectively. The second inequality follows from homogeneous Strichartz estimates (4.3.4). By almost orthogonality in time and the fact that $p \geq 2$, we have

$$\begin{aligned}\|f_1(h^2P)v\|_{L^p(\mathbb{R},L^q)} &\leq C\Big(\sum_{k\in\mathbb{Z}}\|f_1(h^2P)v_k\|^2_{L^p(\mathbb{R},L^q)}\Big)^{1/2}\\ &\leq Ch^{-\kappa_{p,q}}\Big(\sum_{k\in\mathbb{Z}\backslash 0}\| \langle x\rangle^{-1} f(h^2P)e^{-ith^{-1}(h|\nabla_g|)^\sigma}\psi\|^2_{L^2(I_k,L^2)} + \|f(h^2P)\psi\|^2_{L^2}\Big)^{1/2}\\ &\leq Ch^{-\kappa_{p,q}}\Big(\| \langle x\rangle^{-1} f(h^2P)e^{-ith^{-1}(h|\nabla_g|)^\sigma}\psi\|_{L^2(\mathbb{R},L^2)} + \|f(h^2P)\psi\|_{L^2}\Big)\\ &\leq Ch^{-\kappa_{p,q}}\|f(h^2P)\psi\|_{L^2},\end{aligned}$$

the last inequality comes from Proposition 4.1.12 with $N_0 = 1$. By using (4.3.6), we obtain

$$\| \langle x\rangle^{-1} f(h^2P)e^{-ith^{-1}(h|\nabla_g|)^\sigma}\psi\|_{L^p(\mathbb{R},L^q)} \leq Ch^{-\kappa_{p,q}}\|f(h^2P)\psi\|_{L^2}.$$

This implies that for all $\chi \in C_0^\infty(\mathbb{R}^d)$,

$$\Big\|\chi f(h^2P)e^{-ith^{-1}(h|\nabla_g|)^\sigma}\psi\Big\|_{L^p(\mathbb{R},L^q)} \leq Ch^{-\kappa_{p,q}}\|f(h^2P)\psi\|_{L^2}.$$

Therefore, by scaling in time, we get

$$\Big\|\chi f(h^2P)e^{-it|\nabla_g|^\sigma}\psi\Big\|_{L^p(\mathbb{R},L^q)} \leq Ch^{-\gamma_{p,q}}\|f(h^2P)\psi\|_{L^2}.$$

The proof of (4.2.19) is now complete. □

4.4 Strichartz estimates outside compact sets

4.4.1 The Isozaki-Kitada parametrix

Notations and the Hamilton-Jacobi equations. For any $J \Subset (0, +\infty)$ an open interval, any $R > 0$, any $\tau \in (-1, 1)$, we define the outgoing region $\Gamma^+(R, J, \tau)$ and the incoming region $\Gamma^-(R, J, \tau)$ by

$$\Gamma^\pm(R, J, \tau) := \Big\{(x, \xi) \in \mathbb{R}^{2d}, |x| > R, |\xi|^2 \in J, \pm\frac{x\cdot\xi}{|x||\xi|} > \tau\Big\}.$$

Let $\sigma \in (0,\infty)$[1]. We will use the so called Isozaki-Kitada parametrix to give an approximation at **high frequency** of the form

$$e^{-ith^{-1}\omega(h^2P)}Op^h(\chi^\pm) = J_h^\pm(a^\pm(h))e^{-ith^{-1}(h\Lambda)^\sigma}J_h^\pm(b^\pm(h))^\star + R_N^\pm(h), \tag{4.4.1}$$

with $\Lambda = \sqrt{-\Delta}$ where Δ is the free Laplacian operator on $\mathbb{R}^d$ and $\omega(\cdot) = \tilde{f}(\cdot)\sqrt{\cdot}^\sigma \in C_0^\infty(\mathbb{R}\backslash\{0\})$ for some $\tilde{f} \in C_0^\infty(\mathbb{R}\backslash\{0\})$ satisfying $\tilde{f} = 1$ near $\text{supp}(f)$. The functions $\chi^\pm$ are supported in $\Gamma^\pm(R^4, J_4, \tau_4)$ (see Proposition 4.4.6 for the choice of J_4 and τ_4) and

$$J_h^\pm(a^\pm(h)) = \sum_{j=1}^{N-1} h^j J_h^\pm(a_j^\pm),$$

where

$$J_h^\pm(a^\pm)u(x) = (2\pi h)^{-d}\iint_{\mathbb{R}^{2d}} e^{ih^{-1}\left(S_R^\pm(x,\xi)-y\cdot\xi\right)}a^\pm(x,\xi)u(y)dyd\xi, \quad u \in \mathscr{S}(\mathbb{R}^d).$$

The amplitude functions $a_j^\pm$ are supported in $\Gamma^\pm(R, J_1, \tau_1)$ (see Proposition 4.4.1) and the phase functions $S_R^\pm := S_{1,R}^\pm$ will be described later. The same notation for $J_h^\pm(b^\pm(h))$ is used with $b_k^\pm$ in place of $a_j^\pm$.

The Isozaki-Kitada parametrix at **low frequency** is of the form

$$e^{-it\epsilon\omega(\epsilon^{-2}P)}Op_\epsilon(\chi_\epsilon^\pm)\zeta(\epsilon x) = \mathcal{J}_\epsilon^\pm(a_\epsilon^\pm)e^{-it\epsilon\Lambda^\sigma}\mathcal{J}_\epsilon^\pm(b_\epsilon^\pm)^\star + \mathcal{R}_N^\pm(t,\epsilon), \tag{4.4.2}$$

where ω is as above and $\zeta \in C^\infty(\mathbb{R}^d)$ supported outside $B(0,1)$ and equal to 1 near infinity. The functions $\chi_\epsilon^\pm$ are supported in $\Gamma^\pm(R^4, J_4, \tau_4)$ and

$$\mathcal{J}_\epsilon^\pm(a_\epsilon^\pm) = \sum_{j=1}^{N} \mathcal{J}_\epsilon^\pm(a_{\epsilon,j}^\pm),$$

where

$$\mathcal{J}_\epsilon^\pm(a) := D_\epsilon J_\epsilon^\pm(a), \quad \mathcal{J}_\epsilon^\pm(b)^\star := J_\epsilon^\pm(b)^\star D_\epsilon^{-1}, \tag{4.4.3}$$

with D_ϵ as in (4.1.4),

$$J_\epsilon^\pm(a)u(x) := (2\pi)^{-d}\iint_{\mathbb{R}^{2d}} e^{i(S_{\epsilon,R}^\pm(x,\xi)-y\cdot\xi)}a(x,\xi)u(y)dyd\xi,$$

and

$$J_\epsilon^\pm(b)^\star u(x) = (2\pi)^{-d}\iint_{\mathbb{R}^{2d}} e^{i(x\cdot\xi - S_{\epsilon,R}^\pm(y,\xi))}\overline{b(y,\xi)}u(y)dyd\xi.$$

The amplitude functions $a_{\epsilon,j}^\pm$ are supported in $\Gamma^\pm(R, J_1, \tau_1)$ and the phase functions $S_{\epsilon,R}^\pm$ will be described in the next proposition. The same notation for $\mathcal{J}_\epsilon^\pm(b_\epsilon^\pm)$ will be used with $b_\epsilon^\pm$ in place of $a_\epsilon^\pm$.

Proposition 4.4.1. *Fix $J_1 \Subset (0,+\infty)$ and $\tau_1 \in (-1,1)$. Then there exist two families of smooth functions $(S_{\epsilon,R}^\pm)_{R\gg 1}$ satisfying the following Hamilton-Jacobi equation*

$$p_\epsilon(x, \nabla_x S_{\epsilon,R}^\pm(x,\xi)) = |\xi|^2, \tag{4.4.4}$$

for all $(x,\xi) \in \Gamma^\pm(R, J_1, \tau_1)$, where p_ϵ is given in (4.1.5). Moreover, for all $\alpha, \beta \in \mathbb{N}^d$, there exists

[1] The construction of the Isozaki-Kitada parametrix we present here works well for the half-wave equation, i.e. $\sigma = 1$.

$C_{\alpha\beta} > 0$ *such that*

$$\left|\partial_x^\alpha \partial_\xi^\beta \Big(S^\pm_{\epsilon,R}(x,\xi) - x\cdot\xi \Big)\right| \leq C_{\alpha\beta} \min \Big\{ R^{1-\rho-|\alpha|}, \langle x\rangle^{1-\rho-|\alpha|} \Big\}, \tag{4.4.5}$$

for all $x,\xi \in \mathbb{R}^d$*, all* $\epsilon \in (0,1]$ *and* $R \gg 1$.

Remark 4.4.2. From (4.4.5), we see that for $R>0$ large enough, the phase functions satisfy for all $x,\xi\in\mathbb{R}^d$ and all $\epsilon \in (0,1]$,

$$\left\| \nabla_x \cdot \nabla_\xi S^\pm_{\epsilon,R}(x,\xi) - \mathrm{Id}_{\mathbb{R}^d} \right\| \leq \frac{1}{2}, \tag{4.4.6}$$

and for all $|\alpha|\geq 1$ and all $|\beta| \geq 1$,

$$|\partial_x^\alpha\partial_\xi^\beta S^\pm_{\epsilon,R}(x,\xi)| \leq C_{\alpha\beta}. \tag{4.4.7}$$

The estimates (4.4.6) and (4.4.7) are useful in the construction of Isozaki-Kitada parametrix as well as the L^2-boundedness of Fourier integral operators.

Proof of Proposition 4.4.1. We firstly note that the case $\epsilon = 1$ is given in [BT07, Proposition 3.1]. Let $J_1 \Subset J_0 \Subset (0,+\infty)$ and $-1 < \tau_0 < \tau_1 < 1$. By using Remark 4.1.4, in the region $\Gamma^\pm(R/2, J_0, \tau_0)$ which implies that $|x|>1$, we see that the function $p_\epsilon(x,\xi)$ satisfies for all $\alpha,\beta \in \mathbb{N}^d$, there exists $C_{\alpha\beta}>0$ such that for all $(x,\xi)\in \Gamma^\pm(R/2,J_0,\tau_0)$ and all $\epsilon\in(0,1]$,

$$|\partial_x^\alpha \partial_\xi^\beta p_\epsilon(x,\xi)| \leq C_{\alpha\beta}\langle \xi\rangle^{2-|\beta|}.$$

Thanks to this uniform bound, by using the argument given in [Rob94, Proposition 4.1], we can solve (for $R>0$ large enough) the Hamilton-Jacobi equation (4.4.4) in $\Gamma^\pm(R/2,J_0,\tau_0)$ uniformly with respect to $\epsilon\in(0,1]$. We denote such solutions by $\tilde{S}^\pm_\epsilon$. Next, by choosing a special cutoff (see [BT07], see also (4.4.9)) $\chi^\pm_R \in S(0,-\infty)$ such that $\chi^\pm_R(x,\xi) = 1$ for $(x,\xi)\in\Gamma^\pm(R,J_1,\tau_1)$ and $\mathrm{supp}(\chi^\pm_R) \subset \Gamma^\pm(R/2,J_0,\tau_0)$, then the functions

$$S^\pm_{\epsilon,R}(x,\xi) = \chi^\pm_R(x,\xi)\tilde{S}^\pm_\epsilon(x,\xi) + (1-\chi^\pm_R)(x,\xi)\langle x,\xi\rangle$$

satisfy the properties of Proposition 4.4.1, where $\langle x,\xi\rangle = x\cdot\xi$. □

Construction of the parametrix. Let us firstly consider the high frequency case (4.4.1). The construction in the low frequeny case (4.4.2) is similar up to some modifications (see after Theorem 4.4.8). We only treat the outgoing case (+), the incoming one is similar. We start with the following Duhamel formula

$$\begin{aligned} e^{-ith^{-1}\omega(h^2P)} J_h^+(a^+(h)) &= J_h^+(a^+(h)) e^{-ith^{-1}(h\Lambda)^\sigma} \\ &- ih^{-1}\int_0^t e^{-i(t-s)h^{-1}\omega(h^2P)}\Big(\omega(h^2P)J_h^+(a^+(h)) - J_h^+(a^+(h))(h\Lambda)^\sigma\Big) e^{-ish^{-1}(h\Lambda)^\sigma} ds. \end{aligned} \tag{4.4.8}$$

We want the term $\omega(h^2P)J_h^+(a^+(h)) - J_h^+(a^+(h))(h\Lambda)^\sigma$ to have a small contribution. To do so, we firstly introduce a special cutoff. For any $J_2 \Subset J_1 \Subset (0,+\infty)$ and $-1<\tau_1<\tau_2<1$, we define

$$\chi^+_{1\to 2}(x,\xi) = \kappa\left(\frac{|x|}{R^2}\right)\rho_{1\to 2}(|\xi|^2)\theta_{1\to 2}\left(+\frac{x\cdot\xi}{|x||\xi|}\right), \tag{4.4.9}$$

where $\kappa \in C^\infty(\mathbb{R})$ is non-decreasing such that

$$\kappa(t) = \begin{cases} 1 & \text{when } t \geq 1/2 \\ 0 & \text{when } t \leq 1/4 \end{cases},$$

and $\rho_{1\to 2} \in C^\infty(\mathbb{R})$ is non-decreasing such that $\rho_{1\to 2} = 1$ near J_2, supported in J_1 and $\theta_{1\to 2} \in C_0^\infty(\mathbb{R})$ such that

$$\theta_{1\to 2}(t) = \begin{cases} 1 & \text{when } t > \tau_2 - \varepsilon \\ 0 & \text{when } t < \tau_1 + \varepsilon \end{cases},$$

with $\varepsilon \in (0, \tau_2 - \tau_1)$. We see that $\chi^+_{1\to 2} \in S(0, -\infty)$ and for $R \gg 1$,

$$\text{supp}(\chi^+_{1\to 2}) \subset \Gamma^+(R, J_1, \tau_1), \quad \chi^+_{1\to 2} = 1 \text{ near } \Gamma^+(R^2, J_2, \tau_2).$$

Proposition 4.4.3. *Let $S^+_R := S^+_{1,R}$ be the solution of* (4.4.4) *given as in* Proposition 4.4.1. *Let J_2 be an arbitrary open interval such that $J_2 \Subset J_1 \Subset (0, +\infty)$ and τ_2 be an arbitrary real number such that $-1 < \tau_1 < \tau_2 < 1$. Then for $R > 0$ large enough, we can find a sequence of symbols $a^+_j \in S(-j, -\infty)$ supported in $\Gamma^+(R, J_1, \tau_1)$ such that for all $N \geq 1$,*

$$\omega(h^2 P) J^+_h(a^+(h)) - J^+_h(a^+(h))(h\Lambda)^\sigma = h^N R_N(h) J^+_h(a^+(h)) + h^N J^+_h(r^+_N(h)) + J^+_h(\check{a}^+(h)), \tag{4.4.10}$$

$$\sup_{\Gamma^+(R, J_1, \tau_1)} |a^+_0(x, \xi)| \gtrsim 1, \tag{4.4.11}$$

where $a^+(h) = \sum_{j=0}^{N-1} h^j a^+_j$ and $(r^+_N(h))_{h\in(0,1]}$ is bounded in $S(-N, -\infty)$, $R_N(h)$ is as in Proposition 4.1.2, *$(\check{a}^+(h))_{h\in(0,1]}$ is bounded in $S(0, -\infty)$ and is a finite sum depending on N of the form*

$$\check{a}^+(h) = \sum_{|\alpha|\geq 1} \check{a}^+_\alpha(h) \partial^\alpha_x \chi^+_{1\to 2}, \tag{4.4.12}$$

with $(\check{a}^+_\alpha(h))_{h\in(0,1]}$ bounded in $S(0, -\infty)$ and $\chi^+_{1\to 2}$ given in (4.4.9).

Proof. We firstly use the parametrix of $\omega(h^2 P)$ given in Proposition 4.1.2 and get

$$\omega(h^2 P) = Op^h(q(h)) + h^N R_N(h), \tag{4.4.13}$$

where $q(h) = \sum_{k=0}^{N-1} h^k q_k$ and $q_k \in S(-k, -\infty), k = 0, ..., N-1$. Note that $q_0(x, \xi) = \omega(p(x, \xi)) = \tilde{f}(p(x, \xi)) \sqrt{p(x, \xi)}^\sigma$ and $\text{supp}(q_k) \subset \text{supp}(\omega \circ p)$. Up to remainder term, we consider the action of $Op^h(q(h))$ on $J^+_h(a^+(h))$. To do this, we need the following action of a pseudo-differential operator on a Fourier integral operator (see e.g. [Rob87, Theorem IV-19], [Bouc00, Appendix] or [RS11]).

Proposition 4.4.4. *Let $a \in S(\mu_1, -\infty)$ and $b \in S(\mu_2, -\infty)$ and S satisfy* (4.4.6) *and* (4.4.7). *Then*

$$Op^h(a) \circ J_h(S, b) = \sum_{j=0}^{N-1} h^j J_h(S, (a \triangleleft b) j) + h^N J_h(S, r_N(h)),$$

where $(a \triangleleft b)_j$ is a universal linear combination of

$$\partial^\beta_\xi a(x, \nabla_x S(x, \xi)) \partial^{\beta-\alpha}_x b(x, \xi) \partial^{\alpha_1}_x S(x, \xi) \cdots \partial^{\alpha_k}_x S(x, \xi),$$

with $\alpha \leq \beta, \alpha_1 + \cdots + \alpha_k = \alpha$ and $|\alpha_l| \geq 2$ for all $l = 1, ..., k$ and $|\beta| = j$. The maps $(a, b) \mapsto (a \triangleleft b)_j$ and $(a, b) \mapsto r_N(h)$ are continuous from $S(\mu_1, -\infty) \times S(\mu_2, -\infty)$ to $S(\mu_1 + \mu_2 - j, -\infty)$ and $S(\mu_1 + \mu_2 - N, -\infty)$ respectively. In particular, we have

$$(a \triangleleft b)_0(x, \xi) = a(x, \nabla_x S(x, \xi)) b(x, \xi),$$

$$i(a \triangleleft b)_1(x, \xi) = \nabla_\xi a(x, \nabla_x S(x, \xi)) \cdot \nabla_x b(x, \xi) + \frac{1}{2} \text{tr} \left(\nabla^2_\xi a(x, \nabla_x S(x, \xi)) \cdot \nabla^2_x S(x, \xi) \right) b(x, \xi).$$

Using this result, we have

$$Op^h(q(h))J_h^+(a^+(h)) = \sum_{k+j+l=0}^{N-1} h^{k+j+l} J_h^+((q_k \lhd a_j^+)_l) + h^N J_h^+(r_N^+(h)).$$

On the other hand, we have

$$J_h^+(a^+(h))(h\Lambda)^\sigma = J_h^+(a^+(h)|\xi|^\sigma).$$

Thus we get

$$\begin{aligned} \omega(h^2P)J_h^+(a^+(h)) - J_h^+(a^+(h))(h\Lambda)^\sigma = & \sum_{r=0}^{N-1} h^r J_h^+ \Bigg(\sum_{k+j+l=r} (q_k \lhd a_j^+)_l - a_r^+|\xi|^\sigma \Bigg) \\ & + h^N J_h^+(r_N^+(h)) + h^N R_N(h) J_h^+(a^+(h)). \end{aligned}$$

In order to make the left hand side of (4.4.10) small, we need to find $a_j^+ \in S(-j, -\infty)$ supported in $\Gamma^+(R, J_1, \tau_1)$ such that

$$\sum_{k+j+l=r} (q_k \lhd a_j^+)_l - a_r^+|\xi|^\sigma = 0, \quad r = 0, ..., N-1.$$

In particular,

$$\big(q_0(x, \nabla_x S_R^+(x,\xi)) - |\xi|^\sigma\big)\, a_0^+(x,\xi) = 0.$$

By noting that if $p(x,\xi) \in \operatorname{supp}(f)$ (see after (4.4.1)), then $q_0(x,\xi) = \sqrt{p(x,\xi)}^\sigma$. Thus in the region where the Hamilton-Jacobi equation (4.4.4) with $\epsilon = 1$ is satisfied, we need to show the following transport equations

$$(q_0 \lhd a_0^+)_1 + (q_1 \lhd a_0^+)_0 = 0 \tag{4.4.14}$$

$$(q_0 \lhd a_r^+)_1 + (q_1 \lhd a_r^+)_0 = -\sum_{\substack{k+j+l=r+1 \\ j \leq r-1}} (q_k \lhd a_j^+)_l, \quad r = 1, ..., N-1. \tag{4.4.15}$$

Here $(q_0 \lhd a^+)_1 + (q_1 \lhd a^+)_0$ can be written as

$$i\Big[(q_0 \lhd a^+)_1(x,\xi) + (q_1 \lhd a^+)_0(x,\xi)\Big] = \sum_{j=1}^d V_j^+(x,\xi)\partial_{x_j} a^+(x,\xi) + p_0^+(x,\xi)a^+(x,\xi),$$

where

$$\begin{aligned} V_j^+(x,\xi) &= (\partial_{\xi_j} q_0)(x, \nabla_x S_R^+(x,\xi)), \\ p_0^+(x,\xi) &= iq_1(x, \nabla_x S_R^+(x,\xi)) + \frac{1}{2}\operatorname{tr}\Big[\nabla_\xi^2 q_0(x, \nabla_x S_R^+(x,\xi)) \cdot \nabla_x^2 S_R^+(x,\xi)\Big]. \end{aligned}$$

We now consider the flow $X^+(t,x,\xi)$ associated to $V^+ = (V_j^+)_{j=1}^d$ as

$$\left\{ \begin{array}{rcl} \dot{X}^+(t) &=& V^+(X^+(t),\xi), \\ X^+(0) &=& x. \end{array} \right. \tag{4.4.16}$$

We have the following result (see [Bouc04, Proposition 3.2] or [Bouc00, Appendix]).
Proposition 4.4.5. *Let $\sigma \in (0,\infty)$, $J_1 \Subset (0,+\infty)$ and $-1 < \tau_1 < 1$. There exists $R > 0$ large enough and $e_1 > 0$ small enough such that for all $(x,\xi) \in \Gamma^+(R, J_1, \tau_1)$, the solution $X^+(t,x,\xi)$*

to (4.4.16) *is defined for all* $t \geq 0$ *and satisfies*

$$|X^+(t,x,\xi)| \geq e_1(t+|x|), \tag{4.4.17}$$
$$(X^+(t,x,\xi),\xi) \in \Gamma^+(R,J_1,\tau_1). \tag{4.4.18}$$

Moreover, for all $\alpha,\beta \in \mathbb{N}^d$*, there exists* $C_{\alpha\beta} > 0$ *such that for all* $t \geq 0$ *and all* $h \in (0,1]$,

$$|\partial_x^\alpha \partial_\xi^\beta (X^+(t,x,\xi) - x - \sigma t \xi |\xi|^{\sigma-2})| \leq C_{\alpha\beta} \langle t \rangle \langle x \rangle^{-\rho-|\alpha|}, \tag{4.4.19}$$

for all $(x,\xi) \in \Gamma^+(R,J_1,\tau_1)$.

Now, we can define for $(x,\xi) \in \Gamma^+(R,J_1,\tau_1)$ the functions

$$A_0^+(x,\xi) = \exp\Big(\int_0^{+\infty} p_0^+(X^+(t,x,\xi),\xi)dt\Big),$$
$$A_r^+(x,\xi) = \int_0^{+\infty} p_r^+(X^+(t,x,\xi),\xi) \exp\Big(\int_0^t p_0^+(X^+(s,x,\xi),\xi)ds\Big)dt,$$

for $r = 1,...,N-1$, where

$$p_r^+(x,\xi) = i \sum_{\substack{k+j+l=r+1 \\ j \leq r-1}} (q_k \triangleleft A_j^+)_l(x,\xi).$$

Using (4.4.17) and the fact that $p_r^+ \in S(-1-\rho-r,-\infty)$ for $r = 0,...,N-1$, we see that $p_r^+(X^+(t,x,\xi))$ are integrable with respect to t. Hence $A_r^+(x,\xi)$ are well-defined. Moreover, we have (see e.g. [Bouc04, Proposition 3.1]) that for all $(x,\xi) \in \Gamma^+(R,J_1,\tau_1)$,

$$|\partial_x^\alpha \partial_\xi^\beta (A_0^+(x,\xi) - 1)| \leq C_{\alpha\beta} \langle x \rangle^{-|\alpha|}, \tag{4.4.20}$$
$$|\partial_x^\alpha \partial_\xi^\beta A_r^+(x,\xi)| \leq C_{\alpha\beta} \langle x \rangle^{-r-|\alpha|}.$$

We also have that A_0^+, A_r^+ for $r = 1,...,N-1$ solve (4.4.14) and (4.4.15) respectively in $\Gamma^+(R,J_1,\tau_1)$. Now, by setting $a_r^+ = \chi_{1\to 2}^+ A_r^+$ (see (4.4.9)), we see that a_r^+ are globally defined on $\mathbb{R}^{2d}$ and $a_r^+ \in S(-r,-\infty)$. It is easy to see (4.4.11) from (4.4.20). We next insert $a^+(h) = \sum_{j=1}^{N-1} h^j a_j^+$ into the left hand side of (4.4.10) and get

$$\omega(h^2P)J_h^+(a^+(h)) - J_h^+(a^+(h))(h\Lambda)^\sigma = \sum_{r=0}^{N-1} h^r J_h^+ \left(\sum_{k+j+l=r} (q_k \triangleleft \chi_{1\to 2}^+ A_j^+)_l - \chi_{1\to 2}^+ A_r^+ |\xi|^\sigma \right)$$
$$+ h^N J_h^+(r_N^+(h)) + h^N R_N(h) J_h^+(a^+(h)).$$

Using the expression of $(a \triangleleft b)_l$ given in Proposition 2.2.3, we see that

$$(q_k \triangleleft \chi_{1\to 2}^+ A_j^+)_l = \chi_{1\to 2}^+ (q_k \triangleleft A_j^+)_l + \text{ terms in which derivatives fall into } \chi_{1\to 2}^+.$$

This gives (4.4.10) with $\check{a}^+(h)$ as in (4.4.12). The proof is complete. □

We now are able to construct the symbols b_k^+, for $k = 0,...,N-1$.

Proposition 4.4.6. *Let* J_3, J_4 *and* τ_3,τ_4 *be such that* $J_4 \Subset J_3 \Subset J_2$ *and* $-1 < \tau_2 < \tau_3 < \tau_4 < 1$. *Then for* $R > 0$ *large enough and all* χ^+ *supported in* $\Gamma^+(R^4,J_4,\tau_4)$*, there exists a sequence of symbols* $b_k^+ \in S(-k,-\infty)$*, for* $k = 0,...,N-1$*, supported in* $\Gamma^+(R^3,J_3,\tau_3)$ *such that*

$$J_h^+(a^+(h))J_h^+(b^+(h))^\star = Op^h(\chi^+) + h^N Op^h(\tilde{r}_N^+(h)), \tag{4.4.21}$$

where $a^+(h) = \sum_{j=0}^{N-1} h^j a_j^+$ *is given in* Proposition 4.4.3 *and* $b^+(h) = \sum_{k=0}^{N-1} h^k b_k^+$ *and* $(\tilde{r}_N^+(h))_{h\in(0,1]}$

is bounded in $S(-N,-\infty)$.

Before giving the proof, we need the following result (see [Bouc00, Appendix] or [Bouc04, Lemma 3.3]).

Lemma 4.4.7. *Let* $S^+_R := S^+_{1,R}$ *be as in* Proposition 4.4.1. *For* $x, y, \xi \in \mathbb{R}^d$*, we define*

$$\eta^+(R,x,y,\xi) := \int_0^1 \nabla_x S^+_R(y+\lambda(x-y),\xi)d\lambda. \tag{4.4.22}$$

Then for $R>0$ *large enough, we have the following properties.*

i. For all $x, y \in \mathbb{R}^d$*, the map* $\xi \mapsto \eta^+(R,x,y,\xi)$ *is a diffeomorphism from* $\mathbb{R}^d$ *onto itself. Let* $\eta \mapsto \xi^+(R,x,y,\eta)$ *be its inverse.*

ii. There exists $C>1$ *such that for all* $x,y,\eta \in \mathbb{R}^d$,

$$C^{-1}\langle\eta\rangle \leq \left\langle \xi^+(R,x,y,\eta)\right\rangle \leq C\langle\eta\rangle .$$

iii. For all $\alpha,\alpha',\beta \in \mathbb{N}^d$*, there exists* $C_{\alpha\alpha'\beta}>0$ *such that for all* $x,y,\eta \in \mathbb{R}^d$ *and all* $k \leq |\alpha|, k' \leq |\alpha'|$,

$$|\partial_x^\alpha \partial_y^{\alpha'} \partial_\eta^\beta \left(\xi^+(R,x,y,\eta)-\eta\right)| \leq C_{\alpha\alpha'\beta}\langle x\rangle^{-k}\langle y\rangle^{-\rho-k'}\langle x-y\rangle^{\rho+k+k'}.$$

Proof of Proposition 4.4.6. We firstly consider the general term $J^+_h(a^+)J^+_h(b^+)^\star$ and write its kernel as

$$K^+_h(x,y) = (2\pi h)^{-d}\int_{\mathbb{R}^d} e^{ih^{-1}\left(S^+_R(x,\xi)-S^+_R(y,\xi)\right)}a^+(x,\xi)\overline{b^+(y,\xi)}d\xi.$$

By Taylor's formula, we have

$$S^+_R(x,\xi)-S^+_R(y,\xi) = \left\langle x-y, \eta^+(R,x,y,\xi)\right\rangle,$$

where η^+ given in (4.4.22). By change of variable $\xi \mapsto \xi^+(R,x,y,\eta)$, the kernel becomes

$$K^+_h(x,y) = (2\pi h)^{-d}\int_{\mathbb{R}^d} e^{ih^{-1}(x-y)\cdot\eta}a^+(x,\xi^+(R,x,y,\eta))\overline{b^+(y,\xi^+(R,x,y,\eta))}|\det \partial_\eta\xi^+(R,x,y,\eta)|d\eta.$$

Now, using Lemma (4.4.7), the symbolic calculus gives

$$J^+_h(a^+)J^+_h(b^+)^\star = \sum_{l=0}^{N-1} h^l Op^h((a^+ \triangleright b^+)_l) + h^N Op^h(\tilde{r}^+_N(h)),$$

where $(a^+ \triangleright b^+)_l \in S(-l,-\infty)$ is of the form

$$(a^+ \triangleright b^+)_l(x,\eta) = \sum_{|\alpha|=l} \frac{\partial_y^\alpha D_\eta^\alpha c^+(x,y,\eta)\big|_{y=x}}{\alpha!},$$

for $l=0,...,N-1$ with

$$c^+(x,y,\eta) = a^+(x,\xi^+(R,x,y,\eta))\overline{b^+(y,\xi^+(R,x,y,\eta))}|\det\partial_\eta\xi^+(R,x,y,\eta)|,$$

and $(\tilde{r}^+_N(h))_{h\in(0,1]}$ is bounded in $S(-N,-\infty)$. We have now

$$\begin{aligned} J^+_h(a^+(h))J^+_h(b^+(h))^\star &= \sum_{j,k} h^{j+k}J^+_h(a^+_j)J^+_h(b^+_k)^\star \\ &= \sum_{j+k+l=0}^{N-1} h^{j+k+l}Op^h((a^+_j \triangleleft b^+_k)_l) + h^N Op^h(\tilde{r}^+_N(h)). \end{aligned}$$

Compare with (4.4.21), the result follows if we solve the following equations:

$$(a_0^+ \triangleleft b_0^+)_0 = \chi^+,$$
$$(a_0^+ \triangleleft b_r^+)_0 = -\sum_{\substack{j+k+l=r \\ k\leq r-1}} (a_j^+ \triangleleft b_k^+)_l, \quad r = 1, ..., N-1.$$

We can define $b_0^+, ..., b_{N-1}^+$ iteratively by

$$\overline{b_0^+(x,\xi)} = \chi^+(x, \eta^+(R,x,x,\xi))\Big(a_0^+(x,\xi)\left|\det \partial_\eta \xi^+(R,x,x,\eta^+(R,x,x,\xi))\right|\Big)^{-1},$$
$$\overline{b_r^+(x,\xi)} = -\sum_{\substack{j+k+l=r \\ k\leq r-1}} (a_j^+ \triangleleft b_k^+)_l(x,\eta^+(R,x,x,\xi))\Big(a_0^+(x,\xi)\left|\det \partial_\eta \xi(R,x,x,\eta^+(R,x,x,\xi))\right|\Big)^{-1},$$

for $r = 1, ..., N-1$. Note that by (4.4.11) and Lemma 4.4.7, the term in $(\cdots)^{-1}$ cannot vanish on the support of $\chi^+(\cdot, \eta^+(R,\cdot,\cdot,\cdot))$. Thus the above functions are well-defined. Moreover, by choosing $R > 0$ large enough with the fact

$$\eta^+(R,x,x,\xi) = \nabla_x S_R^+(x,\xi) = \xi + O(\min\{R^{-\rho}, \langle x\rangle^{-\rho}\}),$$

we see that the support of $\chi^+(x,\eta^+(R,x,x,\xi))$ is contained in $\Gamma^+(R^3, J_3, \tau_3)$. This completes the proof of Proposition 4.4.6. ☐

By (4.4.8), Proposition 4.4.3 and Proposition 4.4.6, we are able to state the Isozaki-Kitada parametrix for the Schrödinger-type equation at high frequency.

Theorem 4.4.8. *Let $\sigma \in (0,\infty)$. Fix $J_4 \Subset (0,+\infty)$ open interval containing* supp(f) *and $-1 < \tau_4 < 1$. Choose arbitrary open intervals J_1, J_2, J_3 such that $J_4 \Subset J_3 \Subset J_2 \Subset J_1 \Subset (0,+\infty)$ and arbitrary τ_1, τ_2, τ_3 such that $-1 < \tau_1 < \tau_2 < \tau_3 < \tau_4 < 1$. Then for $R > 0$ large enough, we can find sequences of symbols*

$$a_j^\pm \in S(-j,-\infty), \quad \mathrm{supp}(a_j^\pm) \subset \Gamma^\pm(R, J_1, \tau_1),$$

such that for all

$$\chi^\pm \in S(0,-\infty), \quad \mathrm{supp}(\chi^\pm) \subset \Gamma^\pm(R^4, J_4, \tau_4),$$

there exist sequences of symbols

$$b_k^\pm \in S(-k,-\infty), \quad \mathrm{supp}(b_k^\pm) \subset \Gamma^\pm(R^3, J_3, \tau_3),$$

such that for all $N \geq 1$, for all $h \in (0,1]$ and all $\pm t \geq 0$,

$$e^{-ith^{-1}\omega(h^2P)}Op^h(\chi^\pm) = J_h^\pm(a^\pm(h))e^{-ith^{-1}(h\Lambda)^\sigma} J_h^\pm(b^\pm(h))^\star + R_N^\pm(t,h),$$

where the phase functions $S_R^\pm := S_{1,R}^\pm$ are as in Proposition 4.4.1 *and the remainder terms*

$$R_N^\pm(t,h) = R_1^\pm(N,t,h) + R_2^\pm(N,t,h) + R_3^\pm(N,t,h) + R_4^\pm(N,t,h),$$

with

$$R_1^{\pm}(N,t,h) = -h^{N-1}e^{-ith^{-1}\omega(h^2P)}Op^h(\tilde{r}_N^{\pm}(h)),$$
$$R_2^{\pm}(N,t,h) = -ih^{N-1}\int_0^t e^{-i(t-s)h^{-1}\omega(h^2P)}R_N(h)J_h^{\pm}(a^{\pm}(h))e^{-ish^{-1}(h\Lambda)^{\sigma}}J_h^{\pm}(b^{\pm}(h))^{\star}ds,$$
$$R_3^{\pm}(N,t,h) = -ih^{N-1}\int_0^t e^{-i(t-s)h^{-1}\omega(h^2P)}J_h^{\pm}(r_N^{\pm}(h))e^{-ish^{-1}(h\Lambda)^{\sigma}}J_h^{\pm}(b^{\pm}(h))^{\star}ds,$$
$$R_4^{\pm}(N,t,h) = -ih^{-1}\int_0^t e^{-i(t-s)h^{-1}\omega(h^2P)}J_h^{\pm}(\check{a}^{\pm}(h))e^{-ish^{-1}(h\Lambda)^{\sigma}}J_h^{\pm}(b^{\pm}(h))^{\star}ds.$$

Here $(\tilde{r}_N^{\pm}(h))_{h\in(0,1]}, (r_N^{\pm}(h))_{h\in(0,1]}$ *are bounded in* $S(-N,-\infty)$, $R_N(h)$ *is as in* (4.4.13), $(\check{a}^{\pm}(h))_{h\in(0,1]}$ *are bounded in* $S(0,-\infty)$ *and are finite sums depending on* N *of the form*

$$\check{a}^{\pm}(h) = \sum_{|\alpha|\geq 1}\check{a}_{\alpha}^{\pm}(h)\partial_x^{\alpha}\chi_{1\to 2}^{\pm}, \tag{4.4.23}$$

where $(\check{a}_{\alpha}^{\pm}(h))_{h\in(0,1]}$ *are bounded in* $S(0,-\infty)$ *and* $\chi_{1\to 2}^{\pm}$ *are given in* (4.4.9).

We now give the main steps for the construction of the Isozaki-Kitada parametrix at low frequency. For simplicity, we omit the $\pm$ sign. Let us start with the following Duhamel formula

$$e^{-it\epsilon\omega(\epsilon^{-2}P)}\mathcal{J}_{\epsilon}(a_{\epsilon}) = \mathcal{J}_{\epsilon}(a_{\epsilon})e^{-it\epsilon\Lambda^{\sigma}} - i\epsilon\int_0^t e^{-i(t-s)\epsilon\omega(\epsilon^{-2}P)}\Big(\omega(\epsilon^{-2}P)\mathcal{J}_{\epsilon}(a_{\epsilon}) - \mathcal{J}_{\epsilon}(a_{\epsilon})\Lambda^{\sigma}\Big)e^{-is\epsilon\Lambda^{\sigma}}ds.$$

Thanks to the support of a_{ϵ}, we can write

$$\omega(\epsilon^{-2}P)\mathcal{J}_{\epsilon}(a_{\epsilon}) = \omega(\epsilon^{-2}P)\zeta_1(\epsilon x)\mathcal{J}_{\epsilon}(a_{\epsilon}),$$

where $\zeta_1 \in C^{\infty}(\mathbb{R}^d)$ is supported outside $B(0,1)$ and satisfies $\zeta_1(x) = 1$ for $|x| > R$. Using the parametrix of $\omega(\epsilon^{-2}P)\zeta_1(\epsilon x)$ given in Proposition 4.1.7 (by taking the adjoint), we have

$$\omega(\epsilon^{-2}P)\zeta_1(\epsilon x) = \sum_{k=0}^{N-1}\tilde{\zeta}_1(\epsilon x)Op_{\epsilon}(q_{\epsilon,k})\zeta_1(\epsilon x) + R_N(\epsilon),$$

where $q_{\epsilon,0}(x,\xi) = \omega(p_{\epsilon}(x,\xi)) = \tilde{f}(p_{\epsilon}(x,\xi))\sqrt{p_{\epsilon}(x,\xi)}^{\sigma}$, $\mathrm{supp}(q_{\epsilon,k}) \subset \mathrm{supp}(\omega\circ p_{\epsilon})$ and $(R_N(\epsilon))_{\epsilon\in(0,1]}$ satisfies (4.1.8). Here $\tilde{\zeta}_1 \in C^{\infty}(\mathbb{R}^d)$ is supported outside $B(0,1)$ and $\tilde{\zeta}_1 = 1$ near $\mathrm{supp}(\zeta_1)$. We want to find $a_{\epsilon} = \sum_{j=0}^{N-1}a_{\epsilon,j}$ so that the term $\omega(\epsilon^{-2}P)\mathcal{J}_{\epsilon}(a_{\epsilon}) - \mathcal{J}_{\epsilon}(a_{\epsilon})\Lambda^{\sigma}$ has a small contribution. By the choice of cutoff functions and the action of pseudo-differential operators on Fourier integral operators given in Proposition 2.2.3 with $h=1$, we have

$$\begin{aligned}\omega(\epsilon^{-2}P)\mathcal{J}_{\epsilon}(a_{\epsilon}) - \mathcal{J}_{\epsilon}(a_{\epsilon})\Lambda^{\sigma} = &\sum_{r=0}^{N-1}\left(\sum_{k+j+l=r}\mathcal{J}_{\epsilon}((q_{\epsilon,k}\triangleleft a_{\epsilon,j})_l) - \mathcal{J}_{\epsilon}(a_{\epsilon,r}|\xi|^{\sigma})\right)\\ &+R_N(\epsilon)\mathcal{J}_{\epsilon}(a_{\epsilon}) + \mathcal{J}_{\epsilon}(r_N(\epsilon)),\end{aligned} \tag{4.4.24}$$

where $(r_N(\epsilon))_{\epsilon\in(0,1]}$ is bounded in $S(-N,-\infty)$. This implies that we need to find $(a_{\epsilon,j})_{\epsilon\in(0,1]}$ bounded in $S(-j,-\infty)$ supported in $\Gamma(R,J_1,\tau_1)$ such that

$$\sum_{k+j+l=r}(q_{\epsilon,k}\triangleleft a_{\epsilon,j})_l - a_{\epsilon,r}|\xi|^{\sigma} = 0, \quad r = 0,...,N-1.$$

By noting that if $p_{\epsilon}(x,\xi) \in \mathrm{supp}(f)$, then $q_{\epsilon,0}(x,\xi) = \sqrt{p_{\epsilon}(x,\xi)}^{\sigma}$. This leads to the following

Hamilton-Jacobi and transport equations,

$$p_\epsilon(x, \nabla_x S_{\epsilon,R}(x,\xi)) = |\xi|^2, \tag{4.4.25}$$

$$(q_{\epsilon,0} \lhd a_{\epsilon,0})_1 + (q_{\epsilon,1} \lhd a_{\epsilon,0})_0 = 0 \tag{4.4.26}$$

$$(q_{\epsilon,0} \lhd a_{\epsilon,r})_1 + (q_{\epsilon,1} \lhd a_{\epsilon,r})_0 = -\sum_{\substack{k+j+l=r+1 \\ j\leq r-1}} (q_{\epsilon,k} \lhd a_{\epsilon,j})_l, \quad r = 1, ..., N-1. \tag{4.4.27}$$

We can solve (4.4.25) on $\Gamma^{\pm}(R, J_1, \tau_1)$ using Proposition 4.4.1. We then solve (4.4.26), (4.4.27) on $\Gamma^{\pm}(R, J_1, \tau_1)$ and extend solutions globally on $\mathbb{R}^{2d}$. We obtain

$$\omega(\epsilon^{-2}P)\mathcal{J}_\epsilon(a_\epsilon) - \mathcal{J}_\epsilon(a_\epsilon)\Lambda^\sigma = R_N(\epsilon)\mathcal{J}_\epsilon(a_\epsilon) + \mathcal{J}_\epsilon(r_N(\epsilon)) + \mathcal{J}_\epsilon(\check{a}(\epsilon)),$$

where $(\check{a}(\epsilon))_{\epsilon\in(0,1]}$ is bounded in $S(0,-\infty)$ and is a finite sum depending on N of the form

$$\check{a}(\epsilon) = \sum_{|\alpha|\geq 1} \check{a}_\alpha(\epsilon)\partial_x^\alpha \chi_{1\to 2},$$

with $(\check{a}_\alpha(\epsilon))_{\epsilon\in(0,1]}$ bounded in $S(0,-\infty)$ and $\chi_{1\to 2}$ as in (4.4.9).

Next, we can find bounded families of symbols $b_{\epsilon,k} \in S(-k,-\infty)$ for $k = 0, ..., N-1$ supported in $\Gamma(R^3, J_3, \tau_3)$ such that

$$\mathcal{J}_\epsilon(a_\epsilon)\mathcal{J}_\epsilon(b_\epsilon)^\star = Op_\epsilon(\chi_\epsilon)\zeta(\epsilon x) + Op_\epsilon(\tilde{r}_N(\epsilon))\zeta(\epsilon x),$$

where $b_\epsilon = \sum_{k=0}^{N-1} b_{\epsilon,k}$ and $(\tilde{r}_N(\epsilon))_{\epsilon\in(0,1]}$ is bounded in $S(-N,-\infty)$. This is possible by writing for R large enough $\mathcal{J}_\epsilon(b_\epsilon) = \zeta(\epsilon x)\mathcal{J}_\epsilon(b_\epsilon)$ and taking the adjoint. We have the following Isozaki-Kitata parametrix for the Schrödinger-type equation at low frequency.

Theorem 4.4.9. *Let $\sigma \in (0,\infty)$, $\zeta \in C^\infty(\mathbb{R}^d)$ be supported outside $B(0,1)$ and equal to 1 near infinity. Fix $J_4 \Subset (0,+\infty)$ open interval containing* supp(f) *and $-1 < \tau_4 < 1$. Choose arbitrary open intervals J_1, J_2, J_3 such that $J_4 \Subset J_3 \Subset J_2 \Subset J_1 \Subset (0,+\infty)$ and arbitrary τ_1, τ_2, τ_3 such that $-1 < \tau_1 < \tau_2 < \tau_3 < \tau_4 < 1$. Then for $R > 0$ large enough, we can find bounded families of symbols*

$$(a^{\pm}_{\epsilon,j})_{\epsilon\in(0,1]} \in S(-j,-\infty), \quad \text{supp}(a^{\pm}_{\epsilon,j}) \subset \Gamma^{\pm}(R, J_1, \tau_1),$$

such that for all

$$(\chi^{\pm}_\epsilon)_{\epsilon\in(0,1]} \in S(0,-\infty), \quad \text{supp}(\chi^{\pm}_\epsilon) \subset \Gamma^{\pm}(R^4, J_4, \tau_4),$$

there exists families of symbols

$$(b^{\pm}_{\epsilon,k})_{\epsilon\in(0,1]} \in S(-k,-\infty), \quad \text{supp}(b^{\pm}_{\epsilon,k}) \subset \Gamma^{\pm}(R^3, J_3, \tau_3),$$

such that for all $N \geq 1$, for all $\epsilon \in (0,1]$ and all $\pm t \geq 0$,

$$e^{-it\epsilon\omega(\epsilon^{-2}P)}Op_\epsilon(\chi^{\pm}_\epsilon)\zeta(\epsilon x) = \mathcal{J}^{\pm}_\epsilon(a^{\pm}_\epsilon)e^{-it\epsilon\Lambda^\sigma}\mathcal{J}^{\pm}_\epsilon(b^{\pm}_\epsilon)^\star + \mathcal{R}^{\pm}_N(t,\epsilon),$$

where the phase functions $S^{\pm}_{\epsilon,R}$ are given in Proposition 4.4.1 *and the remainder terms*

$$\mathcal{R}^{\pm}_N(t,\epsilon) = \mathcal{R}^{\pm}_1(N,t,\epsilon) + \mathcal{R}^{\pm}_2(N,t,\epsilon) + \mathcal{R}^{\pm}_3(N,t,\epsilon) + \mathcal{R}^{\pm}_4(N,t,\epsilon),$$

with

$$\begin{aligned}
\mathcal{R}_1^\pm(N,t,\epsilon) &= -e^{-it\epsilon\omega(\epsilon^{-2}P)} Op_\epsilon(\tilde{r}_N^\pm(\epsilon))\zeta(\epsilon x),\\
\mathcal{R}_2^\pm(N,t,\epsilon) &= -i\epsilon\int_0^t e^{-i(t-s)\epsilon\omega(\epsilon^{-2}P)} R_N(\epsilon)\mathcal{J}_\epsilon^\pm(a_\epsilon^\pm)e^{-is\epsilon\Lambda^\sigma}\mathcal{J}_\epsilon^\pm(b_\epsilon^\pm)^\star ds,\\
\mathcal{R}_3^\pm(N,t,\epsilon) &= -i\epsilon\int_0^t e^{-i(t-s)\epsilon\omega(\epsilon^{-2}P)} \mathcal{J}_\epsilon^\pm(r_N^\pm(\epsilon))e^{-is\epsilon\Lambda^\sigma}\mathcal{J}_\epsilon^\pm(b_\epsilon^\pm)^\star ds,\\
\mathcal{R}_4^\pm(N,t,\epsilon) &= -i\epsilon\int_0^t e^{-i(t-s)\epsilon\omega(\epsilon^{-2}P)} \mathcal{J}_\epsilon^\pm(\check{a}^\pm(\epsilon))e^{-is\epsilon\Lambda^\sigma}\mathcal{J}_\epsilon^\pm(b_\epsilon^\pm)^\star ds.
\end{aligned}$$

Here $(\tilde{r}_N^\pm(\epsilon))_{\epsilon\in(0,1]}, (r_N^\pm(\epsilon))_{\epsilon\in(0,1]}$ *are bounded in* $S(-N,-\infty)$*,* $(R_N(\epsilon))_{\epsilon\in(0,1]}$ *is given in* Proposition 4.1.7*,* $(\check{a}^\pm(\epsilon))_{\epsilon\in(0,1]}$ *are bounded in* $S(0,-\infty)$ *and are finite sums depending on* N *of the form*

$$\check{a}^\pm(\epsilon) = \sum_{|\alpha|\geq 1} \check{a}_\alpha^\pm(\epsilon)\partial_x^\alpha \chi_{1\to 2}^\pm,$$

where $(\check{a}_\alpha^\pm(\epsilon))_{\epsilon\in(0,1]}$ *are bounded in* $S(0,-\infty)$ *and* $\chi_{1\to2}^\pm$ *are as in* (4.4.9).

We have the following dispersive estimates for the main terms of the Isozaki-Kitada parametrix both at high and low frequencies.

Proposition 4.4.10. *Let* $\sigma\in(0,\infty)\backslash\{1\}$*,* $S_{\epsilon,R}^\pm$ *be as in* Proposition 4.4.1 *and* $(a_\epsilon^\pm)_{\epsilon\in(0,1]}, (b_\epsilon^\pm)_{\epsilon\in(0,1]}$ *be bounded in* $S(0,-\infty)$ *compactly supported in* ξ *away from zero.*

1. *Then for* $R>0$ *large enough, there exists* $C>0$ *such that for all* $t\in\mathbb{R}$ *and all* $h\in(0,1]$*,*

$$\|J_h^\pm(a^\pm)e^{-ith^{-1}(h\Lambda)^\sigma}J_h^\pm(b^\pm)^\star\|_{L^1\to L^\infty} \leq Ch^{-d}(1+|t|h^{-1})^{-d/2}, \tag{4.4.28}$$

where $a^\pm := a_{\epsilon=1}^\pm, b^\pm := b_{\epsilon=1}^\pm$.

2. *Then for* $R>0$ *large enough, there exists* $C>0$ *such that for all* $t\in\mathbb{R}$ *and all* $\epsilon\in(0,1]$*,*

$$\|\mathcal{J}_\epsilon^\pm(a_\epsilon^\pm)e^{-it\epsilon\Lambda^\sigma}\mathcal{J}_\epsilon^\pm(b_\epsilon^\pm)^\star\|_{L^1\to L^\infty} \leq C\epsilon^d(1+\epsilon|t|)^{-d/2}. \tag{4.4.29}$$

Proof. 1. For simplicity, we drop the superscript $\pm$. The kernel of $J_h(a)e^{-ith^{-1}(h\Lambda)^\sigma}J_h(b)^\star$ reads

$$K_h(t,x,y) = (2\pi h)^{-d}\int_{\mathbb{R}^d} e^{ih^{-1}(S_R(x,\xi)-S_R(y,\xi)-t|\xi|^\sigma)}a(x,\xi)\overline{b(y,\xi)}d\xi.$$

The estimates (4.4.28) are in turn equivalent to

$$|K_h(t,x,y)| \leq Ch^{-d}(1+|t|h^{-1})^{-d/2}, \tag{4.4.30}$$

for all $t\in\mathbb{R}, h\in(0,1]$ and $x,y\in\mathbb{R}^d$. We only consider $t\geq 0$, the case $t\leq 0$ is similar. Let us denote the compact support of the amplitude by $\mathcal{K}$. Since a,b are bounded uniformly in $x,y\in\mathbb{R}^d$, we have

$$|K_h(t,x,y)|\leq Ch^{-d},$$

for all $t\in\mathbb{R}$ and all $x,y\in\mathbb{R}^d$. If $0\leq t\leq h$ or $1+th^{-1}\leq 2$, then

$$|K_h(t,x,y)|\leq Ch^{-d}\leq Ch^{-d}(1+th^{-1})^{-d/2}.$$

So, we can assume that $t\geq h$ or $(1+th^{-1})\leq 2th^{-1}$ and denote the phase function

$$\Phi(R,t,x,y,\xi) = (S_R(x,\xi)-S_R(y,\xi))/t - |\xi|^\sigma,$$

and parameter $\lambda = th^{-1}\geq 1$. We can rewrite

$$\Phi(R,t,x,y,\xi) = \langle (x-y)/t, \eta(R,x,y,\xi)\rangle - |\xi|^\sigma,$$

where

$$\eta(R,x,y,\xi) = \int_0^1 \nabla_x S_R(y+\lambda(x-y),\xi)d\lambda.$$

Using the properties of the phase functions S_R given in (4.4.5), we have that

$$\eta(R,x,y,\xi) = \xi + Q(R,x,y,\xi),$$

where $Q(R,x,y,\xi)$ is a vector in $\mathbb{R}^d$ satisfying for $R>0$ large enough,

$$|\partial_\xi^\beta Q(R,x,y,\xi)| \leq C_\beta R^{-\rho}, \tag{4.4.31}$$

for all $x,y\in\mathbb{R}^d$ and $\xi\in\mathcal{K}$. We have

$$\nabla_\xi \Phi(R,t,x,y,\xi) = \frac{x-y}{t}\cdot(\mathrm{Id}_{\mathbb{R}^d} + \nabla_\xi Q(R,x,y,\xi)) - \sigma\xi|\xi|^{\sigma-2}.$$

If $|(x-y)/t| \geq C$ for some constant $C>0$ large enough then for $R>0$ large enough, there exists $C_1>0$,

$$|\nabla_\xi \Phi(R,t,x,y,\xi)| \geq \frac{1}{2}\Big|\frac{x-y}{t}\Big| \geq C_1.$$

Thus the phase is non-stationary. By using integration by parts with respect to ξ together with the fact

$$|\partial_\xi^\beta \Phi(R,t,x,y,\xi)| \leq C_\beta \Big|\frac{x-y}{t}\Big|, \quad |\beta|\geq 2,$$

we have that for all $N\geq 1$,

$$|K_h(t,x,y)| \leq Ch^{-d}(th^{-1})^{-N} \leq Ch^{-d}(1+th^{-1})^{-d/2},$$

provided N is taken bigger than $d/2$. The same result still holds for $|(x-y)/t|\leq c$ for some $c>0$ small enough.

Therefore, we can assume that $c\leq |x-y/t| \leq C$. In this case, we write

$$\nabla^2_\xi \Phi(R,t,x,y,\xi) = \frac{x-y}{t}\cdot \nabla^2_\xi Q(R,x,y,\xi) - \sigma|\eta|^{\sigma-2}\Big(\mathrm{Id}_{\mathbb{R}^d} + (\sigma-2)\frac{\eta\cdot\eta^T}{|\eta|^2}\Big).$$

Using the fact that $\sigma\in(0,\infty)\backslash\{1\}$ and

$$\Big|\det \sigma|\eta|^{\sigma-2}\Big(\mathrm{Id}_{\mathbb{R}^d} + (\sigma-2)\frac{\eta\cdot\eta^T}{|\eta|^2}\Big)\Big| = \sigma^d|\sigma-1||\eta|^{(\sigma-2)d} \geq C$$

and (4.4.31), we see that for $R>0$ large enough, the map $\xi\mapsto\nabla_\xi\Phi(R,t,x,y,\xi)$ is a local diffeomorphism from a neighborhood of $\mathcal{K}$ to its range. Moreover, for all $\beta\in\mathbb{N}^d$ satisfying $|\beta|\geq 1$, we have $|\partial_\xi^\beta\Phi(R,t,x,y,\xi)|\leq C_\beta$. The stationary phase theorem then implies that for $R>0$ large enough, all $t\geq h$ and all $x,y\in\mathbb{R}^d$ satisfying $c\leq|(x-y)/t|\leq C$,

$$|K_h(t,x,y)| \leq Ch^{-d}\lambda^{-d/2} \leq Ch^{-d}(1+th^{-1})^{-d/2}.$$

This gives (4.4.30).

2. We are now in position to show (4.4.29). As above, we drop the superscript $\pm$ for simplicity. We see that up to a conjugation by D_ϵ, the kernel of $\mathcal{J}_\epsilon(a_\epsilon)e^{-it\epsilon\Lambda^\sigma}\mathcal{J}_\epsilon(b_\epsilon)^\star$ reads

$$K_\epsilon(t,x,y) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(S_{\epsilon,R}(x,\xi)-t\epsilon|\xi|^\sigma - S_{\epsilon,R}(y,\xi))}a_\epsilon(x,\xi)\overline{b_\epsilon(y,\xi)}d\xi.$$

The dispersive estimates (4.4.29) follow from

$$|K_\epsilon(t,x,y)| \le C(1+\epsilon|t|)^{-d/2}, \tag{4.4.32}$$

for all $t \in \mathbb{R}$ uniformly in $x, y \in \mathbb{R}^d$, $\epsilon \in (0,1]$ and the fact that

$$\|D_\epsilon\|_{L^\infty \to L^\infty} = \epsilon^{d/2}, \quad \|D_\epsilon^{-1}\|_{L^1\to L^1} = \epsilon^{d/2}.$$

The estimates (4.4.32) are proved by repeating the same line as above. The proof is complete. □

Micro-local propagation estimates. In this paragraph, we will prove some propagation estimates which are useful for our purpose. To do this, we need the following result (see [BT08, Lemma 4.1]).

Lemma 4.4.11. *Let $\tau_+, \tau_- \in (-1,1)$.*

1. *For all $x, y, \xi \in \mathbb{R}^d \backslash \{0\}$ satisfying $\pm x\cdot\xi/|x||\xi| > \tau_\pm$ and $\pm t \ge 0$, we have*

$$\pm \frac{(x+t\xi)\cdot\xi}{|x+t\xi||\xi|} > \tau_\pm \text{ and } |x+t\xi| \ge c_\pm(|x|+|t\xi|), \tag{4.4.33}$$

   where $c_\pm = \sqrt{1+\tau_\pm}/\sqrt{2}$.

2. *If $\tau_- + \tau_+ > 0$, then there exists $c = c(\tau_-, \tau_+) > 0$ such that for all $x, y, \xi \in \mathbb{R}^d\backslash\{0\}$ satisfying $+x\cdot\xi/|x||\xi| > \tau_+$ and $-y\cdot\xi/|y||\xi| > \tau_-$, we have*

$$|x-y| \ge c(|x|+|y|). \tag{4.4.34}$$

We start with the following estimates.

Lemma 4.4.12. *Let $\sigma \in (0,\infty)$ and $\chi \in C_0^\infty(\mathbb{R}^d)$ satisfying $\chi(x) = 1$ for $|x| \le 1$.*

1. *Using the notations given in* Theorem 4.4.8*, if $R > 0$ is large enough, then for all $m \ge 0$, there exists $C > 0$ such that for all $\pm s \ge 0$ and all $h \in (0,1]$,*

$$\|\chi\left(x/R^2\right) J_h^\pm(\check{a}^\pm(h)) e^{-ish^{-1}(h\Lambda)^\sigma} J_h^\pm(b^\pm(h))^\star \langle x\rangle^m \|_{H^{-m}\to H^m} \le Ch^m \langle s\rangle^{-m}. \tag{4.4.35}$$

   Moreover,

$$\| \langle x\rangle^m (1-\chi)\left(x/R^2\right) J_h^\pm(\check{a}^\pm(h)) e^{-ish^{-1}(h\Lambda)^\sigma} J_h^\pm(b^\pm(h))^\star \langle x\rangle^m \|_{H^{-m}\to H^m} \le Ch^m \langle s\rangle^{-m}. \tag{4.4.36}$$

   In particular

$$\| \langle x\rangle^m J_h^\pm(\check{a}^\pm(h)) e^{-ish^{-1}(h\Lambda)^\sigma} J_h^\pm(b^\pm(h))^\star \langle x\rangle^m \|_{H^{-m}\to H^m} \le Ch^m \langle s\rangle^{-m}. \tag{4.4.37}$$

2. *Using the notations given in* Theorem 4.4.9*, if $R > 0$ is large enough, then for all $m \ge 0$, there exists $C > 0$ such that for all $\pm s \ge 0$ and all $\epsilon \in (0,1]$,*

$$\|\chi(\epsilon x/R^2)\mathcal{J}_\epsilon^\pm(\check{a}^\pm(\epsilon)) e^{-is\epsilon\Lambda^\sigma} \mathcal{J}_\epsilon^\pm(b_\epsilon^\pm)^\star \langle \epsilon x\rangle^m \|_{L^2\to L^2} \le C\langle \epsilon s\rangle^{-m}. \tag{4.4.38}$$

   Moreover,

$$\| \langle \epsilon x\rangle^m (1-\chi)(\epsilon x/R^2)\mathcal{J}_\epsilon^\pm(\check{a}^\pm(\epsilon)) e^{-is\epsilon\Lambda^\sigma} \mathcal{J}_\epsilon^\pm(b_\epsilon^\pm)^\star \langle \epsilon x\rangle^m \|_{L^2\to L^2} \le C\langle \epsilon s\rangle^{-m}. \tag{4.4.39}$$

   In particular

$$\| \langle \epsilon x\rangle^m \mathcal{J}_\epsilon^\pm(\check{a}^\pm(\epsilon)) e^{-is\epsilon\Lambda^\sigma} \mathcal{J}_\epsilon^\pm(b_\epsilon^\pm)^\star \langle \epsilon x\rangle^m \|_{L^2\to L^2} \le C\langle \epsilon s\rangle^{-m}. \tag{4.4.40}$$

Proof. 1. We firstly consider the high frequency case. The proof in this case is essentially given in [BT08]. For reader's convenience, we will give a sketch of the proof. The kernel of the operator in the left hand side of (4.4.35) reads

$$K_h^{\pm}(s,x,y) = (2\pi h)^{-d}\chi(x/R^2)\int_{\mathbb{R}^d} e^{ih^{-1}\Phi^{\pm}(R,s,x,y,\xi)}\check{a}^{\pm}(h,x,\xi)\overline{b^{\pm}(h,y,\xi)}d\xi\,\langle y\rangle^m,$$

where the phase $\Phi^{\pm}(R,s,x,y,\xi) = S_R^{\pm}(x,\xi) - s|\xi|^{\sigma} - S_R^{\pm}(y,\xi)$. Using (4.4.5), we have

$$|\nabla_\xi \Phi^{\pm}(R,s,x,y,\xi)| = |x - \sigma s\xi|\xi|^{\sigma-2} - y + O(1)| \geq |\sigma s\xi|\xi|^{\sigma-2} + y| - |x| + O(1),$$

where $|x| \leq CR^2$ and $(y,\xi) \in \Gamma^{\pm}(R^3, J_3, \tau_3)$. We then apply (4.4.33) with $\pm y\cdot\xi/|y||\xi| > \tau_3$ and $\pm t = \pm\sigma s|\xi|^{\sigma-2} \geq 0$ to get

$$|\sigma s\xi|\xi|^{\sigma-2} + y| \geq C(|s| + |y|), \tag{4.4.41}$$

for all $\pm s \geq 0$. We next use $|y| > R^3$ to control $|x| \lesssim R^2$ and obtain

$$|\nabla_\xi\Phi^{\pm}(R,s,x,y,\xi)| \geq C(1+|s|+|x|+|y|),$$

for all $\pm s \geq 0$. By integrations by part with respect to ξ with remark that higher derivatives of $\partial_\xi\Phi^{\pm}$ are controlled by $|\nabla_\xi\Phi^{\pm}|$, we get for all $N \geq 0$,

$$\left|\chi(x/R^2)\int_{\mathbb{R}^d} e^{ih^{-1}\Phi^{\pm}(R,s,x,y,\xi)}\check{a}^{\pm}(h,x,\xi)\overline{b^{\pm}(h,y,\xi)}d\xi\right| \leq Ch^N(1+|s|+|x|+|y|)^{-N}.$$

By choosing N large enough, we can dominate $\langle y\rangle^m$ and get

$$|K_h^{\pm}(s,x,y)| \leq Ch^N(1+|s|+|x|+|y|)^{-N},$$

for all N large enough, therefore for all $N \geq 0$. We do the same for higher derivatives $\partial_x^\alpha\partial_y^\beta K_h(s,x,y)$ and the result follows. The kernel of the operator in the left hand side of (4.4.36) reads

$$K_h^{\pm}(s,x,y) = (2\pi h)^{-d}\langle x\rangle^m(1-\chi)(x/R^2)\int_{\mathbb{R}^d} e^{ih^{-1}\Phi^{\pm}(R,s,x,y,\xi)}\check{a}^{\pm}(h,x,\xi)\overline{b^{\pm}(h,y,\xi)}d\xi\,\langle y\rangle^m.$$

We use the form of $\check{a}^{\pm}(h)$ given in (4.4.23). In the case derivatives fall on $\kappa(x/R^2)$, we have that $|x| \leq CR^2$ and we can proceed as above. Note that we have from (4.4.33) with $\pm y\cdot\xi/|y||\xi| > \sigma_3$ and $\pm t = \pm\sigma s|\xi|^{\sigma-2} \geq 0$ that

$$\pm\frac{(y+\sigma s\xi|\xi|^{\sigma-2})\xi}{|y+\sigma s\xi|\xi|^{\sigma-2}||\xi|} > \sigma_3 \text{ and } |y+\sigma s\xi|\xi|^{\sigma-2}| \geq c_{\pm}(|s|+|y|).$$

In the case derivatives fall on $\theta_{1\to 2}$, we have

$$\tau_1 + \varepsilon \leq \pm\frac{x\cdot\xi}{|x||\xi|} \leq \tau_2 - \varepsilon \text{ or } \mp\frac{x\cdot\xi}{|x||\xi|} \geq -\tau_2 + \varepsilon > -\tau_2 + \varepsilon/2.$$

By choosing $\varepsilon > 0$ small enough such that $\tau_3 - \tau_2 + \varepsilon/2 > 0$, (4.4.34) gives

$$|y+\sigma s\xi|\xi|^{\sigma-2} - x| \geq c\left(|y+\sigma s\xi|\xi|^{\sigma-2}| + |x|\right) \geq C(|s|+|x|+|y|).$$

Thus $|\nabla_\xi\Phi^{\pm}| \geq C(1+|s|+|x|+|y|)$ for $\pm s \geq 0$ and (4.4.36) follows as above.
2. The proof for the low frequency case is the same as above up to the conjugation by the unitary map D_ϵ in $L^2(\mathbb{R}^d)$. For instance, the kernel of the operator in the left hand side of (4.4.38) reads

$$K_\epsilon^{\pm}(s,x,y) = (2\pi)^{-d}\chi(x/R^2)\int_{\mathbb{R}^d} e^{i\Phi_\epsilon^{\pm}(R,s,x,y,\xi)}\check{a}^{\pm}(\epsilon,x,\xi)\overline{b_\epsilon^{\pm}(y,\xi)}d\xi\,\langle y\rangle^m,$$

where the phase $\Phi^\pm_\epsilon(R,s,x,y,\xi) = S^\pm_{\epsilon,R}(x,\xi) - \epsilon s|\xi|^\sigma - S^\pm_{\epsilon,R}(y,\xi)$. $\square$

Lemma 4.4.13. *Let $\sigma \in (0,\infty)$.*

1. *Under the notations of* Theorem 4.4.8*, for all $m \geq 0$ and all N large enough, there exists $C > 0$ such that for all $\pm s \geq 0$ and all $h \in (0,1]$,*

$$\| \langle x\rangle^{N/8} J^\pm_h(r^\pm_N(h))e^{-ish^{-1}(h\Lambda)^\sigma} J^\pm_h(b^\pm(h))^\star \langle x\rangle^{N/4} \|_{H^{-m}\to H^m} \leq Ch^{-d-2m} \langle s\rangle^{-N/4}. \tag{4.4.42}$$

2. *Under the notations of* Theorem 4.4.9*, for all N large enough, there exists $C > 0$ such that for all $\pm s \geq 0$ and all $\epsilon \in (0,1]$,*

$$\| \langle \epsilon x\rangle^{N/8} \mathcal{J}^\pm_\epsilon(r^\pm_N(\epsilon))e^{-i\epsilon s\Lambda^\sigma} \mathcal{J}^\pm_\epsilon(b^\pm_\epsilon)^\star \langle \epsilon x\rangle^{N/4} \|_{L^2\to L^2} \leq C \langle \epsilon s\rangle^{-N/4}. \tag{4.4.43}$$

Proof. We only give the proof for the high frequency case, the low frequency one is similar. The kernel of the operator in the left hand side of (4.4.42) reads

$$K^\pm_h(s,x,y) = (2\pi h)^{-d}\int_{\mathbb{R}^d} e^{ih^{-1}\Phi^\pm(R,s,x,y,\xi)}A^\pm(h,x,y,\xi)d\xi,$$

where the amplitude $A^\pm(h,x,y,\xi) = \langle x\rangle^{N/8} r^\pm_N(h,x,\xi)\overline{b^\pm(h,y,\xi)}\langle y\rangle^{N/4}$ and is compactly supported in ξ. We have from Proposition 4.4.1 and (4.4.41) that $\nabla_\xi\Phi^\pm(R,s,x,y,\xi) = x-\sigma s\xi|\xi|^{\sigma-2}-y+O(1)$ and $|\sigma s\xi|\xi|^{\sigma-2}+y| \geq C(|s|+|y|)$ for all $\pm s \geq 0$. By Peetre's inequality, we see that

$$\left\langle\nabla_\xi\Phi^\pm\right\rangle^{-1} \leq \langle x\rangle \left\langle y+\sigma s\xi|\xi|^{\sigma-2}\right\rangle^{-1} \leq C\langle x\rangle (\langle y\rangle + \langle s\rangle)^{-1}.$$

We next write

$$1 = \chi(\nabla_\xi\Phi^\pm) + (1-\chi)(\nabla_\xi\Phi^\pm),$$

where $\chi \in C^\infty_0(\mathbb{R}^d)$ with $\chi = 1$ near 0. Then $K^\pm_h(s,x,y)$ is split into two terms. For the first term

$$I_1 = (2\pi h)^{-d}\int_{\mathbb{R}^d} e^{ih^{-1}\Phi^\pm(R,s,x,y,\xi)}\chi(\nabla_\xi\Phi^\pm)A^\pm(h,x,y,\xi)d\xi,$$

by using the fact that

$$\begin{aligned}|\chi(\nabla_\xi\Phi^\pm)| &\leq C\left\langle\nabla_\xi\Phi^\pm\right\rangle^{-3N/4} \leq C\langle x\rangle^{3N/4}(\langle y\rangle+\langle s\rangle)^{-3N/4} \\ &\leq C\langle x\rangle^{3N/4}\langle y\rangle^{-N/2}\langle s\rangle^{-N/4},\end{aligned} \tag{4.4.44}$$

and $A^\pm(h,x,y,\xi) = O(\langle x\rangle^{-7N/8}\langle y\rangle^{N/4})$, it is bounded by $Ch^{-d}\langle x\rangle^{-N/8}\langle y\rangle^{-N/4}\langle s\rangle^{-N/4}$. For the second term

$$I_2 = (2\pi h)^{-d}\int_{\mathbb{R}^d} e^{ih^{-1}\Phi^\pm(R,s,x,y,\xi)}(1-\chi)(\nabla_\xi\Phi^\pm)A^\pm(h,x,y,\xi)d\xi,$$

thanks to the support of $(1-\chi)$, we can integrate by parts with respect to $\mathcal{L} := \frac{h\nabla_\xi\Phi^\pm}{i|\nabla_\xi\Phi^\pm|^2}\circ\nabla_\xi$ to get many negative powers of $|\nabla_\xi\Phi^\pm|$ as we wish and estimate as in (4.4.44). Combine two terms and Schur's lemma, we have (4.4.42) for $m = 0$. For $m \geq 1$, we can do the same with $\partial^\alpha_x\partial^\beta_y K^\pm_h(s,x,y)$ with $|\alpha| \leq m, |\beta| \leq m$. This completes the proof. $\square$

Combining Lemma 4.4.12 and Lemma 4.4.13, we have the following result.

Proposition 4.4.14. *1. Using the notations given in* Theorem 4.4.8*, for all $0 \leq m \leq d+1$ and all N large enough, we can write for $k = 2,3,4$,*

$$R^\pm_k(N,t,h) = h^{N/2}\int_0^t e^{-i(t-s)h^{-1}\omega(h^2P)}\langle x\rangle^{-N/8}B^\pm_m(N,s,h)\langle x\rangle^{-N/4}ds,$$

with

$$\|B_m^{\pm}(N,s,h)\|_{H^{-m}\to H^m} \leq C\langle s\rangle^{-N/4}, \tag{4.4.45}$$

for all $\pm s \geq 0$ *and* $h \in (0,1]$.

2. *Using the notations given in* Theorem 4.4.9 *and for all* N *large enough, we can write for* $k = 2,3,4$,

$$\mathcal{R}_k^{\pm}(N,t,\epsilon) = \epsilon \int_0^t e^{-i(t-s)\epsilon\omega(\epsilon^{-2}P)} \langle \epsilon x\rangle^{-N/8} \mathcal{B}_N^{\pm}(s,\epsilon) \langle \epsilon x\rangle^{-N/4} ds,$$

with

$$\|\mathcal{B}_N^{\pm}(s,\epsilon)\|_{L^2\to L^2} \leq C\langle \epsilon s\rangle^{-N/4}, \tag{4.4.46}$$

for all $\pm s \geq 0$ *and all* $\epsilon \in (0,1]$.

Proof. The cases $k = 3,4$ follow immediately from Lemma 4.4.12 and Lemma 4.4.13. It remains to show the case $k = 2$. Let us consider the high frequency case. We can write $R_N(h)E^{\pm}(h)$ as

$$\langle x\rangle^{-N/8}\Big(\langle x\rangle^{N/8} R_N(h)\langle x\rangle^{7N/8}\Big)\Big(\langle x\rangle^{N/8}\langle x\rangle^{-N} E^{\pm}(h)\langle x\rangle^{N/4}\Big)\langle x\rangle^{-N/4},$$

where $E^{\pm}(h) := J_h^{\pm}(a^{\pm}(h))e^{-ish^{-1}(h\Lambda)^{\sigma}} J_h^{\pm}(b^{\pm}(h))^{\star}$. The first bracket is bounded on L^2 using Proposition 4.1.2. The second one is bounded from H^{-m} to H^m using Lemma 4.4.13 with the fact that $\langle x\rangle^{-N} J_h^{\pm}(a^{\pm}(h)) = J_h^{\pm}(\tilde{r}_N^{\pm}(h))$ where $\tilde{r}_N^{\pm}(h)$ are bounded in $S(-N,-\infty)$. The low frequency case is similar using Proposition 4.1.7. □

Next, we have the following micro-local propagation estimates both at high and low frequencies.

Proposition 4.4.15. *Let* $\sigma \in (0,\infty)$, $f \in C_0^{\infty}(\mathbb{R}\backslash\{0\})$, $J_4 \Subset (0,+\infty)$ *be an open interval and* $-1 < \tau_4 < 1$.

1. *Consider* $\mathbb{R}^d, d \geq 2$ *equipped with a smooth metric* g *satisfying* (4.0.2), (4.0.3) *and suppose that* (4.0.11) *is satisfied. Then for* $R > 0$ *large enough and* $\chi^{\pm} \in S(0,-\infty)$ *supported in* $\Gamma^{\pm}(R^4, J_4, \tau_4)$, *we have the following estimates.*
   i. *For all* $m \in \mathbb{N}$ *and all integer* l *large enough, there exists* $C > 0$ *such that for all* $\pm t \leq 0$ *and all* $h \in (0,1]$,

$$\|Op^h(\chi^{\pm})^{\star} e^{-ith^{-1}(h|\nabla_g|)^{\sigma}} f(h^2P)\langle x\rangle^{-l}\|_{L^2\to H^m} \leq Ch^{-m}\langle t\rangle^{-3l/4}. \tag{4.4.47}$$

   ii. *For all* $m \in \mathbb{N}$, *all* $\chi \in C_0^{\infty}(\mathbb{R}^d)$ *and all* $l \geq 1$, *there exists* $C > 0$ *such that for all* $\pm t \leq 0$ *and all* $h \in (0,1]$,

$$\|Op^h(\chi^{\pm})^{\star} e^{-ith^{-1}(h|\nabla_g|)^{\sigma}} f(h^2P)\chi(x/R^2)\|_{L^2\to H^m} \leq Ch^l\langle t\rangle^{-l}. \tag{4.4.48}$$

   iii. *For all* $\tilde{\chi}^{\mp} \in S(0,-\infty)$ *supported in* $\Gamma^{\mp}(R, J_1, \tilde{\tau}_1)$ *with* $-\tau_4 < \tilde{\tau}_1 < 1$ *and* $J_4 \Subset J_1$ *and all* $l \geq 1$, *there exists* $C > 0$ *such that for all* $\pm t \leq 0$ *and all* $h \in (0,1]$,

$$\|Op^h(\chi^{\pm})^{\star} e^{-ith^{-1}(h|\nabla_g|)^{\sigma}} f(h^2P) Op^h(\tilde{\chi}^{\mp})\|_{L^{\infty}\to L^{\infty}} \leq Ch^l\langle t\rangle^{-l}. \tag{4.4.49}$$

2. *Consider* $\mathbb{R}^d, d \geq 3$ *equipped with a smooth metric* g *satisfying* (4.0.2), (4.0.3). *Let* $\zeta \in C^{\infty}(\mathbb{R}^d)$ *be supported outside* $B(0,1)$ *and equal to* 1 *near infinity. Then for* $R > 0$ *large enough and all* $(\chi_{\epsilon}^{\pm})_{\epsilon\in(0,1]}$ *bounded families in* $S(0,-\infty)$ *supported in* $\Gamma^{\pm}(R^4, J_4, \tau_4)$, *we have the following estimates.*
   i. *For all integer* l *large enough, there exists* $C > 0$ *such that for all* $\pm t \leq 0$ *and all*

$\epsilon \in (0,1]$,

$$\|\zeta(\epsilon x)Op_\epsilon(\chi_\epsilon^\pm)^\star e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma} f(\epsilon^{-2}P)\langle \epsilon x\rangle^{-l}\|_{L^2\to L^2} \le C\langle \epsilon t\rangle^{-3l/4}. \quad (4.4.50)$$

ii. For all $\chi \in C_0^\infty(\mathbb{R}^d)$ and all $l \ge 1$, there exists $C>0$ such that for all $\pm t \le 0$ and all $\epsilon \in (0,1]$,

$$\|\zeta(\epsilon x)Op_\epsilon(\chi_\epsilon^\pm)^\star e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma} f(\epsilon^{-2}P)\chi(\epsilon x/R^2)\|_{L^2\to L^2} \le C\langle \epsilon t\rangle^{-l}. \quad (4.4.51)$$

iii. For all $\tilde\zeta \in C^\infty(\mathbb{R}^d)$ supported outside $B(0,1)$ and equal to 1 near infinity and all $(\tilde\chi_\epsilon^\mp)_{\epsilon\in(0,1]}$ bounded families in $S(0,-\infty)$ supported in $\Gamma^\mp(R,J_1,\tilde\tau_1)$ with $-\tau_4<\tilde\tau_1<1$ and $J_4 \Subset J_1$ and all $l\ge 1$, there exists $C>0$ such that for all $\pm t \le 0$ and all $\epsilon\in(0,1]$,

$$\|\zeta(\epsilon x)Op_\epsilon(\chi_\epsilon^\pm)^\star e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma} f(\epsilon^{-2}P)Op_\epsilon(\tilde\chi_\epsilon^\mp)\tilde\zeta(\epsilon x)\|_{L^2\to L^2} \le C\langle \epsilon t\rangle^{-l}. \quad (4.4.52)$$

Proof. We only give the proof for the low frequency case, the proof at high frequency is similar and essentially given in [BT08, Proposition 4.5].

i. We only consider the case χ_ϵ^+ and $t\le 0$, the case χ_ϵ^- and $t \ge 0$ is similar. By taking the adjoint, (4.4.50) is equivalent to

$$\|\langle \epsilon x\rangle^{-l} f(\epsilon^{-2}P)e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma} Op_\epsilon(\chi_\epsilon^+)\zeta(\epsilon x)\|_{L^2\to L^2} \le C\langle \epsilon t\rangle^{-3l/4}, \quad t\ge 0, \quad (4.4.53)$$

uniformly in $\epsilon \in (0,1]$. Thanks to the spectral localization, we can apply the Isozaki-Kitada parametrix given in Theorem 4.4.9 and obtain

$$e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma} Op_\epsilon(\chi_\epsilon^+)\zeta(\epsilon x) = \mathcal{J}_\epsilon^+(a_\epsilon^+)e^{-it\epsilon\Lambda^\sigma}\mathcal{J}_\epsilon^+(b_\epsilon^+)^\star + \mathcal{R}_N^+(t,\epsilon).$$

The main term can be written as

$$\langle \epsilon x\rangle^{-l} f(\epsilon^{-2}P)\langle \epsilon x\rangle^{l}\langle \epsilon x\rangle^{-n}\langle \epsilon x\rangle^{n-l}\mathcal{J}_\epsilon^+(a_\epsilon^+)e^{-it\epsilon\Lambda^\sigma}\mathcal{J}_\epsilon^+(b_\epsilon^+)^\star\langle \epsilon x\rangle^{n}\langle \epsilon x\rangle^{-n}.$$

By using Corollary 4.1.9, we have the terms $\langle \epsilon x\rangle^{-l} f(\epsilon^{-2}P)\langle \epsilon x\rangle^{l}$ and $\langle \epsilon x\rangle^{-n}$ are bounded on L^2. It suffices to show for l large enough,

$$\|\langle \epsilon x\rangle^{n-l}\mathcal{J}_\epsilon^+(a_\epsilon^+)e^{-it\epsilon\Lambda^\sigma}\mathcal{J}_\epsilon^+(b_\epsilon^+)^\star\langle \epsilon x\rangle^{n}\|_{L^2\to L^2} \le C\langle \epsilon t\rangle^{-3l/4}, \quad t\ge 0,$$

uniformly in $\epsilon\in(0,1]$. This expected estimate follows by using the same process as in Lemma 4.4.13. We now study the remainders.

For $k=1$, we have

$$\begin{aligned}\|\langle \epsilon x\rangle^{-l} f(\epsilon^{-2}P)\mathcal{R}_1^+(N,t,\epsilon)\|_{L^2\to L^2} &= \|\langle \epsilon x\rangle^{-l} f(\epsilon^{-2}P)e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma} Op_\epsilon(\tilde r_N^+(\epsilon))\zeta(\epsilon x)\|_{L^2\to L^2}\\ &\le C\langle \epsilon t\rangle^{1-l}.\end{aligned}$$

Here we insert $\langle \epsilon x\rangle^{-l}\langle \epsilon x\rangle^{l}$ in the middle and use (4.1.23) and rescaled pseudo-differential calculus.

For $k=2,3,4$, Item 2 of Proposition 4.4.14 yields

$$\langle \epsilon x\rangle^{-l} f(\epsilon^{-2}P)\mathcal{R}_k^+(N,t,\epsilon) = \epsilon\int_0^t \langle \epsilon x\rangle^{-l} f(\epsilon^{-2}P)e^{-i(t-s)\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma}\langle \epsilon x\rangle^{-N/8}\mathcal{B}_N(s,\epsilon)\langle \epsilon x\rangle^{-N/4}ds.$$

Using again (4.1.23) and the fact that $\langle \epsilon x\rangle^{l-N/8}$ and $\langle \epsilon x\rangle^{-N/4}$ are of size $O_{L^2\to L^2}(1)$ for N large enough and (4.4.46), we obtain

$$\|\langle \epsilon x\rangle^{-l} f(\epsilon^{-2}P)\mathcal{R}_k^+(N,t,\epsilon)\|_{L^2\to L^2} \le C\epsilon\int_0^t \langle \epsilon(t-s)\rangle^{1-l}\langle \epsilon s\rangle^{-N/4}ds \le C\langle \epsilon t\rangle^{1-l}.$$

By choosing l large enough such that $l-1 \geq 3l/4$, it shows (4.4.53).

ii. We do the same for (4.4.51), it is equivalent to show

$$\|\chi(\epsilon x/R^2)f(\epsilon^{-2}P)e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma}Op_\epsilon(\chi^+_\epsilon)\zeta(\epsilon x)\|_{L^2\to L^2} \leq C\langle \epsilon t\rangle^{-l}, \quad t\geq 0, \tag{4.4.54}$$

uniformly in $\epsilon \in (0,1]$. We again use the Isozaki-Kitada parametrix. Let us firstly study remainder terms. We write the first remainder term $\chi(\epsilon x/R^2)f(\epsilon^{-2}P)\mathcal{R}^+_1(N,t,\epsilon)$ as

$$\chi(\epsilon x/R^2)\langle \epsilon x\rangle^{l}\langle \epsilon x\rangle^{-l} f(\epsilon^{-2}P)e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma}\langle \epsilon x\rangle^{-l}\langle \epsilon x\rangle^{l} Op_\epsilon(\tilde{r}^+_N(\epsilon))\zeta(\epsilon x).$$

Using (4.1.23) and the fact that $\chi(\epsilon x/R^2)\langle \epsilon x\rangle^l$ and $\langle \epsilon x\rangle^l Op_\epsilon(\tilde{r}^+_N(\epsilon))\zeta(\epsilon x)$ are bounded on L^2 due to the support property of χ and rescaled pseudo-differential calculus given as in Proposition 4.1.7, we get

$$\|\chi(\epsilon x/R^2)f(\epsilon^{-2}P)\mathcal{R}^+_1(N,t,\epsilon)\|_{L^2\to L^2}\leq C\langle \epsilon t\rangle^{1-l}.$$

For $k=2,3,4$, we have

$$\|\chi(\epsilon x/R^2)f(\epsilon^{-2}P)\mathcal{R}^+_k(N,t,\epsilon)\|_{L^2\to L^2}\leq C\epsilon\int_0^t \langle \epsilon(t-s)\rangle^{1-l}\langle \epsilon s\rangle^{-N/4}ds \leq C\langle \epsilon t\rangle^{1-l}.$$

For the main term, we can write

$$\chi(\epsilon x/R^2)\langle \epsilon x\rangle^{l}\langle \epsilon x\rangle^{-l}f(\epsilon^{-2}P)\langle \epsilon x\rangle^{l}\langle \epsilon x\rangle^{-n}\langle \epsilon x\rangle^{n-l}\mathcal{J}^+_\epsilon(a^+_\epsilon)e^{-it\epsilon\Lambda^\sigma}\mathcal{J}^+_\epsilon(b^+_\epsilon)^\star\langle \epsilon x\rangle^{n}\langle \epsilon x\rangle^{-n}.$$

Thanks to the L^2-boundedness of $\chi(\epsilon x/R^2)\langle \epsilon x\rangle^l$, $\langle \epsilon x\rangle^{-l}f(\epsilon^{-2}P)\langle \epsilon x\rangle^l$, $\langle \epsilon x\rangle^{-n}$, it suffices to prove

$$\|\langle \epsilon x\rangle^{n-l}\mathcal{J}^+_\epsilon(a^+_\epsilon)e^{-it\epsilon\Lambda^\sigma}\mathcal{J}^+_\epsilon(b^+_\epsilon)^\star\langle \epsilon x\rangle^{n}\|_{L^2\to L^2}\leq C\langle \epsilon t\rangle^{-l}, \quad t\geq 0,$$

uniformly in $\epsilon\in(0,1]$. This expected estimate again follows from Lemma 4.4.12 by taking l large enough. This proves (4.4.54).

iii. For (4.4.52), we firstly use the Isozaki-Kitada parametrix for $\tilde{\chi}^-_\epsilon$, namely

$$e^{-it\epsilon\omega(\epsilon^{-2}P)}Op_\epsilon(\tilde{\chi}^-_\epsilon)\tilde{\zeta}(\epsilon x) = \mathcal{J}^-_\epsilon(\tilde{a}^-_\epsilon)e^{-it\epsilon\Lambda^\sigma}\mathcal{J}^-_\epsilon(\tilde{b}^-_\epsilon)^\star + \sum_{k=1}^4 \tilde{\mathcal{R}}^-_k(N,t,\epsilon), \tag{4.4.55}$$

where $\mathrm{supp}(\tilde{a}^-_\epsilon)\subset \Gamma^-(R^{1/4},\tilde{J}_{1/4},\tilde{\tau}_{1/4})$ and $\mathrm{supp}(\tilde{b}^-_\epsilon)\subset\Gamma^-(R^{3/4},\tilde{J}_{3/4},\tilde{\tau}_{3/4})$ with $\tilde{J}_{3/4}\Subset\tilde{J}_{1/4}$ small neighborhood of J_1 and $\tilde{\tau}_{1/4},\tilde{\tau}_{3/4}$ can be chosen so that

$$-1<-\tau_4<\tilde{\tau}_{1/4}<\tilde{\tau}_{3/4}<\tilde{\tau}_1<1.$$

Multiplying $\zeta(\epsilon x)Op_\epsilon(\chi^+_\epsilon)^\star f(\epsilon^{-2}P)$ to the left of (4.4.55), we see that the terms $\zeta(\epsilon x)Op_\epsilon(\chi^+_\epsilon)^\star f(\epsilon^{-2}P)\tilde{\mathcal{R}}^-_k(N,t,\epsilon)$ for $k=1,2,3,4$ satisfy the required estimate using the estimate (4.4.50), Lemma 4.4.12 and (4.4.46). Therefore, it remains to show

$$\|\zeta(\epsilon x)Op_\epsilon(\chi^+_\epsilon)^\star f(\epsilon^{-2}P)\mathcal{J}^-_\epsilon(\tilde{a}^-_\epsilon)e^{-it\epsilon\Lambda^\sigma}\mathcal{J}^-_\epsilon(\tilde{b}^-_\epsilon)^\star\|_{L^2\to L^2}\leq C\langle \epsilon t\rangle^{-l}, \quad \pm t\leq 0,$$

uniformly in $\epsilon\in(0,1]$. Thanks to the support of $\tilde{a}^-_\epsilon$, we can write $\mathcal{J}^-_\epsilon(\tilde{a}^-_\epsilon)=\zeta_1(\epsilon x)\mathcal{J}^-_\epsilon(\tilde{a}^-_\epsilon)$ with $\zeta_1\in C^\infty(\mathbb{R}^d)$ supported outside $B(0,1)$ such that $\zeta_1(x)=1$ for $|x|>R^{1/4}$. The parametrix of $f(\epsilon^{-2}P)\zeta_1(\epsilon x)$ given in Proposition 4.1.7 and symbolic calculus give

$$\zeta(\epsilon x)Op_\epsilon(\chi^+_\epsilon)^\star f(\epsilon^{-2}P)\zeta_1(\epsilon x) = Op_\epsilon(c^+_\epsilon)+B^+_N(\epsilon)\langle \epsilon x\rangle^{-N},$$

where $(c^+_\epsilon)_{\epsilon\in(0,1]}\in S(0,-\infty)$ with $\mathrm{supp}(c^+_\epsilon)\subset\mathrm{supp}(\chi^+_\epsilon)$ and $B^+_N(\epsilon)=O_{L^2\to L^2}(1)$ uniformly in $\epsilon\in(0,1]$. We treat the remainder term by using Lemma 4.4.13. For the main terms, we need to recall the following version of Proposition 4.4.4 which is essentially [2] given in [BT08, Lemma 4.6].

Lemma 4.4.16. *Given $J \Subset (0,+\infty)$, $-1 < \tau < 1$ and the associated families of phase functions $(S^{\pm}_{\epsilon,R})_{R\gg 1}$ as in* Proposition 4.4.1. *Let $(a_\epsilon)_{\epsilon\in(0,1]}$ and $(c_\epsilon)_{\epsilon\in(0,1]}$ be bounded families in $S(0,-\infty)$. Then for all $N \geq 1$,*

$$Op_\epsilon(c_\epsilon)\mathcal{J}^{\pm}_\epsilon(a_\epsilon) = \sum_{j=0}^{N-1} \mathcal{J}^{\pm}_\epsilon(e_{\epsilon,j}) + \mathcal{J}^{\pm}_\epsilon(e_N(\epsilon)),$$

where $(e_{\epsilon,j})_{\epsilon\in(0,1]}$ and $(e_N(\epsilon))_{\epsilon\in(0,1]}$ are bounded families in $S(0,-\infty)$ and $S(-N,-\infty)$ respectively. In particular, for all $\varepsilon > 0$ small enough, by choosing $R > 0$ large enough, we have

$$\mathrm{supp}(c_\epsilon) \subset \Gamma^{\pm}(R,J,\tau) \Longrightarrow \mathrm{supp}(e_{\epsilon,j}) \subset \Gamma^{\pm}(R, J+(-\varepsilon,\varepsilon), \tau-\varepsilon)$$

since $\nabla_x S^{\pm}_{\epsilon,R}(x,\xi) = \xi + O(R^{-\rho})$.

Using this lemma, we expand $Op_\epsilon(c^+_\epsilon)J^-_\epsilon(\tilde{a}^-_\epsilon)$ and treat the remainder terms using again Lemma 4.4.13. It remains to prove the required estimate for the general term, namely

$$\|\mathcal{J}^-_\epsilon(e^+_\epsilon)e^{-it\epsilon\Lambda^\sigma}\mathcal{J}^-_\epsilon(\tilde{b}^-_\epsilon)^\star\|_{L^2\to L^2} \leq C\langle \epsilon t\rangle^{-l}, \quad \pm t \leq 0,$$

uniformly in $\epsilon \in (0,1]$, where $(e^+_\epsilon)_{\epsilon\in(0,1]} \in S(0,-\infty)$ and $\mathrm{supp}(e^+_\epsilon) \in \Gamma^+(R^4, J_4+(-\varepsilon,\varepsilon), \tau_4-\varepsilon)$. Up to the conjugation by D_ϵ, the kernel of the left hand side operator reads

$$K_\epsilon(t,x,y) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i\Phi_\epsilon(R,t,x,y,\xi)} e^+_\epsilon(x,\xi)\overline{\tilde{b}^-_\epsilon(y,\xi)}d\xi,$$

where $\Phi_\epsilon(R,t,x,y,\xi) = S^-_{\epsilon,R}(x,\xi) - \epsilon t|\xi|^\sigma - S^-_{\epsilon,R}(y,\xi)$. Since $\mathrm{supp}(e^+_\epsilon) \subset \Gamma^+(R^4, J_4+(-\varepsilon,\varepsilon), \tau_4-\varepsilon)$ and $\mathrm{supp}(\tilde{b}^-_\epsilon) \subset \Gamma^-(R^{3/4}, \tilde{J}_{3/4}, \tilde{\tau}_{3/4})$, we have

$$\frac{x\cdot\xi}{|x||\xi|} > \tau_4 - \varepsilon, \quad -\frac{y\cdot\xi}{|y||\xi|} > \tilde{\tau}_{3/4}.$$

By choosing $R > 0$ large enough, we have that $\tau_4 - \varepsilon + \tilde{\tau}_{3/4} > 0$. Thus by Item 2 of Lemma 4.4.11, we have

$$|\nabla_\xi \Phi_\epsilon| \geq C(1+\epsilon|t| + |x| + |y|).$$

Using the non-stationary phase argument as in the proof of Lemma 4.4.12, we have

$$\|\mathcal{J}^+_\epsilon(e^+_\epsilon)e^{-it\epsilon\Lambda^\sigma}\mathcal{J}^-_\epsilon(\tilde{b}^-_\epsilon)^\star\|_{L^2\to L^2} \leq C\langle \epsilon t\rangle^{-l}, \quad \pm t \leq 0,$$

uniformly in $\epsilon \in (0,1]$. The proof of Proposition 4.4.15 is now complete. □

4.4.2 Strichartz estimates

High frequencies. In this paragraph, we give the proof of (4.2.21). By scaling in time, it is in turn equivalent to prove

$$\|(1-\chi)e^{-ith^{-1}(h|\nabla_g|)^\sigma} f(h^2P)\psi\|_{L^p(\mathbb{R},L^q)} \leq Ch^{-\kappa_{p,q}}\|f(h^2P)\psi\|_{L^2},$$

where $\kappa_{p,q} = d/2 - d/q - 1/p$. By choosing $\tilde{f} \in C^\infty_0(\mathbb{R}\backslash 0)$ such that $\tilde{f} = 1$ near $\mathrm{supp}(f)$, we can write for all $l \in \mathbb{N}$,

$$(1-\chi)\tilde{f}(h^2P) = \sum_{k=0}^{N-1} h^k Op^h(a_k)^\star + h^N B_N(h)\langle x\rangle^{-l},$$

[2] See (4.1.4), (4.4.3) and use Lemma 4.6 of [BT08] with $h = 1$.

where for $q \geq 2$,

$$\|B_N(h)\|_{L^2 \to L^q} \leq Ch^{-(d/2-d/q)}. \tag{4.4.56}$$

Thus $(1-\chi)e^{-ith^{-1}(h|\nabla_g|)^\sigma} f(h^2P)\psi$ becomes

$$\sum_{k=0}^{N-1} h^k Op^h(a_k)^\star e^{-ith^{-1}(h|\nabla_g|)^\sigma} f(h^2P)\psi + h^N B_N(h) \langle x\rangle^{-l} e^{-ith^{-1}(h|\nabla_g|)^\sigma} f(h^2P)\psi.$$

Using (4.4.56) and (4.1.20), $\|B_N(h)\langle x\rangle^{-l} e^{-ith^{-1}(h|\nabla_g|)^\sigma} f(h^2P)\psi\|_{L^p(\mathbb{R},L^q)}$ is bounded by

$$Ch^{-(d/2-d/q)}\| \langle x\rangle^{-l} e^{-ith^{-1}(h|\nabla_g|)^\sigma} f(h^2P)\psi\|_{L^p(\mathbb{R},L^2)} \leq Ch^{-(d/2-d/q)+(1-N_0)/p}\|f(h^2P)\psi\|_{L^2}.$$

Hence, by taking N large enough, the remainder is bounded by $Ch^{-\kappa_{p,q}}\|f(h^2P)\psi\|_{L^2}$. For the main terms, by choosing $\chi_0 \in C_0^\infty(\mathbb{R}^d)$ such that $\chi_0 = 1$ for $|x| \leq 2$ and setting $\chi(x) = \chi_0(x/R^4)$, we see that $(1-\chi)$ is supported in $\{x \in \mathbb{R}^d, |x| \geq 2R^4 > R^4\}$. For $R > 0$ large enough and $\mathrm{supp}(\tilde{f})$ close enough to $\mathrm{supp}(f)$ and $J_4 \Subset (0,+\infty)$ any open interval containing $\mathrm{supp}(f)$, we have

$$\mathrm{supp}(a_k) \subset \left\{(x,\xi) \in \mathbb{R}^{2d}, |x| > R^4, |\xi|^2 \in J_4\right\}, \quad k = 0, ..., N-1. \tag{4.4.57}$$

We want to show

$$\|Op^h(a_k)^\star e^{-ith^{-1}(h|\nabla_g|)^\sigma} f(h^2P)\psi\|_{L^p(\mathbb{R},L^q)} \leq Ch^{-\kappa_{p,q}}\|f(h^2P)\psi\|_{L^2}, \quad k = 0, ..., N-1.$$

Let us consider a general term, namely $Op^h(a)^\star e^{-ith^{-1}(h|\nabla_g|)^\sigma} f(h^2P)\psi$ with $a \in S(0,-\infty)$ satisfying (4.4.57). Next, by choosing a suitable partition of unity $\theta^- + \theta^+ = 1$ such that $\mathrm{supp}(\theta^-) \subset (-\infty, -\tau_4)$ and $\mathrm{supp}(\theta^+) \subset (\tau_4, +\infty)$ and setting

$$\chi^\pm(x,\xi) = a(x,\xi)\theta^\pm\left(\pm\frac{x\cdot\xi}{|x||\xi|}\right),$$

we have that $\chi^\pm \in S(0,-\infty)$, $\mathrm{supp}(\chi^\pm) \subset \Gamma^\pm(R^4, J_4, \tau_4)$ and

$$Op^h(a)^\star e^{-ith^{-1}(h|\nabla_g|)^\sigma} f(h^2P)\psi = (Op^h(\chi^-)^\star + Op^h(\chi^+)^\star)e^{-ith^{-1}(h|\nabla_g|)^\sigma} f(h^2P)\psi.$$

We only prove the estimate for χ^+, i.e.

$$\|Op^h(\chi^+)^\star e^{-ith^{-1}(h|\nabla_g|)^\sigma} f(h^2P)\psi\|_{L^p(\mathbb{R},L^q)} \leq Ch^{-\kappa_{p,q}}\|f(h^2P)\psi\|_{L^2},$$

the one for χ^- is similar. Since $Op^h(\chi^+)^\star e^{-ith^{-1}(h|\nabla_g|)^\sigma} f(h^2P)$ is bounded on L^2 uniformly in $h \in (0,1]$ and $t \in \mathbb{R}$, by Proposition 4.3.3, it suffices to prove the dispersive estimates, i.e.

$$\|Op^h(\chi^+)^\star e^{-ith^{-1}(h|\nabla_g|)^\sigma} f^2(h^2P)Op^h(\chi^+)\|_{L^1\to L^\infty} \leq Ch^{-d}(1+|t|h^{-1})^{-d/2},$$

for all $t \in \mathbb{R}$ uniformly in $h \in (0,1]$. By taking the adjoint, it reduces to prove

$$\|Op^h(\chi^+)^\star e^{-ith^{-1}(h|\nabla_g|)^\sigma} f^2(h^2P)Op^h(\chi^+)\|_{L^1\to L^\infty} \leq Ch^{-d}(1+|t|h^{-1})^{-d/2}, \tag{4.4.58}$$

for all $t \leq 0$ uniformly in $h \in (0,1]$. We now prove (4.4.58). By using the Isozaki-Kitada parametrix with J_4 and τ_4 as above together with arbitrary open intervals J_1, J_2, J_3 such that $J_4 \Subset J_3 \Subset J_2 \Subset J_1 \Subset (0,+\infty)$ and arbitrary real numbers τ_1, τ_2, τ_3 satisfying $-1 < \tau_1 < \tau_2 < \tau_3 < \tau_4 < 1$, the operator in the left hand side of (4.4.58) is written as

$$Op^h(\chi^+)^\star f^2(h^2P)\left(J_h^+(a^+(h))e^{-ith^{-1}(h\Lambda)^\sigma}J_h^+(b^+(h))^\star+\sum_{k=1}^4 R_k^+(N,t,h)\right).$$

Using the fact that $Op^h(\chi^+)^\star f^2(h^2P)$ is bounded on L^∞ and Proposition 4.4.10, we have

$$\|Op^h(\chi^+)^\star f^2(h^2P)J_h^+(a^+(h))e^{-ith^{-1}(h\Lambda)^\sigma}J_h^+(b^+(h))^\star\|_{L^1\to L^\infty}\leq Ch^{-d}(1+|t|h^{-1})^{-d/2},$$

for all $t\in\mathbb{R}$ and $h\in(0,1]$. It remains to study the remainder terms.

For $k=1$, using the Sobolev embedding with $m>d/2$, (4.4.47) and the fact that $\langle x\rangle^l Op^h(\tilde{r}_N^+(h))$ is of size $O_{H^{-m}\to L^2}(h^{-m})$ by pseudo-differential calculus, we have

$$\|Op^h(\chi^+)^\star f^2(h^2P)R_1^+(N,t,h)\|_{L^1\to L^\infty}\leq Ch^{N-1-2m}\langle t\rangle^{-3l/4}\leq Ch^{-d}(1+|t|h^{-1})^{-d/2},$$

for all $t\leq 0$ and all $h\in(0,1]$. The last estimate follows by taking $l=2d/3$ and N large enough.

For $k=2$, by using (4.4.47) and the Sobolev embedding with $m>d/2$, we have for $t-s\leq 0$,

$$\|Op^h(\chi^+)^\star e^{-i(t-s)h^{-1}(h|\nabla_g|)^\sigma}f^2(h^2P)\langle x\rangle^{-l}\|_{L^2\to L^\infty}\leq Ch^{-m}\langle t-s\rangle^{-3l/4}. \tag{4.4.59}$$

We also have that $\langle x\rangle^l R_N(h)$ is bounded from L^∞ to L^2 due to Proposition 4.1.2 provided $N>l$. Thus for N and l large enough, Proposition 4.4.10 implies that

$$\begin{aligned}\|Op^h(\chi^+)^\star f^2(h^2P)R_2^+(N,t,h)\|_{L^1\to L^\infty}&\\ \leq Ch^{N-1-m-d}\int_0^t\langle t-s\rangle^{-3l/4}(1+|s|h^{-1})^{-d/2}ds&\leq Ch^{-d}(1+|t|h^{-1})^{-d/2}.\end{aligned}$$

For $k=3$, by inserting $\langle x\rangle^{-l}\langle x\rangle^{l-N}\langle x\rangle^N$ and using the fact that $\langle x\rangle^{l-N}=O_{L^\infty\to L^2}(1)$ for N large enough, (4.4.59) and Proposition 4.4.10 with $J_h^+(a^+)=\langle x\rangle^N J_h^+(r_N^+(h))$, we see that this remainder term satisfies the required estimate as for the second one.

For $k=4$, we rewrite $Op^h(\chi^+)^\star f^2(h^2P)R_4^+(N,t,h)$ as $-ih^{-1}$ times

$$\int_0^t Op^h(\chi^+)^\star f^2(h^2P)e^{-i(t-s)h^{-1}(h|\nabla_g|)^\sigma}(\chi+(1-\chi))(x/R^2)J_h^+(\check{a}^+(h))e^{-ish^{-1}(h\Lambda)^\sigma}J_h^+(b^+(h))^\star ds,$$

where $\chi\in C_0^\infty(\mathbb{R}^d)$ satisfying $\chi(x)=1$ for $|x|\leq 2$. The first term can be treated similarly as the second remainder using (4.4.48) instead of (4.4.47). For the second term, we need the following lemma (see [BT08, Proposition 5.2]).

Lemma 4.4.17. *Choose $\tilde{\tau}_1$ such that $-\tau_4<\tilde{\tau}_1<-\tau_2$. If $R>0$ is large enough, we may choose $\tilde{\chi}^-\in S(0,-\infty)$ satisfying $\operatorname{supp}(\tilde{\chi}^-)\subset\Gamma^-(R,J_1,\tilde{\tau}_1)$ such that for all m large enough,*

$$f(h^2P)(1-\chi)(x/R^2)J_h^+(\check{a}^+(h))=Op^h(\tilde{\chi}^-)J_h^+(\tilde{e}_m(h))+h^m\tilde{R}_m(h)$$

where

$$\tilde{R}_m(h)=J_h^+(\tilde{r}_m(h))+\langle x\rangle^{-m/2}R_m(h)\langle x\rangle^{-m/2}J_h^+(\check{a}^+(h)),$$

with $(\tilde{e}_m(h))_{h\in(0,1]}$ and $(\tilde{r}_m(h))_{h\in(0,1]}$ bounded families in $S(0,-\infty)$ and $S(-m,-\infty)$ respectively and $R_m(h)=O_{L^\infty\to L^\infty}(1)$ uniformly in $h\in(0,1]$.

Using this lemma, the second term is written as $-ih^{-1}$ times

$$\int_0^t Op^h(\chi^+)^\star e^{-i(t-s)h^{-1}(h|\nabla_g|)^\sigma}\left(Op^h(\tilde{\chi}^-)J_h^+(\tilde{e}_m(h))+h^m\tilde{R}_m(h)\right)e^{-ish^{-1}(h\Lambda)^\sigma}J_h^+(b^+(h))^\star ds.$$

The remainder terms are treated similarly as the second remainder term using (4.4.47). The term involving $Op^h(\tilde{\chi}^-)J_h^+(\tilde{e}_m(h))$ is studied by the same analysis as the second term using (4.4.49) instead of (4.4.47). This completes the proof. □

Low frequencies. In this paragraph, we will prove (4.2.24). By scaling in time, it is equivalent to show

$$\|(1-\chi)(\epsilon x)f(\epsilon^{-2}P)e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma}\psi\|_{L^p(\mathbb{R},L^q)} \leq C\epsilon^{\kappa_{p,q}}\|f(\epsilon^{-2}P)\psi\|_{L^2},$$

where $\kappa_{p,q}=d/2-d/q-1/p$. By choosing $\tilde{\tilde{f}}\in C_0^\infty(\mathbb{R}\backslash 0)$ such that $\tilde{\tilde{f}}=1$ near $\mathrm{supp}(f)$, we can write $(1-\chi)(\epsilon x)f(\epsilon^{-2}P)=(1-\chi)(\epsilon x)\tilde{\tilde{f}}(\epsilon^{-2}P)f(\epsilon^{-2}P)$. Next, we choose $\zeta\in C^\infty(\mathbb{R}^d)$ supported in $\mathbb{R}^d\backslash B(0,1)$ such that $\zeta=1$ near $\mathrm{supp}(1-\chi)$ and use Proposition 4.1.7 to have

$$(1-\chi)(\epsilon x)\tilde{\tilde{f}}(\epsilon^{-2}P)=\sum_{k=0}^{N-1}\zeta(\epsilon x)Op_\epsilon(a_{\epsilon,k})^\star+R_N(\epsilon),$$

where $R_N(\epsilon)=\zeta(\epsilon x)(\epsilon^{-2}P+1)^{-N}B_N(\epsilon)\langle\epsilon x\rangle^{-N}$ with $(B_N(\epsilon))_{\epsilon\in(0,1]}$ bounded on L^2. Thus $(1-\chi)(\epsilon x)f(\epsilon^{-2}P)e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma}\psi$ reads

$$\sum_{k=0}^{N-1}\zeta(\epsilon x)Op_\epsilon(a_{\epsilon,k})^\star e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma}f(\epsilon^{-2}P)\psi+R_N(\epsilon)e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma}f(\epsilon^{-2}P)\psi.$$

We firstly consider the remainder term.

Proposition 4.4.18. *Let $N\geq(d-1)/2+1$. Then for all (p,q) Schrödinger admissible with $q<\infty$, there exists $C>0$ such that for all $\epsilon\in(0,1]$,*

$$\|R_N(\epsilon)e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma}f(\epsilon^{-2}P)\psi\|_{L^p(\mathbb{R},L^q)}\leq C\epsilon^{\kappa_{p,q}}\|\psi\|_{L^2}.$$

Proof. This result follows from the $TT^\star$ criterion given in Proposition 4.3.3 with ϵ^{-1} in place of h and $T(t)=R_N(\epsilon)e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma}f(\epsilon^{-2}P)$. The $L^2\to L^2$ bounds of $T(t)$ are obvious. Thus we need to prove the dispersive estimates. Using (4.1.6) with $q=\infty$ and (4.1.23) with $N\geq d/2+1$, we have

$$\begin{aligned}\|T(t)T(s)^\star\|_{L^1\to L^\infty}&\leq C\epsilon^d\|\langle\epsilon x\rangle^{-N}e^{-i(t-s)\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma}f^2(\epsilon^{-2}P)\langle\epsilon x\rangle^{-N}\|_{L^2\to L^2}\\&\leq C\epsilon^d\langle\epsilon(t-s)\rangle^{1-N}\leq C\epsilon^d(1+\epsilon|t-s|)^{-d/2}.\end{aligned}$$

This completes the proof. □

For the main terms, by choosing $\chi_0\in C_0^\infty(\mathbb{R}^d)$ such that $\chi_0=1$ for $|x|\leq 2$ and setting $\chi(x)=\chi_0(x/R^4)$, we see that $(1-\chi)$ is supported in $\{x\in\mathbb{R}^d,|x|>R^4\}$. For $R>0$ large enough and $\mathrm{supp}(\tilde{f})$ close enough to $\mathrm{supp}(f)$ and $J_4\Subset(0,+\infty)$ any open interval containing $\mathrm{supp}(f)$, we have

$$\mathrm{supp}(a_{\epsilon,k})\subset\left\{(x,\xi)\in\mathbb{R}^{2d},|x|>R^4,|\xi|^2\in J_4\right\},\quad k=0,...,N-1. \tag{4.4.60}$$

We want to show for $k=0,...,N-1$,

$$\|\zeta(\epsilon x)Op_\epsilon(a_{\epsilon,k})^\star e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma}f(\epsilon^{-2}P)\psi\|_{L^p(\mathbb{R},L^q)}\leq C\epsilon^{\kappa_{p,q}}\|f(\epsilon^{-2}P)\psi\|_{L^2}.$$

Let us consider the general term, namely $\zeta(\epsilon x)Op_\epsilon(a_\epsilon)^\star e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma}f(\epsilon^{-2}P)\psi$ with $(a_\epsilon)_{\epsilon\in(0,1]}\in S(0,-\infty)$ satisfying (4.4.60). Next, by choosing a suitable partition of unity $\theta^-+\theta^+=1$ such that $\mathrm{supp}(\theta^-)\subset(-\infty,-\tau_4)$ and $\mathrm{supp}(\theta^+)\subset(\tau_4,+\infty)$ and setting

$$\chi_\epsilon^\pm(x,\xi)=a_\epsilon(x,\xi)\theta^\pm\left(\pm\frac{x\cdot\xi}{|x||\xi|}\right),$$

we have that $(\chi_\epsilon^\pm)_{\epsilon\in(0,1]} \in S(0,-\infty)$, $\mathrm{supp}(\chi_\epsilon^\pm) \subset \Gamma^\pm(R^4, J_4, \tau_4)$ and

$$\zeta(\epsilon x)Op_\epsilon(a_\epsilon)^\star e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma} f(\epsilon^{-2}P)\psi = \zeta(\epsilon x)(Op_\epsilon(\chi_\epsilon^-)^\star + Op_\epsilon(\chi_\epsilon^+)^\star)e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma} f(\epsilon^{-2}P)\psi.$$

We only prove the estimate for χ_ϵ^+, i.e.

$$\|\zeta(\epsilon x)Op_\epsilon(\chi_\epsilon^+)^\star e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma} f(\epsilon^{-2}P)\psi\|_{L^p(\mathbb{R},L^q)} \leq C\epsilon^{\kappa_{p,q}}\|f(\epsilon^{-2}P)\psi\|_{L^2},$$

the one for χ_ϵ^- is similar. By $TT^\star$ criterion and that $T(t) := \zeta(\epsilon x)Op_\epsilon(\chi_\epsilon^+)^\star e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma} f(\epsilon^{-2}P)$ is bounded on L^2 for all $t \in \mathbb{R}$ and all $\epsilon \in (0,1]$, it suffices to prove dispersive estimates, i.e.

$$\|\zeta(\epsilon x)Op_\epsilon(\chi_\epsilon^+)^\star e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma} f^2(\epsilon^{-2}P)Op_\epsilon(\chi_\epsilon^+)\zeta(\epsilon x)\|_{L^1\to L^\infty} \leq C\epsilon^d(1+\epsilon|t|)^{-d/2},$$

for all $t \in \mathbb{R}$ uniformly in $\epsilon \in (0,1]$. By taking the adjoint, it reduces to prove

$$\|\zeta(\epsilon x)Op_\epsilon(\chi_\epsilon^+)^\star e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma} f^2(\epsilon^{-2}P)Op_\epsilon(\chi_\epsilon^+)\zeta(\epsilon x)\|_{L^1\to L^\infty} \leq C\epsilon^d(1+\epsilon|t|)^{-d/2}, \qquad (4.4.61)$$

for all $t \leq 0$ uniformly in $\epsilon \in (0,1]$. Let us prove (4.4.61). For simplicity, we set

$$A_\epsilon^+ := \zeta(\epsilon x)Op_\epsilon(\chi_\epsilon^+)^\star f^2(\epsilon^{-2}P).$$

Using the Isozaki-Kitada parametrix given in Theorem 4.4.9, we see that

$$A_\epsilon^+ e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma} Op_\epsilon(\chi_\epsilon^+)\zeta(\epsilon x) = A_\epsilon^+\left(\mathcal{J}_\epsilon^+(a_\epsilon^+)e^{-it\epsilon\Lambda^\sigma}\mathcal{J}_\epsilon^+(b_\epsilon^+)^\star + \sum_{k=1}^4 \mathcal{R}_k^+(N,t,\epsilon)\right).$$

We firstly note that A_ϵ^+ is bounded on L^∞. Indeed, we write

$$\zeta(\epsilon x)Op_\epsilon(\chi_\epsilon^+)^\star f^2(\epsilon^{-2}P) = \zeta(\epsilon x)Op_\epsilon(\chi_\epsilon^+)^\star\zeta_1(\epsilon x)f^2(\epsilon^{-2}P),$$

where $\zeta_1 \in C^\infty(\mathbb{R}^d)$ is supported outside $B(0,1)$ satisfying $\zeta_1(x) = 1$ for $|x| > R^4$. This is possible since $Op_\epsilon(\chi_\epsilon^+) = \zeta_1(\epsilon x)Op_\epsilon(\chi_\epsilon^+)$. The factors $\zeta(\epsilon x)Op_\epsilon(\chi_\epsilon^+)^\star$ and $\zeta_1(\epsilon x)f^2(\epsilon^{-2}P)$ are bounded in $\mathcal{L}(L^\infty)$ by the rescaled pseudo-differential operator and Corollary 4.1.8 respectively. Thanks to the $\mathcal{L}(L^\infty)$-bound of A_ϵ^+ and (4.4.29), we have dispersive estimates for the main terms. It remains to prove dispersive estimates for remainder terms. By rescaled pseudo-differential calculus, we can write for $l > d/2$,

$$A_\epsilon^+ = \tilde{\zeta}(\epsilon x)(\epsilon^{-2}P+1)^{-l}\left(\zeta(\epsilon x)Op_\epsilon(\tilde{\chi}_\epsilon^+)^\star + \tilde{B}_l^+(\epsilon)\langle\epsilon x\rangle^{-l}\right)f^2(\epsilon^{-2}P),$$

where $\tilde{\zeta} \in C^\infty(\mathbb{R}^d)$ is supported outside $B(0,1)$ and equal to 1 near $\mathrm{supp}(\zeta)$ and $(\tilde{\chi}_\epsilon^+)_{\epsilon\in(0,1]} \in S(0,-\infty)$ satisfying $\mathrm{supp}(\tilde{\chi}_\epsilon^+) \subset \mathrm{supp}(\chi_\epsilon^+)$ and $\tilde{B}_l^+(\epsilon) = O_{L^2\to L^2}(1)$ uniformly in $\epsilon \in (0,1]$. This follows by expanding $(\epsilon^{-2}P+1)^l\zeta(\epsilon x)Op_\epsilon(\chi_\epsilon^+)^\star$ by rescaled pseudo-differential calculus.

For $k = 1$, using the Proposition 4.1.7, we can write

$$\mathcal{R}_1^+(N,t,\epsilon) = e^{-it\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma}\langle\epsilon x\rangle^{-N} B_N^+(\epsilon)(\epsilon^{-2}P+1)^{-N}\zeta(\epsilon x),$$

where $B_N^+(\epsilon) = O_{L^2\to L^2}(1)$ uniformly in $\epsilon \in (0,1]$. Then, using Proposition 4.1.6 with $q = \infty$ and (4.4.50), we have

$$\begin{aligned}\|\tilde{\zeta}(\epsilon x)(\epsilon^{-2}P+1)^{-l}\zeta(\epsilon x)Op_\epsilon(\tilde{\chi}_\epsilon^+)^\star f^2(\epsilon^{-2}P)\mathcal{R}_1^+(N,t,\epsilon)\|_{L^1\to L^\infty} &\leq C\epsilon^d\langle\epsilon t\rangle^{-3N/4} \\ &\leq C\epsilon^d(1+\epsilon|t|)^{-d/2},\end{aligned}$$

for all $t \leq 0$ and all $\epsilon \in (0,1]$ provided N is taken large enough. Moreover, using again Proposition

4.1.6 and (4.1.23), we also have

$$\begin{aligned}\|\tilde{\zeta}(\epsilon x)(\epsilon^{-2}P+1)^{-l}\tilde{B}_l(\epsilon)\langle \epsilon x\rangle^{-l} f^2(\epsilon^{-2}P)\mathcal{R}_1^+(N,t,\epsilon)\|_{L^1\to L^\infty} &\leq C\epsilon^d\langle \epsilon t\rangle^{1-l}\\ &\leq C\epsilon^d(1+\epsilon|t|)^{-d/2},\end{aligned}$$

for all $t\leq 0$ and all $\epsilon\in(0,1]$ provided l and N are taken large enough. This implies

$$\|A_\epsilon^+\mathcal{R}_1^+(N,t,\epsilon)\|_{L^1\to L^\infty}\leq C\epsilon^d(1+\epsilon|t|)^{-d/2},$$

for all $t\leq 0$ and all $\epsilon\in(0,1]$.

Next, thanks to the support of b_ϵ^+, we can write

$$\mathcal{J}_\epsilon^+(b_\epsilon^+)^\star=\mathcal{J}_\epsilon^+(\tilde{b}_\epsilon^+)^\star(\epsilon^{-2}P+1)^{-N}\zeta_1(\epsilon x), \tag{4.4.62}$$

where $(\tilde{b}_\epsilon^+)_{\epsilon\in(0,1]}\in S(0,-\infty)$, $\mathrm{supp}(\tilde{b}_\epsilon^+)\subset\Gamma^+(R^3,J_3,\sigma_3)$ and $\zeta_1\in C^\infty(\mathbb{R}^d)$ is supported outside $B(0,1)$ such that $\zeta_1(x)=1$ for $|x|>R^3$. Indeed, we write for $\tilde{\zeta}_1\in C^\infty(\mathbb{R}^d)$ supported outside $B(0,1)$ and $\tilde{\zeta}_1=1$ in $\mathrm{supp}(\zeta_1)$,

$$\mathcal{J}_\epsilon^+(b_\epsilon^+)^\star=\mathcal{J}_\epsilon^+(b_\epsilon^+)^\star\tilde{\zeta}_1(\epsilon x)(\epsilon^{-2}P+1)^N\left((\epsilon^{-2}P+1)^{-N}\zeta_1(\epsilon x)\right).$$

We have (4.4.62) by taking the adjoint of $(\epsilon^{-2}P+1)^N\tilde{\zeta}_1(\epsilon x)\mathcal{J}_\epsilon^+(b_\epsilon^+)=\mathcal{J}_\epsilon^+(\tilde{b}_\epsilon^+)$.

For $k=2$, using (4.1.6) and its adjoint, (4.4.50), (4.4.62), $\langle\epsilon x\rangle^l R_N(\epsilon)\langle \epsilon x\rangle^{N-l}=O_{L^2\to L^2}(1)$ and estimating as in Lemma 4.4.13, we have

$$\begin{aligned}&\|\tilde{\zeta}(\epsilon x)(\epsilon^{-2}P+1)^{-l}\zeta(\epsilon x)Op_\epsilon(\tilde{\chi}_\epsilon^+)^\star f^2(\epsilon^{-2}P)\mathcal{R}_2^+(N,t,\epsilon)\|_{L^1\to L^\infty}\\ &\qquad\leq C\epsilon^d\epsilon\int_0^t\langle\epsilon(t-s)\rangle^{-3l/4}\langle\epsilon s\rangle^{-N/4}ds\leq C\epsilon^d(1+\epsilon|t|)^{-d/2},\end{aligned}$$

for $t\leq 0$ provided that l and N are taken large enough. Moreover, using (4.1.23) instead of (4.4.50), we have

$$\begin{aligned}&\|\tilde{\zeta}(\epsilon x)(\epsilon^{-2}P+1)^{-l}\tilde{B}_l(\epsilon)\langle\epsilon x\rangle^{-l}f^2(\epsilon^{-2}P)\mathcal{R}_2^+(N,t,\epsilon)\|_{L^1\to L^\infty}\\ &\qquad\leq C\epsilon^d\epsilon\int_0^t\langle\epsilon(t-s)\rangle^{1-l}\langle\epsilon s\rangle^{-N/4}ds\leq C\epsilon^d(1+\epsilon|t|)^{-d/2},\end{aligned}$$

for all $t\leq 0$ and all $\epsilon\in(0,1]$. This implies

$$\|A_\epsilon^+\mathcal{R}_2^+(N,t,\epsilon)\|_{L^1\to L^\infty}\leq C\epsilon^d(1+\epsilon|t|)^{-d/2},\quad \forall t\leq 0,\epsilon\in(0,1].$$

The third remainder term is treated similarly as the second one. It remains to study the last remainder term. To do so, we split

$$A_\epsilon^+\mathcal{R}_4^+(N,t,\epsilon)=-i\int_0^t A_\epsilon^+e^{-i(t-s)\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma}(\chi+(1-\chi))(\epsilon x/R^2)\mathcal{J}_\epsilon^+(\check{a}^+(\epsilon))e^{-is\epsilon\Lambda^\sigma}\mathcal{J}_\epsilon^+(b_\epsilon^+)^\star ds,$$

where $\chi\in C_0^\infty(\mathbb{R}^d)$ satisfying $\chi(x)=1$ for $|x|\leq 2$. The first term can be treated similarly as the second remainder using (4.4.51) instead of (4.4.50) and Lemma 4.4.12. For the second term, we need the following lemma (see [BT08, Proposition 5.2]).

Lemma 4.4.19. *Choose $\tilde{\tau}_1$ such that $-\tau_4<\tilde{\tau}_1<-\tau_2$. If $R>0$ is large enough, we may choose a bounded family of symbols $\tilde{\chi}_\epsilon^-\in S(0,-\infty)$ satisfying* $\mathrm{supp}(\tilde{\chi}_\epsilon^-)\subset\Gamma^-(R,J_1,\tilde{\tau}_1)$ *and $\tilde{\zeta}_2\in C^\infty(\mathbb{R}^d)$ supported outside $B(0,1)$ satisfying $\tilde{\zeta}_2=1$ on* $\mathrm{supp}(1-\chi)$ *such that for all m large enough,*

$$f(\epsilon^{-2}P)(1-\chi)(\epsilon x/R^2)\mathcal{J}_\epsilon^+(\check{a}^+(\epsilon))=Op_\epsilon(\tilde{\chi}_\epsilon^-)\tilde{\zeta}_2(\epsilon x)\mathcal{J}_\epsilon^+(\tilde{e}_m(\epsilon))+\tilde{R}_m(\epsilon),$$

where

$$\tilde{R}_m(\epsilon) = \mathcal{J}^+_\epsilon(\tilde{r}_m(\epsilon)) + \langle \epsilon x\rangle^{-m/2} R_m(\epsilon) \langle \epsilon x\rangle^{-m/2} \mathcal{J}^+_\epsilon(\check{a}^+(\epsilon)),$$

with $(\tilde{e}_m(\epsilon))_{\epsilon\in(0,1]}$ *and* $(\tilde{r}_m(\epsilon))_{\epsilon\in(0,1]}$ *bounded families in* $S(0,-\infty)$ *and* $S(-m,-\infty)$ *respectively and* $R_m(\epsilon) = O_{L^2\to L^2}(1)$ *uniformly in* $\epsilon \in (0,1]$.

We set

$$A^+_\epsilon = (A^+_{\epsilon,1} + A^+_{\epsilon,2}) f(\epsilon^{-2}P),$$

where

$$A^+_{\epsilon,1} = \tilde{\zeta}(\epsilon x)(\epsilon^{-2}P+1)^{-l}\zeta(\epsilon x) Op_\epsilon(\tilde{\chi}^+_\epsilon)^\star f(\epsilon^{-2}P),$$
$$A^+_{\epsilon,2} = \tilde{\zeta}(\epsilon x)(\epsilon^{-2}P+1)^{-l}\tilde{B}_l(\epsilon)\langle \epsilon x\rangle^{-l} f(\epsilon^{-2}P).$$

Using Lemma 4.4.19, we firstly consider

$$-ih^{-1}\int_0^t A^+_{\epsilon,1} e^{-i(t-s)\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma}\left(Op_\epsilon(\tilde{\chi}^-_\epsilon)\tilde{\zeta}_2(\epsilon x)\mathcal{J}^+_\epsilon(\tilde{e}_m(\epsilon)) + \tilde{R}_m(\epsilon)\right) e^{-is\epsilon\Lambda^\sigma}\mathcal{J}^+_\epsilon(b^+_\epsilon)^\star ds.$$

The remainder terms are treated similarly as the second remainder term using (4.4.50) and Lemma 4.4.13. The term involving $Op_\epsilon(\tilde{\chi}^-_\epsilon)\tilde{\zeta}_2(\epsilon x)\mathcal{J}^+_\epsilon(\tilde{e}_m(\epsilon))$ is studied by the same analysis as the second term using (4.4.52) instead of (4.4.47). For the term

$$-ih^{-1}\int_0^t A^+_{\epsilon,2} e^{-i(t-s)\epsilon(\epsilon^{-1}|\nabla_g|)^\sigma}\left(Op_\epsilon(\tilde{\chi}^-_\epsilon)\tilde{\zeta}_2(\epsilon x)\mathcal{J}^+_\epsilon(\tilde{e}_m(\epsilon)) + \tilde{R}_m(\epsilon)\right) e^{-is\epsilon\Lambda^\sigma}\mathcal{J}^+_\epsilon(b^+_\epsilon)^\star ds,$$

the required estimate follows by using (4.1.23) and Lemma 4.4.13. This completes the proof. □

4.5 Inhomogeneous Strichartz estimates

In this section, we will give the proofs of Proposition 4.0.6 and Proposition 4.0.8. The main tool is the homogeneous Strichartz estimates (4.0.14) and the so called Christ-Kiselev Lemma. To do so, we recall the following result (see [CK01] or [Sog93]).

Lemma 4.5.1. *Let* X *and* Y *be Banach spaces and assume that* $K(t,s)$ *is a continuous function taking its values in the bounded operators from* Y *to* X*. Suppose that* $-\infty \le c < d \le \infty$*, and set*

$$Af(t) = \int_c^d K(t,s)f(s)ds.$$

Assume that

$$\|Af\|_{L^q([c,d],X)} \le C\|f\|_{L^p([c,d],Y)}.$$

Define the operator $\tilde{A}$ *as*

$$\tilde{A}f(t) = \int_c^t K(t,s)f(s)ds,$$

Then for $1 \le p < q \le \infty$*, there exists* $\tilde{C} > 0$ *such that*

$$\|\tilde{A}f\|_{L^q([c,d],X)} \le \tilde{C}\|f\|_{L^p([c,d],Y)}.$$

We are now able to prove the inhomogeneous Strichartz estimates (4.0.16) and (4.0.20).

Inhomogeneous Strichartz estimates for Schrödinger-type equation. We give the proof of Proposition 4.0.6 by following a standard argument (see e.g. [Zha15]). Let u be the solution to (4.0.1). By Duhamel formula, we have

$$u(t) = e^{-it|\nabla_g|^\sigma}\psi - i\int_0^t e^{-i(t-s)|\nabla_g|^\sigma}F(s)ds =: u_{\rm hom}(t) + u_{\rm inh}(t).$$

Using (4.0.14), we have

$$\|u_{\rm hom}\|_{L^p(\mathbb{R},L^q)} \leq C\|\psi\|_{\dot{H}^{\gamma_{p,q}}_g}.$$

It remains to prove the inhomogeneous part, namely

$$\Big\| \int_0^t e^{-i(t-s)|\nabla_g|^\sigma} F(s) ds \Big\|_{L^p(\mathbb{R},L^q)} \leq C\|F\|_{L^{a'}(\mathbb{R},L^{b'})},$$

where $(p,q),(a,b)$ are Schrödinger admissible pairs with $q<\infty$ and $b<\infty$ satisfying $(p,a)\ne(2,2)$ and the gap condition (4.0.17). By the Christ-Kiselev Lemma, it suffices to prove

$$\Big\| \int_{\mathbb{R}} e^{-i(t-s)|\nabla_g|^\sigma} F(s) ds \Big\|_{L^p(\mathbb{R},L^q)} \leq C\|F\|_{L^{a'}(\mathbb{R},L^{b'})}, \tag{4.5.1}$$

for all Schrödinger admissible pairs (p,q) and (a,b) with $q<\infty$ and $b<\infty$ satisfying (4.0.17) excluding the case $p=a'=2$. We now prove (4.5.1). Define

$$T_{\gamma_{p,q}} : \psi \in \mathscr{L}_g \mapsto |\nabla_g|^{-\gamma_{p,q}} e^{-it|\nabla_g|^\sigma}\psi \in L^p(\mathbb{R},L^q).$$

Thanks to (4.0.14), we see that $T_{\gamma_{p,q}}$ is a bounded operator. Similar result holds for $T_{\gamma_{a,b}}$. Next, we take the adjoint for $T_{\gamma_{a,b}}$ and obtain a bounded operator

$$T^\star_{\gamma_{a,b}} : F \in L^{a'}(\mathbb{R},L^{b'}) \mapsto \int_{\mathbb{R}} |\nabla_g|^{-\gamma_{a,b}} e^{is|\nabla_g|^\sigma} F(s) ds \in \mathscr{L}'_g,$$

where $\mathscr{L}'_g$ is the dual space of $\mathscr{L}_g$. Using (4.0.17) or $\gamma_{a,b} = -\gamma_{a',b'}-\sigma = -\gamma_{p,q}$, we have

$$\Big\| \int_{\mathbb{R}} e^{-i(t-s)|\nabla_g|^\sigma} F(s) ds \Big\|_{L^p(\mathbb{R},L^q)} = \|T_{\gamma_{p,q}} T^\star_{\gamma_{a,b}} F\|_{L^p(\mathbb{R},L^q)} \leq C\|F\|_{L^{a'}(\mathbb{R},L^{b'})},$$

and (4.5.1) follows.

Next, we prove

$$\|u\|_{L^\infty(\mathbb{R},\dot{H}^{\gamma_{p,q}}_g)} \leq C\Big(\|\psi\|_{\dot{H}^{\gamma_{p,q}}_g} + \|F\|_{L^{a'}(\mathbb{R},L^{b'})}\Big).$$

By using the homogeneous Strichartz estimate for a Schrödinger admissible pair $(\infty,2)$ with $\gamma_{\infty,2}=0$ and that $\|u\|_{L^\infty(\mathbb{R},\dot{H}^{\gamma_{p,q}}_g)} = \||\nabla_g|^{\gamma_{p,q}}u\|_{L^\infty(\mathbb{R},L^2)}$, we have

$$\|u\|_{L^\infty(\mathbb{R},\dot{H}^{\gamma_{p,q}}_g)} \leq C\Big(\||\nabla_g|^{\gamma_{p,q}}\psi\|_{L^2} + \Big\| \int_0^t |\nabla_g|^{\gamma_{p,q}} e^{-i(t-s)|\nabla_g|^\sigma} F(s) ds \Big\|_{L^\infty(\mathbb{R},L^2)}\Big).$$

Using the Christ-Kiselev Lemma, it suffices to prove

$$\Big\| \int_{\mathbb{R}} |\nabla_g|^{\gamma_{p,q}} e^{-i(t-s)|\nabla_g|^\sigma} F(s) ds \Big\|_{L^\infty(\mathbb{R},L^2)} \leq C\|F\|_{L^{a'}(\mathbb{R},L^{b'})}.$$

Using the above notation, we have

$$\begin{aligned}\Big\| \int_{\mathbb{R}} |\nabla_g|^{\gamma_{p,q}} e^{-i(t-s)|\nabla_g|^\sigma} F(s) ds \Big\|_{L^\infty(\mathbb{R},L^2)} &= \|T_0 T^\star_{\gamma_{a,b}} F\|_{L^\infty(\mathbb{R},L^2)} \\ &\leq C\|T^\star_{\gamma_{a,b}}F\|_{L^2} \leq C\|F\|_{L^{a'}(\mathbb{R},L^{b'})}.\end{aligned}$$

This completes the proof of Proposition 4.0.6. □

Inhomogeneous Strichartz estimates for wave-type equation. We give the proof of Proposition 4.0.8. Let v be the solution to (4.0.19). By Duhamel formula, we have

$$v(t)=\cos t|\nabla_g|^\sigma\psi+\frac{\sin t|\nabla_g|^\sigma}{|\nabla_g|^\sigma}u_1+\int_0^t\frac{\sin(t-s)|\nabla_g|^\sigma}{|\nabla_g|^\sigma}F(s)ds=:v_{\text{hom}}(t)+v_{\text{inh}}(t),$$

where v_{hom} is the sum of first two terms and v_{inh} is the last one. We firstly prove

$$\|v\|_{L^p(\mathbb{R},L^q)}\leq C\Big(\|v_0\|_{\dot H_g^{\gamma_{p,q}}}+\|v_1\|_{\dot H_g^{\gamma_{p,q}-\sigma}}+\|F\|_{L^{a'}(\mathbb{R},L^{b'})}\Big).$$

By observing that

$$\cos t|\nabla_g|^\sigma=\frac{e^{it|\nabla_g|^\sigma}+e^{-it|\nabla_g|^\sigma}}{2},\quad \sin t|\nabla_g|^\sigma=\frac{e^{it|\nabla_g|^\sigma}-e^{-it|\nabla_g|^\sigma}}{2i},$$

and using (4.0.14), we have

$$\|v_{\text{hom}}\|_{L^p(\mathbb{R},L^q)}\leq C\Big(\|v_0\|_{\dot H_g^{\gamma_{p,q}}}+\|v_1\|_{\dot H_g^{\gamma_{p,q}-\sigma}}\Big).$$

Let us prove the inhomogeneous part which is in turn equivalent to

$$\Big\|\int_0^t\frac{e^{-i(t-s)|\nabla_g|^\sigma}}{|\nabla_g|^\sigma}F(s)ds\Big\|_{L^p(\mathbb{R},L^q)}\leq C\|F\|_{L^{a'}(\mathbb{R},L^{b'})},\tag{4.5.2}$$

where $(p,q),(a,b)$ are Schrödinger admissible with $q<\infty$ and $b<\infty$ satisfying the gap condition (4.0.21). We define the operator

$$T_{\gamma_{p,q}}:\psi\in\mathscr{L}_g\mapsto|\nabla_g|^{-\gamma_{p,q}}e^{-it|\nabla_g|^\sigma}\psi\in L^p(\mathbb{R},L^q).$$

Thanks to (4.0.14), we see that $T_{\gamma_{p,q}}$ is a bounded operator. Next, we take the adjoint for $T_{\gamma_{a,b}}$ and obtain a bounded operator

$$T^\star_{\gamma_{a,b}}:F\in L^{a'}(\mathbb{R},L^{b'})\mapsto\int_{\mathbb{R}}|\nabla_g|^{-\gamma_{a,b}}e^{is|\nabla_g|^\sigma}F(s)ds\in\mathscr{L}'_g.$$

Using (4.0.21) or $\gamma_{a,b}=-\gamma_{a',b'}-\sigma=-\gamma_{p,q}+\sigma$, we have

$$\Big\|\int_{\mathbb{R}}\frac{e^{-i(t-s)|\nabla_g|^\sigma}}{|\nabla_g|^\sigma}F(s)ds\Big\|_{L^p(\mathbb{R},L^q)}=\|T_{\gamma_{p,q}}T^\star_{\gamma_{a,b}}F\|_{L^p(\mathbb{R},L^q)}\leq C\|F\|_{L^{a'}(\mathbb{R},L^{b'})}.$$

As in the proof of the inhomogeneous Strichartz estimates for the Schrödinger-type equations, the Christ-Kiselev Lemma implies (4.5.2) for all Schrödinger admissible pairs (p,q) and (a,b) with $q<\infty$ and $b<\infty$ satisfying the gap condition (4.0.21) excluding the case $p=a'=2$.

Next, we prove

$$\|v\|_{L^\infty(\mathbb{R},\dot H_g^{\gamma_{p,q}})}\leq C\Big(\|v_0\|_{\dot H_g^{\gamma_{p,q}}}+\|v_1\|_{\dot H_g^{\gamma_{p,q}-\sigma}}+\|F\|_{L^{a'}(\mathbb{R},L^{b'})}\Big).$$

By using the homogeneous Strichartz estimate for a Schrödinger admissible pair $(\infty,2)$ with $\gamma_{\infty,2}=0$ and that $\|v\|_{L^\infty(\mathbb{R},\dot H_g^{\gamma_{p,q}})}=\||\nabla_g|^{\gamma_{p,q}}v\|_{L^\infty(\mathbb{R},L^2)}$, we have

$$\begin{aligned}\|v\|_{L^\infty(\mathbb{R},\dot H_g^{\gamma_{p,q}})}\leq C\Big(&\||\nabla_g|^{\gamma_{p,q}}v_0\|_{L^2}+\||\nabla_g|^{\gamma_{p,q}}v_1\|_{\dot H_g^{-\sigma}}\\&+\Big\|\int_0^t|\nabla_g|^{(\gamma_{p,q}-\sigma)}\sin{(t-s)}|\nabla_g|^\sigma F(s)ds\Big\|_{L^\infty(\mathbb{R},L^2)}\Big).\end{aligned}$$

Using the Christ-Kiselev Lemma, it suffices to prove

$$\Big\| \int_{\mathbb{R}} |\nabla_g|^{(\gamma_{p,q}-\sigma)} e^{-i(t-s)|\nabla_g|^{\sigma}} F(s)ds \Big\|_{L^\infty(\mathbb{R},L^2)} \leq C\|F\|_{L^{a'}(\mathbb{R},L^{b'})}.$$

Using the above notation, we have

$$\begin{aligned}\Big\| \int_{\mathbb{R}} |\nabla_g|^{(\gamma_{p,q}-\sigma)} e^{-i(t-s)|\nabla_g|^{\sigma}} F(s)ds \Big\|_{L^\infty(\mathbb{R},L^2)} &= \|T_0 T^\star_{\gamma_{a,b}} F\|_{L^\infty(\mathbb{R},L^2)} \\ &\leq C\|T^\star_{\gamma_{a,b}} F\|_{L^2} \leq C\|F\|_{L^{a'}(\mathbb{R},L^{b'})}.\end{aligned}$$

We repeat the same process for $\partial_t v$ and obtain

$$\|\partial_t v\|_{L^\infty(\mathbb{R},\dot{H}_g^{\gamma_{p,q}-\sigma})} \leq C\Big(\|v_0\|_{\dot{H}_g^{\gamma_{p,q}}} + \|v_1\|_{\dot{H}_g^{\gamma_{p,q}-\sigma}} + \|F\|_{L^{a'}(\mathbb{R},L^{b'})}\Big).$$

This completes the proof of Proposition 4.0.8. □

Part II

Nonlinear Schrödinger-type equations

Chapter 5

Local well-posedness for nonlinear Schrödinger-type equations in Sobolev spaces

Contents

In this chapter, we study the local well-posedness in Sobolev spaces for the power-type nonlinear Schrödinger-type equations, namely

$$\left\{ \begin{array}{rcl} i\partial_t u(t,x) + |\nabla|^\sigma u(t,x) &=& -\mu(|u|^{\nu-1}u)(t,x), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ u(0,x) &=& \psi(x), \quad x \in \mathbb{R}^d, \end{array} \right. \tag{NLST}$$

with $\sigma \in (0,\infty), \nu > 1$ and $\mu \in \{\pm 1\}$. When $\sigma = 1$, we use the notation (NLHW) instead of (NLST). The operator $|\nabla| = \sqrt{-\Delta}$ is the Fourier multiplier by $|\xi|$ where $\Delta = \sum_{j=1}^d \partial_j^2$ is the free Laplace operator on $\mathbb{R}^d$. The number $\mu = 1$ (resp. $\mu = -1$) corresponds to the defocusing case (resp. focusing case).

Before stating the main results, we recall some useful facts about (NLST). By a standard approximation argument, the following quantities are conserved under the flow of (NLST):

$$M(u(t)) = \int |u(t,x)|^2 dx = M(\psi), \quad E(u(t)) = \int \frac{1}{2}||\nabla|^{\sigma/2}u(t,x)|^2 + \frac{\mu}{\nu+1}|u(t,x)|^{\nu+1}dx = E(\psi).$$

Moreover, if we set for $\lambda > 0$,

$$u_\lambda(t,x) = \lambda^{-\frac{\sigma}{\nu-1}} u(\lambda^{-\sigma}t, \lambda^{-1}x),$$

then (NLST) is invariant under this scaling, that is for $T \in (0,+\infty]$,

$$u \text{ solves (NLST) on } (-T,T) \Longleftrightarrow u_\lambda \text{ solves (NLST) on } (-\lambda^\sigma T, \lambda^\sigma T).$$

We also have

$$\|u_\lambda(0)\|_{\dot{H}^\gamma} = \lambda^{\frac{d}{2}-\frac{\sigma}{\nu-1}-\gamma}\|\psi\|_{\dot{H}^\gamma}.$$

From this, we define the critical regularity exponent for (NLST) by

$$\gamma_{\rm c} = \frac{d}{2} - \frac{\sigma}{\nu-1}. \tag{5.0.1}$$

In this chapter, we are interested in the well-posedness results for (NLST) when $\gamma \geq \gamma_c$. Since we are working in Sobolev spaces of fractional order γ, γ_c, we need the nonlinearity $F(z) = -\mu|z|^{\nu-1}z$ to have enough regularity. When ν is an odd integer, $F \in C^\infty(\mathbb{C}, \mathbb{C})$ (in the real sense). When ν is not an odd integer, we need the following assumption

$$\lceil \gamma \rceil, \lceil \gamma_c \rceil \leq \nu, \tag{5.0.2}$$

where $\lceil \gamma \rceil$ is the smallest integer greater than or equal to γ, similarly for $\lceil \gamma_c \rceil$.

In order to study the local well-posedness of (NLST) in Sobolev spaces, we need two important tools: linear estimates (or Strichartz estimates) and nonlinear estimates. Strichartz estimates for the linear Schrödinger-type equation are derived in Chapter 1. Note that in the case $\sigma \in (0,2)\backslash\{1\}$, we always have $\gamma_{p,q} > 0$ (see (1.0.7)) for all admissible pairs except $(p,q) = (\infty, 2)$. This shows that Strichartz estimates for the linear Schrödinger-type equation given in Corollary 1.1.3 have a loss of derivatives. That is if we use Strichartz estimates at H^γ-level, then we need the initial data at $H^{\gamma+\gamma_{p,q}}$-level except $(p,q) = (\infty, 2)$. This loss of derivatives leads to restrictions (and hence weak results) compared to the those in the case $\sigma \in [2, \infty)$. Therefore, we will consider three cases $\sigma \in (0,2)\backslash\{1\}$, $\sigma = 1$ and $\sigma \in [2,\infty)$ separately. We also recall some nonlinear estimates as follows.

Nonlinear estimates. Let us start with the following Kato-Ponce inequality (or fractional Leibniz rule).

Proposition 5.0.1. *Let $\gamma \geq 0, 1 < r < \infty$ and $1 < p_1, p_2, q_1, q_2 \leq \infty$ satisfying $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. Then there exists $C = C(d, \gamma, r, p_1, q_1, p_2, q_2) > 0$ such that for all $u, v \in \mathscr{S}$,*

$$\||\nabla|^\gamma (uv)\|_{L^r} \leq C\Big(\||\nabla|^\gamma u\|_{L^{p_1}} \|v\|_{L^{q_1}} + \|u\|_{L^{p_2}} \||\nabla|^\gamma v\|_{L^{q_2}}\Big), \tag{5.0.3}$$

$$\|\langle \nabla \rangle^\gamma (uv)\|_{L^r} \leq C\Big(\|\langle \nabla \rangle^\gamma u\|_{L^{p_1}} \|v\|_{L^{q_1}} + \|u\|_{L^{p_2}} \|\langle \nabla \rangle^\gamma v\|_{L^{q_2}}\Big). \tag{5.0.4}$$

We refer to [GO14] (and references therein) for the proof of above inequalities and more general results. We also have the following fractional chain rule.

Proposition 5.0.2. *Let $F \in C^1(\mathbb{C}, \mathbb{C})$ and $G \in C(\mathbb{C}, \mathbb{R}^+)$ such that $F(0) = 0$ and*

$$|F'(\theta z + (1-\theta)\zeta)| \leq \mu(\theta)(G(z) + G(\zeta)), \quad z, \zeta \in \mathbb{C}, \quad 0 \leq \theta \leq 1,$$

where $\mu \in L^1((0,1))$. Then for $\gamma \in (0,1)$ and $1 < r, p < \infty$, $1 < q \leq \infty$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, there exists $C = C(d, \mu, \gamma, r, p, q) > 0$ such that for all $u \in \mathscr{S}$,

$$\||\nabla|^\gamma F(u)\|_{L^r} \leq C\|F'(u)\|_{L^q} \||\nabla|^\gamma u\|_{L^p}, \tag{5.0.5}$$

$$\|\langle \nabla \rangle^\gamma F(u)\|_{L^r} \leq C\|F'(u)\|_{L^q} \|\langle \nabla \rangle^\gamma u\|_{L^p}. \tag{5.0.6}$$

We refer to [CW91] (see also [Sta95]) for the proof of (5.0.5) and [Tay00, Proposition 5.1] for (5.0.6). A direct consequence of the fractional Leibniz rule and the fractional chain rule is the following fractional derivative estimates.

Corollary 5.0.3. *Let $F \in C^k(\mathbb{C}, \mathbb{C}), k \in \mathbb{N}\backslash\{0\}$. Assume that there is $\nu \geq k$ such that*

$$|D^i F(z)| \leq C|z|^{\nu - i}, \quad z \in \mathbb{C}, \quad i = 1, 2, ..., k.$$

Then for $\gamma \in [0,k]$ and $1 < r, p < \infty$, $1 < q \leq \infty$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{\nu-1}{q}$, there exists $C = C(d, \nu, \gamma, r, p, q) > 0$ such that for all $u \in \mathscr{S}$,

$$\||\nabla|^\gamma F(u)\|_{L^r} \leq C\|u\|_{L^q}^{\nu-1} \||\nabla|^\gamma u\|_{L^p}, \tag{5.0.7}$$

$$\|\langle \nabla \rangle^\gamma F(u)\|_{L^r} \leq C\|u\|_{L^q}^{\nu-1} \|\langle \nabla \rangle^\gamma u\|_{L^p}. \tag{5.0.8}$$

The reader can find the proof of (5.0.7) in [Kat95, Lemma A.3]. The one of (5.0.8) follows from (5.0.7), the Hölder inequality and the fact that

$$\| \langle \nabla \rangle^\gamma u\|_{L^r} \sim \|u\|_{L^r} + \||\nabla|^\gamma u\|_{L^r},$$

for $1 < r < \infty, \gamma > 0$. Another consequence of the fractional Leibniz rule given in Proposition 5.0.1 is the following result.

Corollary 5.0.4. *Let $F(z)$ be a homogeneous polynomial in $z, \overline{z}$ of degree $\nu \geq 1$. Then* (5.0.7) *and* (5.0.8) *hold true for any $\gamma \geq 0$ and r, p, q as in Corollary* 5.0.3.

Corollary 5.0.5. *Let $F(z) = |z|^{\nu-1}z$ with $\nu > 1$, $\gamma \geq 0$ and $1 < r, p < \infty$, $1 < q \leq \infty$ satisfying $\frac{1}{r} = \frac{1}{p} + \frac{\nu-1}{q}$.*

i. *If ν is an odd integer or $\lceil \gamma \rceil \leq \nu$ otherwise, then there exists $C = C(d, \nu, \gamma, r, p, q) > 0$ such that for all $u \in \mathscr{S}$,*

$$\|F(u)\|_{\dot{H}^\gamma_r} \leq C\|u\|^{\nu-1}_{L^q}\|u\|_{\dot{H}^\gamma_p}.$$

A similar estimate holds with $\dot{H}^\gamma_r, \dot{H}^\gamma_p$-norms are replaced by H^γ_r, H^γ_p-norms respectively.

ii. *If ν is an odd integer or $\lceil \gamma \rceil \leq \nu - 1$ otherwise, then there exists $C = C(d, \nu, \gamma, r, p, q) > 0$ such that for all $u, v \in \mathscr{S}$,*

$$\begin{aligned} \|F(u) - F(v)\|_{\dot{H}^\gamma_r} \leq C\Big(&(\|u\|^{\nu-1}_{L^q} + \|v\|^{\nu-1}_{L^q})\|u - v\|_{\dot{H}^\gamma_p} \\ &+ (\|u\|^{\nu-2}_{L^q} + \|v\|^{\nu-2}_{L^q})(\|u\|_{\dot{H}^\gamma_p} + \|v\|_{\dot{H}^\gamma_p})\|u - v\|_{L^q}\Big). \end{aligned}$$

A similar estimate holds with $\dot{H}^\gamma_r, \dot{H}^\gamma_p$-norms are replaced by H^γ_r, H^γ_p-norms respectively.

Proof. Item 1 is an immediate consequence of Corollary 5.0.3 and Corollary 5.0.4. For Item 2, we firstly write

$$F(u) - F(v) = \int_0^1 \partial_z F(v + \theta(u - v))(u - v) + \partial_{\overline{z}} F(v + \theta(u - v))(\overline{u} - \overline{v})d\theta,$$

and use the fractional Leibniz rule given in Proposition 5.0.1. Then the results follow by applying the fractional derivative estimates given in Corollary 5.0.3 and Corollary 5.0.4. □

5.1 Local well-posedness for Schrödinger-type equations in Sobolev spaces when $\sigma \in (0, 2)\backslash\{1\}$

5.1.1 Local well-posedness in the subcritical case

Let us start with the following local well-posedness in the subcritical case.

Theorem 5.1.1. *Given $\sigma \in (0, 2)\backslash\{1\}$ and $\nu > 1$. Let $\gamma \geq 0$ be such that*

$$\left\{ \begin{array}{ll} \gamma > 1/2 - \sigma/\max(\nu - 1, 4) & \text{when } d = 1, \\ \gamma > d/2 - \sigma/\max(\nu - 1, 2) & \text{when } d \geq 2, \end{array} \right. \tag{5.1.1}$$

and also, if ν is not an odd integer, (5.0.2). *Let*

$$\left\{ \begin{array}{ll} p > \max(\nu - 1, 4) & \text{when } d = 1 \\ p > \max(\nu - 1, 2) & \text{when } d \geq 2 \end{array} \right. \tag{5.1.2}$$

be such that $\gamma > \frac{d}{2} - \frac{\sigma}{p}$. Then for all $\psi \in H^\gamma$, there exist $T^* \in (0, \infty]$ and a unique solution to (NLST) *satisfying*

$$u \in C([0, T^*), H^\gamma) \cap L^p_{\text{loc}}([0, T^*), L^\infty).$$

Moreover, the following properties hold:

i. *If $T^* < \infty$, then $\|u(t)\|_{H^\gamma} \to \infty$ as $t \to T^*$.*

ii. *u depends continuously on ψ in the following sense. There exists $T \in (0,T^*)$ such that if $\psi_n \to \psi$ in H^γ and if u_n denotes the solution of* (NLST) *with initial data ψ_n, then $0 < T < T^*(\psi_n)$ for all n sufficiently large and u_n is bounded in $L^a([0,T], H_b^{\gamma-\gamma_{a,b}})$ for any Schrödinger admissible pair (a,b) with $b < \infty$. Moreover, $u_n \to u$ in $L^a([0,T], H_b^{-\gamma_{a,b}})$ as $n \to \infty$. In particular, $u_n \to u$ in $C([0,T], H^{\gamma-\epsilon})$ for all $0 < \epsilon < \gamma$.*

iii. *Let $\beta > \gamma$ be such that if ν is not an odd integer, $\lceil \beta \rceil \leq \nu$. If $\psi \in H^\beta$, then $u \in C([0,T^*), H^\beta)$.*

The local well-posedness in Sobolev spaces for the nonlinear Schrödinger-type equation in the case $\sigma \in (0,2)\backslash\{1\}$ was first established by Hong-Sire in [HS15]. Theorem 5.1.1 improves the one in [HS15] at the point that Hong-Sire only give the local well-posedness for $\nu \geq 2$ when $d = 1$ and $\nu \geq 3$ when $d \geq 2$. This result also covers the one in [CHKL15] when $d = 1$ and in [GH13] when $d \geq 2$, where the authors considered the cubic Schrödinger-type equation with $\sigma \in (1,2)$.

Proof of Theorem 5.1.1. We follow the standard process (see e.g. [Caz03, Chapter 4] or [BGT04]) by using the fixed point argument in a suitable Banach space. Let p be as in (5.1.2). We then choose $q \in [2,\infty)$ such that

$$\frac{2}{p} + \frac{d}{q} \leq \frac{d}{2}.$$

Step 1. Existence. Let us consider

$$X := \Big\{ u \in L^\infty(I, H^\gamma) \cap L^p(I, H_q^{\gamma-\gamma_{p,q}}) \mid \|u\|_{L^\infty(I,H^\gamma)} + \|u\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \leq M \Big\},$$

equipped with the distance

$$d(u,v) := \|u-v\|_{L^\infty(I,L^2)} + \|u-v\|_{L^p(I,H_q^{-\gamma_{p,q}})},$$

where $I = [0,T]$ and $M,T > 0$ to be chosen later. The persistence of regularity (see e.g. [Caz03, Theorem 1.2.5]) shows that (X,d) is a complete metric space. By the Duhamel formula, it suffices to prove that the functional

$$\Phi(u)(t) = e^{it|\nabla|^\sigma}\psi + i\mu \int_0^t e^{i(t-s)|\nabla|^\sigma} |u(s)|^{\nu-1} u(s) ds \tag{5.1.3}$$

is a contraction on (X,d). The local Strichartz estimate (1.1.18) gives

$$\|\Phi(u)\|_{L^\infty(I,H^\gamma)} + \|\Phi(u)\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \lesssim \|\psi\|_{H^\gamma} + \|F(u)\|_{L^1(I,H^\gamma)},$$
$$\|\Phi(u)-\Phi(v)\|_{L^\infty(I,L^2)} + \|\Phi(u)-\Phi(v)\|_{L^p(I,H_q^{-\gamma_{p,q}})} \lesssim \|F(u)-F(v)\|_{L^1(I,L^2)},$$

where $F(u) = |u|^{\nu-1}u$. By our assumptions on ν, Corollary 5.0.5 gives

$$\|F(u)\|_{L^1(I,H^\gamma)} \lesssim \|u\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)} \|u\|_{L^\infty(I,H^\gamma)} \lesssim T^{1-\frac{\nu-1}{p}} \|u\|^{\nu-1}_{L^p(I,L^\infty)} \|u\|_{L^\infty(I,H^\gamma)}, \tag{5.1.4}$$

$$\begin{aligned} \|F(u)-F(v)\|_{L^1(I,L^2)} &\lesssim \Big(\|u\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)} + \|v\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)} \Big) \|u-v\|_{L^\infty(I,L^2)} \\ &\lesssim T^{1-\frac{\nu-1}{p}} \Big(\|u\|^{\nu-1}_{L^p(I,L^\infty)} + \|v\|^{\nu-1}_{L^p(I,L^\infty)} \Big) \|u-v\|_{L^\infty(I,L^2)}. \end{aligned} \tag{5.1.5}$$

Using that $\gamma - \gamma_{p,q} > d/q$, the Sobolev embedding implies $L^p(I, H_q^{\gamma-\gamma_{p,q}}) \subset L^p(I, L^\infty)$. Thus, we get

$$\|\Phi(u)\|_{L^\infty(I,H^\gamma)} + \|\Phi(u)\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \lesssim \|\psi\|_{H^\gamma} + T^{1-\frac{\nu-1}{p}} \|u\|^{\nu-1}_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \|u\|_{L^\infty(I,H^\gamma)},$$

and

$$d(\Phi(u),\Phi(v)) \lesssim T^{1-\frac{\nu-1}{p}} \Big(\|u\|^{\nu-1}_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} + \|v\|^{\nu-1}_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \Big) \|u-v\|_{L^\infty(I,L^2)}.$$

This shows that for all $u, v \in X$, there exists $C > 0$ independent of $\psi \in H^\gamma$ such that

$$\|\Phi(u)\|_{L^\infty(I,H^\gamma)} + \|\Phi(u)\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \leq C\|\psi\|_{H^\gamma} + CT^{1-\frac{\nu-1}{p}} M^\nu,$$
$$d(\Phi(u), \Phi(v)) \leq CT^{1-\frac{\nu-1}{p}} M^{\nu-1} d(u,v).$$

Therefore, if we set $M = 2C\|\psi\|_{H^\gamma}$ and choose $T > 0$ small enough so that $CT^{1-\frac{\nu-1}{p}} M^{\nu-1} \leq \frac{1}{2}$, then X is stable by Φ and Φ is a contraction on X. By the fixed point theorem, there exists a unique $u \in X$ so that $\Phi(u) = u$.

Step 2. Uniqueness. Consider $u, v \in C(I, H^\gamma) \cap L^p(I, L^\infty)$ two solutions of (NLST). Since the uniqueness is a local property (see [Caz03, Chapter 4]), it suffices to show $u = v$ for T small. We have from (5.1.5) that

$$d(u,v) \leq CT^{1-\frac{\nu-1}{p}} \Big(\|u\|^{\nu-1}_{L^p(I,L^\infty)} + \|v\|^{\nu-1}_{L^p(I,L^\infty)}\Big) d(u,v).$$

We see that if $T > 0$ is small enough, then

$$d(u,v) \leq \frac{1}{2} d(u,v) \text{ or } u = v.$$

Step 3. Item i. Since the time of existence constructed in Step 1 only depends on $\|\psi\|_{H^\gamma}$. The blowup alternative follows by standard argument (see again [Caz03, Chapter 4]).

Step 4. Item ii. Let $\psi_n \to \psi$ in H^γ and $C, T = T(\psi)$ be as in Step 1. Set $M = 4C\|\psi\|_{H^\gamma}$. It follows that $2C\|\psi_n\|_{H^\gamma} \leq M$ for sufficiently large n. Thus the solution u_n constructed in Step 1 belongs to X with $T = T(\psi)$ for n large enough. We have from Strichartz estimate (1.1.18) and (5.1.4) that

$$\|u\|_{L^a(I,H_b^{\gamma-\gamma_{a,b}})} \lesssim \|\psi\|_{H^\gamma} + T^{1-\frac{\nu-1}{p}} \|u\|^{\nu-1}_{L^p(I,L^\infty)} \|u\|_{L^\infty(I,H^\gamma)},$$

provided that (a, b) is Schrödinger admissible and $b < \infty$. This shows the boundedness of u_n in $L^a(I, H_b^{\gamma-\gamma_{a,b}})$. We also have from (5.1.5) and the choice of T that

$$d(u_n, u) \leq C\|\psi_n - \psi\|_{L^2} + \frac{1}{2} d(u_n, u) \text{ or } d(u_n, u) \leq 2C\|\psi_n - \psi\|_{L^2}.$$

This yields that $u_n \to u$ in $L^\infty(I, L^2) \cap L^p(I, H_q^{-\gamma_{p,q}})$. Strichartz estimate (1.1.18) again implies that $u_n \to u$ in $L^a(I, H_b^{-\gamma_{a,b}})$ for any Schrödinger admissible pair (a, b) with $b < \infty$. The convergence in $C(I, H^{\gamma-\epsilon})$ follows from the boundedness in $L^\infty(I, H^\gamma)$, the convergence in $L^\infty(I, L^2)$ and that $\|u\|_{H^{\gamma-\epsilon}} \leq \|u\|^{1-\frac{\epsilon}{\gamma}}_{H^\gamma} \|u\|^{\frac{\epsilon}{\gamma}}_{L^2}$.

Step 5. Item iii. If $\psi \in H^\beta$ for some $\beta > \gamma$ satisfying $\lceil \beta \rceil \leq \nu$ if $\nu > 1$ is not an odd integer, then Step 1 shows the existence of H^β solution defined on some maximal interval $[0, T)$. Since H^β solution is also a H^γ solution, thus $T \leq T^*$. Suppose that $T < T^*$. Then the unitary property of $e^{it|\nabla|^\sigma}$ and Corollary 5.0.5 imply that

$$\|u(t)\|_{H^\beta} \leq \|\psi\|_{H^\beta} + C \int_0^t \|u(s)\|^{\nu-1}_{L^\infty} \|u(s)\|_{H^\beta} ds,$$

for all $0 \leq t < T$. The Gronwall's inequality then gives

$$\|u(t)\|_{H^\beta} \leq \|\psi\|_{H^\beta} \exp\Big(C \int_0^t \|u(s)\|^{\nu-1}_{L^\infty} ds\Big),$$

for all $0 \leq t < T$. Using the fact that $u \in L^{\nu-1}_{\text{loc}}([0, T^*), L^\infty)$, we see that $\limsup \|u(t)\|_{H^\beta} < \infty$ as $t \to T$ which is a contradiction to the blowup alternative in H^β. □

Remark 5.1.2. If we assume that $\nu > 1$ is an odd integer or

$$\lceil \gamma \rceil \leq \nu - 1 \tag{5.1.6}$$

otherwise, then the continuous dependence holds in $C([0,T],H^\gamma)$. Indeed, if the above condition holds true, then the continuous dependence holds in $C(I,H^\gamma)$. To see this, we consider X as above equipped with the following metric

$$d(u,v) := \|u-v\|_{L^\infty(I,H^\gamma)} + \|u-v\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})}.$$

By Item ii of Corollary 5.0.5, we have

$$\begin{aligned}\|F(u)-F(v)\|_{L^1(I,H^\gamma)} &\lesssim (\|u\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)} + \|v\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)})\|u-v\|_{L^\infty(I,H^\gamma)} \\ &+ (\|u\|^{\nu-2}_{L^{\nu-1}(I,L^\infty)} + \|v\|^{\nu-2}_{L^{\nu-1}(I,L^\infty)})(\|u\|_{L^\infty(I,H^\gamma)} + \|v\|_{L^\infty(I,H^\gamma)})\|u-v\|_{L^{\nu-1}(I,L^\infty)}.\end{aligned}$$

Using the Sobolev embedding, we see that for all $u,v \in X$,

$$d(\Phi(u),\Phi(v)) \lesssim T^{1-\frac{\nu-1}{p}} M^{\nu-1} d(u,v).$$

Therefore, the continuity in $C(I,H^\gamma)$ follows as in Step 4.

Proposition 5.1.3. *Let*

$$\begin{cases} \sigma \in (2/3,1) & \text{when } d=1, \\ \sigma \in (1,2) & \text{when } d=2, \\ \sigma \in (3/2,2) & \text{when } d=3. \end{cases} \tag{5.1.7}$$

and $\nu > 1$ be such that $\sigma/2 > \gamma_c$, and also, if ν is not an odd integer, $\lceil \sigma/2 \rceil \leq \nu$. Then for any $\psi \in H^{\sigma/2}$, the solution to (NLST) *given in Theorem* 5.1.1 *can be extended to the whole $\mathbb{R}$ if one of the following is satisfied:*

i. $\mu = 1$.
ii. $\mu = -1, \nu < 1 + 2\sigma/d$.
iii. $\mu = -1, \nu = 1 + 2\sigma/d$ *and* $\|\psi\|_{L^2}$ *is small.*
iv. $\mu = -1$ *and* $\|\psi\|_{H^{\sigma/2}}$ *is small.*

Proof. The assumption (5.1.7) allows us to apply Theorem 5.1.1 with $\gamma = \sigma/2$ and obtain the local well-posedness in $H^{\sigma/2}$. We now prove the global extension using the blowup alternative. Item i follows from the conservation of mass and energy. For Item ii and Item iii, we firstly use Gagliardo-Nirenberg's inequality (see e.g. [Tao06, Proposition A.3]) with the fact that

$$\frac{1}{\nu+1} = \frac{1}{2} - \frac{\theta\sigma}{2d} \text{ or } \theta = \frac{d(\nu-1)}{\sigma(\nu+1)}$$

and the conservation of mass to get

$$\|u(t)\|^{\nu+1}_{L^{\nu+1}} \lesssim \||\nabla|^{\sigma/2}u(t)\|^{\frac{d(\nu-1)}{\sigma}}_{L^2} \|u(t)\|^{\nu+1-\frac{d(\nu-1)}{\sigma}}_{L^2} = \|u(t)\|^{\frac{d(\nu-1)}{\sigma}}_{\dot{H}^{\sigma/2}} \|\psi\|^{\nu+1-\frac{d(\nu-1)}{\sigma}}_{L^2}.$$

Note that here the assumption $\nu \leq 1 + 2\sigma/d$ ensures that $\theta \in (0,1)$. The conservation of mass then gives

$$\frac{1}{2}\|u(t)\|^2_{\dot{H}^{\sigma/2}} = E(u(t)) + \frac{1}{\nu+1}\|u(t)\|^{\nu+1}_{L^{\nu+1}} \lesssim E(\psi) + \frac{1}{\nu+1}\|u(t)\|^{\frac{d(\nu-1)}{\sigma}}_{\dot{H}^{\sigma/2}} \|\psi\|^{\nu+1-\frac{d(\nu-1)}{\sigma}}_{L^2}.$$

If $\nu \in (1, 1+2\sigma/d)$ or $\frac{d(\nu-1)}{\sigma} \in (0,2)$, then $\|u(t)\|_{\dot{H}^{\sigma/2}} \leq C$. This together with the conservation of mass implies the boundedness of $\|u(t)\|_{H^{\sigma/2}}$ and Item ii follows. Item iii is treated similarly with $\|\psi\|_{L^2}$ is small. It remains to show Item iv. By Sobolev embedding with $\frac{1}{2} \leq \frac{1}{\nu+1} + \frac{\sigma}{2d}$, we have

$$\frac{1}{2}\|u(t)\|^2_{\dot{H}^{\sigma/2}} = E(u(t)) + \frac{1}{\nu+1}\|u(t)\|^{\nu+1}_{L^{\nu+1}} \leq E(\psi) + \frac{C}{\nu+1}\|u(t)\|^{\nu+1}_{H^{\sigma/2}}, \tag{5.1.8}$$

for all $t \in [0, T^\star)$. Similarly, we use the Sobolev embedding to bound

$$E(\psi) \leq \frac{1}{2}\|\psi\|^2_{H^{\sigma/2}} + \frac{C}{\nu+1}\|\psi\|^{\nu+1}_{H^{\sigma/2}}.$$

Since $\nu + 1 > 2$, it follows that $E(\psi) \leq \|\psi\|^2_{H^{\sigma/2}}$ provided $\|\psi\|_{H^{\sigma/2}}$ is small enough. Denote

$$\tau := \sup\left\{t \in [0, T^\star) \ : \ \|u(s)\|_{H^{\sigma/2}} \leq 2\|u_0\|_{H^{\sigma/2}}, \ \forall s \leq t\right\}.$$

We want to show $\tau = T^\star$. Indeed, if $\tau < T^\star$, then by the continuity of $t \mapsto \|u(t)\|_{H^{\sigma/2}}$, we have $\|u(\tau)\|_{H^{\sigma/2}} = 2\|u_0\|_{H^{\sigma/2}}$. Inserting it into (5.1.8), we get

$$2\|u_0\|^2_{H^{\sigma/2}} \leq E(u_0) + \frac{C}{\nu+1}(2\|u_0\|_{H^{\sigma/2}})^{\nu+1} \leq \|u_0\|^2_{H^{\sigma/2}} + \frac{C}{\nu+1}(2\|u_0\|_{H^{\sigma/2}})^{\nu+1}.$$

This inequality is not possible for $\|u_0\|_{H^{\sigma/2}}$ is small enough. The proof is complete. □

5.1.2 Local well-posedness in the critical case

We now turn to the local well-posedness and scattering with small data for (NLST) in the critical case.

Theorem 5.1.4. *Let $\sigma \in (0,2)\backslash\{1\}$ and*

$$\begin{cases} \nu > 5 \text{ when } d = 1, \\ \nu > 3 \text{ when } d \geq 2 \end{cases} \tag{5.1.9}$$

be such that $\gamma_{\mathrm{c}} \geq 0$, and also, if ν is not an odd integer, (5.0.2). *Then for all $\psi \in H^{\gamma_{\mathrm{c}}}$, there exist $T^* \in (0,\infty]$ and a unique solution to* (NLST) *satisfying*

$$u \in C([0,T^*), H^{\gamma_{\mathrm{c}}}) \cap L^p_{\mathrm{loc}}([0,T^*), B^{\gamma_{\mathrm{c}}-\gamma_{p,q}}_q),$$

where $p = 4, q = \infty$ when $d = 1$; $2 < p < \nu - 1, q = p^\star = 2p/(p-2)$ when $d = 2$ and $p = 2, q = 2^\star = 2d/(d-2)$ when $d \geq 3$. Moreover, if $\|\psi\|_{\dot{H}^{\gamma_{\mathrm{c}}}} < \varepsilon$ for some $\varepsilon > 0$ small enough, then $T^* = \infty$ and the solution is scattering in $H^{\gamma_{\mathrm{c}}}$, i.e. there exists $\psi^+ \in H^{\gamma_{\mathrm{c}}}$ such that

$$\lim_{t\rightarrow+\infty} \|u(t) - e^{it|\nabla|^\sigma}\psi^+\|_{H^{\gamma_{\mathrm{c}}}} = 0.$$

This theorem is a modification of Theorem 1.2 and Theorem 1.3 in [HS15] where the authors proved the global well-posedness and scattering for small inhomogeneous data. Note that for $\sigma \in (0,2)$, Strichartz estimates for the unitary group $e^{it|\nabla|^\sigma}$ have a loss of derivatives. In the sub-critical case $\gamma > \gamma_{\mathrm{c}}$, the derivative loss is compensated for by using Sobolev embeddings. In the critical case $\gamma = \gamma_{\mathrm{c}}$, the Sobolev embedding does not help. To remove the derivative loss, we use Strichartz norms localized in dyadic pieces, and then sum up in a ℓ^2-fashion. It needs a delicate estimate on $L^{\nu-1}_t L^\infty_x$ (see [HS15, Lemma 3.5]). The range $\nu \in (1,5]$ when $d = 1$ and $\nu \in (1,3]$ still remains open, and it requires another technique rather than Strichartz estimate.

In order to prove Theorem 5.1.4, we need the following estimates which control the nonlinearity.

Lemma 5.1.5 ([HS15])**.** *Let $\sigma \in (0,2)\backslash\{1\}$, ν be as in* (5.1.9)*, γ_{c} as in* (5.0.1)*. Then we have*

$$\|u\|^{\nu-1}_{L^{\nu-1}(\mathbb{R},L^\infty)} \lesssim \begin{cases} \|u\|^4_{L^4(\mathbb{R},\dot{B}^{\gamma_{\mathrm{c}}-\gamma_{4,\infty}}_\infty)}\|u\|^{\nu-5}_{L^\infty(\mathbb{R},\dot{B}^{\gamma_{\mathrm{c}}}_2)} \text{ when } d = 1, \\ \|u\|^p_{L^p(\mathbb{R},\dot{B}^{\gamma_{\mathrm{c}}-\gamma_{p,p^\star}}_{p^\star})}\|u\|^{\nu-1-p}_{L^\infty(\mathbb{R},\dot{B}^{\gamma_{\mathrm{c}}}_2)} \text{ where } \nu-1 > p > 2 \text{ when } d = 2, \\ \|u\|^2_{L^2(\mathbb{R},\dot{B}^{\gamma_{\mathrm{c}}-\gamma_{2,2^\star}}_{2^\star})}\|u\|^{\nu-3}_{L^\infty(\mathbb{R},\dot{B}^{\gamma_{\mathrm{c}}}_2)} \text{ when } d \geq 3, \end{cases}$$

where $p^\star = 2p/(p-2)$ *and* $2^\star = 2d/(d-2)$.

This result is a slight modification of [HS15, Lemma 3.5] which generalizes Lemma 3.1 in [CKSTT5]. The main difference is the exponent power in $\mathbb{R}^2$. For the reader's convenience, we recall some details.

Proof of Lemma 5.1.5. The proof is essentially given in [HS15, Lemma 3.5] which uses a trick of [CKSTT5]. For the reader's convenience, we only give the details for $d=2$. The case $d=1$ and $d\geq 3$ are treated similarly. We refer to [HS15] for the proof when $d \geq 3$ (see also Lemma 5.2.3). By interpolation, we can assume that $\nu - 1 = m/n > 2, m, n \in \mathbb{N}$ with $(\nu-1-p)n \geq 1$. We proceed as in [HS15] and set

$$c_N(t) = N^{\gamma_{\rm c}-\gamma_{p,p^\star}} \|P_N u(t)\|_{L^{p^\star}(\mathbb{R}^2)}, \quad c'_N(t) = N^{\gamma_{\rm c}} \|P_N u(t)\|_{L^2(\mathbb{R}^2)}.$$

Remark that in this case $(p,p^\star)$ is a Schrödinger admissible pair, $\gamma_{\rm c} = 1-\sigma n/m$ and $\gamma_{p,p^\star} = 1 - 2/p^\star - \sigma/p$. By Bernstein's inequality, we have

$$\begin{aligned} \|P_N u(t)\|_{L^\infty(\mathbb{R}^2)} &\lesssim N^{\frac{\sigma n}{m}-\frac{\sigma}{p}} c_N(t), \\ \|P_N u(t)\|_{L^\infty(\mathbb{R}^2)} &\lesssim N^{\frac{\sigma n}{m}} c'_N(t). \end{aligned} \tag{5.1.10}$$

This implies that for $\theta \in (0,1)$ which will be chosen later,

$$\|P_N u(t)\|_{L^\infty(\mathbb{R}^2)} \lesssim N^{\frac{\sigma n}{m}-\frac{\sigma\theta}{p}} (c_N(t))^\theta (c'_N(t))^{1-\theta}. \tag{5.1.11}$$

We next use

$$A(t) := \Big(\sum_{N\in 2^{\mathbb{Z}}} \|P_N u(t)\|_{L^\infty(\mathbb{R}^2)}\Big)^m \lesssim \sum_{N_1\geq \cdots \geq N_m} \prod_{j=1}^m \|P_{N_j} u(t)\|_{L^\infty(\mathbb{R}^2)}.$$

Here the first equality follows from the Sobolev embedding with the fact that $(\gamma_{\rm c} - \gamma_{p,p^\star})p^\star = 2 + (\sigma/p - \sigma/(\nu-1))p^\star > 2$. Estimating the n highest frequencies by (5.1.10) and the rest by (5.1.11), we get

$$A(t) \lesssim \sum_{N_1\geq\cdots\geq N_m} \Big(\prod_{j=1}^n N_j^{\frac{\sigma n}{m}-\frac{\sigma}{p}} c_{N_j}(t)\Big)\Big(\prod_{j=n+1}^m N_j^{\frac{\sigma n}{m}-\frac{\sigma\theta}{p}} (c_{N_j}(t))^\theta (c'_{N_j}(t))^{1-\theta}\Big).$$

For an arbitrary $\delta > 0$, we set

$$\tilde{c}_N(t) = \sum_{N'\in 2^{\mathbb{Z}}} \min(N/N', N'/N)^\delta c_{N'}(t), \quad \tilde{c}'_N(t) = \sum_{N'\in 2^{\mathbb{Z}}} \min(N/N', N'/N)^\delta c'_{N'}(t).$$

Using the fact that $c_N(t) \leq \tilde{c}_N(t)$ and $\tilde{c}_{N_j}(t) \lesssim (N_1/N_j)^\delta \tilde{c}_{N_1}(t)$ for $j=2,...,m$ and similarly for primes, we see that

$$A(t) \lesssim \sum_{N_1\geq\cdots\geq N_m} \Big(\prod_{j=1}^n N_j^{\frac{\sigma n}{m}-\frac{\sigma}{p}} (N_1/N_j)^\delta \tilde{c}_{N_1}(t)\Big)\Big(\prod_{j=n+1}^m N_j^{\frac{\sigma n}{m}-\frac{\sigma\theta}{p}} (N_1/N_j)^\delta (\tilde{c}_{N_1}(t))^\theta (\tilde{c}'_{N_1}(t))^{1-\theta}\Big).$$

We can rewrite the above quantity in the right hand side as

$$\sum_{N_1\geq\cdots\geq N_m} \Big(\prod_{j=n+1}^m N_j^{\frac{\sigma n}{m}-\frac{\sigma\theta}{p}-\delta}\Big)\Big(\prod_{j=2}^n N_j^{\frac{\sigma n}{m}-\frac{\sigma}{p}-\delta}\Big) N_1^{\frac{\sigma n}{m}-\frac{\sigma}{p}+(m-1)\delta} (\tilde{c}_{N_1}(t))^{n+(m-n)\theta} (\tilde{c}'_{N_1}(t))^{(m-n)(1-\theta)}.$$

Next, we choose $\theta = (p-1)/(\nu-2) \in (0,1)$ and $\delta > 0$ such that

$$\frac{\sigma n}{m} - \frac{\sigma\theta}{p} - \delta > 0, \quad \frac{\sigma n}{m} - \frac{\sigma}{p} + (m-1)\delta < 0 \quad \text{or} \quad \delta < \frac{\sigma(m-np)}{pm(m-1)}.$$

Here condition $\nu > 3$ allows us to choose $p > 2$ such that $m - np > 0$. Summing in N_m, then in N_{m-1},..., then in N_2, we have

$$A(t) \lesssim \sum_{N_1 \in 2^{\mathbb{Z}}} (\tilde{c}_{N_1}(t))^{pn} (\tilde{c}'_{N_1}(t))^{(\nu-1-p)n}.$$

The Hölder inequality with the fact that $(\nu - 1 - p)n \geq 1$ implies

$$\begin{aligned} A(t) &\lesssim \|(\tilde{c}(t))^{pn}\|_{\ell^2(2^{\mathbb{Z}})} \|(\tilde{c}'(t))^{(\nu-1-p)n}\|_{\ell^2(2^{\mathbb{Z}})} \\ &= \|\tilde{c}(t)\|^{pn}_{\ell^{2pn}(2^{\mathbb{Z}})} \|\tilde{c}'(t)\|^{(\nu-1-p)n}_{\ell^{2(\nu-1-p)n}(2^{\mathbb{Z}})} \leq \|\tilde{c}(t)\|^{pn}_{\ell^2(2^{\mathbb{Z}})} \|\tilde{c}'(t)\|^{(\nu-1-p)n}_{\ell^2(2^{\mathbb{Z}})}, \end{aligned}$$

where $\|\tilde{c}(t)\|_{\ell^q(2^{\mathbb{Z}})} := \Big(\sum_{N \in 2^{\mathbb{Z}}} |\tilde{c}_N(t)|^q \Big)^{1/q}$ and similarly for prime. The Minkowski inequality then implies

$$A(t) \lesssim \|c(t)\|^{pn}_{\ell^2(2^{\mathbb{Z}})} \|c'(t)\|^{(\nu-1-p)n}_{\ell^2(2^{\mathbb{Z}})}.$$

This implies that $A(t) < \infty$ for amost every where t, hence that $\sum_N \|P_N u(t)\|_{L^\infty(\mathbb{R}^d)} < \infty$. Therefore $\sum_N P_N u(t)$ converges in $L^\infty(\mathbb{R}^d)$. Since it converges to u in the ditribution sense, so the limit is $u(t)$. Thus

$$\begin{aligned} \|u\|^{\nu-1}_{L^{\nu-1}(\mathbb{R}, L^\infty(\mathbb{R}^2))} &= \int_{\mathbb{R}} \|u(t)\|^{m/n}_{L^\infty(\mathbb{R}^2)} dt \lesssim \int_{\mathbb{R}} \|c(t)\|^p_{\ell^2(2^{\mathbb{Z}})} \|c'(t)\|^{\nu-1-p}_{\ell^2(2^{\mathbb{Z}})} dt \\ &\lesssim \|c\|^p_{L^p_{\mathbb{R}} \ell^2(2^{\mathbb{Z}})} \|c'\|^{\nu-1-p}_{L^\infty_{\mathbb{R}} \ell^2(2^{\mathbb{Z}})} = \|u\|^p_{L^p(\mathbb{R}, \dot{B}^{\gamma_c - \gamma_{p,p^\star}}_{p^\star}(\mathbb{R}^2))} \|u\|^{\nu-1-p}_{L^\infty(\mathbb{R}, \dot{B}^{\gamma_c}_2(\mathbb{R}^2))}. \end{aligned}$$

The proof is complete. □

Proof of Theorem 5.1.4. As in the proof of Theorem 5.1.1, we proceed in several steps.

Step 1. Existence. We only treat for $d \geq 3$, the ones for $d = 1, d = 2$ are completely similar. Let us consider

$$X := \Big\{ u \in L^\infty(I, H^{\gamma_c}) \cap L^2(I, B^{\gamma_c - \gamma_{2,2^\star}}_{2^\star}) \mid \|u\|_{L^\infty(I, \dot{H}^{\gamma_c})} \leq M, \|u\|_{L^2(I, \dot{B}^{\gamma_c - \gamma_{2,2^\star}}_{2^\star})} \leq N \Big\},$$

equipped with the distance

$$d(u, v) := \|u - v\|_{L^\infty(I, L^2)} + \|u - v\|_{L^2(I, \dot{B}^{-\gamma_{2,2^\star}}_{2^\star})},$$

where $I = [0, T]$ and $T, M, N > 0$ will be chosen later. One can check (see again [CW90] or [Caz03, Chapter 4]) that (X, d) is a complete metric space. Using the Duhamel formula

$$\Phi(u)(t) = e^{it|\nabla|^\sigma} \psi + i\mu \int_0^t e^{i(t-s)|\nabla|^\sigma} |u(s)|^{\nu-1} u(s) ds =: u_{\text{hom}}(t) + u_{\text{inh}}(t), \tag{5.1.12}$$

the Strichartz estimate (1.1.4) yields

$$\|u_{\text{hom}}\|_{L^2(I, \dot{B}^{\gamma_c - \gamma_{2,2^\star}}_{2^\star})} \lesssim \|\psi\|_{\dot{H}^{\gamma_c}}.$$

A similar estimate holds for $\|u_{\text{hom}}\|_{L^\infty(I, \dot{H}^{\gamma_c})}$. We see that $\|u_{\text{hom}}\|_{L^2(I, \dot{B}^{\gamma_c - \gamma_{2,2^\star}}_{2^\star})} \leq \varepsilon$ for some $\varepsilon > 0$ small enough which will be chosen later, provided that either $\|\psi\|_{\dot{H}^{\gamma_c}}$ is small or it is satisfied for some $T > 0$ small enough. Therefore, we can take $T = \infty$ in the first case and T be this small time in the second. On the other hand, using again (1.1.4), we have

$$\|u_{\text{inh}}\|_{L^2(I, \dot{B}^{\gamma_c - \gamma_{2,2^\star}}_{2^\star})} \lesssim \|F(u)\|_{L^1(I, \dot{H}^{\gamma_c})}.$$

The same estimate holds for $\|u_{\text{inh}}\|_{L^\infty(I,\dot H^{\gamma_c})}$. Corollary 5.0.5 and Lemma 5.1.5 give

$$\|F(u)\|_{L^1(I,\dot H^{\gamma_c})} \lesssim \|u\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)}\|u\|_{L^\infty(I,\dot H^{\gamma_c})} \lesssim \|u\|^2_{L^2(I,\dot B^{\gamma_c-\gamma_{2,2^\star}}_{2^\star})}\|u\|^{\nu-2}_{L^\infty(I,\dot H^{\gamma_c})}. \tag{5.1.13}$$

Similarly, we have

$$\begin{aligned}\|F(u)-F(v)\|_{L^1(I,L^2)} &\lesssim \Big(\|u\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)} + \|v\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)}\Big)\|u-v\|_{L^\infty(I,L^2)} \\ &\lesssim \Big(\|u\|^2_{L^2(I,\dot B^{\gamma_c-\gamma_{2,2^\star}}_{2^\star})}\|u\|^{\nu-3}_{L^\infty(I,\dot H^{\gamma_c})} + \|v\|^2_{L^2(I,\dot B^{\gamma_c-\gamma_{2,2^\star}}_{2^\star})}\|v\|^{\nu-3}_{L^\infty(I,\dot H^{\gamma_c})}\Big)\|u-v\|_{L^\infty(I,L^2)}.\end{aligned} \tag{5.1.14}$$

This implies for all $u, v \in X$, there exists $C > 0$ independent of $\psi \in H^{\gamma_c}$ such that

$$\begin{aligned}\|\Phi(u)\|_{L^2(I,\dot B^{\gamma_c-\gamma_{2,2^\star}}_{2^\star})} &\leq \varepsilon + CN^2M^{\nu-2}, \\ \|\Phi(u)\|_{L^\infty(I,\dot H^{\gamma_c})} &\leq C\|\psi\|_{\dot H^{\gamma_c}} + CN^2M^{\nu-2}, \\ d(\Phi(u),\Phi(v)) &\leq CN^2M^{\nu-3}d(u,v).\end{aligned}$$

Now by setting $N = 2\varepsilon$ and $M = 2C\|\psi\|_{\dot H^{\gamma_c}}$ and choosing $\varepsilon > 0$ small enough such that $CN^2M^{\nu-3} \leq \min\{1/2, \varepsilon/M\}$, we see that X is stable by Φ and Φ is a contraction on X. By the fixed point theorem, there exists a unique solution $u \in X$ to (NLST). Note that when $\|\psi\|_{\dot H^{\gamma_c}}$ is small enough, we can take $T = \infty$.

Step 2. Uniqueness. The uniqueness in $C^\infty(I, H^{\gamma_c}) \cap L^2(I, B^{\gamma_c-\gamma_{2,2^\star}}_{2^\star})$ follows as in Step 2 of the proof of Theorem 5.1.1 using (5.1.14). Here $\|u\|_{L^2(I,\dot B^{\gamma_c-\gamma_{2,2^\star}}_{2^\star})}$ can be small as T is small.

Step 3. Scattering. The global existence when $\|\psi\|_{\dot H^{\gamma_c}}$ is small is given in Step 1. It remains to show the scattering property. Thanks to (5.1.13), we see that

$$\begin{aligned}\|e^{-it_2|\nabla|^\sigma}u(t_2) - e^{-it_1|\nabla|^\sigma}u(t_1)\|_{\dot H^{\gamma_c}} &= \Big\|i\mu\int_{t_1}^{t_2} e^{-is|\nabla|^\sigma}(|u|^{\nu-1}u)(s)ds\Big\|_{\dot H^{\gamma_c}} \\ &\leq \|F(u)\|_{L^1([t_1,t_2],\dot H^{\gamma_c})} \lesssim \|u\|^2_{L^2([t_1,t_2],\dot B^{\gamma_c-\gamma_{2,2^\star}}_{2^\star})}\|u\|^{\nu-2}_{L^\infty([t_1,t_2],\dot H^{\gamma_c})} \to 0\end{aligned} \tag{5.1.15}$$

as $t_1, t_2 \to +\infty$. We have from (5.1.14) that

$$\|e^{-it_2|\nabla|^\sigma}u(t_2) - e^{-it_1|\nabla|^\sigma}u(t_1)\|_{L^2} \lesssim \|u\|^2_{L^2([t_1,t_2],\dot B^{\gamma_c-\gamma_{2,2^\star}}_{2^\star})}\|u\|^{\nu-3}_{L^\infty([t_1,t_2],\dot H^{\gamma_c})}\|u\|_{L^\infty([t_1,t_2],L^2)}, \tag{5.1.16}$$

which also tends to zero as $t_1, t_2 \to +\infty$. This implies that the limit

$$\psi^+ := \lim_{t\to+\infty} e^{-it|\nabla|^\sigma}u(t)$$

exists in H^{γ_c}. Moreover, we have

$$u(t) - e^{it|\nabla|^\sigma}\psi^+ = -i\mu\int_t^{+\infty} e^{i(t-s)|\nabla|^\sigma}F(u(s))ds.$$

The unitary property of $e^{it|\nabla|^\sigma}$ in L^2, (5.1.15) and (5.1.16) imply that $\|u(t) - e^{it|\nabla|^\sigma}\psi^+\|_{H^{\gamma_c}} \to 0$ when $t \to +\infty$. This completes the proof of Theorem 5.1.4. □

5.2 Local well-posedness for nonlinear half-wave equation in Sobolev spaces

5.2.1 Local well-posedness in the subcritical case

We have the following local well-posedness in the subcritical case.

Theorem 5.2.1. *Let $\gamma \geq 0$ and $\nu > 1$ be such that*

$$\begin{cases} \gamma > 1 - 1/\max(\nu - 1, 4) & \text{when } d = 2, \\ \gamma > d/2 - 1/\max(\nu - 1, 2) & \text{when } d \geq 3, \end{cases} \tag{5.2.1}$$

and also, if ν is not an odd integer, (5.0.2). *Let*

$$\begin{cases} p > \max(\nu - 1, 4) & \text{when } d = 2 \\ p > \max(\nu - 1, 2) & \text{when } d \geq 3 \end{cases} \tag{5.2.2}$$

be such that $\gamma > \frac{d}{2} - \frac{1}{p}$. Then for all $\psi \in H^\gamma$, there exist $T^* \in (0, \infty]$ and a unique solution to (NLHW) *satisfying*

$$u \in C([0, T^*), H^\gamma) \cap L^p_{\mathrm{loc}}([0, T^*), L^\infty).$$

Moreover, the following properties hold:

(i) *If $T^* < \infty$, then $\|u(t)\|_{H^\gamma} \to \infty$ as $t \to T^*$.*

(ii) *u depends continuously on ψ in the following sense. There exists $T \in (0, T^*)$ such that if $\psi_n \to \psi$ in H^γ and if u_n denotes the solution of* (NLHW) *with initial data ψ_n, then $0 < T < T^*(\psi_n)$ for all n sufficiently large and u_n is bounded in $L^a([0, T], H_b^{\gamma - \gamma_{a,b}})$ for any wave admissible pair (a, b) with $b < \infty$. Moreover, $u_n \to u$ in $L^a([0, T], H_b^{-\gamma_{a,b}})$ as $n \to \infty$. In particular, $u_n \to u$ in $C([0, T], H^{\gamma - \epsilon})$ for all $0 < \epsilon < \gamma$.*

(iii) *Let $\beta > \gamma$ be such that if ν is not an odd integer, $\lceil \beta \rceil \leq \nu$. If $\psi \in H^\beta$, then $u \in C([0, T^*), H^\beta)$.*

As in Remark 5.1.2, the continuous dependence can be improved to hold in $C([0, T], H^\gamma)$ if we assume that $\nu > 1$ is an odd integer or $\lceil \gamma \rceil \leq \nu - 1$ otherwise.

Proof of Theorem 5.2.1. The proof is similar to the one for Theorem 5.1.4 by using Strichartz estimates for the linear half-wave equation. For the reader's convenience, we give a sketch of the proof for the local existence. Let p be as in (5.2.2) and then choose $q \in [2, \infty)$ such that

$$\frac{2}{p} + \frac{d-1}{q} \leq \frac{d-1}{2}.$$

Let us consider

$$X := \Big\{ u \in L^\infty(I, H^\gamma) \cap L^p(I, H_q^{\gamma - \gamma_{p,q}}) \mid \|u\|_{L^\infty(I, H^\gamma)} + \|u\|_{L^p(I, H_q^{\gamma - \gamma_{p,q}})} \leq M \Big\},$$

equipped with the distance

$$d(u, v) := \|u - v\|_{L^\infty(I, L^2)} + \|u - v\|_{L^p(I, H_q^{-\gamma_{p,q}})},$$

where $I = [0, T]$ and $M, T > 0$ to be chosen later. By the Duhamel formula, it suffices to prove that the functional

$$\Phi(u)(t) = e^{it|\nabla|}\psi + i\mu \int_0^t e^{i(t-s)|\nabla|} |u(s)|^{\nu - 1} u(s) ds$$

is a contraction on (X, d). The Strichartz estimate (1.2.10) yields

$$\|\Phi(u)\|_{L^\infty(I, H^\gamma)} + \|\Phi(u)\|_{L^p(I, H_q^{\gamma - \gamma_{p,q}})} \lesssim \|\psi\|_{H^\gamma} + \|F(u)\|_{L^1(I, H^\gamma)},$$

$$\|\Phi(u) - \Phi(v)\|_{L^\infty(I, L^2)} + \|\Phi(u) - \Phi(v)\|_{L^p(I, H_q^{-\gamma_{p,q}})} \lesssim \|F(u) - F(v)\|_{L^1(I, L^2)},$$

where $F(u) = |u|^{\nu-1}u$ and similarly for $F(v)$. By our assumptions on ν, Corollary 5.0.5 gives

$$\|F(u)\|_{L^1(I,H^\gamma)} \lesssim \|u\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)}\|u\|_{L^\infty(I,H^\gamma)} \lesssim T^{1-\frac{\nu-1}{p}}\|u\|^{\nu-1}_{L^p(I,L^\infty)}\|u\|_{L^\infty(I,H^\gamma)}, \tag{5.2.3}$$

$$\begin{aligned}\|F(u)-F(v)\|_{L^1(I,L^2)} &\lesssim \Big(\|u\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)} + \|v\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)}\Big)\|u-v\|_{L^\infty(I,L^2)} \\ &\lesssim T^{1-\frac{\nu-1}{p}}\Big(\|u\|^{\nu-1}_{L^p(I,L^\infty)} + \|v\|^{\nu-1}_{L^p(I,L^\infty)}\Big)\|u-v\|_{L^\infty(I,L^2)}. \end{aligned} \tag{5.2.4}$$

The Sobolev embedding with the fact that $\gamma - \gamma_{p,q} > d/q$ implies $L^p(I, H^{\gamma-\gamma_{p,q}}_q) \subset L^p(I, L^\infty)$. Thus, we get

$$\|\Phi(u)\|_{L^\infty(I,H^\gamma)} + \|\Phi(u)\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} \lesssim \|\psi\|_{H^\gamma} + T^{1-\frac{\nu-1}{p}}\|u\|^{\nu-1}_{L^p(I,H_q^{\gamma-\gamma_{p,q}})}\|u\|_{L^\infty(I,H^\gamma)},$$

and

$$d(\Phi(u),\Phi(v)) \lesssim T^{1-\frac{\nu-1}{p}}\Big(\|u\|^{\nu-1}_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} + \|v\|^{\nu-1}_{L^p(I,H_q^{\gamma-\gamma_{p,q}})}\Big)\|u-v\|_{L^\infty(I,L^2)}.$$

This shows that for all $u, v \in X$, there exists $C > 0$ independent of $\psi \in H^\gamma$ and T such that

$$\begin{aligned}\|\Phi(u)\|_{L^\infty(I,H^\gamma)} + \|\Phi(u)\|_{L^p(I,H_q^{\gamma-\gamma_{p,q}})} &\leq C\|\psi\|_{H^\gamma} + CT^{1-\frac{\nu-1}{p}}M^\nu, \\ d(\Phi(u),\Phi(v)) &\leq CT^{1-\frac{\nu-1}{p}}M^{\nu-1}d(u,v).\end{aligned}$$

Therefore, if we set $M = 2C\|\psi\|_{H^\gamma}$ and choose $T > 0$ small enough so that $CT^{1-\frac{\nu-1}{p}}M^{\nu-1} \leq \frac{1}{2}$, then X is stable by Φ and Φ is a contraction on X. By the fixed point theorem, there exists a unique $u \in X$ so that $\Phi(u) = u$. □

5.2.2 Local well-posedness in the critical case

We also have the following local well-posedness and scattering with small data for (NLHW) which is similar to (NLST) with $\sigma \in (0,2)\backslash\{1\}$ in the critical case.

Theorem 5.2.2. *Let*

$$\left\{\begin{array}{ll} \nu > 5 & \text{when } d = 2, \\ \nu > 3 & \text{when } d \geq 3, \end{array}\right. \tag{5.2.5}$$

and also, if ν is not an odd integer, (5.0.2). *Then for all $\psi \in H^{\gamma_c}$, there exist $T^* \in (0,\infty]$ and a unique solution to* (NLHW) *satisfying*

$$u \in C([0,T^*), H^{\gamma_c}) \cap L^p_{\text{loc}}([0,T^*), B^{\gamma_c-\gamma_{p,q}}_q),$$

where $p = 4, q = \infty$ when $d = 2$; $2 < p < \nu - 1, q = p^\star = 2p/(p-2)$ when $d = 3$; $p = 2, q = 2^\star = 2(d-1)/(d-3)$ when $d \geq 4$. Moreover, if $\|\psi\|_{\dot H^{\gamma_c}} < \varepsilon$ for some $\varepsilon > 0$ small enough, then $T^* = \infty$ and the solution is scattering in H^{γ_c}, i.e. there exists $\psi^+ \in H^{\gamma_c}$ such that

$$\lim_{t\to+\infty} \|u(t) - e^{it|\nabla|}\psi^+\|_{H^{\gamma_c}} = 0.$$

In order to prove Theorem 5.2.2, we need the following estimates which control the nonlinearity.

Lemma 5.2.3. *Let ν be as in Theorem* 5.2.2 *and γ_c as in* (5.0.1). *Then*

$$\|u\|^{\nu-1}_{L^{\nu-1}(\mathbb{R},L^\infty)} \lesssim \left\{\begin{array}{ll} \|u\|^4_{L^4(\mathbb{R},\dot B^{\gamma_c-\gamma_{4,\infty}}_\infty)}\|u\|^{\nu-5}_{L^\infty(\mathbb{R},\dot B^{\gamma_c}_2)} & \text{when } d = 2, \\ \|u\|^p_{L^p(\mathbb{R},\dot B^{\gamma_c-\gamma_{p,p^\star}}_{p^\star})}\|u\|^{\nu-1-p}_{L^\infty(\mathbb{R},\dot B^{\gamma_c}_2)} & \text{where } 2 < p < \nu-1 \text{ when } d = 3, \\ \|u\|^2_{L^2(\mathbb{R},\dot B^{\gamma_c-\gamma_{2,2^\star}}_{2^\star})}\|u\|^{\nu-3}_{L^\infty(\mathbb{R},\dot B^{\gamma_c}_2)} & \text{when } d \geq 4, \end{array}\right.$$

where $p^\star = 2p/(p-2)$ and $2^\star = 2(d-1)/(d-3)$.

The above lemma follows the same spirit as [HS15, Lemma 3.5] using the argument of [CKSTT5, Lemma 3.1]. The proof is similar to Lemma 5.1.5, we thus omit the details.

Proof of Theorem 5.2.2. As before, we use the standard contraction mapping argument. The proof is done in several steps.

Step 1. Existence. We only treat for $d \geq 4$, the ones for $d = 2, d = 3$ are completely similar. Let us consider

$$X := \Big\{ u \in L^\infty(I, H^{\gamma_c}) \cap L^2(I, B^{\gamma_c - \gamma_{2,2^\star}}_{2^\star}) \mid \|u\|_{L^\infty(I, \dot{H}^{\gamma_c})} \leq M, \|u\|_{L^2(I, \dot{B}^{\gamma_c - \gamma_{2,2^\star}}_{2^\star})} \leq N \Big\},$$

equipped with the distance

$$d(u,v) := \|u - v\|_{L^\infty(I, L^2)} + \|u - v\|_{L^2(I, \dot{B}^{-\gamma_{2,2^\star}}_{2^\star})},$$

where $I = [0, T]$ and $T, M, N > 0$ will be chosen later. One can check (see e.g. [CW90] or [Caz03, Chapter 4]) that (X, d) is a complete metric space. We will show that the functional

$$\Phi(u)(t) = e^{it|\nabla|}\psi + i\mu \int_0^t e^{i(t-s)|\nabla|} |u(s)|^{\nu-1} u(s) ds =: u_{\text{hom}}(t) + u_{\text{inh}}(t),$$

is a contraction on (X, d). The Strichartz estimate (1.2.3) yields

$$\|u_{\text{hom}}\|_{L^2(I, \dot{B}^{\gamma_c - \gamma_{2,2^\star}}_{2^\star})} \lesssim \|\psi\|_{\dot{H}^{\gamma_c}}. \tag{5.2.6}$$

We see that $\|u_{\text{hom}}\|_{L^2(I, \dot{B}^{\gamma_c - \gamma_{2,2^\star}}_{2^\star})} \leq \varepsilon$ for some $\varepsilon > 0$ small enough which will be chosen later, provided that either $\|\psi\|_{\dot{H}^{\gamma_c}}$ is small or it is satisfied for some $T > 0$ small enough by the dominated convergence theorem. Therefore, we can take $T = \infty$ in the first case and T be this small time in the second. A similar estimate to (5.2.6) holds for $\|u_{\text{hom}}\|_{L^\infty(I, \dot{H}^{\gamma_c})}$. On the other hand, using again (1.2.3), we have

$$\|u_{\text{inh}}\|_{L^2(I, \dot{B}^{\gamma_c - \gamma_{2,2^\star}}_{2^\star})} \lesssim \|F(u)\|_{L^1(I, \dot{H}^{\gamma_c})}.$$

The same estimate holds for $\|u_{\text{inh}}\|_{L^\infty(I, \dot{H}^{\gamma_c})}$. Corollary 5.0.5 and Lemma 5.2.3 give

$$\|F(u)\|_{L^1(I, \dot{H}^{\gamma_c})} \lesssim \|u\|^{\nu-1}_{L^{\nu-1}(I, L^\infty)} \|u\|_{L^\infty(I, \dot{H}^{\gamma_c})} \lesssim \|u\|^2_{L^2(I, \dot{B}^{\gamma_c - \gamma_{2,2^\star}}_{2^\star})} \|u\|^{\nu-2}_{L^\infty(I, \dot{H}^{\gamma_c})}. \tag{5.2.7}$$

Similarly, we have

$$\begin{aligned} \|F(u) - F(v)\|_{L^1(I, L^2)} &\lesssim \Big(\|u\|^{\nu-1}_{L^{\nu-1}(I, L^\infty)} + \|v\|^{\nu-1}_{L^{\nu-1}(I, L^\infty)} \Big) \|u - v\|_{L^\infty(I, L^2)} \\ &\lesssim \Big(\|u\|^2_{L^2(I, \dot{B}^{\gamma_c - \gamma_{2,2^\star}}_{2^\star})} \|u\|^{\nu-3}_{L^\infty(I, \dot{H}^{\gamma_c})} + \|v\|^2_{L^2(I, \dot{B}^{\gamma_c - \gamma_{2,2^\star}}_{2^\star})} \|v\|^{\nu-3}_{L^\infty(I, \dot{H}^{\gamma_c})} \Big) \|u - v\|_{L^\infty(I, L^2)}. \end{aligned} \tag{5.2.8}$$

This implies for all $u, v \in X$, there exists $C > 0$ independent of $\psi \in H^{\gamma_c}$ such that

$$\begin{aligned} \|\Phi(u)\|_{L^2(I, \dot{B}^{\gamma_c - \gamma_{2,2^\star}}_{2^\star})} &\leq \varepsilon + CN^2 M^{\nu-2}, \\ \|\Phi(u)\|_{L^\infty(I, \dot{H}^{\gamma_c})} &\leq C\|\psi\|_{\dot{H}^{\gamma_c}} + CN^2 M^{\nu-2}, \\ d(\Phi(u), \Phi(v)) &\leq CN^2 M^{\nu-3} d(u,v). \end{aligned}$$

Now by setting $N = 2\varepsilon$ and $M = 2C\|\psi\|_{\dot{H}^{\gamma_c}}$ and choosing $\varepsilon > 0$ small enough such that $CN^2 M^{\nu-3} \leq \min\{1/2, \varepsilon/M\}$, we see that X is stable by Φ and Φ is a contraction on X. By the fixed point theorem, there exists a unique solution $u \in X$ to (NLHW). Note that when $\|\psi\|_{\dot{H}^{\gamma_c}}$ is small enough, we can take $T = \infty$.

Step 2. Uniqueness. The uniqueness in $C^\infty(I, H^{\gamma_c}) \cap L^2(I, B^{\gamma_c - \gamma_{2,2^\star}}_{2^\star})$ follows as in Step 2 of the proof of Theorem 5.2.1 using (5.2.8). Here $\|u\|_{L^2(I,\dot{B}^{\gamma_c-\gamma_{2,2^\star}}_{2^\star})}$ can be small as T is small.

Step 3. Scattering. The global existence for $\|\psi\|_{\dot{H}^{\gamma_c}}$ small is given in Step 1. It remains to show the scattering property. Thanks to (5.2.7), we see that

$$\begin{aligned}\|e^{-it_2|\nabla|}u(t_2) - e^{-it_1|\nabla|}u(t_1)\|_{\dot{H}^{\gamma_c}} &= \Big\| i\mu \int_{t_1}^{t_2} e^{-is|\nabla|}(|u|^{\nu-1}u)(s)ds \Big\|_{\dot{H}^{\gamma_c}} \\ &\leq \|F(u)\|_{L^1([t_1,t_2],\dot{H}^{\gamma_c})} \lesssim \|u\|^2_{L^2([t_1,t_2],\dot{B}^{\gamma_c-\gamma_{2,2^\star}}_{2^\star})}\|u\|^{\nu-2}_{L^\infty([t_1,t_2],\dot{H}^{\gamma_c})} \to 0 \end{aligned} \tag{5.2.9}$$

as $t_1,t_2 \to +\infty$. We have from (5.2.8) that

$$\|e^{-it_2|\nabla|}u(t_2) - e^{-it_1|\nabla|}u(t_1)\|_{L^2} \lesssim \|u\|^2_{L^2([t_1,t_2],\dot{B}^{\gamma_c-\gamma_{2,2^\star}}_{2^\star})}\|u\|^{\nu-3}_{L^\infty([t_1,t_2],\dot{H}^{\gamma_c})}\|u\|_{L^\infty([t_1,t_2],L^2)}, \tag{5.2.10}$$

which also tends to zero as $t_1,t_2 \to +\infty$. This implies that the limit

$$\psi^+ := \lim_{t\to+\infty} e^{-it|\nabla|}u(t)$$

exists in H^{γ_c}. Moreover, we have

$$u(t) - e^{it|\nabla|}\psi^+ = -i\mu\int_t^{+\infty} e^{i(t-s)|\nabla|}F(u(s))ds.$$

The unitary property of $e^{it|\nabla|}$ in L^2, (5.2.9) and (5.2.10) imply that $\|u(t) - e^{it|\nabla|}\psi^+\|_{H^{\gamma_c}} \to 0$ when $t \to +\infty$. This completes the proof of Theorem 5.2.2. □

5.3 Local well-posedness for Schrödinger-type equations in Sobolev spaces when $\sigma \in [2,\infty)$

In this case, due to better Strichartz estimates, we can obtain the local well-posedness for (NLST) in H^γ with $\gamma \geq 0$. Our first result concerns the local well-posedness of (NLST) in H^γ with $\gamma \in [0,d/2)$ in both subcritical and critical cases.

Theorem 5.3.1. *Given $\sigma \in [2,\infty)$ and $\nu > 1$. Let $\gamma \in [0,d/2)$ be such that $\gamma \geq \gamma_c$, and also, if ν is not an odd integer,* (5.0.2). *Let*

$$p = \frac{2\sigma(\nu+1)}{(\nu-1)(d-2\gamma)}, \quad q = \frac{d(\nu+1)}{d+(\nu-1)\gamma}. \tag{5.3.1}$$

Then for all $\psi \in H^\gamma$, there exist $T^* \in (0,\infty]$ and a unique solution to (NLST) *satisfying*

$$u \in C([0,T^*),H^\gamma) \cap L^p_{\rm loc}([0,T^*),H^\gamma_q).$$

Moreover, the following properties hold:

i. *$u \in L^a_{\rm loc}([0,T^*),H^\gamma_b)$ for any Schrödinger admissible pair (a,b) with $b<\infty$ and $\gamma_{a,b}=0$.*
ii. *If $\gamma > \gamma_c$ and $T^*<\infty$, then $\|u(t)\|_{\dot{H}^\gamma} \to \infty$ as $t \to T^*$.*
iii. *If $\gamma = \gamma_c$ and $T^*<\infty$, then $\|u\|_{L^p([0,T^*),H^{\gamma_c}_q)} = \infty$.*
iv. *u depends continuously on ψ in the following sense. There exists $T \in (0,T^*)$ such that if $\psi_n \to \psi$ in H^γ and if u_n denotes the solution of* (NLST) *with initial data ψ_n, then $0<T<T^*(\psi_n)$ for all n sufficiently large and u_n is bounded in $L^a([0,T],H^\gamma_b)$ for any Schrödinger admissible pair (a,b) with $\gamma_{a,b}=0$ and $b<\infty$. Moreover, $u_n \to u$ in $L^a([0,T],L^b)$ as $n\to\infty$. In particular, $u_n \to u$ in $C([0,T],H^{\gamma-\epsilon})$ for all $0<\epsilon<\gamma$.*
v. *If $\gamma=\gamma_c$ and $\|\psi\|_{\dot{H}^{\gamma_c}} < \varepsilon$ for some $\varepsilon>0$ small enough, then $T^*=\infty$ and the solution is*

scattering in H^{γ_c}, i.e. there exists $\psi^+ \in H^{\gamma_c}$ such that

$$\lim_{t\to+\infty} \|u(t) - e^{it|\nabla|^\sigma}\psi^+\|_{H^{\gamma_c}} = 0.$$

Proof. Let us firstly comment about the choice of (p,q) given in (5.3.1). It is easy to see that (p,q) is Schrödinger admissible and $\gamma_{p,q} = 0 = \gamma_{p',q'} + 4$. This allows us to use Strichartz estimate (1.1.17) for (p,q). Moreover, if we choose (m,n) so that

$$\frac{1}{p'} = \frac{1}{m} + \frac{\nu-1}{p}, \quad \frac{1}{q'} = \frac{1}{q} + \frac{\nu-1}{n}, \tag{5.3.2}$$

Thanks to this choice of n, we have the Sobolev embedding $\dot{H}^\gamma_q \hookrightarrow L^n$ since

$$q \leq n = \frac{dq}{d-\gamma q}.$$

Step 1. Existence. Let us consider

$$X := \Big\{ u \in L^p(I, H^\gamma_q) \mid \|u\|_{L^p(I,\dot{H}^\gamma_q)} \leq M \Big\},$$

equipped with the distance

$$d(u,v) = \|u-v\|_{L^p(I,L^q)},$$

where $I = [0,T]$ and $M, T > 0$ to be chosen later. It is easy to verify (see e.g. [CW90] or [Caz03, Chapter 4]) that (X,d) is a complete metric space. By the Duhamel formula, it suffices to prove that the functional (5.1.3) is a contraction on (X,d).

Let us firstly consider the case $\gamma > \gamma_c$. In this case, we have $1 < m < p$ and

$$\frac{1}{m} - \frac{1}{p} = 1 - \frac{(\nu-1)(d-2\gamma)}{2\sigma} =: \theta > 0. \tag{5.3.3}$$

Using Strichartz estimate (1.1.17), we obtain

$$\begin{aligned} \|\Phi(u)\|_{L^p(I,\dot{H}^\gamma_q)} &\lesssim \|\psi\|_{\dot{H}^\gamma} + \|F(u)\|_{L^{p'}(I,\dot{H}^\gamma_{q'})}, \\ \|\Phi(u)-\Phi(v)\|_{L^p(I,L^q)} &\lesssim \|F(u)-F(v)\|_{L^{p'}(I,L^{q'})}, \end{aligned}$$

where $F(u) = |u|^{\nu-1}u$ and similarly for $F(v)$. It then follows from Corollary 5.0.3, (5.3.2), Sobolev embedding and (5.3.3) that

$$\|F(u)\|_{L^{p'}(I,\dot{H}^\gamma_{q'})} \lesssim T^\theta \|u\|^\nu_{L^p(I,\dot{H}^\gamma_q)}, \tag{5.3.4}$$

$$\|F(u)-F(v)\|_{L^{p'}(I,L^{q'})} \lesssim T^\theta \Big(\|u\|^{\nu-1}_{L^p(I,\dot{H}^\gamma_q)} + \|v\|^{\nu-1}_{L^p(I,\dot{H}^\gamma_q)} \Big) \|u-v\|_{L^p(I,L^q)}. \tag{5.3.5}$$

This shows that for all $u, v \in X$, there exists $C > 0$ independent of T and $\psi \in H^\gamma$ such that

$$\begin{aligned} \|\Phi(u)\|_{L^p(I,\dot{H}^\gamma_q)} &\leq C\|\psi\|_{\dot{H}^\gamma} + CT^\theta M^\nu, \\ d(\Phi(u),\Phi(v)) &\leq CT^\theta M^{\nu-1} d(u,v). \end{aligned} \tag{5.3.6}$$

If we set $M = 2C\|\psi\|_{\dot{H}^\gamma}$ and choose $T > 0$ so that

$$CT^\theta M^{\nu-1} \leq \frac{1}{2},$$

then Φ is a strict contraction on (X,d).

We now turn to the case $\gamma = \gamma_c$. Using (5.1.12), we have from Strichartz estimate (1.1.17) that

$$\|u_{\rm hom}\|_{L^p(I,\dot{H}^{\gamma_c}_q)} \lesssim \|\psi\|_{\dot{H}^{\gamma_c}}.$$

This shows that $\|u_{\rm hom}\|_{L^p(I,\dot{H}^{\gamma_c}_q)} \leq \varepsilon$ for some $\varepsilon > 0$ small enough provided that T is small or $\|\psi\|_{\dot{H}^{\gamma_c}}$ is small. We also have from (1.1.17) that

$$\|u_{\rm inh}\|_{L^p(I,\dot{H}^{\gamma_c}_q)} \lesssim \|F(u)\|_{L^{p'}(I,\dot{H}^{\gamma_c}_{q'})}.$$

Corollary (5.0.3), (5.3.2) and Sobolev embedding (note that in this case $m = p$) then yield that

$$\|F(u)\|_{L^{p'}(I,\dot{H}^{\gamma_c}_{q'})} \lesssim \|u\|^{\nu}_{L^p(I,\dot{H}^{\gamma_c}_q)}, \tag{5.3.7}$$

$$\|F(u)-F(v)\|_{L^{p'}(I,L^{q'})} \lesssim \Big(\|u\|^{\nu-1}_{L^p(I,\dot{H}^{\gamma_c}_q)} + \|v\|^{\nu-1}_{L^p(I,\dot{H}^{\gamma_c}_q)}\Big)\|u-v\|_{L^p(I,L^q)}. \tag{5.3.8}$$

This implies that for all $u, v \in X$, there exists $C > 0$ independent of T and $\psi \in H^{\gamma_c}$ such that

$$\|\Phi(u)\|_{L^p(I,\dot{H}^{\gamma_c}_q)} \leq \varepsilon + CM^{\nu},$$
$$d(\Phi(u),\Phi(v)) \leq CM^{\nu-1} d(u,v).$$

If we choose ε and M small so that

$$CM^{\nu-1} \leq \frac{1}{2}, \quad \varepsilon + \frac{M}{2} \leq M,$$

then Φ is a contraction on (X, d).

Therefore, in both subcritical and critical cases, Φ has a unique fixed point in X. Moreover, since $\psi \in H^{\gamma}$ and $u \in L^p(I, H^{\gamma}_q)$, the Strichartz estimate shows that $u \in C(I, H^{\gamma})$ (see e.g. [CW90] or [Caz03, Chapter 4]). This shows the existence of solution $u \in C(I, H^{\gamma}) \cap L^p(I, H^{\gamma}_q)$ to (NLST). Note that in the case $\gamma = \gamma_c$, if $\|\psi\|_{\dot{H}^{\gamma_c}}$ is small enough, then we can take $T = \infty$.
Step 2. Uniqueness. It follows easily from (5.3.5) and (5.3.8) using the fact that $\|u\|_{L^p(I,\dot{H}^{\gamma}_q)}$ can be small if T is small.
Step 3. Item i. Let $u \in C(I, H^{\gamma}) \cap L^p(I, H^{\gamma}_q)$ be a solution to (NLFS) where $I = [0, T]$ and (a, b) a Schrödinger admissible pair with $b < \infty$ and $\gamma_{a,b} = 0$. Then Strichartz estimate (1.1.17) implies

$$\|u\|_{L^a(I,L^b)} \lesssim \|\psi\|_{L^2} + \|F(u)\|_{L^{p'}(I,L^{q'})}, \tag{5.3.9}$$

$$\|u\|_{L^a(I,\dot{H}^{\gamma}_b)} \lesssim \|\psi\|_{\dot{H}^{\gamma}} + \|F(u)\|_{L^{p'}(I,\dot{H}^{\gamma}_{q'})}. \tag{5.3.10}$$

It then follows from (5.3.4) and (5.3.7) that $u \in L^a(I, H^{\gamma}_b)$.
Step 4. Item ii. The blowup alternative in subcritical case is easy since the time of existence depends only on $\|\psi\|_{\dot{H}^{\gamma}}$.
Step 5. Item iii. It also follows from a standard argument (see e.g. [CW90]). Indeed, if $T^* < \infty$ and $\|u\|_{L^p([0,T^*),H^{\gamma_c}_q)} < \infty$, then Strichartz estimate (1.1.17) implies that $u \in C([0, T^*], H^{\gamma_c})$. Thus, one can extend the solution to (NLST) beyond T^*. It leads to a contradiction with the maximality of T^*.
Step 6. Item iv. We use the argument given in [CW90]. From Step 1, in the subcritical case, we can choose T and M so that the fixed point argument can be carried out on X for any initial data with $\dot{H}^{\gamma}$ norm less than $2\|\psi\|_{\dot{H}^{\gamma}}$. In the critical case, there exist T, M and an $\dot{H}^{\gamma_c}$ neighborhood U of ψ such that the fixed point argument can be carried out on X for all initial data in U. Now let $\psi_n \to \psi$ in H^{γ}. In both subcritical and critical cases, we see that $T < T^*(\psi)$, $\|u\|_{L^p([0,T],\dot{H}^{\gamma}_q)} \leq M$, and that for sufficiently large n, $T < T^*(\psi_n)$ and $\|u_n\|_{L^p([0,T],\dot{H}^{\gamma}_q)} \leq M$. Thus, (5.3.9) and (5.3.10) together with (5.3.4) and (5.3.7) yield that u_n is bounded in $L^a([0, T], H^{\gamma}_b)$ for any Schrödinger admissible pair (a, b) with $b < \infty$ and $\gamma_{a,b} = 0$. We also have from (5.3.5), (5.3.8) and the choice

of T that

$$d(u_n, u) \leq C\|\psi_n - \psi\|_{L^2} + \frac{1}{2} d(u_n, u) \text{ or } d(u_n, u) \leq 2C\|\psi_n - \psi\|_{L^2}.$$

This shows that $u_n \to u$ in $L^p([0,T], L^q)$. Again (5.3.10) together with (5.3.5) and (5.3.8) implies that $u_n \to u$ in $L^a([0,T], L^b)$ for any Schrödinger admissible pair (a,b) with $b < \infty$ and $\gamma_{a,b} = 0$. The convergence in $C(I, H^{\gamma-\epsilon})$ follows from the boundedness in $L^\infty(I, H^\gamma)$ and the convergence in $L^\infty(I, L^2)$ and that $\|u\|_{H^{\gamma-\epsilon}} \leq \|u\|_{H^\gamma}^{1-\frac{\epsilon}{\gamma}} \|u\|_{L^2}^{\frac{\epsilon}{\gamma}}$.

Step 7. Item vi. As mentioned in Step 1, when $\|\psi\|_{\dot{H}^{\gamma_c}}$ is small, we can take $T^* = \infty$. It remains to prove the scattering property. To do so, we make use of the adjoint estimate to the homogeneous Strichartz estimate, namely $L^2 \ni \psi \mapsto e^{it|\nabla|^\sigma}\psi \in L^p(\mathbb{R}, L^q)$ to obtain

$$\begin{aligned} \|e^{-it_2|\nabla|^\sigma} u(t_2) - e^{-it_1|\nabla|^\sigma} u(t_1)\|_{\dot{H}^{\gamma_c}} &= \Big\| i\mu \int_{t_1}^{t_2} e^{-is|\nabla|^\sigma} F(u)(s) ds \Big\|_{\dot{H}^{\gamma_c}} \\ &= \Big\| i\mu \int_{t_1}^{t_2} |\nabla|^{\gamma_c} e^{-is|\nabla|^\sigma} (\mathbb{1}_{[t_1,t_2]} F(u))(s) ds \Big\|_{L^2} \\ &\lesssim \|F(u)\|_{L^{p'}([t_1,t_2], \dot{H}^{\gamma_c}_{q'})}. \end{aligned} \tag{5.3.11}$$

Similarly,

$$\|e^{-it_2|\nabla|^\sigma} u(t_2) - e^{-it_1|\nabla|^\sigma} u(t_1)\|_{L^2} \lesssim \|F(u)\|_{L^{p'}([t_1,t_2], L^{q'})}. \tag{5.3.12}$$

Thanks to (5.3.7) and (5.3.8), we get

$$\|e^{-it_2|\nabla|^\sigma} u(t_2) - e^{-it_1|\nabla|^\sigma} u(t_1)\|_{H^{\gamma_c}} \to 0,$$

as $t_1, t_2 \to +\infty$. This implies that the limit

$$\psi^+ := \lim_{t \to +\infty} e^{-it|\nabla|^\sigma} u(t)$$

exists in H^{γ_c}. Moreover,

$$u(t) - e^{it|\nabla|^\sigma}\psi^+ = -i\mu \int_t^{+\infty} e^{i(t-s)|\nabla|^\sigma} F(u(s)) ds.$$

Using again (5.3.11) and (5.3.12) together with (5.3.7) and (5.3.8), we have

$$\lim_{t \to +\infty} \|u(t) - e^{it|\nabla|^\sigma}\psi^+\|_{H^{\gamma_c}} = 0.$$

This completes the proof of Theorem 5.3.1. □

We also have the following local well-posedness in the critical Sobolev space $H^{d/2}$, where the embedding into L^∞ breaks down.

Theorem 5.3.2. *Given $\sigma \in [2, \infty)$ and $\gamma = d/2$. Let $\nu > 1$ be an odd integer or* (5.0.2) *otherwise. Let*

$$\begin{cases} p > \max(\nu - 1, 4) & \text{when } d = 1, \\ p > \max(\nu - 1, 2) & \text{when } d \geq 2. \end{cases} \tag{5.3.13}$$

Then for all $\psi \in H^{d/2}$, there exists $T^* \in (0, \infty]$ and a unique solution to (NLST) *satisfying*

$$u \in C([0, T^*), H^{d/2}) \cap L^p_{\text{loc}}([0, T^*), L^\infty),$$

for some $p > \max(\nu - 1, 4)$ when $d = 1$ and some $p > \max(\nu - 1, 2)$ when $d \geq 2$. Moreover, the following properties hold:

i. $u \in L^a_{\text{loc}}([0,T^*), H_b^{d/2})$ *for any Schrödinger admissible pair* (a,b) *with* $b<\infty$ *and* $\gamma_{a,b}=0$.
ii. *If* $T^*<\infty$, *then* $\|u(t)\|_{H^{d/2}} \to \infty$ *as* $t\to T^*$.
iii. *u depends continuously on* ψ *in the sense of* Theorem 5.3.1

The continuous dependence can be improved (see Remark 5.3.3) if we assume that $\nu>1$ is an odd integer or $\lceil d/2 \rceil \leq \nu-1$. Concerning the well-posedness of the nonlinear Schrödinger equation in this critical space, we refer to [Kat95] and [NO98]. Note that in [NO98], the global well-posedness with small data is proved with exponential-type nonlinearity but not the local well-posedness without size restriction on the initial data.

Proof of Theorem 5.3.2. Let p be as in (5.3.13) and then choose $q\in[2,\infty)$ such that

$$\frac{2}{p}+\frac{d}{q}\leq \frac{d}{2}.$$

Step 1. Existence. We will show that Φ defined in (5.1.12) is a contraction on

$$X := \Big\{ u \in L^\infty(I,H^{d/2}) \cap L^p(I, H_q^{d/2-\gamma_{p,q}}) \ | \ \|u\|_{L^\infty(I,H^{d/2})} + \|u\|_{L^p(I,H_q^{d/2-\gamma_{p,q}})} \leq M \Big\},$$

equipped with the distance

$$d(u,v) := \|u-v\|_{L^\infty(I,L^2)} + \|u-v\|_{L^p(I,H^{-\gamma_{p,q}})},$$

where $I=[0,T]$ and $M,T>0$ to be determined. The local Strichartz estimate (1.1.16) gives

$$\begin{aligned} \|\Phi(u)\|_{L^\infty(I,H^{d/2})} + \|\Phi(u)\|_{L^p(I,H_q^{d/2-\gamma_{p,q}})} &\lesssim \|\psi\|_{H^{d/2}} + \|F(u)\|_{L^1(I,H^{d/2})}, \\ \|\Phi(u)-\Phi(v)\|_{L^\infty(I,L^2)} + \|\Phi(u)-\Phi(v)\|_{L^p(I,H_q^{-\gamma_{p,q}})} &\lesssim \|F(u)-F(v)\|_{L^1(I,L^2)}. \end{aligned}$$

Thanks to the assumptions on ν, Corollary 5.0.3 implies

$$\|F(u)\|_{L^1(I,H^{d/2})} \lesssim \|u\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)}\|u\|_{L^\infty(I,H^{d/2})} \lesssim T^\theta \|u\|^{\nu-1}_{L^p(I,L^\infty)}\|u\|_{L^\infty(I,H^{d/2})}, \quad (5.3.14)$$

$$\begin{aligned} \|F(u)-F(v)\|_{L^1(I,L^2)} &\lesssim \Big(\|u\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)} + \|v\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)}\Big)\|u-v\|_{L^\infty(I,L^2)} \\ &\lesssim T^\theta \Big(\|u\|^{\nu-1}_{L^p(I,L^\infty)} + \|v\|^{\nu-1}_{L^p(I,L^\infty)}\Big)\|u-v\|_{L^\infty(I,L^2)}, \end{aligned} \quad (5.3.15)$$

where $\theta = 1-\frac{\nu-1}{p}>0$. Using the fact that $d/2-\gamma_{p,q}>d/q$, the Sobolev embedding implies $H_q^{d/2-\gamma_{p,q}} \hookrightarrow L^\infty$. Thus,

$$\begin{aligned} \|\Phi(u)\|_{L^\infty(I,H^{d/2})} + \|\Phi(v)\|_{L^p(I,H_q^{d/2-\gamma_{p,q}})} &\lesssim \|\psi\|_{H^{d/2}} + T^\theta\|u\|^{\nu-1}_{L^p(I,H_q^{d/2-\gamma_{p,q}})}\|u\|_{L^\infty(I,H^{d/2})}, \\ d(\Phi(u),\Phi(v)) &\lesssim T^\theta\Big(\|u\|^{\nu-1}_{L^p(I,H_q^{d/2-\gamma_{p,q}})} + \|v\|^{\nu-1}_{L^p(I,H_q^{d/2-\gamma_{p,q}})}\Big) d(u,v). \end{aligned}$$

Thus for all $u,v\in X$, there exists $C>0$ independent of $\psi\in H^{d/2}$ such that

$$\begin{aligned} \|\Phi(u)\|_{L^\infty(I,H^{d/2})} + \|\Phi(v)\|_{L^p(I,H_q^{d/2-\gamma_{p,q}})} &\leq C\|\psi\|_{H^{d/2}} + CT^\theta M^\nu, \\ d(\Phi(u),\Phi(v)) &\leq CT^\theta M^{\nu-1} d(u,v). \end{aligned}$$

If we set $M=2C\|\psi\|_{H^{d/2}}$ and choose $T>0$ small enough so that $CT^\theta M^{\nu-1}\leq \frac{1}{2}$, then Φ is a contraction on X.

Step 2. Uniqueness. It is easy using (5.3.15) since $\|u\|_{L^p(I,L^\infty)}$ is small if T is small.

Step 3. Item i. It follows easily from Step 1 and Strichartz estimate (1.1.16) that for any Schrödinger admissible pair (a,b) with $b<\infty$ and $\gamma_{a,b}=0$,

$$\|u\|_{L^a(I,H_b^{d/2})} \lesssim \|\psi\|_{H^{d/2}} + \|F(u)\|_{L^1(I,H^{d/2})}.$$

Step 4. Item ii. The blowup alternative is obvious since the time of existence depends only on $\|\psi\|_{H^{d/2}}$.
Step 5. Item iii. The continuous dependence is similar to Step 7 of the proof of Theorem 5.3.1 using (5.3.15). ☐

Remark 5.3.3. If we assume that $\nu > 1$ is an odd integer or $\lceil d/2 \rceil \leq \nu - 1$ otherwise, then the continuous dependence holds in $C(I, H^{d/2})$. Indeed, we consider X as above equipped with the following metric

$$d(u,v) := \|u - v\|_{L^\infty(I,H^{d/2})} + \|u - v\|_{L^p(I,H_q^{d/2-\gamma_{p,q}})}.$$

Thanks to the assumptions on ν, we are able to apply the fractional derivative estimates given in Corollary 5.0.3 to have

$$\begin{aligned}\|F(u) - F(v)\|_{L^1(I,H^{d/2})} &\lesssim (\|u\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)} + \|v\|^{\nu-1}_{L^{\nu-1}(I,L^\infty)})\|u - v\|_{L^\infty(I,H^{d/2})} \\ &+ (\|u\|^{\nu-2}_{L^{\nu-1}(I,L^\infty)} + \|v\|^{\nu-2}_{L^{\nu-1}(I,L^\infty)})(\|u\|_{L^\infty(I,H^{d/2})} + \|v\|_{L^\infty(I,H^{d/2})})\|u - v\|_{L^{\nu-1}(I,L^\infty)}.\end{aligned}$$

The Sobolev embedding then implies that for all $u, v \in X$,

$$d(\Phi(u), \Phi(v)) \lesssim T^\theta M^{\nu-1} d(u,v).$$

The continuous dependence in $C(I, H^{d/2})$ follows as Step 7 of the proof of Theorem 5.3.1.

It is well-known that (see [Caz03, Chapter 4], [Kat95] or [Tao06, Chapter 3]) that for $\gamma > d/2$, the nonlinear Schrödinger equation is locally well-posed provided the nonlinearity has enough regularity. It is not a problem to extend this result for the nonlinear fourth-order Schrödinger equation. For the sake of completeness, we state (without proof) the local well-posedness for (NLST) in this range.

Theorem 5.3.4. *Given $\sigma \in [2, \infty)$. Let $\gamma > d/2$ be such that if $\nu > 1$ is not an odd integer,* (5.0.2)*. Then for all $\psi \in H^\gamma$, there exist $T^* \in (0, \infty]$ and a unique solution $u \in C([0, T^*), H^\gamma)$ to* (NLST)*. Moreover, the following properties hold:*

i. *$u \in L^a_{\text{loc}}([0, T^*), H^\gamma_b)$ for any Schrödinger admissible pair (a, b) with $b < \infty$ and $\gamma_{a,b} = 0$.*
ii. *If $T^* < \infty$, then $\|u(t)\|_{H^\gamma} \to \infty$ and $\limsup \|u(t)\|_{L^\infty} \to \infty$ as $t \to T^*$.*
iii. *u depends continuously on ψ in the following sense. There exists $T \in (0, T^*)$ such that if $\psi_n \to \psi$ in H^γ and if u_n is the solution of* (NLST) *with the initial data ψ_n, then $u_n \to u$ in $C([0, T], H^\gamma)$.*

Combining Theorem 5.3.1 with the conservation of mass, we have the following global well-posedness in L^2 for (NLST) in the case $\sigma \in [2, \infty)$.

Corollary 5.3.5. *Let $\sigma \in [2, \infty)$ and $\nu \in (1, 1 + 2\sigma/d)$. Then for all $\varphi \in L^2$, there exists a unique global solution to* (NLST) *satisfying $u \in C(\mathbb{R}, L^2) \cap L^p_{\text{loc}}(\mathbb{R}, L^q)$, where (p, q) given in* (5.3.1)*.*

In the energy space $H^{\sigma/2}$, we have the following global well-posedness result. The proof follows by the same lines as in Proposition 5.1.3.

Proposition 5.3.6. *Let $\sigma \in [2, \infty)$ and $\nu \in (1, 1 + 2\sigma/(d - \sigma))$ for $d > \sigma$ and $\nu > 1$ for $d \leq \sigma$. Then for any $\psi \in H^{\sigma/2}$, the solution to* (NLST) *given in* Theorem 5.3.1, Theorem 5.3.2 and Theorem 5.3.4 *can be extended to the whole $\mathbb{R}$ if one of the following is satisfied:*

i. *$\mu = 1$.*
ii. *$\mu = -1, \nu < 1 + 2\sigma/d$.*
iii. *$\mu = -1, \nu = 1 + 2\sigma/d$ and $\|\psi\|_{L^2}$ is small.*
iv. *$\mu = -1$ and $\|\psi\|_{H^{\sigma/2}}$ is small.*

Our next result concerns with the regularity of solutions of (NLST) in the subcritical case.

Theorem 5.3.7. *Given $\sigma \in [2, \infty)$. Let $\beta > \gamma \geq 0$ be such that $\gamma \geq \gamma_c$, and also, if $\nu > 1$ is not an odd integer,* (5.0.2)*. Let $\psi \in H^\gamma$ and u be the corresponding H^γ solution of* (NLST) *given in* Theorem 5.3.1, Theorem 5.3.2, Theorem 5.3.4*. If $\psi \in H^\beta$, then $u \in C([0, T^*), H^\beta)$.*

The following result is a direct consequence of Theorem 5.3.7 and the global well-posedness in Corollary 5.3.5 and Proposition 5.3.6.

Corollary 5.3.8. $\sigma \in [2,\infty)$.

i. *Let* $\gamma \geq 0$ *and* $\nu \in (1, 1+2\sigma/d)$ *be such that if* ν *is not an odd integer,* (5.0.2). *Then* (NLST) *is globally well-posed in* H^γ.
ii. *Let* $\gamma \geq \sigma/2$, $\nu \in [1+2\sigma/d, 1+2\sigma/(d-\sigma))$ *for* $d \geq \sigma$ *and* $\nu \in [1+2\sigma/d, \infty)$ *for* $d \leq \sigma$ *be such that if* ν *is not an odd integer,* (5.0.2). *Then* (NLST) *is globally well-posed in* H^γ *provided one of conditions* (i), (iii), (iv) *in* Proposition 5.3.6 *is satisfied.*

Proof of Theorem 5.3.7. We follow the argument given in Chapter 5 of [Caz03]. To do so, we will consider three cases $\gamma \in [0, d/2)$, $\gamma = d/2$ and $\gamma > d/2$.

The case $\gamma \in [0, d/2)$**.** Let $\beta > \gamma$. If $\psi \in H^\beta$, then Theorem 5.3.1 or Theorem 5.3.2 or Theorem 5.3.4 shows that there exists a maximal solution to (NLST) satisfying $u \in C([0,T), H^\beta) \cap L^a_{\text{loc}}([0,T), H^\beta_b)$ for any Schrödinger admissible pair (a,b) with $b < \infty$ and $\gamma_{a,b} = 0$. Since H^β-solution is in particular an H^γ-solution, the uniqueness implies that $T \leq T^*$. We will show that T is actually equal to T^*. Suppose that $T < T^*$, then the blowup alternative implies

$$\|u(t)\|_{H^\beta} \to \infty \text{ as } t \to T. \tag{5.3.16}$$

Moreover, since $T < T^*$, we have

$$\|u\|_{L^p((0,T),H^\gamma_q)} + \sup_{0\leq t\leq T} \|u(t)\|_{H^\gamma} < \infty,$$

where (p,q) given in (5.3.1). Using Strichartz estimate, we have for any interval $I \subset (0,T)$,

$$\|u\|_{L^\infty(I,L^2)} + \|u\|_{L^p(I,L^q)} \lesssim \|\psi\|_{L^2} + \|F(u)\|_{L^{p'}(I,L^{q'})},$$
$$\|u\|_{L^\infty(I,\dot H^\beta)} + \|u\|_{L^p(I,\dot H^\beta_q)} \lesssim \|\psi\|_{\dot H^\beta} + \|F(u)\|_{L^{p'}(I,\dot H^\beta_{q'})}.$$

Now, let (m,n) be as in (5.3.2). Corollary 5.0.3, (5.3.2) and Sobolev embedding then give

$$\|F(u)\|_{L^{p'}(I,L^{q'})} \lesssim \|u\|^{\nu-1}_{L^p(I,L^n)} \|u\|_{L^m(I,L^q)} \lesssim \|u\|^{\nu-1}_{L^p(I,\dot H^\gamma_q)} \|u\|_{L^m(I,L^q)} \lesssim \|u\|_{L^m(I,L^q)},$$
$$\|F(u)\|_{L^{p'}(I,\dot H^\beta_{q'})} \lesssim \|u\|^{\nu-1}_{L^p(I,L^n)} \|u\|_{L^m(I,\dot H^\beta_q)} \lesssim \|u\|^{\nu-1}_{L^p(I,\dot H^\gamma_q)} \|u\|_{L^m(I,\dot H^\beta_q)} \lesssim \|u\|_{L^m(I,\dot H^\beta_q)}.$$

Here we use the fact that $\|u\|_{L^p((0,T),H^\gamma_q)}$ is bounded. This shows that

$$\|u\|_{L^\infty(I,H^\beta)} + \|u\|_{L^p(I,H^\beta_q)} \lesssim \|\psi\|_{H^\beta} + \|u\|_{L^m(I,H^\beta_q)},$$

for every interval $I \subset (0,T)$. Now let $0 < \epsilon < T$ and consider $I = (0,\tau)$ with $\epsilon < \tau < T$. We have

$$\|u\|_{L^m(I,H^\beta_q)} \leq \|u\|_{L^m((0,\tau-\epsilon),H^\beta_q)} + \|u\|_{L^m((\tau-\epsilon,\tau),H^\beta_q)} \leq C_\epsilon + \epsilon^\theta \|u\|_{L^p(I,H^\beta_q)},$$

where θ given in (5.3.3). Here we also use the fact that $u \in L^p_{\text{loc}}([0,T), H^\beta_q)$ since $\gamma_{p,q} = 0$. Thus,

$$\|u\|_{L^\infty(I,H^\beta)} + \|u\|_{L^p(I,H^\beta_q)} \leq C + C_\epsilon + \epsilon^\theta C \|u\|_{L^p(I,H^\beta_q)},$$

where the various constants are independent of $\tau < T$. By choosing ϵ small enough, we have

$$\|u\|_{L^\infty(I,H^\beta)} + \|u\|_{L^p(I,H^\beta_q)} \leq C,$$

where C is independent of $\tau < T$. Let $\tau \to T$, we get a contradiction with (5.3.16).

The case $\gamma = d/2$**.** Since $\psi \in H^{d/2}$, Theorem 5.3.2 shows that there exists a unique, maximal solution to (NLST) satisfying $u \in C([0,T^*), H^{d/2}) \cap L^p_{\text{loc}}([0,T^*), L^\infty)$ for some $p > \max(\nu-1, 4)$

when $d = 1$ and $p > \max(\nu - 1, 2)$ when $d \geq 2$. This implies in particular that

$$u \in L^{\nu-1}_{\text{loc}}([0, T^*), L^\infty). \tag{5.3.17}$$

Now let $\beta > \gamma$. If $\psi \in H^\beta$, then we know that u is an H^β solution defined on some maximal interval $[0, T)$ with $T \leq T^*$. Suppose that $T < T^*$. Then the unitary property of $e^{it|\nabla|^\sigma}$ and Corollary 5.0.3 imply that

$$\|u(t)\|_{H^\beta} \leq \|\psi\|_{H^\beta} + \int_0^t \|F(u)(s)\|_{H^\beta} ds \leq \|\psi\|_{H^\beta} + C \int_0^t \|u(s)\|^{\nu-1}_{L^\infty} \|u(s)\|_{H^\beta} ds,$$

for all $0 \leq t < T$. The Gronwall's inequality then yields

$$\|u(t)\|_{H^\beta} \leq \|\psi\|_{H^\beta} \exp\Big(C \int_0^t \|u(s)\|^{\nu-1}_{L^\infty} ds \Big)$$

for all $0 \leq t < T$. Using (5.3.17), we see that $\limsup \|u(t)\|_{H^\beta} < \infty$ as $t \to T$. This is a contradiction with the blowup alternative in H^β.

The case $\gamma > d/2$. Let $\beta > \gamma$. If $\psi \in H^\beta$, then Theorem 5.3.4 shows that there is a unique maximal solution $u \in C([0, T), H^\beta)$ to (NLST). By the uniqueness, we have $T \leq T^*$. Suppose $T < T^*$. Then

$$\sup_{0 \leq t \leq T} \|u(t)\|_{H^\beta} < \infty,$$

and hence

$$\sup_{0 \leq t \leq T} \|u(t)\|_{L^\infty} < \infty.$$

This is a contradiction with the fact that $\limsup \|u(t)\|_{L^\infty} = \infty$ as $t \to T$. The proof of Theorem 5.3.7 is now complete. □

We end this section with the following remark. In [PS10], the authors proved the global existence for the L^2-critical nonlinear fourth-order Schrödinger equation (NL4S), i.e. $\sigma = 4$ and $\nu - 1 = 8/d$, in higher dimensions $d \geq 5$. More precisely, they proved that the equation is globally well-posed in L^2

- for any initial data in L^2 in the defocusing case;
- for initial data in L^2 satisfying $\|\psi\|_{L^2} < \|Q\|_{L^2}$ in the focusing case, where Q is the solution to the elliptic equation

$$\Delta^2 Q + Q = |Q|^{\frac{8}{d}} Q. \tag{5.3.18}$$

Moreover, in both cases, the following uniform bound holds true

$$\|u\|_{L^{2+\frac{8}{d}}(\mathbb{R}, L^{2+\frac{8}{d}})} \leq C(\|\psi\|_{L^2}).$$

With this uniform bound, we have the following global existence for the L^2-critical (NL4S) in dimensions $d \geq 5$.

Proposition 5.3.9. *Let $d \geq 5, \nu = 1+8/d$ and $\beta > 0$ be such that if $d \neq 1, 2, 4$, then $\lceil \beta \rceil \leq 1+8/d$. Let $\psi \in H^\beta$ be such that if $\mu = -1$, $\|\psi\|_{L^2} < \|Q\|_{L^2}$, where Q is the solution to* (5.3.18)*. Then the L^2-critical* (NL4S) *is globally well-posed in H^β.*

Proof. Let $\beta > 0$ and $\psi \in H^\beta$ be such that if $\mu = -1$, $\|\psi\|_{L^2} < \|Q\|_{L^2}$, where Q is the solution to (5.3.18). We learn from the result of Pausader-Shao [PS10] that the L^2-critical (NL4S) is globally well-posed in L^2. Moreover, the solutions enjoy the uniform bound

$$\|u\|_{L^{2+\frac{8}{d}}(\mathbb{R}, L^{2+\frac{8}{d}})} \leq C(\|\psi\|_{L^2}).$$

Since $\psi \in H^\beta$, we have from Theorem 5.3.1, Theorem 5.3.2 and Theorem 5.3.4 that there exists a maximal solution to the L^2-critical (NL4S) satisfying $C([0,T), H^\beta) \cap L^a_{\text{loc}}([0,T), H^\beta_b)$ for any Schrödinger admissible pair (a,b) with $b < \infty$ and $\gamma_{a,b} = 0$. By the blowup alternative, it suffices to show that $\|u\|_{L^\infty((0,T),H^\beta)} < \infty$. Let $p = 2 + 8/d$. It is easy to see that (p,p) is a Schrödinger admissible pair with $\gamma_{p,p} = 0$. Since $\|u\|_{L^p((0,T),L^p)} < \infty$, we decompose $(0,T)$ into a finite number of subintervals I_k so that $\|u\|_{L^p(I_k,L^p)} < \epsilon$ for some $\epsilon > 0$ to be chosen later. By Strichartz estimates,

$$\begin{aligned}
\|u\|_{L^\infty(I_k,H^\beta)} + \|u\|_{L^p(I_k,H^\beta_p)} &\lesssim \|\psi\|_{H^\beta} + \|F(u)\|_{L^{p'}(I_k,H^\beta_{p'})} \\
&\lesssim \|\psi\|_{H^\beta} + \|u\|^{\frac{8}{d}}_{L^p(I_k,L^p)} \|u\|_{L^p(I_k,H^\beta_p)} \\
&\lesssim \|\psi\|_{H^\beta} + \epsilon^{\frac{8}{d}} \|u\|_{L^p(I_k,H^\beta_p)}.
\end{aligned}$$

By choosing $\epsilon > 0$ small enough, we get $\|u\|_{L^\infty(I_k,H^\beta)} \leq C$ for some constant C independent of I_k. By summing over all subintervals I_k, we obtain $\|u\|_{L^\infty((0,T),H^\beta)} < \infty$. The proof is complete. □

Chapter 6

Global well-posedness for the defocusing mass-critical nonlinear fourth-order Schrödinger equation below the energy space

Contents

In this chapter, we consider the defocusing mass-critical nonlinear fourth-order Schrödinger equation, namely

$$\left\{ \begin{array}{rcl} i\partial_t u(t,x) + \Delta^2 u(t,x) &=& -(|u|^{\frac{8}{d}} u)(t,x), \quad t \geq 0, x \in \mathbb{R}^d, \\ u(0,x) &=& \psi(x) \in H^\gamma(\mathbb{R}^d), \end{array} \right. \tag{dNL4S}$$

where $u(t,x)$ is a complex valued function in $\mathbb{R}^+ \times \mathbb{R}^d$.

The fourth-order Schrödinger equation was introduced by Karpman [Kar96] and Karpman-Shagalov [KS00] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. The study of nonlinear fourth-order Schrödinger equation has attracted a lot of interest in the past several years (see [Pau1], [Pau2], [HHW06], [HHW07], [HJ05], [MXZ09], [MXZ11], [MWZ15] and references therein).

As in the previous chapter, we see that the (dNL4S) is locally well-posed in $H^\gamma(\mathbb{R}^d)$ for $\gamma > 0$ satisfying, in the case $d \neq 1, 2, 4$,

$$\lceil \gamma \rceil \leq 1 + \frac{8}{d}. \tag{6.0.1}$$

Here $\lceil \gamma \rceil$ is the smallest integer greater than or equal to γ. This condition ensures the nonlinearity to have enough regularity. The time of existence depends only on the H^γ-norm of initial data. Moreover, the local solution enjoys mass conservation, i.e.

$$M(u(t)) := \|u(t)\|^2_{L^2(\mathbb{R}^d)} = \|\psi\|^2_{L^2(\mathbb{R}^d)},$$

and H^2-solution has conserved energy, i.e.

$$E(u(t)) := \int_{\mathbb{R}^d} \frac{1}{2}|\Delta u(t,x)|^2 + \frac{d}{2d+8}|u(t,x)|^{\frac{2d+8}{d}} dx = E(\psi).$$

The conservations of mass and energy together with the persistence of regularity (see Theorem 5.3.7) yield the global well-posedness for the (dNL4S) in $H^\gamma(\mathbb{R}^d)$ with $\gamma \geq 2$ satisfying for $d \neq 1, 2, 4$, (6.0.1). We also have the local well-posedness for the (dNL4S) with initial data $\psi \in L^2(\mathbb{R}^d)$ but the time of existence depends on the profile of ψ instead of its L^2-norm. The global existence holds for small L^2-norm initial data. For large L^2-norm initial data, the conservation of mass does not immediately give the global well-posedness in $L^2(\mathbb{R}^d)$. For the global well-posedness with large L^2-norm initial data, we refer the reader to [PS10] where the authors established the global well-posedness and scattering for the (dNL4S) in $L^2(\mathbb{R}^d), d \geq 5$.

The main goal of this chapter is to prove the global well-posedness for the (dNL4S) in low regularity spaces $H^\gamma(\mathbb{R}^d), d \geq 4$ with $0 < \gamma < 2$. Since we are working with low regularity data, the conservation of energy does not hold. In order to overcome this problem, we make use of the I-method introduced by [CKSTT1] and the interaction Morawetz inequality (which is available for $d \geq 5$). We thus consider separately two cases $d = 4$ and $d \geq 5$. In the case $d = 4$, we use I-method in Bourgain space, which is an adaptation of the one given in [CKSTT1] for proving the low regularity global well-posedness of the defocusing cubic nonlinear Schrödinger equation on $\mathbb{R}^2$. In this consideration the nonlinearity is algebraic. It allows to write explicitly the commutator between the I-operator and the nonlinearity by means of the Fourier transform, and then control it by multi-linear analysis. In the case $d \geq 5$, the nonlinearity is no longer algebraic, so the above method does not work. We thus rely purely on Strichartz and interaction Morawetz inequalities.

After submitting a paper concerning the global well-posedness for the (dNL4S) below the energy space in dimensions $5 \leq d \leq 7$, the author was informed that a better result (see Proposition 5.3.9) follows from the work of Pausader-Shao [PS10]. Indeed, in [PS10], the authors showed that the (dNL4S) is globally well-posed in L^2. Moreover the global solution scatters in L^2 and satisfies the uniform bound

$$\|u\|_{L^{\frac{2(d+4)}{d}}(\mathbb{R}\times\mathbb{R}^d)} < \infty.$$

It follows from the regularity given in Theorem 5.3.7 that the (dNL4S) is globally well-posed in H^γ for any $0 < \gamma < 2$. However, we decide to keep our proof in the case $5 \leq d \leq 7$ because it will be used in the next chapter to study dynamics of blowup solutions for the focusing mass-critical NL4S.

We end this introduction by recalling some known results about the global existence below the energy space for the nonlinear fourth-order Schrödinger equation. To our knowledge, the first result to address this problem belongs to Guo in [Guo10], where the author considered a more general fourth-order Schrödinger equation, namely

$$i\partial_t u + \lambda\Delta u + \mu\Delta^2 u + \nu|u|^{2m}u = 0,$$

and established the global existence in $H^\gamma(\mathbb{R}^d)$ for $1 + \frac{md-9+\sqrt{(4m-md+7)^2+16}}{4m} < \gamma < 2$ where m is an integer satisfying $4 < md < 4m + 2$. The proof is based on the I-method which is a modification of the one invented by I-Team [CKSTT1] in the context of nonlinear Schrödinger equation. Later, Miao-Wu-Zhang studied the defocusing cubic fourth-order Schrödinger equation, namely

$$i\partial_t u + \Delta^2 u + |u|^2 u = 0,$$

and proved the global well-posedness and scattering in $H^\gamma(\mathbb{R}^d)$ with $\gamma(d) < \gamma < 2$ where $\gamma(5) = \frac{16}{11}, \gamma(6) = \frac{16}{9}$ and $\gamma(7) = \frac{45}{23}$. The proof relies on the combination of I-method and a new interaction Morawetz inequality.

6.1 Global well-posedness for the 4D defocusing mass-critical NL4S below the energy space

Our main result in this section is the following global existence for the (dNL4S) in the fourth dimensional spatial space.

Theorem 6.1.1. *Let $d = 4$. The initial value problem* (dNL4S) *is globally well-posed in $H^\gamma(\mathbb{R}^4)$ for any $2 > \gamma > \overline{\gamma} := \frac{60}{53}$. Moreover, the solution satisfies*

$$\|u(T)\|_{H^\gamma(\mathbb{R}^4)} \leq C(1+T)^{\frac{15(2-\gamma)}{53\gamma-60}+},$$

for $|T| \to \infty$, where the constant C depends only on $\|\psi\|_{H^\gamma(\mathbb{R}^4)}$.

The proof of this theorem is based on the I-method, which is similar to [CKSTT1] (see also [Guo10]). It is thus convenient to recall techniques and known results about the low regularity defocusing cubic Schrödinger equation on $\mathbb{R}^2$. The first attempt to solve this problem is due to Bourgain in [Bou3] where he used a "Fourier truncation" approach to prove the global existence for $\gamma > 3/5$. It was then improved for $\gamma > 4/7$ by I-Team in [CKSTT1]. The proof is based on the almost conservation of a modified energy functional. The idea is to replace the conserved energy $E(u)$, which is not available when $\gamma < 1$, by an "almost conserved" quantity $E(I_N u)$ with $N \gg 1$ where I_N is a smoothing operator which behaves like the identity for low frequencies $|\xi| \leq N$ and like a fractional integral operator of order $1-\gamma$ for high frequencies $|\xi| \geq 2N$. Since $I_N u$ is not a solution to the equation, we may expect an energy increment. The key idea is to show that on the time interval of local existence, the increment of the modified energy $E(I_N u)$ decays with respect to a large parameter N. This allows to control $E(I_N u)$ on time interval where the local solution exists, and we can iterate this estimate to obtain a global in time control of the solution by means of the bootstrap argument. Fang-Grillakis then upgraded this result to $\gamma \geq 1/2$ in [FG07]. Later, Colliander-Grillakis-Tzirakis improved for $\gamma > 2/5$ in [CGT07] using an almost interaction Morawetz inequality. Subsequent paper [CR11] has decreased the necessary regularity to $\gamma > 1/3$. Afterwards, Dodson established in [Dod1] the global existence for the equation when $\gamma > 1/4$. The proof combines the almost conservation law and an improved interaction Morawetz estimate. Recently, Dodson in [Dod2] proved the global well-posedness and scattering for the equation with initial data $\psi \in L^2(\mathbb{R}^2)$ using the bilinear estimate and a frequency localized interaction Morawetz estimate. To prove Theorem 6.1.1, we shall consider a modified I-operator and show a suitable "almost conservation law" for the fourth-order Schrödinger equation.

6.1.1 Preliminaries

Littlewood-Paley decomposition. Let φ be a smooth, real-valued, radial function in $\mathbb{R}^d$ such that $\varphi(\xi) = 1$ for $|\xi| \leq 1$ and $\varphi(\xi) = 0$ for $|\xi| \geq 2$. Let $M = 2^k, k \in \mathbb{Z}$. We denote the Littlewood-Paley operators by

$$\widehat{P_{\leq M} f}(\xi) := \varphi(M^{-1}\xi)\hat{f}(\xi),$$
$$\widehat{P_{>M} f}(\xi) := (1-\varphi(M^{-1}\xi))\hat{f}(\xi),$$
$$\widehat{P_M f}(\xi) := (\varphi(M^{-1}\xi) - \varphi(2M^{-1}\xi))\hat{f}(\xi),$$

where $\hat{\cdot}$ is the spatial Fourier transform. We similarly define

$$P_{<M} := P_{\leq M} - P_M, \quad P_{\geq M} := P_{>M} + P_M,$$

and for $M_1 \leq M_2$,

$$P_{M_1 < \cdot \leq M_2} := P_{\leq M_2} - P_{\leq M_1} = \sum_{M_1 < M \leq M_2} P_M.$$

We have the following so called Bernstein's inequalities (see e.g. [BCD11, Chapter 2] or [Tao06, Appendix]).

Lemma 6.1.2. *Let* $\gamma \geq 0$ *and* $1 \leq p \leq q \leq \infty$.

$$\begin{aligned} \|P_{\geq M} f\|_{L^p} &\lesssim M^{-\gamma} \||\nabla|^\gamma P_{\geq M} f\|_{L^p}, \\ \|P_{\leq M} |\nabla|^\gamma f\|_{L^p} &\lesssim M^\gamma \|P_{\leq M} f\|_{L^p}, \\ \|P_M |\nabla|^{\pm\gamma} f\|_{L^p} &\sim M^{\pm\gamma} \|P_M f\|_{L^p}, \\ \|P_{\leq M} f\|_{L^q} &\lesssim M^{d/p-d/q} \|P_{\leq M} f\|_{L^p}, \\ \|P_M f\|_{L^q} &\lesssim M^{d/p-d/q} \|P_M f\|_{L^p}. \end{aligned}$$

Norms and Strichartz estimates. Let $\gamma, b \in \mathbb{R}$. The Bourgain space $X^{\gamma,b}_{\tau=|\xi|^4}$ is the closure of space-time Schwartz space $\mathscr{S}_{t,x}$ under the norm

$$\|u\|_{X^{\gamma,b}_{\tau=|\xi|^4}} := \| \langle \xi \rangle^\gamma \langle \tau - |\xi|^4 \rangle^b \tilde{u} \|_{L^2_\tau L^2_\xi},$$

where $\tilde{\cdot}$ is the space-time Fourier transform,i.e.

$$\tilde{u}(\tau, \xi) := \iint e^{-i(t\tau + x\cdot\xi)} u(t,x) dt dx.$$

We shall use $X^{\gamma,b}$ instead of $X^{\gamma,b}_{\tau=|\xi|^4}$ when there is no confusion. We recall a following special property of $X^{\gamma,b}$ space (see e.g. [Tao06, Lemma 2.9]).
Lemma 6.1.3. *Let* $\gamma, \gamma_1, \gamma_2 \in \mathbb{R}$ *and* Y *be a Banach space of functions on* $\mathbb{R} \times \mathbb{R}^4$. *If*

$$\|e^{it\tau} e^{it\Delta^2} f\|_Y \lesssim \|f\|_{H^\gamma},$$

for all $f \in H^\gamma$ *and all* $\tau \in \mathbb{R}$, *then*

$$\|u\|_Y \lesssim \|u\|_{X^{\gamma,1/2+}},$$

for all $u \in \mathscr{S}_{t,x}$. *Moreover, if*

$$\|[e^{it\tau} e^{it\Delta^2} f_1][e^{it\zeta} e^{it\Delta^2} f_2]\|_Y \lesssim \|f_1\|_{H^{\gamma_1}} \|f_2\|_{H^{\gamma_2}},$$

for all $f_1 \in H^{\gamma_1}, f_2 \in H^{\gamma_2}$ *and all* $\tau, \zeta \in \mathbb{R}$, *then*

$$\|u_1 u_2\|_Y \lesssim \|u_1\|_{X^{\gamma_1,1/2+}} \|u_2\|_{X^{\gamma_2,1/2+}},$$

for all $u_1, u_2 \in \mathscr{S}_{t,x}$.

We refer the reader to Lemma $A.2.7$ for the proof of this result.

Throughout this section, a pair (p,q) is called admissible in $\mathbb{R}^4$ if

$$(p,q) \in [2,\infty]^2, \quad (q,p) \neq (2,\infty), \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}. \tag{6.1.1}$$

We recall the following Strichartz estimate given in Corollary 1.1.3 with $\sigma = 4$. It is obvious that for (p,q) a admissible pair (6.1.1), $\gamma_{p,q} = 0$.
Proposition 6.1.4. *Let* u *be a solution to*

$$i\partial_t u(t,x) + \Delta^2 u(t,x) = F(t,x), \quad u(0,x) = \psi(x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^4.$$

Then for all (p,q) *and* (a,b) *admissible pairs,*

$$\|u\|_{L^p(\mathbb{R},L^q)} \lesssim \|\psi\|_{L^2} + \|F\|_{L^{a'}(\mathbb{R},L^{b'})}. \tag{6.1.2}$$

Here (a,a') *and* (b,b') *are conjugate exponents.*

A direct consequence of Lemma 6.1.3 and Proposition 6.1.4 is the following linear estimate in $X^{\gamma,b}$ space.

Corollary 6.1.5. *Let (p,q) be an admissible pair. Then*

$$\|u\|_{L^p(\mathbb{R},L^q)} \lesssim \|u\|_{X^{0,1/2+}}, \tag{6.1.3}$$

for all $u \in \mathscr{S}_{t,x}$.

We also have the following bilinear estimate in $\mathbb{R}^4$.

Proposition 6.1.6. *Let $M_1, M_2 \in 2^{\mathbb{Z}}$ be such that $M_1 \leq M_2$. Then*

$$\|[e^{it\Delta^2} P_{M_1}\psi][e^{it\Delta^2} P_{M_2}\phi]\|_{L^2(\mathbb{R},L^2)} \lesssim (M_1/M_2)^{3/2} \|\psi\|_{L^2}\|\phi\|_{L^2}.$$

We refer the reader to Theorem $A.3.1$ for the proof of this bilinear estimate.

The following result is another application of Lemma 6.1.3 and Proposition 6.1.6.

Corollary 6.1.7. *Let $u_1, u_2 \in X^{0,1/2+}$ be supported on spatial frequencies $|\xi| \sim M_1, M_2$ respectively. Then for $M_1 \leq M_2$,*

$$\|u_1u_2\|_{L^2(\mathbb{R},L^2)} \lesssim (M_1/M_2)^{3/2}\|u_1\|_{X^{0,1/2+}}\|u_2\|_{X^{0,1/2+}}. \tag{6.1.4}$$

A similar estimate holds for $\overline{u}_1u_2$ or $u_1\overline{u}_2$.

***I*-operator.** For $0 \leq \gamma < 2$ and $N \gg 1$, we define the Fourier multiplier I_N by

$$\widehat{I_N f}(\xi) := m_N(\xi)\hat{f}(\xi), \tag{6.1.5}$$

where m is a smooth, radially symmetric, non-increasing function such that

$$m_N(\xi) := \begin{cases} 1 & \text{if } |\xi| \leq N, \\ (N^{-1}|\xi|)^{\gamma-2} & \text{if } |\xi| \geq 2N. \end{cases} \tag{6.1.6}$$

For simplicity, we shall drop the N from the notation and write I and m instead of I_N and m_N. The operator I is the identity on low frequencies $|\xi| \leq N$ and behaves like a fractional integral operator of order $2-\gamma$ on high frequencies $|\xi| \geq 2N$. We collect some basic properties of the I-operator in the following lemma.

Lemma 6.1.8. *Let $0 \leq \sigma \leq \gamma < 2$ and $1 < q < \infty$. Then*

$$\|If\|_{L^q} \lesssim \|f\|_{L^q}, \tag{6.1.7}$$

$$\||\nabla|^\sigma P_{>N} f\|_{L^q} \lesssim N^{\sigma-2}\|\Delta I f\|_{L^q}, \tag{6.1.8}$$

$$\|\langle\nabla\rangle^\sigma f\|_{L^q} \lesssim \|\langle\Delta\rangle If\|_{L^q}, \tag{6.1.9}$$

$$\|f\|_{H^\gamma} \lesssim \|If\|_{H^2} \lesssim N^{2-\gamma}\|f\|_{H^\gamma}, \tag{6.1.10}$$

$$\|If\|_{\dot{H}^2} \lesssim N^{2-\gamma}\|f\|_{\dot{H}^\gamma}. \tag{6.1.11}$$

Proof. The estimate (6.1.7) is a direct consequence of the Hörmander-Mikhlin multiplier theorem (see e.g. [Gra14, Theorem 6.2.7]). To prove (6.1.8), we write

$$\||\nabla|^\sigma P_{>N} f\|_{L^q} = \||\nabla|^\sigma P_{>N}(\Delta I)^{-1}\Delta I f\|_{L^q}.$$

The desired estimate (6.1.8) follows again from the Hörmander-Mikhlin multiplier theorem. In order to get (6.1.9), we estimate

$$\|\langle\nabla\rangle^\sigma f\|_{L^q} \leq \|P_{\leq N}\langle\nabla\rangle^\sigma f\|_{L^q} + \|P_{>N}f\|_{L^q} + \|P_{>N}|\nabla|^\sigma f\|_{L^q}.$$

Thanks to the fact that the I-operator is the identity at low frequency $|\xi| \leq N$, the multiplier theorem and (6.1.8) imply

$$\| \langle \nabla \rangle^\sigma f\|_{L^q} \lesssim \| \langle \Delta \rangle If\|_{L^q} + \|\Delta If\|_{L^q}.$$

This proves (6.1.9). Finally, by the definition of the I-operator and (6.1.8), we have

$$\begin{aligned}\|f\|_{H^\gamma} &\lesssim \|P_{\leq N} f\|_{H^\gamma} + \|P_{>N} f\|_{L^2} + \||\nabla|^\gamma P_{>N} f\|_{L^2} \\ &\lesssim \|P_{\leq N} If\|_{H^\gamma} + N^{-2}\|\Delta If\|_{L^2} + N^{\gamma-2}\|\Delta If\|_{L^2} \lesssim \|If\|_{H^2}.\end{aligned}$$

This shows the first inequality in (6.1.10). For the second inequality in (6.1.10), we estimate

$$\|If\|_{H^2} \lesssim \|P_{\leq N} \langle \nabla \rangle^2 If\|_{L^2} + \|P_{>N} \langle \nabla \rangle^2 If\|_{L^2} \lesssim N^{2-\gamma}\|f\|_{H^\gamma}.$$

Here we use the definition of I-operator to get

$$\|P_{\leq N} I \langle \nabla \rangle^{2-\gamma} \|_{L^2\to L^2}, \quad \|P_{>N} I \langle \nabla \rangle^{2-\gamma} \|_{L^2\to L^2} \lesssim N^{2-\gamma}.$$

The estimate (6.1.11) is proved as for the second estimate in (6.1.10). The proof is complete. □

6.1.2 Almost conservation law

As mentioned in the introduction, the equation (dNL4S) is locally well-posed in H^γ for any $\gamma > 0$. Moreover, the time of existence depends only on the H^γ-norm of the initial data. Thus, the global well-posedness will follow from a global $L^\infty(\mathbb{R}, H^\gamma)$ bound of the solution by the usual iterative argument. For H^γ solution with $\gamma \geq 2$, one can obtain easily the $L^\infty(\mathbb{R}, H^\gamma)$ bound of solution using the persistence of regularity and the conserved quantities of mass and energy. But it is not the case for H^γ solution with $\gamma < 2$ since the energy is no longer conserved. However, it follows from (6.1.10) that the H^γ-norm of the solution u can be controlled by the H^2-norm of Iu. It leads to consider the following modified energy functional

$$E(Iu(t)) := \frac{1}{2}\|Iu(t)\|^2_{\dot H^2} + \frac{1}{4}\|Iu(t)\|^4_{L^4}. \tag{6.1.12}$$

Since Iu is not a solution to (dNL4S), we can expect an energy increment. We have the following "almost conservation law".

Proposition 6.1.9. *Let $2 > \gamma > \overline{\gamma} := \frac{60}{53}$ and $N \gg 1$. If the initial data $\psi \in C^\infty(\mathbb{R}^4)$ satisfies $E(I\psi) \leq 1$, then there exists $\delta = \delta(\|\psi\|_{L^2}) > 0$ so that the solution $u \in C([0, \delta], H^\gamma(\mathbb{R}^4))$ of* (dNL4S) *satisfies*

$$E(Iu(t)) = E(I\psi) + O(N^{-\gamma_0+}), \tag{6.1.13}$$

where $\gamma_0 := \frac{46}{15}$ for all $t \in [0, \delta]$.

Remark 6.1.10. This proposition tells us that the modified energy $E(Iu(t))$ decays with respect to the parameter N. We will see in Section 6.1.3 that if we can replace the increment $N^{-\gamma_0+}$ in the right hand side of (6.1.13) with $N^{-\gamma_1+}$ for some $\gamma_1 > \gamma_0$, then the global existence can be improved for all $\gamma > \frac{8}{4+\gamma_1}$. In particular, if $\gamma_1 = \infty$, then $E(Iu(t))$ is conserved, and the global well-posedness holds for all $\gamma > 0$.

In order to prove Proposition 6.1.9, we recall the following interpolation result (see [CKSTT4, Lemma 12.1]). Let η be a smooth, radial, decreasing function which equals 1 for $|\xi| \leq 1$ and equals $|\xi|^{-1}$ for $|\xi| \geq 2$. For $N \geq 1$ and $\alpha \in \mathbb{R}$, we define the spatial Fourier multiplier J^α_N by

$$\widehat{J^\alpha_N f}(\xi) := (\eta(N^{-1}\xi))^\alpha \hat{f}(\xi). \tag{6.1.14}$$

The operator J^α_N is a smoothing operator of order α, and it is the identity on the low frequencies $|\xi| \leq N$.

Lemma 6.1.11 (Interpolation [CKSTT4]). *Let $\alpha_0 > 0$ and $n \geq 1$. Suppose that $Z, X_1, ..., X_n$ are* *translation invariant Banach spaces* [1] *and T is a translation invariant n-linear operator such that*

$$\|J_1^\alpha T(u_1,...,u_n)\|_Z \lesssim \prod_{i=1}^n \|J_1^\alpha u_i\|_{X_i},$$

for all $u_1,...,u_n$ and all $0\leq\alpha\leq\alpha_0$. Then one has

$$\|J_N^\alpha T(u_1,...,u_n)\|_Z \lesssim \prod_{i=1}^n \|J_N^\alpha u_i\|_{X_i},$$

for all $u_1,...,u_n$, all $0\leq\alpha\leq\alpha_0$, and $N\geq 1$, with the implicit constant independent of N.

Using this interpolation lemma, we are able to prove the following modified version of the usual local well-posedness result.

Proposition 6.1.12. *Let $\gamma\in(2/3,2)$ and $\psi\in H^\gamma(\mathbb{R}^4)$ be such that $E(I\psi)\leq 1$. Then there is a constant $\delta=\delta(\|\psi\|_{L^2})$ so that the solution u to* (dNL4S) *satisfies*

$$\|Iu\|_{X_\delta^{2,1/2+}} \lesssim 1. \tag{6.1.15}$$

Here $X_\delta^{\gamma,b}$ is the space of restrictions of elements of $X^{\gamma,b}$ endowed with the norm

$$\|u\|_{X_\delta^{\gamma,b}} := \inf\{\|w\|_{X^{\gamma,b}} \mid w_{|[0,\delta]\times\mathbb{R}^4}=u\}. \tag{6.1.16}$$

Proof. We recall the following estimates involving the $X^{\gamma,b}$ spaces which are proved in the Appendix $A.2$. Let $\gamma\in\mathbb{R}$ and $\psi\in C_0^\infty(\mathbb{R})$ be such that $\psi(t)=1$ for $t\in[-1,1]$. One has

$$\|\psi(t)e^{it\Delta^2}\psi\|_{X^{\gamma,b}} \lesssim \|\psi\|_{H^\gamma}, \tag{6.1.17}$$

$$\Big\|\psi_\delta(t)\int_0^t e^{i(t-s)\Delta^2}F(s)ds\Big\|_{X^{\gamma,b}} \lesssim \delta^{1-b-b'}\|F\|_{X^{\gamma,-b'}}, \tag{6.1.18}$$

where $\psi_\delta(t):=\psi(\delta^{-1}t)$ provided $0<\delta\leq 1$ and

$$0<b'<1/2<b, \quad b+b'<1. \tag{6.1.19}$$

Note that the implicit constants are independent of δ. This implies for $0<\delta\leq 1$ and b,b' as in (6.1.19) that

$$\|e^{it\Delta^2}\psi\|_{X_\delta^{\gamma,b}} \lesssim \|\psi\|_{H^\gamma}, \tag{6.1.20}$$

$$\Big\|\int_0^t e^{i(t-s)\Delta^2}F(s)ds\Big\|_{X_\delta^{\gamma,b}} \lesssim \delta^{1-b-b'}\|F\|_{X_\delta^{\gamma,-b'}}. \tag{6.1.21}$$

By the Duhamel principle, we have

$$\|Iu\|_{X_\delta^{2,b}} = \Big\|e^{it\Delta^2}I\psi + \int_0^t e^{i(t-s)\Delta^2}I(|u|^2u)(s)ds\Big\|_{X_\delta^{2,b}} \lesssim \|I\psi\|_{H^2} + \delta^{1-b-b'}\|I(|u|^2u)\|_{X_\delta^{2,-b'}}.$$

By the definition of restriction norm (6.1.16),

$$\|Iu\|_{X_\delta^{2,b}} \lesssim \|I\psi\|_{H^2} + \delta^{1-b-b'}\|I(|w|^2w)\|_{X^{2,-b'}},$$

[1] A Banach space X of space functions on Ω is said to be translation invariant if

$$\|u(\cdot-y)\|_X = \|u\|_X, \quad \forall u\in X, \forall y\in\Omega.$$

where w agrees with u on $[0,\delta]\times\mathbb{R}^4$ and

$$\|Iu\|_{X^{2,b}_\delta}\sim\|Iw\|_{X^{2,b}}.$$

Let us assume for the moment that

$$\|I(|w|^2w)\|_{X^{2,-b'}}\lesssim\|Iw\|^3_{X^{2,b}}. \tag{6.1.22}$$

This implies that

$$\|Iu\|_{X^{2,b}_\delta}\lesssim\|I\psi\|_{H^2}+\delta^{1-b-b'}\|Iu\|^3_{X^{2,b}_\delta}.$$

Note that

$$\|I\psi\|_{H^2}\sim\|I\psi\|_{\dot H^2}+\|I\psi\|_{L^2}\leq 1+\|\psi\|_{L^2}. \tag{6.1.23}$$

As $\|Iu\|_{X^{2,b}_\delta}$ is continuous in the δ variable, the bootstrap argument (see e.g. [Tao06, Section 1.3]) yields

$$\|Iu\|_{X^{2,b}_\delta}\lesssim 1.$$

This proves (6.1.15). It remains to show (6.1.22). We will take the advantage of interpolation Lemma 6.1.11. Note that the I-operator defined in (6.1.5) is equal to J^α_N defined in (6.1.14) with $\alpha=2-\gamma$. Thus, by Lemma 6.1.11, (6.1.22) is proved once there is $\alpha_0>0$ so that

$$\|J^\alpha_1(|w|^2w)\|_{X^{2,-b'}}\lesssim\|J^\alpha_1 w\|^3_{X^{2,b}},$$

for all $0\leq\alpha\leq\alpha_0$. Splitting w to low and high frequency parts $|\xi|\lesssim 1$ and $|\xi|\gg 1$ respectively and using definition of J^α_1, it suffices to show

$$\||w|^2w\|_{X^{\gamma,-b'}}\lesssim\|w\|^3_{X^{\gamma,b}}, \tag{6.1.24}$$

for all $\gamma\in[\overline{\gamma},2]$. By duality and the Leibniz rule, (6.1.24) follows from

$$\Big|\iint_{\mathbb{R}\times\mathbb{R}^4}(\langle\nabla\rangle^\gamma w_1)\overline{w}_2w_3w_4dtdx\Big|\lesssim\|w_1\|_{X^{\gamma,b}}\|w_2\|_{X^{\gamma,b}}\|w_3\|_{X^{\gamma,b}}\|w_4\|_{X^{0,b'}}. \tag{6.1.25}$$

Note that the last term is written more precisely as $\|w_4\|_{X^{0,b'}_{\tau=-|\xi|^4}}$ but it does not affect our estimate. Using Hölder's inequality, we can bound the left hand side of (6.1.25) as

$$\text{LHS}(6.1.25)\leq\|\langle\nabla\rangle^\gamma w_1\|_{L^4(\mathbb{R},L^4)}\|w_2\|_{L^4(\mathbb{R},L^4)}\|w_3\|_{L^6(\mathbb{R},L^6)}\|w_4\|_{L^3(\mathbb{R},L^3)}.$$

Since $(4,4)$ is an admissible pair, Corollary 6.1.5 gives

$$\|\langle\nabla\rangle^\gamma w_1\|_{L^4(\mathbb{R},L^4)}\lesssim\|w_1\|_{X^{\gamma,b}},\quad \|w_2\|_{L^4(\mathbb{R},L^4)}\lesssim\|w_2\|_{X^{0,b}}\leq\|w_2\|_{X^{\gamma,b}}.$$

Similarly, Sobolev embedding and Corollary 6.1.5 yield

$$\|w_3\|_{L^6(\mathbb{R},L^6)}\lesssim\|\langle\nabla\rangle^{2/3}w_3\|_{L^6(\mathbb{R},L^3)}\lesssim\|w_3\|_{X^{2/3,b}}\leq\|w_3\|_{X^{\gamma,b}}.$$

The last estimate comes from the fact that $\gamma>2/3$. Finally, we interpolate between $\|w_4\|_{L^2(\mathbb{R},L^2)}=\|w_4\|_{X^{0,0}}$ and $\|w_4\|_{L^4(\mathbb{R},L^4)}\lesssim\|w_4\|_{X^{0,1/2+}}$ to get

$$\|w_4\|_{L^3(\mathbb{R},L^3)}\lesssim\|w_4\|_{X^{0,b'}}.$$

Combing these estimates, we have (6.1.25). The proof of Proposition 6.1.12 is now complete. □

We are now able to prove the almost conservation law.

Proof of Proposition 6.1.9**.** By the assumption $E(I\psi) \leq 1$, Proposition 6.1.12 shows that there exists $\delta = \delta(\|\psi\|_{L^2})$ such that the solution u to (dNL4S) satisfies (6.1.15). We firstly note that the usual energy satisfies

$$\begin{aligned}\frac{d}{dt}E(u(t)) &= \text{Re } \int_{\mathbb{R}^4} \overline{\partial_t u(t,x)}(|u(t,x)|^2 u(t,x) + \Delta^2 u(t,x))dx \\ &= \text{Re } \int_{\mathbb{R}^4} \overline{\partial_t u(t,x)}(|u(t,x)|^2 u(t,x) + \Delta^2 u(t,x) + i\partial_t u(t,x))dx = 0.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{d}{dt}E(Iu(t)) &= \text{Re } \int_{\mathbb{R}^4} \overline{I\partial_t u(t,x)}(|Iu(t,x)|^2 Iu(t,x) + \Delta^2 Iu(t,x) + i\partial_t Iu(t,x))dx \\ &= \text{Re } \int_{\mathbb{R}^4} \overline{I\partial_t u(t,x)}(|Iu(t,x)|^2 Iu(t,x) - I(|u(t,x)|^2 u(t,x)))dx.\end{aligned}$$

Here the second line follows by applying I to both sides of (dNL4S). Integrating in time and applying the Parseval formula, we obtain

$$E(Iu(\delta)) - E(I\psi) = \text{Re } \int_0^\delta \int_{\sum_{j=1}^4 \xi_j = 0} \Big(1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}\Big)\widehat{\overline{I\partial_t u}}(\xi_1)\widehat{Iu}(\xi_2)\widehat{\overline{Iu}}(\xi_3)\widehat{Iu}(\xi_4)dt.$$

Here $\int_{\sum_{j=1}^4 \xi_j = 0}$ denotes the integration with respect to the hyperplane's measure $\delta_0(\xi_1 + ... + \xi_4)d\xi_1...d\xi_4$. Using that $iI\partial_t u = -\Delta^2 Iu - I(|u|^2 u)$, we have

$$|E(Iu(t)) - E(I\psi)| \leq \text{Term}_1 + \text{Term}_2,$$

where

$$\text{Term}_1 = \Big|\int_0^\delta \int_{\sum_{j=1}^4 \xi_j = 0} \mu(\xi_2,\xi_3,\xi_4)\widehat{\overline{\Delta^2 Iu}}(\xi_1)\widehat{Iu}(\xi_2)\widehat{\overline{Iu}}(\xi_3)\widehat{Iu}(\xi_4)dt\Big|,$$

and

$$\text{Term}_2 = \Big|\int_0^\delta \int_{\sum_{j=1}^4 \xi_j = 0} \mu(\xi_2,\xi_3,\xi_4)\widehat{\overline{I(|u|^2 u)}}(\xi_1)\widehat{Iu}(\xi_2)\widehat{\overline{Iu}}(\xi_3)\widehat{Iu}(\xi_4)dt\Big|,$$

with

$$\mu(\xi_2,\xi_3,\xi_4) := 1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)}.$$

Our purpose is to prove

$$\text{Term}_1 + \text{Term}_2 \lesssim N^{-\gamma_0+}.$$

Let us consider the first term (Term_1). To do so, we decompose $u = \sum_{M\geq 1} P_M u =: \sum_{M \geq 1} u_M$ with the convention $P_1 u := P_{\leq 1} u$ and write Term_1 as a sum over all dyadic pieces. By the symmetry of μ in ξ_2, ξ_3, ξ_4 and the fact that the bilinear estimate (6.1.4) allows complex conjugations on either factors, we may assume that $M_2 \geq M_3 \geq M_4$. Thus,

$$\text{Term}_1 \lesssim \sum_{\substack{M_1,M_2,M_3,M_4 \geq 1 \\ M_2 \geq M_3 \geq M_4}} A(M_1,M_2,M_3,M_4),$$

where

$$A(M_1,M_2,M_3,M_4) := \Big|\int_0^\delta \int_{\sum_{j=1}^4 \xi_j = 0} \mu(\xi_2,\xi_3,\xi_4)\widehat{\overline{\Delta^2 Iu_{M_1}}}(\xi_1)\widehat{Iu_{M_2}}(\xi_2)\widehat{\overline{Iu_{M_3}}}(\xi_3)\widehat{Iu_{M_4}}(\xi_4)dt\Big|.$$

To simplify the notation, we will drop the dependence of M_1, M_2, M_3, M_4 and write A instead of $A(M_1, M_2, M_3, M_4)$. In order to have $\text{Term}_1 \lesssim N^{-\gamma_0+}$, it suffices to prove

$$A \lesssim N^{-\gamma_0+} M_2^{0-}. \tag{6.1.26}$$

To show (6.1.26), we will break the frequency interactions into three cases due to the comparison of N with M_j. It is worth to notice that $M_1 \lesssim M_2$ due to the fact that $\sum_{j=1}^4 \xi_j = 0$.
Case 1. $N \gg M_2$. In this case, we have $|\xi_2|, |\xi_3|, |\xi_4| \ll N$ and $|\xi_2 + \xi_3 + \xi_4| \ll N$, hence

$$m(\xi_2 + \xi_3 + \xi_4) = m(\xi_2) = m(\xi_3) = m(\xi_4) = 1 \text{ and } \mu(\xi_2, \xi_3, \xi_4) = 0.$$

Thus (6.1.26) holds trivially.
Case 2. $M_2 \gtrsim N \gg M_3 \geq M_4$. Since $\sum_{j=1}^4 \xi_j = 0$, we get $M_1 \sim M_2$. We also have from the mean value theorem that

$$|\mu(\xi_2, \xi_3, \xi_4)| = \Big|1 - \frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)}\Big| \lesssim \frac{|\nabla m(\xi_2) \cdot (\xi_3 + \xi_4)|}{m(\xi_2)} \lesssim \frac{M_3}{M_2}.$$

The pointwise bound, Hölder's inequality, Plancherel theorem and bilinear estimate (6.1.4) yield

$$\begin{aligned}
A &\lesssim \frac{M_3}{M_2} \|\Delta^2 I u_{M_1} I u_{M_3}\|_{L^2(\mathbb{R}, L^2)} \|I u_{M_2} I u_{M_4}\|_{L^2(\mathbb{R}, L^2)} \\
&\lesssim \frac{M_3}{M_2} \Big(\frac{M_3}{M_1}\Big)^{3/2} \Big(\frac{M_4}{M_2}\Big)^{3/2} M_1^4 \prod_{j=1}^4 \|I u_{M_j}\|_{X^{0,1/2+}} \\
&\lesssim \frac{M_3}{M_2} \Big(\frac{M_3}{M_1}\Big)^{3/2} \Big(\frac{M_4}{M_2}\Big)^{3/2} \frac{M_1^2}{M_2^2 \langle M_3 \rangle^2 \langle M_4 \rangle^2} \prod_{j=1}^4 \|I u_{M_j}\|_{X^{2,1/2+}} \\
&= \Big(\frac{M_3}{N}\Big)^{1/2} \Big(\frac{M_1}{M_2}\Big)^{1/2} \Big(\frac{N}{M_2}\Big)^{4-} N^{-7/2+} M_2^{0-} \prod_{j=1}^4 \|I u_{M_j}\|_{X^{2,1/2+}} \\
&\lesssim N^{-7/2+} M_2^{0-} \prod_{j=1}^4 \|I u_{M_j}\|_{X^{2,1/2+}}.
\end{aligned} \tag{6.1.27}$$

Using (6.1.15) and the fact that $\gamma_0 < 7/2$, we have (6.1.26).
Case 3. $M_2 \geq M_3 \gtrsim N$. In this case, we simply bound

$$|\mu(\xi_2, \xi_3, \xi_4)| \lesssim \frac{m(\xi_1)}{m(\xi_2) m(\xi_3) m(\xi_4)}.$$

Here we use that $m(\xi_1) \gtrsim m(\xi_2)$ and $m(\xi_3) \leq m(\xi_4) \leq 1$ due to the fact that $M_1 \lesssim M_2$ and $M_3 \geq M_4$.
Subcase 3a. $M_2 \gg M_3 \gtrsim N$. We see that $M_1 \sim M_2$ since $\sum_{j=1}^4 \xi_j = 0$. The pointwise bound, Hölder's inequality, Plancherel theorem and bilinear estimate (6.1.4) again give

$$\begin{aligned}
A &\lesssim \frac{m(M_1)}{m(M_2) m(M_3) m(M_4)} \|\overline{\Delta^2 I u_{M_1}} I u_{M_4}\|_{L^2(\mathbb{R}, L^2)} \|I u_{M_2} \overline{I u_{M_3}}\|_{L^2(\mathbb{R}, L^2)} \\
&\lesssim \frac{m(M_1)}{m(M_2) m(M_3) m(M_4)} \Big(\frac{M_4}{M_1}\Big)^{3/2} \Big(\frac{M_3}{M_2}\Big)^{3/2} \frac{M_1^2}{M_2^2 M_3^2 \langle M_4 \rangle^2} \prod_{j=1}^4 \|I u_{M_j}\|_{X^{2,1/2+}}.
\end{aligned}$$

Thanks to (6.1.15), we only need to show

$$\frac{m(M_1)}{m(M_2)m(M_3)m(M_4)}\Big(\frac{M_4}{M_1}\Big)^{3/2}\Big(\frac{M_3}{M_2}\Big)^{3/2}\frac{M_1^2}{M_2^2M_3^2\langle M_4\rangle^2}\lesssim N^{-\gamma_0+}M_2^{0-}. \tag{6.1.28}$$

Remark that the function $m(\lambda)\lambda^\alpha$ is increasing, and $m(\lambda)\langle\lambda\rangle^\alpha$ is bounded below for any $\alpha+\gamma-2>0$ due to

$$(m(\lambda)\lambda^\alpha)'=\begin{cases}\alpha\lambda^{\alpha-1} & \text{if } 1\leq\lambda\leq N,\\ N^{2-\gamma}(\alpha+\gamma-2)\lambda^{\alpha+\gamma-3} & \text{if } \lambda\geq 2N.\end{cases}$$

We shall shortly choose an appropriate value of α, says $\overline{\alpha}$, so that

$$m(M_4)\langle M_4\rangle^{\overline{\alpha}}\gtrsim 1,\quad m(M_3)M_3^{\overline{\alpha}}\gtrsim m(N)N^{\overline{\alpha}}=N^{\overline{\alpha}}. \tag{6.1.29}$$

Using that $m(M_1)\sim m(M_2)$, we have

$$\begin{aligned}\text{LHS}(6.1.28)&\lesssim\frac{M_3^{\overline{\alpha}-1/2}\langle M_4\rangle^{\overline{\alpha}-1/2}M_1^{1/2}}{m(M_3)M_3^{\overline{\alpha}}m(M_4)\langle M_4\rangle^{\overline{\alpha}}M_2^{7/2}}\\ &\lesssim\frac{1}{N^{\overline{\alpha}}M_2^{4-2\overline{\alpha}}}\Big(\frac{M_3}{M_2}\Big)^{\overline{\alpha}-1/2}\Big(\frac{\langle M_4\rangle}{M_2}\Big)^{\overline{\alpha}-1/2}\Big(\frac{M_1}{M_2}\Big)^{1/2}\\ &\lesssim N^{-(4-\overline{\alpha})+}M_2^{0-}.\end{aligned}$$

Therefore, if we choose $\overline{\alpha}$ so that $\gamma_0=4-\overline{\alpha}$ or $\overline{\alpha}=4-\gamma_0=\frac{14}{15}$, then we get (6.1.26). Note that $\overline{\alpha}+\overline{\gamma}-2\geq 0$, hence (6.1.29) holds.

Subcase 3b. $M_2\sim M_3\gtrsim N$. In this case, we see that $M_1\lesssim M_2$. Arguing as in Subcase 3a, we obtain

$$\begin{aligned}A&\lesssim\frac{m(M_1)}{m(M_2)m(M_3)m(M_4)}\|\overline{\Delta^2 Iu_{M_1}}Iu_{M_2}\|_{L^2(\mathbb{R},L^2)}\|Iu_{M_3}\overline{Iu_{M_4}}\|_{L^2(\mathbb{R},L^2)}\\ &\lesssim\frac{m(M_1)}{m(M_2)m(M_3)m(M_4)}\Big(\frac{M_1}{M_2}\Big)^{3/2}\Big(\frac{M_4}{M_3}\Big)^{3/2}\frac{\langle M_1\rangle^2}{M_2^2M_3^2\langle M_4\rangle^2}\prod_{j=1}^4\|Iu_{M_j}\|_{X^{2,1/2+}}.\end{aligned}$$

As in Subcase 3a, our aim is to prove

$$\frac{m(M_1)}{m(M_2)m(M_3)m(M_4)}\Big(\frac{M_1}{M_2}\Big)^{3/2}\Big(\frac{M_4}{M_3}\Big)^{3/2}\frac{\langle M_1\rangle^2}{M_2^2M_3^2\langle M_4\rangle^2}\lesssim N^{-\gamma_0+}M_2^{0-}. \tag{6.1.30}$$

We use (6.1.29) to get

$$\begin{aligned}\text{LHS}(6.1.30)&\lesssim\frac{m(M_1)}{m(M_2)m(M_3)m(M_4)\langle M_4\rangle^{1/2}M_3^{7/2}}\\ &\lesssim\frac{m(M_1)M_2^{\alpha}\langle M_4\rangle^{\alpha-1/2}}{m(M_2)M_2^{\alpha}m(M_3)M_3^{\overline{\alpha}}m(M_4)\langle M_4\rangle^{\overline{\alpha}}M_3^{4-\overline{\alpha}-1/2}}\\ &\lesssim\frac{1}{N^{2\overline{\alpha}}}\Big(\frac{M_2}{M_3}\Big)^{\overline{\alpha}}\Big(\frac{\langle M_4\rangle}{M_3}\Big)^{\overline{\alpha}-1/2}\frac{1}{M_3^{4-3\overline{\alpha}}}\\ &\lesssim N^{-(4-\overline{\alpha})+}M_2^{0-}.\end{aligned}$$

Choosing $\overline{\alpha}$ as in Subcase 3a, we get (6.1.26).

We now consider the second term (Term$_2$). We again decompose u in dyadic frequencies, $u=\sum_{M\geq 1}u_M$. By the symmetry, we can assume that $M_2\geq M_3\geq M_4$. We can assume further that $M_2 \gtrsim N$ since $\mu(\xi_2, \xi_3, \xi_4)$ vanishes otherwise. Thus,

$$\text{Term}_2 \lesssim \sum_{\substack{M_1, M_2, M_3, M_4 \geq 1 \\ M_2 \geq M_3 \geq M_4}} B(M_1, M_2, M_3, M_4),$$

where

$$B(M_1, M_2, M_3, M_4) := \Big| \int_0^\delta \int_{\sum_{j=1}^4 \xi_j = 0} \mu(\xi_2, \xi_3, \xi_4) \overline{\widehat{P_{M_1} I(|u|^2 u)}}(\xi_1) \widehat{Iu_{M_2}}(\xi_2) \overline{\widehat{Iu_{M_3}}}(\xi_3) \widehat{Iu_{M_4}}(\xi_4) dt \Big|.$$

As for the Term_1, we will use the notation B instead of $B(M_1, M_2, M_3, M_4)$. Using the trivial bound

$$|\mu(\xi_2, \xi_3, \xi_4)| \lesssim \frac{m(M_1)}{m(M_2)m(M_3)m(M_4)},$$

Hölder's inequality and Plancherel theorem, we bound

$$B \lesssim \frac{m(M_1)}{m(M_2)m(M_3)m(M_4)} \|P_{M_1} I(|u|^2 u)\|_{L^2(\mathbb{R}, L^2)} \|Iu_{M_2}\|_{L^4(\mathbb{R}, L^4)} \|Iu_{M_3}\|_{L^4(\mathbb{R}, L^4)} \|Iu_{M_4}\|_{L^\infty(\mathbb{R}, L^\infty)}.$$

Lemma 6.1.13. *We have*

$$\|P_{M_1} I(|u|^2 u)\|_{L^2(\mathbb{R}, L^2)} \lesssim \frac{1}{\langle M_1 \rangle^2} \|Iu\|^3_{X^{2,1/2+}}, \tag{6.1.31}$$

$$\|Iu_{M_j}\|_{L^4(\mathbb{R}, L^4)} \lesssim \frac{1}{\langle M_j \rangle^2} \|Iu_{M_j}\|_{X^{2,1/2+}}, \quad j = 2, 3, \tag{6.1.32}$$

$$\|Iu_{M_4}\|_{L^\infty(\mathbb{R}, L^\infty)} \lesssim \|Iu_{M_4}\|_{X^{2,1/2+}}. \tag{6.1.33}$$

Proof. The estimate (6.1.31) is in turn equivalent to

$$\| \langle \nabla \rangle^2 P_{M_1} I(|u|^2 u)\|_{L^2(\mathbb{R}, L^2)} \lesssim \|Iu\|^3_{X^{2,1/2+}}.$$

Since $\langle \nabla \rangle^2 I$ obeys a Leibniz rule, it suffices to prove

$$\|P_{M_1}((\langle \nabla \rangle^2 Iu_1) u_2 u_3)\|_{L^2(\mathbb{R}, L^2)} \lesssim \prod_{j=1}^3 \|Iu_j\|_{X^{2,1/2+}}. \tag{6.1.34}$$

The Littlewood-Paley theorem and Hölder's inequality imply

$$\text{LHS(6.1.34)} \lesssim \| \langle \nabla \rangle^2 Iu_1\|_{L^4(\mathbb{R}, L^4)} \|u_2\|_{L^8(\mathbb{R}, L^8)} \|u_3\|_{L^8(\mathbb{R}, L^8)}.$$

We have from Strichartz estimate (6.1.2) that

$$\| \langle \nabla \rangle^2 Iu_1\|_{L^4(\mathbb{R}, L^4)} \lesssim \| \langle \nabla \rangle^2 Iu_1\|_{X^{0,1/2+}} = \|Iu_1\|_{X^{2,1/2+}}.$$

Combining Sobolev embedding and Strichartz estimate (6.1.2) yield

$$\|u_2\|_{L^8(\mathbb{R}, L^8)} \lesssim \| \langle \nabla \rangle u_2\|_{L^8(\mathbb{R}, L^{8/3})} \lesssim \| \langle \nabla \rangle u_2\|_{X^{0,1/2+}} \lesssim \|Iu_2\|_{X^{2,1/2+}},$$

where the last estimate follows from (6.1.10). Similarly for $\|u_3\|_{L^8(\mathbb{R}, L^8)}$. This shows (6.1.34). The estimate (6.1.32) follows easily from Strichartz estimate. For (6.1.33), we use Sobolev embedding and Strichartz estimate to get

$$\|Iu_{M_4}\|_{L^\infty(\mathbb{R}, L^\infty)} \lesssim \| \langle \nabla \rangle^2 Iu_{M_4}\|_{L^\infty(\mathbb{R}, L^2)} \lesssim \| \langle \nabla \rangle^2 Iu_{M_4}\|_{X^{0,1/2+}} = \|Iu_{M_4}\|_{X^{2,1/2+}}.$$

The proof is complete. ◻

We use Lemma 6.1.13 to bound

$$B \lesssim \frac{m(M_1)}{m(M_2)m(M_3)m(M_4)} \frac{1}{\langle M_1\rangle^2 \langle M_2\rangle^2 \langle M_3\rangle^2} \|Iu\|^3_{X^{2,1/2+}} \prod_{j=2}^4 \|Iu_{M_j}\|_{X^{2,1/2+}},$$

with $M_2 \geq M_3 \geq M_4$ and $M_2 \gtrsim N$. Using (6.1.15), the estimate (6.1.26) follows once we have

$$\frac{m(M_1)}{m(M_2)m(M_3)m(M_4)} \frac{1}{\langle M_1\rangle^2 \langle M_2\rangle^2 \langle M_3\rangle^2} \lesssim N^{-\gamma_0+} M_2^{0-}. \tag{6.1.35}$$

We now break the frequency interactions into two cases: $M_2 \sim M_3$ and $M_2 \sim M_1$ since $\sum_{j=1}^4 \xi_j = 0$.

Case 1. $M_2 \sim M_3, M_2 \geq M_3 \geq M_4$ and $M_2 \gtrsim N$. We see that

$$\begin{aligned}\text{LHS}(6.1.35) &\sim \frac{m(M_1)}{(m(M_2))^2 m(M_4)} \frac{1}{\langle M_1\rangle^2 \langle M_2\rangle^4} \lesssim \frac{m(M_1)}{N^{2\overline{\alpha}} m(M_4) \langle M_1\rangle^2 \langle M_2\rangle^{4-2\overline{\alpha}}} \\ &\lesssim \frac{1}{N^{2\overline{\alpha}}} \frac{1}{m(M_4)\langle M_2\rangle^{4-2\overline{\alpha}}} \lesssim \frac{1}{N^{2\overline{\alpha}}} \frac{1}{M_2^{4-3\overline{\alpha}}} \lesssim N^{-(4-\overline{\alpha})+} M_2^{0-}.\end{aligned}$$

Here we use that $m(M_2)\langle M_2\rangle^{\overline{\alpha}} \geq m(N)N^{\overline{\alpha}} = N^{\overline{\alpha}}, m(M_1) \lesssim \langle M_1\rangle^2$ and that $m(y)\langle x\rangle^{\overline{\alpha}} \gtrsim 1$ for all $1 \leq y \leq x$.

Case 2. $M_2 \sim M_1$, $M_2 \geq M_3 \geq M_4$ and $M_2 \gtrsim N$. We have

$$\begin{aligned}\text{LHS}(6.1.35) &\lesssim \frac{1}{m(M_3)m(M_4)} \frac{1}{\langle M_2\rangle^4 \langle M_3\rangle^2} \\ &\lesssim \frac{1}{m(M_3)\langle M_3\rangle^{\overline{\alpha}}} \frac{1}{m(M_4)\langle M_2\rangle^{\overline{\alpha}}} \frac{1}{\langle M_2\rangle^{4-\overline{\alpha}} \langle M_3\rangle^{2-\overline{\alpha}}} \\ &\lesssim N^{-(4-\overline{\alpha})+} M_2^{0-}.\end{aligned}$$

Here we use again $m(M_3)\langle M_3\rangle^{\overline{\alpha}}, m(M_4)\langle M_2\rangle^{\overline{\alpha}} \gtrsim 1$. By choosing $\overline{\alpha}$ as in Subcase 3a, we prove (6.1.35). The proof of Proposition 6.1.9 is now complete. □

Remark 6.1.14. Let us now comment on the choices of $\overline{\alpha}$ and γ_0. As mentioned in Remark 6.1.10, if the increment of the modified energy is $N^{-\gamma_0}$, then we can show (see Section 6.1.3, after (6.1.40)) that the global well-posedness holds for data in $H^\gamma(\mathbb{R}^4)$ with $\gamma > \frac{8}{4+\gamma_0} =: \overline{\gamma}$. We learn from (6.1.27) that $\gamma_0 \leq 7/2$, hence $\overline{\gamma} \geq \frac{16}{15}$. On the other hand, in Subcase 3a, we need $\overline{\alpha}+\gamma-2 > 0$ and $\overline{\alpha} = 4-\gamma_0$. Since $\gamma > \overline{\gamma}$, we have $\overline{\alpha}+\gamma-2 > \overline{\alpha}+\overline{\gamma}-2 \geq \overline{\alpha}+\frac{16}{15}-2$. We thus choose $\overline{\alpha} := 2-\frac{16}{15} = \frac{14}{15}$, hence $\gamma_0 = 4-\overline{\alpha} = \frac{46}{15}$.

6.1.3 The proof of Theorem 6.1.1

We now are able to show the global existence given in Theorem 6.1.1. We only consider positive time, the negative one is treated similarly. The conservation of mass and Lemma 6.1.8 give

$$\|u(t)\|^2_{H^\gamma} \lesssim \|Iu(t)\|^2_{H^2} \sim \|Iu(t)\|^2_{\dot{H}^2} + \|Iu(t)\|^2_{L^2} \lesssim E(Iu(t)) + \|\psi\|^2_{L^2}. \tag{6.1.36}$$

By density argument, we may assume that $\psi \in C_0^\infty(\mathbb{R}^4)$. Let u be a global solution to (dNL4S) with initial data ψ. As $E(I\psi)$ is not necessarily small, we will use the scaling

$$u_\lambda(t,x) := \lambda^{-2} u(\lambda^{-4}t, \lambda^{-1}x), \quad \lambda > 0$$

to make the energy of rescaled initial data small in order to apply the almost conservation law given in Proposition 6.1.9. We have

$$E(Iu_\lambda(0)) = \frac{1}{2}\|Iu_\lambda(0)\|^2_{\dot{H}^2} + \frac{1}{4}\|Iu_\lambda(0)\|^4_{L^4}. \tag{6.1.37}$$

We then estimate

$$\|Iu_\lambda(0)\|^2_{\dot{H}^2} \lesssim N^{2(2-\gamma)}\|u_\lambda(0)\|^2_{\dot{H}^\gamma} = N^{2(2-\gamma)}\lambda^{-2\gamma}\|\psi\|^2_{\dot{H}^\gamma},$$

and

$$\|Iu_\lambda(0)\|^4_{L^4} \lesssim \|u_\lambda(0)\|^4_{L^4} = \lambda^{-4}\|\psi\|^4_{L^4} \lesssim \lambda^{-4}\|\psi\|^4_{H^\gamma}.$$

Note that $\gamma > \overline{\gamma} \geq 1$ allows us to use Sobolev embedding in the last inequality. Thus, (6.1.37) gives for $\lambda \gg 1$,

$$E(Iu_\lambda(0)) \lesssim (N^{2(2-\gamma)}\lambda^{-2\gamma} + \lambda^{-4})(1 + \|\psi\|_{H^\gamma})^4 \leq C_0 N^{2(2-\gamma)}\lambda^{-2\gamma}(1 + \|\psi\|_{H^\gamma})^4.$$

We now choose

$$\lambda := N^{\frac{2-\gamma}{\gamma}}\Big(\frac{1}{2C_0}\Big)^{-\frac{1}{2\gamma}}(1 + \|\psi\|_{H^\gamma})^{\frac{2}{\gamma}} \tag{6.1.38}$$

so that $E(Iu_\lambda(0)) \leq 1/2$. We then apply Proposition 6.1.9 for $u_\lambda(0)$. Note that we may reapply this proposition until $E(Iu_\lambda(t))$ reaches 1, that is at least $C_1 N^{\gamma_0-}$ times. Therefore,

$$E(Iu_\lambda(C_1 N^{\gamma_0-}\delta)) \sim 1. \tag{6.1.39}$$

Now given any $T \gg 1$, we choose $N \gg 1$ so that

$$T \sim \frac{N^{\gamma_0-}}{\lambda^4}C_1\delta.$$

Using (6.1.38), we see that

$$T \sim N^{\frac{(\gamma_0+4)\gamma-8}{\gamma}-}. \tag{6.1.40}$$

Here $\gamma > \overline{\gamma} = \frac{8}{\gamma_0+4}$, hence the power of N is positive and the choice of N makes sense for arbitrary $T \gg 1$. A direct computation shows

$$E(Iu(t)) = \lambda^4 E(Iu_\lambda(\lambda^4 t)).$$

Thus, we have from (6.1.38), (6.1.39) and (6.1.40) that

$$\begin{aligned} E(Iu(T)) &= \lambda^4 E(Iu_\lambda(\lambda^4 T)) = \lambda^4 E(Iu_\lambda(C_1 N^{\gamma_0-}\delta)) \\ &\sim \lambda^4 \leq N^{\frac{4(2-\gamma)}{\gamma}} \sim T^{\frac{4(2-\gamma)}{(\gamma_0+4)\gamma-8}+}. \end{aligned}$$

This shows that there exists $C_2 = C_2(\|\psi\|_{H^\gamma}, \delta)$ such that

$$E(Iu(T)) \leq C_2 T^{\frac{4(2-\gamma)}{(\gamma_0+4)\gamma-8}+},$$

for $T \gg 1$. This together with (6.1.36) show that

$$\|u(T)\|_{H^\gamma} \lesssim C_3 T^{\frac{2(2-\gamma)}{(\gamma_0+4)\gamma-8}+} + C_4 = C_3 T^{\frac{15(2-\gamma)}{53\gamma-60}+} + C_4,$$

where C_3, C_4 depend only on $\|\psi\|_{H^\gamma}$. The proof of Theorem 6.1.1 is complete.

6.2 Global well-posedness for the defocusing mass-critical NL4S below the energy space in dimensions $5 \leq d \leq 7$.

This section is devoted to the following result. As mentioned in the introduction of Chapter 6, this result can follow directly from the work of Pausader-Shao [PS10]. However, the proof we present below has its own interest and will be used in Chapter 7.

Theorem 6.2.1. *Let $d = 5, 6, 7$. The initial value problem* (dNL4S) *is globally well-posed in $H^\gamma(\mathbb{R}^d)$, for any $\gamma(d) < \gamma < 2$, where $\gamma(5) = \frac{8}{5}, \gamma(6) = \frac{5}{3}$ and $\gamma(7) = \frac{13}{7}$.*

The proof of the above theorem is based on the combination of the I-method and the interaction Morawetz inequality which is similar to those given in [DPST07]. The key is to show that the modified energy $E(Iu)$ is an "almost conserved" quantity in the sense that the time derivative of $E(Iu)$ decays with respect to a large parameter N (see Section 6.1.1 for the definition of I and N). To do so, we need delicate estimates on the commutator between the I-operator and the nonlinearity. Note that in our setting, the nonlinearity is not algebraic. Thus we can not apply the Fourier transform technique. Fortunately, thanks to a special Strichartz estimate (6.2.11), we are able to apply the technique given in [VZ09] to control the commutator. The interaction Morawetz inequality for the nonlinear fourth-order Schrödinger equation was first introduced in [Pau2] for $d \geq 7$, and was extended for $d \geq 5$ in [MWZ15]. With this estimate, the interpolation argument and Sobolev embedding give for any compact interval J,

$$\|u\|_{M(J)} := \|u\|_{L^{\frac{8(d-3)}{d}}(J, L^{\frac{2(d-3)}{d-4}})} \lesssim |J|^{\frac{d-4}{8(d-3)}} \|\psi\|_{L^2}^{\frac{1}{d-3}} \|u\|_{L^\infty(J, \dot{H}^{\frac{1}{2}})}^{\frac{d-4}{d-3}}. \tag{6.2.1}$$

As a byproduct of Strichartz estimates and the I-method, we show the almost conservation law for the modified energy of (dNL4S), that is if u is a smooth solution to (dNL4S) on a time interval $J = [0, T]$, and satisfies $\|I\psi\|_{H^2} \leq 1$ and if u satisfies in addition the a priori bound $\|u\|_{M(J)} \leq \mu$ for some small constant $\mu > 0$, then

$$\sup_{t \in [0,T]} |E(Iu(t)) - E(I\psi)| \lesssim N^{-(2-\gamma+\delta)}.$$

for $\max\left\{3 - \frac{8}{d}, \frac{8}{d}\right\} < \gamma < 2$ and $0 < \delta < \gamma + \frac{8}{d} - 3$.

We now briefly outline the idea of the proof. Let u be a global in time solution to (dNL4S). Observe that for any $\lambda > 0$,

$$u_\lambda(t, x) := \lambda^{-\frac{d}{2}} u(\lambda^{-4}t, \lambda^{-1}x) \tag{6.2.2}$$

is also a solution to (dNL4S). By choosing

$$\lambda \sim N^{\frac{2-\gamma}{\gamma}}, \tag{6.2.3}$$

and using some harmonic analysis, we can make $E(Iu_\lambda(0)) \leq \frac{1}{4}$ by taking λ sufficiently large depending on $\|\psi\|_{H^\gamma}$ and N. Fix an arbitrary large time T. The main goal is to show

$$E(Iu_\lambda(\lambda^4 T)) \leq 1. \tag{6.2.4}$$

With this bound, we can easily obtain the growth of $\|u(T)\|_{H^\gamma}$, and the global well-posedness in $H^\gamma(\mathbb{R}^d)$ follows immediately. In order to get (6.2.4), we claim that

$$\|u_\lambda\|_{M([0,t])} \leq K t^{\frac{d-4}{8(d-3)}}, \quad \forall t \in [0, \lambda^4 T],$$

for some constant K. If it is not so, then there exists $T_0 \in [0, \lambda^4 T]$ such that

$$\|u_\lambda\|_{M([0,T_0])} > K T_0^{\frac{d-4}{8(d-3)}}, \tag{6.2.5}$$

$$\|u_\lambda\|_{M([0,T_0])} \leq 2K T_0^{\frac{d-4}{8(d-3)}}. \tag{6.2.6}$$

Using (6.2.6), we can split $[0, T_0]$ into L subintervals $J_k, k = 1, ..., L$ so that

$$\|u_\lambda\|_{M(J_k)} \leq \mu.$$

The number L must satisfy

$$L \sim T_0^{\frac{d-4}{d}}. \tag{6.2.7}$$

Thus we can apply the almost conservation law to get

$$\sup_{[0,T_0]} E(Iu_\lambda(t)) \leq E(Iu_\lambda(0)) + N^{-(2-\gamma+\delta)} L.$$

Since $E(Iu_\lambda(0)) \leq \frac{1}{4}$, in order to have $E(Iu_\lambda(t)) \leq 1$ for all $t \in [0, T_0]$, we need

$$N^{-(2-\gamma+\delta)} L \ll \frac{1}{4}. \tag{6.2.8}$$

Combining (6.2.3), (6.2.7) and (6.2.8), we obtain the condition on γ. Next, using (6.2.1) together with some harmonic analysis, we estimate

$$\|u_\lambda\|_{M([0,T_0])} \lesssim T_0^{\frac{d-4}{8(d-3)}} \|\psi\|_{L^2}^{\frac{1}{d-3}} \sup_{[0,T_0]} \left(\|\psi\|^{\frac{3}{4}} \|Iu_\lambda(t)\|_{\dot{H}^2}^{\frac{1}{4}} + N^{-\frac{3}{4}} \|Iu_\lambda(t)\|_{\dot{H}^2} \right)^{\frac{d-4}{d-3}}.$$

Since $\|Iu_\lambda(t)\|_{\dot{H}^2} \lesssim E(Iu_\lambda(t)) \leq 1$ for all $t \in [0, T_0]$, we get

$$\|u_\lambda\|_{M([0,T_0])} \leq C T_0^{\frac{d-4}{8(d-3)}},$$

for some constant $C > 0$. This leads to a contradiction to (6.2.5) for an appropriate choice of K. Thus we have the claim and also

$$E(Iu_\lambda(t)) \leq 1, \quad \forall t \in [0, \lambda^4 T].$$

For more details, we refer the reader to Section 6.2.3.

6.2.1 Preliminaries

Nonlinearity. Let $F(z) := |z|^{\frac{8}{d}} z, d = 5, 6, 7$ be the function that defines the nonlinearity in (dNL4S). The derivative $F'(z)$ is defined as a real-linear operator acting on $w \in \mathbb{C}$ by

$$F'(z) \cdot w := w \partial_z F(z) + \overline{w} \partial_{\overline{z}} F(z),$$

where

$$\partial_z F(z) = \frac{2d+8}{2d} |z|^{\frac{8}{d}}, \quad \partial_{\overline{z}} F(z) = \frac{4}{d} |z|^{\frac{8}{d}} \frac{z}{\overline{z}}.$$

We shall identify $F'(z)$ with the pair $(\partial_z F(z), \partial_{\overline{z}} F(z))$, and define its norm by

$$|F'(z)| := |\partial_z F(z)| + |\partial_{\overline{z}} F(z)|.$$

It is clear that $|F'(z)| = O(|z|^{\frac{8}{d}})$. We also have the following chain rule

$$\partial_k F(u) = F'(u)\partial_k u,$$

for $k \in \{1, \cdots, d\}$. In particular, we have

$$\nabla F(u) = F'(u)\nabla u.$$

We next recall the fractional chain rule to estimate the nonlinearity.

Lemma 6.2.2. *Suppose that $G \in C^1(\mathbb{C}, \mathbb{C})$, and $\alpha \in (0,1)$. Then for $1 < q \leq q_2 < \infty$ and $1 < q_1 \leq \infty$ satisfying $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$,*

$$\||\nabla|^\alpha G(u)\|_{L^q} \lesssim \|G'(u)\|_{L^{q_1}} \||\nabla|^\alpha u\|_{L^{q_2}}.$$

We refer the reader to [CW91, Proposition 3.1] for the proof of the above estimate when $1 < q_1 < \infty$, and to [KPV93, Theorem A.6] for the proof when $q_1 = \infty$.

When G is no longer C^1, but Hölder continuous, we have the following fractional chain rule.

Lemma 6.2.3. *Suppose that $G \in C^{0,\beta}(\mathbb{C}, \mathbb{C}), \beta \in (0,1)$. Then for every $0 < \alpha < \beta, 1 < q < \infty$, and $\frac{\alpha}{\beta} < \rho < 1$,*

$$\||\nabla|^\alpha G(u)\|_{L^q} \lesssim \||u|^{\beta - \frac{\alpha}{\rho}}\|_{L^{q_1}} \||\nabla|^\rho u\|^{\frac{\alpha}{\rho}}_{L^{\frac{\alpha}{\rho} q_2}},$$

provided $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ and $\left(1 - \frac{\alpha}{\beta\rho}\right) q_1 > 1$.

The reader can find the proof of this result in [Vis06, Proposition A.1].

Strichartz estimates. Let $I \subset \mathbb{R}$ and $p, q \in [1, \infty]$. We define the mixed norm

$$\|u\|_{L^p(I, L^q)} := \Big(\int_I \Big(\int_{\mathbb{R}^d} |u(t,x)|^q dx \Big)^{\frac{p}{q}} \Big)^{\frac{1}{p}}$$

with a usual modification when either p or q are infinity.

In this section, we denote for $(p,q) \in [1, \infty]^2$,

$$\gamma_{p,q} = \frac{d}{2} - \frac{d}{q} - \frac{4}{p}.$$

Definition 6.2.4. A pair (p,q) is called **biharmonic admissible**, for short $(p,q) \in B$, if (p,q) is Schrödinger admissible satisfying

$$\gamma_{p,q} = 0.$$

We recall Strichartz estimates for the linear fourth-order Schrödinger equation given in Theorem 1.1.2 (see also Corollary 1.1.3) with $\sigma = 4$.

Proposition 6.2.5. *Let $\gamma \in \mathbb{R}$ and u be a (weak) solution to the linear fourth-order Schrödinger equation namely*

$$u(t) = e^{it\Delta^2}\psi + \int_0^t e^{i(t-s)\Delta^2} F(s) ds,$$

for some data ψ, F. Then for all (p,q) and (a,b) Schrödinger admissible with $q < \infty$ and $b < \infty$,

$$\||\nabla|^\gamma u\|_{L^p(\mathbb{R}, L^q)} \lesssim \||\nabla|^{\gamma + \gamma_{p,q}} \psi\|_{L^2} + \||\nabla|^{\gamma + \gamma_{p,q} - \gamma_{a',b'} - 4} F\|_{L^{a'}(\mathbb{R}, L^{b'})}. \tag{6.2.9}$$

Here (a,a') and (b,b') are conjugate pairs, and $\gamma_{p,q}, \gamma_{a',b'}$ are defined as in (1.0.7).

Note that the estimate (6.2.9) is exactly the one given in [MZ07], [Pau1] or [Pau2] where the author considered (p,q) and (a,b) are either sharp Schrödinger admissible (see (0.0.1)) or biharmonic admissible. The proof of Strichartz estimates proved by [MZ07, Pau1, Pau2] are based on delicate dispersive estimates of [BKS00] for the fundamental solution of the homogeneous fourth-order Schrödinger equation.

The following result is a direct consequence of (6.2.9).

Corollary 6.2.6. *Let u be a (weak) solution to the linear fourth-order Schrödinger equation for some data ψ, F. Then for all (p,q) and (a,b) biharmonic admissible satisfying $q<\infty$ and $b<\infty$,*

$$\|u\|_{L^p(\mathbb{R},L^q)} \lesssim \|\psi\|_{L^2} + \|F\|_{L^{a'}(\mathbb{R},L^{b'})}, \tag{6.2.10}$$

and

$$\|\Delta u\|_{L^p(\mathbb{R},L^q)} \lesssim \|\Delta\psi\|_{L^2} + \|\nabla F\|_{L^2(\mathbb{R},L^{\frac{2d}{d+2}})}. \tag{6.2.11}$$

Commutator estimates Let I be as in Subsection 6.1.1. When the nonlinearity $F(u)$ is algebraic, one can use the Fourier transform to write the commutator like $F(Iu)-IF(u)$ as a product of Fourier transforms of u and Iu, and then measure the frequency interactions. However, when $d \geq 5$, the nonlinearity is no longer algebraic, we thus need the following rougher estimate which is a modified version of the Schrödinger context (see [VZ09]).

Lemma 6.2.7. *Let $1<\gamma<2, 0<\delta<\gamma-1$ and $1<q,q_1,q_2<\infty$ be such that $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$. Then*

$$\|I(fg)-(If)g\|_{L^q} \lesssim N^{-(2-\gamma+\delta)}\|If\|_{L^{q_1}}\|\langle\nabla\rangle^{2-\gamma+\delta} g\|_{L^{q_2}}. \tag{6.2.12}$$

The proof is a slight modification of the one given in Lemma 2.5 of [VZ09]. We thus only give a sketch of the proof.

Sketch of the proof. By the Littlewood-Paley decomposition, we write

$$\begin{aligned}
I(fg)-(If)g &= I(fP_{\leq 1}g)-(If)P_{\leq 1}g + \sum_{M>1}[I(P_{\lesssim M}fP_Mg)-(IP_{\lesssim M}f)P_Mg] \\
&\quad + \sum_{M>1}[I(P_{\gg M}fP_Mg)-(IP_{\gg M}f)P_Mg] \\
&= I(P_{\gtrsim N}fP_{\leq 1}g)-(IP_{\gtrsim N}f)P_{\leq 1}g + \sum_{M\gtrsim N}[I(P_{\lesssim M}fP_Mg)-(IP_{\lesssim M}f)P_Mg] \\
&\quad + \sum_{M>1}[I(P_{\gg M}fP_Mg)-(IP_{\gg M}f)P_Mg] \\
&= \text{Term}_1+\text{Term}_2+\text{Term}_3.
\end{aligned}$$

Here we use the definition of the I-operator to get

$$I(P_{\ll N}fP_{\leq 1}g) = (IP_{\ll N}f)P_{\leq 1}g, \quad I(P_{\lesssim M}fP_Mg) = (IP_{\lesssim M}f)P_Mg,$$

for all $M \ll N$.

For the second term, using Lemma 6.1.2 and Lemma 6.1.8, we estimate

$$\begin{aligned}
\|I(P_{\lesssim M}fP_Mg)-(IP_{\lesssim M}f)P_Mg\|_{L^q} &\lesssim \|P_{\lesssim M}f\|_{L^{q_1}}\|P_Mg\|_{L^{q_2}}, \quad M\gtrsim N \\
&\lesssim \Big(\frac{M}{N}\Big)^{2-\gamma}\|If\|_{L^{q_1}}\|P_Mg\|_{L^{q_2}} \\
&\lesssim M^{-\delta}N^{-(2-\gamma)}\|If\|_{L^{q_1}}\||\nabla|^{2-\gamma+\delta}g\|_{L^{q_2}}.
\end{aligned}$$

Summing over all $N \lesssim M \in 2^{\mathbb{Z}}$, we get

$$\|\text{Term}_2\|_{L^q} \lesssim N^{-(2-\gamma+\delta)}\|If\|_{L^{q_1}}\||\nabla|^{2-\gamma+\delta}g\|_{L^{q_2}}.$$

For the third term, we write

$$\begin{aligned} I(P_{\gg M} f P_M g) - (IP_{\gg M} f) P_M g &= \sum_{1 \ll k \in \mathbb{N}} [I(P_{2^k M} f P_M g) - (IP_{2^k M} f) P_M g] \\ &= \sum_{\substack{1 \ll k \in \mathbb{N} \\ N \lesssim 2^k M}} I(P_{2^k M} f P_M g) - (IP_{2^k M} f) P_M g]. \end{aligned}$$

We note that

$$[I(P_{2^k M} f P_M g) - (IP_{2^k M} f) P_M g]\widehat{\ }(\xi) = \int_{\xi = \xi_1 + \xi_2} (m_N(\xi_1 + \xi_2) - m_N(\xi_1)) \widehat{P_{2^k M} f}(\xi_1) \widehat{P_M g}(\xi_2).$$

For $|\xi_1| \sim 2^k M \gtrsim N$ and $|\xi_2| \sim M$, the mean value theorem implies

$$|m_N(\xi_1 + \xi_2) - m_N(\xi_2)| \lesssim |\nabla m_N(\xi_1)| |\xi_2| \lesssim 2^{-k} \Big(\frac{2^k M}{N}\Big)^{\gamma - 2}.$$

The Coifman-Meyer multiplier theorem (see e.g. [CM75, CM91]) then yields

$$\begin{aligned} \|I(P_{2^k M} f P_M g) - (IP_{2^k M} f) P_M g\|_{L^q} &\lesssim 2^{-k} \Big(\frac{2^k M}{N}\Big)^{\gamma - 2} \|P_{2^k M} f\|_{L^{q_1}} \|P_M g\|_{L^{q_2}} \\ &\lesssim 2^{-k} M^{-(2-\gamma+\delta)} \|If\|_{L^{q_1}} \||\nabla|^{2-\gamma+\delta} g\|_{L^{q_2}}. \end{aligned}$$

By rewriting $2^{-k} M^{-(2-\gamma+\delta)} = 2^{-k(\gamma-1-\delta)} (2^k M)^{-(2-\gamma+\delta)}$, we sum over all $k \gg 1$ with $\gamma - 1 > \delta$ and $N \lesssim 2^k M$ to get

$$\|\mathrm{Term}_3\|_{L^q} \lesssim N^{-(2-\gamma+\delta)} \|If\|_{L^{q_1}} \||\nabla|^{2-\gamma+\delta} g\|_{L^{q_2}}.$$

Finally, we consider the first term. It is proved by the same argument as for the third term. We estimate

$$\begin{aligned} \|\mathrm{Term}_1\|_{L^q} &\lesssim \sum_{k \in \mathbb{N}, 2^k \gtrsim N} \|I(P_{2^k} f P_{\leq 1} g) - (IP_{2^k} f) P_{\leq 1} g\|_{L^q} \\ &\lesssim \sum_{k \in \mathbb{N}, 2^k \gtrsim N} 2^{-k} \|If\|_{L^{q_1}} \|g\|_{L^{q_2}} \\ &\lesssim N^{-1} \|If\|_{L^{q_1}} \|g\|_{L^{q_2}}. \end{aligned}$$

Note that the condition $\gamma - 1 > \delta$ ensures that $N^{-1} \lesssim N^{-(2-\gamma+\delta)}$. This completes the proof. □

As a direct consequence of Lemma 6.2.7 with the fact that

$$\nabla F(u) = F'(u) \nabla u,$$

we have the following corollary. Note that the I-operator commutes with ∇.

Corollary 6.2.8. *Let $1 < \gamma < 2, 0 < \delta < \gamma - 1$ and $1 < q, q_1, q_2 < \infty$ be such that $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. Then*

$$\|\nabla IF(u) - (I\nabla u) F'(u)\|_{L^q} \lesssim N^{-(2-\gamma+\delta)} \|\nabla Iu\|_{L^{q_1}} \| \langle \nabla \rangle^{2-\gamma+\delta} F'(u)\|_{L^{q_2}}. \tag{6.2.13}$$

Interaction Morawetz inequality. We end this section by recalling the interaction Morawetz inequality for the nonlinear fourth-order Schrödinger equation. This estimate was first established by Pausader in [Pau2] for $d \geq 7$. Later, Miao-Wu-Zhang in [MWZ15] extended this interaction Morawetz estimate to $d \geq 5$.

Proposition 6.2.9 (Interaction Morawetz inequality [Pau2], [MWZ15])**.** *Let $d \geq 5$, J be a compact time interval and u a solution to* (dNL4S) *on the spacetime slab $J \times \mathbb{R}^d$. Then we have the following* *a priori estimate:*

$$\||\nabla|^{-\frac{d-5}{4}}u\|_{L^4(J,L^4)} \lesssim \|\psi\|^{\frac{1}{2}}_{L^2}\|u\|^{\frac{1}{2}}_{L^\infty(J,\dot{H}^{\frac{1}{2}})}. \tag{6.2.14}$$

By interpolating (6.2.14) and the trivial estimate

$$\|u\|_{L^\infty(J,\dot{H}^{\frac{1}{2}})} \leq \|u\|_{L^\infty(J,\dot{H}^{\frac{1}{2}})},$$

we obtain

$$\|u\|_{L^{2(d-3)}(J,L^{\frac{2(d-3)}{d-4}})} \lesssim \left(\|\psi\|_{L^2}\|u\|_{L^\infty(J,\dot{H}^{\frac{1}{2}})}\right)^{\frac{1}{d-3}}\|u\|^{\frac{d-5}{d-3}}_{L^\infty(J,\dot{H}^{\frac{1}{2}})} = \|\psi\|^{\frac{1}{d-3}}_{L^2}\|u\|^{\frac{d-4}{d-3}}_{L^\infty(J,\dot{H}^{\frac{1}{2}})}.$$

Using Sobolev embedding in time, we get

$$\|u\|_{M(J)} := \|u\|_{L^{\frac{8(d-3)}{d}}(J,L^{\frac{2(d-3)}{d-4}})} \lesssim |J|^{\frac{d-4}{8(d-3)}}\|\psi\|^{\frac{1}{d-3}}_{L^2}\|u\|^{\frac{d-4}{d-3}}_{L^\infty(J,\dot{H}^{\frac{1}{2}})}. \tag{6.2.15}$$

Here $\left(\frac{8(d-3)}{d}, \frac{2(d-3)}{d-4}\right)$ is a biharmonic admissible pair.

6.2.2 Almost conservation law

For any spacetime slab $J \times \mathbb{R}^d$, we define

$$Z_I(J) := \sup_{(p,q)\in B} \|\langle\Delta\rangle Iu\|_{L^p(J,L^q)}.$$

Note that in our consideration $5 \leq d \leq 7$, the biharmonic admissible condition $(p,q) \in B$ ensures $q < \infty$. Let us start with the following commutator estimates.

Lemma 6.2.10. *Let $5 \leq d \leq 7, 1 < \gamma < 2$ and $0 < \delta < \gamma - 1$. Then*

$$\|\nabla IF(u) - (I\nabla u)F'(u)\|_{L^2(J,L^{\frac{2d}{d+2}})} \lesssim N^{-(2-\gamma+\delta)}(Z_I(J))^{1+\frac{8}{d}}, \tag{6.2.16}$$

$$\|\nabla IF(u)\|_{L^2(J,L^{\frac{2d}{d+2}})} \lesssim \|u\|^{\frac{8}{d}}_{M(J)}Z_I(J) + N^{-(2-\gamma+\delta)}(Z_I(J))^{1+\frac{8}{d}}, \tag{6.2.17}$$

where $\|u\|_{M(J)}$ is given in (6.2.15)*. In particular,*

$$\|\nabla IF(u)\|_{L^2(J,L^{\frac{2d}{d+2}})} \lesssim (Z_I(J))^{1+\frac{8}{d}}. \tag{6.2.18}$$

Proof. We apply (6.2.13) with $q = \frac{2d}{d+2}, q_1 = \frac{2d(d-3)}{d^2-9d+22}$ and $q_2 = \frac{d(d-3)}{2(2d-7)}$ to get

$$\|\nabla IF(u) - (I\nabla u)F'(u)\|_{L^{\frac{2d}{d+2}}} \lesssim N^{-(2-\gamma+\delta)}\|\nabla Iu\|_{L^{\frac{2d(d-3)}{d^2-9d+22}}}\|\langle\nabla\rangle^{2-\gamma+\delta}F'(u)\|_{L^{\frac{d(d-3)}{2(2d-7)}}}.$$

We then apply Hölder's inequality to have

$$\|\nabla IF(u)-(I\nabla u)F'(u)\|_{L^2(J,L^{\frac{2d}{d+2}})} \lesssim N^{-\alpha}\|\nabla Iu\|_{L^{\frac{2(d-3)}{d-4}}(J,L^{\frac{2d(d-3)}{d^2-9d+22}})}\|\langle\nabla\rangle^{\alpha}F'(u)\|_{L^{2(d-3)}(J,L^{\frac{d(d-3)}{2(2d-7)}})},$$

where $\alpha = 2 - \gamma + \delta \in (0,1)$ by our assumptions. For the first factor in the right hand side, we use the Sobolev embedding to obtain

$$\|\nabla Iu\|_{L^{\frac{2(d-3)}{d-4}}(J,L^{\frac{2d(d-3)}{d^2-9d+22}})} \lesssim \|\Delta Iu\|_{L^{\frac{2(d-3)}{d-4}}(J,L^{\frac{2d(d-3)}{d^2-7d+16}})} \lesssim Z_I, \tag{6.2.19}$$

where $\left(\frac{2(d-3)}{d-4}, \frac{2d(d-3)}{d^2-7d+16}\right)$ is a biharmonic admissible pair. For the second factor, we estimate

$$\| \langle \nabla \rangle^\alpha F'(u)\|_{L^{2(d-3)}(J, L^{\frac{d(d-3)}{2(2d-7)}})} \lesssim \|F'(u)\|_{L^{2(d-3)}(J, L^{\frac{d(d-3)}{2(2d-7)}})} + \||\nabla|^\alpha F'(u)\|_{L^{2(d-3)}(J, L^{\frac{d(d-3)}{2(2d-7)}})}. \tag{6.2.20}$$

Since $F'(u) = O(|u|^{\frac{8}{d}})$, we use (6.1.9) to have

$$\|F'(u)\|_{L^{2(d-3)}(J, L^{\frac{d(d-3)}{2(2d-7)}})} \lesssim \|u\|^{\frac{8}{d}}_{L^{\frac{16(d-3)}{d}}(J, L^{\frac{4(d-3)}{2d-7}})} \lesssim Z_I^{\frac{8}{d}}, \tag{6.2.21}$$

where $\left(\frac{16(d-3)}{d}, \frac{4(d-3)}{2d-7}\right)$ is biharmonic admissible. In order to treat the second term in (6.2.20), we apply Lemma 6.2.2 with $q = \frac{d(d-3)}{2(2d-7)}, q_1 = \frac{2d(d-3)}{-d^2+11d-26}$ and $q_2 = \frac{2d(d-3)}{d^2-3d-2}$ to get

$$\||\nabla|^\alpha F'(u)\|_{L^{2(d-3)}(J, L^{\frac{d(d-3)}{2(2d-7)}})} \lesssim \|F''(u)\|_{L^{4(d-3)}(J, L^{\frac{2d(d-3)}{-d^2+11d-26}})} \||\nabla|^\alpha u\|_{L^{4(d-3)}(J, L^{\frac{2d(d-3)}{d^2-3d-2}})}.$$

As $F''(u) = O(|u|^{\frac{8}{d}-1})$, we have

$$\|F''(u)\|_{L^{4(d-3)}(J, L^{\frac{2d(d-3)}{-d^2+11d-26}})} \lesssim \|u\|^{\frac{8}{d}-1}_{L^{\frac{4(8-d)(d-3)}{d}}(J, L^{\frac{2(8-d)(d-3)}{-d^2+11d-26}})} \lesssim Z_I^{\frac{8}{d}-1}. \tag{6.2.22}$$

Note that the above estimate is valid for d at most 7. Here $\left(\frac{4(8-d)(d-3)}{d}, \frac{2(8-d)(d-3)}{-d^2+11d-26}\right)$ is biharmonic admissible. Since $\left(4(d-3), \frac{2d(d-3)}{d^2-3d-2}\right)$ is also a biharmonic admissible, we have from (6.1.9) that

$$\||\nabla|^\alpha u\|_{L^{4(d-3)}(J, L^{\frac{2d(d-3)}{d^2-3d-2}})} \lesssim Z_I. \tag{6.2.23}$$

Note that $\alpha < 1 < \gamma$. Collecting (6.2.19), (6.2.21), (6.2.22) and (6.2.23), we prove (6.2.16).

We now prove (6.2.17). We have from (6.2.16) and the triangle inequality that

$$\|\nabla IF(u)\|_{L^2(J, L^{\frac{2d}{d+2}})} \lesssim \|(\nabla Iu)F'(u)\|_{L^2(J, L^{\frac{2d}{d+2}})} + N^{-(2-\gamma+\delta)} Z_I^{1+\frac{8}{d}}. \tag{6.2.24}$$

The Hölder inequality gives

$$\|(\nabla Iu)F'(u)\|_{L^2(J, L^{\frac{2d}{d+2}})} \lesssim \|\nabla Iu\|_{L^{\frac{2(d-3)}{d-5}}(J, L^{\frac{2d(d-3)}{d^2-9d+26}})} \|F'(u)\|_{L^{d-3}(JL^{\frac{d(d-3)}{4(d-4)}})}.$$

We use the Sobolev embedding to estimate

$$\|\nabla Iu\|_{L^{\frac{2(d-3)}{d-5}}(J, L^{\frac{2d(d-3)}{d^2-9d+26}})} \lesssim \|\Delta Iu\|_{L^{\frac{2(d-3)}{d-5}}(J, L^{\frac{2d(d-3)}{d^2-7d+20}})} \lesssim Z_I. \tag{6.2.25}$$

Here $\left(\frac{2(d-3)}{d-5}, \frac{2d(d-3)}{d^2-7d+20}\right)$ is biharmonic admissible. Since $F'(u) = O(|u|^{\frac{8}{d}})$, we have

$$\|F'(u)\|_{L^{d-3}(J, L^{\frac{d(d-3)}{4(d-4)}})} \lesssim \|u\|^{\frac{8}{d}}_{L^{\frac{8(d-3)}{d}}(J, L^{\frac{2(d-3)}{d-4}})} = \|u\|^{\frac{8}{d}}_M. \tag{6.2.26}$$

Combining (6.2.24), (6.2.25) and (6.2.26), we obtain (6.2.17). The estimate (6.2.18) follows directly from (6.2.17) and (6.1.9). Note that $\left(\frac{8(d-3)}{d}, \frac{2(d-3)}{d-4}\right)$ is biharmonic admissible. The proof is complete. □

We are now able to prove the almost conservation law for the modified energy functional $E(Iu)$,

where

$$E(Iu(t)) = \frac{1}{2}\|Iu(t)\|^2_{\dot{H}^2} + \frac{d}{2d+8}\|Iu(t)\|^{\frac{2d+8}{d}}_{L^{\frac{2d+8}{d}}}.$$

Proposition 6.2.11. *Let* $5 \leq d \leq 7, \max\left\{3-\frac{8}{d}, \frac{8}{d}\right\} < \gamma < 2$ *and* $0 < \delta < \gamma + \frac{8}{d} - 3$. *Assume that* u *is a smooth solution to* (dNL4S) *on a time interval* $J = [0,T]$, *and satisfies* $\|I\psi\|_{H^2} \leq 1$. *Assume in addition that* u *satisfies the a priori bound*

$$\|u\|_{M(J)} \leq \mu,$$

for some small constant $\mu > 0$. *Then, for* N *sufficiently large,*

$$\sup_{t\in[0,T]} |E(Iu(t)) - E(I\psi)| \lesssim N^{-(2-\gamma+\delta)}. \tag{6.2.27}$$

Here the implicit constant depends only on the size fo $E(I\psi)$.

Proof. Our first step is to control the size of Z_I. Applying I, ΔI to (dNL4S), and then using Strichartz estimates (6.2.10), (6.2.11), we have

$$Z_I \lesssim \|I\psi\|_{H^2} + \|IF(u)\|_{L^2(J,L^{\frac{2d}{d+4}})} + \|\nabla IF(u)\|_{L^2(J,L^{\frac{2d}{d+2}})}. \tag{6.2.28}$$

Using (6.2.17), we have

$$\|\nabla IF(u)\|_{L^2(J,L^{\frac{2d}{d+2}})} \lesssim \|u\|_M^{\frac{8}{d}} Z_I + N^{-(2-\gamma+\delta)} Z_I^{1+\frac{8}{d}} \lesssim \mu^{\frac{8}{d}} Z_I + N^{-(2-\gamma+\delta)} Z_I^{1+\frac{8}{d}}. \tag{6.2.29}$$

We next drop the I-operator (see (6.1.7)) and use Hölder's inequality to estimate

$$\begin{aligned}
\|IF(u)\|_{L^2(J,L^{\frac{2d}{d+4}})} &\lesssim \||u|^{\frac{8}{d}}\|_{L^{d-3}(J,L^{\frac{d(d-3)}{4(d-4)}})} \|u\|_{L^{\frac{2(d-3)}{d-5}}(J,L^{\frac{2d(d-3)}{d^2-7d+20}})} \\
&\lesssim \|u\|^{\frac{8}{d}}_{L^{\frac{8(d-3)}{d}}(J,L^{\frac{2(d-3)}{d-4}})} \|u\|_{L^{\frac{2(d-3)}{d-5}}(J,L^{\frac{2d(d-3)}{d^2-7d+20}})} \\
&\lesssim \|u\|_M^{\frac{8}{d}} Z_I \lesssim \mu^{\frac{8}{d}} Z_I.
\end{aligned} \tag{6.2.30}$$

The last inequality follows from (6.1.9) and the fact $\left(\frac{2(d-3)}{d-5}, \frac{2d(d-3)}{d^2-7d+20}\right)$ is biharmonic admissible. Collecting from (6.2.28) to (6.2.30), we obtain

$$Z_I \lesssim \|I\psi\|_{H^2} + \mu^{\frac{8}{d}} Z_I + N^{-(2-\gamma+\delta)} Z_I^{1+\frac{8}{d}}.$$

By taking μ sufficiently small and N sufficiently large, the continuity argument gives

$$Z_I \lesssim \|I\psi\|_{H^2} \leq 1. \tag{6.2.31}$$

Next, we have from a direct computation that

$$\partial_t E(Iu(t)) = \mathrm{Re} \int \overline{I\partial_t u}(\Delta^2 Iu + F(Iu))dx.$$

By the Fundamental Theorem of Calculus,

$$E(Iu(t)) - E(I\psi) = \int_0^t \partial_s E(Iu(s))ds = \mathrm{Re} \int_0^t \int \overline{I\partial_s u}(\Delta^2 Iu + F(Iu))dxds.$$

Using $I\partial_t u = i\Delta^2 Iu + iIF(u)$, we see that

$$\begin{aligned}
E(Iu(t)) - E(I\psi) &= \mathrm{Re} \int_0^t \int \overline{I\partial_s u}(F(Iu) - IF(u))dxds \\
&= \mathrm{Im} \int_0^t \int \overline{\Delta^2 Iu + IF(u)}(F(Iu) - IF(u))dxds \\
&= \mathrm{Im} \int_0^t \int \overline{\Delta Iu}\Delta(F(Iu) - IF(u))dxds \\
&\quad + \mathrm{Im} \int_0^t \int \overline{IF(u)}(F(Iu) - IF(u))dxds.
\end{aligned}$$

We next write

$$\begin{aligned}
\Delta(F(Iu) - IF(u)) &= (\Delta Iu)F'(Iu) + |\nabla Iu|^2 F''(Iu) - I(\Delta u F'(u)) - I(|\nabla u|^2 F''(u)) \\
&= (\Delta Iu)(F'(Iu) - F'(u)) + |\nabla Iu|^2 (F''(Iu) - F''(u)) \\
&\quad + \nabla Iu \cdot (\nabla Iu - \nabla u)F''(u) + (\Delta Iu)F'(u) - I(\Delta u F'(u)) \\
&\quad + (I\nabla u) \cdot \nabla u F''(u) - I(\nabla u \cdot \nabla u F''(u)).
\end{aligned}$$

Therefore,

$$E(Iu(t)) - E(I\psi) = \mathrm{Im} \int_0^t \int \overline{\Delta Iu}\Delta Iu(F'(Iu) - F'(u))dxds \tag{6.2.32}$$

$$+\mathrm{Im} \int_0^t \int \overline{\Delta Iu}|\nabla Iu|^2(F''(Iu) - F''(u))dxds \tag{6.2.33}$$

$$+\mathrm{Im} \int_0^t \int \overline{\Delta Iu}\nabla Iu \cdot (\nabla Iu - \nabla u)F''(u)dxds \tag{6.2.34}$$

$$+\mathrm{Im} \int_0^t \int \overline{\Delta Iu}[(\Delta Iu)F'(u) - I(\Delta u F'(u))]dxds \tag{6.2.35}$$

$$+\mathrm{Im} \int_0^t \int \overline{\Delta Iu}[(I\nabla u) \cdot \nabla u F''(u) - I(\nabla u \cdot \nabla u F''(u))]dxds \tag{6.2.36}$$

$$+\mathrm{Im} \int_0^t \int \overline{IF(u)}(F(Iu) - IF(u))dxds. \tag{6.2.37}$$

Let us consider (6.2.32). By Hölder's inequality, we estimate

$$\begin{aligned}
|(6.2.32)| &\lesssim \|\Delta Iu\|^2_{L^4(J, L^{\frac{2d}{d-2}})} \|F'(Iu) - F'(u)\|_{L^2(J, L^{\frac{d}{2}})} \\
&\lesssim Z_I^2 \||Iu - u|(|Iu| + |u|)^{\frac{8}{d}-1}\|_{L^2(J, L^{\frac{d}{2}})} \\
&\lesssim Z_I^2 \|P_{>N} u\|_{L^{\frac{16}{d}}(J, L^4)} \|u\|^{\frac{8}{d}-1}_{L^{\frac{16}{d}}(J, L^4)}.
\end{aligned} \tag{6.2.38}$$

By (6.1.8), we bound

$$\|P_{>N} u\|_{L^{\frac{16}{d}}(J, L^4)} \lesssim N^{-2} \|\Delta Iu\|_{L^{\frac{16}{d}}(J, L^4)} \lesssim N^{-2} Z_I, \tag{6.2.39}$$

where $\left(\frac{16}{d}, 4\right)$ is biharmonic admissible. Similarly, we have from (6.1.9) that

$$\|u\|_{L^{\frac{16}{d}}(J, L^4)} \lesssim Z_I. \tag{6.2.40}$$

Combining (6.2.38) – (6.2.40), we get

$$|(6.2.32)| \lesssim N^{-2} Z_I^{2+\frac{8}{d}}. \tag{6.2.41}$$

We next bound

$$\begin{aligned}
|(6.2.33)| &\lesssim \|\Delta I u\|_{L^4(J,L^{\frac{2d}{d-2}})} \||\nabla I u|^2\|_{L^{\frac{16}{11}}(J,L^{\frac{4d}{4d-11}})} \|F''(Iu) - F''(u)\|_{L^{16}(J,L^{\frac{4d}{15-2d}})} \\
&\lesssim \|\Delta I u\|_{L^4(J,L^{\frac{2d}{d-2}})} \|\nabla I u\|^2_{L^{\frac{32}{11}}(J,L^{\frac{8d}{4d-11}})} \|F''(Iu) - F''(u)\|_{L^{16}(J,L^{\frac{4d}{15-2d}})} \\
&\lesssim Z_I^3 \||Iu - u|^{\frac{8}{d}-1}\|_{L^{16}(J,L^{\frac{4d}{15-2d}})} \\
&\lesssim Z_I^3 \|P_{>N} u\|^{\frac{8}{d}-1}_{L^{\frac{16(8-d)}{d}}(J,L^{\frac{4(8-d)}{15-2d}})} \\
&\lesssim N^{-2\left(\frac{8}{d}-1\right)} Z_I^{2+\frac{8}{d}}.
\end{aligned} \tag{6.2.42}$$

Here we drop the I-operator and apply (6.1.9) with the fact $\gamma > 1$ to get the third line. We also use the fact that for $5 \leq d \leq 7$,

$$|F''(z) - F''(\zeta)| \lesssim |z - \zeta|^{\frac{8}{d}-1}, \quad \forall z, \zeta \in \mathbb{C}.$$

The last estimate uses (6.2.39). Note that $\left(\frac{32}{11}, \frac{8d}{4d-11}\right)$ and $\left(\frac{16(8-d)}{d}, \frac{4(8-d)}{15-2d}\right)$ are biharmonic admissible. Similarly, we estimate

$$\begin{aligned}
|(6.2.34)| &\lesssim \|\Delta I u\|_{L^4(J,L^{\frac{2d}{d-2}})} \|\nabla I u\|_{L^{\frac{32}{11}}(J,L^{\frac{8d}{4d-11}})} \|\nabla I u - \nabla u\|_{L^{\frac{32}{11}}(J,L^{\frac{8d}{4d-11}})} \|F''(u)\|_{L^{16}(J,L^{\frac{4d}{15-2d}})} \\
&\lesssim Z_I^2 \|\nabla P_{>N} u\|_{L^{\frac{32}{11}}(J,L^{\frac{8d}{4d-11}})} \|F''(u)\|_{L^{16}_t(J,L^{\frac{4d}{15-2d}})}.
\end{aligned}$$

We next use (6.1.8) to have

$$\|\nabla P_{>N} u\|_{L^{\frac{32}{11}}(J,L^{\frac{8d}{4d-11}})} \lesssim N^{-1} \|\Delta I u\|_{L^{\frac{32}{11}}(J,L^{\frac{8d}{4d-11}})} \lesssim N^{-1} Z_I.$$

As $F''(u) = O(|u|^{\frac{8}{d}-1})$, we use (6.1.9) to get

$$\|F''(u)\|_{L^{16}(J,L^{\frac{4d}{15-2d}})} \lesssim \|u\|^{\frac{8}{d}-1}_{L^{\frac{16(8-d)}{d}}(J,L^{\frac{4(8-d)}{15-2d}})} \lesssim Z_I^{\frac{8}{d}-1}. \tag{6.2.43}$$

We thus obtain

$$|(6.2.34)| \lesssim N^{-1} Z_I^{2+\frac{8}{d}}. \tag{6.2.44}$$

By Hölder's inequality,

$$|(6.2.35)| \lesssim \|\Delta I u\|_{L^2(J,L^{\frac{2d}{d-4}})} \|(\Delta I u) F'(u) - I(\Delta u F'(u))\|_{L^2(J,L^{\frac{2d}{d+4}})}.$$

We then apply Lemma 6.2.7 with $q = \frac{2d}{d+4}, q_1 = \frac{2d(d-3)}{d^2-7d+16}$ and $q_2 = \frac{d(d-3)}{2(2d-7)}$ to get

$$\|(\Delta I u) F'(u) - I(\Delta u F'(u))\|_{L^{\frac{2d}{d+4}}} \lesssim N^{-\alpha} \|\Delta I u\|_{L^{\frac{2d(d-3)}{d^2-7d+16}}} \| \langle \nabla \rangle^{\alpha} F'(u)\|_{L^{\frac{d(d-3)}{2(2d-7)}}},$$

where $\alpha = 2 - \gamma + \delta$. The Hölder inequality then implies

$$\|(\Delta I u)F'(u) - I(\Delta u F'(u))\|_{L^2(J, L^{\frac{2d}{d+4}})} \lesssim N^{-\alpha}\|\Delta I u\|_{L^{\frac{2(d-3)}{d-4}}(J, L^{\frac{2d(d-3)}{d^2-7d+16}})} \times \| \langle \nabla \rangle^{\alpha} F'(u)\|_{L^{2(d-3)}(J, L^{\frac{d(d-3)}{2(2d-7)}})}.$$

We have from (6.2.20), (6.2.21), (6.2.22) and (6.2.23) that

$$\| \langle \nabla \rangle^{\alpha} F'(u)\|_{L^{2(d-3)}(J, L^{\frac{d(d-3)}{2(2d-7)}})} \lesssim Z_I^{\frac{8}{d}}.$$

Thus

$$|(6.2.35)| \lesssim N^{-(2-\gamma+\delta)} Z_I^{2+\frac{8}{d}}. \tag{6.2.45}$$

Similarly, we bound

$$|(6.2.36)| \lesssim \|\Delta I u\|_{L^4(J, L^{\frac{2d}{d-2}})} \|(I\nabla u)\cdot \nabla u F''(u) - I(\nabla u \cdot \nabla u F''(u))\|_{L^{\frac{4}{3}}(J, L^{\frac{2d}{d+2}})}. \tag{6.2.46}$$

Applying Lemma 6.2.7 with $q = \frac{2d}{d+2}, q_1 = \frac{8d}{4d-11}$ and $q_2 = \frac{8d}{19}$ and using Hölder inequality, we have

$$\|(I\nabla u)\cdot \nabla u F''(u) - I(\nabla u \cdot \nabla u F''(u))\|_{L^{\frac{4}{3}}(J, L^{\frac{2d}{d+2}})} \lesssim N^{-\alpha}\|I\nabla u\|_{L^{\frac{32}{11}}(J, L^{\frac{8d}{4d-11}})} \times \| \langle \nabla \rangle^{\alpha} (\nabla u F''(u))\|_{L^{\frac{8}{5}}(J, L^{\frac{8d}{19}})}. \tag{6.2.47}$$

The fractional chain rule implies

$$\| \langle \nabla \rangle^{\alpha} (\nabla u F''(u))\|_{L^{\frac{8}{5}}(J, L^{\frac{8d}{19}})} \lesssim \| \langle \nabla \rangle^{\alpha+1} u\|_{L^{\frac{32}{11}}(J, L^{\frac{8d}{4d-11}})} \|F''(u)\|_{L^{16}(J, L^{\frac{4d}{15-2d}})} + \|\nabla u\|_{L^{\frac{32}{11}}(J, L^{\frac{8d}{4d-11}})} \| \langle \nabla \rangle^{\alpha} F''(u)\|_{L^{16}(J, L^{\frac{4d}{15-2d}})}. \tag{6.2.48}$$

By our assumptions on γ and δ, we see that $\alpha + 1 < \gamma$. By (6.1.9) (and dropping the I-operator if necessary) and (6.2.43),

$$\begin{aligned} \|I\nabla u\|_{L^{\frac{32}{11}}(J, L^{\frac{8d}{4d-11}})}, \quad \|\nabla u\|_{L^{\frac{32}{11}}(J, L^{\frac{8d}{4d-11}})}, \quad \| \langle \nabla \rangle^{\alpha+1} u\|_{L^{\frac{32}{11}}(J, L^{\frac{8d}{4d-11}})} &\lesssim Z_I, \\ \|F''(u)\|_{L^{16}(J, L^{\frac{4d}{15-2d}})} &\lesssim Z_I^{\frac{8}{d}-1}. \end{aligned} \tag{6.2.49}$$

Here $\left(\frac{32}{11}, \frac{8d}{4d-11}\right)$ is biharmonic admissible. It remains to bound $\| \langle \nabla \rangle^{\alpha} F''(u)\|_{L^{16}(J, L^{\frac{4d}{15-2d}})}$. To do so, we use

$$\| \langle \nabla \rangle^{\alpha} F''(u)\|_{L^{16}(J, L^{\frac{4d}{15-2d}})} \lesssim \|F''(u)\|_{L^{16}(J, L^{\frac{4d}{15-2d}})} + \||\nabla|^{\alpha} F''(u)\|_{L^{16}(J, L^{\frac{4d}{15-2d}})}. \tag{6.2.50}$$

The first term in the right hand side is treated in (6.2.43). For the second term in the right hand side, we make use of the fractional chain rule given in Lemma 6.2.3 with $\beta = \frac{8}{d} - 1$, $\alpha = 2 - \gamma + \delta$, $q = \frac{4d}{15-2d}$ and q_1, q_2 satisfying

$$\Big(\frac{8}{d} - 1 - \frac{\alpha}{\rho}\Big) q_1 = \frac{\alpha}{\rho} q_2 = \frac{4(8-d)}{15-2d},$$

and $\frac{\alpha}{\frac{8}{d}-1} < \rho < 1$. Note that the choice of ρ is possible since $\alpha < \frac{8}{d} - 1$ by our assumptions. With these choices, we have

$$\Big(1-\frac{\alpha}{\beta\rho}\Big)q_1=\frac{4d}{15-2d}>1,$$

for $5\leq d\leq 7$. Then,

$$\||\nabla|^\alpha F''(u)\|_{L^{\frac{4d}{15-2d}}}\lesssim \||u|^{\frac{8}{d}-1-\frac{\alpha}{\rho}}\|_{L^{q_1}}\||\nabla|^\rho u\|^{\frac{\alpha}{\rho}}_{L^{\frac{\alpha}{\rho}q_2}}\lesssim \|u\|^{\frac{8}{d}-1-\frac{\alpha}{\rho}}_{L^{(\frac{8}{d}-1-\frac{\alpha}{\rho})q_1}}\||\nabla|^\rho u\|^{\frac{\alpha}{\rho}}_{L^{\frac{\alpha}{\rho}q_2}}.$$

By Hölder's inequality,

$$\begin{aligned}\||\nabla|^\alpha F''(u)\|_{L^{16}(J,L^{\frac{4d}{15-2d}})}&\lesssim \|u\|^{\frac{8}{d}-1-\frac{\alpha}{\rho}}_{L^{(\frac{8}{d}-1-\frac{\alpha}{\rho})p_1}(J,L^{(\frac{8}{d}-1-\frac{\alpha}{\rho})q_1})}\||\nabla|^\rho u\|^{\frac{\alpha}{\rho}}_{L^{\frac{\alpha}{\rho}p_2}(J,L^{\frac{\alpha}{\rho}q_2})}\\&=\|u\|^{\frac{8}{d}-1-\frac{\alpha}{\rho}}_{L^{\frac{16(8-d)}{d}}(J,L^{\frac{4(8-d)}{15-2d}})}\||\nabla|^\rho u\|^{\frac{\alpha}{\rho}}_{L^{\frac{16(8-d)}{d}}(J,L^{\frac{4(8-d)}{15-2d}})},\end{aligned}$$

provided

$$\Big(\frac{8}{d}-1-\frac{\alpha}{\rho}\Big)p_1=\frac{\alpha}{\rho}p_2=\frac{16(8-d)}{d}.$$

Since $\Big(\frac{16(8-d)}{d},\frac{4(8-d)}{15-2d}\Big)$ is biharmonic admissible, we have from (6.1.9) with the fact $0<\rho<1<\gamma$ that

$$\||\nabla|^\alpha F''(u)\|_{L^{16}(J,L^{\frac{4d}{15-2d}})}\lesssim Z_I^{\frac{8}{d}-1}. \tag{6.2.51}$$

Collecting from (6.2.46) to (6.2.51), we get

$$|(6.2.36)|\lesssim N^{-(2-\gamma+\delta)}Z_I^{2+\frac{8}{d}}. \tag{6.2.52}$$

Finally, we consider (6.2.37). We bound

$$\begin{aligned}|(6.2.37)|&\lesssim \||\nabla|^{-1}IF(u)\|_{L^2(J,L^{\frac{2d}{d-2}})}\|\nabla(F(Iu)-IF(u))\|_{L^2_t(J,L^{\frac{2d}{d+2}})}\\&\lesssim \|\nabla IF(u)\|_{L^2(J,L^{\frac{2d}{d+2}})}\|\nabla(F(Iu)-IF(u))\|_{L^2(J,L^{\frac{2d}{d+2}})}.\end{aligned} \tag{6.2.53}$$

By (6.2.18),

$$\|\nabla IF(u)\|_{L^2(J,L^{\frac{2d}{d+2}})}\lesssim Z_I^{1+\frac{8}{d}}.$$

By the triangle inequality, we estimate

$$\begin{aligned}\|\nabla(F(Iu)-IF(u))\|_{L^2(J,L^{\frac{2d}{d+2}})}\lesssim&\ \|(\nabla Iu)(F'(Iu)-F'(u))\|_{L^2(J,L^{\frac{2d}{d+2}})}\\&+\|(\nabla Iu)F'(u)-\nabla IF(u)\|_{L^2(J,L^{\frac{2d}{d+2}})}.\end{aligned}$$

We firstly use Hölder's inequality and estimate as in (6.2.38) to get

$$\begin{aligned}\|(\nabla Iu)(F'(Iu)-F'(u))\|_{L^2(J,L^{\frac{2d}{d+2}})}&\lesssim \|\nabla Iu\|_{L^\infty(J,L^{\frac{2d}{d-2}})}\|F'(Iu)-F'(u)\|_{L^2(J,L^{\frac{d}{2}})}\\&\lesssim \|\Delta Iu\|_{L^\infty(J,L^2)}\|P_{>N}u\|_{L^{\frac{16}{d}}(J,L^4)}\|u\|^{\frac{8}{d}-1}_{L^{\frac{16}{d}}(J,L^4)}\\&\lesssim N^{-2}Z_I^{1+\frac{8}{d}}.\end{aligned} \tag{6.2.54}$$

By (6.2.16),

$$\|(\nabla Iu)F'(u)-\nabla IF(u)\|_{L^2(J,L^{\frac{2d}{d+2}})}\lesssim N^{-(2-\gamma+\delta)}Z_I^{1+\frac{8}{d}}. \tag{6.2.55}$$

Combining (6.2.53) – (6.2.55), we get

$$|(6.2.37)| \lesssim Z_I^{1+\frac{8}{d}}(N^{-2}Z_I^{1+\frac{8}{d}} + N^{-(2-\gamma+\delta)}Z_I^{1+\frac{8}{d}}). \tag{6.2.56}$$

The desired estimate (6.2.27) follows from (6.2.41), (6.2.42), (6.2.44), (6.2.45), (6.2.56) and (6.2.31). The proof is complete. □

6.2.3 Global well-posedness

Let us now show the global existence given in Theorem 6.2.1. By density argument, we assume that $\psi \in C_0^\infty(\mathbb{R}^d)$. Let u be a global solution to (dNL4S) with initial data ψ. In order to apply the almost conservation law, we need the modified energy of initial data to be small. Since $E(I\psi)$ is not necessarily small, we will use the scaling (6.2.2) to make $E(Iu_\lambda(0))$ small. We have

$$E(Iu_\lambda(0)) = \frac{1}{2}\|Iu_\lambda(0)\|^2_{\dot{H}^2} + \frac{d}{2d+8}\|Iu_\lambda(0)\|^{\frac{2d+8}{d}}_{L^{\frac{2d+8}{d}}}. \tag{6.2.57}$$

We use (6.1.11) to estimate

$$\|Iu_\lambda(0)\|_{\dot{H}^2} \lesssim N^{2-\gamma}\|u_\lambda(0)\|_{\dot{H}^\gamma} = N^{2-\gamma}\lambda^{-\gamma}\|\psi\|_{\dot{H}^\gamma}. \tag{6.2.58}$$

In order to make $\|Iu_\lambda(0)\|_{\dot{H}^2} \leq \frac{1}{8}$, we choose

$$\lambda \approx N^{\frac{2-\gamma}{\gamma}}. \tag{6.2.59}$$

We next bound $\|Iu_\lambda(0)\|_{L^{\frac{2d+8}{d}}}$. Using the Gagliardo-Nirenberg inequality, we have

$$\|Iu_\lambda(0)\|^{\frac{2d+8}{d}}_{L^{\frac{2d+8}{d}}} \lesssim \|Iu_\lambda(0)\|^{\frac{8}{d}}_{L^2}\|Iu_\lambda(0)\|^2_{\dot{H}^2}.$$

By (6.1.7), the scaling invariance, the conservation of mass and (6.2.58), it follows that

$$\|Iu_\lambda(0)\|_{L^{\frac{2d+8}{d}}} \lesssim (\|Iu_\lambda(0)\|_{\dot{H}^2})^{\frac{d}{d+4}} \lesssim \left(N^{2-\gamma}\lambda^{-\gamma}\|\psi\|_{\dot{H}^\gamma}\right)^{\frac{d}{d+4}}. \tag{6.2.60}$$

Therefore, it follows from (6.2.57), (6.2.58), (6.2.59) and (6.2.60) by taking λ sufficiently large depending on $\|\psi\|_{H^\gamma}$ and N (which will be chosen later and depends only on $\|\psi\|_{H^\gamma}$) that

$$E(Iu_\lambda(0)) \leq \frac{1}{4}.$$

Now let T be arbitrarily large. We define

$$X := \{0 \leq t \leq \lambda^4 T \mid \|u_\lambda\|_{M([0,t])} \leq Kt^{\frac{d-4}{8(d-3)}}\},$$

with K a constant to be chosen later. Here $M(J)$ is given in (6.2.15). We claim that $X = [0, \lambda^4 T]$. Assume by contradiction that it is not so. Since $\|u_\lambda\|_{M([0,t])}$ is a continuous function of time, there exists $T_0 \in [0, \lambda^4 T]$ such that

$$\|u_\lambda\|_{M([0,T_0])} > KT_0^{\frac{d-4}{8(d-3)}}, \tag{6.2.61}$$

$$\|u_\lambda\|_{M([0,T_0])} \leq 2KT_0^{\frac{d-4}{8(d-3)}}. \tag{6.2.62}$$

Using (6.2.62), we are able to split $[0, T_0]$ into subintervals $J_k, k = 1, ..., L$ in such a way that

$$\|u_\lambda\|_{M(J_k)} \leq \mu,$$

where μ is as in Proposition 6.2.11. The number L of possible subinterval must satisfy

$$L \sim \Big(\frac{2KT_0^{\frac{d-4}{8(d-3)}}}{\mu}\Big)^{\frac{8(d-3)}{d}} \sim T_0^{\frac{d-4}{d}}. \tag{6.2.63}$$

Next, thanks to Proposition 6.2.11, we see that for $1 < \gamma < 2$ and any $0 < \delta < \gamma - 1$,

$$\sup_{[0,T_0]} E(Iu_\lambda(t)) \lesssim E(Iu_\lambda(0)) + N^{-(2-\gamma+\delta)}L,$$

for $\max\left\{3 - \frac{8}{d}, \frac{8}{d}\right\} < \gamma < 2$ and $0 < \delta < \gamma + \frac{8}{d} - 3$. Since $E(Iu_\lambda(0)) \leq \frac{1}{4}$, we need

$$N^{-(2-\gamma+\delta)}L \ll \frac{1}{4} \tag{6.2.64}$$

in order to guarantee

$$E(Iu_\lambda(t)) \leq 1, \tag{6.2.65}$$

for all $t \in [0, T_0]$. As $T_0 \leq \lambda^4 T$, we have from (6.2.63) and (6.2.64) and the choice of λ given in (6.2.59) that

$$N^{-(2-\gamma+\delta)} N^{\frac{4(2-\gamma)(d-4)}{\gamma d}} T^{\frac{d-4}{d}} \ll \frac{1}{4}, \tag{6.2.66}$$

or

$$\frac{4(2-\gamma)(d-4)}{\gamma d} < 2 - \gamma + \delta, \tag{6.2.67}$$

for $\max\left\{3 - \frac{8}{d}, \frac{8}{d}\right\} < \gamma < 2$ and $0 < \delta < \gamma + \frac{8}{d} - 3$. Since $2 - \gamma + \delta < \frac{8}{d} - 1$, the condition (6.2.67) is possible if we have

$$\frac{4(2-\gamma)(d-4)}{\gamma d} < \frac{8}{d} - 1.$$

This implies $\gamma > \frac{8(d-4)}{3d-8}$. Thus

$$\gamma > \max\left\{3 - \frac{8}{d}, \frac{8}{d}, \frac{8(d-4)}{3d-8}\right\}.$$

Next, by (6.2.15),

$$\|u_\lambda\|_{M([0,T_0])} \lesssim T_0^{\frac{d-4}{8(d-3)}} \|\psi\|_{L^2}^{\frac{1}{d-3}} \|u_\lambda\|_{L_t^\infty([0,T_0],\dot{H}^{\frac{1}{2}})}^{\frac{d-4}{d-3}}.$$

We use (6.1.8) and the definition of the I-operator to estimate

$$\begin{aligned}
\|u_\lambda(t)\|_{\dot{H}^{\frac{1}{2}}} &\leq \|P_{\leq N} u_\lambda(t)\|_{\dot{H}^{\frac{1}{2}}} + \|P_{>N} u_\lambda(t)\|_{\dot{H}^{\frac{1}{2}}} \\
&\lesssim \|P_{\leq N} u_\lambda(t)\|_{L^2}^{\frac{3}{4}} \|P_{\leq N} u_\lambda(t)\|_{\dot{H}^2}^{\frac{1}{4}} + N^{-\frac{3}{2}} \|Iu_\lambda(t)\|_{\dot{H}^2} \\
&\lesssim \|\psi\|_{L^2}^{\frac{3}{4}} \|Iu_\lambda(t)\|_{\dot{H}^2}^{\frac{1}{4}} + N^{-\frac{3}{2}} \|Iu_\lambda(t)\|_{\dot{H}^2}.
\end{aligned}$$

Thus,

$$\|u_\lambda\|_{M([0,T_0])} \lesssim T_0^{\frac{d-4}{8(d-3)}} \|\psi\|_{L^2}^{\frac{1}{d-3}} \sup_{[0,T_0]} \Big(\|\psi\|_{L^2}^{\frac{3}{4}} \|Iu_\lambda(t)\|_{\dot{H}^2}^{\frac{1}{4}} + N^{-\frac{3}{2}} \|Iu_\lambda(t)\|_{\dot{H}^2}\Big)^{\frac{d-4}{d-3}}. \tag{6.2.68}$$

Since $\|Iu_\lambda(t)\|_{\dot{H}^2} \lesssim \sqrt{E(Iu_\lambda(t))}$, we obtain from (6.2.65) and (6.2.68),

$$\|u_\lambda\|_{M([0,T_0])} \leq CT_0^{\frac{d-4}{8(d-3)}},$$

for some constant $C > 0$. This contradicts with (6.2.61) for an appropriate choice of K. We get $X = [0, \lambda^4 T]$ with T arbitrarily large and

$$E(Iu_\lambda(\lambda^4 T)) \leq 1. \tag{6.2.69}$$

Note that under the condition of γ, we see from (6.2.66) that the choice of N makes sense for arbitrarily large T. Now, by the conservation of mass and (6.2.69), we bound

$$\begin{aligned}
\|u(T)\|_{H^\gamma} &\lesssim \|u(T)\|_{L^2} + \|u(T)\|_{\dot{H}^\gamma} \lesssim \|\psi\|_{L^2} + \lambda^\gamma \|u_\lambda(\lambda^4 T)\|_{\dot{H}^\gamma} \\
&\lesssim \|\psi\|_{L^2} + \lambda^\gamma \|Iu_\lambda(\lambda^4 T)\|_{H^2} \\
&\lesssim \lambda^\gamma \lesssim N^{2-\gamma} \lesssim T^{\alpha(\gamma,d)},
\end{aligned}$$

where $\alpha(\gamma, d)$ is a positive number that depends on γ and d. This a priori bound gives the global existence in H^γ. The proof is now complete.

Chapter 7

Blowup for the focusing mass-critical nonlinear fourth-order Schrödinger equation below the energy space

Contents

In this chapter, we consider the focusing mass-critical nonlinear fourth-order Schrödinger equation, namely

$$\left\{ \begin{array}{rcl} i\partial_t u(t,x) + \Delta^2 u(t,x) &=& (|u|^{\frac{8}{d}} u)(t,x), \quad t \geq 0, x \in \mathbb{R}^d, \\ u(0,x) &=& \psi(x) \in H^\gamma(\mathbb{R}^d), \end{array} \right. \tag{fNL4S}$$

where $u(t,x)$ is a complex valued function in $\mathbb{R}^+ \times \mathbb{R}^d$. The (fNL4S) is a special case of the generalized nonlinear fourth-order Schrödinger equation

$$i\partial_t u + \Delta^2 u + \varepsilon \Delta u + \mu |u|^{\nu-1} u = 0, \quad u(0) = \psi, \tag{7.0.1}$$

where $\varepsilon \in \{0, \pm 1\}, \mu \in \{\pm 1\}$ and $\nu > 1$. The equation (7.0.1) was introduced by Karpman [Kar96] and Karpman-Shagalov [KS00] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity.

The (fNL4S) enjoys a natural scaling invariance, that is if u solves (fNL4S), then for any $\lambda > 0$,

$$u_\lambda(t,x) := \lambda^{-\frac{d}{2}} u(\lambda^{-4} t, \lambda^{-1} x) \tag{7.0.2}$$

solves the same equation with initial data $u_\lambda(0,x) = \lambda^{-\frac{d}{2}} \psi(\lambda^{-1} x)$. This scaling also preserves the L^2-norm, i.e. $\|u_\lambda(0)\|_{L^2} = \|\psi\|_{L^2}$. As in the previous Chapter, (fNL4S) is locally well-posed in $H^\gamma(\mathbb{R}^d)$ for $\gamma \geq 0$ satisfying, in the case $d \neq 1, 2, 4$, (6.0.1). Moreover, for $u_0 \in H^2$, the unique solution enjoys mass and energy conservation laws. In the sub-critical regime, i.e. $\gamma > 0$, the time of existence depends only on the H^γ-norm of the initial data. Let T^* be the maximal time of existence. The local well-posedness gives the following blowup alternative criterion: either

$T^* = \infty$ or

$$T^* < \infty, \quad \lim_{t \to T^*} \|u(t)\|_{H^\gamma} = \infty.$$

The study of blowup solutions for the focusing nonlinear fourth-order Schrödinger equation has attracted a lot of interest in a past decade (see e.g. [FIP02], [BFM10], [ZYZ10], [ZYZ11], [BL17] and references therein). It is closely related to ground states Q of (fNL4S) which are solutions to the elliptic equation

$$\Delta^2 Q + Q - |Q|^{\frac{8}{d}} Q = 0. \tag{7.0.3}$$

The equation (7.0.3) is obtained by considering solitary solutions (standing waves) of (fNL4S) of the form $u(t,x) = Q(x)e^{-it}$. The existence of solutions to (7.0.3) is proved in [ZYZ10], but the uniqueness of the solution is still an open problem. In the case $\|\psi\|_{L^2} < \|Q\|_{L^2}$, using the sharp Gagliardo-Nirenberg inequality (see [FIP02] or [ZYZ10]), namely

$$\|u\|^{2+\frac{8}{d}}_{L^{2+\frac{8}{d}}} \leq C(d)\|u\|^{\frac{8}{d}}_{L^2}\|\Delta u\|^2_{L^2}, \quad C(d) := \frac{1+\frac{4}{d}}{\|Q\|^{\frac{8}{d}}_{L^2}}, \tag{7.0.4}$$

together with the energy conservation, Fibich-Ilan-Papanicolaou in [FIP02] (see also [BFM10]) proved that (fNL4S) is globally well-posed in H^2. Moreover, the authors in [FIP02] also provided some numerical observations showing that the H^2-solution to (fNL4S) may blowup if the initial data satisfies $\|\psi\|_{L^2} \geq \|Q\|_{L^2}$. Baruch-Fibich-Mandelbaum in [BFM10] proved some dynamical properties of radially symmetric blowup solutions such as blowup rate, L^2-concentration. Later, Zhu-Yang-Zhang in [ZYZ10] removed the radially symmetric assumption and established the profile decomposition, the existence of the ground state of elliptic equation (7.0.3) and the following concentration compactness property for (fNL4S).

Theorem 7.0.1 (Concentration compactness [ZYZ10])**.** *Let $(v_n)_{n\geq 1}$ be a bounded family of H^2 functions such that*

$$\limsup_{n\to\infty} \|\Delta v_n\|_{L^2} \leq M < \infty \quad \text{and} \quad \limsup_{n\to\infty} \|v_n\|_{L^{2+\frac{8}{d}}} \geq m.$$

Then there exists a sequence $(x_n)_{n\geq 1}$ of $\mathbb{R}^d$ such that up to a subsequence

$$v_n(\cdot + x_n) \rightharpoonup V \text{ weakly in } H^2 \text{ as } n \to \infty,$$

with $\|V\|^{\frac{8}{d}}_{L^2} \geq \frac{\|Q\|^{\frac{8}{d}}_{L^2} m^{2+\frac{8}{d}}}{(1+\frac{4}{d})M^2}$, where Q is the solution to the ground state equation (7.0.3).

Consequently, the authors in [ZYZ11] established the limiting profile and the L^2-concentration for (fNL4S) with initial data $\psi \in H^\gamma(\mathbb{R}^4)$, $\frac{9+\sqrt{721}}{20} < \gamma < 2$. Recently, Boulenger-Lenzmann in [BL17] proved a general result on finite-time blowup for the focusing generalized nonlinear fourth-order Schrödinger equation (i.e. (7.0.1) with $\mu = 1$) with radial data in H^2.

Our main purpose in this chapter is to lower the required regularity of [ZYZ11] for (fNL4S) in the fourth dimensional case and to extend the results of [ZYZ11] to higher dimensions $d \geq 5$.

7.1 Blowup for the focusing mass-critical nonlinear fourth-order Schrödinger equation below the energy space when $d = 4$

In this section, we lower the required regularity in [ZYZ11]. To do so, we make use of the analysis performed in Subsection 6.1. More precisely, our main results in this section are as follows.

Theorem 7.1.1. *Let $\psi \in H^\gamma(\mathbb{R}^4)$ with $\frac{67+\sqrt{40489}}{150} < \gamma < 2$. If the corresponding solution to the* (fNL4S) *blows up in finite time $0 < T^* < \infty$, then there exists a function $U \in H^2(\mathbb{R}^4)$ such that*

$\|U\|_{L^2(\mathbb{R}^4)} \geq \|Q\|_{L^2(\mathbb{R}^4)}$ *and there exist sequences* $(t_n, \lambda_n, x_n)_{n\geq 1} \in \mathbb{R}^+ \times \mathbb{R}^+_* \times \mathbb{R}^4$ *satisfying*

$$t_n \nearrow T^* \text{ as } n \to \infty \quad \text{and} \quad \lambda_n \lesssim (T^* - t_n)^{\frac{\gamma}{8}}, \quad \forall n \geq 1$$

such that

$$\lambda_n^2 u(t_n, \lambda_n \cdot + x_n) \rightharpoonup U \text{ weakly in } H^{\tilde{a}(\gamma)-}(\mathbb{R}^4) \text{ as } n \to \infty,$$

where

$$\tilde{a}(\gamma) := \frac{30\gamma^2 + 74\gamma + 120}{97\gamma + 120 - 30\gamma^2}, \tag{7.1.1}$$

and Q is the solution of the ground state equation (7.0.3).

This result improves the regularity requirement of [ZYZ11] where the authors proved the above result for $\frac{9+\sqrt{721}}{20} < \gamma < 2$. This improvement is due to a better bilinear estimate (6.1.4), hence a better energy increment (see Proposition 7.1.8).

As a consequence of Theorem 7.1.1, we have the following mass concentration property.

Theorem 7.1.2. *Let $\psi \in H^\gamma(\mathbb{R}^4)$ with $\frac{67+\sqrt{40489}}{150} < \gamma < 2$. Assume that the corresponding solution u to the* (fNL4S) *blows up in finite time $0 < T^* < \infty$. If $\alpha(t) > 0$ is an arbitrary function such that*

$$\lim_{t \nearrow T^*} \frac{(T^* - t)^{\frac{\gamma}{8}}}{\alpha(t)} = 0,$$

then there exists a function $x(t) \in \mathbb{R}^4$ such that

$$\limsup_{t \nearrow T^*} \int_{|x - x(t)| \leq \alpha(t)} |u(t,x)|^2 dx \geq \int_{\mathbb{R}^4} |Q(x)|^2 dx,$$

where Q is the solution to the ground state equation (7.0.3).

When the mass of the initial data equals to the mass of the solution of the ground state equation (7.0.3), we have the following improvement of Theorem 7.1.1. Note that in the below result, we assume that there exists a unique solution to the ground state equation (7.0.3) which is a delicate open problem.

Theorem 7.1.3. *Let $\psi \in H^\gamma(\mathbb{R}^4)$ with $\frac{67+\sqrt{40489}}{150} < \gamma < 2$ be such that $\|\psi\|_{L^2(\mathbb{R}^4)} = \|Q\|_{L^2(\mathbb{R}^4)}$. If the corresponding solution u to the* (fNL4S) *blows up in finite time $0 < T^* < \infty$, then there exist sequences $(t_n, e^{i\theta_n}, \lambda_n, x_n)_{n\geq 1} \in \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{R}^+_* \times \mathbb{R}^4$ satisfying*

$$t_n \nearrow T^* \text{ as } n \to \infty \quad \text{and} \quad \lambda_n \lesssim (T^* - t_n)^{\frac{\gamma}{8}}, \quad \forall n \geq 1$$

such that

$$\lambda_n^2 e^{i\theta_n} u(t_n, \lambda_n \cdot + x_n) \to Q \text{ strongly in } H^{\tilde{a}(\gamma)-}(\mathbb{R}^4) \text{ as } n \to \infty,$$

where $\tilde{a}(\gamma)$ is as in (7.1.1) *and Q is the unique solution to the ground state equation* (7.0.3).

7.1.1 Modified local well-posedness

We firstly recall the local theory for (fNL4S) in Sobolev spaces (see Theorem 5.3.1 with $\sigma = 4$).

Proposition 7.1.4 (Local well-posedness)**.** *Let $0 < \gamma < 2$ and $\psi \in H^\gamma(\mathbb{R}^4)$. Then the equation* (fNL4S) *is locally well-posed on $[0, T_{\text{lwp}}]$ with*

$$T_{\text{lwp}} \sim \|\psi\|_{H^\gamma}^{-\frac{4}{\gamma}}.$$

Corollary 7.1.5 (Blowup criterion)**.** *Let $0 < \gamma < 2$ and $\psi \in H^\gamma(\mathbb{R}^4)$. Assume that the unique solution u to* (fNL4S) *blows up at time $0 < T^* < \infty$. Then,*

$$\|u(t)\|_{H^\gamma} \gtrsim (T^* - t)^{-\frac{\gamma}{4}}, \tag{7.1.2}$$

for all $0 < t < T^*$.

Proof. We follow the argument of [CW90]. Let $0 < t < T^*$. If we consider (fNL4S) with initial data $u(t)$, then it follows from (5.3.6) with $\sigma = 4, \nu = 3$ and the fixed point argument that if for some $M > 0$

$$C\|u(t)\|_{H^\gamma} + C(T-t)^{\frac{\gamma}{2}} M^3 \leq M,$$

then $T < T^*$. Thus,

$$C\|u(t)\|_{H^\gamma} + C(T^*-t)^{\frac{\gamma}{2}} M^3 > M,$$

for all $M > 0$. Choosing $M = 2C\|u(t)\|_{H^\gamma}$, we see that

$$(T^*-t)^{\frac{\gamma}{2}} \|u(t)\|^2_{H^\gamma} > C.$$

This proves (7.1.2) and the proof is complete. □

We have the following modified local well-posedness which is essentially given in Proposition 6.1.12

Proposition 7.1.6 (Modified local well-posedness)**.** *Let* $\gamma \in (2/3, 2)$ *and* $\psi \in H^\gamma(\mathbb{R}^4)$. *Let*

$$\delta = c\|I\psi\|_{H^2}^{-\frac{4}{\gamma}},$$

for a small constant $c = c(\gamma) > 0$. *Then the* (fNL4S) *is locally well-posed on* $[0, \delta]$ *and the unique solution satisfies for* N *large enough,*

$$\|Iu\|_{X^{2,1/2+}_\delta} \lesssim \|I\psi\|_{H^2}. \tag{7.1.3}$$

Here $X^{\gamma,b}_\delta$ *is defined as in* (6.1.16).

Proof. Since $\|\psi\|_{H^\gamma} \lesssim \|I\psi\|_{H^2}$, we see that for $c > 0$ small enough,

$$\delta = c\|I\psi\|_{H^2}^{-\frac{4}{\gamma}} \lesssim c\|\psi\|_{H^\gamma}^{-\frac{4}{\gamma}} \leq T_{\text{lwp}}.$$

Here T_{lwp} is as in Proposition 7.1.4. This shows that (fNL4S) is locally well-posed on $[0, \delta]$. It remains to prove (7.1.3). This bound follows by the same lines as in the proof of Proposition 6.1.12. The proof is complete. □

7.1.2 Modified energy increment

In this subsection, we study the modified energy increment. More precisely, we will show that the modified energy, namely $E(Iu)$ grows much slower than the modified kinetic of u, namely $\|\Delta Iu\|^2_{L^2}$. It is crucial to prove the limiting profile for blowup solutions.

Proposition 7.1.7 (Local increment of modified energy)**.** *Let* $\frac{60}{53} < \gamma < 2$ *and* $\psi \in H^\gamma(\mathbb{R}^4)$. *Let*

$$\delta = c\|I\psi\|_{H^2}^{-\frac{4}{\gamma}},$$

for a small constant $c = c(\gamma) > 0$. *Then for* N *sufficiently large,*

$$\sup_{t\in[0,\delta]} |E(Iu(t)) - E(I\psi)| \lesssim N^{-\frac{46}{15}+} \left(\|I\psi\|^4_{H^2} + \|I\psi\|^6_{H^2}\right). \tag{7.1.4}$$

Here the implicit contant depends only on γ *and* $\|\psi\|_{H^\gamma}$.

Proof. By Proposition 7.1.6, (fNL4S) is local well-posed on $[0, \delta]$ and the unique solution satisfies (7.1.3). By the proof of Proposition 6.2.11, we see that for N sufficiently large,

$$\sup_{t\in[0,\delta]} |E(Iu(t)) - E(I\psi)| \lesssim N^{-\frac{46}{15}+}\Big(\|Iu\|^4_{X^{2,1/2+}_\delta} + \|Iu\|^6_{X^{2,1/2+}_\delta}\Big).$$

This estimate together with (7.1.3) show (7.1.4). The proof is complete. $\square$

We next introduce

$$\Lambda(t) := \sup_{0\leq s\leq t} \|u(s)\|_{H^\gamma}, \quad \Sigma(t) := \sup_{0\leq s\leq t} \|I_N u(s)\|_{H^2}. \tag{7.1.5}$$

Proposition 7.1.8 (Increment of the modified energy)**.** *Let* $\frac{67+\sqrt{40489}}{150} < \gamma < 2$*. Let* $\psi \in H^\gamma(\mathbb{R}^4)$ *be such that the corresponding solution* u *to the* (fNL4S) *blows up at time* $0 < T^* < \infty$*. Let* $0 < T < T^*$*. Then for*

$$N(T) \sim \Lambda(T)^{\frac{a(\gamma)}{2(2-\gamma)}}, \tag{7.1.6}$$

we have

$$|E(I_{N(T)}u(T))| \lesssim \Lambda(T)^{a(\gamma)}. \tag{7.1.7}$$

Here the implicit constants depend only on γ, T^* *and* $\|\psi\|_{H^\gamma}$*, and* $0 < a(\gamma) < 2$ *is given by*

$$a(\gamma) := \frac{2(2-\gamma)\left(6+\frac{4}{\gamma}\right)}{\left[\frac{46}{15} - (2-\gamma)\left(4+\frac{4}{\gamma}\right)\right]-}. \tag{7.1.8}$$

Proof. Let $\delta := c\Sigma(T)^{-\frac{4}{\gamma}}$ for some constant $c = c(\gamma) > 0$ small enough. For $N(T)$ sufficiently large, Proposition 7.1.6 shows that

$$\|I_{N(T)}u\|_{X^{2,1/2+}([t,t+\delta])} \lesssim \|I_{N(T)}u(t)\|_{H^2} \lesssim \Sigma(T), \tag{7.1.9}$$

uniformly in t provided that $[t, t+\delta] \subset [0,T]$. We split $[0,T]$ into $O(T/\delta)$ subintervals and apply Proposition 7.1.7 on each of these intervals together with (7.1.9) to have for $\frac{60}{53} < \gamma < 2$,

$$\begin{aligned}\sup_{t\in[0,T]} |E(I_{N(T)}u(t))| &\lesssim |E(I_{N(T)}\psi)| + \frac{T}{\delta} N(T)^{-\frac{46}{15}+} \left(\Sigma^4(T) + \Sigma^6(T)\right) \\ &\lesssim |E(I_{N(T)}\psi)| + N(T)^{-\frac{46}{15}+} \left(\Sigma^{4+\frac{4}{\gamma}}(T) + \Sigma^{6+\frac{4}{\gamma}}(T)\right). \end{aligned}\tag{7.1.10}$$

Using (6.1.10), we see that

$$\Sigma(T) \lesssim N(T)^{2-\gamma}\Lambda(T). \tag{7.1.11}$$

By the Gagliardo-Nirenberg inequality and (6.1.11),

$$\begin{aligned}|E(I_{N(T)}\psi)| &\lesssim \|\Delta I_{N(T)}\psi\|^2_{L^2} + \|I_{N(T)}\psi\|^4_{L^4} \\ &\lesssim \|\Delta I_{N(T)}\psi\|^2_{L^2} + \|I_{N(T)}\psi\|^2_{L^2}\|\Delta I_{N(T)}\psi\|^2_{L^2} \\ &\lesssim N(T)^{2(2-\gamma)}\left(\|\psi\|^2_{H^\gamma} + \|\psi\|^4_{H^\gamma}\right) \lesssim N(T)^{2(2-\gamma)}. \end{aligned}\tag{7.1.12}$$

Substituting (7.1.11) and (7.1.12) into (7.1.10), we get

$$\begin{aligned}\sup_{t\in[0,T]} |E(I_{N(T)}u(t))| \lesssim N(T)^{2(2-\gamma)} &+ N(T)^{(2-\gamma)\left(4+\frac{4}{\gamma}\right)-\frac{46}{15}+}\Lambda(T)^{4+\frac{4}{\gamma}} \\ &+ N(T)^{(2-\gamma)\left(6+\frac{4}{\gamma}\right)-\frac{46}{15}+}\Lambda(T)^{6+\frac{4}{\gamma}}. \end{aligned}\tag{7.1.13}$$

Optimizing (7.1.13), we observe that if

$$N(T)^{2(2-\gamma)} \sim N(T)^{(2-\gamma)\left(6+\frac{4}{\gamma}\right)-\frac{46}{15}+}\Lambda(T)^{6+\frac{4}{\gamma}}$$

or

$$N(T) \sim \Lambda(T)^{\frac{6+\frac{4}{\gamma}}{\left[\frac{46}{15}-(2-\gamma)\left(4+\frac{4}{\gamma}\right)\right]-}},$$

then

$$\sup_{t\in[0,T]} |E(I_{N(T)}u(t))| \lesssim N(T)^{2(2-\gamma)} \lesssim \Lambda(T)^{a(\gamma)},$$

where $a(\gamma)$ is given in (7.1.8). In order to make $0 < a(\gamma) < 2$, we need

$$\begin{cases} \frac{46}{15} - (2-\gamma)\left(4+\frac{4}{\gamma}\right) & > 0, \\ \quad\; (2-\gamma)\left(6+\frac{4}{\gamma}\right) & < \frac{46}{15} - (2-\gamma)\left(4+\frac{4}{\gamma}\right). \end{cases}$$

Solving the above inequalities, we obtain $\frac{67+\sqrt{40489}}{150} < \gamma < 2$. The proof is complete. □

7.1.3 Limiting profile

Proof of Theorem 7.1.1 As the solution blows up at time $0 < T^* < \infty$, the blowup alternative allows us to choose a sequence of times $(t_n)_{n\geq 1}$ such that $t_n \to T^*$ as $n \to \infty$ and $\|u(t_n)\|_{H^\gamma} = \Lambda(t_n) \to \infty$ as $n \to \infty$ (see (7.1.5) for the notation). Denote

$$v_n(x) := \lambda_n^2 I_{N(t_n)} u(t_n, \lambda_n x),$$

where $N(t_n)$ is given as in (7.1.6) with $T = t_n$ and the parameter λ_n is given by

$$\lambda_n^2 := \frac{\|\Delta Q\|_{L^2}}{\|\Delta I_{N(t_n)} u(t_n)\|_{L^2}}. \tag{7.1.14}$$

By (6.1.10) and the blowup criterion given in Corollary 7.1.5, we see that

$$\lambda_n^2 \lesssim \frac{\|\Delta Q\|_{L^2}}{\|u(t_n)\|_{H^\gamma}} \lesssim (T^* - t_n)^{\frac{\gamma}{4}} \text{ or } \lambda_n \lesssim (T^* - t_n)^{\frac{\gamma}{8}}.$$

On the other hand, $(v_n)_{n\geq 1}$ is bounded in $H^2(\mathbb{R}^4)$. Indeed,

$$\begin{aligned} \|v_n\|_{L^2} &= \|I_{N(t_n)} u(t_n)\|_{L^2} \leq \|u(t_n)\|_{L^2} = \|\psi\|_{L^2}, \\ \|\Delta v_n\|_{L^2} &= \lambda_n^2 \|\Delta I_{N(t_n)} u(t_n)\|_{L^2} = \|\Delta Q\|_{L^2}. \end{aligned} \tag{7.1.15}$$

By Proposition 7.1.8 with $T = t_n$, we have

$$E(v_n) = \lambda_n^4 E(I_{N(t_n)} u(t_n)) \lesssim \lambda_n^4 \Lambda(t_n)^{a(\gamma)} \lesssim \Lambda(t_n)^{a(\gamma)-2}.$$

As $0 < a(\gamma) < 2$ for $\frac{67+\sqrt{40489}}{150} < \gamma < 2$, we see that $E(v_n) \to 0$ as $n \to \infty$. Therefore, the expression of the modified energy and (7.1.15) give

$$\|v_n\|_{L^4}^4 \to 2\|\Delta Q\|_{L^2}^2, \tag{7.1.16}$$

as $n \to \infty$. Applying Theorem 7.0.1 to the sequence $(v_n)_{n\geq 1}$ with $M = \|\Delta Q\|_{L^2}$ and $m = \left(2\|\Delta Q\|_{L^2}^2\right)^{\frac{1}{4}}$, there exist a sequence $(x_n)_{n\geq 1} \subset \mathbb{R}^4$ and a function $U \in H^2(\mathbb{R}^4)$ such that $\|U\|_{L^2} \geq \|Q\|_{L^2}$ and up to a subsequence,

$$v_n(\cdot + x_n) \rightharpoonup U \text{ weakly in } H^2(\mathbb{R}^4),$$

as $n \to \infty$. That is

$$\lambda_n^2 I_{N(t_n)} u(t_n, \lambda_n \cdot + x_n) \rightharpoonup U \text{ weakly in } H^2(\mathbb{R}^4), \tag{7.1.17}$$

as $n \to \infty$. To conclude Theorem 7.1.1, we need to remove $I_{N(t_n)}$ from (7.1.17). To do so, we consider for any $0 \leq \sigma < \gamma$,

$$\begin{aligned} \|\lambda_n^2(u - I_{N(t_n)}u)(t_n, \lambda_n \cdot + x_n)\|_{\dot{H}^\sigma} &= \lambda_n^\sigma \|P_{\geq N(t_n)} u(t_n)\|_{\dot{H}^\sigma} \\ &\lesssim \lambda_n^\sigma N(t_n)^{\sigma-\gamma} \|P_{\geq N(t_n)} u(t_n)\|_{\dot{H}^\gamma} \\ &\lesssim \Lambda(t_n)^{-\frac{\sigma}{2}} \Lambda(t_n)^{\frac{(\sigma-\gamma)a(\gamma)}{2(2-\gamma)}} \|P_{\geq N(t_n)} u(t_n)\|_{H^\gamma} \\ &\lesssim \Lambda(t_n)^{1-\frac{\sigma}{2}+\frac{(\sigma-\gamma)a(\gamma)}{2(2-\gamma)}}. \end{aligned} \tag{7.1.18}$$

Using the explicit expression of $a(\gamma)$ given in (7.1.8), we find that for

$$\sigma < \tilde{a}(\gamma) := \frac{30\gamma^2 + 74\gamma + 120}{97\gamma + 120 - 30\gamma^2},$$

the exponent of $\Lambda(t_n)$ in (7.1.18) is negative. Note that an easy computation shows that the condition $\tilde{a}(\gamma) < \gamma$ requires

$$\frac{7 + \sqrt{7249}}{60} < \gamma < 2,$$

which is satisfied since $\frac{67+\sqrt{40489}}{150} < \gamma < 2$. Thus,

$$\|\lambda_n^2(u - I_{N(t_n)}u)(t_n, \lambda_n \cdot + x_n)\|_{H^{\tilde{a}(\gamma)-}} \to 0, \tag{7.1.19}$$

as $n \to \infty$. Combining (7.1.17) and (7.1.19), we prove

$$\lambda_n^2 u(t_n, \lambda_n \cdot + x_n) \rightharpoonup U \text{ weakly in } H^{\tilde{a}(\gamma)-}(\mathbb{R}^4),$$

as $n \to \infty$. The proof is complete. □

Proof of Theorem 7.1.2 By Theorem 7.1.1, there exists a blowup profile $U \in H^2(\mathbb{R}^4)$ with $\|U\|_{L^2} \geq \|Q\|_{L^2}$ and there exist sequences $(t_n, \lambda_n, x_n)_{n\geq 1} \subset \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}^4$ such that $t_n \to T^*$,

$$\frac{\lambda_n}{(T^* - t_n)^{\frac{\gamma}{8}}} \lesssim 1, \tag{7.1.20}$$

for all $n \geq 1$ and $\lambda_n^2 u(t_n, \lambda_n \cdot + x_n) \rightharpoonup U$ weakly in $H^{\tilde{a}(\gamma)-}(\mathbb{R}^4)$ (hence in $L^2(\mathbb{R}^4)$) as $n \to \infty$. Thus for any $R > 0$, we have

$$\liminf_{n\to\infty} \lambda_n^4 \int_{|x|\leq R} |u(t_n, \lambda_n x + x_n)|^2 dx \geq \int_{|x|\leq R} |U(x)|^2 dx.$$

By change of variables, we get

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^4} \int_{|x-y|\leq R\lambda_n} |u(t_n, x)|^2 dx \geq \int_{|x|\leq R} |U(x)|^2 dx.$$

Using the assumption $\frac{(T^*-t_n)^{\frac{\gamma}{8}}}{\alpha(t_n)} \to 0$ as $n \to \infty$, we have from (7.1.20) that $\frac{\lambda_n}{\alpha(t_n)} \to 0$ as $n \to \infty$. We thus obtain for any $R > 0$,

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^4} \int_{|x-y|\leq \alpha(t_n)} |u(t_n, x)|^2 dx \geq \int_{|x|\leq R} |U(x)|^2 dx.$$

Let $R \to \infty$, we obtain

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^4} \int_{|x-y|\leq \alpha(t_n)} |u(t_n, x)|^2 dx \geq \|U\|_{L^2}^2.$$

This implies

$$\limsup_{t\nearrow T^*} \sup_{y\in\mathbb{R}^4} \int_{|x-y|\leq \alpha(t)} |u(t,x)|^2 dx \geq \|Q\|^2_{L^2}.$$

Sine for any fixed time t, the map $y \mapsto \int_{|x-y|\leq\alpha(t)} |u(t,x)|^2 dx$ is continuous and goes to zero as $|y|\to\infty$, there exists $x(t)\in\mathbb{R}^4$ such that

$$\sup_{y\in\mathbb{R}^4} \int_{|x-y|\leq\alpha(t)} |u(t,x)|^2 dx = \int_{|x-x(t)|\leq\alpha(t)} |u(t,x)|^2 dx.$$

This shows

$$\limsup_{t\nearrow T^*} \int_{|x-x(t)|\leq\alpha(t)} |u(t,x)|^2 dx \geq \|Q\|^2_{L^2}.$$

The proof is complete. □

Proof of Theorem 7.1.3 We firstly recall the following variational characterization of the solution to the ground state equation (7.0.3). Note that the uniqueness up to translations in space, phase and dilations of solution to this ground state equation is assumed here.

Lemma 7.1.9 (Variation characterization of the ground state [ZYZ10])**.** *If $v\in H^2(\mathbb{R}^d)$ is such that $\|v\|_{L^2}=\|Q\|_{L^2}$ and $E(u)=0$, then v is of the form*

$$v(x)=e^{i\theta}\lambda^{\frac{d}{2}}Q(\lambda x+x_0),$$

for some $\theta\in\mathbb{R},\lambda>0$ and $x_0\in\mathbb{R}^d$, where Q is the unique solution to the ground state equation (7.0.3).

Using the notation in the proof of Theorem 7.1.1 and the assumption $\|\psi\|_{L^2}=\|Q\|_{L^2}$, we have

$$\|v_n\|_{L^2}\leq\|\psi\|_{L^2}=\|Q\|_{L^2}\leq\|U\|_{L^2}.$$

Sine $v_n(\cdot+x_n)\rightharpoonup U$ weakly in $L^2(\mathbb{R}^4)$, the semi-continuity of weak convergence implies

$$\|U\|_{L^2}\leq\liminf_{n\to\infty}\|v_n\|_{L^2}\leq\|Q\|_{L^2}.$$

Thus,

$$\|U\|_{L^2}=\|Q\|_{L^2}=\lim_{n\to\infty}\|v_n\|_{L^2}. \tag{7.1.21}$$

Hence up to a subsequence

$$v_n(\cdot+x_n)\to U \text{ strongly in } L^2(\mathbb{R}^d), \tag{7.1.22}$$

as $n\to\infty$. On the other hand, using (7.1.15), the Gagliardo-Nirenberg inequality (7.0.4) implies $v_n(\cdot+x_n)\to U$ strongly in $L^4(\mathbb{R}^4)$. Indeed, by (7.1.15),

$$\begin{aligned}\|v_n(\cdot+x_n)-U\|^4_{L^4} &\lesssim \|\psi(\cdot+x_n)-U\|^2_{L^2}\|\Delta(v_n(\cdot+x_n)-U\|^2_{L^2}\\ &\lesssim (\|\Delta Q\|_{L^2}+\|\Delta U\|_{L^2})^2\|\psi(\cdot+x_n)-U\|^2_{L^2}\to 0,\end{aligned}$$

as $n\to\infty$. Moreover, using (7.1.16) and (7.1.21), the sharp Gagliardo-Nirenberg inequality (7.0.4) also gives

$$\|\Delta Q\|^2_{L^2}=\frac{1}{2}\|U\|^4_{L^4}\leq\Big(\frac{\|U\|_{L^2}}{\|Q\|_{L^2}}\Big)^2\|\Delta U\|^2_{L^2}=\|\Delta U\|^2_{L^2},$$

or $\|\Delta Q\|_{L^2}\leq\|\Delta U\|_{L^2}$. By the semi-continuity of weak convergence and (7.1.15),

$$\|\Delta U\|_{L^2}\leq\liminf_{n\to\infty}\|\Delta v_n\|_{L^2}=\|\Delta Q\|_{L^2}.$$

Therefore,

$$\|\Delta U\|_{L^2} = \|\Delta Q\|_{L^2} = \lim_{n\to\infty} \|\Delta v_n\|_{L^2}. \tag{7.1.23}$$

Combining (7.1.21), (7.1.23) and using the fact $v_n(\cdot + x_n) \rightharpoonup U$ weakly in $H^2(\mathbb{R}^4)$, we conclude that $v_n(\cdot + x_n) \to U$ strongly in $H^2(\mathbb{R}^4)$. In particular,

$$E(U) = \lim_{n\to\infty} E(v_n) = 0,$$

as $n \to \infty$. This shows that there exists $U \in H^2(\mathbb{R}^4)$ satisfying

$$\|U\|_{L^2} = \|Q\|_{L^2}, \quad \|\Delta U\|_{L^2} = \|\Delta Q\|_{L^2}, \quad E(U) = 0.$$

Applying the variational characterization given in Lemma 7.1.9, we have (taking $\lambda = 1$),

$$U(x) = e^{i\theta} Q(x + x_0),$$

for some $(\theta, x_0) \in \mathbb{R} \times \mathbb{R}^4$. Hence

$$\lambda_n^2 I_{N(t_n)} u(t_n, \lambda_n \cdot + x_n) \to e^{i\theta} Q(\cdot + x_0) \text{ strongly in } H^2(\mathbb{R}^4),$$

as $n \to \infty$. Using (7.1.19), we prove

$$\lambda_n^2 u(t_n, \lambda_n \cdot + x_n) \to e^{i\theta} Q(\cdot + x_0) \text{ strongly in } H^{\tilde{a}(\gamma)-}(\mathbb{R}^4),$$

as $n \to \infty$. The proof is complete. □

7.2 Blowup for the focusing mass-critical nonlinear fourth-order Schrödinger equation below the energy space when $5 \leq d \leq 7$

In this section, we extend the results of [ZYZ11] to higher dimensions $d \geq 5$. Since we are working with low regularity data, the energy argument does not work. In order to overcome this problem, we make use of the I-method. Due to the high-order term $\Delta^2 u$, we require the nonlinearity to have at least two orders of derivatives in order to successfully establish the almost conservation law. We thus restrict to space of dimensions $d = 5, 6, 7$. Our main results are as follows.

Theorem 7.2.1. *Let $d = 5, 6, 7$ and $\psi \in H^\gamma(\mathbb{R}^d)$ with $\frac{56-3d+\sqrt{137d^2+1712d+3136}}{2(2d+32)} < \gamma < 2$. If the corresponding solution to the* (fNL4S) *blows up in finite time $0 < T^* < \infty$, then there exists a function $U \in H^2(\mathbb{R}^d)$ such that $\|U\|_{L^2(\mathbb{R}^d)} \geq \|Q\|_{L^2(\mathbb{R}^d)}$ and there exist sequences $(t_n, \lambda_n, x_n)_{n\geq 1} \in \mathbb{R}^+ \times \mathbb{R}^+_* \times \mathbb{R}^d$ satisfying*

$$t_n \nearrow T^* \text{ as } n \to \infty \quad \text{and} \quad \lambda_n \lesssim (T^* - t_n)^{\frac{\gamma}{8}}, \quad \forall n \geq 1$$

such that

$$\lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \rightharpoonup U \text{ weakly in } H^{a(d,\gamma)-}(\mathbb{R}^d) \text{ as } n \to \infty,$$

where

$$a(d,\gamma) := \frac{4d\gamma^2 + (2d+48)\gamma + 16d}{16d + (56-3d)\gamma - 16\gamma^2},$$

and Q is the solution of the ground state equation (7.0.3).

The proof of the above theorem is based on the combination of the I-method and the concentration compactness property given in Theorem 7.0.1 which is similar to those given in [VZ07] and [ZYZ11]. The key is to show that on intervals of local well-posedness, the modified energy

$E(Iu)$ is an "almost conserved" quantity and grows much slower than the modified kinetic energy $\|\Delta Iu\|^2_{L^2(\mathbb{R}^d)}$. To do so, we need delicate estimates on the commutator between the I-operator and the nonlinearity. Note that when $d = 4$, the nonlinearity is algebraic, one can use the Fourier transform technique to write the commutator explicitly and then control it by multi-linear analysis. In our setting, the nonlinearity is not algebraic. Thus we can not apply the Fourier transform technique. Fortunately, thanks to a special Strichartz estimate (6.2.11), we are able to apply the technique given in [VZ07] to control the commutator. The concentration compactness property given in Theorem 7.0.1 is very useful to study the dynamical properties of blowup solutions for the nonlinear fourth-order Schrödinger equation. With the help of this property, Zhu-Yang-Zhang proved in [ZYZ10] the L^2-concentration of blowup solutions and the limiting profile of minimal-mass blowup solutions with non-radial data in $H^2(\mathbb{R}^d)$. In [ZYZ11], they extended these results for non-radial data below the energy space in the fourth dimensional space.

As a consequence of Theorem 7.2.1, we have the following mass concentration property.

Theorem 7.2.2. *Let $d = 5, 6, 7$ and $\psi \in H^\gamma(\mathbb{R}^d)$ with $\frac{56-3d+\sqrt{137d^2+1712d+3136}}{2(2d+32)} < \gamma < 2$. Assume that the corresponding solution u to the* (fNL4S) *blows up in finite time $0 < T^* < \infty$. If $\alpha(t) > 0$ is an arbitrary function such that*

$$\lim_{t \nearrow T^*} \frac{(T^* - t)^{\frac{\gamma}{8}}}{\alpha(t)} = 0,$$

then there exists a function $x(t) \in \mathbb{R}^d$ such that

$$\limsup_{t \nearrow T^*} \int_{|x-x(t)| \leq \alpha(t)} |u(t,x)|^2 dx \geq \int_{\mathbb{R}^d} |Q(x)|^2 dx,$$

where Q is the solution to the ground state equation (7.0.3)*.*

When the mass of the initial data equals the mass of the solution of the ground state equation (7.0.3), we have the following improvement of Theorem 7.2.1. Note that in the result below, we assume that there exists a unique solution to the ground state equation (7.0.3) which is a delicate open problem.

Theorem 7.2.3. *Let $d = 5, 6, 7$ and $\psi \in H^\gamma(\mathbb{R}^d)$ with $\frac{56-3d+\sqrt{137d^2+1712d+3136}}{2(2d+32)} < \gamma < 2$ be such that $\|\psi\|_{L^2(\mathbb{R}^d)} = \|Q\|_{L^2(\mathbb{R}^d)}$. If the corresponding solution u to the* (fNL4S) *blows up in finite time $0 < T^* < \infty$, then there exist sequences $(t_n, e^{i\theta_n}, \lambda_n, x_n)_{n\geq 1} \in \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{R}^+_* \times \mathbb{R}^d$ satisfying*

$$t_n \nearrow T^* \text{ as } n \to \infty \quad \text{and} \quad \lambda_n \lesssim (T^* - t_n)^{\frac{\gamma}{8}}, \quad \forall n \geq 1$$

such that

$$\lambda_n^{\frac{d}{2}} e^{i\theta_n} u(t_n, \lambda_n \cdot + x_n) \to Q \text{ strongly in } H^{a(d,\gamma)-}(\mathbb{R}^d) \text{ as } n \to \infty,$$

where

$$a(d,\gamma) := \frac{4d\gamma^2 + (2d+48)\gamma + 16d}{16d + (56-3d)\gamma - 16\gamma^2},$$

and Q is the unique solution to the ground state equation (7.0.3)*.*

7.2.1 Modified local well-posedness

We firstly recall the local well-posedness in Sobolev spaces for (fNL4S) given in Theorem 5.3.1 with $\sigma = 4$.

Proposition 7.2.4 (Local well-posedness)**.** *Let $d \geq 5, 0 < \gamma < 2$ and $\psi \in H^\gamma(\mathbb{R}^d)$. Then the equation* (fNL4S) *is locally well-posed on $[0, T_{\text{lwp}}]$ with*

$$T_{\text{lwp}} \sim \|\psi\|_{H^\gamma}^{-\frac{4}{\gamma}}.$$

Moreover,

$$\sup_{(a,b)\in B} \|u\|_{L^a([0,T_{\text{lwp}}], W^{\gamma,b})} \lesssim \|\psi\|_{H^\gamma}.$$

The implicit constants depend only on the dimension d and the regularity γ.

We also have the following blowup rate which is essentially proven in Corollary 7.1.5.

Corollary 7.2.5 (Blowup rate)**.** *Let $d \geq 5, 0 < \gamma < 2$ and $\psi \in H^\gamma(\mathbb{R}^d)$. Assume that the unique solution u to* (fNL4S) *blows up at time $0 < T^* < \infty$. Then,*

$$\|u(t)\|_{H^\gamma} \gtrsim (T^* - t)^{-\frac{\gamma}{4}}, \tag{7.2.1}$$

for all $0 < t < T^*$.

We next define for any spacetime slab $J \times \mathbb{R}^d$,

$$Z_I(J) := \sup_{(p,q)\in B} \| \langle \Delta \rangle Iu\|_{L^p(J,L^q)}.$$

Note that in our consideration $d \geq 5$, for any admissible pair $(p, q) \in B$, we always have $q < \infty$. Let us start with the following commutator estimates.

Lemma 7.2.6. *Let $5 \leq d \leq 7, 1 < \gamma < 2, 0 < \delta < \gamma - 1$ and J a compact interval. Then*

$$\|IF(u)\|_{L^2(J,L^{\frac{2d}{d+4}})} \lesssim |J|^{\frac{2\gamma}{d}} (Z_I(J))^{1+\frac{8}{d}}, \tag{7.2.2}$$

$$\|\nabla IF(u) - (I\nabla u)F'(u)\|_{L^2(J,L^{\frac{2d}{d+2}})} \lesssim N^{-(2-\gamma+\delta)}(Z_I(J))^{1+\frac{8}{d}}, \tag{7.2.3}$$

$$\|\nabla IF(u)\|_{L^2(J,L^{\frac{2d}{d+2}})} \lesssim |J|^{\frac{2\gamma}{d}} (Z_I(J))^{1+\frac{8}{d}} + N^{-(2-\gamma+\delta)}(Z_I(J))^{1+\frac{8}{d}}, \tag{7.2.4}$$

$$\|\nabla IF(u)\|_{L^2(J,L^{\frac{2d}{d+4}})} \lesssim (Z_I(J))^{1+\frac{8}{d}}. \tag{7.2.5}$$

Proof. We firstly note that the estimates (7.2.3) and (7.2.5) are given in Lemma 6.2.10. Let us consider (7.2.2). By (6.1.7) and Hölder's inequality,

$$\begin{aligned}\|IF(u)\|_{L^2(J,L^{\frac{2d}{d+4}})} &\lesssim \|F(u)\|_{L^2(J,L^{\frac{2d}{d+4}})} \\ &\lesssim \|u\|_{L^{\frac{2(d+8)}{d-4\gamma}}(J,L^{\frac{2d(d+8)}{d^2+4d+16\gamma}})} \|F'(u)\|_{L^{\frac{d+8}{2(2+\gamma)}}(J,L^{\frac{d(d+8)}{4d+16-8\gamma}})}.\end{aligned}$$

Since $F'(u) = O(|u|^{\frac{8}{d}})$, the Sobolev embedding implies

$$\begin{aligned}\|IF(u)\|_{L^2(J,L^{\frac{2d}{d+4}})} &\lesssim \|u\|_{L^{\frac{2(d+8)}{d-4\gamma}}(J,L^{\frac{2d(d+8)}{d^2+4d+16\gamma}})} \|u\|^{\frac{8}{d}}_{L^{\frac{4(d+8)}{d(2+\gamma)}}(J,L^{\frac{2(d+8)}{d+4-2\gamma}})} \\ &\lesssim |J|^{\frac{2\gamma}{d}} \|u\|_{L^{\frac{2(d+8)}{d-4\gamma}}(J,L^{\frac{2d(d+8)}{d^2+4d+16\gamma}})} \|u\|^{\frac{8}{d}}_{L^{\frac{2(d+8)}{d-4\gamma}}(J,L^{\frac{2(d+8)}{d+4-2\gamma}})} \\ &\lesssim |J|^{\frac{2\gamma}{d}} \|u\|_{L^{\frac{2(d+8)}{d-4\gamma}}(J,L^{\frac{2d(d+8)}{d^2+4d+16\gamma}})} \||\nabla|^\gamma u\|_{L^{\frac{2(d+8)}{d-4\gamma}}(J,L^{\frac{2d(d+8)}{d^2+4d+16\gamma}})} \\ &\lesssim |J|^{\frac{2\gamma}{d}} \| \langle \nabla \rangle^\gamma u\|^{1+\frac{8}{d}}_{L^{\frac{2(d+8)}{d-4\gamma}}(J,L^{\frac{2d(d+8)}{d^2+4d+16\gamma}})} \\ &\lesssim |J|^{\frac{2\gamma}{d}} (Z_I(J))^{1+\frac{8}{d}}.\end{aligned}$$

Here we use (6.1.9) and the fact $\left(\frac{2(d+8)}{d-4\gamma}, \frac{2d(d+8)}{d^2+4d+16\gamma}\right)$ is biharmonic admissible to get the last estimate.

It remains to prove (7.2.4). We have from (6.2.17) and the triangle inequality that

$$\|\nabla IF(u)\|_{L^2(J,L^{\frac{2d}{d+2}})} \lesssim \|(\nabla Iu)F'(u)\|_{L^2(J,L^{\frac{2d}{d+2}})} + N^{-(2-\gamma+\delta)}(Z_I(J))^{1+\frac{8}{d}}. \tag{7.2.6}$$

By Hölder's inequality,

$$\|(\nabla Iu)F'(u)\|_{L^2(J,L^{\frac{2d}{d+2}})} \lesssim \|\nabla Iu\|_{L^{\frac{2(d+8)}{d-4\gamma}}(J,L^{\frac{2d(d+8)}{d^2+2d+16(\gamma-1)}})} \|F'(u)\|_{L^{\frac{d+8}{2(2+\gamma)}}(J,L^{\frac{d(d+8)}{4d+16-8\gamma}})}. \tag{7.2.7}$$

We use the Sobolev embedding to estimate

$$\|\nabla Iu\|_{L^{\frac{2(d+8)}{d-4\gamma}}(J,L^{\frac{2d(d+8)}{d^2+2d+16(\gamma-1)}})} \lesssim \|\Delta Iu\|_{L^{\frac{2(d+8)}{d-4\gamma}}(J,L^{\frac{2d(d+8)}{d^2+4d+16\gamma}})} \lesssim Z_I(J). \tag{7.2.8}$$

Here $\left(\frac{2(d+8)}{d-4\gamma}, \frac{2d(d+8)}{d^2+4d+16\gamma}\right)$ is biharmonic admissible. Since $F'(u) = O(|u|^{\frac{8}{d}})$, the Sobolev embedding again gives

$$\begin{aligned}
\|F'(u)\|_{L^{\frac{d+8}{2(2+\gamma)}}(J,L^{\frac{d(d+8)}{4d+16-8\gamma}})} &\lesssim \|u\|^{\frac{8}{d}}_{L^{\frac{4(d+8)}{d(2+\gamma)}}(J,L^{\frac{2(d+8)}{d+4-2\gamma}})} \\
&\lesssim |J|^{\frac{2\gamma}{d}} \|u\|^{\frac{8}{d}}_{L^{\frac{2(d+8)}{d-4\gamma}}(J,L^{\frac{2(d+8)}{d+4-2\gamma}})} \\
&\lesssim |J|^{\frac{2\gamma}{d}} \||\nabla|^\gamma u\|^{\frac{8}{d}}_{L^{\frac{2(d+8)}{d-4\gamma}}(J,L^{\frac{2d(d+8)}{d^2+4d+16\gamma}})} \\
&\lesssim |J|^{\frac{2\gamma}{d}} (Z_I(J))^{\frac{8}{d}}.
\end{aligned} \tag{7.2.9}$$

Collecting (6.2.19) – (6.2.23), we obtain (7.2.4). The proof is complete. □

Proposition 7.2.7 (Modified local well-posedness)**.** *Let* $5 \leq d \leq 7, 1 < \gamma < 2, 0 < \delta < \gamma - 1$ *and* $\psi \in H^\gamma(\mathbb{R}^d)$*. Let*

$$\widetilde{T}_{\text{lwp}} := c\|I\psi\|_{H^2}^{-\frac{4}{\gamma}},$$

for a small constant $c = c(d,\gamma) > 0$*. Then* (fNL4S) *is locally well-posed on* $[0, \widetilde{T}_{\text{lwp}}]$*. Moreover, for* N *sufficiently large,*

$$Z_I([0, \widetilde{T}_{\text{lwp}}]) \lesssim \|I\psi\|_{H^2}. \tag{7.2.10}$$

Proof. By (6.1.10), $\|\psi\|_{H^\gamma} \lesssim \|I\psi\|_{H^2}$. Thus,

$$\widetilde{T}_{\text{lwp}} = c\|I\psi\|_{H^2}^{-\frac{4}{\gamma}} \lesssim c\|\psi\|_{H^\gamma}^{-\frac{4}{\gamma}} \leq T_{\text{lwp}},$$

provided c is small enough. Here T_{lwp} is as in Proposition 7.2.4. This shows that (fNL4S) is locally well-posed on $[0, \widetilde{T}_{\text{lwp}}]$. It remains to prove (7.2.10). Denote $J = [0, \widetilde{T}_{\text{lwp}}]$. By Strichartz estimates (6.2.10) and (6.2.9),

$$\begin{aligned}
Z_I(J) &\lesssim \sup_{(p,q)\in B} \|Iu\|_{L^p(J,L^q)} + \sup_{(p,q)\in B} \|\Delta Iu\|_{L^p(J,L^q)} \\
&\lesssim \|I\psi\|_{L^2} + \|IF(u)\|_{L^2(J,L^{\frac{2d}{d+4}})} + \|\Delta I\psi\|_{L^2} + \|\nabla IF(u)\|_{L^2(J,L^{\frac{2d}{d+2}})} \\
&\lesssim \|I\psi\|_{H^2} + \|IF(u)\|_{L^2(J,L^{\frac{2d}{d+4}})} + \|\nabla IF(u)\|_{L^2(J,L^{\frac{2d}{d+2}})}.
\end{aligned}$$

We next use (6.2.16) and (6.2.18) to have

$$Z_I(J) \lesssim \|I\psi\|_{H^2} + \left(|J|^{\frac{2\gamma}{d}} + N^{-(2-\gamma+\delta)}\right)(Z_I(J))^{1+\frac{8}{d}}.$$

By taking $c = c(d,\gamma)$ small enough (or $|J|$ is small) and N large enough, the continuity argument shows (7.2.10). The proof is complete. □

7.2.2 Modified energy increment

Lemma 7.2.8 (Local increment of the modified energy)**.** *Let* $5 \leq d \leq 7, \max\{3 - \frac{8}{d}, \frac{8}{d}\} < \gamma < 2, 0 < \delta < \gamma + \frac{8}{d} - 3$ *and* $\psi \in H^\gamma(\mathbb{R}^d)$*. Let*

$$\widetilde{T}_{\rm lwp} := c\|I\psi\|_{H^2}^{-\frac{4}{\gamma}},$$

for some small constant $c = c(d,\gamma) > 0$*. Then, for* N *sufficiently large,*

$$\sup_{t\in[0,\widetilde{T}_{\rm lwp}]} |E(Iu(t)) - E(I\psi)| \lesssim N^{-(2-\gamma+\delta)}\Big(\|I\psi\|_{H^2}^{2+\frac{8}{d}} + \|I\psi\|_{H^2}^{2+\frac{16}{d}}\Big). \tag{7.2.11}$$

Here the implicit constant depends only on γ *and* $\|\psi\|_{H^\gamma}$*.*

Proof. By Proposition 7.2.7, the equation (fNL4S) is locally well-posed on $J = [0, \widetilde{T}_{\rm lwp}]$ and the unique solution u satisfies

$$Z_I(J) \lesssim \|I\psi\|_{H^2}. \tag{7.2.12}$$

As in the proof of Proposition 6.2.11, we see that

$$\sup_{t\in[0,\widetilde{T}_{lwp}]} |E(Iu(t)) - E(I\psi)| \lesssim N^{-(2-\gamma+\delta)}\Big(Z_I^{2+\frac{8}{d}}(J) + Z_I^{2+\frac{16}{d}}(J)\Big).$$

This estimate together with (7.2.12) proves (7.2.11). The proof is complete. $\square$

We next introduce some notations. We define

$$\Lambda(t) := \sup_{0\leq s\leq t}\|u(s)\|_{H^\gamma}, \quad \Sigma(t) := \sup_{0\leq s\leq t}\|I_N u(s)\|_{H^2}. \tag{7.2.13}$$

Proposition 7.2.9 (Increment of the modified energy)**.** *Let* $5 \leq d \leq 7$ *and* $\frac{56-3d+\sqrt{137d^2+1712d+3136}}{2(2d+32)} < \gamma < 2$*. Let* $\psi \in H^\gamma(\mathbb{R}^d)$ *be such that the corresponding solution* u *to* (fNL4S) *blows up at time* $0 < T^* < \infty$*. Let* $0 < T < T^*$*. Then for*

$$N(T) \sim \Lambda(T)^{\frac{a(\gamma)}{2(2-\gamma)}}, \tag{7.2.14}$$

we have

$$|E(I_{N(T)}u(T))| \lesssim \Lambda(T)^{a(\gamma)}.$$

Here the implicit constants depend only on γ, T^* *and* $\|\psi\|_{H^\gamma}$*, and* $0 < a(\gamma) < 2$ *is given by*

$$a(\gamma) := \frac{2\left(2 + \frac{16}{d} + \frac{4}{\gamma}\right)(2-\gamma)}{\left[\frac{8}{d} - 1 - (2-\gamma)\left(\frac{16}{d} + \frac{4}{\gamma}\right)\right]-}. \tag{7.2.15}$$

Proof. Let $\tau := c\Sigma(T)^{-\frac{4}{\gamma}}$ for some constant $c = c(d,\gamma) > 0$ small enough. For $N(T)$ sufficiently large, Proposition 7.2.7 shows the local existence and the unique solution satisfies

$$Z_{I_{N(T)}}([t, t+\tau]) \lesssim \|I_{N(T)}u(t)\|_{H^2} \lesssim \Sigma(T),$$

uniformly in t provided that $[t, t+\tau] \subset [0, T]$. We next split $[0, T]$ into $O(T/\tau)$ subintervals and apply Lemma 7.2.8 on each of these intervals to have

$$\sup_{t\in[0,T]} |E(I_{N(T)}u(t))| \lesssim |E(I_{N(T)}\psi)| + \frac{T}{\tau} N(T)^{-(2-\gamma+\delta)} \Big(\Sigma(T)^{2+\frac{8}{d}} + \Sigma(T)^{2+\frac{16}{d}}\Big) \tag{7.2.16}$$

$$\lesssim |E(I_{N(T)}\psi)| + N(T)^{-(2-\gamma+\delta)} \Big(\Sigma(T)^{2+\frac{8}{d}+\frac{4}{\gamma}} + \Sigma(T)^{2+\frac{16}{d}+\frac{4}{\gamma}}\Big), \tag{7.2.17}$$

for $\max\left\{3-\frac{8}{d}, \frac{8}{d}\right\} < \gamma < 2$ and $0 < \delta < \gamma + \frac{8}{d} - 3$. Next, by (6.1.10), we have

$$\Sigma(T) \lesssim N(T)^{2-\gamma}\Lambda(T). \tag{7.2.18}$$

Moreover, the Gagliardo-Nirenberg inequality (7.0.4) together with (6.1.11) imply

$$\begin{aligned}
|E(I_{N(T)}\psi)| &\lesssim \|\Delta I_{N(T)}\psi\|^2_{L^2} + \|I_{N(T)}\psi\|^{2+\frac{8}{d}}_{L^{2+\frac{8}{d}}} \\
&\lesssim \|\Delta I_{N(T)}\psi\|^2_{L^2} + \|I_{N(T)}\psi\|^{\frac{8}{d}}_{L^2}\|\Delta I_{N(T)}\psi\|^2_{L^2} \\
&\lesssim N(T)^{2(2-\gamma)}\Big(\|\psi\|^2_{H^\gamma} + \|\psi\|^{2+\frac{8}{d}}_{H^\gamma}\Big) \\
&\lesssim N^{2(2-\gamma)}.
\end{aligned} \tag{7.2.19}$$

Substituting (7.2.18) and (7.2.19) to (7.2.17), we get

$$\begin{aligned}
\sup_{t\in[0,T]} |E(I_{N(T)}u(t))| \lesssim{}& N(T)^{2(2-\gamma)} + N(T)^{-(2-\gamma+\delta)+(2-\gamma)\left(2+\frac{8}{d}+\frac{4}{\gamma}\right)}\Lambda(T)^{2+\frac{8}{d}+\frac{4}{\gamma}} \\
&+ N(T)^{-(2-\gamma+\delta)+(2-\gamma)\left(2+\frac{16}{d}+\frac{4}{\gamma}\right)}\Lambda(T)^{2+\frac{16}{d}+\frac{4}{\gamma}}.
\end{aligned} \tag{7.2.20}$$

Optimizing (7.2.20), we observe that if we take

$$N(T)^{2(2-\gamma)} \sim N(T)^{-(2-\gamma+\delta)+(2-\gamma)\left(2+\frac{16}{d}+\frac{4}{\gamma}\right)}\Lambda(T)^{2+\frac{16}{d}+\frac{4}{\gamma}},$$

or

$$N(T) \sim \Lambda(T)^{\frac{2+\frac{16}{d}+\frac{4}{\gamma}}{(2-\gamma+\delta)-(2-\gamma)\left(\frac{16}{d}+\frac{4}{\gamma}\right)}},$$

then

$$\sup_{t\in[0,T]} |E(I_{N(T)}u(t))| \lesssim N(T)^{2(2-\gamma)} \sim \Lambda(T)^{\frac{2\left(2+\frac{16}{d}+\frac{4}{\gamma}\right)(2-\gamma)}{(2-\gamma+\delta)-(2-\gamma)\left(\frac{16}{d}+\frac{4}{\gamma}\right)}}.$$

Denote

$$a(\gamma) := \frac{2\left(2+\frac{16}{d}+\frac{4}{\gamma}\right)(2-\gamma)}{(2-\gamma+\delta)-(2-\gamma)\left(\frac{16}{d}+\frac{4}{\gamma}\right)}.$$

Since $2-\gamma+\delta < \frac{8}{d}-1$, we see that

$$a(\gamma) = \frac{2\left(2+\frac{16}{d}+\frac{4}{\gamma}\right)(2-\gamma)}{\left[\frac{8}{d}-1-(2-\gamma)\left(\frac{16}{d}+\frac{4}{\gamma}\right)\right]-}.$$

In order to make $0 < a(\gamma) < 2$, we need

$$\left\{\begin{array}{rcl}
\frac{8}{d}-1-(2-\gamma)\Big(\frac{16}{d}+\frac{4}{\gamma}\Big) &>& 0, \\
\Big(2+\frac{16}{d}+\frac{4}{\gamma}\Big)(2-\gamma) &<& \frac{8}{d}-1-(2-\gamma)\Big(\frac{16}{d}+\frac{4}{\gamma}\Big).
\end{array}\right. \tag{7.2.21}$$

Solving (7.2.21), we obtain

$$\gamma > \frac{56-3d+\sqrt{137d^2+1712d+3136}}{2(2d+32)}.$$

This completes the proof. □

7.2.3 Limiting profile

Proof of Theorem 7.2.1 As the solution blows up at time $0<T^*<\infty$, the blowup alternative allows us to choose a sequence of times $(t_n)_{n\geq 1}$ such that $t_n \to T^*$ as $n\to\infty$ and $\|u(t_n)\|_{H^\gamma} = \Lambda(t_n)\to\infty$ as $n\to\infty$ (see (7.2.13) for the notation). Denote

$$v_n(x) := \lambda_n^{\frac{d}{2}} I_{N(t_n)} u(t_n, \lambda_n x),$$

where $N(t_n)$ is given as in (7.2.14) with $T=t_n$ and the parameter λ_n is given by

$$\lambda_n^2 := \frac{\|\Delta Q\|_{L^2}}{\|\Delta I_{N(t_n)} u(t_n)\|_{L^2}}. \tag{7.2.22}$$

By (6.1.10) and the blowup criterion given in Corollary 7.2.5, we see that

$$\lambda_n^2 \lesssim \frac{\|\Delta Q\|_{L^2}}{\|u(t_n)\|_{H^\gamma}} \lesssim (T^*-t_n)^{\frac{\gamma}{4}} \text{ or } \lambda_n \lesssim (T^*-t_n)^{\frac{\gamma}{8}}.$$

On the other hand, $(v_n)_{n\geq 1}$ is bounded in $H^2(\mathbb{R}^d)$. Indeed,

$$\begin{aligned} \|v_n\|_{L^2} &= \|I_{N(t_n)}u(t_n)\|_{L^2} \leq \|u(t_n)\|_{L^2} = \|\psi\|_{L^2}, \\ \|\Delta v_n\|_{L^2} &= \lambda_n^2 \|\Delta I_{N(t_n)} u(t_n)\|_{L^2} = \|\Delta Q\|_{L^2}. \end{aligned} \tag{7.2.23}$$

By Proposition 7.2.9 with $T=t_n$, we have

$$E(v_n) = \lambda_n^4 E(I_{N(t_n)}u(t_n)) \lesssim \lambda_n^4 \Lambda(t_n)^{a(\gamma)} \lesssim \Lambda(t_n)^{a(\gamma)-2}.$$

As $0<a(\gamma)<2$ for $\frac{56-3d+\sqrt{137d^2+1712d+3136}}{2(2d+32)} < \gamma < 2$, we see that $E(v_n)\to 0$ as $n\to\infty$. Therefore, the expression of the modified energy and (7.2.23) give

$$\|v_n\|^{2+\frac{8}{d}}_{L^{2+\frac{8}{d}}} \to \Big(1+\frac{4}{d}\Big)\|\Delta Q\|^2_{L^2}, \tag{7.2.24}$$

as $n\to\infty$. Applying Theorem 7.0.1 to the sequence $(v_n)_{n\geq 1}$ with $M=\|\Delta Q\|_{L^2}$ and $m = \big(\big(1+\frac{4}{d}\big)\|\Delta Q\|^2_{L^2}\big)^{\frac{d}{2d+8}}$, there exist a sequence $(x_n)_{n\geq 1}\subset\mathbb{R}^d$ and a function $U\in H^2(\mathbb{R}^d)$ such that $\|U\|_{L^2}\geq \|Q\|_{L^2}$ and up to a subsequence,

$$v_n(\cdot + x_n) \rightharpoonup U \text{ weakly in } H^2(\mathbb{R}^d),$$

as $n\to\infty$. That is

$$\lambda_n^{\frac{d}{2}} I_{N(t_n)} u(t_n, \lambda_n \cdot + x_n) \rightharpoonup U \text{ weakly in } H^2(\mathbb{R}^d), \tag{7.2.25}$$

as $n \to \infty$. To conclude Theorem 7.2.1, we need to remove $I_{N(t_n)}$ from (7.2.25). To do so, we consider for any $0 \leq \sigma < \gamma$,

$$\begin{aligned}
\|\lambda_n^{\frac{d}{2}}(u - I_{N(t_n)}u)(t_n, \lambda_n \cdot + x_n)\|_{\dot{H}^\sigma} &= \lambda_n^\sigma \|P_{\geq N(t_n)}u(t_n)\|_{\dot{H}^\sigma} \\
&\lesssim \lambda_n^\sigma N(t_n)^{\sigma-\gamma}\|P_{\geq N(t_n)}u(t_n)\|_{\dot{H}^\gamma} \\
&\lesssim \Lambda(t_n)^{-\frac{\sigma}{2}}\Lambda(t_n)^{\frac{(\sigma-\gamma)a(\gamma)}{2(2-\gamma)}}\|P_{\geq N(t_n)}u(t_n)\|_{H^\gamma} \\
&\lesssim \Lambda(t_n)^{1-\frac{\sigma}{2}+\frac{(\sigma-\gamma)a(\gamma)}{2(2-\gamma)}}. \qquad (7.2.26)
\end{aligned}$$

Using the explicit expression of $a(\gamma)$ given in (7.2.15), we find that for

$$\sigma < a(d,\gamma) := \frac{4d\gamma^2 + (2d+48)\gamma + 16d}{16d + (56-3d)\gamma - 16\gamma^2},$$

the exponent of $\Lambda(t_n)$ in (7.2.26) is negative. Note that an easy computation shows that the condition $a(d,\gamma) < \gamma$ requires

$$\frac{24 - 3d + \sqrt{9d^2 + 368d + 576}}{32} < \gamma < 2,$$

which is satisfied by our assumption on γ. Thus,

$$\|\lambda_n^{\frac{d}{2}}(u - I_{N(t_n)}u)(t_n, \lambda_n \cdot + x_n)\|_{H^{a(d,\gamma)-}} \to 0, \qquad (7.2.27)$$

as $n \to \infty$. Combining (7.2.25) and (7.2.27), we prove

$$\lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \rightharpoonup U \text{ weakly in } H^{a(d,\gamma)-}(\mathbb{R}^d),$$

as $n \to \infty$. The proof is complete. □

Proof of Theorem 7.2.2 By Theorem 7.2.1, there exists a blowup profile $U \in H^2(\mathbb{R}^d)$ with $\|U\|_{L^2} \geq \|Q\|_{L^2}$ and there exist sequences $(t_n, \lambda_n, x_n)_{n\geq 1} \subset \mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}^d$ such that $t_n \to T^*$,

$$\frac{\lambda_n}{(T^* - t_n)^{\frac{\gamma}{8}}} \lesssim 1, \qquad (7.2.28)$$

for all $n \geq 1$ and $\lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \rightharpoonup U$ weakly in $H^{a(d,\gamma)-}(\mathbb{R}^d)$ (hence in $L^2(\mathbb{R}^d)$) as $n \to \infty$. Thus for any $R > 0$, we have

$$\liminf_{n\to\infty} \lambda_n^d \int_{|x|\leq R} |u(t_n, \lambda_n x + x_n)|^2 dx \geq \int_{|x|\leq R} |U(x)|^2 dx.$$

By change of variables, we get

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^d} \int_{|x-y|\leq R\lambda_n} |u(t_n, x)|^2 dx \geq \int_{|x|\leq R} |U(x)|^2 dx.$$

Using the assumption $\frac{(T^*-t_n)^{\frac{\gamma}{8}}}{\alpha(t_n)} \to 0$ as $n \to \infty$, we have from (7.2.28) that $\frac{\lambda_n}{\alpha(t_n)} \to 0$ as $n \to \infty$. We thus obtain for any $R > 0$,

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^d} \int_{|x-y|\leq \alpha(t_n)} |u(t_n, x)|^2 dx \geq \int_{|x|\leq R} |U(x)|^2 dx.$$

Let $R \to \infty$, we obtain

$$\liminf_{n\to\infty} \sup_{y\in\mathbb{R}^d} \int_{|x-y|\leq \alpha(t_n)} |u(t_n,x)|^2 dx \geq \|U\|^2_{L^2}.$$

This implies

$$\limsup_{t\nearrow T^*} \sup_{y\in\mathbb{R}^d} \int_{|x-y|\leq \alpha(t)} |u(t,x)|^2 dx \geq \|Q\|^2_{L^2}.$$

Sine for any fixed time t, the map $y \mapsto \int_{|x-y|\leq\alpha(t)} |u(t,x)|^2 dx$ is continuous and goes to zero as $|y| \to \infty$, there exists $x(t) \in \mathbb{R}^d$ such that

$$\sup_{y\in\mathbb{R}^d} \int_{|x-y|\leq \alpha(t)} |u(t,x)|^2 dx = \int_{|x-x(t)|\leq \alpha(t)} |u(t,x)|^2 dx.$$

This shows

$$\limsup_{t\nearrow T^*} \int_{|x-x(t)|\leq \alpha(t)} |u(t,x)|^2 dx \geq \|Q\|^2_{L^2}.$$

The proof is complete. $\square$

Proof of Theorem 7.2.3 Note that the uniqueness up to translations in space, phase and dilations of solution to this ground state equation is assumed here. Using the notation in the proof of Theorem 7.2.1 and the assumption $\|\psi\|_{L^2} = \|Q\|_{L^2}$, we have

$$\|v_n\|_{L^2} \leq \|\psi\|_{L^2} = \|Q\|_{L^2} \leq \|U\|_{L^2}.$$

Sine $v_n(\cdot + x_n) \rightharpoonup U$ weakly in $L^2(\mathbb{R}^d)$, the semi-continuity of weak convergence implies

$$\|U\|_{L^2} \leq \liminf_{n\to\infty} \|v_n\|_{L^2} \leq \|Q\|_{L^2}.$$

Thus,

$$\|U\|_{L^2} = \|Q\|_{L^2} = \lim_{n\to\infty} \|v_n\|_{L^2}. \tag{7.2.29}$$

Hence up to a subsequence

$$v_n(\cdot + x_n) \to U \text{ strongly in } L^2(\mathbb{R}^d), \tag{7.2.30}$$

as $n \to \infty$. On the other hand, using (7.2.23), the Gagliardo-Nirenberg inequality (7.0.4) implies $v_n(\cdot + x_n) \to U$ strongly in $L^{2+\frac{8}{d}}(\mathbb{R}^d)$. Indeed, by (7.2.23),

$$\begin{aligned}\|v_n(\cdot + x_n) - U\|^{2+\frac{8}{d}}_{L^{2+\frac{8}{d}}} &\lesssim \|\psi(\cdot + x_n) - U\|^{\frac{8}{d}}_{L^2} \|\Delta(v_n(\cdot + x_n) - U\|^2_{L^2} \\ &\lesssim (\|\Delta Q\|_{L^2} + \|\Delta U\|_{L^2})^2 \|\psi(\cdot + x_n) - U\|^{\frac{8}{d}}_{L^2} \to 0,\end{aligned}$$

as $n \to \infty$. Moreover, using (7.2.24) and (7.2.29), the sharp Gagliardo-Nirenberg inequality (7.0.4) also gives

$$\|\Delta Q\|^2_{L^2} = \frac{1}{1+\frac{4}{d}} \|U\|^{2+\frac{8}{d}}_{L^{2+\frac{8}{d}}} \leq \Big(\frac{\|U\|_{L^2}}{\|Q\|_{L^2}}\Big)^{\frac{8}{d}} \|\Delta U\|^2_{L^2} = \|\Delta U\|^2_{L^2},$$

or $\|\Delta Q\|_{L^2} \leq \|\Delta U\|_{L^2}$. By the semi-continuity of weak convergence and (7.2.23),

$$\|\Delta U\|_{L^2} \leq \liminf_{n\to\infty} \|\Delta v_n\|_{L^2} = \|\Delta Q\|_{L^2}.$$

Therefore,

$$\|\Delta U\|_{L^2} = \|\Delta Q\|_{L^2} = \lim_{n\to\infty} \|\Delta v_n\|_{L^2}. \tag{7.2.31}$$

Combining (7.2.29), (7.2.31) and using the fact $v_n(\cdot + x_n) \rightharpoonup U$ weakly in $H^2(\mathbb{R}^d)$, we conclude that $v_n(\cdot + x_n) \to U$ strongly in $H^2(\mathbb{R}^d)$. In particular,

$$E(U) = \lim_{n\to\infty} E(v_n) = 0,$$

as $n \to \infty$. This shows that there exists $U \in H^2(\mathbb{R}^d)$ satisfying

$$\|U\|_{L^2} = \|Q\|_{L^2}, \quad \|\Delta U\|_{L^2} = \|\Delta Q\|_{L^2}, \quad E(U) = 0.$$

Applying the variational characterization given in Lemma 7.1.9, we have (taking $\lambda = 1$),

$$U(x) = e^{i\theta} Q(x + x_0),$$

for some $(\theta, x_0) \in \mathbb{R} \times \mathbb{R}^d$. Hence

$$\lambda_n^{\frac{d}{2}} I_{N(t_n)} u(t_n, \lambda_n \cdot + x_n) \to e^{i\theta} Q(\cdot + x_0) \text{ strongly in } H^2(\mathbb{R}^d),$$

as $n \to \infty$. Using (7.2.27), we prove

$$\lambda_n^{\frac{d}{2}} u(t_n, \lambda_n \cdot + x_n) \to e^{i\theta} Q(\cdot + x_0) \text{ strongly in } H^{a(d,\gamma)-}(\mathbb{R}^d),$$

as $n \to \infty$. The proof is complete. □

Appendix A

Appendices

Contents

A.1 Hamilton-Jacobi equation

In this appendix, we will recall how to construct the standard Hamilton-Jacobi equation (see e.g. [Rob87, Théorème IV.14]). Let us consider the following Hamilton-Jacobi equation

$$\left\{ \begin{array}{rcl} \partial_t S(t,x,\xi) + H(x, \nabla_x S(t,x,\xi)) &=& 0, \\ S(0,x,\xi) &=& x\cdot\xi, \end{array} \right. \tag{A.1.1}$$

where $H \in C^\infty(\mathbb{R}^{2d})$ satisfies that for all $\alpha, \beta \in \mathbb{N}^d$ with $|\alpha+\beta| \geq 2$, there exists $C_{\alpha\beta} > 0$ such that for all $x, \xi \in \mathbb{R}^d$,

$$|\partial_x^\alpha \partial_\xi^\beta H(x,\xi)| \leq C_{\alpha\beta}. \tag{A.1.2}$$

The Hamiltonian flow associated to H is denoted by $\Phi_H(t,x,\xi) := (X(t,x,\xi), \Xi(t,x,\xi))$ where

$$\left\{ \begin{array}{rcl} \dot{X}(t) &=& \nabla_\xi H(X(t),\Xi(t)), \\ \dot{\Xi}(t) &=& -\nabla_x H(X(t),\Xi(t)), \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rcl} X(0) &=& x, \\ \Xi(0) &=& \xi. \end{array} \right.$$

Let us start with the following bound on derivatives of the Hamiltonian flow.

Lemma A.1.1. *Let $t_0 \geq 0$ and $\alpha, \beta \in \mathbb{N}^d$ be such that $|\alpha+\beta| \geq 1$. Then there exists $C_{\alpha\beta t_0} > 0$ such that for all $t \in [-t_0, t_0]$ and all $(x,\xi) \in \mathbb{R}^{2d}$,*

$$|\partial_x^\alpha \partial_\xi^\beta (\Phi_H(t,x,\xi) - (x,\xi)| \leq C_{\alpha\beta t_0}|t|.$$

Proof. The proof is essentially given in [Rob87, Lemme IV.9]. We assume first $|\alpha+\beta| = 1$ and denote

$$Z(t) = \begin{pmatrix} \nabla_x X(t) & \nabla_\xi X(t) \\ \nabla_x \Xi(t) & \nabla_\xi \Xi(t) \end{pmatrix}.$$

By direct computation, we have

$$\frac{d}{dt} Z(t) = A(t) Z(t), \tag{A.1.3}$$

where

$$A(t) = \begin{pmatrix} \nabla_x \nabla_\xi H(X(t),\Xi(t)) & \nabla_\xi^2 H(X(t),\Xi(t)) \\ -\nabla_x^2 H(X(t),\Xi(t)) & -\nabla_\xi \nabla_x H(X(t),\Xi(t)) \end{pmatrix}.$$

This implies that

$$\|Z(t) - I_{\mathbb{R}^{2d}}\| \leq \int_0^t \|A(s)\|\|Z(s)\|ds \leq N|t| + \int_0^t N\|Z(s) - I_{\mathbb{R}^{2d}}\|ds,$$

where $N := \sup_{(t,x,\xi)\in[-t_0,t_0]\times\mathbb{R}^{2d}} \|A(t)\|$. Here $\|\cdot\|$ is the $\mathbb{R}^{2d\times 2d}$-matrix norm. Using Gronwall inequality, we have

$$\|Z(t) - I_{\mathbb{R}^{2d}}\| \leq N|t|e^{Nt} \leq Ne^{Nt_0}|t|.$$

For $|\alpha + \beta| \geq 2$, we take the derivative of $(A.1.3)$ and apply again the Gronwall inequality. □

Lemma A.1.2. *There exists $t_0 > 0$ small enough such that for all $t \in [-t_0, t_0]$ and all $\xi \in \mathbb{R}^d$, the map $x \mapsto X(t, x, \xi)$ is a diffeomorphism from $\mathbb{R}^d$ onto itself. Moreover, if we denote $x \mapsto Y(t, x, \xi)$ the inverse map, then for all $t \in [-t_0, t_0]$ and all $\alpha, \beta \in \mathbb{N}^d$ satisfying $|\alpha + \beta| \geq 1$, there exists $C_{\alpha\beta} > 0$ such that for all $x, \xi \in \mathbb{R}^d$,*

$$|\partial_x^\alpha \partial_\xi^\beta (Y(t, x, \xi) - x)| \leq C_{\alpha\beta}|t|.$$

Proof. By Lemma $A.1.1$, there exists $t_0 > 0$ small enough such that

$$\|\nabla_x X(t) - I_{\mathbb{R}^d}\| \leq \frac{1}{2},$$

for all $t \in [-t_0, t_0]$. By Hadamard global inversion theorem, the map $x \mapsto X(t, x, \xi)$ is a diffeomorphism from $\mathbb{R}^d$ onto itself. Let $x \mapsto Y(t, x, \xi)$ be its inverse. By taking derivative $\partial_x^\alpha \partial_\xi^\beta$ with $|\alpha + \beta| = 1$ of the following equality

$$x = X(t, Y(t, x, \xi), \xi), \tag{A.1.4}$$

we have

$$(\nabla_x X)(t, Y(t, x, \xi), \xi)\partial_x^\alpha \partial_\xi^\beta (Y(t, x, \xi) - x) = -\partial_y^\alpha \partial_\eta^\beta (X(t, y, \eta) - y)|_{(y,\eta)=(Y(t,x,\xi),\xi)}.$$

By choosing t_0 small enough, we see that the matrix $(\partial_x X)(t, Y(t, x, \xi), \xi)$ is invertible and its inverse is bounded uniformly in $t \in [-t_0, t_0]$ and $x, \xi \in \mathbb{R}^d$. This implies that

$$|\partial_x^\alpha \partial_\xi^\beta (Y(t, x, \xi) - x)| \leq C|\partial_y^\alpha \partial_\eta^\beta (X(t, y, \eta) - y)| \leq C_{\alpha\beta}|t|.$$

For higher derivatives, we differentiate $(A.1.4)$ and use an induction on $|\alpha + \beta|$. This completes the proof. □

Now, we are able to solve the Hamilton-Jacobi equation $(A.1.1)$ and have the following result.

Proposition A.1.3. *Let t_0 be as in* Lemma $A.1.2$. *Then there exists a unique function $S \in C^\infty([-t_0, t_0] \times \mathbb{R}^{2d})$ such that S solves the Hamilton-Jacobi equation $(A.1.1)$. The solution S is given by*

$$S(t, x, \xi) = Y(t, x, \xi) \cdot \xi + \int_0^t (\xi \cdot \nabla_\xi H - H) \circ \Phi_H(s, Y(t, x, \xi), \xi)ds, \tag{A.1.5}$$

and S satisfies

$$\nabla_\xi S(t) = Y(t), \quad \nabla_x S(t) = \Xi(t, Y(t), \xi), \quad \Phi_H(t, \nabla_\xi S(t), \xi) = (x, \nabla_x S(t)), \tag{A.1.6}$$

where $S(t) := S(t, x, \xi)$ and $Y(t) := Y(t, x, \xi)$. Moreover, for all $\alpha, \beta \in \mathbb{N}^d$, there exists $C_{\alpha\beta} > 0$ such that for all $t \in [-t_0, t_0]$ and all $x, \xi \in \mathbb{R}^d$,

$$|\partial_x^\alpha \partial_\xi^\beta (S(t, x, \xi) - x \cdot \xi)| \leq C_{\alpha\beta}|t|, \quad |\alpha + \beta| \geq 1, \tag{A.1.7}$$

$$|\partial_x^\alpha \partial_\xi^\beta (S(t, x, \xi) - x \cdot \xi + tH(x, \xi))| \leq C_{\alpha\beta}|t|^2. \tag{A.1.8}$$

Proof. It is well-known (see [Rob87, Théorème IV.14]) that the function S defined in (A.1.5) is the unique solution to (A.1.1) and satisfies (A.1.6). It remains to prove (A.1.7) and (A.1.8). By (A.1.6) and the conservation of energy, we have

$$H(x,\nabla_x S(t)) = H\circ\Phi_H(t,\nabla_\xi S(t),\xi) = H(\nabla_\xi S(t),\xi) = H(Y(t),\xi).$$

This implies that

$$S(t,x,\xi) - x\cdot\xi = t\int_0^1 \partial_t S(\theta t,x,\xi)d\theta = -t\int_0^1 H(Y(\theta t,x,\xi),\xi)d\theta.$$

Using (A.1.2) and Lemma A.1.2, we have (A.1.7). Next, we compute

$$\begin{aligned}\partial_t^2 S(t) &= -\partial_t\left[H(Y(t),\xi)\right] = -(\nabla_x H)(Y(t),\xi)\cdot\partial_t Y(t)\\ &= -(\nabla_x H)(Y(t),\xi)\cdot\nabla_\xi\left[\partial_t S(t)\right] = -(\nabla_x H)(Y(t),\xi)\cdot\nabla_\xi\left[-H(Y(t),\xi)\right]\\ &= (\nabla_x H)^2(Y(t),\xi)\cdot\nabla_\xi Y(t) + (\nabla_x H\cdot\nabla_\xi H)(Y(t),\xi). \end{aligned}\tag{A.1.9}$$

The Taylor formula gives

$$\begin{aligned}S(t,x,\xi) &= x\cdot\xi - tH(x,\xi)\\ &\quad + t^2\int_0^1 (1-\theta)\left[(\nabla_x H)^2(Y(\theta t),\xi)\cdot\nabla_\xi Y(\theta t) + (\nabla_x H\cdot\nabla_\xi H)(Y(\theta t),\xi)\right]d\theta.\end{aligned}$$

Using again (A.1.2) and Lemma A.1.2, we have (A.1.8). □

A.2 Bourgain $X^{\gamma,b}$ spaces

In this appendix, we recall some basic properties of Bourgain spaces $X^{\gamma,b}$ which are used in Subsection 6.1.

Definition A.2.1 ($X^{\gamma,b}$-space)**.** Let $h : \mathbb{R}^d \to \mathbb{R}$ be a continuous function, and let $\gamma, b \in \mathbb{R}$. The space $X^{\gamma,b}_{h(\xi)}(\mathbb{R} \times \mathbb{R}^d)$, abbreviated $X^{\gamma,b}$ is defined to be the closure of the Schwartz space $\mathscr{S}_{t,x}(\mathbb{R} \times \mathbb{R}^d)$ under the norm

$$\|u\|_{X^{\gamma,b}_{h(\xi)}(\mathbb{R}\times\mathbb{R}^d)} := \| \langle \tau - h(\xi) \rangle^b \langle \xi \rangle^\gamma \tilde{u}(\tau,\xi)\|_{L^2_\tau L^2_\xi(\mathbb{R}\times\mathbb{R}^d)},$$

where $\tilde{u}$ (or $\mathfrak{F}(u)$) is the space time Fourier transform

$$\tilde{u}(\tau,\xi) := (2\pi)^{-(d+1)} \iint_{\mathbb{R}\times\mathbb{R}^d} e^{-i(t\tau + x\cdot\xi)} u(t,x)dxdt.$$

When $b = 0$, $X^{\gamma,0} = L^2_t H^\gamma_x$, when $h \equiv 0$, $X^{\gamma,b} = H^b_t H^\gamma_x$ and when $\gamma = b = 0$, $X^{0,0} = L^2_t L^2_x$. We now recall some basic properties of $X^{\gamma,b}$-space.

Proposition A.2.2. *Let $\gamma, b \in \mathbb{R}$. The Bourgain space $X^{\gamma,b}$ satisfies the following properties:*

i. *$X^{s,b}$ is a Banach space.*
ii. *If $\gamma_1 \leq \gamma_2$ and $b_1 \leq b_2$, then $X^{\gamma_2,b_2} \subset X^{\gamma_1,b_1}$.*
iii. $\|\overline{u}\|_{X^{\gamma,b}_{h(\xi)}} = \|u\|_{X^{\gamma,b}_{-h(-\xi)}}$.
iv. $(X^{\gamma,b}_{h(\xi)})^\star = X^{-\gamma,-b}_{-h(-\xi)}$.
v. *Let $\gamma_1 \leq \gamma \leq \gamma_2, b_1 \leq b \leq b_2$ be such that $\gamma = \theta\gamma_1 + (1-\theta)\gamma_2, b = \theta b_1 + (1-\theta)b_2$ for some $\theta \in [0,1]$. If $u \in X^{\gamma_1,b_1} \cap X^{\gamma_2,b_2}$, then $u \in X^{\gamma,b}$. In particular,*

$$\|u\|_{X^{\gamma,b}} \leq \|u\|^\theta_{X^{\gamma_1,b_1}} \|u\|^{1-\theta}_{X^{\gamma_2,b_2}}.$$

vi. *($X^{\gamma,b}$ vs $H^b_t H^\gamma_x$)*

$$\|e^{-ith(D)}u\|_{H^b_t H^\gamma_x} = \|u\|_{X^{\gamma,b}}.$$

Proof. (i) The completeness of $X^{\gamma,b}$ follows from the completeness of $L^2_\tau L^2_\xi$. (ii) It is obvious by the definition. (iii) A direct computation shows

$$\tilde{\overline{u}}(\tau,\xi) = \overline{\tilde{u}}(-\tau,-\xi).$$

By definition and a simple change of variables, we have (iii). (iv) This follows from the fact that the bilinear functional

$$B : \mathscr{S}_{t,x} \times \mathscr{S}_{t,x} \ni (\phi,\varphi) \mapsto \langle \phi,\varphi \rangle_{L^2_t L^2_x} := \iint_{\mathbb{R}\times\mathbb{R}^d} \phi(t,x)\varphi(t,x)dtdx \in \mathbb{C}$$

can be extended to a continuous bilinear functional on $X^{-\gamma,-b}_{-h(-\xi)} \times X^{\gamma,b}_{h(\xi)}$. We also have that if L is a continuous linear functional on $X^{\gamma,b}_{h(\xi)}$, then there exists a unique $u \in X^{-\gamma,-b}_{-h(-\xi)}$ such that

$$\forall \varphi \in X^{\gamma,b}_{h(\xi)}, \quad \langle L,\varphi \rangle = B(u,\varphi).$$

Moreover,

$$\|L\|_{(X^{\gamma,b}_{h(\xi)})^\star} = \|u\|_{X^{-\gamma,-b}_{-h(-\xi)}}.$$

Indeed, by Parseval's identity and Cauchy-Schwarz inequality, we have

$$\begin{aligned}\Big|\iint_{\mathbb{R}\times\mathbb{R}^d}\overline{\phi(t,x)}\varphi(t,x)dtdx\Big| &\sim \Big|\iint_{\mathbb{R}\times\mathbb{R}^d}\tilde{\phi}(-\tau,-\xi)\tilde{\varphi}(\tau,\xi)d\tau d\xi\Big|\\ &\sim \Big|\iint_{\mathbb{R}\times\mathbb{R}^d}\langle\tau-h(\xi)\rangle^{-b}\langle\xi\rangle^{-\gamma}\tilde{\phi}(-\tau,-\xi)\langle\tau-h(\xi)\rangle^{b}\langle\xi\rangle^{\gamma}\tilde{\varphi}(\tau,\xi)d\tau d\xi\Big|\\ &\lesssim \|\phi\|_{X^{-\gamma,-b}_{-h(-\xi)}}\|\varphi\|_{X^{\gamma,b}_{h(\xi)}}.\end{aligned}$$

Since $\mathscr{S}_{t,x}$ is dense in $X^{\gamma,b}$, the bilinear functional B can be extended to $X^{-\gamma,-b}_{-h(-\xi)}\times X^{\gamma,b}_{h(\xi)}$. Now let $L\in(X^{\gamma,b}_{h(\xi)})^{\star}$, i.e. a linear functional on $X^{\gamma,b}_{h(\xi)}$. Consider

$$\begin{aligned}L_{\gamma,b}:\quad L^2_\tau L^2_\xi &\to \mathbb{C}\\ f &\mapsto \Big\langle L,\mathfrak{F}^{-1}(\langle\tau-h(\xi)\rangle^{-b}\langle\xi\rangle^{-\gamma}f)\Big\rangle.\end{aligned}$$

We then have

$$\begin{aligned}\sup_{\|f\|_{L^2_\tau L^2_\xi}=1}|\langle L_{\gamma,b},f\rangle| &= \sup_{\|f\|_{L^2_\tau L^2_\xi}=1}\Big|\Big\langle L,\mathfrak{F}^{-1}(\langle\tau-h(\xi)\rangle^{-b}\langle\xi\rangle^{-\gamma}f)\Big\rangle\Big|\\ &= \sup_{\|\varphi\|_{X^{\gamma,b}_{h(\xi)}}=1}|\langle L,\varphi\rangle| = \|L\|_{(X^{\gamma,b}_{h(\xi)})^\star}.\end{aligned}$$

Here $\varphi=\mathfrak{F}^{-1}(\langle\tau-h(\xi)\rangle^{-b}\langle\xi\rangle^{-\gamma}f)$ and $\|\varphi\|_{X^{\gamma,b}_{h(\xi)}}=\|f\|_{L^2_\tau L^2_\xi}=1$. The Riesz representation theorem then implies that there exists $g\in L^2_\tau L^2_\xi$ such that

$$\forall h\in L^2_\tau L^2_\xi,\quad \langle L_{\gamma,b},h\rangle=B(g,h).$$

Now define $u:=\mathfrak{F}(\langle\tau-h(\xi)\rangle^{b}\langle\xi\rangle^{\gamma}g)$. It is easy to see that $u\in X^{-\gamma,-b}_{-h(-\xi)}$. This shows that for all $\varphi\in\mathscr{S}_{t,x}$,

$$\begin{aligned}B(u,\varphi)=\iint\mathfrak{F}^{-1}u(\tau,\xi)\tilde{\varphi}(\tau,\xi)d\tau d\xi &= \iint g(\tau,\xi)\langle\tau-h(\xi)\rangle^{b}\langle\xi\rangle^{\gamma}\tilde{\varphi}(\tau,\xi)d\tau d\xi\\ &= \Big\langle L_{\gamma,b},\langle\tau-h(\xi)\rangle^{b}\langle\xi\rangle^{\gamma}\tilde{\varphi}\Big\rangle=\langle L,\varphi\rangle.\end{aligned}$$

This shows (iv). (v) It follows from that

$$\begin{aligned}\|u\|_{X^{\gamma,b}} &= \|\langle\tau-h(\xi)\rangle^{b}\langle\xi\rangle^{\gamma}\tilde{u}\|_{L^2_\tau L^2_\xi}\\ &\le \|\langle\tau-h(\xi)\rangle^{\theta b_1}\langle\xi\rangle^{\theta\gamma_1}|\tilde{u}|^{\theta}\|_{L^{2/\theta}_\tau L^{2/\theta}_\xi}\|\langle\tau-h(\xi)\rangle^{(1-\theta)b_2}\langle\xi\rangle^{(1-\theta)\gamma_2}|\tilde{u}|^{(1-\theta)}\|_{L^{2/(1-\theta)}_\tau L^{2/(1-\theta)}_\xi}\\ &\le \|\langle\tau-h(\xi)\rangle^{b_1}\langle\xi\rangle^{\gamma_1}\tilde{u}\|^{\theta}_{L^2_\tau L^2_\xi}\|\langle\tau-h(\xi)\rangle^{b_2}\langle\xi\rangle^{\gamma_2}\tilde{u}\|^{1-\theta}_{L^2_\tau L^2_\xi}\\ &= \|u\|^{\theta}_{X^{\gamma_1,b_1}}\|u\|^{1-\theta}_{X^{\gamma_2,b_2}}.\end{aligned}$$

(vi) We note that

$$\mathfrak{F}(e^{-ith(D)}u)(\tau,\xi)=\iint e^{-i(t\tau+x\cdot\xi)}e^{-ith(D)}u(t,x)dtdx=\int e^{-it\tau}e^{-ith(\xi)}\hat{u}(t,\xi)dt=\tilde{u}(\tau+h(\xi),\xi).$$

This implies that

$$\|e^{-ith(D)}u\|_{H_t^b H_x^\gamma} = \| \langle\tau\rangle^b \langle\xi\rangle^\gamma \mathfrak{F}(e^{-ith(D)}u)(\tau,\xi)\|_{L_\tau^2 L_\xi^2} = \| \langle\tau\rangle^b \langle\xi\rangle^\gamma \tilde{u}(\tau + h(\xi),\xi)\|_{L_\tau^2 L_\xi^2}$$
$$= \| \langle\tau - h(\xi)\rangle^b \langle\xi\rangle^\gamma \tilde{u}(\tau,\xi)\|_{L_\tau^2 L_\xi^2} = \|u\|_{X^{\gamma,b}}.$$

The proof is complete. □

Lemma A.2.3. *Let $\gamma, b \in \mathbb{R}$ and $u_0 \in H_x^\gamma$. Then for any $\psi \in C_0^\infty(\mathbb{R})$,*

$$\|\psi(t)e^{ith(D)}u_0\|_{X^{\gamma,b}} \lesssim \|u_0\|_{H_x^\gamma}.$$

Proof. A direct computation shows that

$$\mathfrak{F}(\psi(t)e^{ith(D)}u_0)(\tau,\xi) = \hat{\psi}(\tau - h(\xi))\hat{u}_0(\xi).$$

By definition, we have

$$\begin{aligned}\|\psi(t)e^{ith(D)}u_0\|_{X^{\gamma,b}} &= \| \langle\tau - h(\xi)\rangle^b \langle\xi\rangle^\gamma \mathfrak{F}(\psi(t)e^{ith(D)}u_0)\|_{L_\tau^2 L_\xi^2} \\ &= \| \langle\tau - h(\xi)\rangle^b \langle\xi\rangle^\gamma \hat{\psi}(\tau - h(\xi))\hat{u}_0(\xi)\|_{L_\tau^2 L_\xi^2} \\ &\lesssim \| \langle\xi\rangle^\gamma \hat{u}_0(\xi)\|_{L_\xi^2} = \|u_0\|_{H_x^\gamma}.\end{aligned}$$

Here we use the fact that $\hat{\psi}$ is rapidly decreasing, hence

$$\int \langle\tau - h(\xi)\rangle^{2b} |\psi(\tau - h(\xi))|^2 d\tau < \infty.$$

□

Lemma A.2.4. *Let $b > 1/2, \gamma \in \mathbb{R}$ and Y be a Banach space of functions on $\mathbb{R} \times \mathbb{R}^d$ with the following property that*

$$\|e^{it\tau}e^{ith(D)}f\|_Y \lesssim \|f\|_{H_x^\gamma},$$

for all $f \in H_x^\gamma$ and all $\tau \in \mathbb{R}$. Then we have

$$\|u\|_Y \lesssim_b \|u\|_{X^{\gamma,b}},$$

for all $u \in \mathscr{S}_{t,x}$.

Proof. Set $f(\tau) := \mathcal{F}_t(e^{-ith(D)}u)(\tau)$. We have

$$\begin{aligned}u(t) = e^{ith(D)}e^{-ith(D)}u(t) = e^{ith(D)}\mathcal{F}_t^{-1}\mathcal{F}_t(e^{-ith(D)}u(t)) &= \frac{1}{2\pi}e^{ith(D)}\int_{\mathbb{R}} e^{it\tau}\mathcal{F}_t(e^{-ith(D)}u)(\tau)d\tau \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\tau}e^{ith(D)}f(\tau)d\tau.\end{aligned}$$

Taking Y-norm and using Minkowski's inequality and the hypothesis on Y, we obtain

$$\|u\|_Y \lesssim \int_{\mathbb{R}} \|f(\tau)\|_{H_x^\gamma} d\tau \leq \| \langle\tau\rangle^{-b} \|_{L_\tau^2} \Big(\int_{\mathbb{R}} \langle\tau\rangle^{2b} \|f(\tau)\|_{H_x^\gamma}^2 d\tau\Big)^{1/2} \lesssim \Big(\int_{\mathbb{R}} \langle\tau\rangle^{2b} \|f(\tau)\|_{H_x^\gamma}^2 d\tau\Big)^{1/2}.$$

Here $\| \langle\tau\rangle^{-b} \|_{L_\tau^2}$ is bounded since $b > 1/2$. Using the Parseval's identity, the right hand side of the above quantity can be written as

$$\Big(\int_{\mathbb{R}} \langle\tau\rangle^{2b} \| \langle\xi\rangle^\gamma \mathcal{F}_x f(\tau)\|_{L_\xi^2}^2 d\tau\Big)^{1/2} = \|e^{-ith(D)}u\|_{H_t^b H_x^\gamma} = \|u\|_{X^{\gamma,b}}.$$

The result then follows. □

Corollary A.2.5. *Let $b > 1/2, \gamma \in \mathbb{R}$. Then for any $u \in X^{\gamma,b}$, we have*

$$\|u\|_{C_t^0 H_x^\gamma} \lesssim_b \|u\|_{X^{\gamma,b}}.$$

Proof. Applying Lemma $A.2.4$ for $Y = C_t^0 H_x^\gamma$, we immediately have the desired estimate. □

Corollary A.2.6. *Let $b > 1/2$ and (p,q) be a Schrödinger admissible pair and let $h(\xi) = |\xi|^\sigma$ with $\sigma \in (0,2]\backslash\{1\}$. Then*

$$\|u\|_{L_t^p L_x^q} \lesssim \|u\|_{X^{\gamma_{p,q},b}},$$

where

$$\gamma_{p,q} = \frac{d}{2} - \frac{d}{q} - \frac{\sigma}{p}.$$

Proof. We firstly recall Strichartz estimates for $e^{ith(D)}$ with $h(\xi) = |\xi|^\sigma, \sigma \in (0,2]\backslash\{1\}$ (see Corollary 1.1.4),

$$\|e^{ith(D)} f\|_{L_t^p L_x^q} \lesssim \|f\|_{H_x^{\gamma_{p,q}}}.$$

Note that when $\sigma \in (2,\infty)$, the above estimate holds locally in time. We then apply Lemma $A.2.4$ with $Y = L_t^p L_x^q$. Note that the space $L_t^p L_x^q$ is invariant under multiplication by phases such as $e^{it\tau}$. □

Lemma A.2.7. *Let $b_1, b_2 > 1/2, \gamma_1, \gamma_2 \in \mathbb{R}$ and Y be a Banach space of functions on $\mathbb{R} \times \mathbb{R}^d$ with the following property that*

$$\|[e^{it\tau} e^{ith(D)} f_1][e^{it\zeta} e^{ith(D)} f_2]\|_Y \lesssim \|f_1\|_{H_x^{\gamma_1}} \|f_2\|_{H_x^{\gamma_2}},$$

for all $f_1 \in H_x^{\gamma_1}, f_2 \in H_x^{\gamma_2}$ and all $\tau, \zeta \in \mathbb{R}$. Then we have

$$\|u_1 u_2\|_Y \lesssim_{b_1,b_2} \|u_1\|_{X^{\gamma_1,b_1}} \|u_2\|_{X^{\gamma_2,b_2}},$$

for all $u_1, u_2 \in \mathscr{S}_{t,x}$.

Proof. The proof is similar to the one of Lemma $A.2.4$. Set

$$f_1(\tau) := \mathcal{F}_t(e^{-ith(D)} u_1(t)), \quad f_2(\zeta) := \mathcal{F}_t(e^{-ith(D)} u_2(t)).$$

We see that

$$u_1(t) = \frac{1}{2\pi} e^{ith(D)} \int e^{it\tau} f_1(\tau) d\tau, \quad u_2(t) = \frac{1}{2\pi} e^{ith(D)} \int e^{it\zeta} f_2(\zeta) d\zeta.$$

Thus

$$\begin{aligned}\|u_1 u_2\|_Y &\lesssim \Big(\int \|f_1(\tau)\|_{H_x^{\gamma_1}} d\tau\Big)\Big(\int \|f_2(\zeta)\|_{H_x^{\gamma_2}} d\zeta\Big) \\ &\lesssim \|\langle\tau\rangle^{b_1}\|_{L^2_\tau} \Big(\int \langle\tau\rangle^{2b_1} \|f_1(\tau)\|^2_{H_x^{\gamma_1}} d\tau\Big)^{1/2} \|\langle\zeta\rangle^{b_2}\|_{L^2_\zeta} \Big(\int \langle\zeta\rangle^{2b_2} \|f_2(\zeta)\|^2_{H_x^{\gamma_2}} d\zeta\Big)^{1/2} \\ &\lesssim \|e^{-ith(D)} u_1\|_{H_t^{b_1} H_x^{\gamma_1}} \|e^{-ith(D)} u_2\|_{H_t^{b_2} H_x^{\gamma_2}} = \|u_1\|_{X^{\gamma_1,b_1}} \|u_2\|_{X^{\gamma_2,b_2}}.\end{aligned}$$

This completes the proof. □

A direct application of localized bilinear estimate given Theorem $A.3.1$ and Theorem $A.3.3$ is the following result.

Corollary A.2.8. *Let $\sigma \geq 2$ and $d > \sigma/2$ and $h(\xi) = |\xi|^\sigma$. Let $u_1 \in X^{0,b_1}, u_2 \in X^{0,b_2}$ with $b_1, b_2 > 1/2$ be supported on spatial frequencies $|\xi| \sim M, N$ respectively. Then for $M \geq N$, one has*

$$\|u_1 u_2\|_{L_t^2 L_x^2} \lesssim_{b_1,b_2} M^{(d-1)/2} N^{-(\sigma-1)/2} \|u_1\|_{X^{0,b_1}} \|u_2\|_{X^{0,b_2}}.$$

Lemma A.2.9. *Let $\gamma, b \in \mathbb{R}$ and ψ a Schwartz function in time. Then*

$$\|\psi(t) u\|_{X^{\gamma,b}} \lesssim \|u\|_{X^{\gamma,b}}.$$

Moreover, if $0 < \delta \le 1$*, then*

$$\|\psi_\delta(t)u\|_{X^{\gamma,b}} \lesssim \delta^{-|b|}\|u\|_{X^{\gamma,b}},$$

where $\psi_\delta(t) = \psi(t/\delta)$*. In the case* $b > 1/2$*, we have the following improvement*

$$\|\psi_\delta(t)u\|_{X^{\gamma,b}} \lesssim \delta^{1/2-b}\|u\|_{X^{\gamma,b}}.$$

Proof. Let us firstly understand how the $X^{\gamma,b}$-space behave with respect to temporal frequency modulation $u(t,x) \mapsto e^{it\tau_0}u(t,x)$. Note that

$$\mathfrak{F}(e^{it\tau_0}u)(\tau,\xi) = \tilde{u}(\tau-\tau_0,\xi).$$

By definition, a simple change of variable and Peetre's inequality, we have

$$\|e^{it\tau_0}u\|_{X^{\gamma,b}} = \| \langle \tau+\tau_0-h(\xi)\rangle^b \langle\xi\rangle^\gamma \tilde{u}\|_{L^2_\tau L^2_\xi} \lesssim_b \langle\tau_0\rangle^{|b|} \| \langle \tau-h(\xi)\rangle^b \langle\xi\rangle^\gamma \tilde{u}\|_{L^2_\tau L^2_\xi} = \langle\tau_0\rangle^{|b|}\|u\|_{X^{\gamma,b}}.$$

By writting $\psi(t) = \int \hat{\psi}(\tau_0)e^{it\tau_0}d\tau_0$, and use Minkowski's inequality, we have

$$\|\psi(t)u\|_{X^{\gamma,b}} \lesssim_b \Big(\int |\hat{\psi}(\tau_0)|\langle\tau_0\rangle^{|b|}d\tau_0\Big)\|u\|_{X^{\gamma,b}}.$$

Since $\hat{\psi}$ is rapidly decreasing, the first claim follows. Similarly, we have

$$\|\psi_\delta(t)u\|_{X^{\gamma,b}} \lesssim_b \Big(\int |\hat{\psi}_\delta(\tau_0)|\langle\tau_0\rangle^{|b|}d\tau_0\Big)\|u\|_{X^{\gamma,b}}.$$

Using that $\hat{\psi}_\delta(\tau) = \delta\hat{\psi}(\delta\tau)$, a change of variable and that $\langle\delta^{-1}\tau_0\rangle \le \delta^{-1}\langle\tau_0\rangle$, we obtain

$$\|\psi_\delta(t)u\|_{X^{\gamma,b}} \lesssim_b \delta^{-|b|}\|u\|_{X^{\gamma,b}}.$$

This proves the second claim. In the case $b > 1/2$, we have

$$\|\psi_\delta(t)u\|_{X^{\gamma,b}} = \|e^{-ith(D)}\psi_\delta(t)u\|_{H^\gamma_x H^b_t} = \| \langle\xi\rangle^\gamma \|e^{-ith(\xi)}\psi_\delta(t)\hat{u}\|_{H^b_t}\|_{L^2_\xi}.$$

We now use the Leibniz rule and Sobolev embedding with $b > 1/2$ to get

$$\|e^{-ith(\xi)}\psi_\delta(t)\hat{u}\|_{H^b_t} \le \|\psi_\delta\|_{H^b_t}\|e^{-ith(\xi)}\hat{u}\|_{L^\infty_t} + \|\psi_\delta\|_{L^\infty_t}\|e^{-ith(\xi)}\hat{u}\|_{H^b_t} \le \|\psi_\delta\|_{H^b_t}\|e^{-ith(\xi)}\hat{u}\|_{H^b_t}.$$

This shows that

$$\|\psi_\delta(t)u\|_{X^{\gamma,b}} \le \|\psi_\delta\|_{H^b_t}\| \langle\xi\rangle^\gamma \|e^{-ith(\xi)}\hat{u}\|_{H^b_t}\|_{L^2_\xi} \lesssim \delta^{1/2-b}\|e^{-ith(D)}u\|_{H^\gamma_x H^b_t} = \delta^{1/2-b}\|u\|_{X^{\gamma,b}}.$$

This completes the proof. □

Lemma A.2.10. *Let* $\gamma, b \in \mathbb{R}$ *and* ψ *a Schwartz function in time. Then for all* $u_0 \in H^\gamma_x$*,*

$$\|\psi(t)e^{ith(D)}u_0\|_{X^{\gamma,b}} \lesssim \|u_0\|_{H^\gamma_x}.$$

Moreover, if $b > 1/2$ *and* $0 < \delta \le 1$*, then*

$$\|\psi_\delta(t)e^{ith(D)}u_0\|_{X^{\gamma,b}} \lesssim \delta^{1/2-b}\|u_0\|_{H^\gamma_x}.$$

Proof. We have from Item (vi) of Proposition $A.2.2$ that $\|u\|_{X^{\gamma,b}} = \|e^{-ith(D)}u\|_{H^b_t H^\gamma_x}$. This implies

$$\|\psi(t)e^{ith(D)}u_0\|_{X^{\gamma,b}} = \|e^{-ith(D)}\psi(t)e^{ith(D)}u_0\|_{H^b_t H^\gamma_x} = \|\psi(t)u_0\|_{H^b_t H^\gamma_x} = \|\psi\|_{H^b_t}\|u_0\|_{H^\gamma_x} \lesssim \|u_0\|_{H^\gamma_x}.$$

The second claim follows by using the fact that

$$\|\psi_\delta\|_{H^b_t} = \| \langle\tau\rangle^b \hat{\psi}_\delta(\tau)\|_{L^2_\tau} = \Big(\int \langle\tau\rangle^{2b} |\hat{\psi}_\delta(\tau)|^2 d\tau\Big)^{1/2} \lesssim \delta^{1/2-b}\|\psi\|_{H^b_t}.$$

□

Lemma A.2.11. *Let $\gamma \in \mathbb{R}, 0 < \delta \leq 1, 0 < b' < 1/2 < b$ and $b + b' < 1$. Let ψ be a Schwartz function in time. Then*

$$\Big\|\psi_\delta(t)\int_0^t g(s)ds\Big\|_{H^b_t} \lesssim \delta^{1-(b+b')}\|g\|_{H^{-b'}_t},$$

and

$$\Big\|\psi_\delta(t)\int_0^t e^{i(t-s)h(D)}F(s)ds\Big\|_{X^{\gamma,b}} \lesssim \delta^{1-(b+b')}\|F\|_{X^{\gamma,-b'}}.$$

Proof. We firstly write

$$\begin{aligned}
\psi_\delta(t)\int_0^t g(s)ds &= \psi_\delta(t)\int_0^t \Big(\int_{\mathbb{R}} e^{i\tau s}\hat{g}(\tau)d\tau\Big)ds \\
&= \psi_\delta(t)\int_{\mathbb{R}}\Big(\int_0^t e^{i\tau s}ds\Big)\hat{g}(\tau)d\tau = \psi_\delta(t)\int_{\mathbb{R}}\frac{e^{it\tau}-1}{i\tau}\hat{g}(\tau)d\tau \\
&= \psi_\delta(t)\sum_{k\geq 1}\frac{t^k}{k!}\int_{|\delta\tau|\leq 1}(i\tau)^{k-1}\hat{g}(\tau)d\tau - \psi_\delta(t)\int_{|\delta\tau|\geq 1}(i\tau)^{-1}\hat{g}(\tau)d\tau \\
&+ \psi_\delta(t)\int_{|\delta\tau|\geq 1}(i\tau)^{-1}e^{it\tau}\hat{g}(\tau)d\tau =: I + II + III.
\end{aligned}$$

Let us consider the first term. The Cauchy-Schwarz inequality gives

$$\|I\|_{H^b_t} \leq \sum_{k\geq 1}\frac{1}{k!}\|t^k\psi_\delta\|_{H^b_t}\delta^{1-k}\|g\|_{H^{-b'}_t}\Big(\int_{|\delta\tau|\leq 1}\langle\tau\rangle^{2b'}d\tau\Big)^{1/2}.$$

Using that $t^k\psi_\delta(t) = \delta^k\varphi_k(t/\delta)$ where $\varphi_k(t) = t^k\psi(t)$, we have

$$\|t^k\psi_\delta\|_{H^b_t} = \delta^k\|\varphi_k(t/\delta)\|_{H^b_t} = \delta^k\Big(\int_{\mathbb{R}}\langle\tau\rangle^{2b}\delta^2|\hat{\varphi}_k(\delta\tau)|^2d\tau\Big)^{1/2} \lesssim \delta^k\delta^{1/2-b}\|\varphi_k\|_{H^b_t}.$$

We also have

$$\int_{|\delta\tau|\leq 1}\langle\tau\rangle^{2b'}d\tau = \int_{|\tau|\leq 1}\langle\delta^{-1}\tau\rangle^{2b'}\delta^{-1}d\tau \lesssim \delta^{-1-2b'}.$$

We then have

$$\|I\|_{H^b_t} \lesssim \sum_{k\geq 1}\frac{1}{k!}\delta^k\delta^{1/2-b}\delta^{1-k}\|g\|_{H^{-b'}_t}\delta^{-1/2-b'} \lesssim \delta^{1-(b+b')}\|g\|_{H^{-b'}_t}.$$

For the second term, we use a same argument to have

$$\|II\|_{H^b_t} \lesssim \|\psi_\delta\|_{H^b_t}\|g\|_{H^{-b'}_t}\Big(\int_{|\delta\tau|\geq 1}|\tau|^{-2}\langle\tau\rangle^{2b'}d\tau\Big)^{1/2}.$$

We then use that $\|\psi_\delta\|_{H^b_t} \lesssim \delta^{1/2-b}\|\psi\|_{H^b_t} \lesssim \delta^{1/2-b}$ and

$$\int_{|\delta\tau|\geq 1} |\tau|^{-2} \langle\tau\rangle^{2b'} d\tau = \int_{|\tau|\geq 1} |\delta^{-1}\tau|^{-2} \left\langle\delta^{-1}\tau\right\rangle^{2b'} \delta^{-1} d\tau \leq \delta^{1-2b'} \int_{|\tau|\geq 1} |\tau|^{-2} \langle\tau\rangle^{2b'} d\tau$$
$$\lesssim \delta^{1-2b'} \int_{|\tau|\geq 1} |\tau|^{-2(1-b')} d\tau \lesssim \delta^{1-2b'}.$$

Here $b' < 1/2$ hence $2(1-b') > 1$ implies the last integral is convergent. This shows that

$$\|II\|_{H^b_t} \lesssim \delta^{1-(b+b')}\|g\|_{H^{-b'}_t}.$$

We next treat the third term as follows. Set

$$J(t) := \int_{|\delta\tau|\geq 1} (i\tau)^{-1}\hat{g}(\tau)e^{it\tau} d\tau.$$

We see that

$$\hat{J}(\zeta) = \int_{|\delta\tau|\geq 1} (i\tau)^{-1}\hat{g}(\tau)\delta_0(\zeta-\tau) d\tau.$$

Note that the Fourier transform of $e^{it\tau}$ is $\delta_0(\zeta-\tau)$. This implies that

$$\|J\|_{H^b_t} = \Big(\int \langle\zeta\rangle^{2b} |\hat{J}(\zeta)|^2 d\zeta\Big)^{1/2} = \Big(\int_{|\delta\tau|\geq 1} \langle\tau\rangle^{2b} |\tau|^{-2}|\hat{g}(\tau)|^2 d\tau\Big)^{1/2}$$
$$\leq \|g\|_{H^{-b'}_t} \sup_{|\delta\tau|\geq 1} |\tau|^{-1} \langle\tau\rangle^{b+b'} \lesssim \delta^{1-(b+b')}\|g\|_{H^{-b'}_t}.$$

Similarly,

$$\|J\|_{L^2_t} \lesssim \delta^{1-b'}\|g\|_{H^{-b'}_t}.$$

Thus, the Young's inequality gives

$$\|III\|_{H^b_t} = \| \langle\tau\rangle^b (\hat{\psi}_\delta \star \hat{J})\|_{L^2_\tau} \lesssim \||\tau|^b\hat{\psi}_\delta\|_{L^1_\tau}\|\hat{J}\|_{L^2_\tau} + \|\hat{\psi}_\delta\|_{L^1_\tau}\| \langle\tau\rangle^b \hat{J}\|_{L^2_\tau}$$
$$\lesssim T^{1-(b+b')}\|g\|_{H^{-b'}_t}.$$

Here we use the fact that $\langle\tau\rangle^b \lesssim |\tau-\zeta|^b + \langle\zeta\rangle^b$ to write

$$\langle\tau\rangle^b (\hat{\psi}_\delta \star \hat{J}) = (|\tau|^b\hat{\psi}_\delta) \star \hat{J} + \hat{\psi}_\delta \star (\langle\tau\rangle^b \hat{J}).$$

This proves the first claim. For the second estimate, we remark from Item (vi) of Proposition $A.2.2$ that it is equivalent to

$$\Big\|\psi_\delta(t)\int_0^t G(s)ds\Big\|_{H^b_t H^\gamma_x} \lesssim \|G\|_{H^{-b'}_t H^\gamma_x}. \tag{A.2.1}$$

We now apply the first estimate for $g(s) = \mathcal{F}_x G(s,\xi)$ with ξ fixed to have

$$\Big\|\psi_\delta(t)\int_0^t \mathcal{F}_x G(s,\xi)ds\Big\|_{H^b_t} \lesssim \delta^{1-(b+b')}\|\mathcal{F}_x G(t,\xi)\|_{H^{-b'}_t}. \tag{A.2.2}$$

If we denote

$$H(t,x) := \psi_\delta(t)\int_0^t G(s,x)ds,$$

then $(A.2.2)$ becomes

$$\|\mathcal{F}_x H(t,\xi)\|_{H_t^b} \lesssim \delta^{1-(b+b')}\|\mathcal{F}_x G(t,\xi)\|_{H_t^{-b'}}.$$

Squaring the above estimate, multiplying both sides with $\langle\xi\rangle^{2\gamma}$ and integrating over $\mathbb{R}^d$, we get

$$\int_{\mathbb{R}^d}\langle\xi\rangle^{2\gamma}\Big(\int_{\mathbb{R}}\langle\tau\rangle^{2b}|\mathcal{F}_t\mathcal{F}_x H(\tau,\xi)|^2 d\tau\Big)d\xi \lesssim \delta^{2(1-(b+b'))}\int_{\mathbb{R}^d}\langle\xi\rangle^{2\gamma}\Big(\int_{\mathbb{R}}\langle\tau\rangle^{-2b'}|\mathcal{F}_t\mathcal{F}_x G(\tau,\xi)|^2 d\tau\Big)d\xi.$$

This shows that

$$\|H\|_{H_t^b H_x^\gamma} \lesssim \delta^{1-(b+b')}\|G\|_{H_t^{-b'}H_x^\gamma},$$

and $(A.2.1)$ follows. □

A.3 Bilinear Strichartz estimates

A.3.1 Bilinear Strichartz estimates for Schrödinger equation

Let us firstly consider the homogeneous Schrödinger equation, namely

$$i\partial_t u + \Delta u = 0, \quad u_{|t=0} = \psi. \tag{A.3.1}$$

The solution of above equation is given by $u(t,x) = e^{it\Delta}\psi(x)$. We recall the following properties:

$$\|e^{it\Delta}\psi\|_{L^2} = \|\psi\|_{L^2}, \tag{A.3.2}$$
$$\|e^{it\Delta}\psi\|_{L^\infty} \lesssim |t|^{-d/2}\|\psi\|_{L^1}, \quad t \neq 0. \tag{A.3.3}$$

The L^2-estimate $(A.3.2)$ and dispersive estimate $(A.3.3)$ give the following Strichartz estimates (see [KT98]):

$$\|e^{it\Delta}\psi\|_{L^p(\mathbb{R},L^q)} \lesssim \|\psi\|_{L^2},$$

provided that (p,q) satisfies the sharp Schrödinger admissible condition (see 0.0.1)). Moreover, if we consider the inhomogeneous linear equation, i.e.

$$i\partial_t u + \Delta u = F, \quad u_{|t=0} = \psi, \tag{A.3.4}$$

then we have (see again [KT98])

$$\|u\|_{L^p(\mathbb{R},L^q)} \lesssim \|\psi\|_{L^2} + \|F\|_{L^{a'}(\mathbb{R},L^{b'})}, \tag{A.3.5}$$

provided that (p,q) and (a,b) are sharp Schrödinger admissible. Strichartz estimates $(A.3.5)$ are also called linear estimates. We now are interested in bilinear estimates for the Schrödinger equation. In order to do so, we introduce some notation. Let $\varphi_0 \in C_0^\infty(\mathbb{R}^d)$ be such that $\varphi_0(\xi) = 1$ for $|\xi| \leq 1$ and $\varphi_0(\xi) = 0$ for $|\xi| \geq 2$ and set $\varphi(\xi) = \varphi_0(\xi) - \varphi_0(2\xi)$. It is easy to see that $\varphi \in C_0^\infty(\mathbb{R}^d)$ and $\text{supp}(\varphi) \subset \{\xi \in \mathbb{R}^d, 1/2 \leq |\xi| \leq 2\}$. For $N \in 2^{\mathbb{Z}}$, i.e. $N = 2^k$ with $k \in \mathbb{Z}$, we define the Littlewood-Paley projection as

$$\widehat{P_N f}(\xi) := \varphi_N(\xi)\hat{f}(\xi), \quad \varphi_N(\xi) = \varphi(N^{-1}\xi).$$

Note that $\text{supp}(\widehat{P_N f}) \subset \{\xi \in \mathbb{R}^d, N/2 \leq |\xi| \leq 2N\}$. We have the following indentity

$$f = \sum_{N \in 2^{\mathbb{Z}}} P_N f,$$

for all Schwartz function f. We also have the following properties with $\gamma \geq 0$ and $1 \leq q \leq r \leq \infty$:

$$\||\nabla|^{\pm\gamma} P_N f\|_{L^q} \sim N^{\pm\gamma}\|P_N f\|_{L^q}, \tag{A.3.6}$$
$$\|P_N f\|_{L^r} \lesssim N^{d/p-d/r}\|P_N f\|_{L^q}. \tag{A.3.7}$$

Let us begin with the following localized bilinear estimate (see [KTV14], Theorem 2.9).

Theorem A.3.1 (Localized bilinear estimate)**.** *Let $d \geq 2$ and $M, N \in 2^{\mathbb{Z}}, M \leq N$. Then*

$$\|[e^{it\Delta}P_M f][e^{it\Delta}P_N g]\|_{L^2(\mathbb{R},L^2)} \lesssim M^{(d-1)/2}N^{-1/2}\|f\|_{L^2}\|g\|_{L^2}. \tag{A.3.8}$$

When $d = 1$, (A.3.8) holds provided $M \ll N$.

Proof. We firstly note that for $M \sim N$, $(A.3.8)$ follows from Strichartz estimate for the pair

$(p,q) = (4, 2d/(d-1))$. Indeed, the Hölder inequality implies

$$\begin{aligned}\|[e^{it\Delta}P_M f][e^{it\Delta}P_N g]\|_{L^2(\mathbb{R},L^2)} &\le \|e^{it\Delta}P_M g\|_{L^4(\mathbb{R},L^{2d})}\|e^{it\Delta}P_N g\|_{L^4(\mathbb{R},L^{2d/(d-1)})}\\ &\lesssim \|e^{it\Delta}P_M g\|_{L^4(\mathbb{R},L^{2d})}\|P_N g\|_{L^2}.\end{aligned}$$

Here $(4, 2d/(d-1))$ is a sharp Schrödinger admissible pair. We next use Bernstein's inequality and Strichartz estimate to have

$$\|e^{it\Delta}P_M f\|_{L^4(\mathbb{R},L^{2d})} \lesssim M^{(d-1)/2-1/2}\|e^{it\Delta}P_M f\|_{L^4(\mathbb{R},L^{2d/(d-1)})} \lesssim M^{(d-1)/2-1/2}\|P_M f\|_{L^2}.$$

Therefore, we have

$$\|[e^{it\Delta}P_M f][e^{it\Delta}P_N g]\|_{L^2(\mathbb{R},L^2)} \lesssim M^{(d-1)/2-1/2}\|P_M f\|_{L^2}\|P_N g\|_{L^2} \sim M^{(d-1)/2}N^{-1/2}\|f\|_{L^2}\|g\|_{L^2}.$$

Here we use that

$$M^{(d-1)/2-1/2} = M^{(d-1)/2}N^{-1/2}(N/M)^{1/2} \sim M^{(d-1)/2}N^{-1/2}.$$

Let us consider the case $M \ll N$. Using the fact

$$\text{LHS}(A.3.8) = \sup_{\|G\|_{L^2(\mathbb{R},L^2)}=1}\Big|\left\langle G, [e^{it\Delta}P_M f][e^{it\Delta}P_N g]\right\rangle_{L^2(\mathbb{R},L^2)}\Big|,$$

where

$$\langle G, H\rangle_{L^2_t(\mathbb{R},L^2)} = \iint_{\mathbb{R}\times\mathbb{R}^d}\overline{G(t,x)}H(t,x)dtdx = \int_{\mathbb{R}}\langle F(t), H(t)\rangle_{L^2}\,dt.$$

By Parseval's identity, we have

$$\text{LHS}(A.3.8) = \sup_{\|G\|_{L^2(\mathbb{R},L^2)}=1}\int_{\mathbb{R}}\Big\langle \hat{G}(t), \mathcal{F}\Big([e^{it\Delta}P_M f][e^{it\Delta}P_N g]\Big)\Big\rangle_{L^2_\xi}dt, \tag{A.3.9}$$

where

$$\mathcal{F}\Big([e^{it\Delta}P_M f][e^{it\Delta}P_N g]\Big) = \int_{\mathbb{R}^d} e^{-it|\xi-\eta|^2}\widehat{P_M f}(\xi-\eta)e^{-it|\eta|^2}\widehat{P_N g}(\eta)d\eta.$$

Here the notation $\hat{\cdot}$ or $\mathcal{F}$ stands for the space Fourier transform. Thus,

$$\begin{aligned}\text{RHS}(A.3.9) &= \int_{\mathbb{R}}\Big\langle \hat{G}(t,\cdot), \int_{\mathbb{R}^d} e^{-it(|\cdot-\eta|^2+|\eta|^2)}\widehat{P_M f}(\cdot-\eta)\widehat{P_N g}(\eta)d\eta\Big\rangle_{L^2_\xi}dt\\ &= \int_{\mathbb{R}^d}\Big\langle \tilde{G}(|\cdot-\eta|^2+|\eta|^2,\cdot), \widehat{P_M f}(\cdot-\eta)\widehat{P_N g}(\eta)\Big\rangle_{L^2_\xi}d\eta\\ &= \iint_{\mathbb{R}^d\times\mathbb{R}^d}\overline{\tilde{G}(|\xi-\eta|^2+|\eta|^2,\xi)}\widehat{P_M f}(\xi-\eta)\widehat{P_N g}(\eta)d\xi d\eta\\ &= \iint_{\mathbb{R}^d\times\mathbb{R}^d}\overline{\tilde{G}(|\xi|^2+|\eta|^2,\xi+\eta)}\widehat{P_M f}(\xi)\widehat{P_N g}(\eta)d\xi d\eta,\end{aligned}$$

where $\tilde{G}$ is the space-time Fourier transform. Hence $(A.3.8)$ is in turn equivalent to

$$\Big|\iint_{\mathbb{R}^d\times\mathbb{R}^d}F(|\xi|^2+|\eta|^2,\xi+\eta)\widehat{P_M f}(\xi)\widehat{P_N g}(\eta)d\xi d\eta\Big| \lesssim M^{(d-1)/2}N^{-1/2}\|F\|_{L^2_\tau L^2_\xi}\|\hat{f}\|_{L^2_\xi}\|\hat{g}\|_{L^2_\xi}. \tag{A.3.10}$$

By renaming components, we may assume that $|\xi_1| \sim |\xi| \sim M$ and $|\eta_1| \sim |\eta| \sim N$, where $\xi = (\xi_1, \underline{\xi}), \eta = (\eta_1, \underline{\eta})$ with $\underline{\xi}, \underline{\eta} \in \mathbb{R}^{d-1}$. We make the change of variables $\tau = |\xi|^2 + |\eta|^2, \zeta = \xi + \eta$ and $d\tau d\zeta = Jd\xi_1 d\underline{\eta}$. A calculation shows that $J = |2(\xi_1 \pm \eta_1)| \sim |\eta_1| \sim N$. The Cauchy-Schwarz inequality and the fact that $|\underline{\xi}| \lesssim M$ then imply

$$\begin{aligned}
\text{LHS}(A.3.10) &= \Big| \iiint_{\mathbb{R}\times\mathbb{R}^{d-1}\times\mathbb{R}^d} F(\tau,\zeta)\widehat{P_M f}(\xi)\widehat{P_N g}(\eta) J^{-1} d\tau d\underline{\xi} d\zeta \Big| \\
&\leq \|F\|_{L^2_\tau L^2_\xi} \int_{\mathbb{R}^{d-1}} \Big(\iint_{\mathbb{R}\times\mathbb{R}^d} |\widehat{P_M f}(\xi)|^2 |\widehat{P_N g}(\eta)|^2 J^{-2} d\tau d\zeta \Big)^{1/2} d\underline{\xi} \\
&\leq \|F\|_{L^2_\tau L^2_\xi} M^{(d-1)/2} \Big(\iiint_{\mathbb{R}\times\mathbb{R}^{d-1}\times\mathbb{R}^d} |\widehat{P_M f}(\xi)|^2 |\widehat{P_N g}(\eta)|^2 J^{-2} d\tau d\zeta d\underline{\xi} \Big)^{1/2} \\
&\leq \|F\|_{L^2_\tau L^2_\xi} M^{(d-1)/2} \Big(\iint_{\mathbb{R}^d\times\mathbb{R}^d} |\widehat{P_M f}(\xi)|^2 |\widehat{P_N g}(\eta)|^2 J^{-1} d\xi d\eta \Big)^{1/2} \\
&\leq \|F\|_{L^2_\tau L^2_\xi} M^{(d-1)/2} N^{-1/2} \|\widehat{P_M f}\|_{L^2_\xi} \|\widehat{P_N g}\|_{L^2_\xi}.
\end{aligned}$$

This gives $(A.3.10)$ and the result follows. In order to see that $(A.3.8)$ is false when $d = 1$ and $M \sim N$, we proceed as follows. The space-time Fourier transforms of $u_M := e^{it\partial_x^2} P_M f$ and $v_N := e^{it\partial_x^2} P_N g$ read

$$\tilde{u}_M(\tau,\xi) = \widehat{P_M f}(\xi)\delta_0(\tau+\xi^2), \quad \tilde{v}_N(\tau,\xi) = \widehat{P_N g}(\xi)\delta_0(\tau+\xi^2),$$

where δ_0 is the Dirac function. We then have

$$\widetilde{u_M v_N}(\tau,\xi) = \int_{\xi_1+\xi_2=\xi} \widehat{P_M f}(\xi_1)\widehat{P_N g}(\xi_2)\delta_0(\tau+\xi_1^2+\xi_2^2) d\xi_1,$$

which gives

$$\widetilde{u_M v_N}(\tau,\xi) = \frac{1}{2|\xi_1-\xi_2|}(\widehat{P_M f}(\xi_1)\widehat{P_N g}(\xi_2) + \widehat{P_M f}(\xi_2)\widehat{P_N g}(\xi_1)),$$

where ξ_1 and ξ_2 are the solution to

$$-\xi_1^2 - \xi_2^2 = \tau, \quad \xi_1 + \xi_2 = \xi.$$

We also have

$$d\tau d\xi = 2|\xi_1 - \xi_2| d\xi_1 d\xi_2,$$

and then

$$\|u_M v_N\|^2_{L^2(\mathbb{R},L^2)} = \|\widetilde{u_M v_N}\|^2_{L^2_\tau L^2_\xi} = \iint \frac{1}{2|\xi_1-\xi_2|} |\widehat{P_M f}(\xi_1)\widehat{P_N g}(\xi_2) + \widehat{P_M f}(\xi_2)\widehat{P_N g}(\xi_1)|^2 d\xi_1 d\xi_2.$$

We see that if $|\xi_1| \sim M$ and $|\xi_2| \sim N$ and $|\xi_1 - \xi_2| \ll 1$, the integral fails to be convergent. □

Theorem A.3.2 (Bilinear estimate [CKSTT5], [Vis07])**.** *Let $d \geq 2$ and u, v be solutions to $(A.3.4)$ with initial data ψ, ϕ respectively. For any $\delta > 0$, we have*

$$\begin{aligned}
\|uv\|_{L^2(\mathbb{R},L^2)} \leq C(\delta)\Big(\|\psi\|_{\dot{H}^{(d-1)/2-\delta}} + \||\nabla|^{(d-1)/2-\delta}(i\partial_t+\Delta)u\|_{L^{p'}(\mathbb{R},L^{q'})}\Big) \\
\times \Big(\|\phi\|_{\dot{H}^{-1/2+\delta}} + \||\nabla|^{-1/2+\delta}(i\partial_t+\Delta)v\|_{L^{a'}(\mathbb{R},L^{b'})}\Big), \quad \text{(A.3.11)}
\end{aligned}$$

for any sharp Schrödinger admissible pairs (p,q) and (a,b) satisfying $p, a > 2$.

Proof. Fix $\delta > 0$ and allow our implicit constants to depend on δ. We firstly consider the homogeneous case, i.e. $u(t) = e^{it\Delta}\psi$ and $v(t) = e^{it\Delta}\phi$. Let us consider the general estimate

$$\|uv\|_{L^2(\mathbb{R},L^2)} \lesssim \|\psi\|_{\dot{H}^{\gamma_1}}\|\phi\|_{\dot{H}^{\gamma_2}}. \quad \text{(A.3.12)}$$

By the scaling invariance, the above estimate requires $\gamma_1 + \gamma_2 = d/2 - 1$. Indeed, for $\lambda > 0$ we consider $u_\lambda(t,x) = u(\lambda^{-2}t, \lambda^{-1}x)$. It is easy to see that if u solves $(A.3.1)$, then u_λ also satisfies $(A.3.1)$ with initial data $u_\lambda(0)$. By change of variable, we have

$$\|u_\lambda v_\lambda\|^2_{L^2(\mathbb{R},L^2)} = \iint_{\mathbb{R}\times\mathbb{R}^d} |u_\lambda(t,x)v_\lambda(t,x)|^2 dtdx = \lambda^{2+d}\|uv\|^2_{L^2(\mathbb{R},L^2)}.$$

We also have $\widehat{u_\lambda(0)}(\xi) = \lambda^d\hat{\psi}(\lambda\xi)$ and then

$$\|u_\lambda(0)\|^2_{\dot{H}^{\gamma_1}} = \int_{\mathbb{R}^d_\xi} |\xi|^{2\gamma_1}|\widehat{u_\lambda(0)}(\xi)|^2 d\xi = \lambda^{d-2\gamma_1}\|\psi\|^2_{\dot{H}^{\gamma_1}}.$$

A similar equality holds for $\|v_\lambda(0)\|_{\dot{H}^{\gamma_2}}$. Therefore,

$$\begin{aligned}\|u_\lambda v_\lambda\|^2_{L^2(\mathbb{R},L^2)} = \lambda^{2+d}\|uv\|^2_{L^2(\mathbb{R},L^2)} &\lesssim \lambda^{2+d}\|\psi\|^2_{\dot{H}^{\gamma_1}}\|\phi\|^2_{\dot{H}^{\gamma_2}} \\ &= \lambda^{2+d}\lambda^{-d+2\gamma_1}\lambda^{-d+2\gamma_2}\|u_\lambda(0)\|^2_{\dot{H}^{\gamma_1}}\|v_\lambda(0)\|^2_{\dot{H}^{\gamma_2}} \\ &= \lambda^{2-d+2(\gamma_1+\gamma_2)}\|u_\lambda(0)\|^2_{\dot{H}^{\gamma_1}}\|v_\lambda(0)\|^2_{\dot{H}^{\gamma_2}}.\end{aligned}$$

This shows that $\gamma_1+\gamma_2 = d/2-1$ as required. We will prove $(A.3.12)$ with $\gamma_1 = (d-1)/2-\delta$ and $\gamma_2 = -1/2+\delta$. The estimate $(A.3.12)$ may be recast using duality and renormalization as

$$\Big|\iint_{\mathbb{R}^d\times\mathbb{R}^d} F(|\xi|^2+|\eta|^2, \xi+\eta)|\xi|^{-\gamma_1}\hat{\psi}(\xi)|\eta|^{-\gamma_2}\hat{\phi}(\eta)d\xi d\eta\Big| \lesssim \|F\|_{L^2_\tau L^2_\xi}\|\hat{\psi}\|_{L^2_\xi}\|\hat{\phi}\|_{L^2_\xi}.$$

Since $\gamma_1 \geq \gamma_2$, we may restrict attention to the interactions with $|\xi| \leq |\eta|$. The remaining case can be reduced to the case under consideration by multiplying by $(|\xi|/|\eta|)^{\gamma_1-\gamma_2} \geq 1$. Moreover, we may further restrict attention to the case $|\xi| \ll |\eta|$ since, in the other case, we can move the frequencies between the two factors and reduce to the case where $\gamma_1 = \gamma_2$, which can be treated by $L^4(\mathbb{R}, L^4)$ Strichartz estimates when $d \geq 2$. Next, we decompose $|\eta|$ dyadically and $|\xi|$ in dyadic multiplies of the size of $|\eta|$ by rewriting the quantity to be controlled as (K, N dyadic):

$$\sum_K\sum_N \iint_{\mathbb{R}^d\times\mathbb{R}^d} P_K F(|\xi|^2+|\eta|^2, \xi+\eta)|\xi|^{-\gamma_1}\widehat{P_{NK}\psi}(\xi)|\eta|^{-\gamma_2}\widehat{P_K\phi}(\eta)d\xi d\eta. \tag{A.3.13}$$

Note that $|\eta| \sim K, |\xi| \sim NK$, hence $|\xi+\eta| \sim K$. This explains why F may be so localized. By remaning components, we may assume that $|\xi_1| \sim |\xi|$ and $|\eta_1| \sim |\eta|$ where $\xi = (\xi_1, \underline{\xi}), \eta = (\eta_1, \underline{\eta})$ with $\underline{\xi}, \underline{\eta} \in \mathbb{R}^{d-1}$. We now change variables by writting $\tau = \xi+\eta, \zeta = |\xi|^2+|\eta|^2$ and $d\tau d\zeta = Jd\xi_1 d\eta$. A calculation shows that $J = 2|\xi_1 \pm \eta_1| \sim |\eta_1| \sim K$. The left hand side of $(A.3.13)$ becomes

$$\Big|\sum_K K^{-\gamma_2}\sum_{N\leq 1}(NK)^{-\gamma_1}\iiint_{\mathbb{R}^{d-1}\times\mathbb{R}\times\mathbb{R}^d} P_K F(\tau,\zeta)\widehat{P_{NK}\psi}(\xi)\widehat{P_K\phi}(\eta)J^{-1}d\tau d\zeta d\underline{\xi}\Big|.$$

We apply Cauchy-Schwarz inequality and change back to the original variables to get

$$\begin{aligned}\mathrm{LHS}(A.3.13) &\leq \sum_K K^{-\gamma_2}\|P_K F\|_{L^2_\tau L^2_\xi}\sum_{N\leq 1}(NK)^{-\gamma_1} \\ &\qquad\times\int_{\mathbb{R}^{d-1}}\Big(\iint_{\mathbb{R}\times\mathbb{R}^d}|\widehat{P_{NK}\psi}(\xi)|^2|\widehat{P_K\phi}(\eta)|^2 J^{-2}d\tau d\zeta\Big)^{1/2}d\underline{\xi} \\ &\leq \sum_K K^{-\gamma_2}\|P_K F\|_{L^2_\tau L^2_\xi}\sum_{N\leq 1}(NK)^{-\gamma_1+(d-1)/2} \\ &\qquad\times\Big(\iiint_{\mathbb{R}^{d-1}\times\mathbb{R}\times\mathbb{R}^d}|\widehat{P_{NK}\psi}(\xi)|^2|\widehat{P_K\phi}(\eta)|^2 J^{-2}d\tau d\zeta d\underline{\xi}\Big)^{1/2}\end{aligned}$$

$$\begin{aligned}\text{LHS}(A.3.13) &\leq \sum_K K^{-\gamma_2}\|P_K F\|_{L^2_\tau L^2_\xi} \sum_{N\leq 1}(NK)^{-\gamma_1+(d-1)/2} \\ &\qquad \times\Big(\iint_{\mathbb{R}^d\times\mathbb{R}^d}|\widehat{P_{NK}\psi}(\xi)|^2|\widehat{P_K\phi}(\eta)|^2 J^{-1}d\xi d\eta\Big)^{1/2} \\ &\leq \sum_K K^{-\gamma_2-1/2}\|P_K F\|_{L^2_\tau L^2_\xi}\sum_{N\leq 1}(NK)^{-\gamma_1+(d-1)/2}\|\widehat{P_{NK}\psi}\|_{L^2_\xi}\|\widehat{P_K\phi}\|_{L^2_\xi}.\end{aligned}$$

We now choose $\gamma_2 = -1/2+\delta$ and $\gamma_1 = (d-1)/2-\delta$ with $\delta > 0$ to obtain

$$\text{LHS}(A.3.13) \lesssim \sum_K \|P_K F\|_{L^2_\tau L^2_\xi}\|\widehat{P_K\phi}\|_{L^2_\xi}\sum_{N\leq 1}N^\delta\|\widehat{P_{NK}\psi}\|_{L^2_\xi} \lesssim \|F\|_{L^2_\tau L^2_\xi}\|\hat{\psi}\|_{L^2_\xi}\|\hat{\phi}\|_{L^2_\xi}.$$

This gives the homogeneous bilinear estimate ($A.3.12$). We turn now our attention to the inhomogeneous estimate ($A.3.11$). Let us introduce

$$\|u\|_{S_{\gamma,p,q}} := \|\psi\|_{\dot{H}^\gamma} + \||\nabla|^\gamma(i\partial_t+\Delta)u\|_{L^{p'}(\mathbb{R},L^{q'})}, \tag{A.3.14}$$

and

$$S_{\gamma,p,q} := \{u \in C(I,\mathscr{S}) \mid \|u\|_{S_{\gamma,p,q}} < \infty\}. \tag{A.3.15}$$

The estimate ($A.3.11$) is in turn equivalent to

$$\|uv\|_{L^2(\mathbb{R},L^2)} \leq C(\delta)\|u\|_{S_{(d-1)/2-\delta,p,q}}\|v\|_{S_{-1/2+\delta,a,b}}. \tag{A.3.16}$$

We firstly note that the homogeneous bilinear estimate reads

$$\|e^{it\Delta}\psi e^{it\Delta}\phi\|_{L^2(\mathbb{R},L^2)} \leq C(\delta)\|\psi\|_{\dot{H}^{(d-1)/2-\delta}}\|\phi\|_{\dot{H}^{-1/2+\delta}}. \tag{A.3.17}$$

Now, let (p,q) and (a,b) be Schrödinger admissible pairs with $p,a>2$. Using Duhamel's formula for u, we have

$$\|uv\|_{L^2(\mathbb{R},L^2)} \leq \|e^{it\Delta}\psi v\|_{L^2(\mathbb{R},L^2)} + \Big\|\Big(\int_0^t e^{i(t-s)\Delta}(i\partial_s+\Delta)u(s)ds\Big)v\Big\|_{L^2(\mathbb{R},L^2)}.$$

Let us consider the first term. Thanks to Duhamel's formula for v, we get

$$\|e^{it\Delta}\psi v\|_{L^2(\mathbb{R},L^2)} \leq \|e^{it\Delta}\psi e^{it\Delta}\phi\|_{L^2(\mathbb{R},L^2)} + \Big\|e^{it\Delta}\psi\Big(\int_0^t e^{i(t-s)\Delta}(i\partial_s+\Delta)v(s)ds\Big)\Big\|_{L^2(\mathbb{R},L^2)}. \tag{A.3.18}$$

The homogeneous bilinear estimate ($A.3.17$) implies

$$\begin{aligned}\text{RHS}(A.3.18) &\leq C(\delta)\|\psi\|_{\dot{H}^{(d-1)/2-\delta}}\|\phi\|_{\dot{H}^{-1/2+\delta}} \\ &\qquad + C(\delta)\|\psi\|_{\dot{H}^{(d-1)/2-\delta}}\Big\|\int_0^t e^{-is\Delta}(i\partial_s+\Delta)v(s)ds\Big\|_{\dot{H}^{-1/2+\delta}} \\ &\leq C(\delta)\|u\|_{S_{(d-1)/2-\delta,p,q}}\Big(\|\phi\|_{\dot{H}^{-1/2+\delta}} + \Big\|\int_0^t e^{-is\Delta}(i\partial_s+\Delta)v(s)ds\Big\|_{\dot{H}^{-1/2+\delta}}\Big).\end{aligned}$$

Moreover, the adjoint to the linear Strichartz estimate also gives

$$\Big\|\int_{\mathbb{R}} e^{-is\Delta}|\nabla|^{-1/2+\delta}(i\partial_s+\Delta)v(s)ds\Big\|_{L^2} \lesssim \||\nabla|^{-1/2+\delta}(i\partial_s+\Delta)v\|_{L^{a'}(\mathbb{R},L^{b'})}.$$

The Christ-Kiselev Lemma 4.5.1 then implies

$$\Big\| \int_0^t e^{-is\Delta}(i\partial_s+\Delta)v(s)ds\Big\|_{\dot H^{-1/2+\delta}} \lesssim \||\nabla|^{-1/2+\delta}(i\partial_s+\Delta)v\|_{L^{a'}(\mathbb{R},L^{b'})},$$

and therefore

$$\|e^{it\Delta}\psi v\|_{L^2(\mathbb{R},L^2)} \leq \|u\|_{S_{(d-1)/2-\delta,p,q}}\|v\|_{S_{-1/2+\delta,a,b}}.$$

It remains to show

$$\Big\|\Big(\int_0^t e^{i(t-s)\Delta}(i\partial_s+\Delta)u(s)ds\Big)v\Big\|_{L^2(\mathbb{R},L^2)} \leq C(\delta)\|u\|_{S_{(d-1)/2-\delta,p,q}}\|v\|_{S_{-1/2+\delta,a,b}}.$$

By Christ-Kiselev Lemma 4.5.1, it suffices to prove

$$\Big\|\Big(\int_{\mathbb{R}} e^{i(t-s)\Delta}(i\partial_s+\Delta)u(s)ds\Big)v\Big\|_{L^2(\mathbb{R},L^2)} \leq C(\delta)\|u\|_{S_{(d-1)/2-\delta,p,q}}\|v\|_{S_{-1/2+\delta,a,b}}.$$

Using again Duhamel's formula for v and repeating the above argument for the first term, we obtain

$$\Big\|e^{it\Delta}\Big(\int_{\mathbb{R}} e^{-is\Delta}(i\partial_s+\Delta)u(s)ds\Big)v\Big\|_{L^2(\mathbb{R},L^2)} \leq C(\delta)\Big\|\int_{\mathbb{R}} e^{-is\Delta}(i\partial_s+\Delta)u(s)ds\Big\|_{\dot H^{(d-1)/2-\delta}}\|v\|_{S_{-1/2+\delta,a,b}}.$$

The adjoint to the linear Strichartz estimate again gives

$$\Big\|\int_{\mathbb{R}} e^{-is\Delta}(i\partial_s+\Delta)u(s)ds\Big\|_{\dot H^{(d-1)/2-\delta}} \lesssim \||\nabla|^{(d-1)/2-\delta}(i\partial_t+\Delta)u\|_{L^{p'}(\mathbb{R},L^{q'})} \lesssim \|u\|_{S_{(d-1)/2,p,q}}.$$

This completes the proof. □

A.3.2 Bilinear Strichartz estimate for higher-order Schrödinger equations

Let $\sigma>2$ and consider the homogeneous higher-order Schrödinger equation, namely

$$i\partial_t u - |\nabla|^\sigma u = 0, \quad u_{|t=0}=\psi. \tag{A.3.19}$$

As in Chapter 1, the equation ($A.3.19$) satisfies the following Strichartz estimates

$$\|e^{-it|\nabla|^\sigma}\psi\|_{L^p(\mathbb{R},L^q)} \lesssim \|\psi\|_{\dot H^{\gamma_{p,q}}},$$

where

$$\gamma_{p,q}=\frac{d}{2}-\frac{d}{q}-\frac{\sigma}{p},$$

and (p,q) satisfies the Schrödinger admissible condition (see (1.1.2)). Moreover, if we consider the inhomogeneous linear equation

$$i\partial_t u - |\nabla|^\sigma u = F, \quad u_{|t=0}=\psi, \tag{A.3.20}$$

then we have

$$\|u\|_{L^p(\mathbb{R},L^q)} \lesssim \|\psi\|_{\dot H^{\gamma_{p,q}}} + \|F\|_{L^{a'}(\mathbb{R},L^{b'})},$$

provided that (p,q) and (a,b) are Schrödinger admissible with $q,b<\infty$ and satisfy the gap condition

$$\gamma_{p,q}=\gamma_{a',b'}+\sigma.$$

Note that if (p,q) is a Schrödinger admissible pair satisfying $\gamma_{p,q}=0$ then $\gamma_{p,q}=\gamma_{p',q'}+\sigma$.

We now turn our attention to the bilinear estimate for the higher-order Schrödinger equation.

Let us begin with the following localized bilinear estimate.

Theorem A.3.3 (Localized bilinear estimate)**.** *Let $\sigma > 2$, $d > \sigma/2$ and $M, N \in 2^{\mathbb{Z}}$ such that $M \leq N$. Then*

$$\|[e^{-it|\nabla|^\sigma} P_M f][e^{-it|\nabla|^\sigma} P_N g]\|_{L^2(\mathbb{R},L^2)} \lesssim M^{(d-1)/2} N^{-(\sigma-1)/2} \|f\|_{L^2}\|g\|_{L^2}. \tag{A.3.21}$$

In the case $d \leq \sigma/2$, the estimate (A.3.21) holds provided $M \ll N$.

Proof. Let us firstly consider the case $M \sim N$. The Hölder inequality gives

$$\|[e^{-it|\nabla|^\sigma} P_M f][e^{-it|\nabla|^\sigma} P_N g]\|_{L^2(\mathbb{R},L^2)} \leq \|e^{-it|\nabla|^\sigma} P_M f\|_{L^4_t L^{4d/\sigma}} \|e^{-it|\nabla|^\sigma} P_N g\|_{L^4_t L^{2d/(d-\sigma/2)}}.$$

Note that when $d > \sigma/2$, $(p,q) = (4, 2d/(d-\sigma/2))$ is a Schrödinger admissible pair satisfying $\gamma_{p,q} = 0$. Moreover, using that $\sigma > 2$, it is easy to check that $(p,q) = (4, 4d/\sigma)$ is also a Schrödinger admissible with $\gamma_{p,q} = d/2 - \sigma/2$. Therefore, Strichartz estimate shows that

$$\|[e^{-it|\nabla|^\sigma} P_M f][e^{-it|\nabla|^\sigma} P_N g]\|_{L^2(\mathbb{R},L^2)} \leq \|P_M f\|_{\dot{H}^{d/2-\sigma/2}} \|P_N g\|_{L^2} \sim M^{d/2-\sigma/2}\|P_M f\|_{L^2}\|P_N g\|_{L^2}.$$

Since we are considering the case $M \sim N$, we have

$$\begin{aligned} M^{d/2-\sigma/2} &= M^{(d-1)/2} N^{-(\sigma-1)/2} N^{(\sigma-1)/2} M^{-(\sigma-1)/2} \\ &= M^{(d-1)/2} N^{-(\sigma-1)/2} (N/M)^{(\sigma-1)/2} \sim M^{(d-1)/2} N^{-(\sigma-1)/2}. \end{aligned}$$

This gives $(A.3.21)$ when $M \sim N$. Let us now consider the case $M \ll N$. By duality, it suffices to prove

$$\Big| \iint_{\mathbb{R}^d \times \mathbb{R}^d} G(|\xi|^\sigma + |\eta|^\sigma, \xi+\eta) \widehat{P_M f}(\xi) \widehat{P_N g}(\eta) d\xi d\eta \Big| \lesssim M^{(d-1)/2} N^{-(\sigma-1)/2} \|G\|_{L^2_\tau L^2_\xi} \|\hat{f}\|_{L^2_\xi} \|\hat{g}\|_{L^2_\xi}. \tag{A.3.22}$$

By renaming the components, we can assume that $|\xi| \sim |\xi_1| \sim M$ and $|\eta| \sim |\eta_1| \sim N$, where $\xi = (\xi_1, \underline{\xi}), \eta = (\eta_1, \underline{\eta})$ with $\underline{\xi}, \underline{\eta} \in \mathbb{R}^{d-1}$. We make a change of variables $\tau = |\xi|^\sigma + |\eta|^\sigma, \zeta = \xi + \eta$ and $d\tau d\zeta = J d\xi_1 d\eta$. A calculation shows that $J = |\sigma(|\xi|^{\sigma-2}\xi_1 \pm |\eta|^\sigma \eta_1| \sim |\eta|^{\sigma-1} \sim N^{\sigma-1}$. The Cauchy-Schwarz inequality with the fact $|\underline{\xi}| \lesssim M$ then shows

$$\begin{aligned} \text{LHS}(A.3.22) &= \Big| \iiint_{\mathbb{R}\times\mathbb{R}^{d-1}\times\mathbb{R}^d} G(\tau,\zeta) \widehat{P_M f}(\xi) \widehat{P_N g}(\eta) J^{-1} d\tau d\underline{\xi} d\zeta \Big| \\ &\leq \|G\|_{L^2_\tau L^2_\xi} \int_{\mathbb{R}^{d-1}} \Big(\iint_{\mathbb{R}\times\mathbb{R}^d} |\widehat{P_M f}(\xi)|^2 |\widehat{P_N g}(\eta)|^2 J^{-2} d\tau d\zeta \Big)^{1/2} d\underline{\xi} \\ &\leq \|G\|_{L^2_\tau L^2_\xi} M^{(d-1)/2} \Big(\iiint_{\mathbb{R}\times\mathbb{R}^{d-1}\times\mathbb{R}^d} |\widehat{P_M f}(\xi)|^2 |\widehat{P_N g}(\eta)|^2 J^{-2} d\tau d\underline{\xi} d\zeta \Big)^{1/2} \\ &\leq \|G\|_{L^2_\tau L^2_\xi} M^{(d-1)/2} \Big(\iiint_{\mathbb{R}\times\mathbb{R}^{d-1}\times\mathbb{R}^d} |\widehat{P_M f}(\xi)|^2 |\widehat{P_N g}(\eta)|^2 J^{-1} d\xi d\zeta \Big)^{1/2} \\ &\leq \|G\|_{L^2_\tau L^2_\xi} M^{(d-1)/2} N^{-(\sigma-1)/2} \|\widehat{P_M f}\|_{L^2_\xi} \|\widehat{P_N g}\|_{L^2_\xi}. \end{aligned}$$

This gives the desired estimate. □

We also have the following non-localized bilinear estimate for the higher-order Schrödinger equation.

Theorem A.3.4. *Let $\sigma > 2, d > \sigma/2$ and u, v be solutions to $(A.3.20)$ with initial data ψ, ϕ*

respectively. Then for any $\delta > 0$,

$$\|uv\|_{L^2(\mathbb{R},L^2)} \leq C(\delta)\Big(\|\psi\|_{\dot{H}^{(d-1)/2-\delta}} + \||\nabla|^{(d-1)/2-\delta}(i\partial_t - |\nabla|^\sigma)u\|_{L^{p'}(\mathbb{R},L^{q'})}\Big) \times \Big(\|\phi\|_{\dot{H}^{-(\sigma-1)/2+\delta}} + \||\nabla|^{-(\sigma-1)/2+\delta}(i\partial_t - |\nabla|^\sigma)v\|_{L^{a'}(\mathbb{R},L^{b'})}\Big), \quad \text{(A.3.23)}$$

for any Schrödinger admissible pairs (p,q) *and* (a,b) *satisfying* $\gamma_{p,q} = \gamma_{a,b} = 0, q, b < \infty$ *and* $p, a > 2$.

Proof. The proof is similar to the one of Theorem $A.3.2$. We only give a sketch of a proof. We firstly consider the homogeneous case, namely

$$\|uv\|_{L^2(\mathbb{R},L^2)} \lesssim \|\psi\|_{\dot{H}^{\gamma_1}}\|\phi\|_{\dot{H}^{\gamma_2}}. \quad \text{(A.3.24)}$$

Due to the scaling invariance, we see that the above estimate requires $\gamma_1 + \gamma_2 = d/2 - \sigma/2$. To see this, we consider $u_\lambda(t,x) = u(\lambda^{-\sigma}t, \lambda^{-1}x)$. The homogenous equation $(A.3.19)$ is invariant under this scaling. We have

$$\|u_\lambda v_\lambda\|^2_{L^2(\mathbb{R},L^2)} = \lambda^{\sigma+d}\|uv\|^2_{L^2(\mathbb{R},L^2)}.$$

Using the fact that $\|u_\lambda(0)\|^2_{\dot{H}^{\gamma_1}} = \lambda^{d-2\gamma_1}\|\psi\|^2_{\dot{H}^{\gamma_1}}$ and similarly for $v_\lambda(0)$, we get

$$\|u_\lambda v_\lambda\|^2_{L^2(\mathbb{R},L^2)} \lesssim \lambda^{\sigma-d+2(\gamma_1+\gamma_2)}\|u_\lambda(0)\|^2_{\dot{H}^{\gamma_1}}\|v_\lambda(0)\|^2_{\dot{H}^{\gamma_2}}.$$

We now prove $(A.3.24)$ with $\gamma_1 = (d-1)/2 - \delta$ and $\gamma_2 = -(\sigma-1)/2 + \delta$. The proof of this estimate follows by the same lines of those given in Theorem $A.3.2$. We now turn to the inhomogeneous case. Using the notations introduced in $(A.3.14)$ and $(A.3.15)$, the estimate $(A.3.23)$ is equivalent to

$$\|uv\|_{L^2(\mathbb{R},L^2)} \leq C(\delta)\|u\|_{S_{(d-1)/2-\delta,p,q}}\|v\|_{S_{-(\sigma-1)/2+\delta,a,b}}. \quad \text{(A.3.25)}$$

We will make use of the homogeneous bilinear estimate

$$\|e^{-it|\nabla|^\sigma}\psi e^{-it||\nabla|^\sigma}\phi\|_{L^2(\mathbb{R},L^2)} \leq C(\delta)\|\psi\|_{\dot{H}^{(d-1)/2-\delta}}\|\phi\|_{\dot{H}^{-(\sigma-1)/2+\delta}}. \quad \text{(A.3.26)}$$

Let (p,q) and (a,b) be Schrödinger admissible satisfying $\gamma_{p,q} = \gamma_{a,b} = 0, q, b < \infty$ and $p, a > 2$. Note that when $\gamma_{p,q} = 0$, Strichartz estimate shows that the map $L^2 \ni \psi \mapsto e^{-it|\nabla|^\sigma}\psi \in L^p(\mathbb{R}, L^q)$ is bounded together with its adjoint

$$L^{p'}(\mathbb{R}, L^{q'}) \ni F \mapsto \int_{\mathbb{R}} e^{is|\nabla|^\sigma}F(s)ds \in L^2.$$

Therefore, we can repeat the same argument as in Theorem $A.3.2$ to get a desired estimate. The proof is complete. □

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Abstract — This dissertation is devoted to the study of linear and nonlinear aspects of the Schrödinger-type equations

$$i\partial_t u + |\nabla|^\sigma u = F, \quad |\nabla| = \sqrt{-\Delta}, \quad \sigma \in (0, \infty).$$

When $\sigma = 2$, it is the well-known Schrödinger equation arising in many physical contexts such as quantum mechanics, nonlinear optics, quantum field theory and Hartree-Fock theory. When $\sigma \in (0,2)\backslash\{1\}$, it is the fractional Schrödinger equation, which was discovered by Laskin (see e.g. [Las00] and [Las02]) owing to the extension of the Feynman path integral, from the Brownian-like to Lévy-like quantum mechanical paths. This equation also appears in the water waves model (see e.g. [IP14] and [Ngu16]). When $\sigma = 1$, it is the half-wave equation which arises in water waves model (see [IP14]) and in gravitational collapse (see [ES07], [FL07]). When $\sigma = 4$, it is the fourth-order or biharmonic Schrödinger equation introduced by Karpman [Kar96] and by Karpman-Shagalov [KS00] taking into account the role of small fourth-order dispersion term in the propagation of intense laser beam in a bulk medium with Kerr nonlinearity.

This thesis is divided into two parts. The first part studies Strichartz estimates for Schrödinger-type equations on manifolds including the flat Euclidean space, compact manifolds without boundary and asymptotically Euclidean manifolds. These Strichartz estimates are known to be useful in the study of nonlinear dispersive equation at low regularity. The second part concerns the study of nonlinear aspects such as local well-posedness, global well-posedness below the energy space and blowup of rough solutions for nonlinear Schrödinger-type equations.

In Chapter 1, we discuss Strichartz estimates for Schrödinger-type equations with $\sigma \in (0, \infty)$ on the Euclidean space $\mathbb{R}^d$.

In Chapter 2, we derive Strichartz estimates for Schrödinger-type equations with $\sigma \in (0,\infty)\backslash\{1\}$ on $\mathbb{R}^d$ equipped with a smooth bounded metric g.

In Chapter 3, we make use of Strichartz estimates proved in Chapter 2 to show Strichartz estimates for Schrödinger-type equations with $\sigma \in (0,\infty)\backslash\{1\}$ on compact manifolds without boundary.

In Chapter 4, we prove global in time Strichartz estimates for Schrödinger-type equations with $\sigma \in (0,\infty)\backslash\{1\}$ on asymptotically Euclidean manifolds under the non-trapping condition.

In Chapter 5, we use Strichartz estimates given in Chapter 1 (among other things) to study the local well-posedness of the power-type nonlinear Schrödinger-type equations with $\sigma \in (0, \infty)$ posed on $\mathbb{R}^d$.

In Chapter 6, we study the global well-posedness for the defocusing mass-critical nonlinear fourth-order Schrödinger equation $\sigma = 4$ below the energy space. We will consider separately two cases $d = 4$ and $d \geq 5$ which respectively correspond to the algebraic and non-algebraic nonlinearity.

In Chapter 7, we study the blowup of rough solutions to the focusing mass-critical nonlinear fourth-order Schrödinger equation. As in Chapter 6, we also consider separately two cases $d = 4$ and $d \geq 5$.

Keywords: Nonlinear Schrödinger-type equations; Strichartz estimates; local well-posedness; global well-posedness; blowup; I-method; bilinear Strichartz estimates; Interaction Morawetz inequatlities; compact manifolds without boundary; asymptotically Euclidean manifolds.

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Université Toulouse III Paul Sabatier
Institut de Mathématiques de Toulouse UMR 5219
118 route de Narbonne, 31062 Toulouse Cedex 9, France