Research Article

The Thermal Statistics of Quasi-Probabilities' Analogs in Phase Space

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We focus attention upon the thermal statistics of the classical analogs of quasi-probabilities (QP) in phase space for the important case of quadratic Hamiltonians. We consider the three more important OPs: Wigner's, P-, and Husimi's. We show that, for all of them, the ensuing semiclassical entropy is a function only of the fluctuation product $\Delta x \Delta p$. We ascertain that the semiclassical analog of P-distribution seems to become unphysical at very low temperatures. The behavior of several other information quantifiers reconfirms such an assertion in manifold ways. We also examine the behavior of the statistical complexity and of thermal quantities like the specific heat.

1. Introduction

A quasi-probability distribution is a mathematical construction that resembles a probability distribution but does not necessarily fulfill some of Kolmogorov's axioms for probabilities [1]. Quasi-probabilities exhibit general features of ordinary probabilities. Most importantly, they yield expectation values with respect to the weights of the distribution. However, they disobey the third probability postulate [1] in the sense that regions integrated under them do not represent probabilities of mutually exclusive states. Some quasi-probability distributions exhibit zones of negative probability density. This kind of distributions often arises in the study of quantum mechanics when discussed in a phase space representation of frequent use in quantum optics, time-frequency analysis, and so forth.

One usually considers a density operator $\widehat{\rho}$ defined with respect to a complete orthonormal basis and shows that it can always be written in a diagonal manner, provided that an overcomplete basis is at hand [2]. This is the case of coherent states $|\alpha\rangle$ [3], for which [2]

$$\widehat{\rho} = \int \frac{d^2\alpha}{\pi} P(\alpha, \alpha^*) |\alpha\rangle \langle\alpha| . \tag{1}$$

We have $d^2\alpha/\pi = dx\,dp/2\pi\hbar$, with x and p being phase space variables. Coherent states, right eigenstates of the annihilation operator $\widehat{a}$, serve as the overcomplete basis in such a build-up [2, 3].

There exists a family of different representations, each connected to a different ordering of the creation and destruction operators $\widehat{a}$ and $\widehat{a}^\dagger$. Historically, the first of these is the Wigner quasi-probability distribution W [4], related to symmetric operator ordering. In quantum optics the particle number operator is naturally expressed in normal order and in the pertinent scenario the associated representation of the phase space distribution is the Glauber-Sudarshan P one [3]. In addition to W and P, one may find many other quasi-probability distributions emerging in alternative representations of the phase space distribution [5]. A quite popular representation is the Husimi Q one [6–9], used when operators are in antinormal order. We emphasize that we work here with classical analogs of W, P, and Q. As stated, we will specialize things to the three f-functions associated to a Harmonic Oscillator (HO) of angular frequency ω. In such a scenario the three (classical analog) functions that we call for convenience f_P, f_Q, and f_W are just Gaussians. The pertinent treatment becomes wholly analytical.

1.1. Our Goal. In this paper, we wish to apply *semiclassical information theory tools* associated with these analog P, Q, and W representations (for quadratic Hamiltonians) *in order to describe the associated thermal semiclassical features.* The idea is to gain physical insight from the application of different information quantifiers to classical analogs of quasi-probability distributions. It will be seen that useful insights are in this way gained. WE will discover that, out of the three functions, only f_Q and f_W are sensible analogs, while f_P exhibits problems if the temperature is low enough.

We insist that, in this paper, we will regard quasi-probabilities as semiclassical distributions in phase space, analogs of the quantum quasi-probabilistic distributions, and try to ascertain what physical features they are able to describe at such semiclassical level. One has [10, 11]

$$f_P = \gamma_P e^{-\gamma_P |\alpha|^2}, \quad \gamma_P = e^{\beta\hbar\omega} - 1 \quad (P\text{-function}),$$

$$f_Q = \gamma_Q e^{-\gamma_Q |\alpha|^2}, \quad \gamma_Q = 1 - e^{-\beta\hbar\omega} \quad (Q\text{-function}),$$

$$f_W = \gamma_W e^{-\gamma_W |\alpha|^2}, \tag{2}$$

$$\gamma_W = 2\tanh\left(\frac{\beta\hbar\omega}{2}\right) \quad (W\text{-function}),$$

where $\beta = 1/k_B T$, k_B is the Boltzmann constant, and T is the temperature. They will be used as semiclassical statistical weight functions. Since ours is not a quantum approach, the ordering of HO-creation and destruction operators $\hat{a}$ and $\hat{a}^\dagger$ plays no role whatsoever.

1.2. Historic Considerations and Organization. The thermodynamics properties associated with coherent states have been the subject of much interest. See, for instance, [12, 13]. Notice that HO is a really relevant system that yields useful insights of wide impact. Indeed, HO constitutes much more than a mere example. It is of special relevance for bosonic or fermionic atoms contained in magnetic traps [14–16] and for system that exhibits an equidistant level spacing in the vicinity of the ground state, like nuclei or Luttinger liquids.

For thermal states, the Gaussian HO-quantum phase spaces distributions are known in the literature for applications in quantum optics.

This paper is organized as follows. Section 2 refers to different information quantifiers in a phase space representation for Gaussian distributions. In Section 3 we calculate the classical analog Fano factor. Features of the fluctuations are analyzed in Section 4. Additionally, we discuss the notion of linear entropy. Finally, some conclusions are drawn in Section 5.

2. Semiclassical Information Quantifiers

Consider a general normalized gaussian distribution in phase space

$$f(\alpha) = \gamma e^{-\gamma |\alpha|^2}, \tag{3}$$

whose normalized variance is $1/\gamma$ with γ taking values γ_P, γ_Q, and γ_W. We discuss next, in these terms, some important information theory quantifiers.

2.1. Shift-Invariant Fisher's Information Measure. The information quantifier Fisher's information measure, specialized for families of shift-invariant distributions, which do not change shape under translations, is [17, 18]

$$I = \int dx\, f(x) \left(\frac{\partial \ln f(x)}{\partial x}\right)^2, \tag{4}$$

and, in phase space, it adopts the appearance [19]

$$I = \frac{1}{4} \int \frac{d^2\alpha}{\pi} f(\alpha) \left(\frac{\partial \ln f(\alpha)}{\partial |\alpha|}\right)^2 \tag{5}$$

such that considering $f(\alpha)$ given by (3) we get $I = \gamma$, whose specific values are γ_P, γ_Q, and γ_W for the three functions f_P, f_Q, and f_W. The behavior of these quantities is displayed in Figure 1. The solid line is the case P, the dashed one is the Wigner one, and the dotted curve is assigned to the Husimi case. Now, it is known that in the present scenario the maximum attainable value for I equals 2 [19]. The P-result violates this restriction at low temperatures more precisely at

$$T < T_{\text{crit}} = \frac{(\hbar\omega/k_B)}{\ln 3} \approx 0.91023 \frac{\hbar\omega}{k_B}, \tag{6}$$

with T being expressed in $(\hbar\omega/k_B)$-units.

2.2. Logarithmic Entropy S. The logarithmic Boltzmann's information measure for the the probability distribution (3) is

$$S = - \int \frac{d^2\alpha}{\pi} f(\alpha) \ln f(\alpha) = 1 - \ln\gamma, \tag{7}$$

so that it acquires the particular values

$$S_P = 1 - \ln\left(e^{\beta\hbar\omega} - 1\right), \tag{8}$$

$$S_Q = 1 - \ln\left(1 - e^{-\beta\hbar\omega}\right), \tag{9}$$

$$S_W = 1 - \ln\left(2\tanh\left(\frac{\beta\hbar\omega}{2}\right)\right), \tag{10}$$

for, respectively, the distributions f_P, f_Q, and f_W. These entropies are plotted in Figure 2. Notice that $S_P < 0$ for $T < \hbar\omega/(k_B \ln(1 + e)) \approx 0.76(\hbar\omega/k_B) < T_{\text{crit}}$. Negative classical entropies are well known. As an example, one can cite [20].

Table 1 lists a set of critical temperatures T_{crit} for typical electromagnetic (EM) waves.

We see from Table 1 that serious anomalies are detected for P-distribution in the case of radio waves of high frequency. P becomes negative, which is absurd, for rather high temperatures, where one expects classical physics to reign. Accordingly, one concludes that quasi-probabilities do not exhibit a sensible classical limit in P-case contrary to what happens in both W and Q ones.

TABLE 1: Critical temperatures T_{crit} for typical radio waves, with $h/k_B = 4.799 \ 10^{-11}$ Kelvin per second.

|	Frequency (ν)	Critical temperatures (°K)
Extremely low frequency (ELF)	3–30 Hz	$1.4397 \ 10^{-10}$–$1.4397 \ 10^{-9}$
Super low frequency (SLF)	30–300 Hz	$1.4397 \ 10^{-9}$–$1.4397 \ 10^{-8}$
Ultra low frequency (ULF)	300–3000 Hz	$1.4397 \ 10^{-8}$–$1.4397 \ 10^{-7}$
Very low frequency (VLF)	3–30 kHz	$1.4397 \ 10^{-7}$–$1.4397 \ 10^{-6}$
Low frequency (LF)	30–300 kHz	$1.4397 \ 10^{-6}$–$1.4397 \ 10^{-5}$
Medium frequency (MF)	300 KHz–3 MHz	$1.4397 \ 10^{-5}$–$1.4397 \ 10^{-4}$
High frequency (HF)	3–30 MHz	$1.4397 \ 10^{-4}$–$1.4397 \ 10^{-3}$
Very high frequency (VHF)	30–300 MHz	$1.4397 \ 10^{-3}$–$1.4397 \ 10^{-2}$
Ultra high frequency (UHF)	300 MHz–3 GHz	$1.4397 \ 10^{-2}$–$1.4397 \ 10^{-1}$
Super high frequency (SHF)	3–30 GHz	$1.4397 \ 10^{-1}$–1.4397
Extremely high frequency (EHF)	30–300 GHz	1.4397–14.397
Tremendously high frequency (THF)	300 GHz–3000 GHz	14.397–143.97
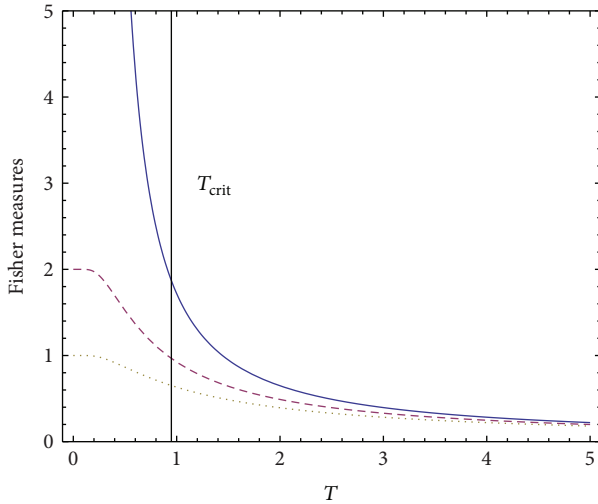
FIGURE 1: Fisher measure versus temperature T expressed in $(\hbar\omega/k_B)$-units. The solid line is the case P, the dashed one is the Wigner one, and the dotted curve is assigned to the Husimi instance. The vertical line represents the critical temperature T_{crit}.

2.3. Statistical Complexity.

The statistical complexity C, according to López-Ruiz et al. [21], is a suitable product of two quantifiers such that C becomes minimal at the extreme situations of perfect order or total randomness. Instead of using the prescription of [21], but without violating its spirit, we will take one of these two quantifiers to be Fisher's measure and take the other to be an entropic form, since it is well known that the two behave in opposite manner [22]. Thus

$$C = SI = \gamma\left(1 - \ln\gamma\right), \tag{11}$$

which vanishes for perfect order or total randomness. For each particular case, we explicitly have

$$C_P = \left(e^{\beta\hbar\omega} - 1\right)\left[1 - \ln\left(e^{\beta\hbar\omega} - 1\right)\right], \tag{12}$$

$$C_Q = \left(1 - e^{-\beta\hbar\omega}\right)\left[1 - \ln\left(1 - e^{-\beta\hbar\omega}\right)\right], \tag{13}$$

$$C_W = 2\tanh\left(\frac{\beta\hbar\omega}{2}\right)\left[1 - \ln\left(2\tanh\left(\frac{\beta\hbar\omega}{2}\right)\right)\right], \tag{14}$$

for, respectively, the distributions f_P, f_Q, and f_W. The maximum of the statistical complexity occurs when $\gamma = 1$ and the associated temperature values are

$$e^{\beta\hbar\omega} - 1 = 1 \Longrightarrow T = \frac{\hbar\omega}{k_B}\ln 2 > T_{\text{crit}}$$

for the f_P-function,

$$1 - e^{-\beta\hbar\omega} = 1 \Longrightarrow T = 0 \quad \text{for the } f_Q\text{-function}, \tag{15}$$

$$2\tanh\left(\frac{\beta\hbar\omega}{2}\right) = 1 \Longrightarrow T = \frac{\hbar\omega}{2k_B}\arctan\left(\frac{1}{2}\right)$$

for the f_W-function.

The statistical complexity C is plotted in Figure 3.

2.4. Linear Entropy.

Another interesting information quantifier is that of the Manfredi-Feix entropy [23], derived from the phase space Tsallis ($q = 2$) entropy [24]. In quantum information this form is referred to as the linear entropy [25]. It reads

$$S_l = 1 - \int\frac{d^2\alpha}{\pi}f^2(\alpha) = 1 - \mathcal{J}, \tag{16}$$

$$\mathcal{J} = \int\frac{d^2\alpha}{\pi}f^2(\alpha) = \frac{\gamma}{2}. \tag{17}$$

Accordingly, we have

$$S_l = 1 - \frac{\gamma}{2}; \quad 0 \le S_l \le 1. \tag{18}$$

This is semiclassical result, which is valid for small γ. In particular,

$$S_{l,P} = 1 - \frac{\gamma_P}{2} = 1 - \frac{e^{\beta\hbar\omega} - 1}{2},$$

$$S_{l,Q} = 1 - \frac{\gamma_Q}{2} = 1 - \frac{1 - e^{-\beta\hbar\omega}}{2}, \tag{19}$$

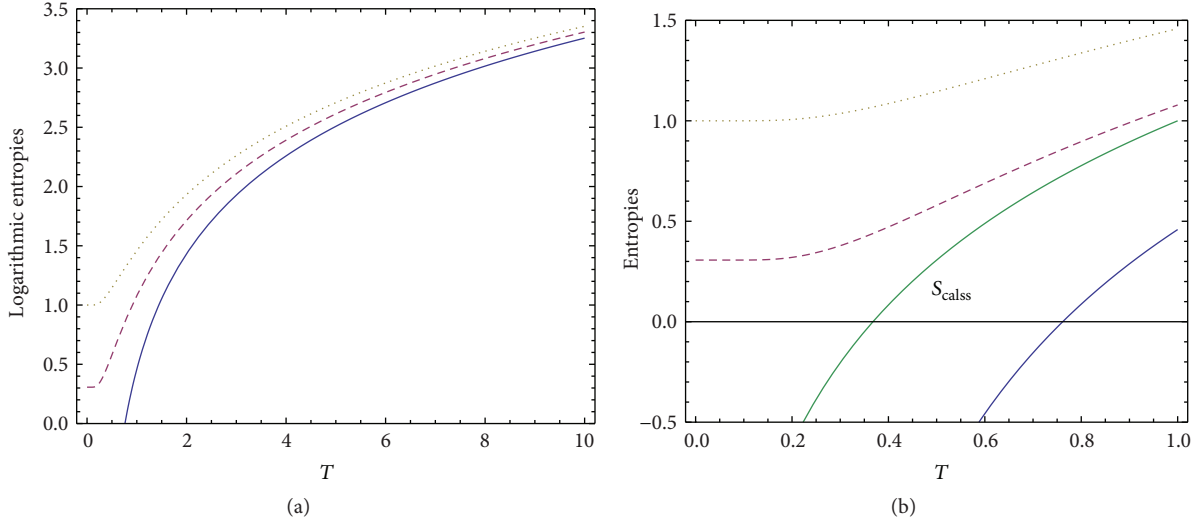$$S_{l,W} = 1 - \frac{\gamma_W}{2} = 1 - \frac{2\tanh(\beta\hbar\omega/2)}{2}.$$

(a)

(b)

FIGURE 2: Left: logarithmic entropies, S_P, S_Q, and S_W, as a function of the temperature T in $(\hbar\omega/k_B)$-units. The solid line is the case P, the dashed one is the Wigner one, and the dotted curve is assigned to the Husimi instance. Right: zoom of the logarithmic entropies as a function of temperature T in $(\hbar\omega/k_B)$-units. Negative values of S_P occur below $T = \hbar\omega/(k_B \ln(1 + e))$, with the critical temperature $< T_{\text{crit}}$. Remaining details are similar to those of left figure. We have added the classical entropy of the harmonic oscillator $S_{\text{class}} = 1 - \ln(\beta\hbar\omega)$.
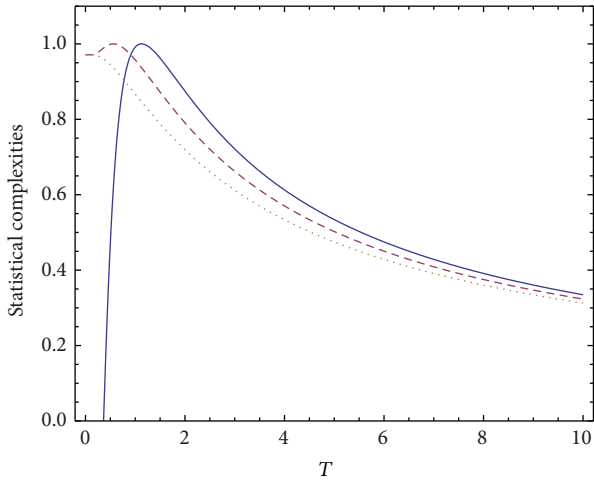

FIGURE 3: Complexities C_P, C_Q, and C_W versus the temperature T in $(\hbar\omega/k_B)$-units. The solid line is the case P, the dashed one is the Wigner one, and the dotted curve is assigned to the Husimi instance.

Note that in the P-instance the linear entropy becomes negative once again for $T < T_{\text{crit}}$. Contrary to what happens for the logarithmic entropy, the linear one can vanish in W representation. The lineal entropies are plotted in Figure 4.

The ensuing statistical complexity that uses S_l becomes

$$C_l = S_l I = I\left(1 - \frac{I}{2}\right) = \gamma\left(1 - \frac{\gamma}{2}\right), \tag{20}$$

vanishing for both $\gamma = 0$ and $\gamma = 2$, the extreme values of γ-physical range (we showed above that γ cannot exceed 2 without violating uncertainty restrictions).

3. Fano Factor's Classical Analog

In general, the Fano factor is the coefficient of dispersion of the probability distribution $p(y)$, which is defined as [26]

$$\mathcal{F} = \frac{\Delta y^2}{\langle y \rangle}, \tag{21}$$

where $\Delta y^2 = \langle y^2 \rangle - \langle y \rangle^2$ is the variance and $\langle x \rangle$ is the mean of a random process y.

If $p(y)$ is a Poisson distribution then one sees that the pertinent Fano factor becomes unity ($\mathcal{F} = 1$) [10, 27]. We remind the reader of two situations:

(1) For $\mathcal{F} < 1$, sub-Poissonian processes occur.

(2) For $\mathcal{F} > 1$, the process is super-Poissonian.

For our Gaussian distribution (3), if one sets now $y = |\alpha|^2$, one has the classical Fano analog

$$\mathcal{F} = \frac{\left\langle |\alpha|^4 \right\rangle_f - \left\langle |\alpha|^2 \right\rangle_f^2}{\left\langle |\alpha|^2 \right\rangle_f}, \tag{22}$$

where the expectation value of the function $\mathcal{A}(\alpha)$ is calculated as

$$\langle \mathcal{A} \rangle_f = \int \frac{d^2\alpha}{\pi} f(\alpha) \mathcal{A}(\alpha), \tag{23}$$

indicating that $f(\alpha)$ is the statistical weight function. Thus, after computing the mean values involved in (22) by taking into account definition (23), the Fano factor becomes

$$\mathcal{F} = \frac{1}{\gamma} = \frac{1}{I}, \tag{24}$$

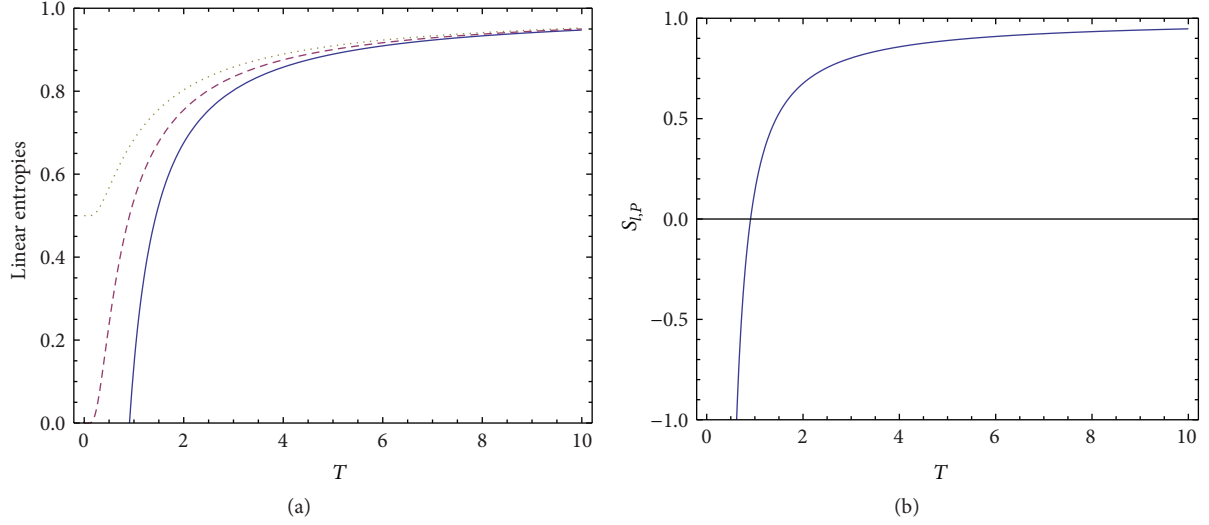(a)   (b)

FIGURE 4: Left: linear entropies $S_{l,P}$, $S_{l,Q}$, and $S_{l,W}$ versus the temperature T in $(\hbar\omega/k_B)$-units. The solid line is the case P, the dashed one is the Wigner one, and the dotted curve is assigned to the Husimi instance. Right: $S_{l,P}$ as a function of the temperature T in $(\hbar\omega/k_B)$-units. We note that the linear entropy $S_{l,P}$ is negative below the T value $t_{\mathrm{crit}} = T_{\mathrm{crit}}k_B/\hbar\omega = \ln 3$.

which, for a Gaussian distribution, links the Fano factor to the distribution's width and to Fisher's measure I. We are speaking of processes that are of a quantum nature and cannot take place in a classical environment. Thus, with reference to the critical temperature defined in (6), we have to deal with

$$\mathscr{F}_P = \frac{1}{e^{\beta\hbar\omega} - 1} \left(= 1 \text{ at } T = \frac{\hbar\omega}{k_B \ln 2} > T_{\mathrm{crit}}\right)$$

for the f_P function,

$$\mathscr{F}_Q = \frac{1}{1 - e^{-\beta\hbar\omega}} (= 1 \text{ at } T = 0)$$

(25)

for the f_Q function,

$$\mathscr{F}_W = \frac{1}{2 \tanh (\beta\hbar\omega/2)} \left(= 1 \text{ at } T = T_{\mathrm{crit}}\right)$$

for the f_W function.

The f_Q-case reaches the *super- to sub-Poissonian transition* only at $T = 0$, while the other two cases reach it at finite temperatures.

4. Fluctuations

We start this section considering the classical Hamiltonian of the harmonic oscillator that reads

$$\mathscr{H}(x, p) = \hbar\omega |\alpha|^2,$$

(26)

where x and p are phase space variables, $|\alpha|^2 = x^2/4\sigma_x^2 + p^2/\sigma_p^2$, $\sigma_x^2 = \hbar/2m\omega$, and $\sigma_p^2 = \hbar m\omega/2$ [28].

Using the definition of the mean value (23), from (26), we immediately find [29]

$$\left\langle \frac{x^2}{2\sigma_x^2} \right\rangle_f = \left\langle \frac{p^2}{2\sigma_p^2} \right\rangle_f = \left\langle |\alpha|^2 \right\rangle_f,$$

(27)

with

$$\left\langle |\alpha|^2 \right\rangle_f = \gamma \int \frac{d^2\alpha}{\pi} e^{-\gamma|\alpha|^2} |\alpha|^2 = \frac{1}{\gamma},$$

(28)

where $\langle x \rangle_f = \langle p \rangle_f = \langle \alpha \rangle_f = 0$, while γ takes the respective values γ_P, γ_Q, and γ_W. The concomitant variances are $\Delta x^2 = \langle x^2 \rangle_f - \langle x \rangle_f^2 = 2\sigma_p^2/\gamma$ and $\Delta p^2 = \langle p^2 \rangle_f - \langle p \rangle_f^2 = 2\sigma_p^2/\gamma$. Hence, for our general Gaussian distribution, one easily establishes that

$$\mathscr{U} = \Delta x \Delta p = \frac{\hbar}{\gamma},$$

(29)

which shows that γ should be constrained by the restriction

$$\gamma \leq 2,$$

(30)

if one wishes the inequality

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

(31)

to hold.

Specializing (29) for our three quasi-probability distributions yields

$$\Delta x \Delta p = \frac{\hbar}{e^{\beta\hbar\omega} - 1}, \text{ for } f_P \text{ function,}$$

$$\Delta x \Delta p = \frac{\hbar}{1 - e^{-\beta\hbar\omega}}, \text{ for } f_Q \text{ function,}$$

(32)

$$\Delta x \Delta p = \frac{\hbar}{2 \tanh (\beta\hbar\omega/2)}, \text{ for } f_W \text{ function.}$$

These fluctuations are plotted in Figure 5. The restriction (31) applied to the P-result entails that it holds if

$$T \geq \frac{\hbar\omega}{\ln 3 k_B} = T_{\text{crit}} \approx 0.91023 \frac{\hbar\omega}{k_B}. \tag{33}$$

Thus, the distribution f_P seems again to become unphysical at temperatures lower than T_{crit}, for which (31) is violated. From (29), we have $\gamma = \hbar/\mathcal{U}$. Accordingly, if we insert this into (7), the logarithmic entropy S can be recast in $\mathcal{U}$-terms via the relation

$$S = 1 - \ln\left(\frac{\hbar}{\Delta x \Delta p}\right), \tag{34}$$

(also demonstrated in [30] to hold for the Wehrl entropy) which vanishes for

$$\Delta x \Delta p = \frac{\hbar}{e}. \tag{35}$$

In the P-instance this happens at

$$T = 0.71463 \frac{\hbar\omega}{k_B}. \tag{36}$$

At this temperature Heisenberg's-like condition (31) is violated. W and Q distributions do not allow for such a circumstance. Actually, in the Wigner case, which is exact, the minimum S-value is attained at $\beta = \infty$, where

$$S_{\text{min}} = 1 - \ln 2 \approx 0.306. \tag{37}$$

The uncertainty restriction (31) seems to impede the phase space entropy to vanish, a sort of quasi-quantum effect. It is clear then that, in phase space, the logarithmic entropy, by itself, is an uncertainty indicator, in agreement with the work, in *other scenarios*, of several authors (see, for instance, [31] and references therein).

Define now the participation ratio's analog as [32, 33]

$$m = \frac{1}{\mathscr{J}} = \frac{2}{\gamma}, \tag{38}$$

where $\mathscr{J}$ is given by (17). This is an important quantity that measures the number of pure states entering the mixture determined by our general Gaussian probability distribution of amplitude γ [32, 33]. We again encounter troubles with the P-distribution in this respect. It is immediately realized by looking at Figure 6 that, for fulfilling the obvious condition $m \geq 1$, one needs a temperature $T \geq T_{\text{crit}}$.

5. Conclusions

We have investigated here the thermal statistics of quasi-probabilities' analogs $f(\alpha)$ in phase space for the important case of quadratic Hamiltonians, focusing attention on the three more important instances, that is, those of Wigner, P-, and Husimi distributions:

(i) We emphasized the fact that for all of them the semi-classical entropy is a function only of the fluctuation
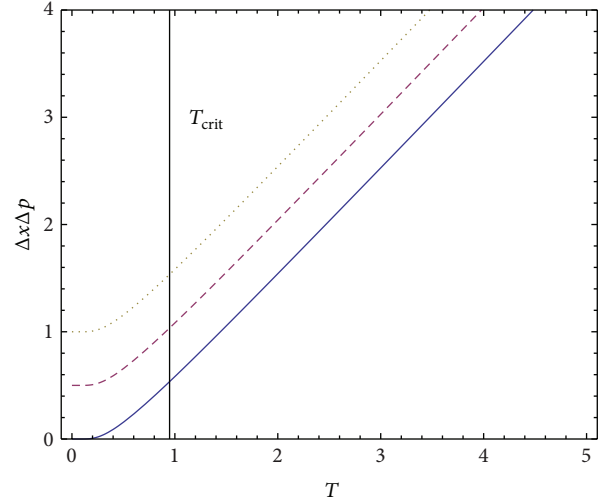

FIGURE 5: Fluctuations versus the temperature T in $(\hbar\omega/k_B)$-units. The solid line is the case P, the dashed one is the Wigner one, and the dotted line is assigned to the Husimi instance.
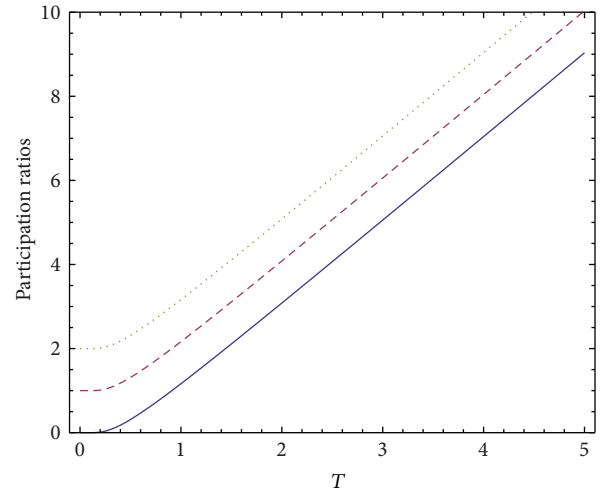

FIGURE 6: Participation ratio m versus temperature T in $(\hbar\omega/k_B)$-units. The solid line is the case P, the dashed one is the Wigner one, and the dotted curve is assigned to the Husimi instance.

product $\Delta x \Delta p$. This fact allows one to ascertain that the analog P-distribution seems to become unphysical at low enough temperatures, smaller than a critical value T_{crit}, because, in such an instance

(1) it would violate Heisenberg's-like principle in such a case. The behavior of other information quantifiers reconfirms such an assertion; that is,

(2) Fisher's measure exceeds its permissible maximum value $I = 2$;

(3) the participation ratio becomes < 1, which is impossible.

(ii) It is also clear then that semiclassical entropy, by itself, in phase space, looks like a kind of "uncertainty" indicator.

(iii) We have determined the temperatures for which the statistical complexity becomes maximal as a signature of the well-known transition between classical and nonclassical states of light whose signature is the transition from super-Poissonian to sub-Poissonian distributions [34].

We have seen that the P-distribution in the case of radio waves of high frequency becomes negative, which is absurd, as the pertinent temperatures are rather high and thus one expects classical physics to reign. Accordingly, one concludes that quasi-probabilities do not exhibit a sensible classical limit in the P-case, contrary to what happens in both W and Q ones.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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