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Research Article

Regularization of $f(T)$ Gravity Theories and Local Lorentz Transformation

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We regularized the field equations of $f(T)$ gravity theories such that the effect of local Lorentz transformation (LLT), in the case of spherical symmetry, is removed. A "general tetrad field," with an arbitrary function of radial coordinate preserving spherical symmetry, is provided. We split that tetrad field into two matrices; the first represents a LLT, which contains an arbitrary function, and the second matrix represents a proper tetrad field which is a solution to the field equations of $f(T)$ gravitational theory (which are not invariant under LLT). This "general tetrad field" is then applied to the regularized field equations of $f(T)$. We show that the effect of the arbitrary function which is involved in the LLT invariably disappears.

1. Introduction

Amended gravitational theories have become very interesting due to their ability to provide an alternative framework for understanding the nature of dark energy. This is done through the modifications of the gravitational Lagrangian so as it renders an arbitrary function of its original argument, for instance, $f(R)$ instead of Ricci scalar R in the Einstein-Hilbert action [1–4].

Indeed there exists an equivalent construction of general relativity (GR) dependent on the concept of parallelism. The idea is initially done by Einstein who had tried to make a unification between electromagnetism and gravity fields using absolute parallelism spacetime [5, 6]. This goal was frustrated by the lack of a Schwarzschild solution. Much later, the theory of absolute parallelism gained much attention as a modification theory of gravity, referred to as "teleparallel equivalent of general relativity" (TEGR) (cf. [7–11]). The basic block in TEGR is the tetrad field. The tetrad field consists of fields of orthonormal bases which belong to the tangent space of the manifold. Note that the contravariant tetrad field, ${h_i}^\mu$, has sixteen components while the metric tensor has only ten. However, the tetrads are invariant under local Lorentz rotations.

The aim of the modification is to treat a more general manifold which comprises in addition to curvature a quantity called "torsion." The curvature tensor, consisting of a part without torsion plus a part with torsion, is vanishing identically. One can generally use either the torsion-free part or the torsion part to represent the gravitational field. The most suitable way is to deal with the covariant tetrad field, ${h^i}_\mu$, and the so-called Weitzenböck spacetime [12]. The tetrad field describes fields of orthonormal bases, which are related to the tangent spacetime of the manifold with spacetime coordinates x^μ. This tangent spacetime is Minkowski spacetime with the metric η_{ij} that can be defined at any given point on the manifold.

Recently, modifications of TEGR have been studied in the domain of cosmology [13–15]. This is known as $f(T)$ gravity and is built from a generalized Lagrangian [13–15]. In such a theory, the gravitational field is not characterized by curved spacetime but with torsion. Moreover, the field equations are only second order unlike the fourth order equations of the $f(R)$ theory.

Many of $f(T)$ gravity theories had been analyzed in [16–28]. It is found that $f(T)$ gravity theory is not dynamically equivalent to TEGR Lagrangian through conformal transformation [29]. Many observational constraints had been studied [30–33]. Large-scale structure in $f(T)$ gravity theory had been analyzed [34, 35]; perturbations in the area of cosmology in $f(T)$ gravity had been demonstrated [36–40]; Birkhoff's theorem in $f(T)$ gravity had been studied [41]. Stationary solutions having spherical symmetry have been derived for $f(T)$ theories [42–45]. Relativistic stars and the cosmic expansion derived in [46, 47].

Nevertheless, a major problem of $f(T)$ gravitational theories is that they are not locally Lorentz invariant and appear to harbour extra degrees of freedom.

The goal of this study is to regularize the field equations of $f(T)$ gravitational theory so that we remove the effect of LLT. We then apply a "general tetrad" field, which consists of two matrices: the first is a solution to the noninvariant field equation of $f(T)$ and the second matrix is a local Lorentz transformation, to the amended field equations, and shows that the effect of LLT disappears.

In Section 2, a brief survey of the $f(T)$ gravitational theory is presented.

In Section 3, a "general tetrad" field, having spherical symmetry with an arbitrary function of the radial coordinate r, is applied to the field equations of $f(T)$ which are not invariant under LLT. It is shown that the arbitrary function has an effect in this application.

In Section 4, we derive the field equations of $f(T)$ which are invariant under LLT. We then apply these amended field equations to a "general tetrad" field. We show that the effect of the arbitrary function invariably disappears.

Section 5 is devoted to discussion.

2. Brief Review of $f(T)$

In the Weitzenböck spacetime, the fundamental field variables describing gravity are a quadruplet of parallel vector fields [12] $h_i{}^\mu$, which we call the tetrad field. This is characterized by

$$D_\nu h_i{}^\mu = \partial_\nu h_i{}^\mu + \Gamma^\mu{}_{\lambda\nu} h_i{}^\lambda = 0, \tag{1}$$

where $\Gamma^\mu{}_{\lambda\nu}$ defines the nonsymmetric affine connection:

$$\Gamma^\lambda{}_{\mu\nu} \stackrel{\text{def}}{=} h_i{}^\lambda h^i{}_{\mu,\nu}, \tag{2}$$

with $h_{i\mu,\nu} = \partial_\nu h_{i\mu}$. (Spacetime indices $\mu, \nu, \ldots$ and SO(3, 1) indices $a, b, \ldots$ run from 0 to 3. Time and space indices are indicated by $\mu = 0, i$, and $a = (0), (i)$.)

Equation (1) leads to the metricity condition and the identical vanishment of the curvature tensor defined by $\Gamma^\lambda{}_{\mu\nu}$, given by (2). The metric tensor $g_{\mu\nu}$ is defined by

$$g_{\mu\nu} \stackrel{\text{def}}{=} \eta_{ij} h^i{}_\mu h^j{}_\nu, \tag{3}$$

with $\eta_{ij} = (-1, +1, +1, +1)$ that is metric of Minkowski spacetime. We note that, associated with any tetrad field $h_i{}^\mu$, there is a metric field defined uniquely by (3), while a given metric $g^{\mu\nu}$ does not determine the tetrad field completely, and any LLT of the tetrad $h_i{}^\mu$ leads to a new set of tetrad which also satisfies (3).

The torsion components and the contortion are defined as

$$\begin{aligned} T^\alpha{}_{\mu\nu} &\stackrel{\text{def}}{=} \Gamma^\alpha{}_{\nu\mu} - \Gamma^\alpha{}_{\mu\nu} = h_a{}^\alpha \left(\partial_\mu h^a{}_\nu - \partial_\nu h^a{}_\mu\right), \\ K^{\mu\nu}{}_\alpha &\stackrel{\text{def}}{=} -\frac{1}{2}\left(T^{\mu\nu}{}_\alpha - T^{\nu\mu}{}_\alpha - T_\alpha{}^{\mu\nu}\right), \end{aligned} \tag{4}$$

where the contortion equals the difference between Weitzenböck and Levi-Civita connection; that is, $K^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\nu\rho} - \left\{{}^{\mu}_{\nu\rho}\right\}$.

One can define the skew-symmetric tensor $S_\alpha{}^{\mu\nu}$ as

$$S_\alpha{}^{\mu\nu} \stackrel{\text{def}}{=} \frac{1}{2}\left(K^{\mu\nu}{}_\alpha + \delta^\mu_\alpha T^{\beta\nu}{}_\beta - \delta^\nu_\alpha T^{\beta\mu}{}_\beta\right), \tag{5}$$

which is skew symmetric in the last two indices. The torsion scalar is defined as

$$T \stackrel{\text{def}}{=} T^\alpha{}_{\mu\nu} S_\alpha{}^{\mu\nu}. \tag{6}$$

Similar to the $f(R)$ theory, one can define the action of $f(T)$ theory as

$$\begin{gathered} \mathscr{L}\left(h^a{}_\mu\right) = \int d^4x h\left[\frac{1}{16\pi} f(T)\right], \\ \text{where } h = \sqrt{-g} = \det\left(h^i{}_\mu\right) \end{gathered} \tag{7}$$

(assuming units in which $G = c = 1$). Considering the action in (7) as a function of the fields $h^i{}_\mu$ and putting the variation of the function with respect to the field $h^i{}_\mu$ to be vanishing, one can obtain the following equations of motion [16, 48]:

$$\begin{aligned} &S_\mu{}^{\rho\nu} T_{,\rho} f(T)_{TT} + \left[h^{-1} h^i{}_\mu \partial_\rho \left(h h_i{}^\alpha S_\alpha{}^{\rho\nu}\right) - T^\alpha{}_{\lambda\mu} S_\alpha{}^{\nu\lambda}\right] f(T)_T \\ &\quad - \frac{1}{4}\delta^\nu{}_\mu f(T) = -4\pi \mathscr{T}^\nu{}_\mu, \end{aligned} \tag{8}$$

where $T_{,\rho} = \partial T/\partial x^\rho$, $f(T)_T = \partial f(T)/\partial T$, $f(T)_{TT} = \partial^2 f(T)/\partial T^2$, and $\mathscr{T}^\nu{}_\mu$ is the energy momentum tensor.

In this study we are interested in studying the vacuum case of $f(T)$ gravity theory; that is, $\mathscr{T}^\nu{}_\mu = 0$.

3. Spherically Symmetric Solution in $f(T)$ Gravity Theory

Assume that the manifold is a stationary and spherically symmetric ($h^i{}_\mu$) having the form:

$$\left(h^{i}{}_{\mu}\right)=\begin{pmatrix} LA+HA_2 & LA_1+HA_3 & 0 & 0 \\ -\left(LA_2+HA\right)\sin\theta\cos\phi & -\left(LA_3+HA_1\right)\sin\theta\cos\phi & -r\cos\theta\cos\phi & r\sin\theta\sin\phi \\ -\left(LA_2+HA\right)\sin\theta\sin\phi & -\left(LA_3+HA_1\right)\sin\theta\sin\phi & -r\cos\theta\sin\phi & -r\sin\theta\cos\phi \\ -\left(LA_2+HA\right)\cos\theta & -\left(LA_3+HA_1\right)\cos\theta & r\sin\theta & 0 \end{pmatrix}, \tag{9}$$

where $A(r)$, $A_1(r)$, $A_2(r)$, and $A_3(r)$ are four unknown functions of the radial coordinate r, $L = L(r) = \sqrt{H(r)^2+1}$, and $H = H(r)$ is an arbitrary function. Tetrad fields (9) transform as

$$\left(h^{i}{}_{\mu}\right)=\left(\Lambda^{i}{}_{j}\right)\left(h^{j}{}_{\mu}\right)_1, \tag{10}$$

where $(h^{j}{}_{\mu})_1$ is given by

$$\left(h^{j}{}_{\mu}\right)_1=\begin{pmatrix} A(r) & A_1(r) & 0 & 0 \\ A_2(r)\sin\theta\cos\phi & A_3(r)\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ A_2(r)\sin\theta\sin\phi & A_3(r)\sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ A_2(r)\cos\theta & A_3(r)\cos\theta & -r\sin\theta & 0 \end{pmatrix}. \tag{11}$$

The tetrad field (11) has been studied [43] and it has been shown that the solution to the $f(T)$ gravitational theory has the form:

$$\begin{aligned} A &= 1-\frac{M}{r}, \\ A_1 &= \frac{M}{r(1-M/r)}, \\ A_2 &= \frac{M}{r}, \\ A_3 &= \frac{1-M/r}{1-2M/r}, \end{aligned} \tag{12}$$

where M is the gravitational mass. Equation (12) is an exact vacuum solution to field equations of $f(T)$ gravitational theory provided that

$$\begin{aligned} f(0) &= 0, \\ f_T(0) &\neq 0, \\ f_{TT} &\neq 0. \end{aligned} \tag{13}$$

The LLT $(\Lambda^{i}{}_{j})$ has the form:

$$\left(\Lambda^{i}{}_{j}\right)=\begin{pmatrix} L & H\sin\theta\cos\phi & H\sin\theta\sin\phi & H\cos\theta \\ -H\sin\theta\cos\phi & 1+H_1\sin\theta^2\cos\phi^2 & H_1\sin\theta^2\sin\phi\cos\phi & H_1\sin\theta\cos\theta\cos\phi \\ -H\sin\theta\sin\phi & H_1\sin\theta^2\sin\phi\cos\phi & 1+H_1\sin\theta^2\sin\phi^2 & H_1\sin\theta\cos\theta\sin\phi \\ -H\cos\theta & H_1\sin\theta\cos\theta\cos\phi & H_1\sin\theta\cos\theta\sin\phi & 1+H_1\cos\theta^2 \end{pmatrix}, \tag{14}$$

where $H_1 = (L-1)$.

From the general spherically symmetric local Lorentz transformation (14), one can generate the previous spherically symmetric solution [36].

Using (12) in (9), one can obtain $h = \det(h^{\mu}{}_{a}) = r^2\sin\theta$ and, with the use of (4) and (5), we obtain the torsion scalar and its derivatives in terms of r:

$$T(r)=\frac{4\left(\left[1-MH'\right]L+HH'\left[M-r\right]-L^2\right)}{r^2L},$$

$$\text{where } H'=\frac{\partial H(r)}{\partial r},$$

$$\begin{aligned} T'(r) &= \frac{\partial T(r)}{\partial r} \\ &= -\left(4\left\{rL^2H''\left[(r-M)H+ML\right]-r(M-r)H'^2\right.\right. \\ &\quad \left.\left.-\ 2MH'L^2\left[L-H\right]+2L^3(1-L)\right\}\right) \\ &\quad \cdot\left(r^3L^3\right)^{-1}. \end{aligned} \tag{15}$$

The field equations (7) have the form

$$
\begin{aligned}
4\pi\mathcal{T}^0_{\ 0} &= -\frac{f_{TT}T'\left[M(2+H)+L(r-M)-r\right]}{r^2} \\
&\quad + \frac{f_T}{r^2L}\left[L\left(1-MH'\right)+HH'(M-r)-L^2\right]+\frac{f}{4}, \\
4\pi\mathcal{T}^0_{\ 1} &= \frac{4f_{TT}T'\left[(M-r)H-ML\right]}{r(r-2M)}, \\
4\pi\mathcal{T}^1_{\ 1} &= \frac{f_T\left\{\left(1-MH'\right)L+HH'(M-r)-L^2\right\}}{r^2L}+\frac{f}{4}, \\
4\pi\mathcal{T}^2_{\ 2} &= 4\pi\mathcal{T}^3_{\ 3} \\
&= -\frac{f_{TT}T'\left\{M(1+H)-r+L(r-M)\right\}}{2r^2} \\
&\quad + \frac{f_T\left\{\left(1-MH'\right)L+(M-r)HH'-L^2\right\}}{r^2L}+\frac{f}{4}.
\end{aligned} \tag{16}
$$

Equations (15)-(16) show that the field equations of $f(T)$ are effected by the inertia which is located in the LLT given by (14). This effect is related to the noninvariance of the field equations of $f(T)$ gravitational theory under LLT.

4. Regularization of $f(T)$ Gravitational Theory under LLT

The tetrad field $(h^i{}_\mu)$ transforms under LLT as:

$$
\left(\overline{h}^i{}_\mu\right)=\left(\Lambda^i{}_j(x)\right)\left(h^j{}_\mu\right). \tag{17}
$$

The derivatives of $(\overline{h}^i{}_\mu)$ have the form:

$$
\begin{aligned}
\frac{\partial\left(\overline{h}^i{}_\mu\right)}{\partial x^\nu} &= \left(\overline{h}^i{}_{\mu,\nu}\right) \\
&= \left(\Lambda^i{}_j(x)\right)_{,\nu}\left(h^j{}_\mu\right)+\left(\Lambda^i{}_j(x)\right)\left(h^j{}_{\mu,\nu}\right).
\end{aligned} \tag{18}
$$

The nonsymmetric affine connection constructed from the tetrad field $(\overline{h}^i{}_\mu)$ has the form

$$
\overline{\Gamma}^\mu{}_{\nu\rho}=\eta^{ij}\left(\overline{h}_i{}^\mu\right)\left(\overline{h}_{j\nu,\rho}\right). \tag{19}
$$

Using (17) and (18) in (19) one gets

$$
\overline{\Gamma}^\mu{}_{\nu\rho}=\Gamma^\mu{}_{\nu\rho}+\left(\Lambda_j{}^i(x)\right)\left(h_i{}^\mu\right)\left(\Lambda^j{}_k(x)\right)_{,\rho}\left(h^k{}_\nu\right), \tag{20}
$$

where $\Gamma^\mu{}_{\nu\rho}$ is the nonsymmetric affine connection constructed from the tetrad field $(h_i{}^\mu)$ which is assumed to satisfy the field equation of $f(T)$. Therefore, for $\overline{\Gamma}^\mu{}_{\nu\rho}$ (which is effected by LLT) to be identical with $\Gamma^\mu{}_{\nu\rho}$ (which is assumed to satisfies the field equation of $f(T)$) we must have

$$
\begin{aligned}
&\left(\overline{\Gamma}^\mu{}_{\nu\rho}\right)_{\text{Regularized}} \\
&\quad=\eta^{ij}\left(\overline{h}_i{}^\mu\right)\left(\overline{h}_{j\nu,\rho}\right) \\
&\quad\quad-\left(\Lambda_j{}^i(x)\right)\left(h_i{}^\mu\right)\left(\Lambda^j{}_k(x)\right)_{,\rho}\left(h^k{}_\nu\right).
\end{aligned} \tag{21}
$$

Equation (21) means that the affine connection is invariant under LLT in the linear case; that is, $f(T)=T$, which means that the extra degrees of freedom, six ones, are controlled. Also (21) breaks the restriction of teleparallelism. From (21) we have

$$
\left(\overline{\Gamma}^\mu_{\nu\rho}\right)_{\text{Regularized}}\equiv\Gamma^\mu_{\nu\rho}. \tag{22}
$$

Therefore, if $\Gamma^\mu_{\nu\rho}$ satisfies the field equations of $f(T)$ then $(\overline{\Gamma}^\mu_{\nu\rho})_{\text{Regularized}}$ need not to be a solution to the field equations of $f(T)$ given by (8). The main reason for this is the second term in (8), that is, $h^{-1}h^i{}_\mu\partial_\rho(hh_i{}^\alpha S_\alpha{}^{\rho\nu})$. This term depends on the choice of the tetrad field. Using (21), the torsion, the contortion, and $\overline{S}_\alpha^{\mu\nu}$ tensors have the form

$$
\begin{aligned}
&\left(\overline{T}^\mu{}_{\nu\rho}\right)_{\text{Regularized}} \\
&\quad=\overline{T}^\mu_{\nu\rho}+\left(h_i{}^\mu\right)\left(\Lambda^i{}_j(x)\right) \\
&\quad\quad\cdot\left\{\left(\Lambda^j{}_k(x)\right)_{,\nu}\left(h^k{}_\rho\right)-\left(\Lambda^j{}_k(x)\right)_{,\rho}\left(h^k{}_\nu\right)\right\}, \\
&\left(\overline{K}_\alpha{}^{\mu\nu}\right)_{\text{Regularized}} \\
&\quad=-\frac{1}{2}\left[\left(\overline{T}_\alpha{}^{\mu\nu}\right)_{\text{Regularized}}-\left(\overline{T}_\alpha{}^{\nu\mu}\right)_{\text{Regularized}}\right. \\
&\quad\quad\left.-\left(\overline{T}_\alpha{}^{\mu\nu}\right)_{\text{Regularized}}\right], \\
&\left(\overline{S}_\alpha{}^{\mu\nu}\right)_{\text{Regularized}} \\
&\quad=\frac{1}{2}\left[\left(\overline{K}_\alpha{}^{\mu\nu}\right)_{\text{Regularized}}+\delta^\mu_\alpha\left(\overline{T}_\beta{}^{\beta\nu}\right)_{\text{Regularized}}\right. \\
&\quad\quad\left.-\delta^\nu_\alpha\left(\overline{T}_\beta{}^{\beta\mu}\right)_{\text{Regularized}}\right].
\end{aligned} \tag{23}
$$

Equation (23) shows that the torsion tensor (and all tensors constructed from it) is invariant under LLT. Using (23) in the field equations of $f(T)$, one can easily see that the first, third, and fourth terms of the field equations (8) will be invariant under LLT, but the second term, $\partial_\rho(h\overline{h}_a{}^\alpha\overline{S}_\alpha{}^{\rho\nu})$, which depends on the derivative, must take the following form:

$$
\begin{aligned}
&\left(\partial_\rho\left[h\overline{h}_a{}^\alpha\overline{S}_\alpha{}^{\rho\nu}\right]\right)_{\text{Regularized}} \\
&\quad=\partial_\rho\left(h\overline{h}_a{}^\alpha\overline{S}_\alpha{}^{\rho\nu}\right)-h\left(\Lambda_a{}^b(x)\right)_{,\rho}\left(\overline{h}_b{}^\alpha\right)\overline{S}_\alpha{}^{\rho\nu}.
\end{aligned} \tag{24}
$$

Using (23) and (24) the invariance field equations of $f(T)$ gravitational theory under LLT take the form:

$$\begin{aligned}&\left(\overline{S}_{\mu}{}^{\rho\nu}\right)_{\text{Regularized}}\left(\overline{T}_{,\rho}\right)_{\text{Regularized}} f\left(\overline{T}\right)_{\overline{TT}}\\&\quad+\left[h^{-1}\overline{h}^{a}{}_{\mu}\left(\partial_{\rho}\left[h\overline{h}_{a}{}^{\alpha}\overline{S}_{\alpha}{}^{\rho\nu}\right]\right)_{\text{Regularized}}\right.\\&\qquad\left.-\left(\overline{T}^{\alpha}{}_{\lambda\mu}\right)_{\text{Regularized}}\left(\overline{S}_{\alpha}{}^{\nu\lambda}\right)_{\text{Regularized}}\right] f\left(\overline{T}\right)_{\overline{T}}\\&\quad+\frac{1}{4}\delta^{\nu}{}_{\mu} f\left(\overline{T}\right)=4\pi\mathcal{T}^{\nu}{}_{\mu},\end{aligned}\tag{25}$$

where $(\overline{T})_{\text{Regularized}}=(\overline{T}^{\alpha}{}_{\mu\nu}\overline{S}_{\alpha}{}^{\mu\nu})_{\text{Regularized}}$.

Let us check if (25) when applied to the tetrad field (9) will indeed remove the effect of the inertia which appears in the LLT (14). Calculating the necessary components of the modified field equations (25) we get a vanishing quantity of the left hand side. This means that the tetrad field (9) is a solution to the $f(T)$ field equations (25) which is invariant under LLT. (The details of calculations of the nonvanishing components of the necessary quantities of the modified field equations (25) are given in the Appendix.)

5. Discussion and Conclusion

In this paper we have addressed the problem of the invariance of the field equations of $f(T)$ gravitational theory under LLT. We first used a “general tetrad field” which contained five unknown functions in r. This tetrad field has been studied [45] and a special solution has been obtained. This solution is characterized by the fact that its scalar torsion vanishes.

We rewrite this tetrad field, “general tetrad field,” into two matrices. The first matrix represents a tetrad field containing four unknown functions in r. This tetrad field has been studied before in [43] and it has been shown that it represents an exact solution within the framework of $f(T)$ gravitational theories. The second matrix represents a LLT that satisfies

$$\left(\Lambda_{i}{}^{j}\right)\eta_{jk}\left(\Lambda^{k}{}_{m}\right)=\eta_{im}\tag{26}$$

and contains an arbitrary $H(r)$.

We have applied the field equations of $f(T)$ which are not invariant under LLT to the general tetrad field. We have obtained a set of nonlinear differential equations which depend on the $H(r)$. Therefore, we have regularized the field equations of $f(T)$ gravitational theory such that it has become invariant under LLT. Then, we have applied these invariant field equations to the generalized tetrad field. We have shown that this general tetrad field is an exact solution to the regularized field equations of $f(T)$ gravitational theory.

The problem of the noninvariance of the field equations of $f(T)$ under LLT is not a trivial task to tackle. The main reason for this is the following: we have the following known relation between the Ricci scalar tensor and the scalar torsion [35]:

$$R=-T-2\nabla^{\mu}T^{\rho}{}_{\mu\rho}=T-\frac{2}{h}\partial^{\mu}\left(hT^{\rho}{}_{\mu\rho}\right).\tag{27}$$

Last term in the right hand side of (27) is a total divergence term which has no effect on the field equations of TEGR; that is, $\mathcal{L}(h^{a}{}_{\mu})=\int d^{4}xh[(1/16\pi)T]$; from this fact comes the well-known name of teleparallel equivalent of general relativity. However, this term, divergence term, is the main reason that makes the field equations of $f(T)$ noninvariance under LLT. Let us explain this for some specific form of $f(T)$. If

$$\begin{aligned}f(R)&=R+R^{2}\\&\equiv\left[-T-2\nabla^{\mu}T^{\rho}{}_{\mu\rho}\right]+\left[-T-2\nabla^{\mu}T^{\rho}{}_{\mu\rho}\right]^{2}\\&=-T-2\nabla^{\mu}T^{\rho}{}_{\mu\rho}+T^{2}+4\left[\nabla^{\mu}T^{\rho}{}_{\mu\rho}\right]^{2}\\&\quad+4T\nabla^{\mu}T^{\rho}{}_{\mu\rho},\end{aligned}\tag{28}$$

last term in the right hand side of (28) is not a total derivative term. This term is responsible to make the quadratic form of $f(T)$ not invariant under LLT. Same discussion can be applied to the general form of $f(R)$ and $f(T)$ which shows in general a difference between the $f(R)$ and $f(T)$ gravitational theories that makes the field equation of $f(R)$ to be of fourth order and invariant under LLT while $f(T)$ is of second order and not invariant under LLT. Here in this study we tackle the problem of the invariance of the field equations of $f(T)$ under LLT for specific symmetry, spherical symmetry. Although the method achieved in this study can be done for any symmetry, however, we do not have the general local Lorentz transformation that has axial symmetry or homogenous and isotropic. This will be study elsewhere.

Appendix

Calculations of the Nonvanishing Components of the Necessary Quantities of the Modified Field Equations (25)

The nonvanishing components of $(\Lambda_{a}{}^{b}(x))_{,\rho}$ are as follows:

$$\begin{aligned}\left(\Lambda^{0}{}_{0,r}\right)&=\frac{H\left(\Lambda^{0}{}_{1,r}\right)}{L\sin\theta\cos\phi}=\frac{H'\left(\Lambda^{0}{}_{1,\theta}\right)}{L\cos\theta\cos\phi}\\&=-\frac{H'\left(\Lambda^{0}{}_{1,\phi}\right)}{L\sin\theta\sin\phi}=\frac{H\left(\Lambda^{0}{}_{2,r}\right)}{L\sin\theta\sin\phi}\\&=-\frac{H'\left(\Lambda^{0}{}_{2,\theta}\right)}{L\cos\theta\sin\phi}=\frac{H'\left(\Lambda^{0}{}_{2,\phi}\right)}{L\sin\theta\cos\phi}\\&=\frac{H\left(\Lambda^{0}{}_{3,r}\right)}{L\cos\theta}=-\frac{H'\left(\Lambda^{0}{}_{3,\theta}\right)}{L\sin\theta}\\&=-\frac{H\left(\Lambda^{1}{}_{0,r}\right)}{L\sin\theta\cos\phi}=-\frac{H'\left(\Lambda^{1}{}_{0,\theta}\right)}{L\cos\theta\cos\phi}\\&=\frac{H'\left(\Lambda^{1}{}_{0,\phi}\right)}{L\sin\theta\sin\phi}=-\frac{\left(\Lambda^{1}{}_{1,r}\right)}{\sin^{2}\theta\cos^{2}\phi}\end{aligned}$$

$$\begin{aligned}
&= -\frac{\left(\Lambda^{1}{}_{2,r}\right)}{\sin^2\theta\cos\phi\sin\phi} = -\frac{H\left(\Lambda^{2}{}_{0,r}\right)}{L\sin\theta\sin\phi} \\
&= -\frac{H'\left(\Lambda^{2}{}_{0,2}\right)}{L\cos\theta\sin\phi} = -\frac{H'\left(\Lambda^{2}{}_{0,\phi}\right)}{L\sin\theta\cos\phi} \\
&= -\frac{\left(\Lambda^{2}{}_{3,r}\right)}{\sin\theta\cos\theta\sin\phi} = -\frac{\left(\Lambda^{2}{}_{1,r}\right)}{\sin^2\theta\cos\phi\sin\phi} \\
&= -\frac{\left(\Lambda^{3}{}_{1,r}\right)}{\sin\theta\cos\theta\cos\phi} = -\frac{\left(\Lambda^{1}{}_{3,r}\right)}{\sin\theta\cos\theta\cos\phi} \\
&= -\frac{\left(\Lambda^{3}{}_{2,r}\right)}{\sin\theta\cos\theta\sin\phi} = -\frac{\left(\Lambda^{3}{}_{3,r}\right)}{\cos^2\theta} \\
&= -\frac{H\left(\Lambda^{3}{}_{0,r}\right)}{L\cos\theta} = -\frac{H'\left(\Lambda^{3}{}_{0,\theta}\right)}{L\sin\theta} = \frac{HH'}{L}, \\
\left(\Lambda^{1}{}_{1,\theta}\right) &= -\cot\theta\cot\phi\left(\Lambda^{1}{}_{1,\phi}\right) = -\cot\theta\cot\phi\left(\Lambda^{2}{}_{2,\phi}\right) \\
&= \cot\phi\left(\Lambda^{1}{}_{2,\theta}\right) = \cot\phi\left(\Lambda^{2}{}_{1,\theta}\right) \\
&= 2\sec 2\phi\cos^2\phi\cot\theta\left(\Lambda^{1}{}_{2,\phi}\right) \\
&= 2\sec 2\phi\cos^2\phi\cot\theta\left(\Lambda^{2}{}_{1,\phi}\right) \\
&= \tan 2\theta\cos\phi\left(\Lambda^{1}{}_{3,\theta}\right) \\
&= -2\cos^2\phi\csc\phi\left(\Lambda^{1}{}_{3,\phi}\right) \\
&= 2\cot^2\phi\cot\theta\left(\Lambda^{2}{}_{2,r}\right) \\
&= \cot^2\phi\left(\Lambda^{2}{}_{2,\theta}\right) = \tan 2\theta\cos^2\phi\csc\phi\left(\Lambda^{2}{}_{3,\theta}\right) \\
&= 2\cos\phi\left(\Lambda^{2}{}_{3,\phi}\right) = \tan 2\theta\cos\phi\left(\Lambda^{3}{}_{1,\theta}\right) \\
&= -2\cos^2\phi\csc\phi\left(\Lambda^{3}{}_{1,\phi}\right) \\
&= 2\tan 2\theta\cos^2\phi\csc\phi\left(\Lambda^{3}{}_{2,\theta}\right) \\
&= 2\cos\phi\left(\Lambda^{3}{}_{2,\phi}\right) = -\cos^2\phi\left(\Lambda^{3}{}_{3,\theta}\right) \\
&= -\sin 2\theta\cos^2\phi\,(L-1)\,.
\end{aligned} \tag{A.1}$$

The nonvanishing components of the nonsymmetric affine connection $(\overline{\Gamma}^{\mu}_{\nu\rho})_{\text{Regularized}}$ are as follows:

$$\begin{aligned}
\left(\overline{\Gamma}^{0}{}_{01}\right)_{\text{Regularized}} &= -\frac{(r-2M)}{r}\overline{\Gamma}^{0}{}_{11} = \frac{1}{r^2}\left(\overline{\Gamma}^{0}{}_{22}\right)_{\text{Regularized}} \\
&= \frac{1}{r^2\sin^2\theta}\left(\overline{\Gamma}^{0}{}_{33}\right)_{\text{Regularized}} = \frac{M}{r(r-2M)}. \\
\left(\overline{\Gamma}^{1}{}_{01}\right)_{\text{Regularized}} &= -\left(\overline{\Gamma}^{2}{}_{02}\right)_{\text{Regularized}} \\
&= -\left(\overline{\Gamma}^{3}{}_{03}\right)_{\text{Regularized}} = -\left(\overline{\Gamma}^{3}{}_{03}\right)_{\text{Regularized}} \\
&= \frac{(r-2M)\left(\overline{\Gamma}^{1}{}_{11}\right)_{\text{Regularized}}}{r} \\
&= \frac{\left(\overline{\Gamma}^{1}{}_{22}\right)_{\text{Regularized}}}{r^2(r-M)} \\
&= \frac{M\left(\overline{\Gamma}^{1}{}_{33}\right)_{\text{Regularized}}}{r^2\sin^2\theta(r-M)} = -\frac{M}{r^2}, \\
\left(\overline{\Gamma}^{2}{}_{12}\right)_{\text{Regularized}} &= \left(\overline{\Gamma}^{3}{}_{13}\right)_{\text{Regularized}} = \frac{r-M}{r(r-2M)}, \\
\left(\overline{\Gamma}^{2}{}_{21}\right)_{\text{Regularized}} &= \left(\overline{\Gamma}^{3}{}_{31}\right)_{\text{Regularized}} = \frac{1}{r}, \\
\left(\overline{\Gamma}^{2}{}_{33}\right)_{\text{Regularized}} &= -\sin\theta\cos\theta, \\
\left(\overline{\Gamma}^{3}{}_{23}\right)_{\text{Regularized}} &= \left(\overline{\Gamma}^{3}{}_{32}\right)_{\text{Regularized}} = \cot\theta.
\end{aligned} \tag{A.2}$$

The nonvanishing components of the torsion $(\overline{T}^{\mu}{}_{\nu\rho})_{\text{Regularized}}$ are as follows:

$$\begin{aligned}
\left(\overline{T}^{1}{}_{10}\right)_{\text{Regularized}} &= -\left(\overline{T}^{1}{}_{01}\right)_{\text{Regularized}} \\
&= -\left(\overline{T}^{2}{}_{20}\right)_{\text{Regularized}} = \left(\overline{T}^{2}{}_{02}\right)_{\text{Regularized}} \\
&= -\left(\overline{T}^{3}{}_{30}\right)_{\text{Regularized}} \\
&= \left(\overline{T}^{3}{}_{03}\right)_{\text{Regularized}} = -\frac{M}{r^2}, \\
\left(\overline{T}^{0}{}_{01}\right)_{\text{Regularized}} &= -\left(\overline{T}^{0}{}_{10}\right)_{\text{Regularized}}\left(\overline{T}^{2}{}_{12}\right)_{\text{Regularized}} \\
&= -\left(\overline{T}^{2}{}_{21}\right)_{\text{Regularized}} \\
&= \left(\overline{T}^{3}{}_{13}\right)_{\text{Regularized}} = -\left(\overline{T}^{3}{}_{31}\right)_{\text{Regularized}} \\
&= -\frac{M}{r(r-2M)}.
\end{aligned} \tag{A.3}$$

The nonvanishing components of $\overline{S}_{\alpha}{}^{\rho\nu}$ are as follows:

$$\begin{aligned}
\left(\overline{S}_{0}{}^{01}\right)_{\text{Regularized}} &= -\left(\overline{S}_{0}{}^{10}\right)_{\text{Regularized}} = -\frac{M}{r^2}, \\
\left(\overline{S}_{1}{}^{10}\right)_{\text{Regularized}} &= -\left(\overline{S}_{1}{}^{10}\right)_{\text{Regularized}} = -\frac{M}{r(r-2M)}.
\end{aligned} \tag{A.4}$$

The nonvanishing components of $\partial_\lambda[h\overline{h}_a{}^{\alpha}\overline{S}_\alpha{}^{\rho\nu}] = N_a{}^{\rho\nu}{}_{,\lambda}$ are as follows:

$$\begin{aligned}
N_0{}^{01}{}_{,\theta} &= -N_0{}^{10}{}_{,\theta} = M(L-H)\cos\theta, \\
N_1{}^{10}{}_{,\theta} &= -N_1{}^{01}{}_{,\theta} = \cot\phi N_2{}^{10}{}_{,\theta} = -\cot\phi N_2{}^{01}{}_{,\theta} \\
&= M(L+1-H)\sin\theta\cos\theta\cos\phi, \\
N_3{}^{10}{}_{,\theta} &= -N_3{}^{01}{}_{,\theta} = M\left(L-\tan^2\theta-H\right)\cos^2\theta, \\
N_1{}^{10}{}_{,\phi} &= -N_1{}^{01}{}_{,\phi} = -\tan\phi N_2{}^{10}{}_{,\phi} = \tan\phi N_2{}^{01}{}_{,\phi} \\
&= -M\sin^2\theta\sin\phi.
\end{aligned} \tag{A.5}$$

Using (A.3) in (24) we get vanishing components of $(\partial_\rho[h\overline{h}_a{}^{\alpha}\overline{S}_\alpha{}^{\rho\nu}])_{\text{Regularized}}()$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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