A Simple Model of Growth Slowdown*

Katsuyuki Shibayama †
School of Economics
University of Kent

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Abstract

This paper studies a simple endogenous growth model to explain growth slowdowns. It is designed to explain, for example, the middle income trap often observed in the south-east Asian countries, the U.K.'s productivity puzzle after the Great Recession and the lost decades of Japan in a unified framework. It is based on the Romer's (1990, JPE) variety expansion model with additional state variable, which we call the R&D environment. The R&D environment is a sort of social capital that captures the research network and culture, society's attitude towards research activities, and so on. Together with the non-negativity constraint of the labour supply, this additional state variable generates multiple steady states (balanced growth paths, BGPs). The model has three BGPs, of which the middle one is unstable (explosive) while the other two satisfy the saddle path stability with high and low R&D activities. Without stochastic shocks, the model exhibits strong initial state dependency, meaning that even only small difference in the initial state could lead to a large difference in the long-run. With stochastic shocks, occasional shifts between two stable BGPs can occur. The model offers an intuitive explanation why a financial shock is particularly important for growth slowdowns. Interestingly, before a growth slowdown, a financial malfunctioning raises the stock return. Finally, our model is fairly realistic in the sense that it allows us to do calibration exercises which are rather standard in the business cycle studies.

KEYWORDS: endogenous growth; growth slowdown; dynamic stochastic general equilibrium model; middle-income trap; natural resource curse; productivity puzzle; R&D; official development assistance

JEL CLASSIFICATION: E3, O3, O4

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† **Corresponding author**: School of Economics, University of Kent, Canterbury, Kent, CT2 7NP, U.K. Phone: +44(0)122782-4714. E-mail: k.shibayama@kent.ac.uk

Non-Technical Summary

After the Great Recession around 2007-8, the U.K. has experienced the slowdown of the labour productivity; known as 'productivity puzzle'. Although it is widely recognized as the stagnation of the labour productivity, not surprisingly, it coincides with the flattening of the total factor productivity (TFP); see Figure 1. Japan also has experienced a similar but much longer and deeper phenomenon after the bubble burst at the beginning of 1990s; see Figure 2. Interestingly, both countries have experienced the growth slowdown after some financial turmoil. This paper aims to explain these growth slowdowns quantitatively in a reasonably realistic economic growth model.

A growth slowdown is a long-lasting, significant decline of growth rate. It is not specific to high-income countries. The economic model developed in this paper also encompasses the "middle income trap"; typical examples include Latin American countries. The standard economic growth theory tells us that low-income countries should grow faster because they tend to accumulate production capital at a faster rate (Solow effect). Actually, many countries has successfully escaped from low income levels, but, out of 101 middle income countries in 1960, only 13 of them is classified as high income countries in 2008; see Larson, Loayza and Woolcock (2016, World Bank).

Extending Romer's (1990) seminal paper, our model has an additional state variable, which we call the R&D environment, to capture social culture (scientists' attitudes toward business, etc.), legal system (including patent laws and property rights), R&D infrastructure (such as innovators' networks and education systems), and so on. We can regard the R&D environment as an intangible social capital, which has the following two properties; (a) society accumulates the R&D environment as an (intangible) asset by conducting R&D; and (b) the R&D activities are more productive when the R&D environment takes a higher value. For (a), an important assumption is such an accumulation of social asset is a positive externality of the R&D activities; i.e., the researchers do not intend to improve the R&D environment, when they engage in R&D. For (b), we want to capture, for example, higher education institutions are better prepared for the commercialization of academic findings when business innovations and inventions are more active.

In our model, there are two stable balanced growth paths (BGPs, long-run equilibria); one with positive R&D activities and the other without them. This is intuitively because of vicious and virtuous cycles. Around the BGP with no R&D, there is a vicious cycle; once R&D becomes inactive, it deteriorates the R&D environment, which itself discourages the R&D activities. Similarly, there is a virtuous cycle in the good BGP.

If there are no shocks (or only small shocks) in the model, then the fate of an economy depends on its initial condition. In our model, depending on the initial state, its final destiny – good BGP or bad BGP – is predetermined. Note importantly that in our model, even an economy moves toward the bad BGP, still it grows at a faster rate in early periods through the capital accumulation (Solow effect); it fails to switch from the capital accumulation as a growth engine to the R&D driven growth. We argue that countries that successfully transited from the middle-income level have had a R&D environment good enough to attain this transition; such as Japan and West Germany after the WWII. This is the explanation of the middle-income trap in our model; see Figure 6.

If there are some shocks in the model, even if they are temporary, still they can have long-run effects. We assume that after a successful innovation, innovators can set up a firm, meaning that the firm value is the reward to innovations. Like Comin and Gertler (2003, AER), because the firm value is the present value of the current and future profits, R&D is more active in booms. Unlike Comin and Gertler's 'medium cycle' effects, however, business cycle fluctuations can have very persistent effects in our model; this is because a shock may push out an economy from one BGP to the other. For example, if an economy experiences a bad external shock, output, firm profit and firm value decline, which in turn discourages innovations. If such a bad shock is large enough and lasts long enough, the dormant R&D activities during the recession may deteriorate significantly, which hampers R&D even after the end of the negative external shock. In our model, this is the mechanism behind the growth slowdown in Japan and the U.K.; see Figure 8.

There are several policy implications. First, to help escape from the middle-income trap and the long-lasting growth slowdown, we need a 'big push'. For example, a large scale ODA as a positive external shock may be required. Second, however, the type of ODA matters. In our model, improving production efficiency and increasing final goods demand do not help escape from the low BGP, because there are two effects that offset each other. On the one hand, these shocks increase the firm profit and firm value, which stimulates the R&D activities. However, on the other hand, the production sector and the R&D sector compete each other in the factor market (labour market in our model). Hence, in our model, if the production efficiency improves, the production sector absorbs more labour, squeezing out the R&D sector from the labour market. There is anecdotal evidence of type of phenomenon; e.g., Wall Street and City have taken many scientists from universities and other research institutes such as NASA. In addition, this story is in parallel with the leading exposition about the natural resource curse; if the extraction of natural resources are very profitable, the natural resource sectors absorb too much production factors such as labour, squeezing out high productivity growth sectors such as manufacturing. Hence, if they only improves production efficiency, building bridges, roads and power plants, for example, has little effect in total. Third, our numerical experiments suggest that the financial sector efficiency has a strong effect. If the financial market is malfunctioning, the firm value may be discounted unduly. In our model, because the firm value is the reward to successful R&D, such a mispricing of the firm value discourages the R&D efforts. This is reminiscent of the fact that the U.K. and Japan have experienced the growth slowdown after the financial market turmoil; see Figure 9.

In conclusion, this paper asserts that a sort of intangible social assets such as culture, legal system, etc. are important for R&D. Our model has multiple BGPs. Even though an economy has experienced a rapid growth via capital accumulation (Solow effect), it may fail to switch to the R&D driven growth (middle-income trap). In addition, even though a country successfully achieved a high-income level, it could slip down from the good BGP, particularly if it suffers from deep and long financial turmoil (like Japan and the U.K.). To regain the sustainable R&D environment, the model suggests the following policy prescriptions; (i) policy measures that directly affect the R&D productivity such as the subsidy to higher education and R&D tax credit and (ii) policies to improve the efficiency in the financial market.

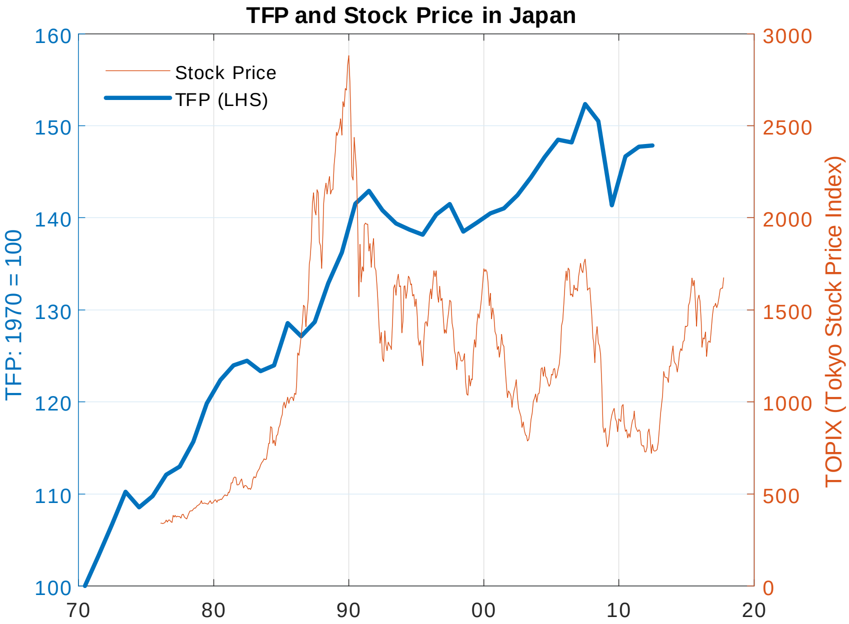

Figure 1: Source: Japan Industrial Productivity Database 2015 (JIP2015). The TFP growth rate is converted into the level. TOPIX means Tokyo Exchange Stock Price Index.

1 Introduction

To explain growth slowdowns, this paper develops a simple model which is an extension of Romer's (1990, JEP) variety expansion model. The key feature of the model is that it has two stable balanced growth paths (BGPs); one with positive endogenous growth and the other without it.[1] Throughout this paper, the BGP with positive endogenous growth is called the high BGP, while that without endogenous growth is called the low BGP (also, 'steady state' is interchangiable to 'BGP' in this paper). Our model explains a growth slowdown as a move from the high BGP to the other, while the Rostow's take-off takes place when an economy moves from the low BGP to the other. The model is fairly realistic in the sense that it allows us a reasonable calibration exercise, which is rather common in the business cycle studies.

Here, a growth slowdown means a significant decline in the rate of economic growth for a certain long period. For example, Eichengreen, Park and Shin (2013, NBER) define growth slowdown as a 2%-point decline or more in the growth rate in successive 7-year period. They find that such growth slowdowns tend to occur in middle income countries, hence called middle income trap. The middle income trap is a concern of policymakers as well as academics. Actually, World Bank, Asian Development Bank, etc. have discussed growth slowdowns (focusing on the middle income trap) repeatedly.[2] But, a growth slowdown can also happen to high income countries. Indeed, in our calibration exercises, what we have in mind is the growth slowdown experienced by Japan in 1990s and 2000s, which is often called "lost decades" after the bubble burst at the end of 1980s, and the

[1] Actually, the model has one more BGP, which is however explosive (locally unstable), between the high and low BGPs. Because the unstable BGP is sandwiched by the other two, the model exhibits global stability.

[2] See, for example, Im and Rosenblatt (2015) and Felipe, Kumar and Galope (2014).

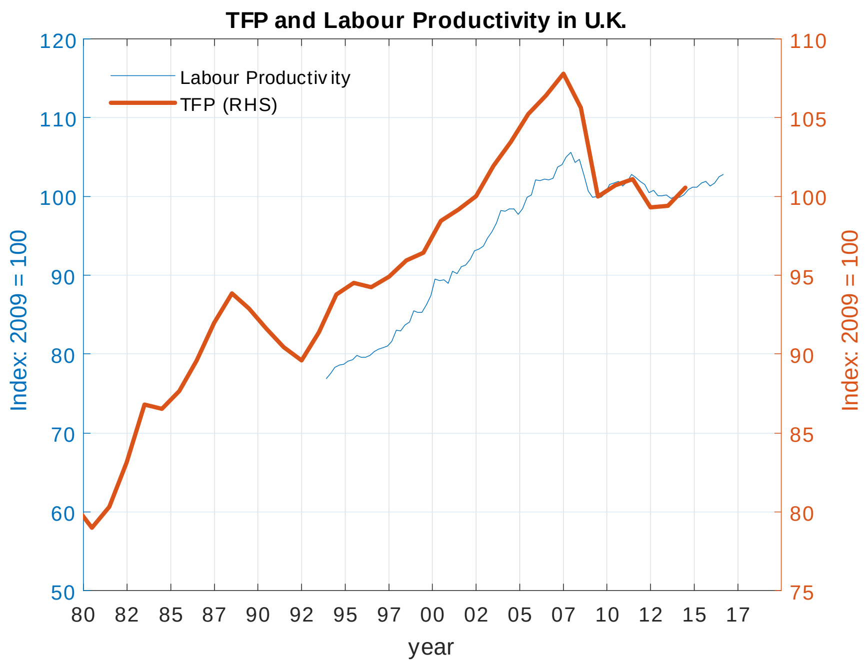

Figure 2: Source: Blunden and Franklin (2016) and ONS. The TFP growth rate is converted into the level.

productivity puzzle in the U.K. after the great recession in late 2000s; see Figures 1 and 2. For the latter, the shift in the TFP trend is one of the major reasons of the slowdown of the U.K. labour productivity growth. The model developed in this paper is however applicable to study the middle income trap and the poverty trap.

Our model extends Romer (1990) model by introducing an additional state variable, which we call the R&D environment. The R&D environment in this paper loosely captures a wide range of social and institutional environment surrounding R&D activities. It includes for example social culture (scientists' attitudes toward business, etc.), legal system (including patent laws and property rights), R&D infrastructure (such as innovators' networks and education systems), and so on. Although there are some controversies in each of them, given so many economic and non-economic papers in this field,[3] their collective importance seems rather obvious. In this paper, rather than cutting into the details of each aspect of these, we consider an abstract state variable (the R&D environment), which has the following two properties; (a) the better the R&D environment is, the higher the R&D productivity is; and (b) society accumulates the R&D environment via R&D activities. Property (b) says, for instance, inventors' research networks tend to be well developed when inventions are more active; higher education institutions are better prepared for the commercialization of academic findings where business innovations and inventions are more active, and so on. In this respect, importantly, we assume an externality; individual inventors do not take into account the effect of their R&D activities on the R&D environment, which is a society-wide variable. This positive feedback (externality) is the key mechanism to generate multiple BGPs in our model. Intuitively, if an economy obtains a high level of the R&D environment, then the higher R&D productivity encourages more R&D activities,

[3] Actually, rather than saying "so many", we could say that there are "too many" to cite. To name a few, see Samila and Sorenson (2017), Audretsch and Keilbach (2004), Beugelsdijk (2010) and Freeman (1991).

which in turn supports the accumulation of the high R&D environment. Contrarily, once R&D activity is disturbed for a long period, the R&D environment deteriorates, which discourages R&D activities. That is, there are virtuous and vicious cycles between individual inventions and society's R&D environment.

Methodologically, we apply quantitative techniques that are rather common in the business cycle studies. As in Comin and Gertler's (2006, AER) "medium-term cycles", the business cycle fluctuations have impacts on R&D activities. In addition to their "medium-term" effect, however, in our model, temporary shocks may also have a "long-term" effect, in the sense that if a shock is either strong enough or long-lasting enough, an economy moves from one BGP to the other. The mechanism behind medium-term effects (transition dynamics generated by a shock) is the same as in Comin and Gertler (2006). For example, consider a temporary productivity improvement. It raises output and hence increases the firm profit, meaning that the firm value also increases. Because this class of models assume that after a successful invention the inventor sets up a new business, a higher firm value means a higher reward to successful inventions. Because it encourages the R&D activities, the TFP growth increases during the period when the productivity is high.[4] In this way, business cycles matter in economic growth.

Our quantitative exercises however find that the productivity has only a little impact on TFP. In our model, as in the original Romer model, the production and the R&D sectors compete each other in the factor markets. More particularly, these two sectors compete in the labour market, because the model assumes that each inventor uses only labour in her R&D activities, given social level of R&D environment and past knowledge accumulation. This generates the second channel; if a higher output level increases labour demand in the production sector, it pushes up wage. A higher wage however increases the cost of R&D, discouraging inventions. Because the improvement in the productivity in the production sector has these two offsetting effects, its impact on R&D is very small. Actually, we have stronger theoretical result in terms of the long-term effect. That is, because these two channels offset each other *exactly*, the production efficiency (exogenous productivity shock) and the government expenditure (exogenous demand shock) are neutral in the BGPs; they have no long-run effects on R&D at all.[5] The main reason why our results are so different from Comin and Gertler (2006), cutting into the gist point, lies on the fact that our shocks last longer than theirs, because we are rather interested in decennial economic fluctuations. Note that, even in our model, the production efficiency and government expenditure have some effects on R&D in the transition periods.

This has stark policy implications. For example, consider an official development assistance (ODA). Certainly, the model suggests that a big-push is necessary to achieve Rostow's take-off, in the sense that there must be something big that pushes out an economy from the low BGP. However, the model also suggests that the type of ODA matters. Suppose an ODA improves the production efficiency of a country forever, by building a new power plant, new road, etc., which presumably improves the production efficiency. Then, the level of the income per capita of this country will

[4] As usual, we define TFP growth as the output growth that cannot be explained by the growth or capital and labour.

[5] Rephrasing this, these two shocks have no effects in comparative statistics.

increase, which however is a one-off lift-up of income. At the same time, working in the production sector becomes more attractive than working in the R&D sector, which discourages R&D activities. This sort of ODA does not support sustainable growth.

Note that this intuition itself is not new. Indeed, one of the leading expositions of the natural resource curse is that a high productivity sector (natural resource extraction) absorbs production factors, squeezing other sectors such as manufacturing where the productivity growth is faster.[6] In our model, the sector squeezed is not the sector with a fast productivity growth, but the R&D sector. Also, unlike the natural resource curse story, a high productivity in the production sector is not negative for R&D. Nonetheless, the competition in the factor markets is the key both in this model and in the natural resource curse story.

Then, what shocks matter for R&D in the long-run? Of course, an exogenous improvement in the R&D productivity has a sharp effect; not surprising. What is interesting is that the R&D is very sensitive to the efficiency in the financial sector. The intuition is as follows. Recall that the firm value (share price) is the reward to successful inventions. Suppose that investors require high risk-premium in the stock investment, because of a financial inefficiencies. Then, given level of firm profit, due to a higher discount rate (a higher risk-premium), the stock price is lower. This lower reward to business inventions discourages R&D activities.

Moreover, the stock market boom is tightly related to growth slowdowns in our model. Consider the scenario, in which an economy starting from the high BGP moves to the low BGP after a long financially inefficient period. In this model, interestingly, during the period of a negative financial shock, the stock return increases (not decreasing), and then this economy goes to the low BGP. We cannot label this stock market boom as a "bubble", because our model has only rational agents. In our model, behind the increase in the stock return, a negative financial efficiency disturbs finding channel to the R&D sector which discourages working in the R&D sector, shifting labour temporarily to production. This increases output per firm and hence firm profit. There is a discrepancy between the aggregate economy and individual firms; while a low level of inventions decreases the aggregate output growth by reducing the speed of the creation of new varieties (creation of new firm in our model), the output and profit of each incumbent firm increases due to the labour shift. This mechanism sheds light on many episodes, where there were financial boom and bust prior to the growth slowdown such as Japan's lost decades and the U.K.'s productivity puzzle. We could include the growth slowdown experienced by the Asian countries after the Asian financial crisis in the mid 1990s.

The plan of this paper is as follows. Section 2 introduces the model, which is the extension of the discrete time version of the Romer's variety expansion model. Section 3 discusses the parameter selection and the steady states. While Section 3 investigates the transition dynamics when there is no shock. Section 4 considers the dynamics with shocks. We particularly focus on the two scenarios; one for a growth slowdown after a long-lasting financially distressed period, and the other for Rostow's take-off. Section 5 discusses several relevant issues. Finally, Section 6 concludes.

[6] See for example Frankel's (2010, NBER) survey.

2 The Model

This paper discusses the discrete time version of the model.[7] The model is a simple extension of Romer's variety expansion model. Although the model is almost standard, to introduce the notations, we show the model setup and the equilibrium equations. A reader who is familiar with Romer (1990) may want to skip Section 2.1 and jump to Section 2.3. See equations (1) and (2) for the equilibrium equations, and Tables 2 and 3 for notations.

2.1 Model Setup and First Order Conditions

The model overview is as follows. The representative household (HH) consumes the consumption goods, saves via bonds and supplies labour. The HH saves only via the bonds issued by the financial intermediary (FI), which in turn invests in capital and equity. The representative final goods producer (F-firm) uses intermediate goods $X_{i,t}$ (M-goods, variety i) and production labour $H_{Y,t}$ to produce final goods Y_t (F-goods), which is allocated into consumption C_t, investment I_t and government expenditure G_t. The intermediate goods producers (M-firms) use only capital K_t to produce $X_{i,t}$, which is sold in the monopolistically competitive M-goods market. The inventors only uses R&D labour $H_{A,t}$. If an invention is successful, a new variety is created, then the successful inventor sets up a new firm. The value of a new firm is V_t, which is financed by the financial intermediary through the stock market. In this version, prices are flexible and the only friction is the exogenously given risk premium on the stock investment.

Setting aside some minor modifications, the only major difference from Romer (1990, JPE) is that we introduce a sort of social capital Z_t, which captures the R&D environment. When Z_t is higher, R&D productivity is higher. In addition, the R&D environment improves when R&D is more active.

2.1.1 Household (HH)

The representative household (HH) inelastically supplies labour $\bar{H}_t$, which (exogenously) grows at $n_t = \bar{H}_t/\bar{H}_{t-1}$; we however set $n_t = 1$ (no population growth) in the calibration. We keep this notation to discuss the effects of population growth later; see Section 6.4. HH allocates $H_{Y,t}$ to production and $H_{A,t}$ to R&D; $\bar{H}_t = H_{Y,t} + H_{A,t}$. This additivity in the labour supply implies that the required wage rates for production and R&D must be the same. The non-negativity constraint of $H_{A,t} \geq 0$ ($H_{Y,t} \leq \bar{H}_t$) may or may not bind, while $H_{A,t} \leq \bar{H}_t$ ($H_{Y,t} \geq 0$) never binds due to the Inada property of the production function. HH also consumes and invests in capital and R&D indirectly via purchasing bonds B_t^N. We denote $R_{t,t+1}^{NB}$ as the gross nominal return of the bonds between time t to $t+1$ issued by the financial intermediary (FI). Likewise, $\Lambda_{t,t+1}$ is the stochastic discount factor (SDF) from time t to $t+1$. Tax $\tau_{G,t}$ and profits from F-firms Π_t^Y and M-firms Π_t^X are lump-sum.

[7] We also have developed the continuous time version (analytical derivations only), which is available upon the request.

Finally, $\zeta_{\beta,t}$ is time preference shock.

$$
\begin{aligned}
&\max E_t \left[\sum_{\tau=t}^{\infty} \beta_{t,\tau} \frac{(C_\tau)^{1-\sigma}}{1-\sigma} \right] \\
&\text{s.t. } C_t + B_{t+1}^N / R_{t,t+1}^{NB} \leq W_t \bar{H}_t + B_t^N + \Pi_t^Y + \Pi_t^X + \tau_{G,t} \\
&\qquad H_{Y,t} + H_{A,t} \leq \bar{H}_t \quad \text{with} \quad H_{Y,t} \leq \bar{H}_t \\
&\qquad \ln \beta_{t,\tau} = (\tau - t) \ln \beta + \textstyle\sum_{i=t+1}^{\tau} \ln \zeta_{\beta,i} \text{ for } \tau \geq t
\end{aligned}
$$

The first order conditions (FOCs) are as follows.

$$
\begin{aligned}
\partial B_{t+1}^N &: 1 = E_t \left[\Lambda_{t,t+1} / \pi_{t,t+1} \right] R_{t,t+1}^{NB} \\
\partial C_t &: \lambda_t = \left(C_t / \bar{H}_t \right)^{-\sigma} \\
\text{SDF} &: \Lambda_{t,s} = \beta_{t,s} \lambda_s / \lambda_t \quad \left(\Lambda_{t,t+1} = \zeta_{\beta,t+1} \beta \left(\frac{C_{t+1}/\bar{H}_{t+1}}{C_t/\bar{H}_t} \right)^{-\sigma} \right) \\
\text{TVC} &: \lim_{t \to \infty} \Lambda_t B_t = 0
\end{aligned}
$$

where λ_t is the Lagrange multiplier on the budget constraint. Note that aggregate consumption C_t is the consumption per head times population. Defining consumption per capital as $c_t = C_t / \bar{H}_t$, the SDF is $\Lambda_{t,t+1} = \zeta_{\beta,t+1} \beta \left(n_{t+1} c_{t+1} / c_t \right)^{-\sigma}$.

2.1.2 Financial Intermediary (FI)

A financial intermediary (FI) raises funds from HH by issuing bonds B_t^N, while it provides funds to intermediate goods producers (M-firms) in two ways. The one is via contingency claims for their capital investment and the other is the purchase of the share of newly established M-firms. In our notation, N_t^X is the number of shares of M-firms, and $R_{t,t+1}^C$ is the return on the claims on capital lending B_{t+1}^K. To match the equity premium in the data, we add risk-premium parameter c_v, which is constant in this version. As a shareholder, FI obtains both income gain (profit per firm $\bar{\Pi}_t^X$ times number N_t^X of shares) and capital gain $(1 - \delta_A) \bar{V}_{t+1}^X / \bar{V}_t^X$, where $\bar{V}_t^X$ is the value of an M-firm, while δ_A is the rate that a variety becomes obsolete.

$$
\begin{aligned}
&\max E_t \left[\sum_{\tau=t}^{\infty} \Lambda_{t,t+\tau} \Pi_{t+\tau}^B \right] \\
&\text{s.t. } \Pi_t^B = B_{t+1}^N - R_{t-1,t}^{NB} B_t^N && \text{(bank profit/cash-flow)} \\
&\qquad + R_{t-1,t}^C B_t^K - B_{t+1}^K && \text{(state contingency claims for } K_t\text{)} \\
&\qquad + \frac{1}{c_v} \left(\bar{\Pi}_t^X N_t^X + (1 - \delta_A) \bar{V}_{t-1}^X N_t^X \right) - \bar{V}_t^X N_{t+1}^X && \text{(equity investment for R\&D)}
\end{aligned}
$$

The first order conditions (FOCs) are as follows.

$$\partial B_{t+1}^{N} : 1 = E_t \left[\Lambda_{t,t+1}\right] R_{t,t+1}^{NB} \qquad \text{(same as the HH's FOC)}$$
$$\partial B_{t+1}^{C} : 1 = E_t \left[\Lambda_{t,t+1} R_{t,t+1}^{C}\right]$$
$$\partial N_{t+1}^{X} : 1 = E_t \left[\frac{\Lambda_{t,t+1}}{c_v} \left\{ \frac{\bar{\Pi}_{t+1}^{X}}{\bar{V}_t^{X}} + (1-\delta_A) \frac{\bar{V}_{t+1}^{X}}{\bar{V}_t^{X}} \right\}\right]$$

2.1.3 Final Goods Firms (F-firms)

The final goods firm (F-firm) purchases intermediate goods $X_{m,t}$ from M-firms, and they aggregate $X_{m,t}$ into $\hat{X}_t$. Then, F-firm combine $\hat{X}_t$ with labour $H_{Y,t}$ to produce final goods Y_t, which is divided into consumption C_t, investment I_t and government expenditure G_t. Parameter ζ_Y shows the productivity level, which is constant in this version. We keep this notation, for the later reference; see Section 2.3.2.

$$\max E_\tau \left[\sum_{t=\tau}^{\infty} \Lambda_{\tau,\tau+t} \left\{ Y_t - W_t H_{Y,t} - \hat{P}_t^X \hat{X}_t \right\} \right]$$
$$\text{s.t. } Y_t = \zeta_Y H_{Y,t}^{1-\alpha} \left(\hat{X}_t\right)^{\alpha}$$
$$Y_t = C_t + I_t + G_t$$
$$\hat{X}_t = \left(\int_0^{A_t} \left(X_{m,t}\right)^{\frac{\theta_x - 1}{\theta_x}} \mathrm{d}m \right)^{\frac{\theta_x}{\theta_x - 1}}$$
$$\hat{P}_t^X = \left(\int_0^{A_t} \left(P_{m,t}^X\right)^{1-\theta_x} \mathrm{d}m \right)^{\frac{1}{1-\theta_x}}$$

Note that $\hat{X}_t$ and $\hat{P}_t^X$ are the Digizt-Stiglitz quantity and price indices of intermediate goods. If elasticity of substitution satisfies $\theta_x = 1/(1-\alpha)$, then $Y_t = \zeta_Y H_{Y,t}^{1-\alpha} \int_0^{A_t} (X_{m,t})^{\alpha} \mathrm{d}m$. That is, the production function reduces to the formulation that is exactly the same as Romer (1990, JPE). But, there is no reason to believe this coincidence.[8]

The first order conditions (FOCs) are as follows.

$$W_t = (1-\alpha) Y_t / H_{Y,t}$$
$$\hat{P}_t^X = \alpha Y_t / \hat{X}_t$$

We shortly show that the equilibrium is symmetric, $X_{m,t} = \bar{X}_t$ and $P_{m,t} = \bar{P}_t$ for all m. Note

[8] See Section 2.3.4 for further discussions on this.

that $\hat{X}_t \neq \bar{X}_t$ and $\hat{P}_t^X \neq \bar{P}_t^X$; instead, we have the following relationships.

$$\hat{X}_t = A_t^{\frac{\theta_x}{\theta_x - 1}} \bar{X}_t$$
$$\hat{P}_t^X = A_t^{\frac{-1}{\theta_x - 1}} \bar{P}_t^X$$
$$\hat{P}_t^X \hat{X}_t = \int_0^{A_t} P_{m,t}^X X_{m,t} \mathrm{d}m = A_t \bar{P}_t^X \bar{X}_t$$

2.1.4 Intermediate Goods Firms (M-Firms)

There are many intermediate goods firms (M-firms), each of which has monopolistic power. Hence, they earn a positive profit, which is the incentive for inventions. Because each M-firm produces a particular variety of M-goods, the "number" of M-firms and the "number" of the varieties of M-goods are both A_t. An M-firm m produces $X_{m,t}$ only by employing capital $K_{m,t}$ with fixed productivity η_K. Capital depreciation is δ_K and capital investment is $I_{m,t}$. Consistently from the previous subsection, firm m finances its capital expenditure by borrowing $B_{m,t}^K$ from FI.

$$\max E_\tau \left[\sum_{t=\tau}^{\infty} \Lambda_{\tau,\tau+t} \Pi_{m,\tau}^X \right]$$
$$\text{s.t. } \Pi_{m,t}^X = P_{m,t}^X X_{m,t} - R_{t-1,t}^C B_{m,t}^K + B_{m,t+1}^K - I_{m,t}$$
$$\frac{X_{m,t}}{\hat{X}_t} = \left(\frac{P_{m,t}^X}{\hat{P}_t^X} \right)^{-\theta_x} \qquad \text{(demand schedule)}$$
$$X_{m,t} = \eta_X K_{m,t} \qquad \text{(M-goods production)}$$
$$B_{m,t+1}^K = K_{m,t+1} \qquad \text{(balance sheet constraint)}$$
$$K_{m,t+1} = (1 - \delta_K) K_{m,t} + I_{m,t} \qquad \text{(law of motion of } K\text{)}$$

The first order conditions (FOCs) are as follows.

$$\partial X_{m,t} : P_{m,t}^X = \frac{\theta_x}{\theta_x - 1} \lambda_{m,t}^X \qquad \text{(mark-up formula)}$$
$$\partial K_{m,t} : \eta_X \lambda_{m,t}^X = R_{t-1,t}^C - (1 - \delta_K) \qquad (\lambda_{m,t}^X \text{ is marginal cost})$$

The latter shows that the marginal cost $\lambda_{m,t}^X$ is the same for all M-firms, and hence the former shows that their sales prices $P_{m,t}^X$ are the same as well; i.e., the equilibrium is symmetric. Define a shorthand notation $\tilde{\alpha} = (\theta_x - 1)/\theta_x$, which is implicitly assumed to be α in Romer's (1990, JPE) parameterization.

Note that, unlike the intermediate goods aggregation, we consider the simple aggregation for capital K_t.

$$K_t = \int_0^{A_t} K_{m,t} \mathrm{d}m = \frac{1}{\eta_K} A_t \bar{X}_t = A_t^{\frac{-1}{\theta_x - 1}} \frac{\hat{X}_t}{\eta_K}$$

Then, from F-firms' FOC of $\hat{X}_t$, we obtain

$$R^C_{t-1,t} = \tilde{\alpha}\alpha Y_t/K_t + (1-\delta_K)$$

Also, substituting $\hat{X}_t$ out from the F-firms' production technology, we obtain the aggregate production function

$$Y_t = \zeta_Y \left(A_t^{\varphi_A} H_{Y,t}\right)^{1-\alpha} \left(\eta_K K_t\right)^{\alpha}$$

where $\varphi_A = \frac{\alpha}{1-\alpha}\frac{1}{\theta_x - 1} = \frac{\alpha}{\tilde{\alpha}}\frac{1-\tilde{\alpha}}{1-\alpha}$, which is one under Romer parameterization $(\theta_x = 1/(1-\alpha))$.

2.1.5 Inventors

Using labour input $H_{A,t}$, inventors create new variety A_t^{new} of M-goods. Once they invent a new variety, they set up a new firm, which has the value of $\bar{V}_t^X$ (value of one firm). In the "production function" of R&D, there are two externalities. First, inventors take the stock of past knowledge A_t as given. This is the classical R&D externality, to generate endogenous growth. In addition, we introduce another externality through R&D environment Z_t. This works as a sort of non-tangible social capital, which captures the society's R&D environment. If Z_t per worker takes a higher value, R&D is more productive. Also, importantly, we assume that Z_t is accumulated via R&D activities; that is, if the share of R&D labour in the total labour is larger, Z_t increases. This says that the R&D environment improves when the R&D is active. We also assume that the R&D environment deteriorates at the rate of δ_Z. Parameter z_L prevents the low BGP from being an absorbing state; Z_t approaches to $z_L > 0$ if $H_{A,t} = 0$ for all t. If Z_t is a constant, the model reduces to Romer (1990, variety expansion). In this version of the model, we consider a shock $\zeta_{\omega_A,t}$ to R&D productivity ω_A.

$$\begin{aligned}
&\max \bar{V}_t^X A_t^{new} - W_t H_{A,t} \\
&\text{s.t. } A_t^{new} = \zeta_{\omega_A,t}\omega_A \frac{Z_t}{\bar{H}_t} H_{A,t} A_t && \text{(lom of } A_t\text{)} \\
&\quad Z_{t+1} = \omega_Z \frac{H_{A,t}}{\bar{H}_t} + \delta_Z z_L + (1-\delta_Z) Z_t && \text{(lom of aggregate } \tilde{Z}_t\text{)} \\
&\quad A_{t+1} = A_t^{new} + (1-\delta_A) A_t \\
&\quad \bar{V}_t^X = E_t\left[\frac{\Lambda_{t,t+1}}{c_v}\left\{\bar{\Pi}_{t+1}^X + (1-\delta_A)\bar{V}_{t+1}^X\right\}\right] && \text{(banks' FOC for reference)}
\end{aligned}$$

The first order conditions (FOCs) are as follows.

$$\partial H_{A,t} : W_t = \zeta_{\omega_A,t}\omega_A \frac{Z_t}{\bar{H}_t} A_t \bar{V}_t^X + \mu_t$$

where the Lagrange multiplier μ_t is 0 only when the non-negativity condition of $H_{A,t}$ is not binding. If it is binding $(H_{A,t} = 0)$, W_t is not equal to the labour productivity in the R&D sector, due to $\mu_t > 0$. Typically this happens when Z_t is low. In the production sector, on the other hand, $W_t =$

$(1-\alpha) Y_t/H_{Y,t}$ holds always. Hence,

$$W_t = (1-\alpha)\,\zeta_Y A_t^{\varphi_A(1-\alpha)} (\eta_X K_t)^{\alpha} (H_{Y,t})^{-\alpha} = \zeta_{\omega_A,t}\omega_A \frac{Z_t}{\bar{H}_t} A_t \bar{V}_t^X \text{ if } \mu_t = 0$$
$$H_{A,t} = 0 \quad \text{and} \quad H_{Y,t} = \bar{H} \text{ otherwise}$$

Anticipating that we will rewrite everything along the BGPs, we rewrite this FOC as

$$\frac{H_{Y,t}}{\bar{H}_t} = \min\left\{ \left(\frac{\zeta_{\omega_A,t}\omega_A}{1-\alpha} \frac{\bar{V}_t^X A_t^{\varphi_A - 1} \bar{H}_t}{\zeta_Y} \frac{Z_t}{\bar{H}_t} \right)^{-1/\alpha} \eta_X \frac{K_t}{A_t^{\varphi_A} \bar{H}_t},\ \ 1 \right\}$$

2.1.6 Fiscal Policy

Finally, we assume that the government expenditure is a constant percentage of the output along the BGPs. Also, in this simple version, prices are flexible and there is no monetary policy.

2.2 Equilibrium Equations

The equilibrium equations are summarized in (1). In our notation, $\gamma_{X,t}$ is the gross growth rate of X_t from $t-1$ to t; $\gamma_{X,t} = X_t/X_{t-1}$.

$$\text{lom of } Z: Z_{t+1} = \omega_Z \frac{H_{A,t}}{\bar{H}_t} + \delta_Z z_L + (1-\delta_Z) Z_t \tag{1a}$$

$$\text{lom of } K: \frac{K_{t+1}}{K_t} = \frac{I_t}{K_t} + (1-\delta_K) \tag{1b}$$

$$\text{lom of } A: \frac{A_{t+1}}{A_t} = \zeta_{\omega_A,t}\omega_A Z_t \frac{H_{A,t}}{\bar{H}_t} + (1-\delta_A) \tag{1c}$$

$$\text{firm value}: \bar{V}_t^X = E_t\left[\frac{\Lambda_{t,t+1}}{c_v} \left\{ \bar{\Pi}_{t+1}^X + (1-\delta_A)\bar{V}_{t+1}^X \right\} \right] \tag{1d}$$

$$\text{flow profit}: \bar{\Pi}_t^X = (1-\tilde{\alpha})\,\alpha \frac{Y_t}{A_t} \tag{1e}$$

$$\text{bond return}: 1 = E_t[\Lambda_{t,t+1}]\, R_{t,t+1}^{NB} \tag{1f}$$

$$\text{SDF}: \Lambda_{t,t+1} = \zeta_{\beta,t+1}\beta\lambda_{t+1}/\lambda_t \tag{1g}$$

$$\text{MU}: \lambda_t = \left(C_t/\bar{H}_t \right)^{-\sigma} \tag{1h}$$

$$\text{production}: Y_t = \zeta_Y H_{Y,t}^{1-\alpha} (\eta_X K_t)^{\alpha} \tag{1i}$$

$$\text{F-goods Mkt}: Y_t = C_t + I_t + G_t \tag{1j}$$

$$\text{MPk}: R_{t-1,t}^K = \tilde{\alpha}\alpha Y_t/K_t \tag{1k}$$

$$\text{wage}: W_t = (1-\alpha) Y_t/H_{Y,t} \tag{1l}$$

$$\text{labour}: \frac{H_{Y,t}}{\bar{H}_t} = \min\left\{ \left(\frac{\zeta_{\omega_A,t}\omega_A}{1-\alpha} \frac{\bar{V}_t^X (A_t/\Gamma_t)}{\zeta_Y} \frac{Z_t}{\bar{H}_t} \right)^{-1/\alpha} \eta_X \frac{K_t}{A_t^{\varphi_A} \bar{H}_t},\ \ 1 \right\} \tag{1m}$$

It is a bit involved to obtain the balanced growth paths (BGPs), not because we have multiple BGPs but because we explicitly take into account the shocks and expectations. The fact that there are three BGPs does not prevent us from rewriting equations only with the stationary variables. Also, note that having multiple BGPs does not prevent the standard time iteration solution algorithm.[9] We have multiple fixed points in the state space, which does not prevent the use of the standard non-linear solution methods. We have a unique fixed point in the function space (space of the policy functions), which is what the standard projection method algorithm seeks. Hence, given an initial state, we can uniquely pin down one equilibrium; i.e., given the initial state, we can uniquely determine a transition path. The detrended equations are as follows.

$$\text{lom of } Z: \tilde{Z}_{t+1} = \omega_Z \tilde{H}_{A,t} + \delta_Z z_L + (1-\delta_Z)\tilde{Z}_t \tag{2a}$$

$$\text{lom of } K: \gamma_{A,t+1}^{\varphi_A}\tilde{K}_{t+1} = \tilde{I}_t + \frac{1-\delta_K}{\gamma_{\bar{H},t}}\tilde{K}_t \tag{2b}$$

$$\text{lom of } A: \gamma_{A,t+1} = \zeta_{\omega_A,t}\omega_A Z_t\left(1-\tilde{H}_{Y,t}\right) + (1-\delta_A) \tag{2c}$$

$$\text{firm value}: \tilde{V}_t^X = E_t\left[\frac{\Lambda_{t,t+1}\gamma_{v,t+1}}{c_v}\left\{\tilde{\Pi}_{t+1}^X + (1-\delta_A)\tilde{V}_{t+1}^X\right\}\right] \tag{2d}$$

$$\text{firm profit}: \tilde{\Pi}_t^X = (1-\tilde{\alpha})\alpha\tilde{Y}_t \tag{2e}$$

$$\text{bond return}: 1 = E_t\left[\Lambda_{t,t+1}\right]\tilde{R}_{t,t+1}^{NB} \tag{2f}$$

$$\text{SDF}: \Lambda_{t,t+1} = \beta\zeta_{\beta,t+1}\gamma_{t+1}^{-\sigma}\tilde{\lambda}_{t+1}/\tilde{\lambda}_t \tag{2g}$$

$$\text{MU}: \tilde{\lambda}_t = \tilde{C}_t^{-\sigma} \tag{2h}$$

$$\text{production}: \tilde{Y}_t = \zeta_{Y,t}\left(\frac{\eta_X}{\gamma_{\bar{H}_t}}\tilde{K}_t\right)^{\alpha}\left(\tilde{H}_{Y,t}\right)^{1-\alpha} \tag{2i}$$

$$\text{F-goods Mkt}: \tilde{Y}_t = \tilde{C}_t + \tilde{I}_t + \tilde{G}_t \tag{2j}$$

$$\text{capital demand}: R_{t-1,t}^K = \gamma_{\bar{H},ss}\tilde{\alpha}\alpha\tilde{Y}_t/\tilde{K}_t \tag{2k}$$

$$\text{labour demand}: \tilde{W}_t = (1-\alpha)\tilde{Y}_t/\tilde{H}_{Y,t} \tag{2l}$$

$$\text{labour mkt}: \tilde{H}_{Y,t} = \min\left\{\left(\frac{\zeta_{\omega_A,t}\omega_A}{1-\alpha}\frac{Z_t}{\zeta_Y}\tilde{V}_t\right)^{-1/\alpha}\frac{\eta_X}{\gamma_{\bar{H}_t}}\tilde{K}_t,\ 1\right\} \tag{2m}$$

To obtain the BGPs, we normalize each endogenous variable by dividing its proper growth factor; see Table 1. We call this detrending and we add tilde on the resultant variables. The steady states of the detrended equations (2) are called BGPs. Later, to discuss the quantitative model behavior, we do reverse detrending to obtain the value of each variable that is comparable to the data.

Table 1 says that basically quantities are divided by $A_t^{\varphi_A}\bar{H}_t$, while prices are unchanged with some exceptions. Wage should not be divided by the population size, because it is of per worker from the first place. Similarly, firm profit and firm value are of one firm, and hence they should not be divided

[9] A less obvious key to the unique equilibrium is the fact that the middle steady state is explosive. If counterfactually the middle steady state were excessively stable (indeterminate), then it would not be possible to pin down a unique equilibrium (transition path). For an indeterminacy model solved by a projection method, see, for example, Benhabib, Schmitt-Grohé and Uribe (2001, JET) "Perils of Taylor Rules".

Table 1: Detrended Variables

types of variables	notes
quantities:	$\tilde{Y}_t = Y_t/\Gamma_t$ and so on, where $\Gamma_t = A_t^{\varphi_A}\bar{H}_t$
capital:	$\tilde{K}_t = K_t/\Gamma_{k,t}$, where$\Gamma_{k,t} = A_t^{\varphi_A}\bar{H}_{t-1} = \Gamma_t/\gamma_{\bar{H}_t}$
prices and Z_t:	$\tilde{R}_t^{NB} = R_t^{NB}$, $\tilde{R}_t^K = R_t^K$ and $\tilde{Z}_t = Z_t$ are unchanged
wage:	$\tilde{W}_t = W_t/\Gamma_{w,t}$, where $\Gamma_{w,t} = \Gamma_t\bar{H}_t$
firm profit and firm value:	$\tilde{\Pi}_t^X = \bar{\Pi}_t^X/\Gamma_{v,t}$ and $\tilde{V}_t^X = \bar{V}_t^X/\Gamma_{v,t}$, where $\Gamma_{v,t} = \Gamma_t A_t$
hours:	$\tilde{H}_{A,t} = H_{A,t}/\bar{H}_t$ and $\tilde{H}_{Y,t} = H_{Y,t}/\bar{H}_t$
others:	marginal utility $\tilde{\lambda}_t = \lambda_t\Gamma_t^\sigma$

by the number of M-firms A_t.[10] Also, we need to pay careful attention to capital. Because K_{t+1} is known at the end of time t, to maintain its "predetermined" nature, we do not want to divide it by $\bar{H}_{t+1}$. We instead divide K_{t+1} by $A_{t+1}^{\varphi_A}\bar{H}_{t-1}$, because A_{t+1} is know at time t. Hence, $\tilde{K}_t$ is always followed by $\gamma_{\bar{H},t} = \bar{H}_t/\bar{H}_{t-1}$. Note that the growth rate of TFP $\gamma_{A,t+1} = A_{t+1}/A_t$ is also a time-t variable in terms of information (A_{t+1} is also predetermined).

2.3 Key Mechanisms and Key Properties

2.3.1 Why Business Cycles Matter

First, we want to discuss the mechanism by which business cycles affect R&D activities, though this is the same as Comin and Gertler (2006, AER). The equation of firm profit (1e) shows that the profit of an M-firm is proportional to the output per firm. Because M-goods play a similar role to capital, if the capital share α is larger, the M-firms' share is also larger. Also, because each M-firm has some monopolistic power, it is more profitable when the demand for M-goods is less elastic (recall $1-\tilde{\alpha} = 1/\theta_x$). Because a successful inventor can establish a new firm to commercialize her invention of a new variety of M-goods, the firm value is the reward to successful inventions. Hence, any shocks that can affect output at least potentially have some effects on R&D activities.

Note however that profit is not the only factor that determines the firm value. The equation of firm value (1d) shows that, because the firm value is the sum of the discounted current and future profits, the discount rate also matters $\Lambda_{t,t+1}/c_v$, where $\Lambda_{t,t+1}$ is stochastic discount factor and c_v is exogenously given risk premium on the stock investment. That is, if the stock market requires high equity premium, given the same current and future profit flow, the price of a firm becomes lower. The efficiency in the financial market matters. We study such a financial shock by considering time-preference shock $\zeta_{\beta,t+1}$; see equation (1g).

2.3.2 Why Financial Shock is Important

In our model, the production and R&D sectors compete with each other in the labour market. This means that, something good for production is not necessarily good for R&D. Consider an

[10] See Section 2.3.4 for further discussions.

improvement in the production efficiency, for example. On the one hand, as discussed above, it increases the reward to R&D activities, through a higher firm profit. On the other hand, however, it increases the cost of R&D labour; that is, given higher efficiency level of production, the production sector absorbs more labour, which raises its offered wage. As a result, the R&D sector is crowded out in the labour market. In our model, these two effects exactly offset in the BGPs. Similarly, an increase in the government expenditure leads to a rise in production. But, though the firm profit increases, the wage cost is also increasing. In the steady states, the government expenditure has no effect on the level of R&D. The following proposition summarizes these results more precisely.

Proposition:

In the system of equations (2), the following three parameters do not affect the growth rate of any balanced growth paths:

- production efficiency level ζ_Y
- government expenditure level G_{ss}/Y_{ss}
- capital depreciation rate δ_K

The proof is, though tedious, straightforward; see Appendix. Note that these parameters do affect the income level, but do not affect the growth rate of income. Note also that this proposition concerns only BGPs; in this sense, this proposition is of comparative statics. Taking into account transition dynamics, a permanent change in some of these parameters has some effect on the income level. But, such an effect through transition periods is very weak because we focus on very persistent shocks. Contrarily, the R&D level is very sensitive to the time preference shock and is even more sensitive to the shock to the equity premium. Although such shocks temporarily affect the output level and hence the firm profit, they directly affect the discount rate and hence affect the firm value. This explains why the growth slowdown often takes place after financial crisis.

2.3.3 Why There are Multiple BGPs

The model has multiple BGPs, because of the R&D environment Z_t. Suppose that, due to some reason, R&D labour $H_{A,t}$ is low for a long period. In this case, the R&D environment deteriorates at the rate of δ_Z. Once Z_t becomes low enough, R&D is not productive anymore, which discourages working in the R&D sector. In a sense, there is a vicious cycle between $H_{A,t}$ and Z_t. On the other hand, suppose that, due to some reason, the R&D productivity increases for a long period. In this case, even though Z_t is low, the R&D may be profitable. Having high $H_{A,t}$, the society accumulates Z_t gradually. Once Z_t becomes high enough, the R&D productivity is high enough even if the R&D productivity returns back to the original level. High Z_t and high $H_{A,t}$ support each other in this scenario. Hence, with shocks, an economy can move from one BGP to the other,but such a shock must be large enough (big-push story). Without shocks, however, an economy is trapped in one of two stable BGPs and cannot escape from it. Or, starting from an arbitrary point in the state space, it goes to one of two stable BGP, which is its destiny. Hence, the model exhibits a stark initial state dependence in the case of no shocks. We shortly discuss these issues in the quantitative experiments.

2.3.4 Differences from Romer (1990, JPE)

In this subsection, we want to clarify rather minor differences from Romer (1990), other than the addition of the R&D environment Z_t. First, we rather focus on the transition dynamics, and hence we have a couple of shocks; shock $\zeta_{\omega_A,t}$ to the R&D productivity and the time-preference shock $\zeta_{\beta,t+1}$ in this version. Second, although we allow the possibility that a variety becomes obsolete in our model setup, we have the depreciation δ_A of A_t to be zero in our calibration exercises. Third, we add the exogenous equity risk premium c_V. This is only for a calibration purpose; without this, although the model dynamics is little affected, the price-earnings ratio is too low, comparing to the data. Financial shocks are caused by $\zeta_{\beta,t+1}$.

Fourth, we assume that the elasticity of substitution among the varieties θ_x is a free parameter. We need to discuss a couple of issues related to this. In Romer (1990), it is implicitly assumed that $\theta_x = 1/(1-\alpha)$, but there is no reason to think that the substitutability among M-goods is determined by the capital share α. Actually, this parameter is quite important to match the steady state values to the data. This is because the incentive for the R&D is the value of a new firm that an inventor can establish after a successful R&D, which in turn is higher when the monopolistic power of the new variety is stronger. A low substitutability of M-goods (low θ_x) implies a high markup and hence a high firm profit. Also, because M-goods play a similar role to capital in Romer model, θ_x affects steady state capital as well. One additional comment is that, when we reverse detrend the variables, the average firm size $\bar{V}_t$ is shrinking over time if $\theta_x < 1/(1-\alpha)$ along the high BGP.[11] It may seem as if it is inconsistent with the data. But, note that our model assumes that one firm produces only one variety, but in reality one firm produces many varieties. That is, the value of one firm $\bar{V}_t$ is actually the business value of one type of M-goods in this model. Because of this, even though the company size is not shrinking in the data, we should not compare it with our $\bar{V}_t$. Unlike firm profit and firm value, the price/earnings ratio and stock return are comparable to the data.

Lastly, our model explicitly takes into account the occasionally binding non-negativity constraint of labour supply, which is not new. Actually, this possibility is already discussed by Romer (1990).

3 Parameter Selection and Steady States

3.1 Parameters

Our basic strategy is to choose the parameter values based on the RBC literature (if they appear in RBC models). These parameters include β, σ_C, α, δ_K and G_{ss}/Y_{ss}.[12] For the parameters specific to this model, we target the several key data moments. Because we solve the model by a fully non-linear method, it is not really easy to define and compute the steady states with shocks. In this

[11] Recall that the profit and the value of an M-firm are detrended as follows; $\tilde{\Pi}_t^X = \bar{\Pi}_t^X / \left(A_t^{\varphi_A - 1}\bar{H}_t\right)$ and $\tilde{V}_t^X = \bar{V}_t^X / \left(A_t^{\varphi_A - 1}\bar{H}_t\right)$. Since $\tilde{\Pi}_t^X$ and $\tilde{V}_t^X$ are constant over time along the BGPs, $\varphi_A < 1$ means that $\bar{\Pi}_t^X$ and $\bar{V}_t^X$ are shrinking as A_t grows. Actually, A_t is growing along the high BGP (A_t is not growing at the low BGP). Note that, because $\varphi_A = \frac{\alpha}{1-\alpha}\frac{1}{\theta_x - 1} = \frac{\alpha}{\tilde{\alpha}}\frac{1-\tilde{\alpha}}{1-\alpha}$ and $\tilde{\alpha} = (\theta_x - 1)/\theta_x$, if $\theta_x = 1/(1-\alpha)$, then $\tilde{\alpha} = \alpha$ and $\varphi_A = 1$.

[12] For the U.S. marco caribration, see Cooley and Prescott (1995) among others.

section, hence, we loosely target the values in the non-stochastic steady states.

Very roughly speaking, there are the following correspondences between parameter values and the steady state values. First, higher R&D productivity ω_A tends to generate higher growth rate $\gamma_{A,ss}$ in the high BGP. Second, given α and β, the elasticity of substitution θ_x among intermediate goods affects capital output ratio in the steady state. Note that $\beta = 0.995$ implies that the annual risk-free rate is 2%. Third, the risk-premium on the stock investment c_V is related to the price/earnings ratio (P/E). For z_L, which is the long-run value of Z_t in the low BGP, we set 0.5, but we just have $z_L > 0$ to avoid $Z_t = 0$; for this purpose, z_L works as long as it is a small positive number. For the depreciation rate δ_Z of Z_t, we have little clue to pin down this parameter value. Actually, unlike z_L, δ_Z plays an important role in the transition dynamics, because it controls the speed of the decrease in Z_t when there is no R&D labour supplied.[13]

Finally, we assume that one period corresponds to one quarter in our parameterization.

Table 2: Parameters

Symbol	Value	Note
β	0.995	discount factor
σ_C	1	intertemporal substitution
α	0.38	capital share
δ_K	0.012	capital depreciation
G_{ss}/Y_{ss}	0.18	govt expenditure share in steady states
ω_A	0.2	productivity in R&D
δ_Z	0.1	depreciation of Z_t
z_L	0.5	lower bound of Z_t
θ_x	2.8	elasticity of substitution of M-goods
c_V	1.025	1 + risk premium on stock investment
ζ_Y	1	production efficiency level
δ_A	0	depreciation of A_t
$\gamma_{\bar{H},t}$	1	1 + population growth rate
η_X	1	productivity in M-production
$\tilde{\alpha}$	0.643	$= (\theta_x - 1)/\theta_x$ (1/mark-up)
φ_A	0.341	$= \frac{\alpha}{1-\alpha}\frac{1}{\theta_x-1} = \frac{\alpha}{\tilde{\alpha}}\frac{1-\tilde{\alpha}}{1-\alpha}$

3.2 Steady States

We have three steady states but because the middle steady state is explosive we focus on the low and high steady states. In our model we assume that the government expenditure is 18% of the output. Given the parameters shown above, consumption is 63% and 65% while investment is 19% and 17% in the high and low steady states, respectively. These numbers are quite reasonable. Capital output ratio is 10 and 14 in the high and low steady states, respectively (it is higher in the

[13] See Footnoe 23.

low steady state). In annual term, they are 2.6 and 3.6, respectively. The value of 2.6 for the high steady state might seem to be slightly too low, in particular comparing to the U.S. data, but recall that, in light of our model structure, some portion of non-tangible assets should be deducted from the total capital. Taking into account this, we think 2.6 is quite reasonable. Also, our P/E ratio is 19 and 32, respectively, which also seem to be reasonable comparing to the market data; see Shiller (2006)[14]. Certainly, the market data is limited to the listed companies, which are usually large and established firms. However, the P/E ratio in the data varies significantly among markets, and hence it seems to be not really fruitful to fine tune the parameter values for this number.

Table 3: Steady State Values

	low	unstable	high	
Z_{ss}	0.5000	0.8783	1.3742	R&D environment
K_{ss}	73.4208	57.7337	35.6347	production capital
V_{ss}	23.0324	16.7273	9.0808	firm value
C_{ss}	3.3148	2.9160	2.2954	consumption
$\gamma_{A,ss}$	1.0000	1.0066	1.0240	growth rate
$H_{Y,ss}$	1.0000	0.9622	0.9126	production labour
Y_{ss}	5.1170	4.5599	3.6735	output
K_{ss}/Y_{ss}	14.3485	12.6611	9.7005	capital/output ratio
C_{ss}/Y_{ss}	0.6478	0.6395	0.6249	consumption/output ratio
I_{ss}/Y_{ss}	0.1722	0.1805	0.1951	investment/output ratio
$R_{K,ss}$	0.0170	0.0193	0.0252	capital return
W_{ss}	3.1725	2.9383	2.4957	wage
Π_{ss}	0.6944	0.6188	0.4985	firm profit
V_{ss}/Π_{ss}	33.1667	27.0297	18.2145	price/earnings ratio

Finally, perhaps the most controversial variable is the labour share of R&D, which is respectively 9% and 0% in the high and low steady states. For the low BGP, it may seem to be strange to have no R&D activities at all. We could easily lift up the lower bound from 0% to some (exogenously specified) positive level, by tinkering the model. But, because such a cosmetic modification of the model does not change the model implications at all, while it brings unnecessary complication, we rather opt to have 0% in the low BGP. For the high BGP, the OECD for example provide a comprehensive R&D data,[15] which says the researchers share is around 1% in many countries and never exceeds 2%. In our model, under a reasonable parameter range, it seems to be very difficult to have such a low R&D labour share, in particular, without making a model extremely sensitive to the external shocks. However, we anyway believe that we should not follow the OECD statistics. There are three reasons. First, in our simplified model, we assume that R&D requires labour input, which is not really true in reality. Often, the research activities require unproportionally large resources. According to the date of the World Bank, the R&D expenditure share in GDP is 2.55% for OECD countries.[16] Second, we should take into account the research activities in the higher education and

[14] For the U.S. price-earnings ratio, see http://www.econ.yale.edu/~shiller/data.htm.
[15] For the share of RnD labour, see http://www.oecd.org/innovation/inno/researchanddevelopmentstatisticsrds.htm
[16] See https://data.worldbank.org/indicator/GB.XPD.RSDV.GD.ZS.

non-profit research institutions. Finally and most importantly, we define our R&D more broadly than the official statistics. Although the exact data coverage in the World Bank and OECD statistics may differ among countries, picking up the U.S. as an example, the definition of R&D is often followed by the word of "systematic studies". Here, in this model, we would like to study more broader activities, which are less systematic and less formal R&D activities. Indeed, for example, U.K.'s innovation survey regards innovation as a much wider concept than their R&D definition.[17] Our position is more or less close to it; our R&D includes not only formal and systematic studies, but also includes less formal business enhancement activities.

Table 4: Roots around Steady States

low	unstable	high
0.9000	0.9841	0.9853
0.9845	1.0105	0.9853
1.0302	1.0344	1.0442
1.0305	4.3467	3.4835

3.3 Stability

Table 4 shows the roots of the detrended dynamic model (2) around its non-stochastic steady states. In our formulation, while we have two state variables (production capital $\tilde{K}_t$ and R&D environment $\tilde{Z}_t$), there are also two dynamic jump variables consumption $\tilde{C}_t$ and firm value $\tilde{V}_t$, the dynamics of which are expressed by the expectational equations. Hence, only the low and high steady states are saddle-path stable. The middle steady state is explosive (unstable).

To see why the middle BGP is explosive, consider Figure 3, which shows the competition between the production and R&D sectors in the labour market. In this figure, we (i) we do not impose the wage equalization between the R&D and production sectors in equations (2), and (ii) treat $\tilde{H}_{A,t}$ as a parameter. Note that these labour demand curves are not monotonically downward sloping, because as labour share changes the capital level and other variables change. As expected, the points at which the wage equalization holds ($\tilde{W}_{A,t} = \tilde{W}_{Y,t}$) are the steady states (BGPs). In addition, the upper left circle also indicates the steady state, at which the non-negativity constraint on the R&D labour is binding ($\tilde{H}_{A,t} = 0$) and hence the wage equalization does not need to hold ($\tilde{W}_{A,t} < \tilde{W}_{Y,t}$). Around the middle steady state, the slope of $\tilde{W}_{A,t}$ is steeper than that of $\tilde{W}_{Y,t}$. This means, for example, if $\tilde{H}_{A,t}$ exceeds its middle steady state value slightly, the R&D sector offers a higher wage than the production sector, which attracts more $\tilde{H}_{A,t}$. This occurs due to the externality; if more people engages in the R&D, the R&D productivity improves which in turn attracts more R&D labour. In contrast, around the high steady state, the opposite happens. Unless the deviation from the high steady state is not too large, there is a mechanism that forces an economy to return back to the high steady state.

[17] For the U.K. innovation survey, see https://www.ons.gov.uk/surveys/informationforbusinesses/businesssurveys/ukinnovationsurvey

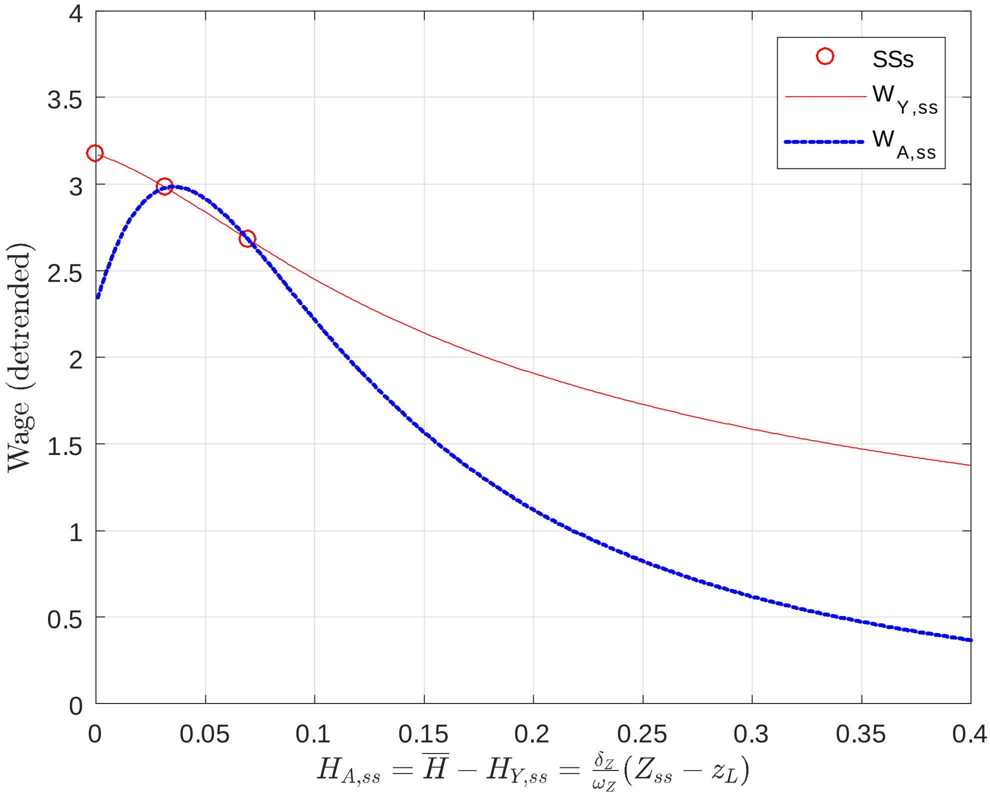

Figure 3: Labour Demand by RnD and Production Sectors

4 Dynamics in Non-Stochastic Case

Even without shocks, the model has transition dynamics. In this case, the model exhibits the stark initial state dependency. To understand the model behavior, look at Figure 4. It is a sort of phase diagram.[18] The red thin lines are the nullclines of $\tilde{Z}_t$, on which $\tilde{Z}_{t+1} = \tilde{Z}_t$. Similarly, the blue dotted line is the nullcline of $\tilde{K}_t$, on which $\tilde{K}_{t+1} = \tilde{K}_t$. The transition of capital is simple in the sense that if $\tilde{K}_t$ is below its nullcline, it increases, and vice versa. The transition of R&D environment $\tilde{Z}_t$ is a bit complicated. If it is very low, more specifically if it is lower than z_L, which is 0.5 under our parameterization, $\tilde{Z}_t$ increases toward z_L, which is the autonomous increase even with zero R&D labour $\tilde{H}_{A,t} = 0$. Above z_L, $\tilde{Z}_t$ decreases if it is too low or too high. In particular, for low $\tilde{K}_t$, $\tilde{Z}_t$ decreases for its entire range. This is because R&D is not profitable enough, if capital accumulation is too low (because firm output and profit are low as well). For $\tilde{K}_t$ high enough, $\tilde{Z}_t$ is increasing if it is in the middle range; $\tilde{Z}_t$ is increasing within the tongue shape area surrounded by the red curved thin line in the upper middle area. In this area, capital is accumulated enough, and $\tilde{Z}_t$ is not too low or not too high. If $\tilde{Z}_t$ is too low, it supports only low level of R&D labour $\tilde{H}_{A,t}$, which is not enough to offset the depreciation of $\delta_Z \tilde{Z}_t$, while if it is too high, though $\tilde{H}_{A,t}$ is high, its depreciation is also large.

[18] We say "a sort of", because, unlike the standard phase diagrams in economics, we omit dynamic jump variables (consumption $\tilde{C}_t$ and firm value $\tilde{V}_t$). Together with two state variables capital K_t and R&D environment Z_t, we have four dimensions, which is not possible to plot. Instead, Figure 4 is the projection of this 4D space into the state space, which is 2D.

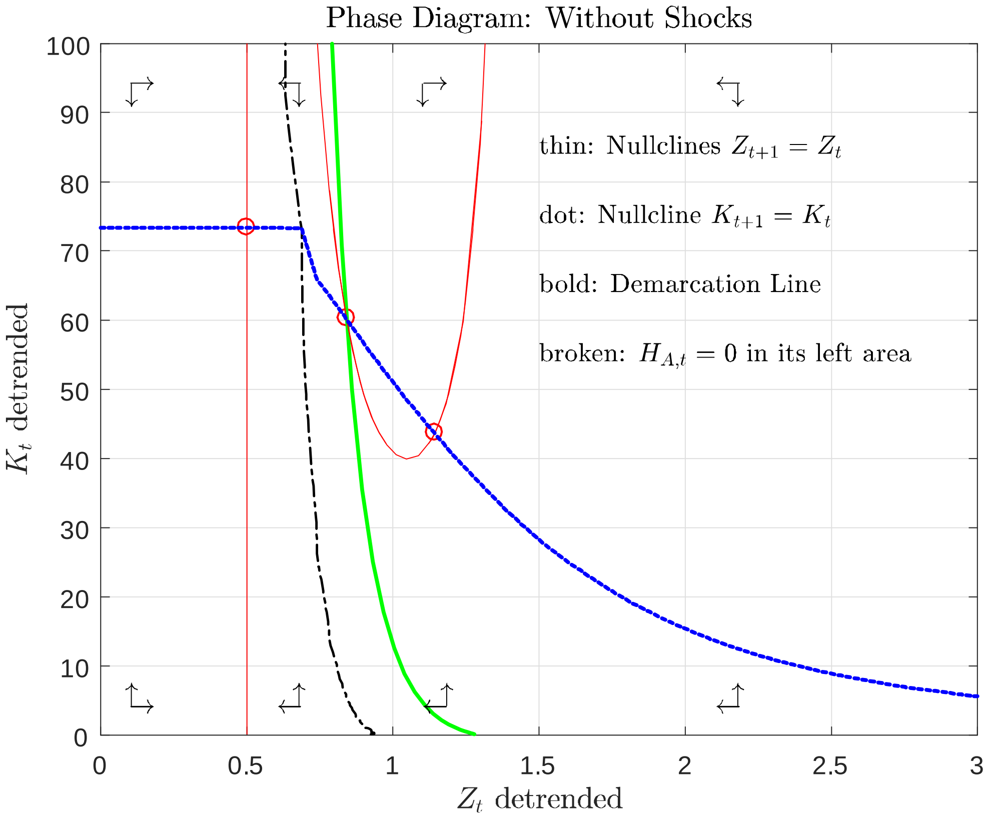

Figure 4: Nullclines and Demarcation Line. In the area left of the black broken line, there is no RnD activities; $H_{A,t} = 0$.

The points that satisfies both $\tilde{Z}_{t+1} = \tilde{Z}_t$ and $\tilde{K}_{t+1} = \tilde{K}_t$ are steady states in the detrended variables (i.e., BGPs). The upper left circle shows the low BGP, in which $\tilde{H}_{A,t} = 0$. Actually, at any point left of the black broken line, $\tilde{H}_{A,t} = 0$. Under our model assumptions (i.e., unless we assume some exogenous growth engine), this implies that $\gamma_{A,t} = 0$; no endogenous growth takes place. In the neighborhood of this point, the model behavior is quite similar to the Cass-Koopmans neoclassical model without endogenous growth engine. The lower right circle shows the high BGP, in which $\tilde{H}_{A,t} > 0$. In the high BGP, $\tilde{Z}_t$ is high enough to support $\tilde{H}_{A,t} > 0$, which in turn maintains a certain level of $\tilde{Z}_t$. Hence, the economy grows endogenously. Near the high BGP, the model behaves similarly to the Romer model. The low and high BGPs satisfy the saddle path stability condition (i.e., stable), while the middle BGP is explosive. For the middle steady state, only if an economy starts exactly at that point, it can stay there. However, even with a very tiny shock, it starts moving toward either of the two stable states.

Starting from any point other than three steady states, an economy shows transition dynamics toward the low or high BGP; see Figure 5. The distance of two successive dots shows the change in the state in one period (one quarter). It tends to be shorter and shorter as a locus moves to the stable BGPs, implying an economy approaches to a BGP only asymptotically.[19] As this figure

[19] Passing the black broken line (the border for $\tilde{H}_{A,t} = 0$) from right to left, the loci are actually accelerating. This is because a mass of labour shifts from the R&D sector to the production sector; see the middle right panel of Figure 6. Along this black broken line, the policy functions have discontinuity; see the Appendix.

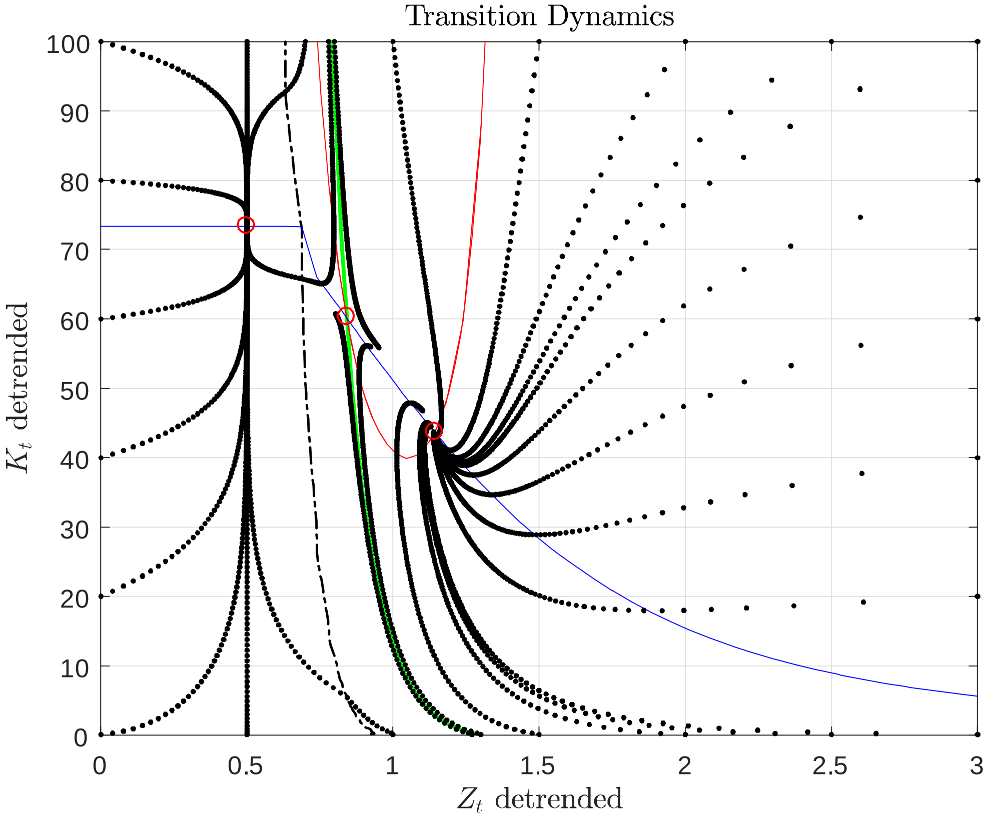

Figure 5: Sample Paths. All dotted lines start from the rim of the figure.

shows, around the high BGP, we see many over and under shootings. Also, as discussed above, without shocks, the model exhibits the initial state dependency. More specifically, if an economy starts from the right of the green bold line (demarcation line), it goes to the high BGP in the end (asymptotically). On the contrary, if it starts from the left of the demarcation line, it goes to the low BGP.

The initial state dependency is most stark, if we compare two economies starting near the point with $\tilde{K}_t$ being almost zero and $\tilde{Z}_t$ being around 1.3; see Figure 6. On the upper left panel, both economies move up along the demarcation line; one on the right side, the other on the left side. Their movements are very close until they come near the middle steady state, which means that in the early periods of their growth experience, they grow in a similar way. For this early period, the growth is led by the capital accumulation, which we call Solow effect. However, in the end, one starting the right side of the demarcation line grows endogenously forever, while the other economy stops R&D and it ends up with no endogenous growth. Looking into these dynamics in the actual level (i.e., reverse-detrended variables), the upper right panel shows that due to the Solow effect, both economies experience rapid growth for the first 80 periods (20 years) or so. The growth rate of the high growth economy gradually declines, but after some time it successfully shifts its growth engine from the capital accumulation to R&D. Contrarily, the growth rate of the low growth economy keeps declining. Around $t = 240$ (60 years), the low growth economy shows a temporary growth boom,

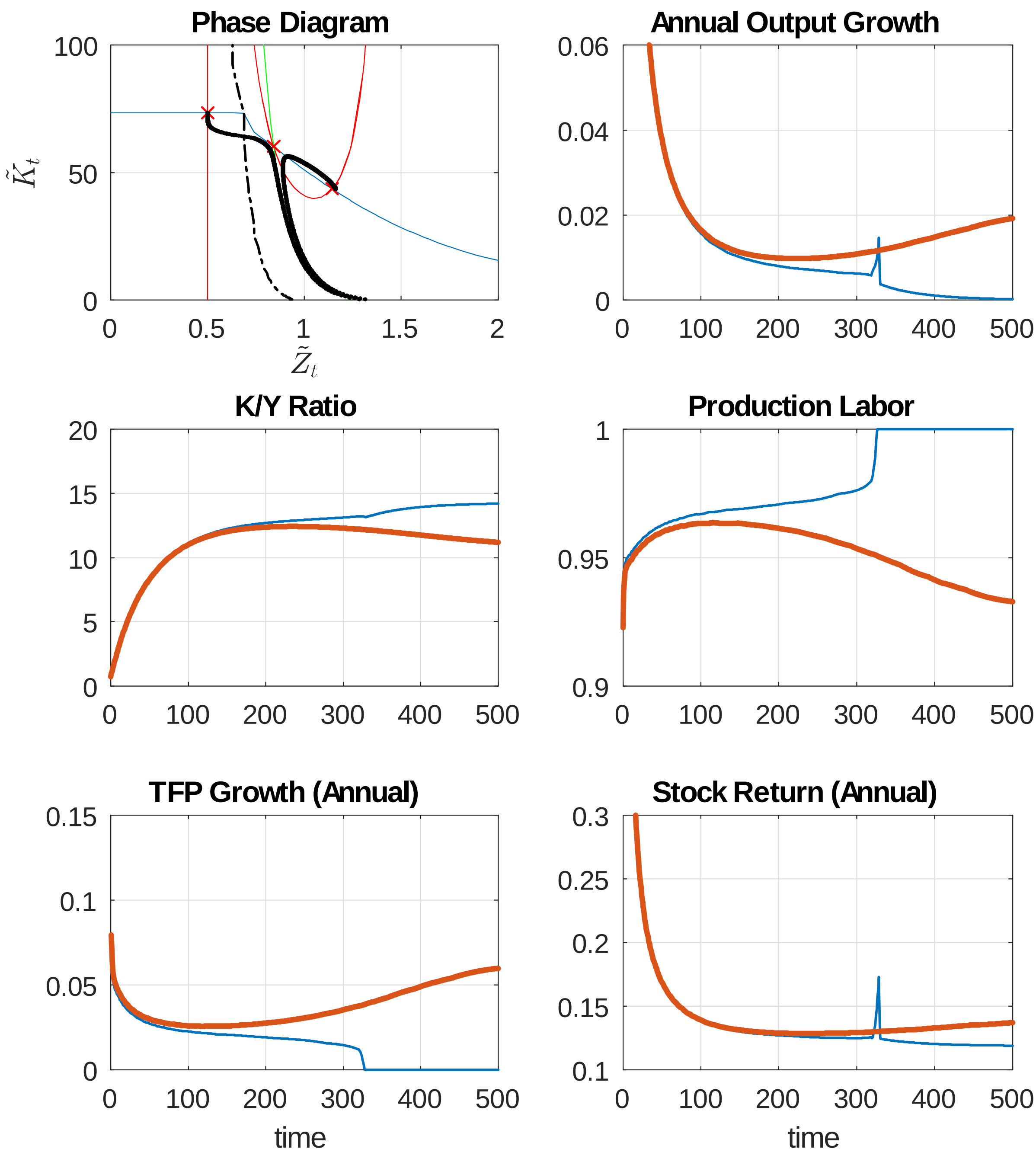

Figure 6: Transition Paths of Non-Stochastic Version. Except for the upper left panel, all variables are in level (reverse detrended variables). For the panels other than the upper left, the bold line shows the economy that is moving toward the high steady state (lower right "×"), while the thin line shows the economy moving to the low steady state.

which is caused by the mass shift of the labour from R&D to production;[20][21] see the middle right panel. This also leads to the stock market boom, because such a labour shift causes an increase in the profit of the incumbent firms; see the bottom right panel. In the long-run, capital to output ratio is lower for the high growth economy, reflecting a higher capital productivity.

5 Dynamics in Stochastic Case

In this version, we consider only two shocks; R&D productivity shock $\zeta_{\omega_A,t}$ and time-preference shock $\zeta_{\beta,t}$.

Table 5: Transition Probabilities

ζ_β: Time-Preference Shock		
from\to	1.0	0.9799
1.0 (normal)	303/304	1/304
0.9799 (crisis)	1/16	15/16

$\zeta_{\omega_A,t}$: R&D Productivity Shock		
from\to	1.0	1.5
1.0 (normal)	979/980	1/980
1.5 (R&D boom)	1/20	19/20

Table 6: Properties of Markov Transition Matrix

R&D Financial Mkt	normal normal	boom normal	normal malfunction	boom malfunction
average duration	232.2 quarters	18.8 quarters	15.8 quarters	9.1 quarters
share in time	93.1%	1.9%	4.9%	0.1%

5.1 Transition Matrix

We assume that R&D productivity shock $\zeta_{\omega_A,t}$ and time-preference shock $\zeta_{\beta,t}$ are uncorrelated each other, and each of them follows a two-point Markov process. In an R&D boom, the R&D productivity is 50% higher than the normal level. More specifically, the R&D productivity is 0.2 in normal periods (see Table 2), while it is 0.3 in the R&D boom; that is, $\omega_A = 0.2$ and $\zeta_{\omega_A,t}$ is either 1.0 or 1.5. In the periods of financial malfunctioning, we assume that the annualized discount rate is increased from 2% to 10%, which is translated to $\zeta_{\beta,t}$ is either 1.0 or 0.9799.

We also need to determine the transition probabilities; see Table 5. First let π_{nb} be the probability to be R&D boom at $t+1$ conditional that it is normal period at t. Similarly, let π_{bn} be the probability to be normal period at $t+1$ conditional that it is R&D boom at t. Hence, if a R&D boom lasts

[20] This sharp hike in the growth rate takes place in one period. This is because we have flexible labour mobility; the R&D labour quickly shifts to the production labour. We can easily add some labour friction, so that this jump in the output growth rate would be spread out for several periods. To keep the model simple, we however opt not to do such a cosmetic tinkering.

[21] This jump in labour takes place when an economy passes through the black broken line in Figure 4, along which some policy functions have discontinuity; see the Appendix.

20 quarters on average, $20 = 1/\pi_{bn}$, while such a period takes place for only 2% of entire time; $0.02 = \pi_{bn}/(\pi_{bn} + \pi_{nb})$. These imply a typical normal period lasts 980 quarters (245 years). In this case, $\pi_{bn} = 1/20$ and $\pi_{nb} = 1/980$.

Likewise, let π_{nc} be the probability to be financial malfunctioning at $t+1$ conditional that it is normal period at t. Similarly, let π_{bn} be the probability to be normal period at $t+1$ conditional that it is financial malfunctioning at t. Hence, if a financial malfunctioning period lasts 16 quarters on average, $16 = 1/\pi_{cn}$. If financial malfunctioning periods arise for 5% of the entire time, $0.05 = \pi_{nc}/(\pi_{cn} + \pi_{nc})$. These two at the same time imply that on average normal times last 304 quarters (76 years). In this case, $\pi_{cn} = 1/16$ and $\pi_{nc} = 1/304$.

Because these two shocks are not correlated each other, using the Kronecker product,

$$\text{Transition Probability Matrix} = \begin{bmatrix} 1-\pi_{nc} & \pi_{nc} \\ \pi_{cn} & 1-\pi_{cn} \end{bmatrix} \otimes \begin{bmatrix} 1-\pi_{nb} & \pi_{nb} \\ \pi_{bn} & 1-\pi_{bn} \end{bmatrix}$$

where

$$\pi_{nc} = 1/304,\ \pi_{cn} = 1/16,\ \pi_{bn} = 1/20,\ \pi_{nb} = 1/980$$

5.2 Phase Diagrams

Figure 7 shows phase diagrams. Because we have four shock regimes, there are four phase diagrams accordingly. Note that $\tilde{K}_t$ and $\tilde{Z}_t$ are both state variables, meaning that they do not jump even if a shock hits. If a shocks hits, it is the phase diagram that changes.

On the upper left panel, we show the phase diagram in the normal time. The crossing points of the nullclines of $\tilde{K}_t$ and $\tilde{Z}_t$ are slightly different from the non-stochastic steady states, because the rational agents take into account the possibility of the future change in the shock regimes. Under our parameter values, there are multiple steady states in the normal time.

The upper right panel shows the regime of R&D boom, where R&D productivity is 50% higher than the normal regime. In this case, while the nullcline of capital changes little, the tongue shape part of the nullcline of the R&D environment shifts down. As a result, there is only one crossing point (high steady state). In this case, $\tilde{Z}_t$ only decreases either within the tiny triangle area or in the area where $\tilde{Z}_t$ is very high. If this shock regime lasts forever, then, starting from any point in the state space, an economy asymptotically approaches to the high BGP, where there is positive R&D activities.[22]

The lower left panel is the case where the negative time-preference shock hits. In this case, the required rate of return on both capital investment and the equity investment (R&D) is higher. Hence, the tongue share area shrinks and the nullcline of $\tilde{K}_t$ shifts down. As a result, there is only one crossing point, where there is no R&D activities; $\tilde{H}_{A,t} = 0$. Because we do not assume any exogenous growth engine, in this low BGP, the long-run growth rate is zero.

[22]Note that $\tilde{Z}_t$ tends to return back to z_L even when $\tilde{H}_{A,t} = 0$. Hence, as long as $\tilde{H}_{A,t} > 0$ at the crossing point of $\tilde{Z}_t = z_L$ and the nullcline of $\tilde{K}_t$, starting from anywhere in the state plane, an economy moves toward high steady state. Though it is not really discernible, the upper right panes indeed satisfies this.

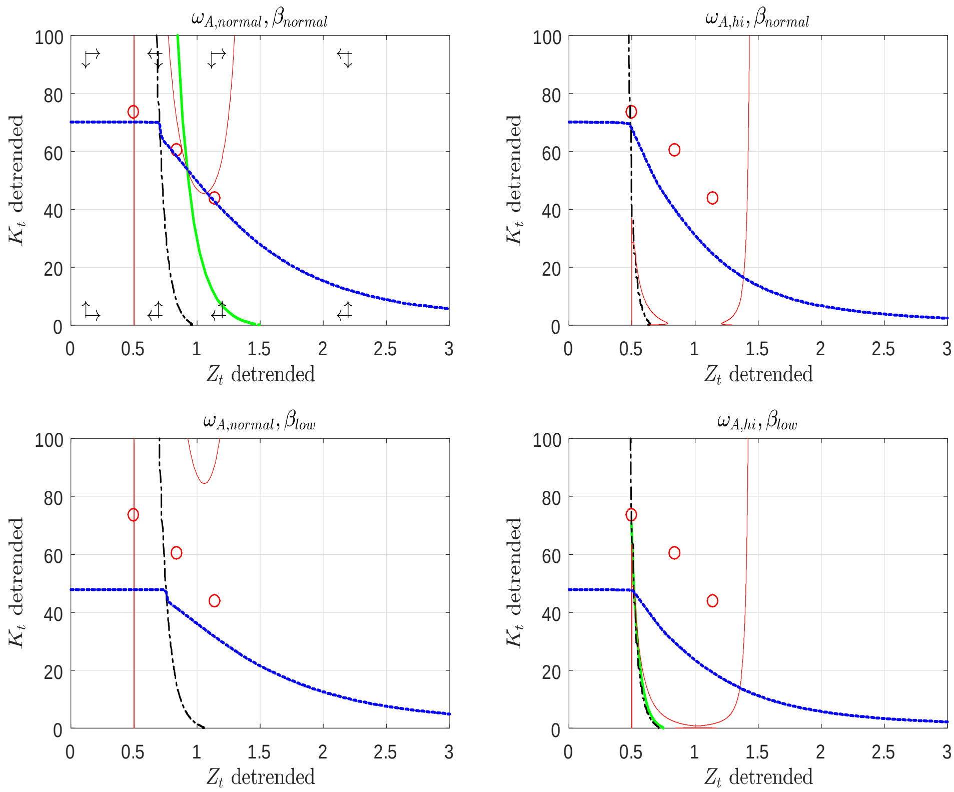

Figure 7: Phase Diagrams with Shocks. The red circles show the steady states without shocks. The demacation line (green bold line) exists only when there are multiple corssing points. It is computed based on the assumption that each shock regime lasts forever.

Finally, in the lower right panel, R&D is in boom but the financial system is malfunctioning. Though this shock regime may sound unusual, under our parameter assumptions, this shock regime is extremely rare (this regime happens only 0.1% of the entire time), and hence it is not very important.

5.3 Financial Malfunctioning

Figure 8 shows the transition dynamics of the economy which starts from the high BGP in the normal shock regime. Here, we assume that, after initial normal regime (initial 50 quarters in the figure), a low time-preference shock hits the economy for 50 quarters[23] (annual interest rate being from 2% to 10%), and then back to the normal shock regime.

Note importantly that the state (a point on the state plane) never jumps. Instead, it is the phase plane that shifts by a shock. Having this time-preference shock, the required rate of return becomes higher for both production and R&D sectors, meaning that the investments in both physical capital and R&D are both less attractive. The tongue shape part of the nullcline of $\tilde{Z}_t$ shifts up, while that of $\tilde{K}_t$ shifts down, and hence the steady states in the normal shock regime are not stationary points anymore; see the upper left panel. In this shock regime, the R&D environment and physical capital both decrease (moving toward southwest). On the day of hitting the negative time-preference shock, a mass of labour shifts from the R&D sector to the production sector, and hence we see a jump in the output growth. During the period with lower time-preference, because the R&D environment deteriorates, the growth rate of TFP A_t decreases; see the middle right panel.

At the end of the financial crisis, it is quite crucial if the economy passes by the demarcation line (green bold line) or not. In this example, the economy passes over the line; see the upper left panel. Hence, even after the time-preference returns back to the normal level (upper right panel), the economy is moving toward the low steady state (unless a large positive shock hits again). If the economy stayed in the right side of the demarcation line, it would go back to the high BGP. Finally, the R&D activities cease totally around $t = 410$. Again, at the timing of the end of R&D, output increases sharply, which is caused by a mass labour shift from the R&D sector to the production sector. Throughout the entire period, the output growth rate shows several spikes, mostly due to the mass labour shifts. This may seem to be unusual, but again if we add some labour friction that prevents an immediate job change, such spikes will be spread out to several periods.

Note that, in computing the policy functions numerically, we have assumed that a typical financial malfunctioning lasts 16 quarters; see Section 5.1. In this experiment, Figure 8 shows the case where the financial malfunctioning lasts 50 quarters, meaning that under the current parameter values, it requires a very long-lasting financial crisis to be trapped by the low growth region. We could interpret this that such a trap can take place only rarely, but unfortunately the required length of the financial disruption sharply depends on δ_Z, which is one of controversial parameter. If we have a larger value for δ_Z, it takes a shorter period to push the economy to the low growth region.

[23] In particular, given near zero $\tilde{H}_{Y,t}$, the depreciation rate of the R&D environment is crucial in determining the speed of the movements in the state plane. If δ_Z is larger, $\tilde{Z}_t$ decreases more quickly, hence it requires shorter periods to pass over the demarcation line. In this example, we assume that the financial malfunction lasts for 50 quarters, but depending on δ_Z, it could be shorter or longer.

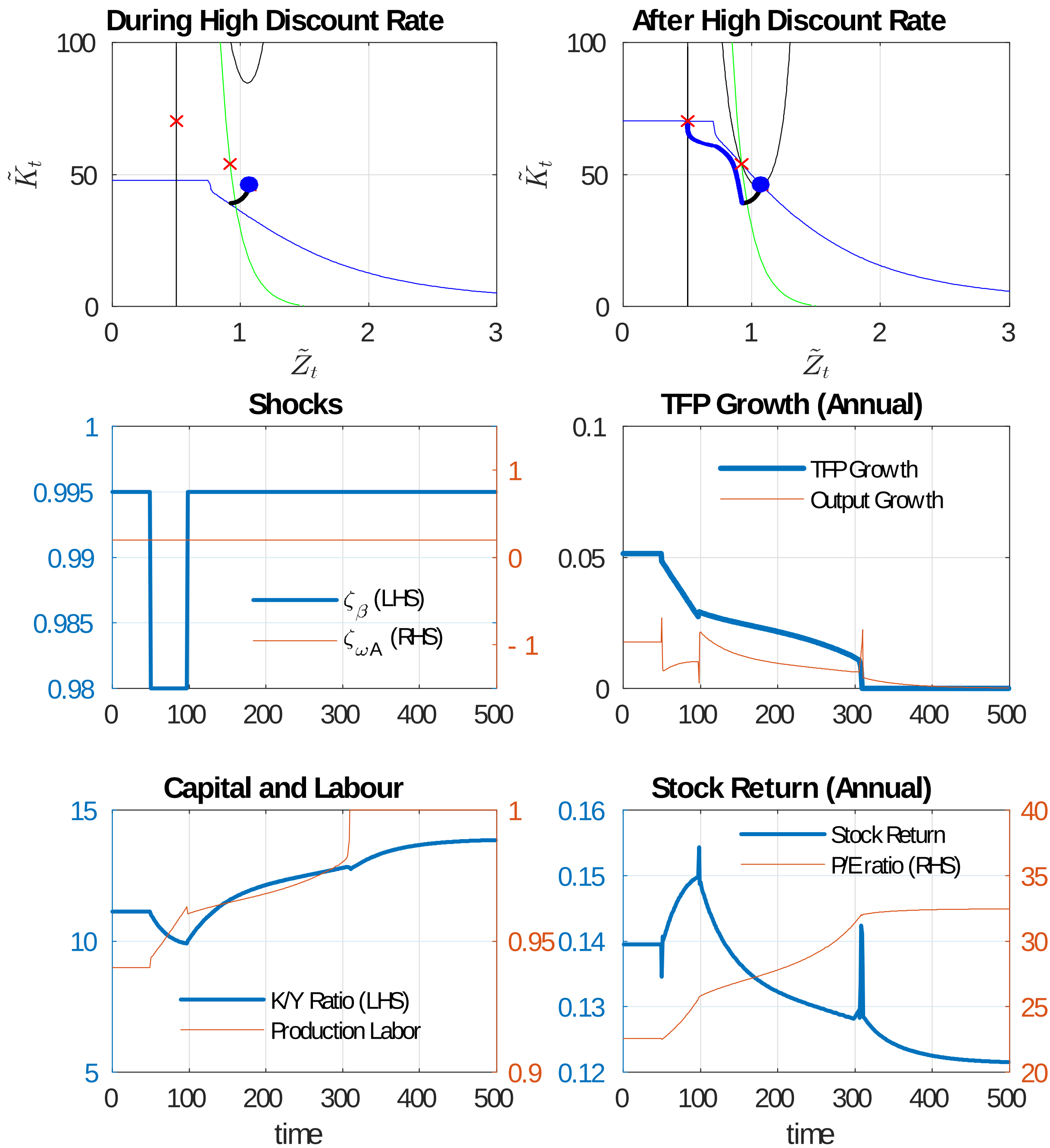

Figure 8: Financial Crisis. This simulation assumes that after initial 50 normal periods, the economy experiences low time-preference shock (high interest rate) for 50 quarters (12.5 years), and then it goes back to the normal shock regime. Except for the upper two panels, all variables are in actual level. For the upper two panels, three "×" show the crossing points of the nullclines under the normal shock regime. In the area left of the black broken line, no labour is dedicated to R&D activities; $\tilde{H}_{A,t} = 0$. This figure plots the same green bold lines in the upper two panels, which is the demarcation line in the normal shock regime. Note that there is no demarcation line for the high discount rate shock regime.

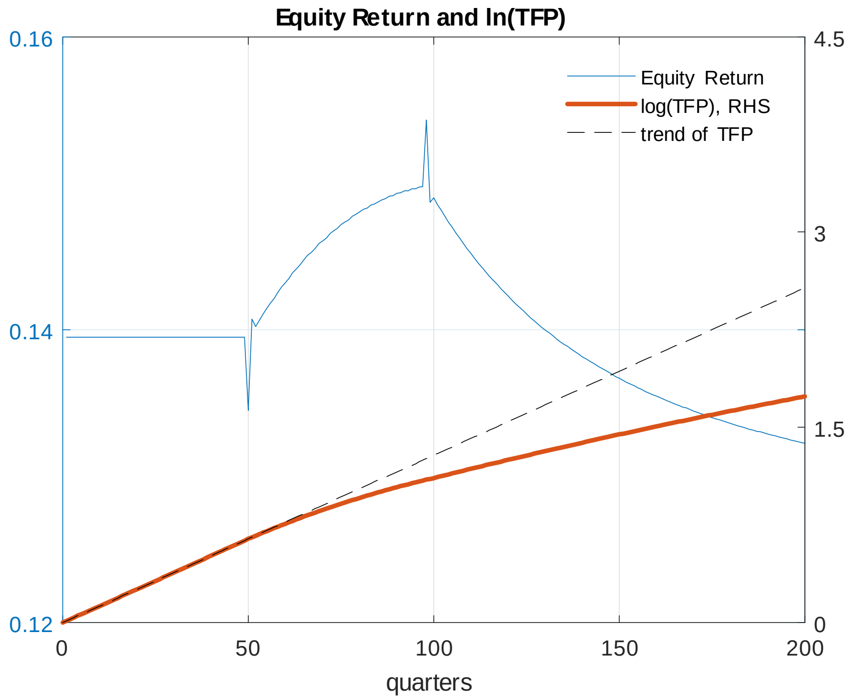

Figure 9: This figure is based on the same simulated data as Figure 8. Equity return is the sum of capital and income gains. The discount rate is 200bp higher than the normal between 51st to 100th quarters. TFP is normalized to be zero at the start of the simulation, and its linear trend is based on the first 50 quarters.

Compare Figure 9 with Figures 2 and 1. It focuses on the stock return and the log of TFP for the first 200 periods, which are generated from the same simulated data as in Figure 8. During the period of high discount rate, the stock return is rather higher, because of (i) the higher required return due to the lower discount factor and (ii) the labour shift from the R&D sector to the production sector. Here, the stock return is defined as the sum of capital and income gains per firm; $\left(\bar{V}_t + \bar{\Pi}_t\right)/\bar{V}_{t-1}$. We can see the downward shift of the TFP growth trend during the stock market boom. Hence, importantly, there is a discrepancy between aggregate economy and individual firms. In aggregate, a low level of inventions decreases the aggregate output growth rate by reducing the growth rate of A_t. In contrast, at the individual firms level, the output and the profit of each incumbent firm increases when they can attract more labour at a lower wage cost.

Note that the log of TFP shows another kink around $t = 410$ (omitted in Figure 9) at which there is a mass labour shift. After that, under our parameter values, the TFP growth rate stays at zero.

5.4 R&D Boom

Motivated by Rostow's takeoff, Figure 10 shows the effect of an improvement of the R&D productivity for a long time period. It plots the transition dynamics of the economy which starts from the low crossing point of the nullclines in the normal shock regime. After initial normal regime (60 quarters), the R&D productivity increases by 50% for 40 quarters, and then back to the normal shock regime.

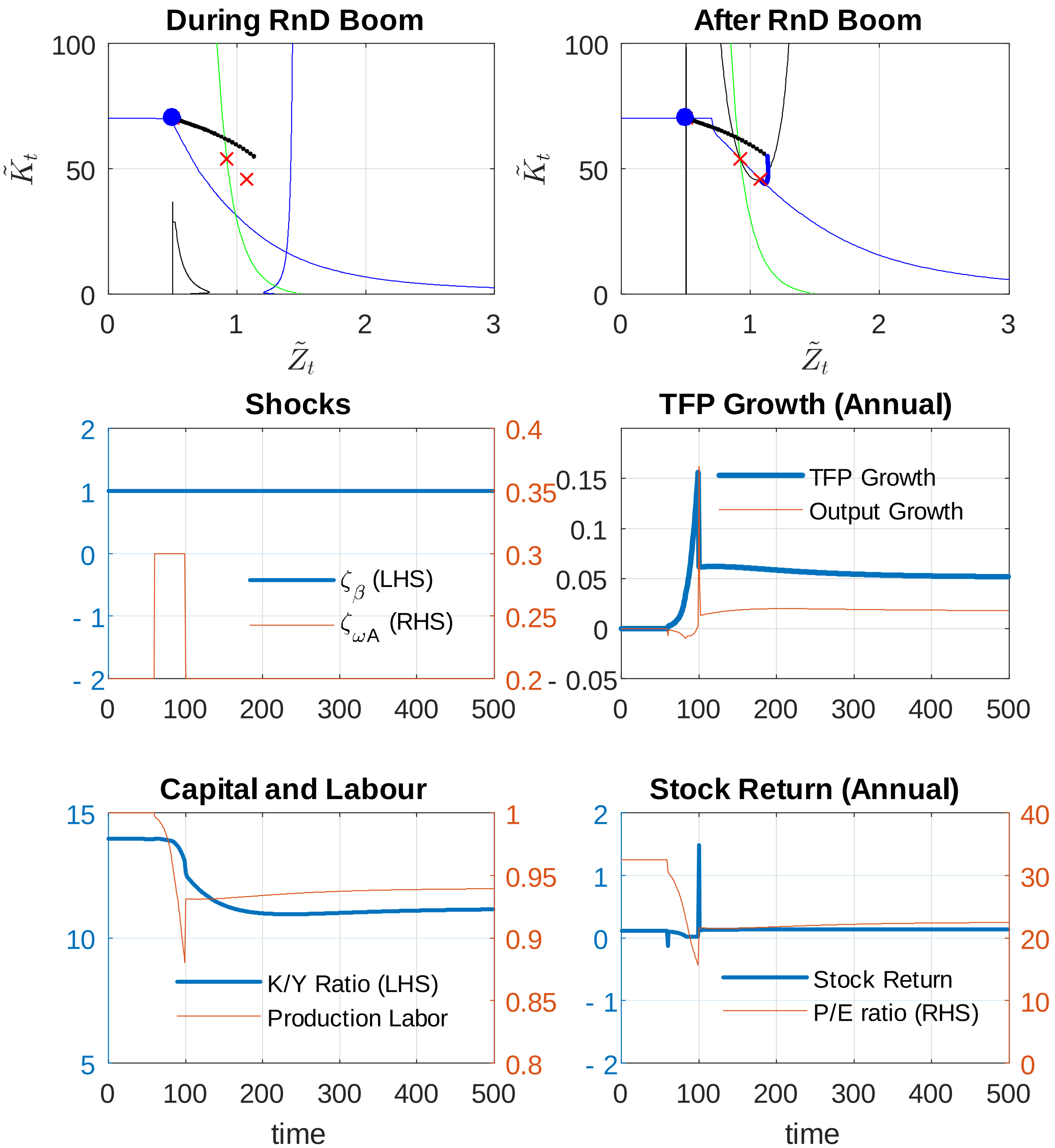

Figure 10: R&D Boom. This simulation assumes that, after intial 60 normal quarters, the high R&D productivity lasts for 40 quarters (10 years), and then back to the normal shock regime. Except for the upper two panels, all variables are in actual level. For the upper two panels, three "×" show the crossing points of the nullclines under the normal shock regime. In the area left of the black broken line, no labour is dedicated to R&D activities; $\tilde{H}_{A,t} = 0$. This figure plots the same green bold lines in the upper two panels, which is the demarcation line in the normal shock regime. Note that there is no demarcation line for the R&D boom regime.

In this scenario, the region where $\tilde{Z}_t$ increases expands, while the nullcline of $\tilde{K}_t$ shifts only a little. Because R&D is more productive, labour shifts to the R&D sector. Capital level relative to output mildly decreases. When the R&D boom ends, some portion of labour shifts back to the production sector leading to a sharp increase in output growth (10%) and stock return (150%). These large numbers may sound implausible, but again recall that our model does not have any labour friction; if we add some mechanism that generates a gradual labour mobility, then these spikes will be spread out over some periods.

Again, what is crucial to escape from the low steady state is that the economy needs to pass over the demarcation line by the time that the R&D boom ends, as in this example (see the upper left panel). If the economy stays in the area left of the demarcation line, then it will go back to the low steady state after the R&D boom.

6 Discussions

6.1 Model Interpretations and Anecdotal Evidence

Our model is very simple and stylized. In addition, we opt to have adjustment devices such as investment adjustment cost as little as possible. The model predicts some extreme values such as zero R&D labour in the low BGP, but we can easily remedy these things by tinkering the model. But, such cosmetic techniques makes the model unnecessary complicated without adding any interesting economic intuitions.

Nonetheless, our model maintains the key intuitions. First, in the model, the R&D and the production sectors compete in the factor markets. As discussed above, this idea itself is not new, in the sense that one of the leading exposition of the natural resource curse is that the profitable sector (natural resource extraction) crowds out the sector with high productivity (or strong spill-over effects) in the factor markets. In our model, the R&D sector competes only with the production sector only in the labour market. In this respect, we could extend the model so that it could be the financial intermediary that squeezes the R&D sector in the labour market. Indeed, anecdotally it is often said that those who obtained the training as a rocket scientist are now engaged in the financial engineering. Second, because of this, something good in the production sector may not be good for R&D, which we shortly come back to this issue in the discussion of policy implications. Third, it is very important to provide the risk-money (funds that can take risks) to the R&D sector. This is because (a) the R&D activities are very risky business; (b) the new idea has a stock value (not a one-off flow value); (c) the supply of the risk-money is very sensitive the required rate of return (risk-premium plus risk-free rate). Hence, the financial market is important for the technological growth. Again, though it is only anecdotal, the country which has the most developed financial markets is the country where innovations and inventions are most active, which is the U.S.

6.2 Policy Implications

Again, the most important implications of this model are (a) production sector may crowd out the R&D activities and (b) financial frictions have very strong impacts on R&D incentives. The following discussions are based on these two.

6.2.1 Fiscal Policy and Monetary Policy

First, like Comin and Gertler (2006, AER), the business cycle shocks can have very persistent effect in our model. Hence, at least potentially, demand control policies such as a counter-cyclical fiscal expenditure and monetary policy also affect the long-run growth. However, quantitatively, their results are very different from ours, because what we are concerned are long-lasting shocks such as financial crisis or R&D boom. In the extreme, ignoring the effect during the transition dynamics, a permanent change in the fiscal expenditure does not affect the BGPs, because, while it enhances the firm value which stimulates R&D activities, it also stimulate the labour demand in the production sector, squeezing R&D labour input. Also, in the case of Comin and Gertler (2006), there is an intertemporal substitution effect. That is, if an improvement in the production productivity is only temporary, then there is an incentive to work more intensively during such a period; such an effect is absent in the comparative statics and is small in our case with long lasting shocks.

For the traditional monetary policy, we can say a similar story. It may sound contradictory because we also find that the time preference shock has a sharp impact on R&D. Although this paper does not include the price stickiness, without expanding the state space dimensionality, we can add (a) Rotemberg type price stickiness in F-goods prices and (b) Taylor rule where nominal interest rate responds only to inflation.[24] In this extended version, if we add a permanent negative monetary policy shock, the steady states change little. The intuition is similar to the standard discussion about discretion vs. commitment policies. That is, because we are interested in, for example, the long-lasting shock that captures a financial crisis type situation, the monetary policy reaction is also long-lasting. In our rational framework, agents rationally anticipate that such an expansionary monetary policy lasts long. Hence, the inflation rate adjusts and such a low inflation rate makes the real interest rate back to a point near the normal level, meaning that the intertemporal substitution does not work. Recall that both real and nominal interest rates are higher during the financial malfunctioning in the quantitative exercise in Section 5.3. Note that, since inflation and deflation are both costly under Rotemberg type price stickiness (which is also true under Calvo pricing), they play a similar role to the negative efficiency shock ζ_Y, but as we have already discussed it does not affect BGPs.

In sum, both fiscal expenditure and traditional interest rate policy are not effective for the long-lasting shocks. If the monetary policy authority can implement a commitment policy, it may prevent from falling into the low BGP, but it is not really clear if such a commitment policy can be maintained for a long time period. Though it is beyond our scope, however, a non-traditional monetary policy

[24] We cannot have a version of Taylor rule, where monetary policy also responds to output growth or output gap, because output growth and output gap both depend on the BGPs. We do not know a single rule (an algebraic expression) which fits to all the BGPs in the same way.

such as quantitative easing may be effective to the extent that it improves the functionality of the financial intermediary as a fund conduit from investors to inventors.

Some long-run industrial policies that do not aim to manage business cycle fluctuations may have a positive effect to support the long-run growth. Although it is counter-intuitive, however, in our model, for example, improving the competitiveness of the industries may have no effect on the long-run TFP growth. Again, if the production sector becomes more productive, then it may absorb the resources that would be otherwise dedicated to the R&D sector. Instead, the model implies that the long-term structural policies that has an affinity to R&D, such as education, are preferable.

6.2.2 Big Push and Poverty Trap

The idea of the big push, though rather a casual idea, is that to escape from the poverty trap a big push, such as a large scale foreign aid, is necessary. Empirically, however, Esterly (2006, J Econ Growth), using a country-base panel data set, finds rather negative evidence against the view that the quality of government, the foreign aid and education spending help the takeoffs or the escape from the poverty trap. Our model indeed tells us that we need a big push to escape from the low BGP. But, at the same time, our model predicts that simple improvement in the production efficiency and an increase in government expenditure as a demand control do not stimulate R&D activities. If a foreign aid for example improves the productivity in the production sector, it rather could squeeze the R&D. That is, a more productive production may absorb resources that could be spent for the R&D. Hence, our model implies that it matters what kind of push it is. Actually, we argue that, depending on its type, in many cases, ODA improves the income of recipient countries but it may not lead to a takeoff.

Related to this, Azariadis and Stachurski (2005, Handbook) mention that "*..., until the last few hundred years no state successfully managed the transition to what we now call modern, self-sustaining growth*". They also discuss the Industrial Revolution in Britain, which could be understood as a big push. As they also mention, however, another puzzle is why such a science-based production regime has not been spread out countries outside Europe. In our model, the explosion of scientific findings prior to the industrial revolution can be interpreted as a large shift in the R&D productivity parameter ω_A. However, an increase in ω_A is only a necessary condition; the takeoffs also require the institutional capital Z_t be high enough. In this respect, Max Weber discussed that capitalism has developed first in the countries such where Calvinism was influential. Certainly, interpreting Calvanism as our Z_t is too heroic, though. Nonetheless, our model suggests that the society's attitude or soft-infrastructure to support the R&D activities are the key to the successful takeoffs.

6.3 Convergence or Divergence

This subsection discusses the convergence. In our model, assuming that the parameters are the same for all countries, (a) there is a convergence mechanism through the Solow effect (capital accumulation) in the transition period; (b) within the group of countries which grow along the same BGP (after the transition period), all of them grow at the same rate, meaning that the gap in the income level is unchanged asymptotically; (c) between the high and the low BGPs, the income is rather diverging. Note that Romer model also shares the same prediction as (a) and (b).

The empirical findings seem to be rather supportive to the divergence. The following discussion owes Durlauf, Johnson and Temple (2004, Handbook). First, in the growth regression type analysis (often called β-convergence), within a group of countries (or regions) which share similar properties (similar parameter values), such as U.S. states, European countries, and Japanese prefectures, there are strong evidence of convergence (absolute convergence). Unfortunately, the standard endogenous growth models, including ours, do not match this finding. The evidence for the β-convergence is mixed if a data set includes a wider range of countries. Second, in the distributional approach (often called σ-convergence),[25] the evidence rather supports the multimodality.[26] Often, polarization, or twin peaks, is found,[27], and, even if a convergence is found in the long-run, the period of twin peaks is estimated to last for a very long period.[28] While the regression type analyses lose a lot of information included in the distribution, the estimation precision tends to be low in the distributional approach. Anyway, though there are some evidence of the convergence as well, all in all it seems that the empirical evidence is rather supportive to the bimodality.

6.4 Scale Effect

As discussed by Jones (1999 AER, 1995 QJE), the class of models such as Romer's variety expansion has the scale effect; that is, the TFP growth rate is increasing in the population size. This is empirically not supported at all. By assuming a constant population level, we avoid considering the effects of population growth in this paper. However, our model does not have the scale effect due to a simple assumption, which some readers may feel is rather an easy tinkering, though. That is, while we assume that (a) it is the institutional capital per capita that determines the labour productivity in the R&D sector, (b) the aggregate capital per capita is increasing in the share of the R&D labour. This can be regarded as an easy tinkering; to see this, consider the case with $\omega_Z = 0$ (still $H_{A,t}$ can be positive) then $Z_t = z_L/\bar{H}_t$, and hence the law of motion of A_t (1c) becomes

$$\frac{A_{t+1}}{A_t} = (\omega_A z_L) \frac{H_{A,t}}{\bar{H}_t} + (1 - \delta_A)$$

That is, the TFP growth is proportional to the share of R&D labour.

We could follow Jones' approach with $0 < \eta_A < 1$ at least potentially.

$$\begin{aligned} A_{t+1} &= A_t^{new} + (1 - \delta_A) A_t \\ A_t^{new} &= \omega_A Z_t H_{A,t} A_t^{1-\eta_A} \end{aligned}$$

We have obtained the equilibrium equations with this formulation. The main drawback is a computational burden. Actually, it turns out that we need to have one more additional state variable,

[25] See for example Anderson 2003 and Anderson and Ge 2004.
[26] See Paap and van Dijik (1998).
[27] See for example Bianchi (1997) and Quah (1997).
[28] See Azariadis and Stachurski (2003).

which is a ratio of population to the technology level $\Xi_t = \bar{H}_{t-1}/A_t^{\eta_A}$. This is followed by a non-expectational dynamic equation (law of motion) $\Xi_{t+1} = \left(\gamma_{\bar{H},t}/\gamma_{A,t+1}^{\eta_A}\right)\Xi_t$. Hence, Jones' approach requires us to increase the dimension of the state space. Another reason why we did not follow Jones' approach is that his method is not really perfect anyway. For example, TFP growth rate $\gamma_{A,t+1}$ is asymptotically determined by the population growth rate $\gamma_{\bar{H},t}$, which is not really plausible. Under our formulation, the TFP growth rate is mainly determined by the R&D population share but not directly determined by neither the population size or the population growth rate. Our modeling device itself is an easy trick but the resultant model property is rather consistent with the empirical regularities.

Anyway, because the scale effect is a rather profound issue in this area, we confine ourselves not to discuss its effect in this paper. One important note related to this is that, depending on the assumptions, the population growth rate may have a very strong effect on the R&D, because the change in the population growth rate affects the effective discount factor. As we have already seen, the discount factor has the most important parameter in our model.

7 Conclusion

This paper studies a simple endogenous growth model to explain growth slowdowns. It is designed to explain, for example, the middle income trap often observed in the south-east Asian countries, the U.K.'s productivity puzzle after the Great Recession and the lost decades of Japan in a unified framework. It is based on the Romer's (1990, JPE) variety expansion model with additional state variable, which we call the R&D environment. Our model is fairly realistic in the sense that it allows us to do calibration exercises which are rather standard in the business cycle studies. Hence, we can apply a similar method to Comin and Gertler (2006), but the quantitative results are very different from theirs, because we are more interested in shocks that last for quite long periods.

The R&D environment in this paper loosely captures a wide range of social and institutional environment surrounding R&D activities. We assume that it has the following two properties, of which the latter generates an externality; (a) the better the R&D environment is, the higher the R&D productivity is; and (b) society accumulates the R&D environment via R&D activities. Together with the non-negativity constraint of the labour supply, this additional state variable generates multiple steady states (balanced growth paths, BGPs).

The main properties of the model are as follows. First, in the steady states, improving the production efficiency and increasing the government expenditure is neutral in the endogenous growth rate, while the R&D is very sensitive to the financial efficiency. The key intuition is that the R&D and production sectors compete in the factor markets (labour market in our model). On the positive side, a higher output level increases firm profit and firm value. But, on the negative side, a higher output means that the production sector absorbs more labour, which raises the wage rate. Because the R&D sector takes labour as an input, a higher wage means a higher cost, which discourages inventions. In our model, these two effects exactly offset each other. In contrast, our numerical simulations show that the R&D activities are very sensitive to the financial market efficiency. Because a higher risk-premium on the equity investment makes the stock price lower, the reward to the invention becomes lower when the interest rate is high.

Second, in terms of the transition dynamics, if an economy has a long financial malfunctioning period, it experiences rises and falls of the stock return, followed by an almost-permanent growth slowdown. The simulation result sheds light on the financial boom and bust often observed prior to the growth slowdowns. In the model, an increase in the stock return is partly a direct result of a high required rate of return caused by the financial malfunctioning. In addition, because discouraged R&D labour shifts to the production sector, output per firm and hence its profit increases. Importantly, there is a discrepancy between aggregate output and the individual firm profits. In aggregate, if the R&D sector attracts more labour, then they invent new products more, implying a faster TFP growth. For the individual firms, on the other hand, output per firm does not increase regardless of the invention of new variety, because a new invention is commercialized by a new firm. A high level of R&D activities rather is harmful for individual firm profit, because the R&D sector absorbs more labour by increasing the labour cost.

These model properties have a couple of policy implications. First, traditional fiscal and monetary polices are not very useful to prevent an economy from falling into the low BGP. Instead, long term structural policies that improves directly the R&D productivity or that improves financial efficiency are effective. Second, a big-push is required for underdeveloped countries to take off. But, classical ODA that improves production efficiency has little effects.

Finally, we admit that this model is fairly stylized and hence are rather reluctant to take all the simulated results at their face value. Nonetheless, some key implications seem to be valid in reality. For example, in purely model-free empirical studies, people often find the importance of the financial deepening in economic growth. One of the messages in this paper is that such an importance of the financial sector is not only for developing countries but also for developed countries such as Japan and the U.K.

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Appendix

A Proof of Proposition

The proof is, though tedious, straightforward. First of all, for the low steady state $H_{A,ss}$, the growth rate does not depend on any parameters simply because it is zero.

In the case of the high steady state, wage equalization holds (interior solution).

$$(2m): \tilde{W}_{ss} = (1-\alpha)\,\lambda_{ss}^{MC}\frac{\tilde{Y}_{ss}}{\tilde{H}_{Y,ss}} = \omega_A \tilde{Z}_{ss}\tilde{V}_{ss}$$

Hence, the equation of firm profit implies

$$(2e): \tilde{\Pi}_{ss} = (1-\tilde{\alpha})\,\alpha\lambda_{ss}^{MC}\tilde{Y}_{ss} = \alpha\frac{1-\tilde{\alpha}}{1-\alpha}\omega_A \tilde{Z}_{ss}\tilde{V}_{ss}$$

Because $1 = \tilde{H}_{Y,ss} + \tilde{H}_{A,ss}$, we find the following equation.

$$1 = \omega_A \alpha \frac{1-\tilde{\alpha}}{1-\alpha}\tilde{Z}_{ss}\left(1-\tilde{H}_{A,ss}\right)\frac{\tilde{V}_{ss}}{\tilde{\Pi}_{ss}}$$

Hence, treating $\tilde{H}_{A,ss}$, $\gamma_{A,ss}$, $\Lambda_{ss}\gamma_{v,ss}$ and $\tilde{\Pi}_{ss}/\tilde{V}_{ss}$ as intermediate variables, we can find $\tilde{Z}_{ss}$ that satisfies $\varepsilon_Z = 0$.

$$\begin{aligned}
(2a): \tilde{H}_{A,ss} &= \frac{\delta_Z}{\omega_Z}\left(\tilde{Z}_{ss} - \tilde{z}_L\right) \\
(2c): \gamma_{A,ss} &= \omega_A \tilde{Z}_{ss}\tilde{H}_{A,ss} + (1-\delta_A) \\
(*): \Lambda_{ss}\gamma_{v,ss} &= \beta\left(\gamma_{\xi,ss}\gamma_{\bar{H},ss}\gamma_{A,ss}^{\varphi_A}\right)^{1-\sigma}/\gamma_{A,ss} \\
(2d): \frac{\tilde{\Pi}_{ss}}{\tilde{V}_{ss}} &= \left(\frac{c_v}{\Lambda_{ss}\gamma_{v,ss}} - (1-\delta_A)\right)\left(\frac{\Lambda_{ss}\gamma_{v,ss}}{c_v}\right)^{\chi_V} \\
\varepsilon_Z &= \omega_A \alpha \frac{1-\tilde{\alpha}}{1-\alpha}\tilde{Z}_{ss}\left(1-\tilde{H}_{A,ss}\right)\frac{\tilde{V}_{ss}}{\tilde{\Pi}_{ss}} - 1
\end{aligned}$$

where $(*)$ defines the discount factor for the firm profit; see (2d) in the main text. Hence, any parameters that do not appear here do not affect none of R&D environment Z_{ss}, R&D labour $\tilde{H}_{A,ss}$, TFP growth $\gamma_{A,ss}$ and price/earnings ratio $(\tilde{V}_{ss}/\tilde{\Pi}_{ss})$ in the high steady state. This completes the proof. Note finally that there are two $\tilde{Z}_{ss}$ that satisfies $\varepsilon_Z = 0$, the lower of which is the R&D environment at the middle explosive steady state.

B Solution Algorithm and Some Policy Functions

To solve the system of equations (2), we employed a Eular equation iteration (a projection method). We first assume arbitrary policy functions of the dynamic jump variables ($\tilde{C}_t$ and $\tilde{V}_t$) as functions of the state variables ($\tilde{Z}_t$ and $\tilde{K}_t$). With these policy functions, we can infer $\tilde{C}_{t+1}$ and $\tilde{V}_{t+1}$ and their expected values by (2). Then, we solve (2) for the dynamic jump variables at each node point on the state plane. We label these solved dynamic jump vairables as $\tilde{C}'_t$ and $\tilde{V}'_t$ with prime "′". In general, $\tilde{C}'_t$ and $\tilde{V}'_t$ are different from $\tilde{C}_t$ and $\tilde{V}_t$. Hence, we update the policy functions ($\tilde{C}_t$ and $\tilde{V}_t$) by using $\tilde{C}'_t$ and $\tilde{V}'_t$. We repeat this until the policy functions coincide with the solved values (time iteration).

Figure 11 shows some selected policy functions for the non-stochastic case. To enhance visibility, in this figure, we use a much fewer grid points than the actual computation. Without shocks, the steady states (the red stars in each panel) must be on the policy functions. Note that the policy functions of $\tilde{Z}_{t+1}$, $\tilde{H}_{A,t}$ and $\gamma_{A,t}$ have discontinuity along the black broken line (border of the area with $\tilde{H}_{A,t} = 0$) in Figure 4, though it is not visible for $\tilde{Z}_{t+1}$. This discontinuity causes a jump in the transition dynamics. While $\tilde{C}_t$ and $\tilde{V}_t$ have kink along the same line, $\tilde{K}_{t+1}$ is smooth in the entire state space.

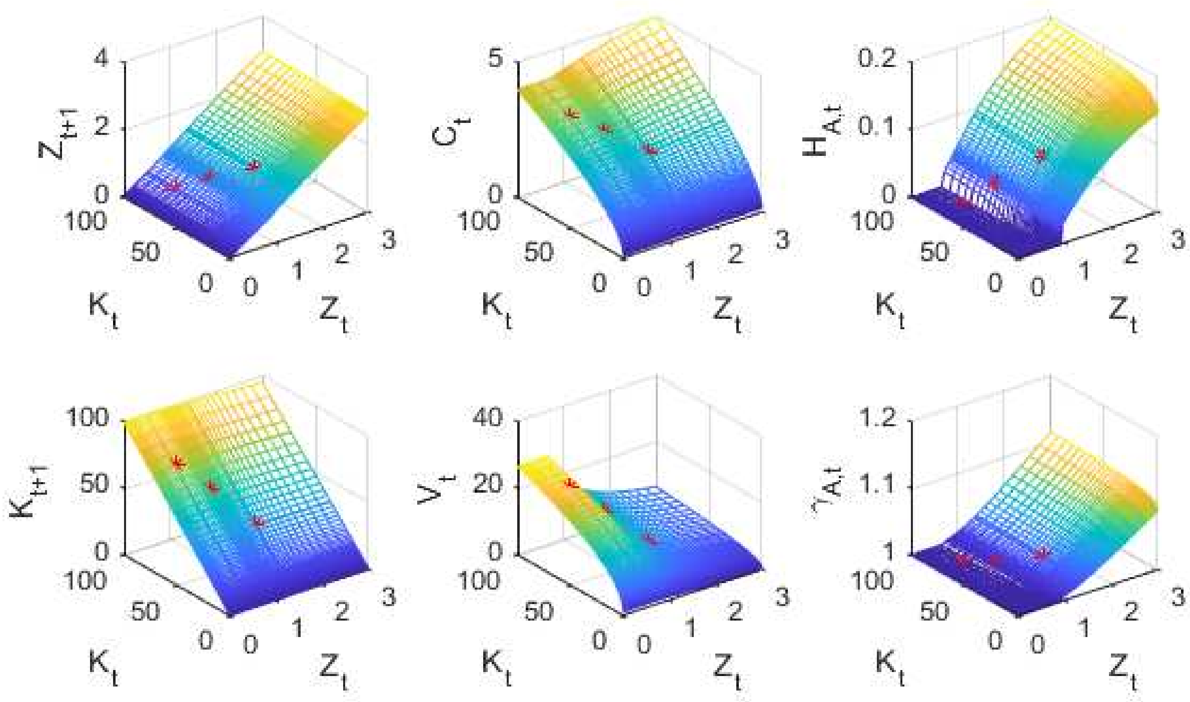

Figure 11: Selected Policy Functions. All variables are detrended. The red stars ($*$) show the balanced growth paths. The grid points are not equi-distanced.