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Research Article

A Morphism Double Category and Monoidal Structure

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We provide a recipe for "fattening" a category that leads to the construction of a double category. Motivated by an example where the underlying category has vector spaces as objects, we show how a monoidal category leads to a law of composition, satisfying certain coherence properties, on the object set of the fattened category.

1. Introduction and Geometric Background

The interaction of point particles through a gauge field can be encoded by means of Feynman diagrams, with nodes representing particles and directed edges carrying an element of the gauge group representing parallel transport along that edge. If the point particles are replaced by extended one-dimensional string-like objects, then the interaction between such objects can be encoded through diagrams of the form

$$\begin{array}{ccc} x_1 & \xrightarrow{f_1} & y_1 \\ {\scriptstyle g_1}\downarrow & \overset{h}{\Downarrow} & \downarrow{\scriptstyle g_2} \\ x_2 & \xrightarrow[f_2]{} & y_2 \end{array} \quad (1)$$

where the labels f_i and g_i describe classical parallel transport and h, which may take values in a different gauge group, describes parallel transport over a space of paths.

We will now give a rapid account of some of the geometric background. We refer to our previous work [1] for further details. This material is not logically necessary for reading the rest of this paper but is presented to indicate the context and motivation for some of the ideas of this paper.

Consider a principal G-bundle $\pi : P \to M$, where M is a smooth finite dimensional manifold and G a Lie group, and a connection $\overline{A}$ on this bundle. In the physical context, M may be spacetime, and $\overline{A}$ describes a gauge field. Now consider the set $\mathcal{P}M$ of piecewise smooth paths on M, equipped with a suitable smooth structure. Then, the space $\mathcal{P}_{\overline{A}}P$ of $\overline{A}$-horizontal paths in P forms a principal G-bundle over $\mathcal{P}M$. We also use a second gauge group H (that governs parallel transport over path space), which is a Lie group along with a fixed smooth homomorphism $\tau : H \to G$ and a smooth map

$$G \times H \longrightarrow H : (g, h) \longmapsto \alpha(g) h \quad (2)$$

such that each $\alpha(g)$ is an automorphism of H, such that

$$\tau(\alpha(g) h) = g\tau(h) g^{-1}, \qquad \alpha(\tau(h)) h' = hh'h^{-1} \quad (3)$$

for all $g \in G$ and $h, h' \in H$. We denote the derivative $\tau'(e)$ by τ, viewed as a map $LH \to LG$, and denote $\alpha'(e)$ by α, to avoid notational complexity. Given also a second connection form A on P and a smooth α-equivariant vertical LH-valued 2-form B on P, it is possible to construct a connection form $\omega_{(A,B)}$ on the bundle $\mathcal{P}_{\overline{A}}P$

$$\omega_{(A,B)} = ev_1^* A + \tau(Z), \quad (4)$$

where Z is the LH-valued 1-form on $\mathcal{P}_{\overline{A}}P$ specified by

$$Z = \int_0^1 B, \tag{5}$$

which is a Chen integral.

Consider a path of paths in P specified through a smooth map

$$\tilde{\Gamma} : [0,1]^2 \longrightarrow P : (t,s) \longmapsto \tilde{\Gamma}(t,s) = \tilde{\Gamma}_s(t) = \tilde{\Gamma}^t(s), \tag{6}$$

where each $\tilde{\Gamma}_s$ is $\overline{A}$-horizontal and the path $s \mapsto \tilde{\Gamma}(0,s)$ is A-horizontal. Let $\Gamma = \pi \circ \tilde{\Gamma}$. The *bi-holonomy* $g(t,s) \in G$ is specified as follows: parallel translate $\tilde{\Gamma}(0,0)$ along $\Gamma_0 \mid [0,t]$ by $\overline{A}$, then up the path $\Gamma^t \mid [0,s]$ by A, back along Γ_s-reversed by $\overline{A}$ and then down $\Gamma^0 \mid [0,s]$ by A, then the resulting point is

$$\tilde{\Gamma}(0,0)\, g(t,s). \tag{7}$$

The following result is proved in [1].

Theorem 1. *Suppose that*

$$\tilde{\Gamma} : [0,1]^2 \longrightarrow P : (t,s) \longmapsto \tilde{\Gamma}(t,s) = \tilde{\Gamma}_s(t) = \tilde{\Gamma}^t(s) \tag{8}$$

is smooth, with each $\tilde{\Gamma}_s$ being $\overline{A}$-horizontal and the path $s \mapsto \tilde{\Gamma}(0,s)$ being A-horizontal. Then, the parallel translate of $\tilde{\Gamma}_0$ by the connection $\omega_{(A,B)}$ along the path $[0,s] \rightarrow \mathcal{P}M : u \mapsto \Gamma_u$, where $\Gamma = \pi \circ \tilde{\Gamma}$, results in

$$\tilde{\Gamma}_s g(1,s)\, \tau(h_0(s)), \tag{9}$$

with $g(1,s)$ being the "bi-holonomy" specified as in (7), *and $s \mapsto h_0(s) \in H$ solving the differential equation*

$$\frac{dh_0(s)}{ds} h_0(s)^{-1} = -\alpha\left(g(1,s)^{-1}\right) \times \int_0^1 B\left(\partial_t \tilde{\Gamma}(t,s), \partial_s \tilde{\Gamma}(t,s)\right) dt \tag{10}$$

with initial condition $h_0(0)$ being the identity in H.

Consider the category $\mathbf{C}_0$ whose objects are fibers of a given vector bundle E over M and whose arrows are piecewise smooth paths in M (up to "backtrack equivalence"; for more on this notion see [2]) along with parallel transport operators, by a connection $\overline{A}$, along such paths. Note that all arrows are invertible. In Figure 1, E_{p_1} is the vector space which is the fiber over the corresponding point p_1. For the path c_1, there is a parallel transport operator $f_1 : E_{p_1} \rightarrow E_{q_1}$. Next, if c_2 is a path from the base of the fiber E_{p_2} to the base of E_{q_2}, then there is a corresponding parallel transport operator $f_2 : E_{p_2} \rightarrow E_{q_2}$.

A "higher" morphism $c_1 \rightarrow c_2$ is obtained from any suitably smooth path of paths, starting with the initial path c_1 and ending with c_2 (again backtracks need to be erased). Using the connection $\overline{A}$, this produces parallel transport operators and paths $E_{p_1} \rightarrow E_{p_2}$ and $E_{q_1} \rightarrow E_{q_2}$. Moreover, another connection A and 2-form B, along with a path of paths lead to a linear map $\mathrm{Mor}_l(E_{p_1}, E_{q_1}) \rightarrow \mathrm{Mor}_l(E_{p_2}, E_{q_2})$, where $\mathrm{Mor}_l(E,F)$ is the vector space of all linear maps $E \rightarrow F$. We view this, in a "first approximation," as a morphism from the object $\mathrm{Mor}(E_{p_1}, E_{q_1})$ to the object $\mathrm{Mor}(E_{p_2}, E_{q_2})$ (say, mapping all paths from p_1 to q_1 to the path c_2). In this paper, we will not develop this framework in full detail (that would build on the theory from our earlier work [1]) but focus on more algebraic aspects and other purely algebraic issues (such as monoidal structures).

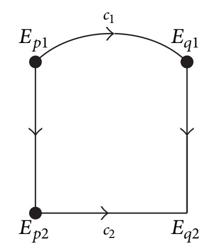


Figure 1: Paths and fibers.

Instead of vector bundles, one could also work with the principal bundle P itself, taking as objects of a category $\mathbf{C}_0$ all the fibers of the bundle P and as morphisms $f : P_p \rightarrow P_q$ the G-equivariant bijections $P_p \rightarrow P_q$, where P_p and P_q are fibers of P, over points p and q, and paths running from p to q.

The interface between gauge theory and category theory, in various forms and cases, has been studied in many works, for instance [1, 3–7]. In the present paper, we extract the abstract essence of some of these structures in a category theory setting, leaving the differential geometry behind as the concrete context. We abstract the process of passing from the point-particle picture to a string-like picture to a functor which generates a category $\mathbb{F}(\mathbf{C})$ from a category $\mathbf{C}$. Proposition 5 describes properties of a natural product operation on the objects of $\mathbb{F}(\mathbf{C})$ when $\mathbf{C}$ is a monoidal category. An excellent review of monoidal categories in relation to topological quantum field theory can be found in [8]. Symmetric monoidal bicategories are discussed in [9] in a context different from ours.

2. The Fat Category

Let $\mathbf{C}$ be a category. We define a new category $\mathbb{F}(\mathbf{C})$ as follows. The objects of $\mathbb{F}(\mathbf{C})$ are the morphisms of $\mathbf{C}$. A morphism in $\mathbb{F}(\mathbf{C})$ from the object $x_1 \xrightarrow{f_1} y_1$ to the object $x_2 \xrightarrow{f_2} y_2$ consists of morphisms $x_1 \xrightarrow{g_1} x_2$ and $y_1 \xrightarrow{g_2} y_2$ in $\mathbf{C}$, along with a set-mapping

$$h : \mathrm{Mor}(x_1, y_1) \longrightarrow \mathrm{Mor}(x_2, y_2), \tag{11}$$

which maps f_1 to f_2 as follows:

$$h(f_1) = f_2. \tag{12}$$

(In a later section we require that the hom-sets $\mathrm{Mor}(x, y)$ themselves also have algebraic structure that should be preserved by such h.) Here is a diagram displaying a morphism u of $\mathbb{F}(\mathbf{C})$:

$$u = \begin{array}{ccc} x_1 & \xrightarrow{f_1} & y_1 \\ {\scriptstyle g_1}\downarrow & \overset{h}{\Downarrow} & \downarrow{\scriptstyle g_2} \\ x_2 & \xrightarrow[f_2]{} & y_2 \end{array} \tag{13}$$

It is clear that this does specify a category, which we call the *fat category* for $\mathbf{C}$ (composition is "vertical," with successive hs composed). Sometimes it will be easier on the eye to write

$$(x, y, f) \tag{14}$$

for $x \xrightarrow{f} y$. Thus, diagram (13) can also be displayed as

$$\begin{array}{c} (x_1, y_1, f_1) \\ \downarrow{\scriptstyle u} \\ (x_2, y_2, f_2) \end{array} \tag{15}$$

The composition $v \circ_V u$ of morphisms in $\mathbb{F}(\mathbf{C})$ is defined "vertically" by drawing the diagram of v below that of u and composing vertically downward.

Commutative diagrams in $\mathbf{C}$ lead to morphisms of $\mathbb{F}(\mathbf{C})$ in a natural way and yield a subcategory of $\mathbb{F}(\mathbf{C})$ that is recognizable as the "category of arrows" [10, §I.4], sometimes denoted as $\mathrm{Arr}(\mathbf{C})$.

Lemma 2. *Any commutative diagram*

$$\begin{array}{ccc} x_1 & \xrightarrow{f_1} & y_1 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ x_2 & \xrightarrow[f_2]{} & y_2 \end{array} \tag{16}$$

in $\mathbf{C}$*, in which* g_1 *is an isomorphism, generates a morphism*

$$(x_1, y_1, f_1) \xrightarrow{u} (x_2, y_2, f_2) \tag{17}$$

in $\mathbb{F}(\mathbf{C})$*,*

$$u = \begin{array}{ccc} x_1 & \xrightarrow{f_1} & y_1 \\ {\scriptstyle g_1}\downarrow & \overset{h_u}{\Downarrow} & \downarrow{\scriptstyle g_2} \\ x_2 & \xrightarrow[f_2]{} & y_2 \end{array} \tag{18}$$

where

$$h_u : \mathrm{Mor}(x_1, y_1) \longrightarrow \mathrm{Mor}(x_2, y_2) : \phi \longmapsto g_2 \phi g_1^{-1}. \tag{19}$$

Moreover, if

$$\begin{array}{ccc} x_1 & \xrightarrow{f_1} & y_1 \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ x_2 & \xrightarrow[f_2]{} & y_2 \\ {\scriptstyle g_1'}\downarrow & & \downarrow{\scriptstyle g_2'} \\ x_3 & \xrightarrow[f_3]{} & y_3 \end{array} \tag{20}$$

is a commutative diagram in $\mathbf{C}$*, where* g_1 *and* g_1' *are isomorphisms, then the composite of the induced morphisms,*

$$\begin{aligned} u &: (x_1, y_1, f_1) \longrightarrow (x_2, y_2, f_2), \\ v &: (x_2, y_2, f_2) \longrightarrow (x_3, y_3, f_3), \end{aligned} \tag{21}$$

is the morphism in $\mathbb{F}(\mathbf{C})$ *induced by the commutative diagram*

$$\begin{array}{ccc} x_1 & \xrightarrow{f_1} & y_1 \\ {\scriptstyle g_1' g_1}\downarrow & & \downarrow{\scriptstyle g_2' g_2} \\ x_2 & \xrightarrow[f_3]{} & z_3 \end{array} \tag{22}$$

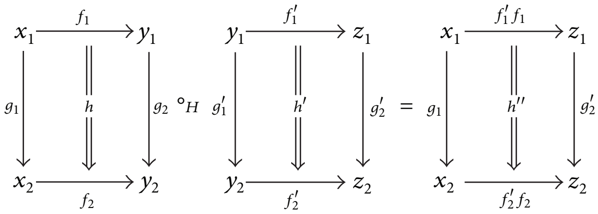
3. A Double Category of Isomorphisms

Let $\mathbb{F}(\mathbf{C})_0$ be the category whose objects are the invertible arrows of $\mathbf{C}$ and whose arrows are the arrows

$$\begin{array}{ccc} x_1 & \xrightarrow{f_1} & y_1 \\ {\scriptstyle g_1}\downarrow & \overset{h}{\Downarrow} & \downarrow{\scriptstyle g_2} \\ x_2 & \xrightarrow[f_2]{} & y_2 \end{array} \tag{23}$$

in $\mathbb{F}(\mathbf{C})$ in which the verticals g_1 and g_2 are also isomorphisms in $\mathbf{C}$. This is, for all purposes here, as good as assuming that all arrows of $\mathbf{C}$ are invertible, since we will only work with such arrows. In the geometric context, the arrows represent parallel transports and so the invertibility assumption is natural. The mapping h is motivated by the "surface" parallel transport mentioned briefly in (10).

Let us define *horizontal composition* of morphisms in $\mathbb{F}(\mathbf{C})_0$ as follows:



(24)

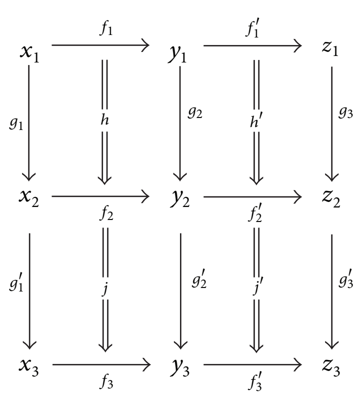
where the composition is defined only when $g_1' = g_2$, and h'' is given by

$$h'' : \operatorname{Mor}(x_1, z_1) \longrightarrow \operatorname{Mor}(x_2, z_2) : f \longmapsto h'\left(ff_1^{-1}\right) \underbrace{h(f_1)}_{f_2}. \tag{25}$$

Note that h'' satisfies

$$h''\left(f_1' f_1\right) = h'\left(f_1'\right) h\left(f_1\right) = f_2' f_2. \tag{26}$$

Consider now the following diagram:



(27)

The morphisms of $\mathbb{F}(\mathbf{C})_0$ thus have two laws of composition: $\circ_V$ and $\circ_H$. As we see below, these compositions obey a consistency condition (28), which thereby specifies a *double category* [10, 11, §I.5].

Proposition 3. *The morphisms of* $\mathbb{F}(\mathbf{C})_0$ *form a double category under the laws of composition* $\circ_V$ *and* $\circ_H$ *in the sense that for diagram* (27)*, with notation as explained above,*

$$\left(u_{j'} \circ_H u_j\right) \circ_V \left(u_{h'} \circ_H u_h\right) = \left(u_{j'} \circ_V u_{h'}\right) \circ_H \left(u_j \circ_V u_h\right), \tag{28}$$

for all morphisms $u_{j'}, u_j, u_h, u_{h'}$ *in* $\operatorname{Mor}(\mathbb{F}(\mathbf{C})_0)$ *for which the compositions on both sides of* (28) *are meaningful.*

Proof. Denote by u_h the morphism of $\mathbb{F}(\mathbf{C})_0$ specified by the upper left square in (27), by $u_{h'}$ the morphism specified by the upper right square, by u_j the morphism specified by the lower left square, and, lastly, by $u_{j'}$ the morphism specified by the lower right square.

Let $f \in \operatorname{Mor}(x_1, z_1)$. Then,

$$\begin{aligned} \left(\left(u_{j'} \circ_H u_j\right) \circ_V \left(u_{h'} \circ_H u_h\right)\right)(f) &= \left(u_{j'} \circ_H u_j\right) \\ &\quad \times \left(h'\left(ff_1^{-1}\right) f_2\right) \\ &= j'\left(h'\left(ff_1^{-1}\right)\right) f_3, \end{aligned} \tag{29}$$

$$\begin{aligned} \left(\left(u_{j'} \circ_V u_{h'}\right) \circ_H \left(u_j \circ_V u_h\right)\right)(f) &= \left(\left(u_{j'} \circ_V u_{h'}\right)\left(ff_1^{-1}\right)\right) f_3 \\ &= j'\left(h'\left(ff_1^{-1}\right)\right) f_3. \end{aligned} \tag{30}$$

Comparing (29) and (30), we have the claimed equality (28). □

Then, $\mathbb{F}(\mathbf{C})_0$ equipped with both laws of composition $\circ_V$ and $\circ_H$ is a *double category* [11]. In the geometric context, this is expressed as a *flatness* condition for the connection $\omega_{\overline{A},A,B}$ described in the Introduction; for more, see, for instance, [1, 3].

4. Enrichment for Morphisms

We continue with the notation and structures as before; $\mathbf{C}$ is a category and $\mathbb{F}(\mathbf{C})$ is the "fat" category described in Section 2. Now let $\mathbb{F}(\mathbf{C})_1$ be a subcategory of $\mathbb{F}(\mathbf{C})_0$, having the same objects but possibly fewer morphisms. The idea is that the hom-sets in $\mathbb{F}(\mathbf{C})$ could have additional structure; for example, if $\mathbf{C}$ has only one object E_p, a fiber of a vector bundle, then $\operatorname{Mor}(E_p, E_p)$ is a group under composition. The morphisms of $\mathbb{F}(\mathbf{C})_1$ could be required to be group automorphisms. We require that for any objects x, y, z of $\mathbf{C}$ and isomorphism $g : y \to x$, the map

$$r_g : \operatorname{Mor}(x, z) \longrightarrow \operatorname{Mor}(y, z) : f \longmapsto fg \tag{31}$$

is a morphism of $\mathbb{F}(\mathbf{C})_1$.

Proposition 4. *Let* $\mathbb{F}(\mathbf{C})_1$ *be any subcategory of* $\mathbb{F}(\mathbf{C})_0$ *having the same objects as* $\mathbb{F}(\mathbf{C})_0$*, and satisfying the condition* (31) *as explained above. Both horizontal and vertical composites of morphisms in* $\mathbb{F}(\mathbf{C})_1$ *are in* $\mathbb{F}(\mathbf{C})_1$*. Thus,* $\mathbb{F}(\mathbf{C})_1$ *is a double category.*

Proof. The consistency condition between horizontal and vertical compositions has already been checked in Proposition 3. Thus, we need only to check that horizontal composition, specified in (25) as

$$h'' : \operatorname{Mor}(x_1, z_1) \longrightarrow \operatorname{Mor}(x_2, z_2) : f \longmapsto h'\left(ff_1^{-1}\right) h\left(f_1\right), \tag{32}$$

is a morphism of $\mathbb{F}(\mathbf{C})_1$, for all invertible $f_1 \in \operatorname{Mor}(x_1, y_1)$ and all $h : \operatorname{Mor}(x_1, y_1) \to \operatorname{Mor}(x_1', y_1')$, $h' \in \operatorname{Mor}(y_1, z_1) \to \operatorname{Mor}(y_1', z_1')$ morphisms in $\mathbb{F}(\mathbf{C})_1$. Observe that

$$h''(f) = h'\left(ff_1^{-1}\right) h\left(f_1\right) = r_{h(f_1)} \circ h' \circ r_{f_1^{-1}}(f), \tag{33}$$

where the notation r_g is as in (31). Thus, h'' is a composite of morphisms in $\mathbb{F}(\mathbf{C})_1$. □

5. Monoidal Structures

In this section we will explore some algebraic structural enhancements of the fattened category $\mathbb{F}(\mathbf{C})_0$. The discussion is motivated by intrinsic algebraic considerations, but we discuss briefly now the relationship with the geometric context.

Consider the very special case where $\mathbf{C}$ is the category with only one object E_o, the fiber over a fixed point o in a vector bundle, and a morphism $f : E_o \to E_o$ is a an ordered pair as follow:

$$f = (c, T), \tag{34}$$

consisting of a piecewise smooth loop c based at o (with backtracks erased) along with a linear map $T : E_o \to E_o$ representing parallel transport around the loop. For $\mathbb{F}(\mathbf{C})_0$ in this special case, a morphism $h : \mathrm{Mor}(E_o, E_o) \to \mathrm{Mor}(E_o, E_o)$ arises from *paths of paths* along with a linear map $\mathrm{End}(E_o) \to \mathrm{End}(E_o)$, where $\mathrm{End}(E_o)$ is the vector space of all linear maps $E_o \to E_o$. Each hom-set $\mathrm{Mor}(E_o, E_o)$ is a monoid: composition

$$\begin{aligned}\mathrm{Mor}\left(E_o, E_o\right) \times \mathrm{Mor}\left(E_o, E_o\right) \longrightarrow \mathrm{Mor}\left(E_o, E_o\right) : \left(f, f'\right) \\ \longmapsto f \otimes f'\end{aligned} \tag{35}$$

is given by concatenation of loops along with ordinary composition of linear maps in $\mathrm{End}(E_o)$:

$$(c, T) \otimes \left(c', T'\right) = \left(c * c', T \circ T'\right), \tag{36}$$

where $c * c'$ is the loop c' followed by the loop c. (Since this discussion is primarily for motivation, we leave out technical details of "backtrack erasure.")

Turning to the abstract setting, we assume henceforth that $\mathbf{C}$ is a monoidal category. This means that there is a bifunctor

$$\otimes : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C} \tag{37}$$

and there is an *identity object* 1 in $\mathbf{C}$ for which certain natural coherence conditions hold as we now describe. In addition, there exists a natural isomorphism α, the *associator*, which associates to any of the objects A, B, C of $\mathbf{C}$ an isomorphism

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) \tag{38}$$

such that the following diagram commutes:

$$\begin{array}{ccccc} ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{A,B,C} \otimes i_D} & (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A,B\otimes C,D}} & A \otimes ((B \otimes C) \otimes D) \\ {\scriptstyle \alpha_{A\otimes B,C,D}} \downarrow & & & & \downarrow {\scriptstyle i_A \otimes \alpha_{B,C,D}} \\ (A \otimes B) \otimes (C \otimes D) & & \xrightarrow[\alpha_{A,B,C\otimes D}]{} & & A \otimes (B \otimes (C \otimes D)) \end{array} \tag{39}$$

There are also natural isomorphisms l and r, the left and right *unitors*, associating to each object A in $\mathbf{C}$ morphisms

$$l_A : 1 \otimes A \longrightarrow A, \qquad r_A : A \otimes 1 \longrightarrow A \tag{40}$$

such that

$$\begin{array}{ccc} (A \otimes 1) \otimes B & \xrightarrow{\alpha_{A,1,B}} & A \otimes (1 \otimes B) \\ & {\scriptstyle r_A \otimes i_B} \searrow \quad \swarrow {\scriptstyle i_A \otimes l_B} & \\ & A \otimes B & \end{array} \tag{41}$$

commutes for all objects A and B in $\mathbf{C}$.

Note that *naturality* means there are certain other conditions as well. For example, that the left unitor is a natural transformation means that for any morphism $x \xrightarrow{f} y$ in $\mathbf{C}$ the diagram

$$\begin{array}{ccc} 1 \otimes x & \xrightarrow{1 \otimes f} & 1 \otimes y \\ {\scriptstyle l_x} \downarrow & & \downarrow {\scriptstyle l_y} \\ x & \xrightarrow{f} & y \end{array} \tag{42}$$

commutes; here, in the upper horizontal arrow, 1 is the unique morphism $i_1 : 1 \to 1$ in $\mathbf{C}$.

We now define a product on the objects of $\mathbb{F}(\mathbf{C})$

$$\mathrm{Obj}\left(\mathbb{F}\left(\mathbf{C}\right)\right) \times \mathrm{Obj}\left(\mathbb{F}\left(\mathbf{C}\right)\right) \longrightarrow \mathrm{Obj}\left(\mathbb{F}\left(\mathbf{C}\right)\right) : (u, v) \longmapsto u \otimes v \tag{43}$$

as follows:

$$\left(x_1 \xrightarrow{f_1} y_1\right) \otimes \left(x_2 \xrightarrow{f_2} y_2\right) \stackrel{\text{def}}{=} x_1 \otimes x_2 \xrightarrow{f_1 \otimes f_2} y_1 \otimes y_2. \tag{44}$$

In the fat category $\mathbb{F}(\mathbf{C})$, we then have associators and unitors as follows. First, the unit is

$$1_{\mathbb{F}} = 1 \xrightarrow{i_1} 1, \tag{45}$$

where 1 denotes the identity object in $\mathbf{C}$ and i_1 the identity map on 1. We will often denote i_1 also simply as 1, the meaning being clear from context. For any object $x \xrightarrow{f} y$, there is the left unitor

$$\begin{array}{c} 1_{\mathbb{F}} \otimes (x, y, f) \\ \Big\downarrow {\scriptstyle l^{\mathbb{F}}_{(x,y,f)}} \\ (x, y, f) \end{array} = \begin{array}{ccc} 1 \otimes x & \xrightarrow{1 \otimes f} & 1 \otimes y \\ {\scriptstyle l_x}\Big\downarrow & \Big\Downarrow {\scriptstyle l_{(x,y,f)}} & \Big\downarrow{\scriptstyle l_y} \\ x & \xrightarrow[f]{} & y \end{array} \tag{46}$$

where the mapping

$$l_{(x,y,f)} : \operatorname{Mor}(1 \otimes x, 1 \otimes y) \longrightarrow \operatorname{Mor}(x, y) : \phi \longmapsto l_y \phi l_x^{-1} \tag{47}$$

takes $1 \otimes f$ to f, as follows from the remarks made above for (42). The right unitor is

$$\begin{array}{ccc} x \otimes 1 & \xrightarrow{f \otimes 1} & y \otimes 1 \\ {\scriptstyle r_x}\Big\downarrow & \Big\Downarrow {\scriptstyle r_{(x,y,f)}} & \Big\downarrow{\scriptstyle r_y} \\ x & \xrightarrow[f]{} & y \end{array} \tag{48}$$

where

$$r_{(x,y,f)} : \operatorname{Mor}(x \otimes 1, y \otimes 1) \longrightarrow \operatorname{Mor}(x, y) : \phi \longmapsto r_y \phi r_x^{-1}. \tag{49}$$

Again, this is indeed a morphism in $\mathbb{F}(\mathbf{C})$ by essentially the same argument that was used above in (46) for the left unitor.

The associator in $\mathbb{F}(\mathbf{C})$ is given as follows. Consider objects $x_i \xrightarrow{f_i} y_i$ in $\mathbb{F}(\mathbf{C})$, for $i \in \{1, 2, 3\}$. The fact that α is a *natural* transformation means that the diagram

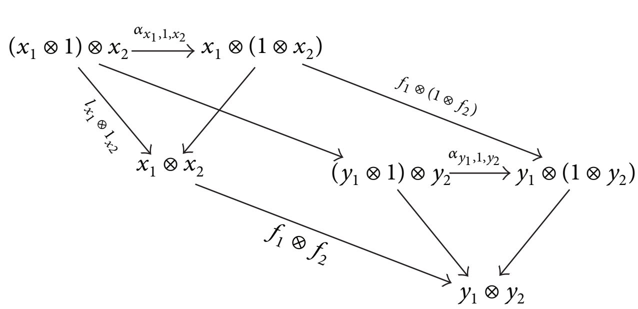
$$\begin{array}{ccc} (x_1 \otimes x_2) \otimes x_3 & \xrightarrow{(f_1 \otimes f_2) \otimes f_3} & (y_1 \otimes y_2) \otimes y_3 \\ {\scriptstyle \alpha_{x_1,x_2,x_3}}\Big\downarrow & & \Big\downarrow{\scriptstyle \alpha_{y_1,y_2,y_3}} \\ x_1 \otimes (x_2 \otimes x_3) & \xrightarrow[f_1 \otimes (f_2 \otimes f_3)]{} & y_1 \otimes (y_2 \otimes y_3) \end{array} \tag{50}$$

is commutative. Hence, by the first half of Lemma 2, this induces a morphism

$$\begin{array}{c} (x_1, y_1, f_1) \otimes (x_2, y_2, f_2) \otimes (x_3, y_3, f_3) \\ \Big\downarrow {\scriptstyle h} \\ (x_1, y_1, f_1) \otimes (x_2, y_2, f_2) \otimes (x_3, y_3, f_3) \end{array} = \begin{array}{ccc} (x_1 \otimes x_2) \otimes x_3 & \xrightarrow{(f_1 \otimes f_2) \otimes f_3} & (y_1 \otimes y_2) \otimes y_3 \\ & \Big\downarrow {\scriptstyle h} & \\ x_1 \otimes (x_2 \otimes x_3) & \xrightarrow[f_1 \otimes (f_2 \otimes f_3)]{} & y_1 \otimes (y_2 \otimes y_3) \end{array} \tag{51}$$

in $\mathbb{F}(\mathbf{C})$. In fact, h is an isomorphism since the vertical arrows in (50) are isomorphisms.

We prove the coherence condition for unitors. For this we have the following diagram:



(52)

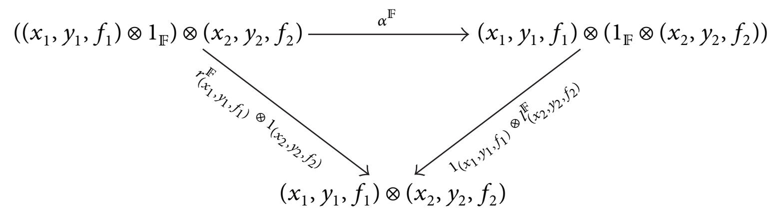
The two triangles at the two ends of this "trough" commute because of coherence in $\mathbf{C}$, the top rectangle also commutes because of the naturality of α. Then, it is entertaining to check that the two rectangular "slanted sides" are also commutative. In fact, the slant side on the left is

$$\begin{aligned} r^{\mathbb{F}}_{(x_1,y_1,f_1)} \otimes 1_{(x_2,y_2,f_2)} : &\left(\left(x_1, y_1, f_1\right) \otimes 1^{\mathbb{F}}\right) \otimes \left(x_2, y_2, f_2\right) \\ &\longrightarrow \left(x_1, y_1, f_1\right) \otimes \left(x_2, y_2, f_2\right) \end{aligned} \tag{53}$$

as a morphism in $\mathbb{F}(\mathbf{C})$, and the slant side on the right is

$$1_{(x_1,y_1,f_1)} \otimes l^{\mathbb{F}}_{(x_2,y_2,f_2)}. \tag{54}$$

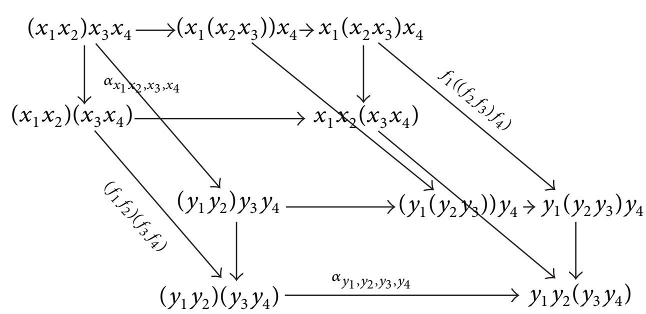
Thus, viewed as a diagram in $\mathbb{F}(\mathbf{C})$, the "trough" looks like



(55)

Since the trough commutes in $\mathbf{C}$, so does its avatar (55) in $\mathbb{F}(\mathbf{C})$, thanks to the second half of Lemma 2. This verifies the coherence property in $\mathbb{F}(\mathbf{C})$ involving the unitors.

Now, we turn to coherence for the associators. In the following diagram, where we leave out the $\otimes$ products for ease of viewing, the slant arrows are all tensor products of the f_i and the horizontal and vertical arrows are various associators:

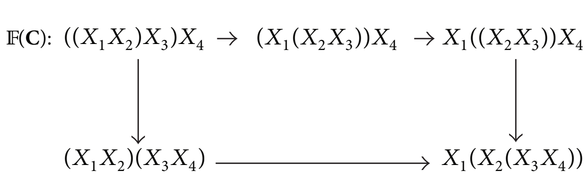


(56)

Coherence in the monoidal category $\mathbf{C}$ implies that the two rectangles at the end of this box are commutative, as mentioned earlier. Naturality of the associator implies that the top, bottom, and sides are also commutative. Thus, the entire diagram is commutative. If we abbreviate the objects in $\mathbb{F}(\mathbf{C})$ as

$$X_i = (x_i, y_i, f_i), \tag{57}$$

for $i \in \{1, 2, 3, 4\}$, we can read the full diagram as a diagram in the category $\mathbb{F}(\mathbf{C})$ as follows:



(58)

As a diagram in $\mathbb{F}(\mathbf{C})$, this is commutative, by Lemma 2. This establishes coherence of the associator in $\mathbb{F}(\mathbf{C})$.

We have completed the proof of Proposition 5.

Proposition 5. *Suppose that* $\mathbf{C}$ *is a monoidal category and let* $\mathbb{F}(\mathbf{C})$ *be the category specified above in the context of* (11). *Then, with tensor product as defined in* (44), $\mathbb{F}(\mathbf{C})$ *satisfies all conditions of a monoidal category at the level of objects.*

6. Concluding Remarks

In this paper, we have presented certain "fattened" categories $\mathbb{F}(\mathbf{C})$, $\mathbb{F}(\mathbf{C})_0$, and $\mathbb{F}(\mathbf{C})_1$ constructed out of a given category $\mathbf{C}$; the morphisms of $\mathbb{F}(\mathbf{C})_0$ form a double category. It is shown how a monoidal structure on $\mathbf{C}$ induces a multiplication on the objects of $\mathbb{F}(\mathbf{C})$ that satisfies certain coherence properties.

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