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Research Article

Second and Secondary Lattice Modules

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Let M be a lattice module over the multiplicative lattice L. A nonzero L-lattice module M is called second if for each $a \in L$, $a1_M = 1_M$ or $a1_M = 0_M$. A nonzero L-lattice module M is called secondary if for each $a \in L$, $a1_M = 1_M$ or $a^n1_M = 0_M$ for some $n > 0$. Our objective is to investigative properties of second and secondary lattice modules.

A multiplicative lattice L is a complete lattice in which there is defined as a commutative, associative multiplication which distributes over arbitrary joins and has the compact greatest element 1_L (least element 0_L) as a multiplicative identity (zero). An element $a \in L$ is said to be proper if $a < 1_L$. An element $p < 1_L$ in L is said to be prime if $ab \leq p$ implies either $a \leq p$ or $b \leq p$. If 0_L is prime, then L is said to be a domain. For $a \in L$, we define $\sqrt{a} = \bigvee\{x \in L : x^n \leq a$ for some integer $n\}$. An element $p < 1_L$ in L is said to be primary if $ab \leq p$ implies either $a \leq p$ or $b \leq \sqrt{p}$.

If a, b belong to L, $(a:_L b)$ is the join of all $c \in L$ such that $cb \leq a$. An element e of L is called meet principal if $a \bigwedge be = ((a:_L e) \bigwedge b)e$ for all $a, b \in L$. An element e of L is called join principal if $((ae \bigvee b):_L e) = a \bigvee (b:_L e)$ for all $a, b \in L$. $e \in L$ is said to be principal if e is both meet principal and join principal. $e \in L$ is said to be weak meet (join) principal if $a \bigwedge e = e(a:_L e)$ $(a \bigvee (0_L:_L e) = (ea:_L e))$ for all $a \in L$. An element a of a multiplicative lattice L is called compact if $a \leq \bigvee b_\alpha$ implies $a \leq b_{\alpha_1} \bigvee b_{\alpha_2} \bigvee \cdots \bigvee b_{\alpha_n}$ for some subsets $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. If each element of L is a join of principal (compact) elements of L, then L is called a PG-lattice (CG-lattice). If L is a CG-lattice and p is a primary element, then $\sqrt{p}$ is prime [1, Lemma 2.1].

Let M be a complete lattice. Recall that M is a lattice module over the multiplicative lattice L or simply an L-module in case there is a multiplication between elements of L and M, denoted by lB for $l \in L$ and $B \in M$, which satisfies the following properties:

(1) $(lb)B = l(bB)$;

(2) $(\bigvee_\alpha l_\alpha)(\bigvee_\beta B_\beta) = \bigvee_{\alpha,\beta} l_\alpha B_\beta$;

(3) $1_L B = B$;

(4) $0_L B = 0_M$;

for all l, l_α, b in L and for all B, B_β in M.

Let M be an L-module. If N, K belong to M, $(N:_L K)$ is the join of all $a \in L$ such that $aK \leq N$. Particularly, $(0_M:_L 1_M)$ is denoted by $ann(M)$. If $a \in L$ and $N \in M$, then $(N:_M a)$ is the join of all $H \in M$ such that $aH \leq N$. An element N of M is called meet principal if $(b \wedge (B:_L N))N = bN \wedge B$ for all $b \in L$ and for all $B \in M$. An element N of M is called join principal if $b \vee (B:_L N) = ((bN \vee B):_L N)$ for all $b \in L$ and for all $B \in M$. N is said to be principal if it is both meet principal and join principal. In special cases, an element N of M is called weak meet principal (weak join principal) if $(B:_L N)N = B \wedge N$ $((bN:_L N) = b \vee (0_M:_L N))$ for all $B \in M$ (for all $b \in L$). N is said to be weak principal if N is both weak meet principal and weak join principal.

Let M be an L-module. An element N in M is called compact if $N \leq \bigvee_\alpha B_\alpha$ implies $N \leq B_{\alpha_1} \vee B_{\alpha_2} \vee \cdots \vee B_{\alpha_n}$ for some subsets $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. The greatest element of M will be denoted by 1_M. If each element of M is a join of principal (compact) elements of M, then M is called a PG-lattice module (CG-lattice module).

Let M be an L-module. An element $N \in M$ is said to be proper if $N < 1_M$. For all elements N of M, $[N, 1_M]$ is a set of all $K \in M$ such that $N \leq K \leq 1_M$ and $[N, 1_M]$ is an L-lattice module with $a \cdot K = aK \vee N$ for all $a \in L$ and $K \in M$ such that $N \leq K$.

For various characterizations of lattice modules, the reader is referred to [2–9].

Definition 1. A nonzero L-lattice module M is called second if for each $a \in L$, $a1_M = 1_M$ or $a1_M = 0_M$.

Definition 2. A nonzero L-lattice module M is called secondary if for each $a \in L$, $a1_M = 1_M$ or $a^n 1_M = 0_M$ for some $n > 0$.

Example 3. Let Z be the integers, let Q be the rational numbers, and let Q be Z-module. Suppose $L = L(Z)$ is the set of all ideals of Z and $M = L(Q)$ is the set of all submodules of Q. Thus, M as L-lattice module is a second module, since for every integer $n \in Z$, $(nZ)Q = Q$ or $(nZ)Q = 0$.

Remark 4. Every second lattice module is a secondary lattice. But the converse is not true. For this, we can give the following example.

Example 5. Let Z be the integers and let Z_4 be Z-module. Suppose that $L = L(Z)$ is the set of all ideals of Z and $M = L(Z_4)$ is the set of all submodules of Z_4. Thus, M as L-lattice module is a secondary lattice module, which is not a second lattice module.

Example 6. Let Z be the integers and $L = L(Z)$ the set of all ideals of Z. Thus, L as L-lattice module is neither a second lattice module nor a secondary lattice module.

Proposition 7. *Let L be a CG-lattice and let M be a nonzero L-lattice module. If for each compact $a \in L$, $a1_M = 1_M$ or $a1_M = 0_M$, then M is a second L-lattice module.*

Proof. Let $r \in L$. Since L is a CG-lattice, then we have $r = \bigvee_i c_i$ such that $c_i{}'$s are compact elements of L. Then, we obtain $r1_M = (\bigvee_i c_i)1_M = \bigvee_i c_i 1_M$. We have two cases.

Case 1. If $c_i 1_M = 0_M$ for each compact $c_i \in L$, then we have $r1_M = (\bigvee_i c_i)1_M = \bigvee_i c_i 1_M = 0_M$.

Case 2. If $c_i 1_M = 1_M$ for some compact $c_i \in L$, then we have $r1_M = (\bigvee_i c_i)1_M = \bigvee_i c_i 1_M = 1_M$.

Hence, $r1_M = 1_M$ or $r1_M = 0_M$ for each $r \in L$. Consequently, M is second. □

Proposition 8. *If M is a second L-lattice module, then $ann(M) = (0_M :_L 1_M) = p$ is a prime element of L. In this case, M is called p-second lattice module.*

Proof. Suppose that M is a second L-lattice module. Clearly, $ann(M) = p$ is a proper element of L. Let $ab \leq p$ and assume that $b \nleq p$; that is, $b1_M \neq 0_M$. But M is a second L-lattice module; then $b1_M = 1_M$. Since $b1_M = 1_M$ and $ab1_M = 0_M$, then $a1_M = 0_M$, which implies that $a \leq p$. □

Proposition 9. *If M is a secondary L-lattice module, then $ann(M)$ is a primary element of L.*

Proof. Suppose that M is a secondary L-lattice module. Let $ab \leq ann(M)$ and $b \nleq \sqrt{ann(M)}$; we prove that $a \leq ann(M)$. Since $ab \leq ann(M)$ and $b \nleq \sqrt{ann(M)}$, we have $ab1_M = 0_M$ and $(b)^n 1_M \neq 0_M$ for each $n > 0$. Since M is secondary, we have $b1_M = 1_M$. Then $ab1_M = a1_M = 0_M$, which implies $a \leq ann(M)$. □

Proposition 10. *Let L be a CG-lattice. If M is a secondary L-lattice module, then $\sqrt{ann(M)} = p$ is a prime element of L. In this case, M is called p-secondary lattice module.*

Proof. Let M be a secondary lattice. Then $ann(M)$ is a primary element of L by Proposition 9. Therefore $\sqrt{ann(M)}$ is prime by [1, Lemma 2.1]. □

Proposition 11. *Let L be a lattice domain and let M be a nonzero L-module. Then M is a second lattice module with $ann(M) = 0_L$ if and only if M is a secondary lattice module with $\sqrt{ann(M)} = 0_L$.*

Proof. ⇒: Since M is a second lattice module, then M is a secondary lattice module. Since L is domain, then $\sqrt{ann(M)} = \sqrt{0_L} = 0_L$.

⇐: Suppose that M is a secondary lattice module with $\sqrt{ann(M)} = 0_L$. Let $a \in L$ and assume that $a1_M \neq 1_M$. Since M is a secondary lattice module, then there exists a positive integer n such that $a^n 1_M = 0_M$; that is, $a \leq \sqrt{ann(M)} = 0_L$. Then $a1_M = 0_M$. Hence, we obtain M is a second lattice. Clearly, $ann(M) = 0_L$. □

Definition 12. Let M be a nonzero L-lattice module. An element $0_M \neq N < 1_M$ is said to be pure element, if $aN = a1_M \bigwedge N$ for all $a \in L$.

Proposition 13. *Let L be a CG-lattice, let M be a nonzero L-lattice module, and let N be a pure element of M. If M is a p-secondary lattice module, then $[N, 1_M]$ and $[0_M, N]$ are both p-secondary lattice modules.*

Proof. Suppose that M is a p-secondary lattice module. Let $s \in L$. Since M is a secondary lattice module, then either $s1_M = 1_M$ and in this case $s \cdot 1_{[N,1_M]} = s \cdot 1_M = s1_M \bigvee N = 1_M \bigvee N = 1_M = 1_{[N,1_M]}$ or there exists a positive integer t, such that $s^t 1_M = 0_M$ and in this case $s^t \cdot 1_{[N,1_M]} = s^t \cdot 1_M = s^t 1_M \bigvee N = 0_M \bigvee N = N = 0_{[N,1_M]}$. Hence, $[N, 1_M]$ is a secondary lattice module.

It remains to show that $\sqrt{ann([N, 1_M])} = p = \sqrt{ann(M)}$. Clearly, $\sqrt{ann(M)} \leq \sqrt{ann([N, 1_M])}$. Let r be compact and $r \leq \sqrt{ann([N, 1_M])}$. Since r is compact, there exists a positive integer n such that $r^n \cdot 1_{[N,1_M]} = 0_{[N,1_M]}$; that is, $r^n 1_M \bigvee N = N$. Hence, we have $r^n 1_M \leq N$. Now we assume that $r \nleq \sqrt{ann(M)}$. Then $r1_M = 1_M$, since M is secondary. Thus, $1_M = r^n 1_M \leq N$; that is, $N = 1_M$, which is a contradiction. Therefore, $r \leq \sqrt{ann(M)}$. Consequently, we obtain $\sqrt{ann([N, 1_M])} \leq \sqrt{ann(M)}$.

Let $a \in L$. Since N is pure, $aN = a1_M \bigwedge N$. As M is a secondary lattice module, then either $a1_M = 1_M$ or there exists a positive integer n such that $a^n 1_M = 0_M$. This implies that either $aN = N$ or $a^n N = a^n 1_M \bigwedge N = 0_M$. Therefore, we have $a \cdot 1_{[0_M,N]} = a \cdot N = aN \bigvee 0_M = aN = N = 1_{[0_M,N]}$ or $a^n \cdot 1_{[0_M,N]} = a^n \cdot N = a^n N \bigvee 0_M = a^n N = 0_M = 0_{[0_M,N]}$. Hence, $[0_M, N]$ is a secondary lattice module.

Now we show that $\sqrt{ann(M)} = \sqrt{ann([0_M, N])}$. Clearly, $\sqrt{ann(M)} \leq \sqrt{ann([0_M, N])}$. Let c be compact and $c \leq \sqrt{ann([0_M, N])}$. Since c is compact, there exists a positive integer k such that $c^k \cdot N = c^k N = 0_{[0_M,N]} = 0_M$. Since N is pure, we have $c^k N = c^k 1_M \bigwedge N = 0_M$. If $c \nleq \sqrt{ann(M)}$, then $c1_M = 1_M$. Thus, $c^k 1_M = 1_M$. This implies that $0_M = c^k N = N \bigwedge c^k 1_M = N \bigwedge 1_M = N$, a contradiction. Therefore, $c \leq \sqrt{ann(M)}$. □

Proposition 14. *Let M be a nonzero L-lattice module and let N be a pure element of M. Then M is a p-second lattice module if and only if $[0_M, N]$ and $[N, 1_M]$ are both p-second lattice modules.*

Proof. ⇒: Suppose that M is a p-second lattice module. Let $a \in L$. Since N is pure, we have $aN = N \bigwedge a1_M$. As M is a second lattice module, then either $a1_M = 0_M$ or $a1_M = 1_M$. This implies that either $aN = 0_M$ or $aN = N$. Hence, $[0_M, N]$ is a second lattice module. Now, we show that $ann(M) = ann([0_M, N])$. Clearly, $ann(M) \leq ann([0_M, N])$. Let $r \leq ann([0_M, N])$. Thus, we have $rN = 0_M$. Now we assume that $r \nleq ann(M)$. Then we obtain $r1_M = 1_M$, since M is a second lattice module. This implies that $0_M = rN = r1_M \bigwedge N = 1_M \bigwedge N = N$, a contradiction. Therefore, $r \leq ann(M)$.

Let $t \in L$. Since M is a second lattice module, either $t1_M = 1_M$ and in this case $t \cdot 1_{[N,1_M]} = t \cdot 1_M = t1_M \bigvee N = 1_M \bigvee N = 1_M = 1_{[N,1_M]}$ or $t1_M = 0_M$ and in this case $t \cdot 1_{[N,1_M]} = t \cdot 1_M = t1_M \bigvee N = 0_M \bigvee N = N = 0_{[N,1_M]}$. Hence, $[N, 1_M]$ is a second lattice module. It remains to show that $ann(M) = ann([N, 1_M])$. Clearly $ann(M) \leq ann([N, 1_M])$. Let $s \leq ann([N, 1_M])$. Thus, $s \cdot 1_{[N,1_M]} = 0_{[N,1_M]}$; that is, $s1_M \bigvee N = N$, and this implies that $s1_M \leq N$. Now we suppose that $s \nleq ann(M)$. Then, we have $s1_M = 1_M$, since M is second. Hence, $1_M = s1_M \leq N$, a contradiction. Therefore, $s \leq ann(M)$.

⇐: Suppose that $[0_M, N]$ and $[N, 1_M]$ are both second lattice modules with $ann([0_M, N]) = ann([N, 1_M]) = p$. Let $r \in L$. We have two cases.

Case 1. If $r \leq ann([0_M, N]) = ann([N, 1_M])$, then $rN = 0_M$ and $r \cdot 1_{[N,1_M]} = r \cdot 1_M = r1_M \bigvee N = N$, which implies $r1_M \leq N$. Thus, $0_M = rN = N \bigwedge r1_M = r1_M$.

Case 2. If $r \nleq ann([0_M, N]) = ann([N, 1_M])$, then $rN = N$ since $[0_M, N]$ is a second lattice module. Hence, we have $N = rN = N \bigwedge r1_M$, that is, $N \leq r1_M$, since N is pure. Because $[N, 1_M]$ is a second lattice module and $r \nleq ann([N, 1_M])$; we obtain $r \cdot 1_{[N,1_M]} = r1_M \bigvee N = 1_{[N,1_M]} = 1_M$. Therefore, we obtain that $r1_M = 1_M$. Consequently, M is a second lattice module.

Now we show that $ann(M) = p$. Clearly $ann(M) \leq ann([N, 1_M])$. Let $s \leq ann([N, 1_M])$. Then we have $s \cdot 1_{[N,1_M]} = 0_{[N,1_M]}$, that is; $s \cdot 1_M = N$. Thus, $s1_M \bigvee N = N$, and so $s1_M \leq N$. Now, we assume that $s \nleq ann(M)$. Then, we have $s1_M = 1_M$, since M is second. Hence, $1_M = s1_M \leq N$, a contradiction. Consequently, we have $s \leq ann(M)$. □

Definition 15. An L-module M is called a multiplication lattice module if for every element $N \in M$, there exists an element $a \in L$, such that $N = a1_M$.

Definition 16. A element N of an L-module M is called prime element if $N \neq 1_M$ and whenever $r \in L$ and $X \in M$ with $rX \leq N$, then $X \leq N$ or $r \leq (N:_L 1_M)$.

Definition 17. A element N of an L-module M is called semiprime element if $N \neq 1_M$ and whenever $r \in L$ and $X \in M$ with $r^2 X \leq N$, then $rX \leq N$.

Remark 18. Let N be a proper element of an L-module M. Then N is a semiprime element if and only if whenever $r \in L$, $X \in M$ and k is a positive integer with $r^k X \leq N$, then $rX \leq N$.

We know that a prime element is semiprime, but the converse is not true in general. The following proposition shows that the converse is true when the module is secondary and multiplication.

Proposition 19. *Let M be a multiplication and secondary L-lattice module. For all element N of M such that $1_M \neq N \in M$, N is a semiprime element of M if and only if N is a prime element of M.*

Proof. ⇒: Suppose that N is a semiprime element of M and let $rX \leq N$, where $r \in L$, $X \in M$. Since M is a secondary lattice module, then either $r^n 1_M = 0_M$ for some positive integer n or $r1_M = 1_M$.

Case 1. If $r^n 1_M = 0_M$, then $r^n 1_M \leq N$. Since N is a semiprime element, we have $r1_M \leq N$.

Case 2. If $r1_M = 1_M$, then we have $X = rX$, since M is a multiplication lattice module. Then we have $rX = X \leq N$.

Therefore, N is a prime element of M.

⇐: It is obvious. □

Definition 20. Let M be an L-lattice module and let N be a proper element of M. N is called a primary element of M, if whenever $a \in L$, $X \in M$ such that $aX \leq N$, then $X \leq N$ or $a \leq \sqrt{(N:_L 1_M)}$. Particularly, if M is nonzero and 0_M is primary, then M is said to be primary lattice module.

Definition 21. An L-lattice module M is said to be simple lattice module if $M = \{0_M, 1_M\}$.

Proposition 22. *Every multiplication secondary lattice module is a primary lattice module.*

Proof. Let M be a multiplication secondary module and $rX = 0_M$ for some $r \in L$, $X \in M$. Now, we assume that $r\sqrt{ann(M)}$. Since M is a secondary module, then we have $r1_M = 1_M$. Because M is a multiplication, then we have $rX = X$. Consequently, we obtain $X = 0_M$. □

Proposition 23. *Every multiplication second lattice module is a simple lattice module.*

Proof. Let M be a multiplication and second module. Since M is a multiplication, for every $N \in M$, there exists $a \in L$ such that $N = a1_M$. Then we obtain $a1_M = 1_M$ or $a1_M = 0_M$, since M is second. Thus, we have $N = 1_M$ or $N = 0_M$ for every $N \in M$; that is, M is simple. □

Definition 24. Let L be a domain and let M be a nonzero L-lattice module. If $r1_M = 1_M$ for every $0_L \neq r \in L$, then M is said to be divisible.

Definition 25. A nonzero L-lattice module M is said to be torsion if there exists $0_L \neq r \in L$ such that $r1_M = 0_M$.

Proposition 26. *Let L be a domain. Let M be a secondary L-lattice module. Then either M is a divisible module or M is a torsion module.*

Proof. Suppose that M is a secondary module over a domain L. If M is not divisible, then there exists $0_L \neq r \in L$ such that $r1_M \neq 1_M$. Since M is a secondary lattice module, then there exists a positive integer n such that $r^n 1_M = 0_M$. Since $0_L \neq r$ and L is a domain, then we have $r^n \neq 0_L$. Consequently, there exists $0_L \neq r^n = s \in L$ such that $s1_M = 0_M$. Therefore, M is a torsion lattice module. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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