Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 805948, 14 pages
doi:10.1155/2011/805948

Research Article

Forward Curvatures on Time Scales

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Received 14 December 2010; Accepted 31 March 2011

Academic Editor: Nobuyuki Kenmochi

We also introduce forward curvature of a curve and give some formulas to calculate forward curvature of a curve on time scales which may be an arbitrary closed subsets of the set of all real numbers. We also introduce the length of a curve parametrized by a time scale parameter in $\mathbb{R}^3$.

1. Introduction

The study of dynamic equations on time scales is an area of mathematics that recently has received a lot of attention. The calculus on time scales has been introduced in order to unify the theories of continuous and discrete processes and in order to extend those theories to a more general class of so-called dynamic equations.

In recent years there have been a few research activities concerning the application of differential geometry on time scales. In [1] Guseinov and Ozyılmaz have defined the notions of forward tangent line, Δ-regular curve, and natural Δ-parametrization. Furthermore, in [2] Bohner and Guseinov, have introduced the concept of a curve parametrized by a time scale parameter and they have given integral formulas for computation of its length in plane. They have established a version of the classical Green formula suitable to time scales. In [3] Ozyılmaz has introduced the directional derivative according to the vector fields.

The general idea of this paper is to study forward curvature of curves where in the parametric equations the parameter varies in a time scale. We present the “differential” part of classical differential geometry on time scale calculus. The new results generalize the well known formulas stated in classical differential geometry. We illustrate our results by applying them to various kinds of time scales.

2. Basic Definitions

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The time scale $\mathbb{T}$ is a complete metric space with the usual metric. We assume throughout that a time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.

For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \tag{2.1}$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}. \tag{2.2}$$

If $\sigma(t) > t$, we say that t is *right-scattered*, while, if $\rho(t) < t$, we say that t is *left-scattered*. Also, if $\sigma(t) = t$, then t is called *right-dense*, and, if $\rho(t) = t$, then t is called *left-dense*. The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t. \tag{2.3}$$

We introduce the set $\mathbb{T}^\kappa$ which is derived from the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. For $a, b \in \mathbb{T}$ with $a \leq b$ we define the interval $[a, b]$ in $\mathbb{T}$ by

$$[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}. \tag{2.4}$$

We will let $[a, b]^\kappa$ denote $[a, \rho(b)]$ if b is left-scattered and $[a, b]$ if b is left-dense.

Definition 2.1 (see [4]). Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function, and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that, given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s] \right| \leq \varepsilon |\sigma(t) - s|, \quad \forall s \in U. \tag{2.5}$$

We call $f^\Delta(t)$ the *delta* (or *Hilger*) *derivative* of f at t. Moreover, we say that f is *delta* (or *Hilger*) *differentiable* on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

Theorem 2.2 (see [4]). *For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$ the following hold.*

(i) *If f is Δ-differentiable at t, then f is continuous at t.*

(ii) *If f is continuous at t and t is right-scattered, then f is Δ-differentiable at t and*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \tag{2.6}$$

(iii) *If t is right-dense, then f is Δ-differentiable at t if and only if the limit*

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s} \tag{2.7}$$

exists as a finite number. In this case $f^{\Delta}(t)$ is equal to this limit.

(iv) *If f is Δ-differentiable at t, then*

$$f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t). \tag{2.8}$$

Theorem 2.3 (see [4]). *If f, g is Δ-differentiable at $t \in \mathbb{T}^{\kappa}$, then*

(i) *$f + g$ is Δ-differentiable at t and*

$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t). \tag{2.9}$$

(ii) *For any constant c, cf is Δ-differentiable at t and*

$$(cf)^{\Delta}(t) = cf^{\Delta}(t). \tag{2.10}$$

(iii) *fg is Δ-differentiable at t and*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)). \tag{2.11}$$

(iv) *If $g(t)g(\sigma(t)) \neq 0$, then f/g is Δ-differentiable at t and*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - g^{\Delta}(t)f(t)}{g(t)g(\sigma(t))}. \tag{2.12}$$

Theorem 2.4 (chain rule [4]). *Assume that $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $w : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\Delta}(t)$ and $w^{\widetilde{\Delta}}(\nu(t))$ exists for $t \in \mathbb{T}^{\kappa}$, then*

$$(w \circ \nu)^{\Delta} = \left(w^{\widetilde{\Delta}} \circ \nu\right)\nu^{\Delta}, \tag{2.13}$$

where one denotes the derivative on $\widetilde{\mathbb{T}}$ by $\widetilde{\Delta}$.

Theorem 2.5 (derivative of the inverse [4]). *Assume that $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Then*

$$\frac{1}{\nu^{\Delta}} = \left(\nu^{-1}\right)^{\widetilde{\Delta}} \circ \nu \tag{2.14}$$

at points where ν^{Δ} is different from zero.

Definition 2.6 (see [4]). A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

Definition 2.7 (see [1]). A *Δ-regular curve (or an arc of a Δ-regular curve)* γ is defined as a mapping $\gamma : [a,b] \subset \mathbb{T} \to \mathbb{R}^3$ that is Δ-differentiable on $[a,b]^{\kappa}$ with rd-continuous Δ-derivatives and

$$\left\|\gamma^{\Delta}(t)\right\| \neq 0, \quad \text{for } t \in [a,b]^{\kappa}. \tag{2.15}$$

Definition 2.8 (see [1]). Let $\gamma : [a,b] \subset \mathbb{T} \to \mathbb{R}^3$ be a curve, Q_0 a point on γ, and L a line through Q_0^{σ}, where

$$Q_0 = \left(\gamma_1(t_0), \gamma_2(t_0), \gamma_3(t_0)\right), \quad Q_0^{\sigma} = \left(\gamma_1(\sigma(t_0)), \gamma_2(\sigma(t_0)), \gamma_3(\sigma(t_0))\right), \quad t_0 \in [a,b]^{\kappa}. \tag{2.16}$$

Take on γ any point Q. Denote by d the distance of the point Q from the point Q_0^{σ}, and by δ the distance of Q from the line L. If $\delta/d \to 0$ as $Q \to Q_0$, $Q \neq Q_0^{\sigma}$, then we say that L is the forward tangent line to the curve γ at the point Q_0.

Theorem 2.9 (see [1]). *Every Δ-regular curve $\gamma : [a,b] \subset \mathbb{T} \to \mathbb{R}^3$ has at any point $Q_0 = (\gamma_1(t_0), \gamma_2(t_0), \gamma_3(t_0))$, $t_0 \in [a,b]^{\kappa}$, the forward tangent line that has the vector $\gamma^{\Delta}(t_0)$ as its direction vector.*

Definition 2.10 (see [1]). Let γ be a Δ-regular curve in $\mathbb{R}^3$ given by the equation

$$\gamma = \gamma(t), \quad t \in [a,b] \subset \mathbb{T}. \tag{2.17}$$

We define the function p by

$$p(t) = \int_a^t \left\|\gamma^{\Delta}(s)\right\| \Delta s, \quad t \in [a,b]. \tag{2.18}$$

The variable p can be used as a parameter for the curve γ. Such a parametrization of a curve we call natural Δ-parametrization.

Theorem 2.11 (see [1]). *In the case of natural Δ-parametrization of the curve γ the forward tangent vector is a unit vector.*

Definition 2.12 (see [2]). Let γ be a continuous curve with equation $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. A partition of $[a,b]$ is any finite ordered set

$$P = \{t_0, t_1, \ldots, t_n\} \subset [a,b], \quad \text{where } a = t_0 < t_1 < \cdots < t_n = b. \tag{2.19}$$

Let us set

$$l(\gamma, P) = \sum_{i=1}^{n} \sqrt{\left[\gamma_1(t_i) - \gamma_1(t_{i-1})\right]^2 + \left[\gamma_2(t_i) - \gamma_2(t_{i-1})\right]^2}. \tag{2.20}$$

The curve γ is rectifiable if

$$\sup_P \{l(\gamma, P) : P \text{ is a partition of } [a,b]\} =: l(\gamma) < \infty. \tag{2.21}$$

The nonnegative number $l(\gamma)$ is called the length of the curve γ. If the supremum does not exist, the curve is said to be nonrectifiable.

Theorem 2.13 (see [2]). *Let the functions γ_1 and γ_2 be continuous on $[a,b]$ and Δ-differentiable on $[a,b)$. If their Δ-derivatives γ_1^{Δ} and γ_2^{Δ} are bounded and Δ-integrable over $[a,b)$, then the curve $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ is rectifiable and its length $l(\gamma)$ can be evaluated by the formula*

$$l(\gamma) = \int_a^b \sqrt{\left[\gamma_1^{\Delta}(t)\right]^2 + \left[\gamma_2^{\Delta}(t)\right]^2}\,\Delta t. \tag{2.22}$$

3. Forward Curvatures on Time Scales

It is easy to see that the notion of rectifiable curve in Definition 2.12 for $\mathbb{R}^2$ can be adapted to $\mathbb{R}^3$.

Definition 3.1. Let γ be a continuous curve in $\mathbb{R}^3$. Let P be a partition of $[a,b]$ as in (2.19), and set

$$l(\gamma, P) = \sum_{i=1}^{n} \sqrt{\sum_{j=1}^{3} \left[\gamma_j(t_i) - \gamma_j(t_{i-1})\right]^2}, \tag{3.1}$$

where $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$. We say that curve γ is rectifiable if

$$\sup_P \{l(\gamma, P) : P \text{ is a partition of } [a,b]\} =: l(\gamma) < \infty. \tag{3.2}$$

In Theorem 2.13, the length of the curve in plane is given. We introduce the length of a curve parametrized by a time scale parameter in $\mathbb{R}^3$ in the following lemma.

Lemma 3.2. *Let the functions γ_1, γ_2, and γ_3 be continuous on $[a,b]$ and Δ-differentiable on $[a,b)$. If their Δ-derivatives γ_1^{Δ}, γ_2^{Δ}, and γ_3^{Δ} are bounded and Δ-integrable over $[a,b)$, then the curve $\gamma(t)=(\gamma_1(t),\gamma_2(t),\gamma_3(t))$ is rectifiable and its length $l(\gamma)$ can be evaluated by the formula*

$$l(\gamma)=\int_a^b\sqrt{\left[\gamma_1^{\Delta}(t)\right]^2+\left[\gamma_2^{\Delta}(t)\right]^2+\left[\gamma_3^{\Delta}(t)\right]^2}\,\Delta t. \tag{3.3}$$

Proof. We show that the curve γ is rectifiable. Let an arbitrary partition of $[a,b]$ be of the form (2.19). Consider $l(\gamma,P)$ defined by (3.1). Applying to each of the functions γ_1,γ_2, and γ_3 the mean value theorem (see [5, Theorem 4.2]) on $[t_{i-1},t_i]$ for $i\in\{1,2,\dots,n\}$, we get that there exist points ξ_i, ξ_i', η_i, η_i' and ζ_i, ζ_i' in $[t_{i-1},t_i)$ such that

$$\gamma_1^{\Delta}(\xi_i)(t_i-t_{i-1})\le\gamma_1(t_i)-\gamma_1(t_{i-1})\le\gamma_1^{\Delta}(\xi_i')(t_i-t_{i-1}), \tag{3.4}$$

$$\gamma_2^{\Delta}(\eta_i)(t_i-t_{i-1})\le\gamma_2(t_i)-\gamma_2(t_{i-1})\le\gamma_2^{\Delta}(\eta_i')(t_i-t_{i-1}), \tag{3.5}$$

$$\gamma_3^{\Delta}(\zeta_i)(t_i-t_{i-1})\le\gamma_3(t_i)-\gamma_3(t_{i-1})\le\gamma_3^{\Delta}(\zeta_i')(t_i-t_{i-1}). \tag{3.6}$$

From (3.4), (3.5), and (3.6) it follows that

$$\begin{aligned}\left|\gamma_1(t_i)-\gamma_1(t_{i-1})\right|&\le A_i(t_i-t_{i-1}),\\ \left|\gamma_2(t_i)-\gamma_2(t_{i-1})\right|&\le B_i(t_i-t_{i-1}),\\ \left|\gamma_3(t_i)-\gamma_3(t_{i-1})\right|&\le C_i(t_i-t_{i-1}),\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}A_i&=\max\left\{\left|\gamma_1^{\Delta}(\xi_i)\right|,\left|\gamma_1^{\Delta}(\xi_i')\right|\right\},\\ B_i&=\max\left\{\left|\gamma_2^{\Delta}(\eta_i)\right|,\left|\gamma_2^{\Delta}(\eta_i')\right|\right\},\\ C_i&=\max\left\{\left|\gamma_3^{\Delta}(\zeta_i)\right|,\left|\gamma_3^{\Delta}(\zeta_i')\right|\right\}.\end{aligned} \tag{3.8}$$

By the assumption of the theorem, the derivatives γ_1^{Δ}, γ_2^{Δ}, and γ_3^{Δ} are bounded on $[a,b)$, so that there is a finite positive constant M such that

$$\left|\gamma_j^{\Delta}(t)\right|\le M,\quad j=1,2,3 \tag{3.9}$$

for all $t\in[a,b)$. Thus

$$\left|\gamma_j(t_i)-\gamma_j(t_{i-1})\right|\le M(t_i-t_{i-1}),\quad j=1,2,3 \tag{3.10}$$

for all $i \in \{1, 2, \dots, n\}$, and we have from (3.1)

$$l(\gamma, P) \leq M\sqrt{3}\sum_{i=1}^{n}(t_i - t_{i-1}) = M\sqrt{3}(b-a), \tag{3.11}$$

so that we get that the curve γ is rectifiable.Now we prove the formula (3.3). Consider the Riemann Δ-sum

$$S = \sum_{i=1}^{n}\sqrt{\left[\gamma_1^{\Delta}(\xi_i)\right]^2 + \left[\gamma_2^{\Delta}(\xi_i)\right]^2 + \left[\gamma_3^{\Delta}(\xi_i)\right]^2}(t_i - t_{i-1}) \tag{3.12}$$

of the Δ-integrable function $\sqrt{\left[\gamma_1^{\Delta}(t)\right]^2 + \left[\gamma_2^{\Delta}(t)\right]^2 + \left[\gamma_3^{\Delta}(t)\right]^2}$, corresponding to the partition P of $[a, b]$ and the choice of intermediate points ξ_i defined in (3.4). For every $\delta > 0$, there exists (see [6, Lemma 5.7]) at least one partition P of $[a, b]$ of the form (2.19) such that for each $i \in \{1, 2, \dots, n\}$ either $t_i - t_{i-1} \leq \delta$ or $t_i - t_{i-1} > \delta$ and $\sigma(t_{i-1}) = t_i$. Let us denote by $\mathcal{P}_{\delta}([a, b])$ the set of all such partitions P of $[a, b]$. For an arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left|l(\gamma, P) - I\right| < \frac{\varepsilon}{2} \quad \forall P \in \mathcal{P}_{\delta}([a, b]), \tag{3.13}$$

where

$$I = \int_b^a \sqrt{\left[\gamma_1^{\Delta}(t)\right]^2 + \left[\gamma_2^{\Delta}(t)\right]^2 + \left[\gamma_3^{\Delta}(t)\right]^2}\Delta t. \tag{3.14}$$

From (3.4), (3.5), and (3.6) we get

$$\begin{aligned}
0 &\leq \gamma_1(t_i) - \gamma_1(t_{i-1}) - \gamma_1^{\Delta}(\xi_i)(t_i - t_{i-1}) \leq \left[\gamma_1^{\Delta}(\xi_i') - \gamma_1^{\Delta}(\xi_i)\right](t_i - t_{i-1}), \\
0 &\leq \gamma_2(\eta_i) - \gamma_2(t_{i-1}) - \gamma_2^{\Delta}(\eta_i)(t_i - t_{i-1}) \leq \left[\gamma_2^{\Delta}(\eta_i') - \gamma_2^{\Delta}(\eta_i)\right](t_i - t_{i-1}), \\
0 &\leq \gamma_3(\zeta_i) - \gamma_3(t_{i-1}) - \gamma_3^{\Delta}(\zeta_i)(t_i - t_{i-1}) \leq \left[\gamma_3^{\Delta}(\zeta_i') - \gamma_3^{\Delta}(\zeta_i)\right](t_i - t_{i-1}),
\end{aligned} \tag{3.15}$$

and, consequently,

$$\begin{aligned}
\gamma_1(t_i) - \gamma_1(t_{i-1}) &= \left[\gamma_1^{\Delta}(\xi_i) + \alpha_i\right](t_i - t_{i-1}), \\
\gamma_2(t_i) - \gamma_2(t_{i-1}) &= \left[\gamma_2^{\Delta}(\eta_i) + \beta_i\right](t_i - t_{i-1}), \\
\gamma_3(t_i) - \gamma_3(t_{i-1}) &= \left[\gamma_3^{\Delta}(\zeta_i) + \omega_i\right](t_i - t_{i-1}),
\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}
0 &\leq \alpha_i \leq \gamma_1^{\Delta}(\xi_i') - \gamma_1^{\Delta}(\xi_i) \leq M_i - m_i, \\
0 &\leq \beta_i \leq \gamma_2^{\Delta}(\eta_i') - \gamma_2^{\Delta}(\eta_i) \leq N_i - n_i, \\
0 &\leq \omega_i \leq \gamma_3^{\Delta}(\zeta_i') - \gamma_3^{\Delta}(\zeta_i) \leq R_i - r_i,
\end{aligned} \tag{3.17}$$

in which M_i and m_i are the supremum and infimum of γ_1^Δ on $[t_{i-1}, t_i)$ and N_i, n_i and R_i, r_i are corresponding numbers for γ_2^Δ and γ_3^Δ, respectively. Using the inequality

$$\left|\sqrt{x^2+y^2+z^2}-\sqrt{x_1^2+y_1^2+z_1^2}\right| \leq |x-x_1|+|y-y_1|+|z-z_1| \tag{3.18}$$

for $x, y, z, x_1, y_1, z_1 \in \mathbb{R}$, we obtain

$$\begin{aligned}
&\left|\sqrt{\left[\gamma_1^\Delta(\xi_i)+\alpha_i\right]^2+\left[\gamma_2^\Delta(\eta_i)+\beta_i\right]^2+\left[\gamma_3^\Delta(\zeta_i)+\omega_i\right]^2}-\sqrt{\left[\gamma_1^\Delta(\xi_i)\right]^2+\left[\gamma_2^\Delta(\xi_i)\right]^2+\left[\gamma_3^\Delta(\xi_i)\right]^2}\right| \\
&\quad \leq |\alpha_i| + \left|\gamma_2^\Delta(\eta_i)+\beta_i-\gamma_2^\Delta(\xi_i)\right| + \left|\gamma_3^\Delta(\zeta_i)+\omega_i-\gamma_3^\Delta(\xi_i)\right| \\
&\quad \leq |\alpha_i| + |\beta_i| + |\omega_i| + \left|\gamma_2^\Delta(\eta_i)-\gamma_2^\Delta(\xi_i)\right| + \left|\gamma_3^\Delta(\zeta_i)-\gamma_3^\Delta(\xi_i)\right| \\
&\quad \leq M_i - m_i + 2(N_i-n_i) + 2(R_i-r_i).
\end{aligned} \tag{3.19}$$

Therefore,

$$\begin{aligned}
|l(\gamma,P)-S| &= \left|\sum_{i=1}^{n}\left(\sqrt{\left[\gamma_1^\Delta(\xi)+\alpha_i\right]^2+\left[\gamma_2^\Delta(\eta_i)+\beta_i\right]^2+\left[\gamma_3^\Delta(\zeta_i)+\omega_i\right]^2}\right)(t_i-t_{i-1})\right| \\
&\leq \sum_{i=1}^{n}\left[M_i-m_i+2(N_i-n_i)+2(R_i-r_i)\right](t_i-t_{i-1}) \\
&= U\left(\gamma_1^\Delta,P\right)-L\left(\gamma_1^\Delta,P\right)+2\left[U\left(\gamma_2^\Delta,P\right)-L\left(\gamma_2^\Delta,P\right)\right] \\
&\quad +2\left[U\left(\gamma_3^\Delta,P\right)-L\left(\gamma_3^\Delta,P\right)\right],
\end{aligned} \tag{3.20}$$

where U and L denote the upper and lower Darboux Δ-sums, respectively. Since the functions $\sqrt{[\gamma_1^\Delta]^2+[\gamma_2^\Delta]^2+[\gamma_3^\Delta]^2}$, γ_1^Δ, γ_2^Δ and γ_3^Δ are Δ-integrable over $[a,b)$, for arbitrary $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|S-I| < \frac{\varepsilon}{12}, \qquad U\left(\gamma_j^\Delta,P\right)-L\left(\gamma_j^\Delta,P\right) < \frac{\varepsilon}{12}, \quad j=1,2,3 \tag{3.21}$$

for all $P \in \mathcal{P}_\delta([a,b])$ (see [6, Theorem 5.9]) and the Riemann definition of Δ-integrability therein), where I is defined by (3.14). Therefore, we get

$$|l(\Gamma,P)-I| \leq |l(\Gamma,P)-S| + |S-I| < \frac{\varepsilon}{12}+2\frac{\varepsilon}{12}+2\frac{\varepsilon}{12}+\frac{\varepsilon}{12} = \frac{\varepsilon}{2}, \tag{3.22}$$

and so the validity of (3.13) is proved. On the other hand, among the partitions P for which (3.13) is satisfied, there is a partition P such that

$$|l(\gamma,P)-l(\gamma)| < \frac{\varepsilon}{2}. \tag{3.23}$$

Indeed, there is a partition P^* of $[a,b]$ such that

$$0 \leq l(\gamma) - l(\gamma, P^*) < \frac{\varepsilon}{2}. \tag{3.24}$$

Next, we refine the partition P^* adding to it new partition points so that we get a partition P that belongs to $\mathcal{P}_\delta([a,b])$. Then by $l(\Gamma, P) \geq l(\Gamma, P^*)$, (3.24) yields

$$0 \leq l(\gamma) - l(\gamma, P) < \frac{\varepsilon}{2}, \tag{3.25}$$

so that (3.23) is shown. By (3.13) and (3.23), we get

$$|l(\Gamma) - I| < \varepsilon. \tag{3.26}$$

Since $\varepsilon > 0$ is arbitrary, we have $l(\Gamma) = I$, and the proof is complete. □

Definition 3.3. The curve $\gamma : [a,b] \subset \mathbb{T} \to \mathbb{R}^3$ is given in the parametric form $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$. Let $P_0 := \gamma(t_0)$, $P_0^\sigma := \gamma(\sigma(t_0))$, for $t_0 \in [a,b]^{\kappa^2}$ and $P := \gamma(t)$ for $t \in [a,b]$. We denote the angle between the forward tangent lines drawn to the curve at P_0 and P by θ_P and the arc length of the segment PP_0 of the curve by $|\widehat{PP_0}|$. The forward curvature of γ at P_0 is defined as

$$\kappa = \lim_{P \to P_0^\sigma} \frac{\theta_P}{\left|\widehat{PP_0}\right|}. \tag{3.27}$$

Theorem 3.4. *Let $\gamma : [a,b] \subset \mathbb{T} \to \mathbb{R}^3$ be a natural Δ-parametrized Δ-regular curve, there exists $\gamma^{\Delta\Delta}(t_0)$ for $t_0 \in [a,b]^{\kappa^2}$. If t_0 is a right-dense point, then*

$$\kappa = \left\| \gamma^{\Delta\Delta}(t_0) \right\|. \tag{3.28}$$

If t_0 is a right-scattered point, then

$$\kappa = \frac{2}{\mu(t_0)} \arcsin\left(\frac{\mu(t_0) \left\| \gamma^{\Delta\Delta}(t_0) \right\|}{2} \right). \tag{3.29}$$

Proof. Let $P_0 := \gamma(t_0)$, $P_0^\sigma := \gamma(\sigma(t_0))$, for a fix point $t_0 \in [a,b]^{\kappa^2}$ and $P := \gamma(t)$ for $t \in [a,b]$. Let the unit vectors on the forward tangent lines to the curve at points P and P_0 be $\tau(t)$ and $\tau(t_0)$, respectively, and assume that $\theta(t)$ is the angle between them. We denote $l(t) = |\widehat{PP_0}|$. Then we have

$$l(t) = \int_{t_0}^{t} \left\| \gamma^\Delta(s) \right\| \Delta s = \int_{t_0}^{t} \Delta s = t - t_0, \tag{3.30}$$

$$\|\tau(t) - \tau(t_0)\| = 2 \sin \frac{\theta(t)}{2}. \tag{3.31}$$

Assume that t_0 is a right-dense point. Then $P \longrightarrow P_0 = P_0^\sigma$ and $\theta(t) \longrightarrow 0$ as $t \longrightarrow t_0$. Since

$$\lim_{t\to t_0} \frac{\theta(t)/(t-t_0)}{\theta(t)/l(t)} = \lim_{t\to t_0} \frac{l(t)}{t-t_0} = 1, \tag{3.32}$$

we obtain

$$\lim_{t\to t_0} \frac{\theta(t)}{t-t_0} = \lim_{t\to t_0} \frac{\theta(t)}{l(t)}. \tag{3.33}$$

By using (3.31) we can write

$$\left\| \frac{\tau(t)-\tau(t_0)}{t-t_0} \right\| = \frac{2}{t-t_0} \sin \frac{\theta(t)}{2} = \frac{\sin(\theta(t)/2)}{\theta(t)/2} \frac{\theta(t)}{t-t_0}. \tag{3.34}$$

Thus, we find

$$\kappa = \lim_{t\to t_0} \frac{\theta(t)}{l(t)} = \left\| \gamma^{\Delta\Delta}(t_0) \right\| \tag{3.35}$$

as $t \longrightarrow t_0$.

Assume that t_0 is a right-scattered point. Then $P \longrightarrow P_0^\sigma$ and $\theta(t) \longrightarrow \theta(\sigma(t_0))$ as $t \longrightarrow \sigma(t_0)$. If the equality (3.31) is divided by $t-t_0$, taking the limit as $t \longrightarrow \sigma(t_0)$, we have

$$\begin{aligned} \left\| \gamma^{\Delta\Delta}(t_0) \right\| &= \left\| \frac{\tau(\sigma(t_0))-\tau(t_0)}{\sigma(t_0)-t_0} \right\| \\ &= \frac{2}{\sigma(t_0)-t_0} \sin \frac{\theta(\sigma(t_0))}{2} \\ &= \frac{2}{\mu(t_0)} \sin \frac{\theta(\sigma(t_0))}{2}. \end{aligned} \tag{3.36}$$

(3.16) On the other hand, we find

$$\kappa = \lim_{t\to \sigma(t_0)} \frac{\theta(t)}{l(t)} = \frac{\theta(\sigma(t_0))}{\sigma(t_0)-t_0} = \frac{\theta(\sigma(t_0))}{\mu(t_0)}. \tag{3.37}$$

It follows from (3.16) and (3.37) that

$$\kappa = \frac{2}{\mu(t_0)} \arcsin\left(\frac{\mu(t_0)\left\|\gamma^{\Delta\Delta}(t_0)\right\|}{2} \right). \tag{3.38}$$

This completes the proof. □

Theorem 3.5. *Let $\gamma : [a,b] \subset \mathbb{T} \to \mathbb{R}^3$ be a curve with arbitrary parameters that is second Δ-differentiable on $[a,b]^{\kappa^2}$ and $\|\gamma^{\Delta}(t)\| \neq 0$ for all $t \in [a,b]^{\kappa^2}$. Moreover assume that the function $h : [a,b]^{\kappa^2} \to \mathbb{R}$ defined by $h(t) := \langle \gamma^{\Delta}(t), \gamma^{\Delta}(t) \rangle$ is strictly increasing. The forward curvature of γ at the right-dense point $t \in [a,b]^{\kappa^2}$ is*

$$\kappa = \frac{\left\| \gamma^{\Delta}(t) \times \gamma^{\Delta\Delta}(t) \right\|}{\left\| \gamma^{\Delta}(t) \right\|^3}. \tag{3.39}$$

Proof. The function $p : [a,b] \to p([a,b]) = \widetilde{\mathbb{T}}$ defined by $s = p(t) = \int_a^t \|\gamma^{\Delta}(v)\| \Delta v$ is continuous and strictly increasing. Therefore $\widetilde{\mathbb{T}}$ will be a time scale (see [1, pages 560-561]). The forward jump operator and the derivative operator on this time scale, will be denoted by $\widetilde{\sigma}$ and $\widetilde{\Delta}$ respectively. Since the curve $\gamma \circ p^{-1}$ is a natural Δ-parametrized, to find the curvature to the γ at the point t, it is sufficient to find that the curvature to the $\gamma \circ p^{-1}$ at the point s.

$$\begin{gathered}
\left(p^{-1}\right)^{\widetilde{\Delta}}(s) = \frac{1}{p^{\Delta}(t)} = \frac{1}{\left\|\gamma^{\Delta}(t)\right\|}, \\
\left(p^{-1}\right)^{\widetilde{\Delta}}(\widetilde{\sigma}(s)) = \frac{1}{p^{\Delta}\left(p^{-1}(\widetilde{\sigma}(s))\right)} = \frac{1}{p^{\Delta}(\sigma(t))} = \frac{1}{\left\|\gamma^{\Delta}(\sigma(t))\right\|} = \frac{1}{\left\|\gamma^{\Delta}(t)\right\|}, \\
\left(p^{-1}\right)^{\widetilde{\Delta}\widetilde{\Delta}}(s) = -\frac{p^{\Delta\Delta}(t)}{\left[p^{\Delta}(t)\right]^2 p^{\Delta}(\sigma(t))} = -\frac{\left\|\gamma^{\Delta}(t)\right\|^{\Delta}}{\left\|\gamma^{\Delta}(t)\right\|^3}.
\end{gathered} \tag{3.40}$$

By using (3.40), one can easily find that

$$\begin{aligned}
\left\| \left(\gamma \circ p^{-1}\right)^{\widetilde{\Delta}\widetilde{\Delta}}(s) \right\| &= \left\| \gamma^{\Delta}(t)\left(p^{-1}\right)^{\widetilde{\Delta}\widetilde{\Delta}}(s) + \gamma^{\Delta\Delta}(t)\left(p^{-1}\right)^{\widetilde{\Delta}}(s)\left(p^{-1}\right)^{\widetilde{\Delta}}(\widetilde{\sigma}(s)) \right\| \\
&= \frac{\left\| \left\|\gamma^{\Delta}(t)\right\| \gamma^{\Delta\Delta}(t) - \left\|\gamma^{\Delta}(t)\right\|^{\Delta} \gamma^{\Delta}(t) \right\|}{\left\|\gamma^{\Delta}(t)\right\|^3}.
\end{aligned} \tag{3.41}$$

As $t \in [a,b]^{\kappa^2}$ is a right-dense point and the function $h(t) := \langle \gamma^{\Delta}(t), \gamma^{\Delta}(t) \rangle$ is strictly increasing, we have

$$\left\|\gamma^{\Delta}(t)\right\|^{\Delta} = \frac{\langle \gamma^{\Delta}(t), \gamma^{\Delta\Delta}(t) \rangle}{\left\|\gamma^{\Delta}(t)\right\|}. \tag{3.42}$$

Substituting (3.42) into (3.41), we obtain

$$\begin{aligned}\kappa &= \frac{\left\|\left(\gamma^{\Delta}(t)\times\gamma^{\Delta\Delta}(t)\right)\times\gamma^{\Delta}(t)\right\|}{\left\|\gamma^{\Delta}(t)\right\|^{4}}\\ &= \frac{\left\|\left(\gamma^{\Delta}(t)\times\gamma^{\Delta\Delta}(t)\right)\right\|\left\|\gamma^{\Delta}(t)\right\|\sin(\pi/2)}{\left\|\gamma^{\Delta}(t)\right\|^{4}}\\ &= \frac{\left\|\left(\gamma^{\Delta}(t)\times\gamma^{\Delta\Delta}(t)\right)\right\|}{\left\|\gamma^{\Delta}(t)\right\|^{3}}.\end{aligned} \tag{3.43}$$

The proof is complete. □

Remark 3.6. It is easy to see that, for the case $\mathbb{T}=\mathbb{R}$, the results (3.28) and (3.39) generalize the following formulas stated in classical differential geometry:

$$\kappa = \left\|\gamma''(t_0)\right\|, \qquad \kappa = \frac{\left\|\gamma'(t)\times\gamma''(t)\right\|}{\left\|\gamma'(t)\right\|^{3}}. \tag{3.44}$$

Theorem 3.7. *Let $\gamma : [a,b]\subset\mathbb{T}\to\mathbb{R}^3$ be a curve with arbitrary parameters that is second Δ differentiable on $[a,b]^{\kappa^2}$ and has continuous Δ derivative on $[a,b]^{\kappa}$ and $\|\gamma^{\Delta}(t)\|\neq 0$ for all $t\in[a,b]^{\kappa}$. Moreover assume that the function $h : [a,b]^{\kappa}\to\mathbb{R}$ defined by $h(t) := \langle\gamma^{\Delta}(t),\gamma^{\Delta}(t)\rangle$ is strictly increasing. The forward curvature of γ at the right-scattered point $t\in[a,b]^{\kappa^2}$ is*

$$\kappa = \frac{2}{\mu(t)\left\|\gamma^{\Delta}(t)\right\|}\arcsin\left(\frac{\mu(t)q_1}{2q_2}\right), \tag{3.45}$$

where

$$\begin{aligned}q_1 &= \left\|\left\|\gamma^{\Delta}(t)\right\|\left(\left\|\gamma^{\Delta}(t)\right\|+\left\|\gamma^{\Delta}(\sigma(t))\right\|\right)\gamma^{\Delta\Delta}(t)-\left\langle\gamma^{\Delta\Delta}(t),\gamma^{\Delta}(t)+\gamma^{\Delta}(\sigma(t))\right\rangle\gamma^{\Delta}(t)\right\|,\\ q_2 &= \left\|\gamma^{\Delta}(t)\right\|\left\|\gamma^{\Delta}(\sigma(t))\right\|\left(\left\|\gamma^{\Delta}(t)\right\|+\left\|\gamma^{\Delta}(\sigma(t))\right\|\right).\end{aligned} \tag{3.46}$$

Proof. Let $t\in[a,b]^{\kappa^2}$ be a right-scattered point. In this case we have

$$\begin{aligned}\widetilde{\mu}(s) &= \widetilde{\sigma}(s)-s\\ &= p(\sigma(t))-p(t)\\ &= \mu(t)\left\|\gamma^{\Delta}(t)\right\|,\end{aligned} \tag{3.47}$$

$$\left\|\gamma^{\Delta}(t)\right\|^{\Delta} = \frac{\left\langle\gamma^{\Delta\Delta}(t),\gamma^{\Delta}(t)+\gamma^{\Delta}(\sigma(t))\right\rangle}{\left\|\gamma^{\Delta}(t)\right\|+\left\|\gamma^{\Delta}(\sigma(t))\right\|}. \tag{3.48}$$

By using (3.41) and (3.48), we find

$$\left\| \left(\gamma \circ p^{-1}\right)^{\tilde{\Delta}\tilde{\Delta}}(s) \right\| = \frac{q_1}{\left\|\gamma^{\Delta}(t)\right\|^2 \left\|\gamma^{\Delta}(\sigma(t))\right\| \left(\left\|\gamma^{\Delta}(t)\right\| + \left\|\gamma^{\Delta}(\sigma(t))\right\|\right)}. \tag{3.49}$$

Substituting (3.49) into (3.29), we obtain

$$\kappa = \frac{2}{\mu(t)\left\|\gamma^{\Delta}(t)\right\|} \arcsin\left(\frac{\mu(t) q_1}{2 q_2}\right). \tag{3.50}$$

□

Example 3.8. Let $\mathbb{T} = \mathbb{Z}$ and $\gamma : \mathbb{T} \to \mathbb{R}^3$, $\gamma(t) = \beta(\cos t, \sin t, t)$.The curve γ satisfies the conditions of Theorem 3.4 for the case $\beta = 1/\sqrt{3 - 2\cos 1}$. In this case we have $\sigma(t) = t + 1$, $\mu(t) = 1$, and

$$\begin{gathered} \gamma^{\Delta}(t) = \beta(\cos(t+1) - \cos t, \sin(t+1) - \sin t, 1), \\ \gamma^{\Delta\Delta}(t) = \beta(\cos(t+2) - 2\cos(t+1) + \cos t, \sin(t+2) - 2\sin(t+1) + \sin t, 0). \end{gathered} \tag{3.51}$$

Since every point of $\mathbb{T}$ is right-scattered point, the curvature of γ at any point t is

$$\begin{aligned} \kappa &= \frac{2}{\mu(t)} \arcsin\left(\frac{\mu(t)\left\|\gamma^{\Delta\Delta}(t)\right\|}{2}\right) \\ &= 2\arcsin\left(2\beta \sin^2\left(\frac{1}{2}\right)\right). \end{aligned} \tag{3.52}$$

This value is the angle between the line through $\gamma(t)$, $\gamma(t+1)$ and the line through $\gamma(t+1)$, $\gamma(t+2)$.

Example 3.9. Assume that $\gamma : \mathbb{T} = \{1, 1/2, 1/3, \ldots\} \cup \{0\} \to \mathbb{R}^3$, $\gamma(t) = (t, t^2, 0)$, is a non-Δ-natural parametrized curve. The only right-dense point of the time scale $\mathbb{T}$ is $t = 0$, and the other points of the time scale are right-scattered. The forward jump operator and the graininess function are

$$\begin{gathered} \sigma(t) = \begin{cases} \dfrac{t}{1-t}, & t \neq 1, \\ 1, & t = 1, \end{cases} \\ \mu(t) = \begin{cases} \dfrac{t^2}{1-t}, & t \neq 1, \\ 0, & t = 1. \end{cases} \end{gathered} \tag{3.53}$$

Furthermore we have

$$\begin{aligned} \gamma^{\Delta}(t) &= \left(1, \frac{t^2-2t}{t-1}, 0\right), \quad \text{for } t \neq 1 \\ \gamma^{\Delta\Delta}(t) &= \left(0, \frac{2t-2}{2t-1}, 0\right), \quad \text{for } t \neq 1,\ t \neq \frac{1}{2}, \end{aligned} \tag{3.54}$$

and, from Theorem 3.5, the forward curvature of the curve γ at the point $t = 0$ is

$$\kappa(0) = \frac{\left\| \gamma^{\Delta}(0) \times \gamma^{\Delta\Delta}(0) \right\|}{\left\| \gamma^{\Delta}(0) \right\|^3} = 2. \tag{3.55}$$

For the right-scattered point $t = 1/3$, by using Theorem 3.7, we find

$$\kappa\left(\frac{1}{3}\right) = \frac{72}{\sqrt{61}} \arcsin\left(\frac{2\sqrt{2\sqrt{793} - 6}}{\sqrt[4]{793}\left(3\sqrt{13} + \sqrt{61}\right)} \right). \tag{3.56}$$

This value is the ratio of the angle between the line through the points $\gamma(1/3)$, $\gamma(1/2)$ and the line through the points $\gamma(1/2)$, $\gamma(1)$ and the distance between the points $\gamma(1/3)$ and $\gamma(1/2)$.

Acknowledgment

The authors wish to express their sincere thanks to the referee for his encouraging attitude and valuable suggestions in the review process of the work.

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