Forum of Mathematics, Sigma (2018), Vol. 6, e11, 66 pages
doi:10.1017/fms.2018.9

MODULAR DECOMPOSITION NUMBERS OF CYCLOTOMIC HECKE AND DIAGRAMMATIC CHEREDNIK ALGEBRAS: A PATH THEORETIC APPROACH

C. BOWMAN[1] and A. G. COX[2]

[1] School of Mathematics, Statistics and Actuarial Science, University of Kent, Canterbury CT2 7NF, UK;
email: C.D.Bowman@kent.ac.uk

[2] Department of Mathematics, City, University of London, London, UK;
email: A.G.Cox@city.ac.uk

Received 12 April 2018; accepted 22 May 2018

This paper is dedicated to Paul Martin on the occasion of his sixtieth birthday.

Abstract

We introduce a path theoretic framework for understanding the representation theory of (quantum) symmetric and general linear groups and their higher-level generalizations over fields of arbitrary characteristic. Our first main result is a 'super-strong linkage principle' which provides degree-wise upper bounds for graded decomposition numbers (this is new even in the case of symmetric groups). Next, we generalize the notion of homomorphisms between Weyl/Specht modules which are 'generically' placed (within the associated alcove geometries) to cyclotomic Hecke and diagrammatic Cherednik algebras. Finally, we provide evidence for a higher-level analogue of the classical Lusztig conjecture over fields of sufficiently large characteristic.

2010 Mathematics Subject Classification: 20C08 (primary); 20G43 (secondary)

1. Introduction

Cyclotomic quiver Hecke algebras (and their quasiherediary covers, the diagrammatic Cherednik algebras) are of central interest in Khovanov homology, knot theory, group theory, and higher representation theory. Rouquier's conjecture [**Rou08b**] (recently solved in a flurry of publications [**RSVV16**, **Los16**, **Web17**]) allows us to understand the complex representation theory of these algebras in terms of Kazhdan–Lusztig theory. This paper seeks to generalize their work to the modular representation theory of these algebras, where almost nothing is known or even conjectured.

Our approach provides new insight even in the classical case; in particular it allows us to provide strong new degree-wise upper bounds for the graded decomposition numbers of symmetric groups. This combinatorial bound is given in terms of folding-up paths in Euclidean space under the action of an affine Weyl group. This seems to be the first result of its kind in the literature, and so we state it now in this simplified form (for the full statement in higher levels, see Theorem 9.1).

THEOREM A (The super-strong linkage principle for symmetric groups). *Let λ, μ be partitions with at most h columns and $\Bbbk$ be a field of characteristic $p > h$. The graded decomposition numbers of the symmetric group are bounded as follows,*

$$[S(\lambda) : D(\mu)\langle k\rangle] \leqslant |\{\mathsf{s} \mid \mathsf{s} \in \mathrm{Path}^{+}(\lambda, \mathsf{t}^{\mu}), \deg(\mathsf{s}) = k\}| \tag{1.1}$$

for $k \in \mathbb{Z}$. In particular if $[S(\lambda) : D(\mu)] \neq 0$, then λ and μ are strongly linked with $\lambda \uparrow \mu$.

Theorem A provides a two-fold strengthening of the famous strong linkage principle for symmetric (and general linear) groups [**And80**]. First, if $\mathrm{Path}^{+}(\lambda, \mathsf{t}^{\mu}) \neq \emptyset$ this implies that $\lambda \uparrow \mu$ and so we obtain infinitely many new zeros of the decomposition matrix not covered by [**And80**, Theorem 1]. Second, (1.1) clearly provides a wealth of new and more complicated bounds on these multiplicities – in addition it incorporates the grading into the picture for the first time. We expect this result to be of independent interest and so we have included illustrative examples in Section 9.

In the case of symmetric groups, it is common practice to restrict ones attention to the representations with at most h columns. In so doing, we obtain a category of representations which remains rich in structure but, thanks to revolutionary work of Riche–Williamson [**RW16**], is now known to stabilize and become understandable over fields of characteristic $p \gg h$. The principal aim of this paper is to identify a higher-level analogue of this category with a similarly rich structure and to generalize the vast array of powerful ideas and results developed by Andersen, Carter, Jantzen, Kleshchev, Koppinen, Lusztig and others over the past forty years (in particular [**And80**, **And98**, **CP80**, **Kop86**, **Kle97**, **Lus80**, **Jan77**]) and hence cast questions concerning the representation theory of these higher-level algebras in terms of their associated alcove geometries.

The quotient algebra of the cyclotomic quiver Hecke algebra $H_n(\kappa)$ (and hence subcategory of $H_n(\kappa)$-mod) of interest to us is

$$\mathcal{Q}_{\ell,h,n}(\kappa) = H_n(\kappa)/\langle e(\underline{i}) \mid \underline{i} \in I^\ell \text{ and } \underline{i}_{k+1} = \underline{i}_k + 1 \text{ for } 1 \leqslant k \leqslant h\rangle$$

for $e > h\ell$ and an h-admissible $\kappa \in I^\ell$ as in Definition 3.3. We shall see that this algebra is Morita equivalent to a certain quotient, $A_h(n, \theta, \kappa)$, of the diagrammatic Cherednik algebra associated to the weighting $\theta = (1, 2, \dots, \ell) \in \mathbb{Z}^\ell$ (over an arbitrary field $\Bbbk$). In particular, the simple representations of both of these algebras are indexed by the set of multipartitions whose components each have at most h columns, denoted by $\mathscr{P}_n^\ell(h)$, and the graded decomposition matrices, $\mathbf{D} = (d_{\lambda\mu}(t))_{\lambda,\mu\in\mathscr{P}_n^\ell}$, coincide. We cast representation-theoretic questions concerning these algebras in the setting of an alcove geometry of type

$$A_{h-1} \times A_{h-1} \times \cdots \times A_{h-1} \subseteq \widehat{A}_{\ell h-1}.$$

We first show that the algebra $A_h(n, \theta, \kappa)$ has a graded cellular basis indexed by orbits of paths in this geometry. For each $\lambda \in \mathscr{P}_n^\ell$, we hence obtain a basis of the Weyl module, $\Delta(\lambda)$, which encodes a great deal of representation-theoretic information. This allows us to provide incredibly simple proofs of a number of new structural results over arbitrary fields. The first of which is our higher-level analogue of the super-strong linkage principle.

We then consider the idea of *generic behaviour*. This generalizes the idea (originally due to Jantzen and later Lusztig [Lus80, Jan77]) that when we are 'sufficiently far away from the walls of the dominant region' representation-theoretic questions simplify greatly. We encounter higher-level analogues of the familiar generic sets of points which are 'close together' in the geometry (for example, points 'around a Steinberg vertex'). In higher levels, there is also a striking new kind of generic behaviour involving points 'as far apart as possible' in the geometry. For such generic sets, one of our main results is the following (see Corollary 10.12 for the full statement).

THEOREM B (Generic homomorphisms). *For (λ, μ) a generic pair, we have that*

$$\dim_t(\mathrm{Hom}_{A_h(n,\theta,\kappa)}(\Delta(\mu), \Delta(\lambda))) = t^{\ell(\mu)-\ell(\lambda)} + \cdots$$

where the other terms are of strictly smaller degree. We provide an explicit construction of these homomorphisms and prove results concerning their composition.

Theorem B generalizes results due to Carter–Lusztig [**CL74**, Section 4] and Koppinen [**Kop86**, Theorem 6.1] for $\ell = 1$ and is utilized in [**CBS18**] in order to construct the first family of BGG resolutions of simple modules of Hecke and Cherednik algebras (indeed the first examples of such resolutions anywhere in modular representation theory) and to generalize and lift all the results of [**Ruf06**, **Kle96**] to a structural level. For higher levels, we find that there are arbitrarily large generic sets (as $n \to \infty$) and we hence obtain arbitrarily long chains of homomorphisms whose composition is nonzero (Section 10.2). Finally, in Theorem 8.2 we obtain a higher-level analogue of the stability for representations of general linear groups obtained by tensoring with the determinant, as follows.

THEOREM C. *Given a partition* $\lambda = (\lambda_1, \lambda_2, \dots)$ *and* $h \in \mathbb{N}$, *we set* $\det_h(\lambda) = (h, \lambda_1, \lambda_2, \dots)$. *We have an injective map* $\det_h : \mathscr{P}^\ell_n(h) \hookrightarrow \mathscr{P}^\ell_{n+h\ell}(h)$ *given by*

$$\det_h(\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(\ell)}) = (\det_h(\lambda^{(1)}), \det_h(\lambda^{(2)}), \dots, \det_h(\lambda^{(\ell)}))$$

and a corresponding injective homomorphism of graded R*-algebras*

$$A_h(n, \theta, \kappa) \hookrightarrow A_h(n + h\ell, \theta, \kappa).$$

In particular, $d_{\lambda,\mu}(t) = d_{\det_h(\lambda),\det_h(\mu)}(t)$ *for all* $\lambda, \mu \in \mathscr{P}^\ell_n(h)$.

Inspired by these results, we provide the first conjectural framework for calculating the (graded) decomposition numbers of cyclotomic Hecke algebras over fields of sufficiently large characteristic in Section 11.2. We verify these conjectures in the cases of (i) maximal finite parabolic orbits with $\Bbbk$ arbitrary – generalizing all the results of [**Kle97**, **CMT08**, **TT13**] (ii) $\ell = 2$ or 3 with $e > n$ and $\Bbbk$ arbitrary – generalizing [**HM15**, Theorems B3 & B5] and [**BS11**, Section 9] (iii) $\ell = 2$ and $h = 1$ and $\Bbbk$ arbitrary [**CGM03**, Section 8] (iv) $\Bbbk = \mathbb{C}$. For $\ell = 1$ the conjecture was formulated by Andersen [**And98**, Section 5] and proved by Riche and Williamson in [**RW16**, Theorem 1.9].

CONJECTURE A. *Let* $e > h\ell$, $\kappa \in I^\ell$ *be an* h*-admissible multicharge, and* $\Bbbk$ *be field of characteristic* $p \gg h\ell$. *The decomposition numbers of* $A_h(n, \theta, \kappa)$ *are given by*

$$d_{\lambda\mu}(t) = n_{\lambda\mu}(t)$$

for $\lambda, \mu \in \mathscr{P}^\ell_n(h)$ *in the first ep-alcove and* $n_{\lambda\mu}(t)$ *the associated affine parabolic Kazhdan–Lusztig polynomial of type* $A_{h-1} \times A_{h-1} \times \cdots \times A_{h-1} \subseteq \widehat{A}_{\ell h-1}$.

Our treatment covers the quotient algebras $\mathcal{Q}_{\ell,h,n}(\kappa)$ for $e \in \mathbb{Z}_{>0}$ uniformly alongside the algebras $H_n(\kappa)$ for $e = \infty$ (of more generally $e > n$). The alcove geometries controlling the latter family of algebras are seen as simple subcases of those controlling the former family of algebras. The following conjecture (a refinement of [KR11, Conjecture 7.3] which was debunked in [Wil14, Section 4.2]) is the corresponding simplification of Conjecture A.

CONJECTURE B. *Let $e = \infty$ or more generally $e > n$ and suppose $\kappa \in I^\ell$ has no repeated entries. Let $\Bbbk$ be field of characteristic $p \gg \ell$. The decomposition numbers of $A(n, \theta, \kappa)$ are given by*

$$d_{\lambda\mu}(t) = n_{\lambda\mu}(t)$$

for $\lambda, \mu \in \mathscr{P}_n^\ell$ and $n_{\lambda\mu}(t)$ the associated parabolic Kazhdan–Lusztig polynomials of type $A_{n-1} \times A_{n-1} \times \cdots \times A_{n-1} \subseteq A_{\ell n-1}$.

Our approach provides the first general framework for studying the modular representation theory of these algebras (for admissible $\kappa \in I^\ell$). Indeed, while cyclotomic Hecke algebras have been extensively studied over the past twenty years, surprisingly little is known about their representation theory over fields of positive characteristic. The blocks of these algebras were determined a decade ago by Lyle and Mathas [LM07]. More recently, homomorphisms between certain pairs of Specht modules were constructed in [LM14] and reduction theorems between certain pairs of Specht modules were given in [FS16, BS16]. Apart from blocks of small weight [Fay06, LR16], nothing else is known or even conjectured concerning the decomposition numbers and homomorphism spaces of these algebras. Other powerful results concerning their representation theory do exist, including explicit branching rules [Kle95, Bru98, Ari06] and a generalized Jantzen sum formula [JM00] but they provide little information about general decomposition numbers.

The paper is structured as follows. The first three sections introduce the main protagonists of this paper. In Section 3 we explicitly review the construction of $\mathcal{Q}_{\ell,h,n}(\kappa)$ in terms of the classical generators of Khovanov–Lauda–Rouquier and its coset-like cellular basis from [Bow16, Section 8]. In Section 4, we recall Webster's definition of the diagrammatic Cherednik algebras. In Section 5 we prove that $A_h(n, \theta, \kappa)$ and $\mathcal{Q}_{\ell,h,n}(\kappa)$ are (graded) Morita equivalent. We also discuss in detail why our choice of weighting is optimal for the purposes of understanding as much of the modular representation theory of $H_n(\kappa)$ as possible. Sections 3 and 5 have been written so as to make the paper intelligible to those studying the classical representation theory of cyclotomic Hecke algebras (without any prior knowledge of how diagrammatic Cherednik algebras fit into the picture). Section 6 provides the crux of this paper: we prove that the algebra $A_h(n, \theta, \kappa)$ possesses a cellular basis indexed by pairs of paths in the alcove geometry of type $A_{h-1} \times A_{h-1} \times \cdots \times A_{h-1} \subseteq \widehat{A}_{\ell h-1}$. In Section 8 we consider the higher-level analogue of tensoring with the determinant. In Section 9, we state and prove the super-strong linkage principle for our algebras and illustrate how it significantly improves on the classical strong linkage principle.

In Section 10, we construct homomorphisms between the Weyl modules for $A_h(n, \theta, \kappa)$ and consider when the composition of these homomorphisms is nonzero. In Section 11 we formulate and provide evidence for our conjectures for calculating the decomposition numbers of $A_h(n, \theta, \kappa)$ over fields of sufficiently large characteristic. We remark that over $\mathbb{C}$, the approach presented here is in the same spirit of our earlier work [**BCS17**] (which covered the case $\mathcal{Q}_{1,\ell,n}(\kappa)$). However our main focus in this paper is the modular case. Finally, in Section 12 we demonstrate the kind of geometries we encounter (in particular, these are more exotic than those seen in classical Lie theory) and illustrate how one can use the tools of this paper to understand a detailed example for the Hecke algebras of type B over an arbitrary field.

The results of this paper were very much inspired by ideas and conjectures of Paul Martin. In particular, his programme of studying non-Lie theoretic algebras via alcove geometries and his use of path theoretic bases for encoding representation-theoretic structures.

2. Graded cellular algebras

We now recall the definition and first properties of graded cellular algebras following [**HM10**, Section 2]. Let R denote an arbitrary commutative integral domain and $\Bbbk$ denote an arbitrary field.

DEFINITION 2.1 [**HM10**, Definition 2.1]. Suppose that A is a $\mathbb{Z}$-graded R-algebra which is of finite rank over R. We say that A is a *graded cellular algebra* if the following conditions hold. The algebra is equipped with a *cell datum* $(\Lambda, \mathcal{T}, C, \deg)$, where $(\Lambda, \trianglerighteq)$ is the *weight poset*. For each $\lambda \in \Lambda$ we have a finite set, denoted by $\mathcal{T}(\lambda)$. There exist maps

$$C : \coprod_{\lambda \in \Lambda} \mathcal{T}(\lambda) \times \mathcal{T}(\lambda) \to A \quad \text{and} \quad \deg : \coprod_{\lambda \in \Lambda} \mathcal{T}(\lambda) \to \mathbb{Z}$$

such that C is injective. We denote $C(\mathsf{S}, \mathsf{T}) = c^\lambda_{\mathsf{ST}}$ for $\mathsf{S}, \mathsf{T} \in \mathcal{T}(\lambda)$, and require:

(1) Each element c^λ_{ST} is homogeneous of degree $(c^\lambda_{\mathsf{ST}}) = \deg(\mathsf{S}) + \deg(\mathsf{T})$, for $\lambda \in \Lambda$, $\mathsf{S}, \mathsf{T} \in \mathcal{T}(\lambda)$.

(2) The set $\{c^\lambda_{\mathsf{ST}} \mid \mathsf{S}, \mathsf{T} \in \mathcal{T}(\lambda), \lambda \in \Lambda\}$ is a R-basis of A.

(3) If $\mathsf{S}, \mathsf{T} \in \mathcal{T}(\lambda)$, for some $\lambda \in \Lambda$, and $a \in A$ then there exist scalars $r_{\mathsf{SU}}(a)$, which do not depend on T, such that

$$ac^\lambda_{\mathsf{ST}} = \sum_{\mathsf{U} \in \mathcal{T}(\lambda)} r_{\mathsf{SU}}(a) c^\lambda_{\mathsf{UT}} \pmod{A^{\rhd \lambda}},$$

where $A^{\rhd\lambda}$ is the R-submodule of A spanned by $\{c^\mu_{\mathsf{QR}} \mid \mu \rhd \lambda \text{ and } \mathsf{Q}, \mathsf{R} \in \mathcal{T}(\mu)\}$.

(4) The R-linear map $* : A \to A$ determined by $(c^\lambda_{\mathsf{ST}})^* = c^\lambda_{\mathsf{TS}}$, for all $\lambda \in \Lambda$ and all $\mathsf{S}, \mathsf{T} \in \mathcal{T}(\lambda)$, is an antiisomorphism of A.

This graded cellular structure allows us to immediately define a natural family of so-called *graded cell modules* as follows. Given any $\lambda \in \Lambda$, the *graded cell module* $\Delta^A(\lambda)$ is the graded left A-module with basis $\{c^\lambda_{\mathsf{S}} \mid \mathsf{S} \in \mathcal{T}(\lambda)\}$. The action of A on $\Delta^A(\lambda)$ is given by

$$ac^\lambda_{\mathsf{S}} = \sum_{\mathsf{U} \in \mathcal{T}(\lambda)} r_{\mathsf{SU}}(a) c^\lambda_{\mathsf{U}}, \tag{2.1}$$

where the scalars $r_{\mathsf{SU}}(a)$ are the scalars appearing in condition (3) of Definition 2.1. Let t be an indeterminate over $\mathbb{Z}_{\geqslant 0}$. If $M = \bigoplus_{z \in \mathbb{Z}} M_z$ is a free graded R-module, then its *graded dimension* is the Laurent polynomial

$$\dim_t(M) = \sum_{k \in \mathbb{Z}} (\dim_{\Bbbk} M_k) t^k.$$

If M is a graded A-module and $k \in \mathbb{Z}$, define $M\langle k \rangle$ to be the same module with $(M\langle k \rangle)_i = M_{i-k}$ for all $i \in \mathbb{Z}$. We call this a *degree shift* by k. If M is a graded A-module and L^A is a graded simple module let $[M : L^A\langle k \rangle]$ be the multiplicity of $L^A\langle k \rangle$ as a graded composition factor of M, for $k \in \mathbb{Z}$.

We now recall the method by which one can, at least in principle, construct all simple modules of a graded cellular algebra. This construction uses only basic linear algebra. Suppose that $\lambda \in \Lambda$. There is a bilinear form $\langle\, , \rangle_\lambda$ on $\Delta^A(\lambda)$ which is determined by

$$c^\lambda_{\mathsf{US}} c^\lambda_{\mathsf{TV}} \equiv \langle c^\lambda_{\mathsf{S}}, c^\lambda_{\mathsf{T}} \rangle_\lambda c^\lambda_{\mathsf{UV}} \pmod{A^{\rhd\lambda}},$$

for any $\mathsf{S}, \mathsf{T}, \mathsf{U}, \mathsf{V} \in \mathcal{T}(\lambda)$. For every $\lambda \in \Lambda$, we let $\langle\, , \rangle_\lambda$ denote the bilinear form on $\Delta^A(\lambda)$ and $\mathrm{rad}\langle\, , \rangle_\lambda$ denote the radical of this bilinear form. Given any $\lambda \in \Lambda$ such that $\mathrm{rad}\langle\, , \rangle_\lambda \neq \Delta^A(\lambda)$, we set $L^A(\lambda) = \Delta^A(\lambda)/\mathrm{rad}\langle\, , \rangle_\lambda$. This module is graded (by [**HM10**, Lemma 2.7]) and simple, and in fact every simple module is of this form, up to grading shift.

Proposition 2.2 [HM10]. *If* $\mu \in \Lambda$ *then* $\dim_t(L^A(\mu)) \in \mathbb{Z}_{\geqslant 0}[t+t^{-1}]$.

The passage between the (graded) cell and simple modules is recorded in the *graded decomposition matrix*, $\mathbf{D}_A(t) = (d^A_{\lambda\mu}(t))$, of A where

$$d^A_{\lambda\mu}(t) = \sum_{k\in\mathbb{Z}} [\Delta^A(\lambda) : L^A(\mu)\langle k\rangle]\, t^k,$$

for $\lambda, \mu \in \Lambda$. This matrix is unitriangular with respect to the partial ordering $\trianglerighteq$ on Λ.

3. Cyclotomic quiver Hecke algebras

We let R denote an arbitrary commutative integral domain. We let $\mathfrak{S}_n$ denote the symmetric group on n letters, with presentation

$$\mathfrak{S}_n = \langle s_1, \ldots, s_{n-1} \mid s_i s_{i+1} s_i - s_{i+1} s_i s_{i+1}, s_i s_j - s_j s_i \text{ for } |i-j| > 1\rangle.$$

We shall be interested in the representation theory (over R) of the reflection groups $(\mathbb{Z}/\ell\mathbb{Z}) \wr \mathfrak{S}_n$ and their deformations. The background material from this section is lifted from [Web17, Section 2] and [Bow16, Section 1].

3.1. The quiver Hecke algebra. Throughout this paper e is a fixed element of the set $\{3, 4, 5, \ldots\} \cup \{\infty\}$. This excludes the case $e = 2$ as we are only interested in 'large' characteristics (where geometries have nonempty alcoves, see Section 6). If $e = \infty$ then we set $I = \mathbb{Z}$, while if $e < \infty$ then we set $I = \mathbb{Z}/e\mathbb{Z}$. We let Γ_e be the quiver with vertex set I and edges $i \longrightarrow i+1$, for $i \in I$. Hence, we are considering either the linear quiver $\mathbb{Z}$ $(e = \infty)$ or a cyclic quiver $(e < \infty)$: to the quiver Γ_e we attach the symmetric Cartan matrix with entries $(a_{i,j})_{i,j\in I}$ defined by $a_{ij} = 2\delta_{ij} - \delta_{i(j+1)} - \delta_{i(j-1)}$.

Following [Kac90, Ch. 1], let $\widehat{\mathfrak{sl}}_e$ be the Kac–Moody algebra of Γ_e with simple roots $\{\alpha_i \mid i \in I\}$, fundamental weights $\{\Lambda_i \mid i \in I\}$, positive weight lattice $P^+ = \bigoplus_{i\in I} \mathbb{Z}_{\geqslant 0}\Lambda_i$ and positive root lattice $Q^+ = \bigoplus_{i\in I} \mathbb{Z}_{\geqslant 0}\alpha_i$. Let $(\cdot,\cdot)$ be the usual invariant form associated with this data, normalized so that $(\alpha_i, \alpha_j) = a_{ij}$ and $(\Lambda_i, \alpha_j) = \delta_{ij}$, for $i, j \in I$. Fix a sequence $\kappa = (\kappa_1, \ldots, \kappa_\ell) \in I^\ell$, the *e-multicharge*, and define $\Lambda = \Lambda(\kappa) = \Lambda_{\kappa_1} + \cdots + \Lambda_{\kappa_\ell}$. Then $\Lambda \in P^+$ is dominant weight of *level* ℓ.

Definition 3.1 [BK09, KL09, Rou08a]. Suppose α is a positive root of height n, and set $I^\alpha = \{i \in I^n \mid \alpha_{i_1} + \cdots + \alpha_{i_n} = \alpha\}$. Define $H^\alpha(\kappa)$ to be the unital, associative R-algebra with generators

$$\{e(\underline{i}) \mid \underline{i} = (i_1, \dots, i_n) \in I^\alpha\} \cup \{y_1, \dots, y_n\} \cup \{\psi_1, \dots, \psi_{n-1}\}, \tag{†}$$

subject to the relations

$$e(\underline{i})e(\underline{j}) = \delta_{\underline{i},\underline{j}} e(\underline{i}); \tag{3.1}$$

$$\sum_{\underline{i} \in I^\alpha} e(\underline{i}) = 1; \tag{3.2}$$

$$y_r e(\underline{i}) = e(\underline{i}) y_r; \tag{3.3}$$

$$\psi_r e(\underline{i}) = e(s_r \underline{i}) \psi_r; \tag{3.4}$$

$$y_r y_s = y_s y_r; \tag{3.5}$$

$$\psi_r y_s = y_s \psi_r \quad \text{if } s \neq r, r+1; \tag{3.6}$$

$$\psi_r \psi_s = \psi_s \psi_r \quad \text{if } |r - s| > 1; \tag{3.7}$$

$$y_r \psi_r e(\underline{i}) = (\psi_r y_{r+1} + \delta_{i_r, i_{r+1}}) e(\underline{i}); \tag{3.8}$$

$$y_{r+1} \psi_r e(\underline{i}) = (\psi_r y_r - \delta_{i_r, i_{r+1}}) e(\underline{i}); \tag{3.9}$$

$$\psi_r^2 e(\underline{i}) = \begin{cases} 0 & \text{if } i_r = i_{r+1}, \\ e(\underline{i}) & \text{if } i_{r+1} \neq i_r, i_r \pm 1, \\ (y_{r+1} - y_r) e(\underline{i}) & \text{if } i_{r+1} = i_r - 1, \\ (y_r - y_{r+1}) e(\underline{i}) & \text{if } i_{r+1} = i_r + 1; \end{cases} \tag{3.10}$$

$$\psi_r \psi_{r+1} \psi_r = \begin{cases} (\psi_{r+1} \psi_r \psi_{r+1} + 1) e(\underline{i}) & \text{if } i_r = i_{r+2} = i_{r+1} + 1, \\ (\psi_{r+1} \psi_r \psi_{r+1} - 1) e(\underline{i}) & \text{if } i_r = i_{r+2} = i_{r+1} - 1, \\ (\psi_{r+1} \psi_r \psi_{r+1}) e(\underline{i}) & \text{otherwise}; \end{cases} \tag{3.11}$$

for all admissible r, s, i, j. Finally, we have the cyclotomic relation

$$y_1^{\langle \Lambda(\kappa) | \alpha_{i_1} \rangle} e(\underline{i}) = 0 \quad \text{for } \underline{i} \in I^\alpha. \tag{3.12}$$

The *quiver Hecke algebra* is the sum $H(\kappa) := \bigoplus_\alpha H^\alpha(\kappa)$ over all positive roots of height n.

THEOREM 3.2 [**BK09**, **KL09**, **Rou08a**]. *We have a grading on $H_n(\kappa)$ given by*

$$\deg(e(\underline{i})) = 0 \quad \deg(y_r) = 2 \quad \deg(\psi_r e(\underline{i})) = \begin{cases} -2 & \textit{if } i_r = i_{r+1}, \\ 1 & \textit{if } i_r = i_{r+1} \pm 1, \\ 0 & \textit{otherwise}. \end{cases}$$

For each $i \in I$ define the i*-string of length* h to be $\alpha_{i,h} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+h-1} \in Q^+$.

DEFINITION 3.3. For $h \in \mathbb{Z}_{>0}$ and $\kappa \in I^\ell$, we say that κ is *h-admissible* if $(\Lambda, \alpha_{h,i}) \leqslant 1$ for all $i \in I$.

DEFINITION 3.4. Let $e > h\ell$ and $\kappa \in I^\ell$ be an h-admissible multicharge. We set

$$\mathcal{Q}_{\ell,h,n}(\kappa) = H_n(\kappa)/\langle e(\underline{i}) \mid \underline{i} \in I^n \text{ and } \underline{i}_{k+1} = \underline{i}_k + 1 \text{ for } 1 \leqslant k \leqslant h\rangle.$$

REMARK 3.5. For $\ell = 1$ and $p = e$ the algebra $\mathcal{Q}_{1,h,n}(\kappa)$ is isomorphic to the image of the symmetric group on n letters in $\mathrm{End}_{\Bbbk}((\Bbbk^h)^{\otimes n})$. Therefore $\mathcal{Q}_{1,h,n}(\kappa)$ is the generalized Temperley–Lieb algebra of [H99̈, Section 1] and is the Ringel dual (see [Erd97, Section 4]) of the classical Schur algebra.

3.2. Weighted standard tableaux.

The background material from this section is lifted from [Web17, Section 2] and [Bow16, Section 1]. Fix integers $\ell, n \in \mathbb{Z}_{\geqslant 0}$ and $e \in \mathbb{Z}_{>0} \cup \{\infty\}$. We define a *weighting* to be any $\theta \in \mathbb{Z}^\ell$ such that $\theta_i - \theta_j \notin \ell\mathbb{Z}$ for $1 \leqslant i < j \leqslant \ell$. Throughout this section, we assume that the weighting $\theta \in \mathbb{Z}^\ell$ and e-multicharge $\kappa \in I^\ell$ are fixed.

We define a *partition*, λ, of n to be a finite weakly decreasing sequence of nonnegative integers $(\lambda_1, \lambda_2, \ldots)$ whose sum, $|\lambda| = \lambda_1 + \lambda_2 + \cdots$, equals n. An ℓ-*multipartition* $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)})$ of n is an ℓ-tuple of partitions such that $|\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| = n$. We will denote the set of ℓ-multipartitions of n by $\mathscr{P}_n^\ell$. Given $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(\ell)}) \in \mathscr{P}_n^\ell$, the *Young diagram* of λ is defined to be the set of nodes,

$$\{(r, c, m) \mid 1 \leqslant c \leqslant \lambda_r^{(m)}\}.$$

We do not distinguish between the multipartition and its Young diagram. We refer to a node (r, c, m) as being in the rth row and cth column of the mth component of λ. Given a node, (r, c, m), we define the *residue* of this node to be $\mathrm{res}(r, c, m) = \kappa_m + c - r \pmod e$. We refer to a node of residue $i \in I$ as an i-node.

Given $\lambda \in \mathscr{P}_n^\ell$, the associated θ-*Russian array* is defined as follows. For each $1 \leqslant m \leqslant \ell$, we place a point on the real line at θ_m and consider the region bounded by half-lines at angles $3\pi/4$ and $\pi/4$. We tile the resulting quadrant with a lattice of squares, each with diagonal of length 2ℓ. We place a box $(1, 1, m) \in \lambda$ at the point θ_m on the real line, with rows going northwest from this node, and columns going northeast. We do not distinguish between λ and its θ-Russian array.

DEFINITION 3.6. Let $\theta \in \mathbb{Z}^\ell$ be a weighting and $\kappa \in I^\ell$. Given $\lambda \in \mathscr{P}_n^\ell$, we define a *tableau* of shape λ to be a filling of the boxes of the θ-Russian array of λ with the numbers $\{1, \ldots, n\}$. We define a *standard tableau* to be a tableau in which the entries increase along the rows and columns of each component. We let $\mathrm{Std}(\lambda)$

denote the set of all standard tableaux of shape $\lambda \in \mathscr{P}_n^\ell$. Given $\mathsf{t} \in \mathrm{Std}(\lambda)$, we set $\mathrm{Shape}(\mathsf{t}) = \lambda$. Given $1 \leqslant k \leqslant n$, we let $\mathsf{t}\downarrow_{\{1,\dots,k\}}$ be the subtableau of t whose entries belong to the set $\{1, \dots, k\}$.

DEFINITION 3.7. Let (r, c, m), (r', c', m') be two i-boxes and $\theta \in \mathbb{Z}^\ell$ be our fixed weighting. We write $(r, c, m) \rhd_\theta (r', c', m')$ if $\theta_m + \ell(r - c) < \theta_{m'} + \ell(r' - c')$ or $\theta_m + \ell(r - c) = \theta_{m'} + \ell(r' - c')$ and $r + c < r' + c'$.

DEFINITION 3.8. Given $\lambda, \mu \in \mathscr{P}_n^\ell$, we say that λ *θ-dominates* μ (and write $\mu \trianglelefteq_\theta \lambda$) if for every i-box $(r, c, m) \in \mu$, there exist at least as many i-boxes $(r', c', m') \in \lambda$ which θ-dominate (r, c, m) than there do i-boxes $(r'', c'', m'') \in \mu$ which θ-dominate (r, c, m).

Given $\lambda \in \mathscr{P}_n^\ell$, we let $\mathrm{Rem}(\lambda)$ (respectively $\mathrm{Add}(\lambda)$) denote the set of all removable (respectively addable) boxes of the Young diagram of λ so that the resulting diagram is the Young diagram of a multipartition. Given $i \in \mathbb{Z}/e\mathbb{Z}$, we let $\mathrm{Rem}_i(\lambda) \subseteq \mathrm{Rem}(\lambda)$ (respectively $\mathrm{Add}_i(\lambda) \subseteq \mathrm{Add}(\lambda)$) denote the subset of boxes of residue $i \in I$.

DEFINITION 3.9. Let $\lambda \in \mathscr{P}_n^\ell$ and $\mathsf{t} \in \mathrm{Std}(\lambda)$. We let $\mathsf{t}^{-1}(k)$ denote the box in t containing the integer $1 \leqslant k \leqslant n$. Given $1 \leqslant k \leqslant n$, we let $\mathcal{A}_\mathsf{t}(k)$, (respectively $\mathcal{R}_\mathsf{t}(k)$) denote the set of all addable $\mathrm{res}(\mathsf{t}^{-1}(k))$-boxes (respectively all removable $\mathrm{res}(\mathsf{t}^{-1}(k))$-boxes) of the multipartition $\mathrm{Shape}(\mathsf{t}\downarrow_{\{1,\dots,k\}})$ which are less than $\mathsf{t}^{-1}(k)$ in the θ-dominance order (that is those which appear to the *right* of $\mathsf{t}^{-1}(k)$).

DEFINITION 3.10. Let $\lambda \in \mathscr{P}_n^\ell$ and $\mathsf{t} \in \mathrm{Std}(\lambda)$. We define the degree of t as follows,

$$\deg(\mathsf{t}) = \sum_{k=1}^{n} (|\mathcal{A}_\mathsf{t}(k)| - |\mathcal{R}_\mathsf{t}(k)|).$$

EXAMPLE 3.11. Let $e = 7$ and $\kappa = (0, 3)$ and $\theta = (0, 1)$. The rightmost tableau $\mathsf{s} \in \mathrm{Std}((3, 2, 1), (3, 2, 2))$ depicted in Figure 1 is of degree 1. The boxes of nonzero degree are $\mathsf{s}^{-1}(k)$ for $k = 5$, 12, and 13. We have that $\deg(\mathsf{t}^{-1}(5)) = 1$, $\deg(\mathsf{s}^{-1}(12)) = -1$, and $\deg(\mathsf{s}^{-1}(13)) = 1$. For example, the box $\mathsf{s}^{-1}(10)$ appears to the right of $\mathsf{s}^{-1}(12)$ and both are of residue $5 \in \mathbb{Z}/7\mathbb{Z}$.

3.3. The graded cellular basis. In this section we recall the construction of the graded cellular basis of the algebra $\mathcal{Q}_{\ell,h,n}(\kappa)$ from [Bow16, Section 8]. For this section, we fix $\theta = (1, 2, \dots, \ell) \in \mathbb{Z}^\ell$.

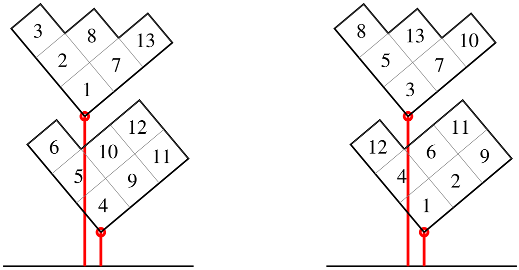


Figure 1. The tableau $\mathsf{s}, \mathsf{t}^{\lambda} \in \mathrm{Std}((3,2,1),(3,2,2))$.

DEFINITION 3.12. Given $\lambda \in \mathscr{P}_n^{\ell}$ we let $\mathsf{t}^{\lambda} \in \mathrm{Std}(\lambda)$ be the tableau obtained by placing the entry n in the least dominant removable box $(r,c,m) \in \lambda$ (in the θ-dominance order) and then placing the entry $n-1$ in the least dominant removable box of $\lambda \setminus \{(r,c,m)\}$ and continuing in this fashion.

EXAMPLE 3.13. We have that $\mathsf{t}^{\lambda} \in \mathrm{Std}(\lambda)$ for $\lambda = ((3,2,1),(3,2,2))$ is the leftmost tableau depicted in Figure 1.

DEFINITION 3.14. Given $\mathsf{t} \in \mathrm{Std}(\lambda)$, we define the associated *residue sequence* as follows,

$$\underline{i}_{\mathsf{t}} = (\mathrm{res}(\mathsf{t}^{-1}(1)), \mathrm{res}(\mathsf{t}^{-1}(2)), \ldots, \mathrm{res}(\mathsf{t}^{-1}(n))) \in I^n.$$

EXAMPLE 3.15. Given $\mathsf{s}, \mathsf{t}^{\lambda} \in \mathrm{Std}(\lambda)$ for $\lambda = (3,2,1),(3,2,2)$, we have that

$$\underline{i}_{\mathsf{t}^{\lambda}} = (0,1,2,3,4,5,6,0,2,3,1,2,5) \quad \underline{i}_{\mathsf{s}} = (3,2,0,4,1,3,6,2,1,5,2,5,0).$$

DEFINITION 3.16. Given $\lambda \in \mathscr{P}_n^{\ell}$ and $\mathsf{s} \in \mathrm{Std}(\lambda)$, we let $d_{\mathsf{s}} \in \mathfrak{S}_n$ denote any element such that $d_{\mathsf{s}}(\mathsf{t}^{\lambda}) = \mathsf{s}$ under the place permutation action of the symmetric group on standard tableaux.

DEFINITION 3.17. Given $\lambda \in \mathscr{P}_n^{\ell}$ and $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda)$ we fix reduced expressions $d_{\mathsf{s}} = s_{i_1} s_{i_2} \ldots s_{i_k}$ and $d_{\mathsf{t}} = s_{j_1} s_{j_2} \ldots s_{j_m}$. We set

$$\psi_{\mathsf{st}}^{\theta,\kappa} = \psi_{\mathsf{s}} e(\underline{i}_{\mathsf{t}^{\lambda}})(\psi_{\mathsf{t}})^*$$

where $\psi_{\mathsf{s}} = \psi_{i_1}\psi_{i_2}\ldots\psi_{i_k}$, $\psi_{\mathsf{t}} = \psi_{j_1}\psi_{j_2}\ldots\psi_{j_m}$.

DEFINITION 3.18. Let $\mathscr{P}_n^\ell(h) \subseteq \mathscr{P}_n^\ell$ denote the subset consisting of all multipartitions with at most h columns in any given component, that is

$$\mathscr{P}_n^\ell(h) = \{\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)}) \mid \lambda_1^{(m)} \leqslant h \text{ for all } 1 \leqslant m \leqslant \ell\}.$$

THEOREM 3.19 [**Bow16**, Theorem 8.2]. *The algebra $\mathcal{Q}_{h,\ell,n}(\kappa)$ admits a graded cellular basis*

$$\{\psi_{\mathsf{st}}^{\theta,\kappa} \mid \lambda \in \mathscr{P}_n^\ell(h), \mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda)\}$$

with respect to the θ-dominance order on $\mathscr{P}_n^\ell(h)$ and the involution $*$. We refer to the resulting cell modules (as in (2.1)) as the Specht modules of $\mathcal{Q}_{h,\ell,n}(\kappa)$ and denote them by

$$S(\lambda) = \{\psi_{\mathsf{s}}^{\theta,\kappa} \mid \mathsf{s} \in \mathrm{Std}(\lambda)\}$$

for $\lambda \in \mathscr{P}_n^\ell(h)$. The modules $S(\lambda)$ lift to modules of $H_n(\kappa)$ and the decomposition matrix $\mathbf{D}^{\mathcal{Q}_{h,\ell,n}(\kappa)}$ appears as a (square) submatrix of $\mathbf{D}^{H_n(\kappa)}$.

REMARK 3.20. As in [**HM10**, Definition 5.1] and [**Bow16**, Definition 7.11], the elements $\psi_{\mathsf{s}}^{\theta,\kappa}$ for $\mathsf{s} \in \mathrm{Std}(\lambda)$ are defined up to a choice of reduced expression for d_{s}, however any such reduced expression can be chosen arbitrarily.

REMARK 3.21. It is shown in [**Bow16**, Corollary 7.3] that for $h = 1$ the algebra $\mathcal{Q}_{1,\ell,n}(\kappa)$ is isomorphic to the generalized blob algebra of [**MW03**, Section 1.2] (in particular $\mathcal{Q}_{1,2,n}(\kappa)$ is isomorphic to the blob algebra of [**MW00**, Section 2]).

4. Diagrammatic Cherednik algebras

We recall the definitions and important properties of diagrammatic Cherednik algebras from [**Web17**]. We first tilt the θ-Russian array of $\lambda \in \mathscr{P}_n^\ell$ ever-so-slightly in the clockwise direction so that the top vertex of the box $(r, c, m) \in \lambda$ has x-coordinate $\mathbf{I}_{(r,c,m)} = \theta_m + \ell(r - c) + (r + c)\varepsilon$ for $\varepsilon \ll 1/n$ (using standard small-angle identities to approximate the coordinate to order ε^2). Given $\lambda \in \mathscr{P}_n^\ell$, we let $\mathbf{I}_\lambda$ denote the disjoint union over the $\mathbf{I}_{(r,c,m)}$ for $(r, c, m) \in \lambda$.

DEFINITION 4.1. Let $\lambda, \mu \in \mathscr{P}_n^\ell$. A λ-tableau of weight μ is a bijective map $\mathsf{T} : \lambda \to \mathbf{I}_\mu$. We say that a tableau T is *semistandard* if it satisfies the following additional properties:

(i) $\mathsf{T}(1, 1, m) > \theta_m$;

(ii) $\mathsf{T}(r, c, m) > \mathsf{T}(r - 1, c, m) + \ell$;

(iii) $\mathsf{T}(r, c, m) > \mathsf{T}(r, c - 1, m) - \ell$.

We denote the set of all semistandard tableaux of shape λ and weight μ by $\mathrm{SStd}_{\theta,\kappa}(\lambda,\mu)$. Given $\mathsf{T} \in \mathrm{SStd}_{\theta,\kappa}(\lambda,\mu)$, we write $\mathrm{Shape}(\mathsf{T}) = \lambda$. When the context is clear we write $\mathrm{SStd}(\lambda,\mu)$ or $\mathrm{SStd}_n(\lambda,\mu)$ for $\mathrm{SStd}_{\theta,\kappa}(\lambda,\mu)$.

DEFINITION 4.2. We let $\mathrm{SStd}_n^+(\lambda,\mu) \subseteq \mathrm{SStd}_n(\lambda,\mu)$ denote the subset of tableaux which respect residues. In other words, if $\mathsf{T}(r,c,m) = \mathbf{I}_{(r',c',m')} \in \mathbf{I}_\mu$ for $(r,c,m) \in \lambda$ and $(r',c',m') \in \mu$, then $\kappa_m + c - r = \kappa_{m'} + c' - r' \pmod e$.

DEFINITION 4.3. We define a *θ-diagram* of type $G(\ell,1,n)$ to be a *frame* $\mathbb{R}\times[0,1]$ with distinguished solid points on the northern and southern boundaries given by $\mathbf{I}_\mu$ and $\mathbf{I}_\lambda$ for some $\lambda,\mu \in \mathscr{P}_n^\ell$ and a collection of *solid strands* each of which starts at a northern point, $\mathbf{I}_{(r,c,m)}$ for $(r,c,m) \in \mu$, and ends at a southern point, $\mathbf{I}_{(r',c',m')}$ for $(r',c',m') \in \lambda$. Each strand carries some residue, $i \in I$ say, and is referred to as a *solid i-strand*. We further require that each solid strand has a mapping diffeomorphically to $[0,1]$ via the projection to the y-axis. Each solid strand is allowed to carry any number of dots. We draw:

- a dashed line ℓ units to the left of each solid i-strand, which we call a *ghost i-strand* or *i-ghost*;
- vertical red lines at $\theta_m \in \mathbb{Z}$ each of which carries a residue κ_m for $1 \leqslant m \leqslant \ell$ which we call a *red κ_m-strand*.

We refer to a point at which two strands cross as a *double point*. We require that there are no triple points (points at which three strands cross) or tangencies (points at which two curves intersect without crossing one another at that point) involving any combination of strands, ghosts or red lines and no dots lie on crossings.

An example of a *θ-diagram* is given in Figure 2.

DEFINITION 4.4 [**Web17**, Definition 4.1]. The *diagrammatic Cherednik algebra*, $\mathbf{A}(n,\theta,\kappa)$, is the R-algebra spanned by all θ-diagrams modulo the following local relations (here a local relation means one that can be applied on a small region of the diagram).

(3.1) Any diagram may be deformed isotopically; that is, by a continuous deformation of the diagram which avoids tangencies, double points and dots on crossings.

(3.2) For $i \neq j$ we have that dots pass through crossings.

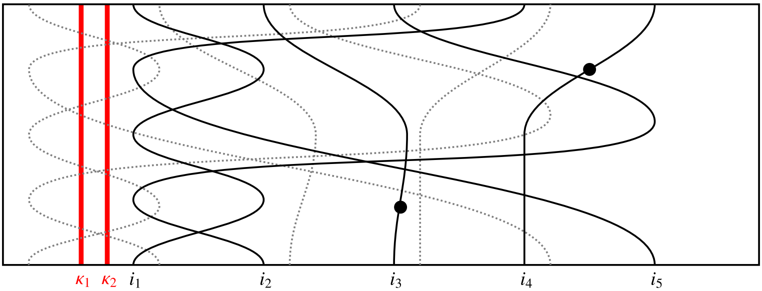
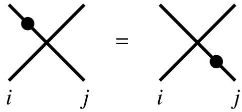
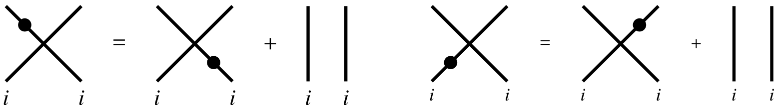


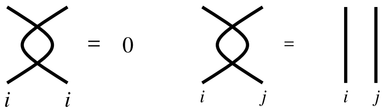
Figure 2. A θ-diagram for $\theta = (0, 1)$ with northern and southern loading given by $\mathbf{I}_\omega$ where $\omega = (\varnothing, (1^5))$.



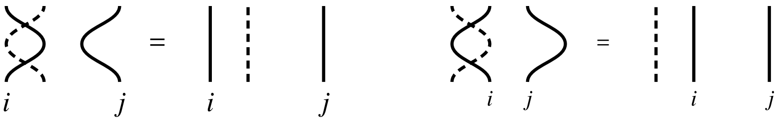
(3.3) For two like-labelled strands we get an error term.



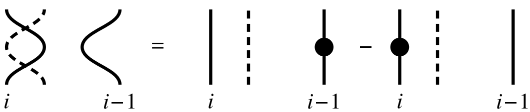
(3.4) For double-crossings of solid strands with $i \neq j$, we have the following.



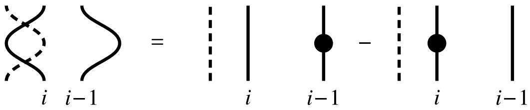
(3.5) If $j \neq i - 1$, then we can pass ghosts through solid strands.



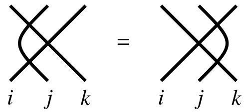
(3.6) On the other hand, in the case where $j = i - 1$, we have the following.



(3.7) We also have the relation below, obtained by symmetry.

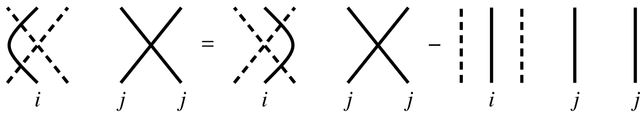


(3.8) Strands can move through crossings of solid strands freely.

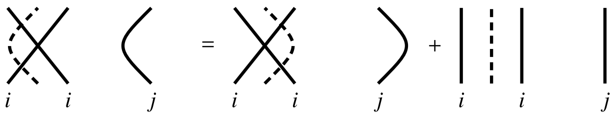


for any $i, j, k \in I$. Similarly, this holds for triple points involving ghosts, except for the following relations when $j = i - 1$.

(3.9)

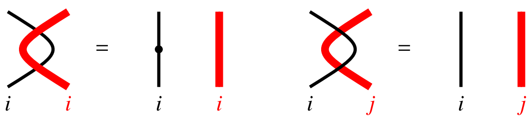


(3.10)



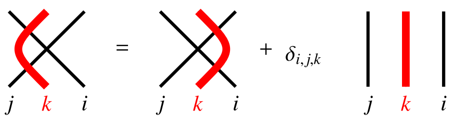
In the diagrams with crossings in (3.9) and (3.10), we say that the solid (respectively ghost) strand bypasses the crossing of ghost strands (respectively solid strands). The ghost strands may pass through red strands freely. For $i \neq j$, the solid i-strands may pass through red j-strands freely. If the red and solid strands have the same label, a dot is added to the solid strand when straightening. Diagrammatically, these relations are given by the following diagrams and their mirror images

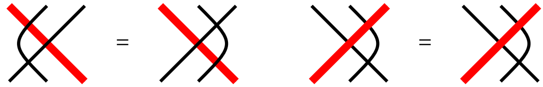
(3.11)

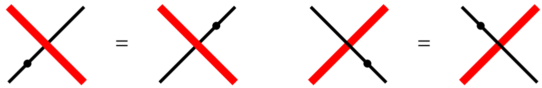


for $i \neq j$. All solid crossings and dots can pass through red strands, with a correction term:

(3.12)



(3.13) 

(3.14) 

Finally, we have the following nonlocal idempotent relation.

(3.15) Any idempotent in which a solid strand is ℓn units to the left of the leftmost red strand is referred to as unsteady and set to be equal to zero.

The product $d_1 d_2$ of two diagrams $d_1, d_2 \in A(n, \theta, \kappa)$ is given by putting d_1 on top of d_2. This product is defined to be 0 unless the southern border of d_1 is given by the same loading as the northern border of d_2 with residues of strands matching in the obvious manner, in which case we obtain a new diagram with loading and labels inherited from those of d_1 and d_2.

PROPOSITION 4.5 [Web17, Section 4]. *There is a $\mathbb{Z}$-grading on the algebra* $\mathbf{A}(n, \theta, \kappa)$ *as follows:* (i) *dots have degree* 2*;* (ii) *the crossing of two strands has degree* 0*, unless they have the same label, in which case it has degree* -2*;* (iii) *the crossing of a solid strand with label i and a ghost has degree* 1 *if the ghost has label* $i - 1$ *and* 0 *otherwise;* (iv) *the crossing of a solid strand with a red strand has degree* 0*, unless they have the same label, in which case it has degree* 1*. In other words,*

deg  $= 2$ deg  $= -2\delta_{i,j}$ deg  $= \delta_{j,i+1}$ deg  $= \delta_{j,i-1}$

i $\quad$ $i\ j$ $\quad$ $i\ j$ $\quad$ $i\ j$

deg  $= \delta_{i,j}$ deg  $= \delta_{j,i}$.

$i\ j$ $\quad$ $i\ j$

Let T be any tableau of shape λ and weight μ. Associated to T, we have a θ-diagram C_{T} with distinguished solid points on the northern and southern borders given by $\mathbf{I}_\lambda$ and $\mathbf{I}_\mu$, respectively; the n solid strands each connect a northern and southern distinguished point and are drawn so that they trace out the bijection determined by T in such a way that we use the minimal number of crossings; the strand terminating at point $(\mathbf{I}_{(r,c,m)}, 0)$ for $(r, c, m) \in \lambda$ carries residue equal to $\mathrm{res}(r, c, m) \in I$. This diagram is not unique up to isotopy, but we can choose one such diagram arbitrarily. Given a pair of semistandard tableaux of the same shape $(\mathsf{S}, \mathsf{T}) \in \mathrm{SStd}(\lambda, \mu) \times \mathrm{SStd}(\lambda, \nu)$, we have a diagram $C_{\mathsf{ST}} = C_{\mathsf{S}} C^*_{\mathsf{T}}$ where C^*_{T} is the diagram obtained from C_{T} by flipping it through the horizontal axis.

THEOREM 4.6 ([Web17, Section 2.6], [Bow16, Theorem 3.19]). *The R-algebra $\mathbf{A}(n, \theta, \kappa)$ is a graded cellular algebra with basis*

$$\{C_{\mathsf{ST}} \mid \mathsf{S} \in \mathrm{SStd}(\lambda, \mu), \mathsf{T} \in \mathrm{SStd}(\lambda, \nu), \lambda, \mu, \nu \in \mathscr{P}_n^\ell\}$$

with respect to the θ-dominance order on $\mathscr{P}_n^\ell(h)$ and the involution $*$.

The *radical* of a finite-dimensional A-module M, denoted by $\mathrm{rad}M$, is the smallest submodule of M such that the corresponding quotient is semisimple. We then let $\mathrm{rad}^2 M = \mathrm{rad}(\mathrm{rad}\, M)$ and inductively define the *radical series*, $\mathrm{rad}^i M$, of M by $\mathrm{rad}^{i+1} M = \mathrm{rad}(\mathrm{rad}^i M)$. We have a finite chain

$$M \supset \mathrm{rad}(M) \supset \mathrm{rad}^2(M) \supset \cdots \supset \mathrm{rad}^i(M) \supset \mathrm{rad}^{i+1}(M) \supset \cdots \supset \mathrm{rad}^s(M) = 0.$$

THEOREM 4.7 [Web17, Theorem 6.2]. *Over $\mathbb{C}$, the diagrammatic Cherednik algebra $\mathbf{A}(n, \theta, \kappa)$ is standard Koszul. The grading coincides with the radical filtration on standard modules as follows,*

$$d_{\lambda\mu}^{\mathbf{A}(n,\theta,\kappa)}(t) = \sum_k [\mathrm{rad}^k(\Delta^{\mathbf{A}(n,\theta,\kappa)}(\lambda))/\mathrm{rad}^{k+1}(\Delta^{\mathbf{A}(n,\theta,\kappa)}(\lambda)) : L^{\mathbf{A}(n,\theta,\kappa)}(\mu)]t^k$$

and hence $d_{\lambda\mu}(t) \in t\mathbb{Z}_{\geqslant 0}[t]$ for $\lambda \neq \mu \in \mathscr{P}_n^\ell$.

5. Modular representations of cyclotomic Hecke and diagrammatic Cherednik algebras

Let $\Bbbk$ be an arbitrary field of characteristic $p > 0$. We wish to understand as much of the representation theory of symmetric groups and cyclotomic Hecke algebras over $\Bbbk$ as possible. As made precise in [Web17, Bow16], this is equivalent to understanding the representation theory of diagrammatic Cherednik algebras for arbitrary weightings $\theta \in \mathbb{Z}^\ell$. A long-standing belief in modular representation theory of algebraic groups is that we should (first) restrict our attention to fields whose characteristic is greater than the Coxeter number of the group. This is equivalent (via Ringel duality) to considering the sub/quotient category of symmetric group representations labelled by partitions with at most h columns over a field of characteristic $p > h$.

DEFINITION 5.1. Let $Q \subseteq \mathscr{P}_n^\ell$. We say that Q is *saturated* if for any $\alpha \in Q$ and $\beta \in \mathscr{P}_n^\ell$ with $\beta \lhd_\theta \alpha$, we have that $\beta \in Q$. We say that Q is *cosaturated* if its complement in $\mathscr{P}_n^\ell$ is saturated. We say that Q is *closed* if it is the intersection of a saturated set and a cosaturated set.

DEFINITION 5.2 (See [BS16, Proposition 2.4]). Let E (respectively F) denote a saturated (respectively cosaturated) subset of $\mathscr{P}^\ell_n$, so that $E \cap F \subseteq \mathscr{P}^\ell_n$ is closed. We let

$$e = \sum_{\mu \in E \cap F} 1_\mu \quad \text{and} \quad f = \sum_{\mu \in F \setminus E} 1_\mu$$

in $\mathbf{A}(n, \theta, \kappa)$. We let $\mathbf{A}_{E\cap F}(n, \theta, \kappa)$ denote the subquotient of $\mathbf{A}(n, \theta, \kappa)$ given by

$$\mathbf{A}_{E\cap F}(n, \theta, \kappa) = e(\mathbf{A}(n, \theta, \kappa)/(\mathbf{A}(n, \theta, \kappa) f \mathbf{A}(n, \theta, \kappa)))e$$

which is cellular with respect to the basis

$$\{C_{\mathsf{ST}} \mid \mathsf{S} \in \mathrm{SStd}(\lambda, \mu), \mathsf{T} \in \mathrm{SStd}(\lambda, \nu), \lambda, \mu, \nu \in E \cap F\},$$

EXAMPLE 5.3. Let $\ell = 1$ and let $\theta \in \mathbb{Z}$ and $\kappa \in I$ be arbitrary. The set $\mathscr{P}^1_n(h) \subseteq \mathscr{P}^1_n$ of partitions with at most h columns is saturated in the θ-dominance ordering. Over fields of characteristic $p > h$ the representation theory of the algebra $\mathbf{A}_{\mathscr{P}^1_n(h)}(n, \theta, \kappa)$ can be understood in terms of the p-canonical basis [RW16, Theorem 1.9]. For $p \gg h$ this simplifies a great deal and can be understood combinatorially in terms of Kazhdan–Lusztig theory. An iterative approach for passing from $p \gg h$ to smaller primes is the subject of [Lus15].

Let $\theta \in \mathbb{Z}^\ell$ be such that $\theta_i - \theta_j > n\ell$ for any $1 \leqslant i < j \leqslant \ell$. We say that such a weighting is *well separated*. For such a weighting, we have that $\mathbf{A}(n, \theta, \kappa)$ is Morita equivalent to the cyclotomic q-Schur algebra of Dipper *et al.* [DJM98, Definition 6.1] (see [Web17, Section 3.3]). This is the most classical example of a θ-dominance ordering. This order is the worst possible ordering for our purposes. This is because if Q is a 'natural' subset of $\mathscr{P}^\ell_n$ which is closed under the θ-dominance order, then understanding the decomposition matrices of $\mathbf{A}_Q(n, \theta, \kappa)$ for $n \in \mathbb{Z}_{\geqslant 0}$ contains as a subproblem that of understanding the entire decomposition matrix of $\mathfrak{S}_n$.

EXAMPLE 5.4. For example let $\ell = 3$ and $\theta \in \mathbb{Z}^3$ be a well-separated weighting and suppose $Q \subseteq \mathscr{P}^\ell_n$ is a closed subset. If $(\lambda, \varnothing, \varnothing), (\varnothing, \varnothing, \nu) \in Q \subseteq \mathscr{P}^3_n$, then $\{(\varnothing, \mu, \varnothing) \mid \mu \in \mathscr{P}^1_n\} \subseteq Q \subseteq \mathscr{P}^3_n$. In particular, for $n \gg p$ we find that understanding $\mathbf{A}_Q(n, \theta, \kappa)$ is at least as difficult as understanding the decomposition numbers of symmetric groups without restriction on the number of columns (and therefore the problem of understanding tilting characters of general linear groups over infinite fields of characteristic strictly less than the Coxeter number [Erd97, Section 4]).

As a remedy to the problem encountered in Example 5.4, we propose studying the largest subset $Q \subseteq \mathscr{P}_n^\ell$ such that no element of Q has more than h columns in any given component. Namely, we consider the set of h-restricted multipartitions $\mathscr{P}_n^\ell(h)$ as in Definition 3.18. We wish to identify a corresponding subcategory of $H_n(\kappa)$-mod. We therefore require a weighting, $\theta \in \mathbb{Z}^\ell$, such that $\mathscr{P}_n^\ell(h) \subseteq \mathscr{P}_n^\ell$ is a saturated subset in the θ-dominance order. This leads us to consider weightings of the form $\theta \in \mathbb{Z}^\ell$ such that $0 \leqslant \theta_i - \theta_j < \ell$ for all $1 \leqslant i < j \leqslant \ell$. We now show that it does not matter which of these weightings we choose.

LEMMA 5.5. *Let $e > h\ell$ and let $\kappa \in I^\ell$ be h-admissible. For a given $i \in I$, an interval of the form $[x, x + (h + 1)\ell] \subseteq \mathbb{R}$ contains at most one i-diagonal of boxes*

$$\{(r, c, m) \mid \mathbf{I}_{(r,c,m)} \in [x, x + h\ell], \kappa_m + c - r = i \in I\}.$$

Proof. This simply follows because the distinct entries of the multicharge differ by at least h and the boxes have width ℓ and $e > h\ell$. $\square$

PROPOSITION 5.6. *Let $\theta = (1, 2, \ldots, \ell)$ and let $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_\ell) \in I^\ell$ be h-admissible. We have that*

$$\mathbf{A}(n, \theta, \kappa + (1^\ell)) \cong \mathbf{A}(n, \theta, \kappa) \cong \mathbf{A}(n, \theta^\sigma, \kappa) \cong \mathbf{A}(n, \theta, \kappa^\sigma)$$

where

$$\kappa^\sigma = (\kappa_{\sigma(1)}, \kappa_{\sigma(2)}, \ldots, \kappa_{\sigma(\ell)}), \quad \theta^\sigma = (\theta_{\sigma(1)}, \theta_{\sigma(2)}, \ldots, \theta_{\sigma(\ell)})$$

and

$$\kappa + (1^\ell) = (\kappa_1 + 1, \kappa_2 + 1, \ldots, \kappa_\ell + 1)$$

for $\sigma \in \mathfrak{S}_\ell$.

Proof. The first isomorphism simply amounts to relabelling the underlying quiver. The isomorphism $\mathbf{A}(n, \theta, \kappa) \cong \mathbf{A}(n, \theta^\sigma, \kappa)$ is trivial. We now consider the isomorphism $A(n, \theta, \kappa) \cong \mathbf{A}(n, \theta, \kappa^\sigma)$. For arbitrary $\kappa \in I^\ell$, it is easy to check that the natural map

$$\mathrm{SStd}_{\theta,\kappa}(\lambda, \mu) \leftrightarrow \mathrm{SStd}_{\theta,\kappa^\sigma}(\lambda^\sigma, \mu^\sigma)$$

is both bijective and degree preserving. Therefore the result holds on the level of graded R-modules. Given $\alpha \in \mathscr{P}_n^\ell(h)$, we let D_σ^α denote any diagram which has solid strands connecting points $(\mathbf{I}_{(r,c,m)}, 0)$ to the points $(\mathbf{I}_{(r,c,\sigma(m))}, 1)$ for $(r, c, m) \in \alpha$ and which has red strands connecting the points $(\theta_m, 0)$ to the points

$(\theta_{\sigma(m)}, 1)$ in such a way as to have the minimal number of crossings. We define $\varphi : A(n, \theta, \kappa) \to A(n, \theta, \kappa^\sigma)$ by $\varphi(C_{\mathsf{S}}) = D^\mu_\sigma C_{\mathsf{S}}(D^\lambda_\sigma)^*$ for $\mathsf{S} \in \mathrm{SStd}_{\theta,\kappa}(\lambda, \mu)$. By Lemma 5.5, any crossings in D_σ involve strands of nonadjacent (and nonequal) residues. Therefore, $C_{\mathsf{S}} D^\lambda_\sigma = C_{\mathsf{S}^\sigma}$ and so this is an algebra isomorphism, as required. □

REMARK 5.7. Recall that we are interested in $\mathbf{A}_{\mathscr{P}^\ell_n(h)}(n, \theta, \kappa)$ (as in Definition 5.2) for $e > h\ell$ and $\kappa \in I^\ell$ h-admissible. By Proposition 5.6, we can assume that $\theta = (1, 2, \dots, \ell)$. We choose to only consider our multicharge up to rotation and shifting residues. Given $e > h\ell$, we can suppose that $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_\ell) \in I^\ell$ is such that

$$\kappa_1 < \kappa_2 < \cdots < \kappa_\ell$$

and such that $\kappa_\ell + h \neq \kappa_1 \pmod e$. (For $e \neq \infty$, we have abused notation by placing an ordering $\mathbb{Z}/e\mathbb{Z}$ via the natural ordering on $\{0, 1, \dots, e-1\}$.)

We let $\omega = (\varnothing, \dots, \varnothing, (1^n))$ and $\underline{i} \in I^n$. We let $\mathsf{E}^{\underline{i}}_\omega$ denote the diagram with northern and southern distinguished points given by $\mathbf{I}_\omega$, no crossing strands, and the kth solid strand decorated with the residue $\underline{i}_k \in I$. We set $\mathsf{E}_\omega = \sum_{\underline{i} \in I^n} \mathsf{E}^{\underline{i}}_\omega$. It is shown in [**Bow16**, Theorem 4.5] that $\mathsf{E}_\omega \mathbf{A}(n, \theta, \kappa)\mathsf{E}_\omega = H_n(\kappa)$ for any weighting $\theta \in \mathbb{Z}^\ell$. For any fixed weighting and multicharge, it is easy to see that there is a corresponding degree-preserving bijective map $\varphi : \mathrm{Std}(\lambda) \to \mathrm{SStd}(\lambda, \omega)$ (see [**Bow16**, Proposition 4.4]).

DEFINITION 5.8. For $\lambda \in \mathscr{P}^\ell_n$, we set $1_\lambda = C_{\mathsf{T}^\lambda \mathsf{T}^\lambda}$ for T^λ the unique element of $\mathrm{SStd}(\lambda, \lambda)$. We set

$$A_h(n, \theta, \kappa) = \left(\sum_{\alpha \in \mathscr{P}^\ell_n} 1_\alpha \right) \mathbf{A}_{\mathscr{P}^\ell_n(h)}(n, \theta, \kappa) \left(\sum_{\alpha \in \mathscr{P}^\ell_n} 1_\alpha \right). \tag{5.1}$$

THEOREM 5.9. *Let $\theta = (1, 2, \dots, \ell)$ and $\kappa \in I^\ell$ be h-admissible. We have that*

$$\mathsf{E}_\omega \mathbf{A}_{\mathscr{P}^\ell_n(h)}(n, \theta, \kappa)\mathsf{E}_\omega \cong \mathcal{Q}_{h,\ell,n}(\kappa) \tag{5.2}$$

and

$$\mathbf{A}_{\mathscr{P}^\ell_n(h)}(n, \theta, \kappa)\mathsf{E}_\omega \otimes_{\mathcal{Q}_{h,\ell,n}(\kappa)} \mathsf{E}_\omega \mathbf{A}_{\mathscr{P}^\ell_n(h)}(n, \theta, \kappa) \cong \mathbf{A}_{\mathscr{P}^\ell_n(h)}(n, \theta, \kappa) \tag{5.3}$$

$$\mathbf{A}_{\mathscr{P}^\ell_n(h)}(n, \theta, \kappa) \left(\sum_{\alpha \in \mathscr{P}^\ell_n} 1_\alpha \right) \otimes_{A_h(n,\theta,\kappa)} \left(\sum_{\alpha \in \mathscr{P}^\ell_n} 1_\alpha \right) \mathbf{A}_{\mathscr{P}^\ell_n(h)}(n, \theta, \kappa)$$
$$\cong \mathbf{A}_{\mathscr{P}^\ell_n(h)}(n, \theta, \kappa) \tag{5.4}$$

as graded R-algebras. Therefore all three algebras are graded Morita equivalent. The cellular structures on the subalgebras are obtained from that of $\mathbf{A}_{\mathscr{P}^\ell_n(h)}(n,\theta,\kappa)$ *by truncation as follows,*

$$\mathsf{E}_\omega C_{\mathsf{ST}} \mathsf{E}_\omega = \begin{cases} \psi^{\theta,\kappa}_{\mathsf{st}} & \textit{if } \varphi(\mathsf{s}) = \mathsf{S} \in \mathrm{SStd}(\lambda,\omega), \varphi(\mathsf{t}) = \mathsf{T} \in \mathrm{SStd}(\lambda,\omega), \\ 0 & \textit{otherwise}, \end{cases}$$

$$\left(\sum_{\lambda \in \mathscr{P}^\ell_n} 1_\alpha\right) C_{\mathsf{ST}} \left(\sum_{\alpha \in \mathscr{P}^\ell_n} 1_\alpha\right) = \begin{cases} C_{\mathsf{ST}} & \textit{if } \mathsf{S} \in \mathrm{SStd}^+(\lambda,\mu), \mathsf{T} \in \mathrm{SStd}^+(\lambda,\nu), \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The truncation of cellular bases follows as in [**Bow16**, Theorem 3.12]. For the isomorphisms in (5.2), see [**Bow16**, Theorem 3.12]. We now prove the isomorphism (5.3) (the isomorphism (5.4) can be proven in an identical fashion). That this map is injective is clear. To prove that the map is surjective, it is enough to check that

$$C^*_{\mathsf{t}^\lambda} C_{\mathsf{t}^\lambda} = C_{\mathsf{T}^\lambda} \tag{5.5}$$

for $\mathsf{t}^\lambda \in \mathrm{Std}(\lambda) = \mathrm{SStd}(\lambda,\omega)$ and $\mathsf{T}^\lambda \in \mathrm{SStd}(\lambda,\lambda)$. In [**Bow16**, Proof of Theorem 8.2] it is shown that $C_{\mathsf{t}^\lambda} C^*_{\mathsf{t}^\lambda}$ is an idempotent, and so $C_{\mathsf{t}^\lambda} C^*_{\mathsf{t}^\lambda} = (C_{\mathsf{t}^\lambda} C^*_{\mathsf{t}^\lambda})(C_{\mathsf{t}^\lambda} C^*_{\mathsf{t}^\lambda}) = C_{\mathsf{t}^\lambda}(C^*_{\mathsf{t}^\lambda} C_{\mathsf{t}^\lambda}) C^*_{\mathsf{t}^\lambda}$ and therefore $C^*_{\mathsf{t}^\lambda} C_{\mathsf{t}^\lambda} = C_{\mathsf{T}^\lambda}$, as required. □

COROLLARY 5.10. *The algebra $A_h(n,\theta,\kappa)$ is a graded cellular algebra with respect to the basis*

$$\{C_{\mathsf{ST}} \mid \mathsf{S} \in \mathrm{SStd}^+(\lambda,\mu), \mathsf{T} \in \mathrm{SStd}^+(\lambda,\nu), \lambda,\mu,\nu \in \mathscr{P}^\ell_n(h)\}$$

with respect to the θ-dominance order on $\mathscr{P}^\ell_n(h)$ and the involution $*$.

We refer to the resulting cell modules (as in (2.1)) as the *Weyl modules* of $A_h(n,\theta,\kappa)$, and denote them by

$$\Delta(\lambda) = \{C_{\mathsf{S}} \mid \mathsf{S} \in \mathrm{SStd}^+(\lambda,\mu) \text{ for some } \mu \in \mathscr{P}^\ell_n(h)\}.$$

REMARK 5.11. We note that the Weyl/simple modules of $A_h(n,\theta,\kappa)$ are indexed by multipartitions with at most h columns (rather than at most h rows) and that $A_h(n,\theta,\kappa)$ is constructed as a quotient (not a subalgebra) of $A(n,\theta,\kappa)$. Specializing to the case that $\ell = 1$ and $e = p$, we have that $A_h(n,\theta,\kappa)$ is the *Ringel dual* of the classical Schur algebra. There is a well-known duality underlying the combinatorics of the Schur algebra and its Ringel dual, given by identifying a partition with its transpose [**Don98**, Ch. 4].

REMARK 5.12. Note that (5.5) does not hold for $\lambda \notin \mathscr{P}^{\ell}_n(h)$. In particular, it does not hold if λ labels a simple module of $\mathbf{A}(n,\theta,\kappa)$ which is killed by the Schur functor.

For the remainder of the paper, we shall develop the combinatorics of $A_h(n,\theta,\kappa)$ and prove results concerning their representation theory. We hence deduce results (though the graded Morita equivalence) concerning the algebras $\mathcal{Q}_{h,\ell,n}(\kappa)$.

6. Alcove geometries and path bases for diagrammatic Cherednik algebras

Following our discussion in Section 5, for the remainder of the paper we set $\theta = (1,\dots,\ell) \in \mathbb{Z}^{\ell}$. In the main result of this section, we prove that $A_h(n,\theta,\kappa)$ has a basis indexed by pairs of paths in a certain alcove geometry. This involves first providing an inductive construction of semistandard tableaux and then embedding these tableaux into Euclidean space via the inductive construction.

6.1. An inductive construction of semistandard tableaux. The purpose of this section is to provide an inductive construction of semistandard tableaux (and hence the basis of $A_h(n,\theta,\kappa)$) under the assumption that $e > h\ell$. We also show that it is impossible to construct tableaux in an inductive fashion for $e < h\ell$ in general.

THEOREM 6.1. *Let $\lambda,\mu \in \mathscr{P}^{\ell}_n(h)$. We let X denote the least dominant element of* $\mathrm{Rem}(\mu)$. *Suppose that* $\mathrm{res}(X) = x \in I$. *We have a bijection*

$$\bigsqcup_{Y \in \mathrm{Rem}_x(\lambda)} \mathrm{SStd}^{+}_{n-1}(\lambda - Y, \mu - X) \to \mathrm{SStd}^{+}_{n}(\lambda,\mu)$$

given by $\mathsf{T} \to \mathsf{T}_Y$ *where*

$$\mathsf{T}_Y(r,c,m) = \begin{cases} X & \text{if } (r,c,m) = Y, \\ \mathsf{T}(r,c,m) & \text{otherwise.} \end{cases}$$

Proof. We have assumed that $X = (r',c',m')$ is the box minimal in the dominance ordering in $\mathrm{Rem}(\mu)$ and that $\mu \in \mathscr{P}^{\ell}_n$. Therefore

$$[\mathbf{I}_X \pm \ell, \infty) \cap \{\mathbf{I}_{(r,c,m)} \mid (r,c,m) \in \mu\} \subset [\mathbf{I}_X \pm \ell, \mathbf{I}_X + h\ell). \qquad (6.1)$$

By assumption, the boxes $\{(r'+i, c'+1+i, m') \mid i \in \mathbb{Z}\}$ and $\{(r'+1+i, c'+i, m') \mid i \in \mathbb{Z}\}$ consist of boxes of residue $x+1$ and $x-1$, respectively. We claim that these are the unique $(x \pm 1)$-diagonals in the region $[\mathbf{I}_X \pm \ell, \infty)$. For the $(x+1)$-diagonal, this follows immediately from Lemma 5.5 and (6.1). We now consider the $(x-1)$-diagonal. By Lemma 5.5 and (6.1), it is enough to show that there is no $(x-1)$-diagonal in the region $[\mathbf{I}_X + h\ell, \mathbf{I}_X + (h+1)\ell)$. Given $X \in \mu$, this region contains an $(x-1)$-diagonal if and only if $X = (r, h, m)$ for $1 \leqslant m < \ell$ and $(r, 1, m+1) \in \mu$, with $\mathrm{res}(r, h, m) = \mathrm{res}(r+1, 1, m+1)$. However, $(r+1, c, m+1) \in \mu$ is to the right of (r, h, m) for all $1 \leqslant c \leqslant h$. This contradicts our assumption on X and so the claim follows.

We first show surjectivity. Namely, we shall let $\mathsf{S} \in \mathrm{SStd}_n^+(\lambda, \mu)$ and we shall show that $\mathsf{S}^{-1}(X) = (r, c, m) \in \mathrm{Rem}(\lambda)$. Assume that $X = (r', c', m') \in \mu$ and that $\mathsf{S}^{-1}(X) = (r, c, m) \in \lambda$ is *not* a removable box. In which case either $(r, c+1, m)$ or $(r+1, c, m)$ is a box in the Young diagram of λ. If $(r, c+1, m)$ (respectively $(r+1, c, m)$) is a box in the Young diagram of λ, then $\mathsf{S}(r, c+1, m) > \mathsf{S}(r, c, m) - \ell$ (respectively $\mathsf{S}(r+1, c, m) > \mathsf{S}(r, c, m) + \ell$) by condition (iii) (respectively condition (ii)) of Definition 4.1. Therefore $\mathsf{S}(r, c+1, m) = (r'+i, c'+1+i, m')$ and $\mathsf{S}(r+1, c, m) = (r'+j, c'+1+j, m')$ for some $i, j \in \mathbb{Z}_{\geqslant 0}$ as these are the unique $(x \pm 1)$-diagonals in the region $[\mathbf{I}_X - \ell, \infty)$ by the above. However, this contradicts our assumption that $X \in \mathrm{Rem}(\mu)$. Therefore we conclude that the box $\mathsf{S}^{-1}(X)$ is indeed a removable box of the partition λ. As $\mathsf{S}^{-1}(X)$ is a removable box of λ (with residue equal to that of X) it is clear that the map is surjective.

It remains to show that the map is injective. Let $Y \in \mathrm{Rem}_x(\lambda)$ and $\mathsf{T} \in \mathrm{SStd}_{n-1}^+(\lambda - Y, \mu - X)$ where $X \in \mathrm{Rem}_x(\mu)$ is the minimal removable box in the dominance ordering. By definition, each tableau T_Y is distinct (and nonzero) and therefore it only remains to check that T_Y satisfies the conditions of being semistandard. We recall that the only $(x \pm 1)$-boxes in the region $[\mathbf{I}_X - \ell, \infty)$ are of the form $\{(r'+1+i, c'+i, m') \mid i \in \mathbb{Z}_{\geqslant 0}\}$ and $\{(r'+j, c'+1+j, m') \mid j \in \mathbb{Z}_{\geqslant 0}\}$, respectively. Therefore condition (ii) (respectively (iii)) of Definition 4.1 is empty because the intersection of these respective sets with μ is empty. Therefore the result follows. $\square$

EXAMPLE 6.2. We now provide a counterexample to the above theorem for $\ell = 1$ and $e \leqslant h$. We let $\lambda = (9, 6^3)$ and $\mu = (6^2, 5^3)$ and $e = 3$ (the multicharge and weighting can be chosen arbitrarily). We let T be the semistandard tableau determined by $\mathsf{T}(r, c, m) = \mathbf{I}_{(r,c,1)}$ for $(r, c, 1) \in \lambda \cap \mu$; and for $x \in \lambda \setminus \lambda \cap \mu$ we set

$$\mathsf{T}(1,7,1) = \mathbf{I}_{(5,2,1)}, \quad \mathsf{T}(1,8,1) = \mathbf{I}_{(5,3,1)}, \quad \mathsf{T}(1,9,1) = \mathbf{I}_{(5,4,1)},$$
$$\mathsf{T}(3,6,1) = \mathbf{I}_{(5,5,1)}, \quad \mathsf{T}(4,6,1) = \mathbf{I}_{(5,1,1)}.$$

We note that there are no removable nodes common to both λ and μ so there is no obvious way to construct the tableau inductively.

Using Theorem 6.1 we are now able to inductively define the component word of both a multipartition μ and a semistandard tableau of weight μ. Our ability to build the tableau inductively according to (the component word of) its weight multipartition will be the key ingredient in our embedding of tableaux into Euclidean space.

DEFINITION 6.3. Let $\mu \in \mathscr{P}_n^\ell(h)$. We define the *component word* of μ to be the series of multipartitions

$$\varnothing = \mu^{(0)} \xrightarrow{+X_1} \mu^{(1)} \xrightarrow{+X_2} \mu^{(2)} \xrightarrow{+X_3} \cdots \xrightarrow{+X_{n-1}} \mu^{(n-1)} \xrightarrow{+X_n} \mu^{(n)} = \mu$$

where $X_k = (r_k, c_k, m_k)$ is the least dominant removable node of the partition $\mu^{(k)} \in \mathscr{P}_k^\ell(h)$. Let $\lambda, \mu \in \mathscr{P}_n^\ell(h)$ and $\mathsf{T} \in \mathrm{SStd}_n^+(\lambda, \mu)$. We define the *component word* of T to be the series of semistandard tableaux

$$\mathsf{T}_{\leqslant 0}, \mathsf{T}_{\leqslant 1}, \ldots, \mathsf{T}_{\leqslant n}$$

such that $\mathsf{T}_{\leqslant k} \in \mathrm{SStd}_k^+(-, \mu^{(k)})$ for $0 \leqslant k < n$ is obtained from $\mathsf{T}_{\leqslant k+1} \in \mathrm{SStd}_{k+1}^+(-, \mu^{(k+1)})$ via the isomorphism of Theorem 6.1. For $0 \leqslant k \leqslant n$, we let $\lambda^{(k)} = \mathrm{Shape}(\mathsf{T}_{\leqslant k})$ and $Y_k \in \mathrm{Rem}_k(\lambda^{(k)})$ be such that $\lambda^{(k-1)} + \{Y_k\} = \lambda^{(k)}$. We shall also refer to the ordered sequence of nodes $(Y_1, Y_2, \ldots, Y_n)$ as the component word of T (as each clearly determines the other: $\mathsf{T}(Y_k) = \mathbf{I}_{X_k}$ for $1 \leqslant k \leqslant n$).

REMARK 6.4. Given $\mu \in \mathscr{P}_n^\ell$, we note that the component word of the multipartition μ is equal to the component word of the tableau T^μ.

EXAMPLE 6.5. Let $e = 6$, $h = 1$, $\ell = 3$, and $\kappa = (0, 2, 4) \in (\mathbb{Z}/7\mathbb{Z})^3$. For $\mu = ((1^5), \varnothing, \varnothing)$, we have that the component word of μ is given by the ordered sequence of nodes

$$((1, 1, 1), (2, 1, 1), (3, 1, 1), (4, 1, 1), (5, 1, 1))$$

for $\lambda = ((1^2), (1), (1^2))$ there is a unique element of $\mathrm{SStd}^+(\lambda, \mu)$ with component word

$$((1, 1, 1), (2, 1, 1), (1, 1, 3), (2, 1, 3), (1, 1, 2)).$$

DEFINITION 6.6. Given $\lambda, \mu \in \mathscr{P}_n^\ell(h)$, we define the *degree* of tableau $\mathsf{T} \in \mathrm{SStd}_n^+(\lambda, \mu)$ via the component word as follows. If

$$[\mathsf{T}_{\leqslant 0}] \xrightarrow{+Y_1} [\mathsf{T}_{\leqslant 1}] \xrightarrow{+Y_2} \cdots \xrightarrow{+Y_n} [\mathsf{T}_{\leqslant n}]$$

is the component word of the tableau T, then we set

$$\begin{aligned}\deg(Y_k) = &\left|\{\text{addable } r_k\text{-nodes of Shape}(\mathsf{T}_{\leqslant k-1}) \text{ to the right of } Y_k\}\right| \\ &- \left|\{\text{removable } r_k\text{-nodes of Shape}(\mathsf{T}_{\leqslant k-1}) \text{ to the right of } Y_k\}\right|\end{aligned}$$

where $r_k \in \mathbb{Z}/e\mathbb{Z}$ is the residue of the node Y_k. We set $\deg(\mathsf{T}) = \sum_{1\leqslant k\leqslant n} \deg(Y_k)$.

PROPOSITION 6.7. *Given* $\mathsf{T} \in \mathrm{SStd}_n^+(\lambda, \mu)$, *we have that* $\deg(\mathsf{T}) = \deg(C_{\mathsf{T}})$.

Proof. We can inductively construct the element C_{T} by drawing each strand, one at a time, according to the ordering of Definition 6.3 (see Section 7). In more detail: at the kth stage, we add a strand to the diagram connecting the southern point $(\mathbf{I}_{Y_k}, 0)$ to the northern point $(\mathbf{I}_{X_k}, 1)$ (making sure that we draw this strand so as to include the minimal number of crossings with strands from earlier in the process). We let A_k denote the strand connecting the southern point $\mathbf{I}_{Y_k}$ to the northern point $\mathbf{I}_{X_k}$. By construction, $\deg(C_{\mathsf{T}_{\leqslant k}}) - \deg(C_{\mathsf{T}_{\leqslant k-1}})$ is equal to the number of crossings of A_k with strands A_i for $1 \leqslant i < k$, each crossing counted with degree given by Proposition 4.5. We shall show that

$$\deg(C_{\mathsf{T}_{\leqslant k}}) - \deg(C_{\mathsf{T}_{\leqslant k-1}}) = \deg(Y_k)$$

and hence deduce the result. We shall set $Y_k = (p_k, q_k, t_k)$ and $X_k = (r_k, c_k, m_k)$ and $\mathrm{res}(A_k) = x_k \in \mathbb{Z}/e\mathbb{Z}$. Clearly, the only crossings in $C_{\mathsf{T}^{(n)}}$ which are not in $C_{\mathsf{T}^{(n-1)}}$ involve strands labelled by $1 \leqslant k < n$ such that either:

(i) $\mathbf{I}_{(p_k,q_k,t_k)} < \mathbf{I}_{(p_n,q_n,t_n)}$ and $\mathbf{I}_{(r_k,c_k,m_k)} > \mathbf{I}_{(r_n,c_n,m_n)}$; or

(ii) $\mathbf{I}_{(p_k,q_k,t_k)} > \mathbf{I}_{(p_n,q_n,t_n)}$ and $\mathbf{I}_{(r_k,c_k,m_k)} < \mathbf{I}_{(r_n,c_n,m_n)}$.

We are only interested in those crossings labelled by a box (p_k, q_k, t_k) of residue $x_n - 1$, x_n, or $x_n + 1 \in I$ by Proposition 4.5. We write

$$C_{\mathsf{T}_{\leqslant n}} = \overline{C}_{\mathsf{T}_{\leqslant n-1}} \times 1_{\lambda}^{\lambda - Y_n + X_n} \tag{6.2}$$

where we obtain $\overline{C}_{\mathsf{T}_{\leqslant n-1}}$ from $C_{\mathsf{T}_{\leqslant n-1}}$ by adding a vertical solid strand with x-coordinate $\mathbf{I}_{X_n}$, and we obtain $1_{\lambda}^{\lambda - Y_n + X_n}$ from $1_{\lambda - Y_n}$ by adding a solid strand from $(\mathbf{I}_{Y_n}, 0)$ to $(\mathbf{I}_{X_n}, 1)$ in such a way as to create no double-crossings.

By construction, any crossing as in (i) (or (ii)) occurs in the diagram $C_{\mathsf{T}_{\leqslant n-1}}$ (or $1_{\lambda}^{\lambda - Y_n + X_n}$) in the factorization of (6.2). Now, recall our assumption that $X_n = (r_n, c_n, m_n)$ is the rightmost removable node of μ. Arguing as in the proof of

Theorem 6.1, we deduce that

$$(\mathbf{I}_{(r_n-1,c_n,m_n)},\infty)\cap\{\mathbf{I}_{(r_k,c_k,m_k)}\mid x_k=x_n+1,1\leqslant k<n\}=\varnothing, \tag{6.3}$$
$$(\mathbf{I}_{(r_n,c_n,m_n)},\infty)\cap\{\mathbf{I}_{(r_k,c_k,m_k)}\mid x_k=x_n,1\leqslant k<n\}=\varnothing, \tag{6.4}$$
$$(\mathbf{I}_{(r_n,c_n-1,m_n)},\infty)\cap\{\mathbf{I}_{(r_k,c_k,m_k)}\mid x_k=x_n-1,1\leqslant k<n\}=\varnothing. \tag{6.5}$$

Therefore if A_k and A_n are crossing strands of adjacent (or equal) residue, then we are in case (ii) above. In particular,

$$\deg(\overline{C}_{\mathsf{T}_{\leqslant k-1}})=\deg(C_{\mathsf{T}_{\leqslant k-1}}).$$

Therefore we only need consider crossings of the strand A_n within the diagram $1_\lambda^{\lambda-Y_n+X_n}$. By (6.3), (6.4), and (6.5), the strands A_n and A_k cross if and only if the box Y_k occurs to the *right* of Y_n. It remains to check that the total degree contribution of these crossings is given by the total number of addable x_n-nodes minus the total number of removable x_n-nodes (to the right of Y_n) as claimed. In order to do this, we first require some notation. Let $1\leqslant m\leqslant \ell$ and let $(r,c,m)\in\lambda\cup\mathrm{Add}(\lambda)$ be a box of residue $x_n\in I$. We refer to the set of nodes

$$\mathbf{D}=\{(a,b,m)\in\lambda^{(k-1)}\mid a-b\in\{r-c-1,r-c,r-c+1\}\}$$

as the associated x_n-*diagonal*. If $a-b$ is greater than, less than, or equal to zero, we say that the x_n-diagonal is to the left of, right of, or centred on θ_m, respectively.

Clearly all boxes of λ of residue x_n-1, x_n, or x_n+1 belong to some x_n-diagonal. We say that the x_n-diagonal containing the node $(r,c,m)\in\lambda\cup\mathrm{Add}(\lambda)$ is to the right of the node Y_n if $\mathbf{I}_{(r,c,m)}>\mathbf{I}_{Y_n}$. We have already seen that each nonzero degree crossing of the A_n-strand (or its ghost) occurs with a strand belonging to an x_n-diagonal to the right of Y_n.

Let $\mathbf{D}$ be an x_n-diagonal in λ to the right of Y_k. Suppose that $\mathbf{D}$ has an addable x_n-node, which we denote by $(r,c,m)\in\lambda$. If (i) $r-c<0$, or (ii) $r-c>0$, (iii) $r-c=0$, then there are:

(i) a total of r distinct solid x_n-strands, $r+1$ distinct ghost (x_n-1)-strands, r distinct solid (x_n+1)-strands, and no red strands;

(ii) a total of c distinct solid x_n-strands, $c+1$ distinct solid (x_n+1)-strands, and c distinct ghost (x_n-1)-strands, and no red strands;

(iii) a total of r distinct solid x_n-strands, r distinct solid (x_n+1)-strands, and r distinct ghost (x_n-1)-strands, and 1 red strand (note that $r=c$),

within the region

$$([\mathbf{I}_{(r,c,m)}-2\ell,\mathbf{I}_{(r,c,m)}+2\ell]\times\{0\})\cap 1_\lambda.$$

The solid A_n-strand crosses all of these strands and the sum over these crossings has total degree equal to 1. The other cases (an x_n-diagonal with a removable x_n-node, or an x_n-diagonal with no addable or removable x_n-node) can be checked in a similar fashion (and have total degree contribution -1 or 0, respectively). $\square$

6.2. The geometry. In this section, we are going to consider a variant of the classical alcove geometries encountered in Lie theory. Fix integers $h, \ell \in \mathbb{Z}_{>0}$ and $e \in \mathbb{Z}_{>0} \cup \{\infty\}$. For each $1 \leqslant i \leqslant h$ and $0 \leqslant m < \ell$ we let ε_{hm+i} denote a formal symbol, and set

$$\mathbb{E}_{h,\ell} = \bigoplus_{\substack{1 \leqslant i \leqslant h \\ 0 \leqslant m < \ell}} \mathbb{R}\varepsilon_{hm+i}$$

to be the associated ℓh-dimensional real vector space. We have an inner product $\langle\, ,\rangle$ given by extending linearly the relations

$$\langle \varepsilon_{hm+i}, \varepsilon_{ht+j} \rangle = \delta_{i,j}\delta_{t,m}$$

for all $1 \leqslant i, j \leqslant h$ and $0 \leqslant m, t < \ell$, where $\delta_{i,j}$ is the Kronecker delta. We let Φ (respectively Φ_0) denote the root system of type $A_{\ell h-1}$ (respectively of type $A_{h-1} \times A_{h-1} \times \cdots A_{h-1}$) consisting of the roots

$$\{\varepsilon_{hm+i} - \varepsilon_{ht+j} \mid 1 \leqslant i, j \leqslant h \text{ and } 0 \leqslant m, t < \ell \text{ with } (i, m) \neq (j, t)\}$$

respectively

$$\{\varepsilon_{hm+i} - \varepsilon_{hm+j} \mid 1 \leqslant i, j \leqslant h \text{ with } i \neq j \text{ and } 0 \leqslant m < \ell\}.$$

Suppose that $e > h\ell$. We identify $\lambda \in \mathscr{P}_n^\ell(h)$ with a point in $\mathbb{E}_{h\ell}$ via the map

$$(\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \mapsto \sum_{\substack{1 \leqslant i \leqslant \ell \\ 1 \leqslant j \leqslant h}} (\lambda^{(m)})_i^T \varepsilon_{h(m-1)+i},$$

(where T denotes the transpose partition). For example, we have that $((2^2, 1^2), (2, 1)) \mapsto 4\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 + \varepsilon_4$. (For Lie theorists who find the appearance of the transpose of the partition peculiar, we refer to Remark 5.11.) For $e \in \mathbb{Z}_{>0}$ (respectively $e = \infty$) we assume that $\kappa \in I^\ell$ is h-admissible (respectively 1-admissible). Given $r \in \mathbb{Z}$ and $\alpha \in \Phi$ we let $s_{\alpha,re}$ denote the reflection which acts on $\mathbb{E}_{h,\ell}$ by

$$s_{\alpha,re}x = x - (\langle x, \alpha\rangle - re)\alpha.$$

Given $e \neq \infty$ we let W^e be the affine reflection group generated by the reflections

$$\mathcal{S} = \{s_{\alpha,re} \mid \alpha \in \Phi, r \in \mathbb{Z}\}$$

and let W_0^e denote the parabolic subgroup generated by

$$\mathcal{S}_0 = \{s_{\alpha,0} \mid \alpha \in \Phi_0\}.$$

If $e = \infty$ then we let W^e be the finite reflection group generated by the reflections

$$\mathcal{S} = \{s_{\alpha,0} \mid \alpha \in \Phi\}$$

and let W_0^e denote the parabolic subgroup generated by

$$\mathcal{S}_0 = \{s_{\alpha,0} \mid \alpha \in \Phi_0\}.$$

We shall consider a shifted action of these groups on $\mathbb{E}_{hl}^{\circledcirc}$ by the element $\rho = (\rho_1, \rho_2, \dots, \rho_\ell) \in I^h$ where

$$\rho_i = (e - \kappa_i, e - \kappa_i - 1, \dots, e - \kappa_i - h + 1) \in I^{h\ell}.$$

Given an element $w \in W^e$, we define the 'dot' action of w on $\mathbb{E}_{h,\ell}$ by

$$w \cdot_\rho x = w(x + \rho) - \rho.$$

We let $\mathbb{E}(\alpha, re)$ denote the affine hyperplane consisting of the points

$$\mathbb{E}(\alpha, re) = \{x \in \mathbb{E}_{h,\ell} \mid s_{\alpha,r} \cdot x = x\}.$$

We say that a point $\lambda \in \mathbb{E}_{h,\ell}$ is e-regular if it does not lie on any hyperplane. Our assumption that $e > h\ell$ implies that e-regular lattice points do exist. In particular, we let $\circledcirc$ denote the origin $(0, \dots, 0) \in \mathbb{E}_{h,\ell}$; note that our choice of ρ ensures that $\circledcirc$ is e-regular.

Given a hyperplane $\mathbb{E}(\alpha, re)$ we remove the hyperplane from $\mathbb{E}_{h,\ell}$ to obtain two distinct subsets $\mathbb{E}^{>}(\alpha, re)$ and $\mathbb{E}^{<}(\alpha, re)$ where $\circledcirc \in \mathbb{E}^{<}(\alpha, re)$. We define

$$\mathbb{E}^{\geqslant}(\alpha, re) = \mathbb{E}(\alpha, re) \cup \mathbb{E}^{>}(\alpha, re)$$

and similarly $\mathbb{E}^{\leqslant}(\alpha, re) = \mathbb{E}(\alpha, re) \cup \mathbb{E}^{<}(\alpha, re)$. The dominant Weyl chamber, denoted by $\mathbb{E}_{h,\ell}^{\circledcirc}$, is set to be

$$\mathbb{E}_{h,\ell}^{\circledcirc} = \bigcap_{\alpha \in \Phi_0} \mathbb{E}^{<}(\alpha, 0).$$

EXAMPLE 6.8. For $\ell = 1$ we obtain the parabolic affine geometry which controls the representation theory of the (quantum) general linear group of $(h \times h)$-matrices in (quantum) characteristic e.

EXAMPLE 6.9. Setting $h = 1$ we have $W_0^e = 1$ and we obtain the (nonparabolic) affine geometry which controls the representation theory of the Kac–Moody algebras of type $\widehat{A}_{\ell-1}$ and the quiver Temperley–Lieb algebras in characteristic e.

DEFINITION 6.10. Let $\lambda \in \mathbb{E}_{h,\ell}$. There are only finitely many hyperplanes lying *strictly* between the point $\lambda \in \mathbb{E}_{h,\ell}^{\circledcirc}$ and the origin $\circledcirc \in \mathbb{E}_{h,\ell}^{\circledcirc}$. For $\alpha \in \Phi$, we let $\ell_\alpha(\lambda)$ denote the total number of these hyperplanes which are perpendicular to $\alpha \in \Phi$. We let $\ell(\lambda) = \sum_{\alpha\in\Phi} \ell_\alpha(\lambda)$.

REMARK 6.11. Note that we do not count any hyperplane upon which λ actually lies.

6.3. Paths in the geometry for $e \neq \infty$.

Let $e \in \mathbb{Z}_{>0}$. We now introduce paths in our Euclidean space $\mathbb{E}_{h,\ell}$; the reader may find it helpful to consider the examples in Section 12. We define a degree function on such paths in terms of the hyperplanes in our geometry. We show how to identify these paths with semi-standard tableaux and hence provide a graded path theoretic basis of $A_h(n, \theta, \kappa)$.

DEFINITION 6.12. Given a map $s : \{1, \ldots, n\} \to \{1, \ldots, \ell h\}$ we define points $\mathsf{s}(k) \in \mathbb{E}_{h,\ell}$ by

$$\mathsf{s}(k) = \sum_{1\leqslant i\leqslant k} \varepsilon_{s(i)}$$

for $1 \leqslant i \leqslant n$. We define the associated path of length n in our alcove geometry $\mathbb{E}_{h,\ell}$ by

$$\mathsf{s} = (\mathsf{s}(0), \mathsf{s}(1), \mathsf{s}(2), \ldots, \mathsf{s}(n)),$$

where we fix all paths to begin at the origin, so that $\mathsf{s}(0) = \circledcirc \in \mathbb{E}_{h,\ell}$. We let $\mathsf{s}_{\leqslant k}$ denote the subpath of s of length k corresponding to the restriction of the map s to the domain $\{1, \ldots, k\} \subseteq \{1, \ldots, n\}$.

DEFINITION 6.13. Given a path $\mathsf{s} = (\mathsf{s}(0), \mathsf{s}(1), \mathsf{s}(2), \ldots, \mathsf{s}(n))$ we set $\deg(\mathsf{s}(0)) = 0$ and define

$$\deg(\mathsf{s}) = \sum_{1\leqslant k\leqslant n} d(\mathsf{s}(k), \mathsf{s}(k-1)),$$

where $d(\mathsf{s}(k), \mathsf{s}(k-1))$ is defined as follows. For $\alpha \in \Phi$ we set $d_\alpha(\mathsf{s}(k), \mathsf{s}(k-1))$ to be

- $+1$ if $\mathsf{s}(k-1) \in \mathbb{E}(\alpha, re)$ and $\mathsf{s}(k) \in \mathbb{E}^{<}(\alpha, re)$;
- -1 if $\mathsf{s}(k-1) \in \mathbb{E}^{>}(\alpha, re)$ and $\mathsf{s}(k) \in \mathbb{E}(\alpha, re)$;
- 0 otherwise.

We let

$$d(\mathsf{s}(k-1), \mathsf{s}(k)) = \sum_{\alpha \in \Phi} d_\alpha(\mathsf{s}(k-1), \mathsf{s}(k)).$$

REMARK 6.14. Let $\delta = ((1^\ell), (1^\ell), \ldots, (1^\ell)) \in \mathscr{P}_n^\ell(h)$. Importantly, there exists a degree zero path from the origin to δ if and only if $e > h\ell$. In Remark 5.7, we chose $\kappa \in I^\ell$ (by applying Proposition 5.6) so that the path $(+\varepsilon_1, +\varepsilon_2, \ldots, +\varepsilon_{h\ell})$ from the origin to δ is of degree zero. We note that every point in this path is e-regular.

DEFINITION 6.15. Let $\mu \in \mathscr{P}_n^\ell(h)$ with component word of μ equal to

$$\varnothing = \mu^{(0)} \xrightarrow{+X_1} \mu^{(1)} \xrightarrow{+X_2} \mu^{(2)} \xrightarrow{+X_3} \cdots \xrightarrow{+X_{n-1}} \mu^{(n-1)} \xrightarrow{+X_n} \mu^{(n)} = \mu.$$

We fix a distinguished path t^μ from the origin to μ given by

$$\mathsf{t}^\mu = (+\varepsilon_{X_1}, +\varepsilon_{X_2}, \ldots, +\varepsilon_{X_n}).$$

Here we have abused notation slightly by identifying the addable box $X_k = (r_k, c_k, m_k)$ with the corresponding $\varepsilon_{X_k} = \varepsilon_{h(m_k-1)+c_k}$.

Let s be a path which passes through a hyperplane $\mathbb{E}_{\alpha,re}$ at point $\mathsf{s}(k)$ (note that k is not necessarily unique). Then, let t be the path obtained from s by applying the reflection $s_{\alpha,re}$ to all the steps in s after the point $\mathsf{s}(k)$. In other words, $\mathsf{t}(i) = \mathsf{s}(i)$ for all $1 \leqslant i \leqslant k$ and $\mathsf{t}(i) = s_{\alpha,re} \cdot \mathsf{s}(i)$ for $k \leqslant i \leqslant n$. We refer to the path t as the reflection of s in $\mathbb{E}_{\alpha,re}$ at point $\mathsf{s}(k)$ and denote this by $s^k_{\alpha,re} \cdot \mathsf{s}$. We write $\mathsf{s} \sim \mathsf{t}$ if the path t can be obtained from s by a series of such reflections.

DEFINITION 6.16. We let $\mathrm{Path}_n(\lambda, \mathsf{t}^\mu)$ denote the set of all paths from the origin to λ which may be obtained by applying repeated reflections to t^μ, in other words

$$\mathrm{Path}_n(\lambda, \mathsf{t}^\mu) = \{\mathsf{s} \mid \mathsf{s}(n) = \lambda, \mathsf{s} \sim \mathsf{t}^\mu\}.$$

We let $\mathrm{Path}^+_n(\lambda, \mathsf{t}^\mu) \subseteq \mathrm{Path}_n(\lambda, \mathsf{t}^\mu)$ denote the set of paths which at no point leave the dominant Weyl chamber, in other words

$$\mathrm{Path}^+_n(\lambda, \mathsf{t}^\mu) = \{\mathsf{s} \in \mathrm{Path}_n(\lambda, \mathsf{t}^\mu) \mid \mathsf{s}(k) \in \mathbb{E}^{\circledcirc}_{h,\ell} \text{ for all } 1 \leqslant k \leqslant n\}.$$

DEFINITION 6.17. Let $\mathsf{T} \in \mathrm{SStd}^+_n(\lambda, \mu)$ be a tableau with component reading word

$$[\mathsf{T}_{\leqslant 0}] \xrightarrow{+Y_1} [\mathsf{T}_{\leqslant 1}] \xrightarrow{+Y_2} \cdots \xrightarrow{+Y_n} [\mathsf{T}_{\leqslant n}].$$

We define a map $\omega : \mathrm{SStd}^+_n(\lambda, \mu) \to \mathrm{Path}^+_n(\lambda, \mathsf{t}^\mu)$ where $\omega(\mathsf{T}) = \mathsf{t}$ is the path in the alcove geometry given by

$$\mathsf{t} = (+\varepsilon_{Y_1}, +\varepsilon_{Y_2}, \dots, +\varepsilon_{Y_n}).$$

REMARK 6.18. Given the unique $\mathsf{T}^\mu \in \mathrm{SStd}^+(\mu, \mu)$, it is clear that $\omega(\mathsf{T}^\mu) = \mathsf{t}^\mu$.

LEMMA 6.19. *Given $\lambda \in \mathscr{P}^\ell_n(h)$, we have that*

$$\langle \lambda + \rho, \varepsilon_{hm+i} - \varepsilon_{hl+j} \rangle = re$$

for some $r \in \mathbb{Z}$, if and only if the nodes $((\lambda^{(m+1)})^T_i, i, m), ((\lambda^{(l+1)})^T_j, j, l) \in \lambda$ have the same residue.

Proof. To see this, note that both statements are equivalent to

$$(\lambda^{(m+1)})^T_i + e - \kappa_i \equiv (\lambda^{(l+1)})^T_j + e - \kappa_j \pmod e. \qquad \square$$

THEOREM 6.20. *Let $e \in \mathbb{Z}_{>0} \cup \{\infty\}$ and $\kappa \in I^\ell$ be h-admissible. For λ, $\mu \in \mathscr{P}^\ell_n(h)$ the map $\omega : \mathrm{SStd}^+_n(\lambda, \mu) \to \mathrm{Path}^+_n(\lambda, \mathsf{t}^\mu)$ is bijective and degree preserving.*

Proof. The result clearly holds for $n = 1$ and so we proceed by induction. We now let $\mu \in \mathscr{P}^\ell_{n+1}(h)$. Let X denote the least dominant element of $\mathrm{Rem}(\mu)$ and suppose this box has residue $x \in I$. By induction and Theorem 6.1, we may assume that

$$\mathrm{SStd}^+_{n+1}(\lambda, \mu) \leftrightarrow \bigsqcup_{Y \in \mathrm{Rem}_x(\lambda)} \mathrm{SStd}^+_n(\lambda - Y, \mu - X) \leftrightarrow \bigsqcup_{Y \in \mathrm{Rem}_x(\lambda)} \mathrm{Path}^+_n(\lambda - Y, \mu - X) \tag{6.6}$$

and that these bijections are degree preserving. Given any $\mathsf{T} \in \mathrm{SStd}^+_{n+1}(\lambda, \mu)$ we can let $Y \in \mathrm{Rem}(\lambda)$ denote the box containing the entry $n + 1$. By induction, the pair

$$\mathsf{T}_{\leqslant n} \in \mathrm{SStd}^+_n(\lambda - Y, \mu - X) \quad \text{and} \quad \omega(\mathsf{T}_{\leqslant n}) \in \mathrm{Path}^+_n(\lambda - Y, \mu - X)$$

is identified under the map 6.6 and the degrees coincide. Moreover given any $\mathsf{T} \in \mathrm{SStd}^+_{n+1}(\lambda, \mu)$, we can write

$$\omega(\mathsf{T}_{\leqslant n}) = s^{(k_t)}_{\varepsilon_{i_t} - \varepsilon_{j_t}, m_t e} \cdots s^{(k_1)}_{\varepsilon_{i_1} - \varepsilon_{j_1}, m_1 e} \cdot \omega(\mathsf{T}^{\mu - X})$$

for some $k_1 \leqslant k_2 \leqslant \cdots \leqslant k_t$. Now, given $\mathsf{t} \in \mathrm{Path}_n^+(\lambda, \mathsf{t}^\mu)$ such that $\mathsf{t}(n) \in \mathbb{E}_{i_n - j_n, m_n e}$ and $\mathsf{t}(n+1) \notin \mathbb{E}_{i_n - j_n, m_n e}$ we have that

$$\begin{aligned}
\mathsf{t} &= s^{(n)}_{\varepsilon_{i_n} - \varepsilon_{j_n}, m_n e}(s^{(k_t)}_{\varepsilon_{i_t} - \varepsilon_{j_t}, m_t e} \ldots s^{(k_1)}_{\varepsilon_{i_1} - \varepsilon_{j_1}, m_1 e}) \cdot \mathsf{t}^\mu \\
&= s^{(n)}_{\varepsilon_{i_n} - \varepsilon_{j_n}, m_n e}(s^{(k_t)}_{\varepsilon_{i_t} - \varepsilon_{j_t}, m_t e} \ldots s^{(k_1)}_{\varepsilon_{i_1} - \varepsilon_{j_1}, m_1 e}) \cdot (\mathsf{t}^\mu \downarrow_{\leqslant n} \circ (+\varepsilon_X)) \\
&= s^{(n)}_{\varepsilon_{i_n} - \varepsilon_{j_n}, m_n e}(s^{(k_t)}_{\varepsilon_{i_t} - \varepsilon_{j_t}, m_t e} \ldots s^{(k_1)}_{\varepsilon_{i_1} - \varepsilon_{j_1}, m_1 e}) \cdot \omega(\mathsf{T}^\mu_{\leqslant n} + \{X\}) \\
&= s^{(n)}_{\varepsilon_{i_n} - \varepsilon_{j_n}, m_{p,q} e} \cdot \omega(\mathsf{T}_{\leqslant n} + \{\sigma(X)\}) \\
&= \omega(\mathsf{T}_{\leqslant n} + \{\sigma'(X)\})
\end{aligned}$$

where $\sigma = s_{\varepsilon_{i_t} - \varepsilon_{j_t}} \ldots s_{\varepsilon_{i_1} - \varepsilon_{j_1}} \in \mathfrak{S}_{h\ell}$, $\sigma' = s_{\varepsilon_{i_n} - \varepsilon_{j_n}} s_{\varepsilon_{i_t} - \varepsilon_{j_t}} \ldots s_{\varepsilon_{i_1} - \varepsilon_{j_1}} \in \mathfrak{S}_{h\ell}$ and $\sigma'(X) = Y \in \mathrm{Rem}_x(\lambda)$. This gives us the required bijection

$$\mathrm{Path}_n^+(\lambda, \mu) \leftrightarrow \bigsqcup_{Y \in \mathrm{Rem}_x(\lambda)} \mathrm{Path}_n^+(\lambda - Y, \mu - X). \tag{6.7}$$

It remains to verify that the bijection is degree preserving. Let $A_{(r,c,m)} \rhd A_{(r',c',m')}$ be two addable nodes of some $\lambda \in \mathscr{P}_n^\ell$ and suppose that $r - c \neq r' - c'$. This implies that the c'th column of the m'th component of λ is *strictly* greater than the cth column of the mth component of λ. Therefore

$$\begin{aligned}
\langle \lambda + A_{(r',c',m')}, \varepsilon_{c'} - \varepsilon_c \rangle &= \langle \lambda, \varepsilon_{c'} - \varepsilon_c \rangle + 1 \\
\langle \lambda + A_{(r,c,m)}, \varepsilon_{c'} - \varepsilon_c \rangle &= \langle \lambda, \varepsilon_{c'} - \varepsilon_c \rangle - 1
\end{aligned} \tag{6.8}$$

are both strictly positive. For any $Y \in \mathrm{Rem}_x(\lambda)$, we let

$$\begin{aligned}
\mathrm{Add}_x(\lambda - Y) &= \{A_{(r_1,c_1,m_1)} \rhd A_{(r_2,c_2,m_2)} \rhd \cdots \rhd A_{(r_a,c_a,m_a)}\} = \mathrm{Add}_x(\lambda - Y) \setminus \{Y\} \\
\mathrm{Rem}_x(\lambda) &= \{R_{(p_1,q_1,t_1)} \rhd R_{(p_2,q_2,t_2)} \rhd \cdots \rhd R_{(p_b,q_b,t_b)} = \mathrm{Rem}_x(\lambda - Y) \setminus \{Y\}.
\end{aligned}$$

We note that $\mathrm{Rem}_z(\lambda - Y) = \mathrm{Rem}_z(\lambda)$ and $\mathrm{Add}_z(\lambda - Y) = \mathrm{Rem}_z(\lambda)$ for all $x \neq z \in I$. We further note that $r_i - c_i \neq r_j - c_j$ (respectively $p_i - q_i \neq p_j - q_j$) for $1 \leqslant i < j \leqslant a$ (respectively $1 \leqslant i < j \leqslant b$) by Lemma 5.5. We set $Y = A_{(r_y,c_y,m_y)} = R_{(p_{y'},q_{y'},t_{y'})}$ for some $1 \leqslant y \leqslant a$ and $1 \leqslant y' \leqslant b$.

If $\lambda \in \mathbb{E}_{\alpha,me}$ and $\lambda + \varepsilon_Y \notin \mathbb{E}_{\alpha,me}$ for some $m \in \mathbb{Z}_{>0}$, then $\alpha = \varepsilon_{c_i} - \varepsilon_{c_j}$ for some $1 \leqslant i < j \leqslant a$. Similarly, if $\lambda \notin \mathbb{E}_{\alpha,me}$ and $\lambda + \varepsilon_Y \in \mathbb{E}_{\alpha,me}$ for some $m \in \mathbb{Z}_{>0}$, then $\alpha = \varepsilon_{q_i} - \varepsilon_{q_j}$ for some $1 \leqslant i < j \leqslant b$. By (6.8), we have that

$$d_{\varepsilon_{c_i} - \varepsilon_{c_j}}(\lambda, \lambda + \varepsilon_Y) = \begin{cases} 1 & \text{if } i = y < j, \\ 0 & \text{otherwise,} \end{cases}$$

$$d_{\varepsilon_{q_i} - \varepsilon_{q_j}}(\lambda, \lambda + \varepsilon_Y) = \begin{cases} -1 & \text{if } i = y' < j, \\ 0 & \text{otherwise,} \end{cases}$$

and summing over all these terms we obtain

$$d(\lambda, \lambda + \varepsilon_Y) = (a - y) - (b - y')$$

which is equal to the number of addable x-boxes to the right of Y minus the number of removable x-boxes to the right of Y, as required. □

THEOREM 6.21. *The R-algebra $A_h(n, \theta, \kappa)$ is a graded cellular algebra with basis*

$$\{C_{\mathsf{st}} \mid \mathsf{s} \in \mathrm{Path}_n^+(\lambda, \mathsf{t}^\mu), \mathsf{t} \in \mathrm{Path}_n^+(\lambda, \mathsf{t}^\nu), \lambda, \mu, \nu \in \mathscr{P}_n^\ell(h)\}$$

with respect to the θ-dominance order on $\mathscr{P}_n^\ell(h)$ and the involution $*$. Here $C_{\mathsf{st}} := C_{\mathsf{ST}}$ for $\omega(\mathsf{S}) = \mathsf{s}$ and $\omega(\mathsf{T}) = \mathsf{t}$.

6.4. The algebras $A(n, \theta, \kappa)$ for $e = \infty$ or $e > n$. We now take a short detour to consider the algebras $A(n, \theta, \kappa)$ with $e = \infty$ (or more generally $e > n$) and $\kappa \in I^\ell$ is 1-admissible (that is, a multicharge with no repeated entries). We can take the condition on the e-multicharge to be less restrictive because of our assumption that $e > n$. With minor technical modifications to the combinatorics, this can be treated in exactly the same way as the algebras $A_h(n, \theta, \kappa)$ for $e > h\ell$ and $\kappa \in I^\ell$ h-admissible. Let t^μ be the path from $\circledcirc$ to μ, defined as above. We have two problems to address:

(i) the path t^μ is not necessarily of degree zero, however

$$\deg(\mathsf{T}^\mu) = \deg(C_{\mathsf{T}^\mu}) = 0;$$

(ii) the map $\varphi : \mathrm{SStd}_n^+(\lambda, \mu) \to \mathrm{Path}_n^+(\lambda, \mu)$ is not necessarily surjective and $\deg(\varphi(\mathsf{S})) = \deg(\mathsf{S}) - \deg(\mathsf{t}^\mu)$ for $\mathsf{S} \in \mathrm{SStd}_n^+(\lambda, \mu)$.

We take care of these problems as follows. Let

$$\Sigma(\mu) = \{k \mid 1 \leqslant k \leqslant n, d_\alpha(\mathsf{t}^\mu(k-1), \mathsf{t}^\mu(k)) \neq 0 \text{ for some } \alpha \in \Phi\}.$$

We let $\mathrm{Path}_n^\infty(\lambda, \mu) \subseteq \mathrm{Path}_n^+(\lambda, \mu)$ denote the subset of paths s such that

$$\mathsf{s} \sim \mathsf{s}^{(1)} \sim \mathsf{s}^{(2)} \sim \cdots \sim \mathsf{s}^{(t)} = \mathsf{t}^\mu$$

where $\mathsf{s}^{(i)} = s_{\alpha,re}^{k_i} \cdot \mathsf{s}^{(i-1)}$ for some $k_i \notin \Sigma(\mu)$. We then define the degree as follows

$$\deg(\mathsf{s}) = \sum_{\substack{1 \leqslant k \leqslant n \\ k \notin \Sigma(\mu)}} d(\mathsf{s}(k), \mathsf{s}(k-1)),$$

for $\mathsf{s} \in \mathrm{Path}_n^\infty(\lambda, \mu)$. In the remainder of the paper, we shall only deal explicitly with the algebras $A_h(n, \theta, \kappa)$ with $e > h\ell$. However all the results can be easily generalized to $A(n, \theta, \kappa)$ for $e > n$ and $\kappa \in I^\ell$ 1-admissible.

THEOREM 6.22. *Let $e > n$ and $\kappa \in I^\ell$ be* 1*-admissible. The R-algebra $A(n, \theta, \kappa)$ is a graded cellular algebra with basis*

$$\{C_{\mathsf{st}} \mid \mathsf{s} \in \mathrm{Path}_n^\infty(\lambda, \mathsf{t}^\mu), \mathsf{t} \in \mathrm{Path}_n^\infty(\lambda, \mathsf{t}^\nu), \lambda, \mu, \nu \in \mathscr{P}_n^\ell(\infty)\}$$

with respect to the θ-dominance order on $\mathscr{P}_n^\ell$ and the involution $*$. Here $C_{\mathsf{st}} := C_{\mathsf{ST}}$ for $\omega(\mathsf{S}) = \mathsf{s}$ and $\omega(\mathsf{T}) = \mathsf{t}$.

REMARK 6.23. We remark that the case $e = \infty$ (or more generally $e > n$) is expected to be far simpler than the case $e \leqslant n$. Indeed, this prompted an optimistic conjecture of Kleshchev–Ram [**KR11**, Conjecture 7.3] (later proven false in [**Wil14**, Section 4.2]). From our point of view, this simplification is a consequence of the fact that the affine Weyl group 'controlling' the alcove geometry for $e = \infty$ is finite (whereas it is infinite for $e < n$). This is reflected in the fact that Theorem 6.22 deals with the entire diagrammatic Cherednik algebra, rather than the quotient considered in Theorem 6.21.

The KLR algebras $H_n(\kappa)$ for $e = \infty$ and $\ell = 2$ have received a great deal of attention from Brundan–Stroppel (see for example [**BS10**, **BS11**]) and some of their results were extended to the case $e > n$ by Mathas–Hu [**HM15**, Appendix B].

REMARK 6.24. Let $e > n$. Up to a trivial re-ordering of the weighting, any two diagrammatic Cherednik algebras for distinct weightings (with $\kappa \in I^\ell$ fixed) are isomorphic as graded R-algebras. (This is certainly not true for $e < n$ and can be seen as another way in which the overall picture simplifies for $e = \infty$.) Therefore, the graded decomposition matrix of $H_n(\kappa)$ does not depend on our choice of weighting by [**Bow16**, Corollary 5.3]. Therefore we can speak of calculating the graded decomposition matrix of $H_n(\kappa)$ (as we shall in Conjecture 11.6) without reference to our chosen weighting.

7. Inductively constructing basis elements from the path

We now pause in order to highlight how one can inductively construct a basis element C_{S} of the algebra directly from the corresponding path $\omega(\mathsf{S}) = \mathsf{s} \in \mathrm{Path}^+(\lambda, \mathsf{t}^\mu)$. Let

$$\mathsf{t}^\mu = (+\varepsilon_{X_1}, +\varepsilon_{X_2}, \ldots, +\varepsilon_{X_n})$$

and let

$$\mathsf{s} = (+\varepsilon_{Y_1}, +\varepsilon_{Y_2}, \ldots, +\varepsilon_{Y_n}).$$

Given $1 \leqslant k \leqslant n$, we obtain $C_{\mathsf{T}_{\leqslant k}}$ from $C_{\mathsf{T}_{\leqslant k-1}}$ by adding a strand connecting the northern point at the end of the i_kth column to the southern point at the end of the j_kth column.

EXAMPLE 7.1. We continue with Example 6.5. Let $e = 7$, $h = 1$, $\ell = 3$, and $\kappa = (0, 2, 4) \in (\mathbb{Z}/7\mathbb{Z})^3$. For $\mu = ((1^5), \varnothing, \varnothing)$, we have that

$$\mathsf{t}^{\mu} = (+\varepsilon_1, +\varepsilon_1, +\varepsilon_1, +\varepsilon_1, +\varepsilon_1)$$

for $\lambda = ((1^2), (1), (1^2))$ we consider the path

$$s^{(4)}_{\varepsilon_2-\varepsilon_3,-e} s^{(2)}_{\varepsilon_1-\varepsilon_3,e} \cdot \mathsf{t}^{\mu} = \mathsf{s} = (+\varepsilon_1, +\varepsilon_1, +\varepsilon_3, +\varepsilon_3, +\varepsilon_2) \in \mathrm{Path}^{+}_{5}(\lambda, \mathsf{t}^{\mu}).$$

For each of the 5 steps in s, the corresponding basis elements (corresponding to $\mathsf{s}{\downarrow}_{\leqslant k}$ and $1 \leqslant k \leqslant 5$) are depicted in Figure 3 below. For each $1 \leqslant k \leqslant 5$, the northern (respectively southern) residue sequence is given by $\mathrm{Shape}(\mathsf{t}^{\mu}{\downarrow}_{\leqslant k})$ (respectively $\mathrm{Shape}(\mathsf{s}{\downarrow}_{\leqslant k})$). For example, if $k = 3$ then $\mathsf{s}{\downarrow}_{\leqslant 3} \in \mathrm{Path}^{+}(((1^2), \varnothing, (1)), ((1^3), \varnothing, \varnothing))$ has northern loading given by $((1^3), \varnothing, \varnothing)$ and southern loading given by $((1^2), \varnothing, (1))$.

8. Tensoring with the determinant

We now identify our higher-level analogue of the stability obtained by 'tensoring with the determinant' for general linear groups. Notice that we are working in the Ringel dual setting and so 'tensoring with the determinant' means 'adding a row' as opposed to 'adding a column'. This has an obvious higher-level generalization, as we shall now see. For the remainder of the paper we shall have to actually multiply diagrams together in order to prove various isomorphisms. We therefore fix some notation regarding the manipulation of diagrams using the relations of Definition 4.4.

REMARK 8.1. We shall refer to the relations (3.1), (3.2), (3.5), (3.8), (3.13) and (3.14) and the latter relation in both (3.4) and (3.11) as *noninteracting relations*. These are the relations given by pulling strands through one another in the naïve fashion (without acquiring error terms or dots or sending the diagram to zero). We refer to a *critical point* as any local neighbourhood in the diagram with nonzero degree. When manipulating diagrams, we focus on the 'critical points' at which we cannot use the noninteracting relations (as this is where things get tricky). Upon reaching a critical point (by manipulating the diagrams as much as possible using the noninteracting relations) we resolve this critical point (if necessary) in order to obtain a linear combination of diagrams.

THEOREM 8.2. *Given a partition* $\lambda = (\lambda_1, \lambda_2, \ldots)$, *let* $\det_h(\lambda) = (h, \lambda_1, \lambda_2, \ldots)$. *We have an injective map of partially ordered sets* $\det_h : \mathscr{P}^{\ell}_{n}(h) \hookrightarrow \mathscr{P}^{\ell}_{n+h\ell}(h)$

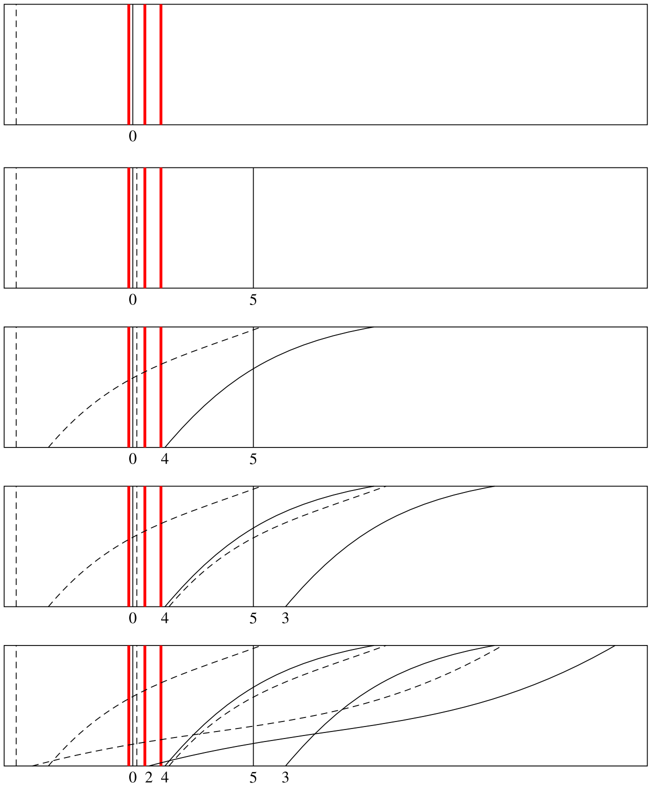


Figure 3. The elements $C_{\mathsf{s}\downarrow_{\leqslant k}}$ for $k = 1, \ldots, 5$.

given by

$$\det_h(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(\ell)}) = (\det_h(\lambda^{(1)}), \det_h(\lambda^{(2)}), \ldots, \det_h(\lambda^{(\ell)})).$$

The image, $\det_h(\mathscr{P}^\ell_n(h))$*, is a closed subset of* $\mathscr{P}^\ell_{n+h\ell}(h)$ *under the* θ*-dominance ordering and we have a degree-preserving bijective map*

$$\det_h : \mathrm{Path}_n(\lambda, \mathsf{t}^\mu) \to \mathrm{Path}_{n+h\ell}(\det_h(\lambda), \mathsf{t}^{\det_h(\mu)})$$

given by

$$\det_h(\mathsf{s}) = (+\varepsilon_1, +\varepsilon_2, \ldots, +\varepsilon_{h\ell}) \circ \mathsf{s} \in \mathrm{Path}_{n+h\ell}(\det_h(\lambda), \mathsf{t}^{\det_h(\mu)})$$

for $\mathsf{s} \in \mathrm{Path}_n(\lambda, \mathsf{t}^\mu)$. *We have an isomorphism of graded* R*-algebras*

$$A_{\det_h(\mathscr{P}^\ell_{n+h\ell}(h))}(n + h\ell, \theta, \kappa) \cong A_{\mathscr{P}^\ell_n(h)}(n, \theta, \kappa).$$

In particular, over an arbitrary field $\Bbbk$ *we have that*

$$d_{\lambda,\mu}(t) = d_{\det_h(\lambda),\det_h(\mu)}(t)$$

for all $\lambda, \mu \in \mathscr{P}_n^\ell(h)$.

Proof. The map $\det_h$ is easily checked to be injective and its image a closed subset under the θ-dominance ordering. Our assumption on $\kappa \in I^\ell$ (see Remark 6.14) implies that $\deg(\mathsf{S}) = \deg(\det_h(\mathsf{S}))$. As remarked in Remark 6.14, the path $(+\varepsilon_1, +\varepsilon_2, \ldots, +\varepsilon_{h\ell})$ does not pass through any W^e-hyperplane and so we obtain the required bijection. Therefore

$$A_{\det_h(\mathscr{P}^\ell_{n+h\ell}(h))}(n + h\ell, \theta, \kappa) \cong A_{\mathscr{P}^\ell_n(h)}(n, \theta, \kappa)$$

on the level of graded R-modules. It remains to prove that the isomorphism holds on the level of R-algebras. Proceeding as in Section 7, we see that the diagram $C_{\det_h(\mathsf{S})\det_h(\mathsf{t})}$ is obtained from that of C_{St} by:

- shifting any solid or ghost strand X (note that we are excluding the case that X is a vertical red strand) rightwards by $(\ell + \varepsilon)$-units (we now refer to this strand as $\det_h(X)$);
- and adding $h\ell$ 'new' vertical solid strands (with their accompanying ghosts) with x-coordinates given by $\mathbf{I}_{(1,c,m)}$ for $1 \leqslant c \leqslant h$ and $1 \leqslant m \leqslant \ell$.

Let X be a strand of residue $i \in I$ in C_{St}. Let A denote any of the $h\ell$ distinct strands in $C_{\det_h(\mathsf{S})\det_h(\mathsf{t})}$ which do not appear in C_{St}. There is no crossing of a solid and red strand of the same residue in either C_{St} or $C_{\det_h(\mathsf{S})\det_h(\mathsf{t})}$, by Lemma 5.5. Again by Lemma 5.5, any crossing of $\det_h(X)$ with a *new* vertical strand, Y, in $C_{\det_h(\mathsf{S})\det_h(\mathsf{t})}$ is of degree zero and can be removed using only the noninteracting relations, that is the relations which do not annihilate the diagram (3.4), change the number of dots on a strand, or which create error terms. Any crossing of strands $\det_h(X)$ and $\det_h(Y)$ in $C_{\det_h(\mathsf{S})\det_h(\mathsf{t})}$ can be removed in exactly the same fashion as X and Y in $C_{\det_h(\mathsf{S})\det_h(\mathsf{t})}$. The R-algebra isomorphism follows. $\square$

9. The super-strong linkage principle

Throughout this section, $\Bbbk$ is an arbitrary field. Let $\lambda, \mu \in \mathbb{E}^{\circledcirc}_{h,\ell}$. We say that λ and μ are *W^e-linked* if they belong to the same orbit under the dot action of the affine Weyl group, that is if $\lambda \in W^e \cdot \mu$. Given two polynomials $f, g \in \mathbb{N}[t, t^{-1}]$, we write $f \leqslant g$ if and only if $f - g \in \mathbb{N}[t, t^{-1}]$.

THEOREM 9.1 (The super-strong linkage principle). *We have that*

$$d_{\lambda\mu}(t) \leqslant \sum_{\mathsf{s} \in \mathrm{Path}^+(\lambda, \mathsf{t}^\mu)} t^{\deg(\mathsf{s})} \tag{9.1}$$

as degree-wise polynomials, in other words for every $k \in \mathbb{Z}$ we have that

$$[\Delta(\lambda) : L(\mu)\langle k \rangle] \leqslant |\{\mathsf{s} \mid \mathsf{s} \in \mathrm{Path}^+(\lambda, \mathsf{t}^\mu), \deg(\mathsf{s}) = k\}|$$

for $\lambda, \mu \in \mathscr{P}_n^\ell(h)$. In particular, if $d_{\lambda,\mu}(t) \neq 0$ then $\mathrm{Path}^+(\lambda, \mathsf{t}^\mu) \neq \emptyset$.

Proof. By definition, we have that

$$d_{\lambda\mu}(t) = \sum_k \dim_{\Bbbk}(\mathrm{Hom}_{A_h(n,\theta,\kappa)}(P(\mu), \Delta(\lambda)\langle k \rangle))t^k.$$

Now, the projective module $P(\mu)$ is a direct summand of $A_h(n, \theta, \kappa)1_\mu$ and so 1_μ acts trivially on the image of any such homomorphism above. For any homomorphism $\varphi \in \mathrm{Hom}_{A_h(n,\theta,\kappa)}(P(\mu), \Delta(\lambda)\langle k \rangle)$, $\varphi(1_\mu) \in 1_\mu \Delta(\lambda)$. Moreover, $P(\mu)$ is cyclic and so φ is determined by $\varphi(1_\mu)$. Therefore

$$\dim_{\Bbbk}(\mathrm{Hom}_{A_h(n,\theta,\kappa)}(P(\mu), \Delta(\lambda)\langle k \rangle)) \leqslant |\{C_\mathsf{S} \mid C_\mathsf{S} \in 1_\mu \Delta(\lambda), \deg(C_\mathsf{S}) = k\}|$$

and so the result follows. □

We now show how our super-strong linkage principle is a (considerable) strengthening of the usual 'strong linkage principle' of [And98].

DEFINITION 9.2. Let $\lambda, \mu \in \mathbb{E}_{h,\ell}^{\circledcirc}$ be such that $\lambda = s_{\alpha,me} \cdot \mu$ for some $\alpha = \varepsilon_i - \varepsilon_j \in \Phi$, $m \in \mathbb{Z}$. We write $\lambda \uparrow_{\alpha,me} \mu$ if $\lambda \in \mathbb{E}^<(\alpha, me)$, $\mu \in \mathbb{E}^>(\alpha, me)$. We write $\lambda \uparrow \mu$ if there exists a sequence

$$\lambda = \lambda^{(0)} \uparrow_{\varepsilon_{i_1} - \varepsilon_{j_1}, m_1 e} \lambda^{(1)} \uparrow_{\varepsilon_{i_2} - \varepsilon_{j_2}, m_2 e} \lambda^{(2)} \uparrow_{\varepsilon_{i_3} - \varepsilon_{j_3}, m_3 e} \cdots \uparrow \lambda^{(k-1)} \uparrow_{\varepsilon_{i_k} - \varepsilon_{j_k}, m_k e} \lambda^{(k)} = \mu$$

for some $k \geqslant 0$. We say that λ and μ are *strongly linked* if $\lambda \uparrow \mu$ or $\mu \uparrow \lambda$.

REMARK 9.3. Note that $\circledcirc$ is the most dominant point in this ordering and that this is the opposite convention to that used in conventional Lie theory.

THEOREM 9.4. *If $\mathrm{Path}_n(\lambda, \mathsf{t}^\mu) \neq \emptyset$, then λ and μ are strongly linked with $\lambda \uparrow \mu$.*

Proof. Given $\mathsf{s} \in \mathrm{Path}_n(\lambda, \mathsf{t}^\mu)$ for $\lambda \neq \mu$, we let $1 \leqslant k \leqslant n$ denote the first integer such that $\mathsf{s}(k) \neq \mathsf{t}^\mu(k)$. By assumption, $\mathsf{s}(k-1) = \mathsf{t}^\mu(k-1) \in \mathbb{E}(\alpha, me)$ and

$\mathsf{s}(k) \notin \mathbb{E}(\alpha, me)$ for some $\alpha \in \Phi$, $m \in \mathbb{Z}$. We let $k < k' \leqslant n$ denote the minimal integer such that $\mathsf{s}(k') \in \mathbb{E}(\alpha, me)$ if such an integer exists, and be undefined otherwise.

By the minimality of both k and k' and the definition of t^μ, we deduce that $\mathsf{s}(j) \in \mathbb{E}^{>}(\alpha, me)$ for all $k \leqslant j \leqslant k'$ if k' is defined and for all $k \leqslant j \leqslant n$ otherwise. We let $\lambda = \lambda^{(1)}$ if k' is defined and set $\lambda \uparrow \lambda^{(1)} = s_{\alpha,me} \cdot \lambda \in \mathbb{E}_{h,\ell}$ otherwise. We let $\mathsf{s}^{(1)} \in \mathrm{Path}_n(\lambda^{(1)}, \mathsf{t}^\mu)$ denote the path

$$\mathsf{s}^{(1)} = \begin{cases} s^{k'}_{(\alpha,me)} s^{k}_{(\alpha,me)} \cdot \mathsf{s} & \text{if } k' \text{ is defined} \\ s^{k}_{(\alpha,me)} \cdot \mathsf{s} & \text{otherwise.} \end{cases}$$

Repeat this procedure with the path $\mathsf{s}^{(1)}$ to obtain a path $\mathsf{s}^{(2)}$. Continuing in this fashion we obtain an ordered sequence of multipartitions

$$\lambda = \lambda^{(0)} \uparrow \lambda^{(1)} \uparrow \lambda^{(2)} \uparrow \cdots \uparrow \lambda^{(k-1)} \uparrow \lambda^{(k)} = \mu.$$

(given by the terminating points of the corresponding paths) as required. □

COROLLARY 9.5 (Strong linkage principle). *If $d_{\lambda,\mu}(t) \neq 0$ for $\lambda, \mu \in \mathscr{P}^\ell_n(h)$, then $\lambda \uparrow \mu$.*

Proof. The result follows by Theorems 9.4 and 9.1 as $\mathrm{Path}^+(\lambda, \mathsf{t}^\mu) \subseteq \mathrm{Path}(\lambda, \mathsf{t}^\mu)$. □

REMARK 9.6. For $\ell = 1$ and $p = e$ the algebra $\mathcal{Q}_{1,h,n}(\kappa)$ is isomorphic to the image of the symmetric group on n letters in $\mathrm{End}_{\Bbbk}((\Bbbk^h)^{\otimes n})$. Therefore the decomposition numbers $d_{\lambda\mu}(t)$ are the (graded) decomposition numbers of symmetric groups and Corollary 9.5 is equivalent to the strong linkage principle for general linear groups for $p > h$ (as Ringel duality preserves the quasihereditary ordering).

We now provide an example which illustrates how (even in level $\ell = 1$) our super-strong linkage principle is a significant strengthening of the usual strong linkage principle.

EXAMPLE 9.7. Let $\Bbbk$ be an arbitrary field. Let $h = 3$ and $\ell = 1$ and let $e = 6$. For $\mu = (2, 1^{12})$, there are six elements of the set

$$\Pi = \{\lambda \in \mathscr{P}^1_{12}(3) \mid \lambda \uparrow \mu\} = \{\lambda \in \mathbb{E}^{\circledcirc}_{3,1} \mid \mathrm{Path}(\lambda, \mathsf{t}^\mu) \neq \emptyset\}.$$

There are a total of eight paths in the set $\{\mathsf{s} \mid \mathsf{s} \in \mathrm{Path}(\lambda, \mathsf{t}^\mu), \lambda \in \Pi\}$. We have pictured one path for each $\lambda \in \Pi$ in the leftmost diagram in Figure 4 below

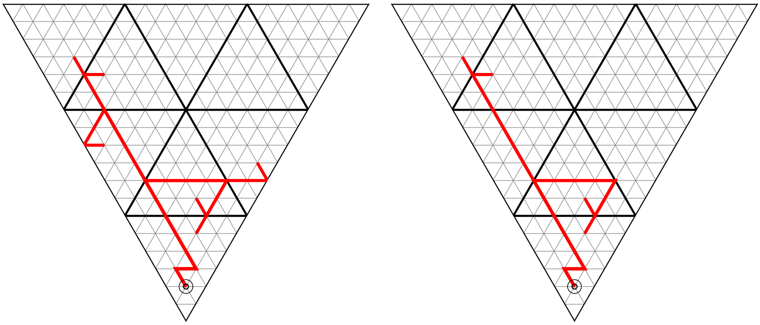


Figure 4. The leftmost diagram depicts six of the eight paths obtainable from t^μ in the geometry of type $A_2 \subseteq \widehat{A}_2$. The rightmost diagram depicts all four *dominant* paths obtainable from t^μ in the geometry of type $A_2 \subseteq \widehat{A}_2$.

(for ease of notation, we do not depict the other two paths). By Corollary 9.5, we deduce that if $\lambda \notin \Pi$, then $d_{\lambda\mu}(t) = 0$. We now wish to see what additional information can be deduced by Theorem 9.1. Notice that only four of the eight paths are dominant. These paths terminate at the points

$$(2, 1^{12}) \quad (2^2, 1^{10}) \quad (3^2, 2^3, 1^2) \quad (3^3, 2^2, 1) \tag{9.2}$$

and are pictured in the rightmost diagram in Figure 4. Therefore we can immediately deduce that $d_{(3^2,1^8),(2,1^{12})} = 0 = d_{(2^7),(2,1^{12})}$. We can also deduce the following bounds on graded decomposition numbers,

$$d_{(2,1^{12}),(2,1^{12})} \leqslant 1 \quad d_{(2^2,1^{10}),(2,1^{12})} \leqslant t^1 \quad d_{(3^3,2^2,1),(2,1^{12})} \leqslant t^2 \quad d_{(3^2,2^3,1^2),(2,1^{12})} \leqslant t^1. \tag{9.3}$$

In fact, we shall see in Example 10.7 that all these bounds are sharp.

Remark 9.8. The two paths which are not depicted in the leftmost diagram of Figure 4 are

$$(+\varepsilon_1, +\varepsilon_2, +\varepsilon_1, +\varepsilon_1, +\varepsilon_1, +\varepsilon_3, +\varepsilon_3, +\varepsilon_2, +\varepsilon_2, +\varepsilon_2, +\varepsilon_2, +\varepsilon_1, +\varepsilon_1, +\varepsilon_3),$$
$$(+\varepsilon_1, +\varepsilon_2, +\varepsilon_1, +\varepsilon_1, +\varepsilon_1, +\varepsilon_3, +\varepsilon_3, +\varepsilon_2, +\varepsilon_2, +\varepsilon_2, +\varepsilon_2, +\varepsilon_1, +\varepsilon_1, +\varepsilon_1).$$

Remark 9.9. We note that Example 9.7 calculates some of the decomposition numbers for symmetric groups labelled by 3-column partitions. These are equal to the decomposition multiplicities for tilting modules for $\mathrm{SL}_3(\Bbbk)$ via Ringel duality [**Don98**, Section 4].

REMARK 9.10. By Theorem 9.4, the simplest case of Theorem 9.1 (the right-hand side of 9.1 is zero) is already a considerable strengthening of the classical strong linkage principle (Corollary 9.5). This is illustrated by our discarding of nondominant paths in Example 9.7. Our super-strong linkage principle is also stronger in the sense that it generalizes the statement of Theorem 9.4 (and hence (Corollary 9.5)) to more complicated upper bounds on decomposition numbers.

REMARK 9.11. It is easy to see, for any e-regular partition (1^n) and any $h \in \mathbb{N}$, that we can obtain a zero of the decomposition matrix generalizing the example $d_{(3^2,1^8),(2,1^{12})} = 0$ in Example 9.7. In particular, we easily obtain infinitely many zeros of the decomposition matrix of $\mathfrak{S}_n$ not covered by [**And80**, Theorem 1].

REMARK 9.12. In Proposition 10.4 below, we shall obtain the converse statement to Corollary 9.5. In Example 10.9 we shall see that the strong and super-strong linkage orderings coincide for *nonparabolic* geometries.

10. Generic behaviour

In this section, we introduce our idea of 'generic behaviour' for diagrammatic Cherednik algebras. It encapsulates the idea that 'generically' the behaviour of a *parabolic* geometry can mimic that of a *nonparabolic* geometry (and hence simplifies). We prove results concerning homomorphisms and decomposition numbers of $A_h(n, \theta, \kappa)$ which are independent of the field $\Bbbk$.

In Sections 10.1 and 10.2, we shall generalize the 'local behaviour' seen in the $\ell = 1$ case (concerning points which are close together in the alcove geometry) to higher levels. In Section 10.3, we shall encounter a new kind of generic behaviour given by relating points which are 'as far away from each other as possible' in the alcove geometry. We refer the reader to Section 12 for examples of this generic behaviour.

DEFINITION 10.1. We say that a subset $\Gamma \subseteq \mathscr{P}_n^\ell(h)$ is *generic* if for every λ, $\mu \in \Gamma$ we have that (i) $\mathrm{Path}_n^+(\lambda, \mathsf{t}^\mu) = \mathrm{Path}_n(\lambda, \mathsf{t}^\mu)$ and (ii) if $\lambda \uparrow \nu \uparrow \mu$, then $\nu \in \Gamma$.

EXAMPLE 10.2. Let $h = 3$, $\ell = 1$, and $e = 4$. The set

$$\Gamma = \{(3^3), (3, 2^2, 1^2), (2^4, 1)\}$$

is not generic. To see this, note that the (unique) path $\mathsf{t} \in \mathrm{Path}_9((3^3), \mathsf{t}^{(2^4,1)})$ given by

$$\mathsf{t} = (+\varepsilon_1 + \varepsilon_2 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_2 + \varepsilon_3 + \varepsilon_1 + \varepsilon_3)$$

does not belong to $\mathrm{Path}_9^+((3^3), \mathsf{t}^{(2^4,1)})$. This is because the point $\mathsf{t}(6) = (2, 3, 1) \notin \mathscr{P}_n^1$ belongs to the $s_{1,2}$-wall of the dominant Weyl chamber. In the literature, one would say that the set Γ is *close to the walls* of the dominant chamber. For a similar example, revisit Example 9.7.

EXAMPLE 10.3. For $\ell = 1$, the points lying 'around the Steinberg weight' form a generic set. For arbitrary $\ell, h \in \mathbb{Z}_{>0}$, any pair of points lying in two adjacent alcoves (of the dominant region) form a generic set (see Definition 10.22 and Example 10.23).

PROPOSITION 10.4. *Let $\mu \in \mathbb{E}_{h,\ell}^{\circledcirc}$, $\lambda \in \mathbb{E}_{h,\ell}$ and suppose $\lambda \uparrow \mu$. We have that*

$$\sum_{\mathsf{s}\in \mathrm{Path}_n(\lambda,\mathsf{t}^\mu)} t^{\deg(s)} = t^{\ell(\mu)-\ell(\lambda)} + \sum_{0<k<\ell(\mu)-\ell(\lambda)} a_k t^{\ell(\mu)-\ell(\lambda)-2k}$$

for coefficients $a_k \in \mathbb{Z}_{\geqslant 0}$.

Proof. Let $\mathsf{s} \in \mathrm{Path}_n(\lambda, \mathsf{t}^\mu)$. The result clearly holds for $n = 1$, we shall assume that the result holds for all paths of length less than or equal to n. Suppose that t^μ is a path of length $n + 1$. We have that

$$\ell(\mathsf{t}^\mu(n+1)) = \ell(\mathsf{t}^\mu(n)) + |\{(\alpha, me) \mid \mathsf{t}^\mu(n) \in \mathbb{E}_{\alpha,me}, \mathsf{t}^\mu(n+1) \notin \mathbb{E}_{\alpha,me}\}|$$

and $\deg(\mathsf{t}^\mu) = \deg(\mathsf{t}^\mu_{\leqslant n}) = 0$. Given $\mathsf{s} \in \mathrm{Path}_{n+1}(\lambda, \mathsf{t}^\mu)$, we have that

$$\begin{aligned}\ell(\mathsf{s}(n+1)) = \ell(\mathsf{s}(n)) &+ |\{(\alpha, me) \mid \mathsf{s}(n) \in \mathbb{E}_{\alpha,me}, \mathsf{s}(n+1) \in \mathbb{E}^{>}_{\alpha,me}\}| \\ &- |\{(\alpha, me) \mid \mathsf{s}(n) \in \mathbb{E}^{>}_{\alpha,me}, \mathsf{s}(n+1) \in \mathbb{E}_{\alpha,me}\}|.\end{aligned}$$

First of all, we note that

$$\begin{aligned}&|\{(\alpha, me) \mid \mathsf{s}(n) \in \mathbb{E}_{\alpha,me}, \mathsf{s}(n+1) \in \mathbb{E}^{<}_{\alpha,me}\}| \\ &\quad = |\{(\alpha, me) \mid \mathsf{s}(n) \in \mathbb{E}_{\alpha,me}, \mathsf{s}(n+1) \notin \mathbb{E}_{\alpha,me}\}| \\ &\qquad - |\{(\alpha, me) \mid \mathsf{s}(n) \in \mathbb{E}_{\alpha,me}, \mathsf{s}(n+1) \in \mathbb{E}^{>}_{\alpha,me}\}| \\ &\quad = |\{(\alpha, me) \mid \mathsf{t}^\mu(n) \in \mathbb{E}_{\alpha,me}, \mathsf{t}^\mu(n+1) \notin \mathbb{E}_{\alpha,me}\}| \\ &\qquad - |\{(\alpha, me) \mid \mathsf{s}(n) \in \mathbb{E}_{\alpha,me}, \mathsf{s}(n+1) \in \mathbb{E}^{>}_{\alpha,me}\}|,\end{aligned}$$

and therefore,

$$\begin{aligned}\ell(\mathsf{t}^\mu(n+1)) - \ell(\mathsf{s}(n+1)) = \ell(\mathsf{t}^\mu(n)) - \ell(\mathsf{s}(n)) &\\ + |\{(\alpha, me) \mid \mathsf{s}(n) \in \mathbb{E}_{\alpha,me}, \mathsf{s}(n+1) \in \mathbb{E}^{<}_{\alpha,me}\}| &\\ + |\{(\alpha, me) \mid \mathsf{s}(n) \in \mathbb{E}^{>}_{\alpha,me}, \mathsf{s}(n+1) \in \mathbb{E}_{\alpha,me}\}| &\end{aligned}$$

and by definition, we have that

$$\begin{aligned}\deg(\mathsf{s}) = \deg(\mathsf{s}{\downarrow}_{\leqslant n}) &+ |\{(\alpha, me) \mid \mathsf{s}(n) \in \mathbb{E}_{\alpha,me}, \mathsf{s}(n+1) \in \mathbb{E}^{<}_{\alpha,me}\}| \\ &- |\{(\alpha, me) \mid \mathsf{s}(n) \in \mathbb{E}^{>}_{\alpha,me}, \mathsf{s}(n+1) \in \mathbb{E}_{\alpha,me}\}|.\end{aligned}$$

Putting these two statements together, we have that

$$\begin{aligned}&\ell(\mathsf{t}^\mu(n+1)) - \ell(\mathsf{s}(n+1)) - \deg(\mathsf{s}) \\ &\quad= (\ell(\mathsf{t}^\mu(n)) - \ell(\mathsf{s}(n)) - \deg(\mathsf{s}{\downarrow}_{\leqslant n})) \\ &\qquad + 2|\{(\alpha, me) \mid \mathsf{s}(n) \in \mathbb{E}^{>}_{\alpha,me}, \mathsf{s}(n+1) \in \mathbb{E}_{\alpha,me}\}|.\end{aligned}$$

The upper degree bound statement and the degree parity follow by induction. The lower bound on degree follows as the first reflection through a hyperplane always increases the degree of the path (by the definition of t^μ). Finally, we note that $\ell(\mathsf{t}^\mu(n+1)) - \ell(\mathsf{s}(n+1)) = \deg(\mathsf{s})$ if and only if both of the following conditions are satisfied

$$|\{(\alpha, me) \mid \mathsf{s}(n) \in \mathbb{E}^{>}(\alpha, me), \mathsf{s}(n+1) \in \mathbb{E}_{\alpha,me}\}| = 0 \quad \text{and} \tag{10.1}$$

$$\deg(\mathsf{s}{\downarrow}_{\leqslant n}) = \ell(\mathsf{t}^\mu(n)) - \ell(\mathsf{s}(n)). \tag{10.2}$$

We now prove that for arbitrary $\lambda \in \mathbb{E}_{h,\ell}$ and $\mu \in \mathbb{E}^{\circledcirc}_{h,\ell}$ there exists a unique (not necessarily dominant) path satisfying both these conditions (and therefore is of degree $\ell(\mu) - \ell(\lambda)$). Set $\mu' = \mathsf{t}^\mu(n)$ and let $\lambda' \in \mathbb{E}_{h,\ell}$ be such that $\lambda' \uparrow \mu'$. We may assume (by induction) that for any such λ', there exists a unique path s satisfying condition (10.2). Now suppose that λ' is such that

$$|\{(\alpha, me) \mid \lambda' = \mathsf{s}(n) \in \mathbb{E}^{<0}(\alpha, me), \mathsf{s}(n+1) \in \mathbb{E}_{\alpha,me}\}| = k \tag{10.3}$$

for $k > 1$. We set $\lambda'' = s_{(\alpha,me)} \cdot \lambda'$ (for any (α, me) in the above set). We have that $\lambda'' \uparrow \lambda'$ and

$$|\{(\alpha, me) \mid \lambda'' = \mathsf{u}(n) \in \mathbb{E}^{<0}(\alpha, me), \mathsf{s}(n+1) \in \mathbb{E}_{\alpha,me}\}| < k \tag{10.4}$$

and indeed the set in (10.4) is a subset of that in (10.3). While the reflection is not unique, there is a unique coset of the stabilizer, $\mathrm{Stab}(\lambda')$, of the point λ' in W^e for $e \in \mathbb{Z}_{>0} \cup \{\infty\}$. Continuing in this fashion, we eventually obtain the unique path s and unique point $\mathsf{s}(n) = \lambda''' \in \mathbb{E}_{h,\ell}$ satisfying (10.1) (where (10.2) is satisfied by our inductive assumption). The result follows. □

DEFINITION 10.5. We let $\mathsf{t}^\mu_\lambda \in \mathrm{Path}(\lambda, \mathsf{t}^\mu)$ denote the unique path of degree $\ell(\mu) - \ell(\lambda)$.

PROPOSITION 10.6. *Let* $\Bbbk$ *be an arbitrary field,* $\lambda, \mu \in \mathbb{E}^{\circledcirc}_{h,\ell}$ *and suppose that* $\lambda \uparrow \mu$*. If*

$$[\Delta(\lambda) : L(\mu)\langle k\rangle] \neq 0 \tag{10.5}$$

this implies that $\ell(\lambda) - \ell(\mu) + 2 \leqslant k \leqslant \ell(\mu) - \ell(\lambda)$*. We have that*

$$[\Delta(\lambda) : L(\mu)\langle \ell(\mu) - \ell(\lambda)\rangle] = 1 \tag{10.6}$$

if $\mathsf{t}^{\mu}_{\lambda} \in \mathrm{Path}^{+}(\lambda, \mathsf{t}^{\mu})$ *and is zero otherwise.*

Proof. Equation (10.5) follows directly from Proposition 10.4. We know that $C_{\mathsf{t}^{\mu}_{\lambda}} \in \Delta(\lambda)$ and that $C_{\mathsf{t}^{\mu}_{\lambda}}$ is a vector belonging to some simple module $L(\nu)$ (with $\lambda \uparrow \nu \uparrow \mu$) appearing in the submodule lattice of $\Delta(\lambda)$. This implies that there exists $a, b \geqslant 0$ such that $d_{\lambda\nu}(t) = t^{a} + \cdots$ and $\dim_t(1_{\mu} L(\nu)) = t^{b} + \cdots$ such that $a + b = \ell(\mu) - \ell(\lambda)$. It remains to show that $\nu = \mu$. We have that

$$(\ell(\mu) - \ell(\nu)) + (\ell(\nu) - \ell(\lambda)) = \ell(\mu) - \ell(\lambda).$$

By Propositions 10.4 and 2.2 we deduce that $a = \ell(\mu) - \ell(\lambda)$ and $b = 0$ and $\nu = \mu$, as required. □

EXAMPLE 10.7. By Proposition 10.6, we immediately deduce that the first three inequalities in (9.3) are actually equalities. For the final inequality, we suppose $\dim_t(1_{(2,1^{12})}\Delta(3^2, 2^3, 1^2)) = t^1 \neq d_{(3^2,2^3,1^2),(2,1^{12})}(t) = 0$. Then there exists ν such that $1_{(2,1^{12})}L(\nu) \neq 0$ and $1_{\nu}L(3^2, 2^3, 1^2) \neq 0$. However, we have already seen that all elements of $\mathrm{Path}^{+}(\nu, \mathsf{t}^{(2,1^{12})})$ are of strictly positive degree. Therefore there does not exist any ν such that $1_{(2,1^{12})}L(\nu) \neq 0$ by Proposition 2.2. Therefore we conclude that $d_{(3^2,2^3,1^2),(2,1^{12})}(t) = t^1$. Notice that while we were unable to apply Proposition 10.6 directly in this final case, we are applying the same argument as in the proof of Proposition 10.6.

COROLLARY 10.8. *Let* $\lambda, \mu \in \mathscr{P}^{\ell}_{n}(h)$ *be a generic pair and suppose* $\lambda \uparrow \mu$*. Then*

$$[\Delta(\lambda) : L(\mu)\langle \ell(\mu) - \ell(\lambda)\rangle] = 1.$$

Proof. By assumption $\mathrm{Path}(\lambda, \mathsf{t}^{\mu}) = \mathrm{Path}^{+}(\lambda, \mathsf{t}^{\mu})$; the result follows by Propositions 10.6 and 10.4. □

EXAMPLE 10.9. Let $h = 1$. In which case, any subset of $\mathbb{E}^{\circledcirc}_{1,\ell}$ is generic. Therefore

$$d_{\lambda\mu}(t) = t^{\ell(\mu)-\ell(\lambda)} + \cdots$$

for all $\lambda \uparrow \mu$. In particular, $d_{\lambda\mu}(t) \neq 0$ if and only if $\lambda \uparrow \mu$. For $h = 1$ and $\ell = 2$ these decomposition numbers were first calculated in [**MW00**, (9.4) Theorem] and [CGM03, Section 8]. For $\Bbbk = \mathbb{C}$ and ℓ arbitrary, this case was studied extensively in [BCS17].

We now present the main result of this section. It will allow us to deduce the existence of many homomorphisms between Weyl and Specht modules. For those not familiar with the diagram combinatorics, we recommend reading the (simpler) proof of Theorem 10.19 below, first. We first require a simple lemma.

LEMMA 10.10. *Let $\lambda, \mu \in \mathbb{E}^+_{h,\ell}$ and suppose $\lambda \uparrow \mu$ and $\ell(\lambda) = \ell(\mu) - 1$. If $(r, c, m) \in \mathrm{Rem}_i(\lambda \cap \mu)$, then there is no crossing of $X_{(r,c,m)}$ (or its ghost) with an i- or $(i+1)$-strand (or its ghost) in $C_{\mathsf{t}^\mu_\lambda}$.*

Proof. The diagram $C_{\mathsf{t}^\mu_\lambda}$ traces out the unique residue preserving bijection between the strip of northern nodes $\mu \setminus (\mu \cap \lambda)$ and the strip of southern nodes $\lambda \setminus (\mu \cap \lambda)$. A necessary condition for a crossing between $X_{(r,c,m)}$ and any other strand Y is that Y is nonvertical (as X is vertical, by assumption) and therefore Y connects the points $(\mathbf{I}_{y_0}, 0)$ and $(\mathbf{I}_{y_1}, 1)$ for some $y_0 \in \lambda \setminus (\mu \cap \lambda)$, $y_1 \in \mu \setminus (\mu \cap \lambda)$. By the definition of the path t^μ_λ, the strand Y is added at a later stage than the strand X in the process outlined in Section 7 (this follows because $\mathbf{I}_{y_1} > \mathbf{I}_{(r,c,m)}$). Therefore we can suppose that X is added at the ath step and Y is added at the bth step for $1 \leqslant a < b \leqslant n$. By induction we can assume that $C_{\mathsf{t}^\mu_\lambda \downarrow_{\leqslant b-1}}$ contains no crossing contradicting the statement of the lemma. If Y is an i-strand which crosses $X_{(r,c,m)}$, then

$$\ell(\mathrm{Shape}(\mathsf{t}^\mu_\lambda \downarrow_{\leqslant b}) + y_0) + 2 = \ell(\mathrm{Shape}(\mathsf{t}^\mu_\lambda \downarrow_{\leqslant b}) + y_1) \leqslant \ell(\mu)$$

as we have stepped onto a hyperplane from above (since we have added an i-node corresponding to y_0 to the left of the removable i-node corresponding to X) and off another hyperplane towards the origin (as we have added an i-node y_0 to the left of the addable i-node y_1). If Y is an $(i-1)$-strand which crosses $X_{(r,c,m)}$, then again

$$\ell(\mathrm{Shape}(\mathsf{t}^\mu_\lambda \downarrow_{\leqslant b}) + y_0) + 2 = \ell(\mathrm{Shape}(\mathsf{t}^\mu_\lambda \downarrow_{\leqslant b}) + y_1) \leqslant \ell(\mu)$$

as we have stepped off two hyperplanes towards the origin (as we have added an $(i-1)$-node y_0 to the left of two addable $(i-1)$-nodes (r, c, m) and y_1). The result follows. □

THEOREM 10.11. *We let*

$$\lambda = \mu^{(t)} \uparrow \mu^{(t-1)} \uparrow \dots \uparrow \mu^{(0)} = \mu \tag{10.7}$$

be a sequence of points in $\mathbb{E}^{\circledcirc}_{h,\ell}$ *such that*

$$s_{\alpha^{(k)},m_k e} \cdot \mu^{(k)} = \mu^{(k-1)}$$

for $1 \leqslant k \leqslant t$. *Suppose that the path* $\mathsf{t}^{\mu}_{\lambda} \in \mathrm{Path}(\lambda, \mathsf{t}^{\mu})$ *is dominant and is obtained by*

$$\mathsf{t}^{\mu}_{\lambda} = s^{i_t}_{\alpha^{(t)},m_t e} \dots s^{i_2}_{\alpha^{(2)},m_2 e} s^{i_1}_{\alpha^{(1)},m_1 e} \cdot \mathsf{t}^{\mu}$$

for some sequence $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_t$. *Then for any* $\nu = \mu^{(k)}$ *for* $1 \leqslant k \leqslant t$ *we have that*

$$\mathsf{t}^{\nu}_{\lambda} = s^{i_t}_{\alpha^{(t)},m_t e} \dots s^{i_{k+1}}_{\alpha^{(k+1)},m_{k+1} e} \cdot \mathsf{t}^{\nu} \qquad \mathsf{t}^{\mu}_{\nu} = s^{i_k}_{\alpha^{(k)},m_k e} \dots s^{i_1}_{\alpha^{(1)},m_1 e} \cdot \mathsf{t}^{\mu}_{\lambda}$$

are both dominant paths and we have that

$$C_{\mathsf{t}^{\mu}_{\nu}} C_{\mathsf{t}^{\nu}_{\lambda}} = C_{\mathsf{t}^{\mu}_{\lambda}}.$$

Proof. The statements concerning paths follow from Theorem 9.4 and Proposition 10.4. For ease of notation, we set

$$\mathsf{t}^k = \mathsf{t}^{\mu^{(k-1)}}_{\mu^{(k)}} \in \mathrm{Path}(\mu^{(k)}, \mu^{(k-1)}) \quad \text{and} \quad \mathsf{t}^k_n = \mathsf{t}^k\!\downarrow_{\leqslant n}$$

for $1 \leqslant k \leqslant t$ and for $n \in \mathbb{Z}_{\geqslant 0}$. We shall now inductively construct the elements

$$C_{\mathsf{t}^i_n} \qquad C_{\mathsf{t}^1_n} C_{\mathsf{t}^2_n} \dots C_{\mathsf{t}^t_n} \quad \text{and} \quad C_{\mathsf{t}^{\mu}_{\lambda}\downarrow_{\leqslant n}} \tag{10.8}$$

for $1 \leqslant i \leqslant t$ simultaneously by induction on $n \in \mathbb{Z}_{\geqslant 0}$ and verify that

$$C_{\mathsf{t}^1} C_{\mathsf{t}^2} \dots C_{\mathsf{t}^t} = C_{\mathsf{t}^{\mu}_{\lambda}} \tag{10.9}$$

For $n = 1$, the t elements and the product in (10.8) are all equal to the same idempotent with one solid strand. Given a fixed n, we obtain each of the diagrams

$$C_{\mathsf{t}^i_{n+1}} \quad \text{and} \quad C_{\mathsf{t}^1_{n+1}} C_{\mathsf{t}^2_{n+1}} \dots C_{\mathsf{t}^t_{n+1}}$$

from those in (10.8) by adding a single strand. For $1 \leqslant i \leqslant t$ we denote this strand by X^i_{n+1} and we let $X_{n+1} = X^1_{n+1} \circ X^2_{n+1} \circ \cdots \circ X^t_{n+1}$ denote the composite strand. We suppose that $\mathsf{t}^{\mu}(n+1) = \mathsf{t}^{\mu}(n) + \varepsilon_c$ and therefore (in the notation of Theorem 10.11) that X^i_{n+1} connects the northern and southern points labelled by boxes added in the

$$c_{k-1} = s_{\alpha^{(k-1)}} \dots s_{\alpha^{(1)}}(c) \quad \text{and} \quad c_k = s_{\alpha^{(k)}} \dots s_{\alpha^{(1)}}(c)$$

columns, respectively. In particular, the elements

$$C_{\mathsf{t}^1_{n+1}} C_{\mathsf{t}^2_{n+1}} \dots C_{\mathsf{t}^t_{n+1}} \quad \text{and} \quad C_{\mathsf{t}^{\mu}_{\lambda}\downarrow_{\leqslant n+1}} \tag{10.10}$$

trace out the same bijections, as required. It remains to show that the product on the left-hand side of (10.10) contains no double-crossings. In fact, it is enough to show that the product contains no double-crossing of strands labelled by *adjacent* or *equal* residues (as all other double-crossings can be trivially removed).

By induction, we can assume there are no double-crossings in $C_{\mathsf{t}^1_n} C_{\mathsf{t}^2_n} \dots C_{\mathsf{t}^t_n}$. Next we suppose that there is a double-crossing of the strands X^{n+1} and X^i in $C_{\mathsf{t}^1_{n+1}} C_{\mathsf{t}^2_{n+1}} \dots C_{\mathsf{t}^t_{n+1}}$ for some $1 \leqslant i \leqslant n$. In which case, there is a crossing both of the strands X^{n+1}_a and X^i_a in the diagram $C_{\mathsf{t}^a_{n+1}}$ and the strands X^{n+1}_b and X^i_b in $C_{\mathsf{t}^b_{n+1}}$ for $1 \leqslant a < b \leqslant t$. We assume that b is minimal with this property. The strands X^i_a and X^{n+1}_b are both vertical. We shall identify strands with the corresponding nodes at which they terminate in the obvious fashion.

Importantly, X^{n+1}_b is a vertical strand corresponding to a *removable* node of $\mu^{(b)} \cap \mu^{(b+1)}$. Therefore, $\mathrm{res}(X^i_b) \neq \mathrm{res}(X^{n+1}_b), \mathrm{res}(X^{n+1}_b) - 1$ by Lemma 10.10. By assumption, the strands X^i and X^{n+1} are of adjacent or equal residue; therefore $\mathrm{res}(X^i) = \mathrm{res}(X^{n+1}) + 1$.

Suppose that X^i_b connects nodes $(r, c, m) \in \mu^{(b)} \setminus \mu^{(b)} \cap \mu^{(b+1)}$ and $(r', c', m') \in \mu^{(b+1)} \setminus \mu^{(b)} \cap \mu^{(b+1)}$. Now, we assume that (r, c, m) is *not* the removable node in the strip $\mu^{(b)} \setminus \mu^{(b)} \cap \mu^{(b+1)}$; in other words we suppose that $(r + 1, c, m) \in \mu^{(b)} \setminus \mu^{(b)} \cap \mu^{(b+1)}$. We let Y^i_b denote the strand connecting points $(\mathbf{I}_{(r+1,c,m)}, 1)$ and $(\mathbf{I}_{(r'+1,c',m')}, 0)$. We have assumed that $\mathrm{res}(r, c, m) - 1 = \mathrm{res}(X^{n+1})$ and $\mathbf{I}_{(r',c',m')}$ is strictly less than the x-coordinate of the vertical strand X^{k+1}_b. Therefore by Lemma 5.5 it follows that $\mathbf{I}_{(r'+1,c',m')} < \mathbf{I}_{(r',c',m')} + h\ell$ is strictly less than the x-coordinate of the vertical strand X^{k+1}_b. Therefore X^{k+1}_b and Y^i_b cross in $C_{\mathsf{t}^b_{n+1}}$ and have the same residue. Therefore we can repeat the argument above to get a contradiction; we hence deduce that X^i_b is a removable node of $\mu^{(b)}$ and $\mu^{(b+1)}$. Therefore we can assume that X^i_b is a removable node of $\mu^{(b)}$ and $\mathrm{res}(X^i_b) = \mathrm{res}(X^{k+1}_b) + 1$.

Now, suppose that X^i_b is not a removable node of $\mu^{(a)} \cap \mu^{(a+1)}$. This implies that there is some reflection labelled by $a < d < b$, which adds a strip at the end of the column containing X^i_b. This results in either a double-crossing between strands in

$$C_{\mathsf{t}^d_k} \dots C_{\mathsf{t}^b_k} \quad \text{or} \quad C_{\mathsf{t}^1_{k+1}} C_{\mathsf{t}^2_{k+1}} \dots C_{\mathsf{t}^d_{k+1}}$$

and hence a contradiction either by induction, or by our assumption of the minimality of b, respectively. Therefore we can assume that X^i_b *is* a removable node of $\mu^{(a)} \cap \mu^{(a+1)}$. Finally, we have that X^i_a is a removable node of the partition $\mu^{(a)}$ and $\mathrm{res}(X^i_a) = \mathrm{res}(X^{k+1}_a) + 1$. Therefore, we obtain a contradiction by Lemma 10.10. Thus we conclude that there are no double-crossings in the product. The result follows. □

COROLLARY 10.12. *If λ, μ are a generic pair such that $\lambda \uparrow \mu$, we have that*

$$\dim_t(\mathrm{Hom}_{A_h(n,\theta,\kappa)}(\Delta(\mu), \Delta(\lambda))) = t^{\ell(\mu)-\ell(\lambda)} + \cdots$$

where the remaining terms are all of strictly smaller degree. This highest-degree homomorphism is given (up to scalar multiplication) by

$$\varphi_\lambda^\mu : C_{\mathsf{t}^\mu} \mapsto C_{\mathsf{t}^\mu_\lambda}.$$

If $\lambda \uparrow \nu \uparrow \mu$ with ν belonging to the sequence (10.7), then

$$\varphi_\lambda^\nu \circ \varphi_\nu^\mu = \varphi_\lambda^\mu.$$

Proof. If $\ell(\lambda) = \ell(\mu) - 1$ and $\lambda \uparrow \mu$, then $d_{\lambda\mu}(t) = t$. Therefore

$$\dim_t(\mathrm{Hom}_{A_h(n,\theta,\kappa)}(P(\mu), \Delta(\lambda))) = t^1.$$

That this homomorphism factors through the projection $P(\mu) \to \Delta(\mu)$ follows from highest weight theory and the fact that there does not exist any α such that $\lambda \uparrow \alpha \uparrow \mu$. Now, let λ and μ be an arbitrary generic pair. By (10.9), we have that the composition of the degree 1 homomorphisms along the sequence (10.7) is itself a homomorphism and equal to φ_λ^μ. The result follows. □

10.1. Maximal parabolic behaviour. We will now consider generic sets whose elements are permuted by some finite group, $\mathfrak{S}_{a+b} \leqslant \widehat{\mathfrak{S}_{h\ell}}$, and such that the stabilizer of any given point $\lambda \in \Gamma$ is a maximal parabolic subgroup $\mathfrak{S}_a \times \mathfrak{S}_b \leqslant \mathfrak{S}_{a+b}$.

DEFINITION 10.13. Let $a, b \in \mathbb{Z}_{>0}$ and $x \in \mathbb{Z}/e\mathbb{Z}$. Let $\gamma \in \mathscr{P}_n^\ell(h)$ be any multipartition with precisely $a+b$ addable x-nodes and zero removable x-nodes. We let $\Gamma_{a,b} \subseteq \mathscr{P}_n^\ell(h)$ denote the set of multipartitions which can be obtained by adding a total of a distinct x-nodes to $\gamma \in \mathscr{P}_n^\ell(h)$.

EXAMPLE 10.14. If $a + b = h$ and $\ell = 1$ then γ is a translate of the Steinberg point $(e-1)\rho$.

EXAMPLE 10.15 (Stepping off a wall). Let $\mu \in \mathbb{E}_{h,\ell}^{\circledcirc}$ be an e-regular point. We have that $\gamma = \mu - \varepsilon_X$ for any $X \in \mathrm{Rem}(\mu)$ belongs to either one or zero hyperplanes. If $\gamma = \mu - \varepsilon_X \in \mathbb{E}_{\alpha,me}$ for some (α, me), then we say that μ is obtained from γ by *stepping off the* (α, me)*-wall*. If μ is obtained from γ by *stepping off the* (α, me)*-wall*, then $\Gamma_{1,1} = \{\mu, s_{(\alpha,me)} \cdot \mu\}$ forms a generic set.

We let $\mathfrak{S}_{a+b} \leqslant \widehat{\mathfrak{S}_{h\ell}}$ denote the group which acts by faithfully permuting the elements of $\Gamma_{a,b}$. We remark that $\mathfrak{S}_{a+b}$ fixes the point $\gamma \in \mathbb{E}^{\circledcirc}_{h,\ell}$. Given any fixed choice $\lambda \in \Gamma_{a,b}$, we let $\mathfrak{S}_{a,b} = \mathfrak{S}_a \times \mathfrak{S}_b \leqslant \mathfrak{S}_{a+b}$ denote the subgroup which fixes $\lambda \in \Gamma_{a,b}$ – this is the subgroup whose elements trivially permute the columns with a removable (respectively addable) x-node amongst themselves.

LEMMA 10.16. *For $\lambda, \mu \in \Gamma_{a,b}$ we have that* $\mathrm{Path}^+_n(\lambda, \mathsf{t}^\mu) = \mathrm{Path}_n(\lambda, \mathsf{t}^\mu)$ *(hence the set $\Gamma_{a,b}$ is generic). Any pair of partitions λ and μ can be written in the form*

$$\mu = \gamma + A_{c_1} + A_{c_2} + \cdots + A_{c_a} \qquad \lambda = \gamma + A_{q_1} + A_{q_2} + \cdots + A_{q_a}$$

where $A_{c_1} \rhd A_{c_2} \rhd \cdots \rhd A_{c_a}$ and $A_{q_1} \rhd A_{q_2} \rhd \cdots \rhd A_{q_a}$. For $\lambda \uparrow \mu$, we have that

$$\mathsf{t}^\mu_\lambda = s^{i_a}_{\varepsilon_{q_a - c_a}, m_r e} \cdots s^{i_2}_{\varepsilon_{q_2 - c_2}, m_2 e} s^{i_1}_{\varepsilon_{q_1 - c_1}, m_1 e} \cdot \mathsf{t}^\mu \tag{10.11}$$

for some $i_1 < i_2 < \cdots < i_a$.

Proof. By construction, any $\mathsf{s} \in \mathrm{Path}(\lambda, \mathsf{t}^\mu)$ is of the form

$$s^{i_{a+b-1}}_{\varepsilon_{\sigma(c_{a+b-1})} - \varepsilon_{c_{a+b}}, m_{a+b-1} e} \cdots s^{i_2}_{\varepsilon_{\sigma(c_2)} - \varepsilon_{c_2}, m_2 e} s^{i_1}_{\varepsilon_{\sigma(c_1)} - \varepsilon_{c_1}, m_1 e} \cdot \mathsf{t}^\mu. \tag{10.12}$$

where $\lambda = \sigma \cdot \mu$ for some $\sigma \in \mathfrak{S}_{a+b}$ (the converse is not true). All such paths are dominant, by construction. Set $\sigma(c_i) = q_i$ for $1 \leqslant i \leqslant a$. The i_kth step in the path (10.12) corresponds to adding a box as far to the left as possible. Therefore by Lemma 5.5, there are no removable x-nodes to the right of A_{q_i}. Hence this step satisfies (10.1). Repeating for every step, we deduce that (10.2) is also satisfied. The result follows. □

THEOREM 10.17. *Let $\Bbbk$ be an arbitrary field. Given $\lambda, \mu \in \Gamma_{a,b} \subseteq \mathscr{P}^\ell_n(h)$, we have that*

$$d_{\lambda,\mu}(t) = n_{\lambda,\mu}(t)$$

is the associated Kazhdan–Lusztig polynomial of type $A_{a-1} \times A_{b-1} \subseteq A_{a+b}$. A closed combinatorial formula for these polynomials is given in [**TT13**, *Section* 3] *and* [**BS10**, *Section* 5].

Proof. Our assumption that $\kappa \in I^\ell$ is h-admissible implies that we can apply [**BS18**, Theorem 4.30]. The result follows. □

REMARK 10.18. These Kazhdan–Lusztig polynomials have recently made two prominent appearances in representation theory. The first is in the work of Kleshchev, Chuang, Miyachi, Tan for $\ell = 1$ and $e \in \mathbb{N}$ [**Kle97**, **CMT08**, **TT13**] and the latter is in the work of Brundan–Stroppel and Mathas–Hu for $\ell = 2$ and $e = \infty$ [**BS11**] and [**HM15**, Theorems B3 & B5]. Theorem 10.17 applies to the former family of results ($\ell = 1$ and $e \in \mathbb{N}$) and generalizes all these results to higher levels. In the latter case ($\ell = 2$ and $e = \infty$) these results follow easily from Theorem 6.22 as the algebra $A(n, \theta, \kappa)$ is a basic algebra. We are unaware of any direct link explaining these two distinct appearances of the same family of Kazhdan–Lusztig polynomials.

As the combinatorics of this case is particularly simple, we are able to prove a strengthened version of Theorem 10.11. Namely, we can understand the composition of *any* chain of these homomorphisms.

THEOREM 10.19. *Let $\Bbbk$ be a field of arbitrary characteristic. For $\lambda, \mu \in \Gamma_{a,b}$ such that $\lambda \uparrow \mu$, we have that*

$$\dim_t(\mathrm{Hom}_{A_h(n,\theta,\kappa)}(\Delta(\mu), \Delta(\lambda))) = t^{\ell(\mu)-\ell(\lambda)} + \cdots$$

where the remaining terms are all of strictly smaller degree. This highest-degree homomorphism is given (up to scalar multiplication) by

$$\varphi_\lambda^\mu : C_{\mathsf{t}^\mu} \mapsto C_{\mathsf{t}_\lambda^\mu}$$

and the composition of these homomorphisms is given by

$$\varphi_\lambda^\nu \circ \varphi_\nu^\mu = \varphi_\lambda^\mu$$

for all $\lambda, \nu, \mu \in \Gamma_{a,b}$ with $\lambda \uparrow \nu \uparrow \mu$.

Proof. If $\ell(\lambda) = \ell(\nu) - 1$ and $\lambda \uparrow \nu$, then the result follows as in Theorem 10.11. To deduce the result, it will now suffice to show that, for *any* sequence of the form

$$\lambda = \mu^{(t)} \uparrow \mu^{(t-1)} \uparrow \cdots \uparrow \mu^{(1)} \uparrow \mu^{(0)} = \mu,$$

that

$$C_{\mathsf{t}_t^{t-1}} C_{\mathsf{t}_{t-1}^{t-2}} \dots C_{\mathsf{t}_1^0} = C_{\mathsf{t}_\lambda^\mu} \quad \text{for } \mathsf{t}_k^{k-1} = \mathsf{t}_{\mu^{(k)}}^{\mu^{(k-1)}} \in \mathrm{Std}(\mu^{(k)}, \mu^{(k-1)}). \tag{10.13}$$

Let $\lambda, \mu \in \Gamma_{a,b}$. We continue with the notation of Lemma 10.16. By (10.11), we have that $\mathsf{t}_\lambda^\mu(A_{q_i}) = \mathbf{I}_{A_{p_i}}$ and $\mathsf{t}_\lambda^\mu(r, c, m) = \mathbf{I}_{(r,c,m)}$ for $(r, c, m) \neq A_{q_i}$ for $1 \leqslant i \leqslant a$. In particular, $C_{\mathsf{t}_\lambda^\mu}$ is obtained from C_{T^γ} by adding a total of a strands connecting the northern points $\{(\mathbf{I}_{A_{c_i}}, 1) \mid 1 \leqslant i \leqslant a\}$ with the southern points $\{(\mathbf{I}_{A_{q_i}}, 0) \mid 1 \leqslant i \leqslant a\}$ in such a fashion that these a strands *do not cross each other* at any point.

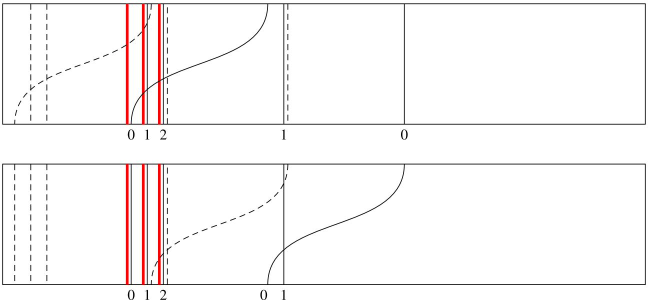


Figure 5. Concatenating basis elements labelled by $\mathsf{t}^{(\varnothing,(1^2),(1^3))}_{((1),(1),(1^3))}$ and $\mathsf{t}^{((1),(1),(1^3))}_{((1),(1^2),(1^2))}$ as in Example 10.20.

Now, composing any number of such diagrams, we clearly obtain a diagram with a strands which do not cross, and trace out the bijection between two such sets. Hence (10.13) and the result follows. □

EXAMPLE 10.20. We let $\gamma = (\varnothing, (1), (1^2)) \in \mathscr{P}^3_4(1)$ with $e = 4$ and $\kappa = (0, 1, 2)$. We have that

$$\Gamma_{2,1} = \{(\varnothing, (1^2), (1^3))((1), (1), (1^3)), ((1), (1^2), (1^2))\}.$$

The graded decomposition matrix for this subquotient is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2 & t & 1 \end{pmatrix}$$

and every decomposition number is given by a homomorphism between Weyl modules. The degree 2 homomorphism is determined as the composition of the two degree 1 homomorphisms, this can be seen by concatenating the two diagrams in Figure 5.

REMARK 10.21. Fix our field to be $\mathbb{C}$. For the type A Temperley–Lieb algebra, $\mathcal{Q}_{2,1,n}(\kappa)$, these homomorphisms provide *all possible* homomorphisms between Weyl modules. For the type B Temperley–Lieb algebra (or blob algebra), $\mathcal{Q}_{1,2,n}(\kappa)$, these homomorphisms are obtained by lifting homomorphisms from $\mathcal{Q}_{2,1,n}(\kappa)$ (and provide 'half' of all homomorphisms between Weyl modules [**MW00**]).

10.2. Nonparabolic finite behaviour. We now consider generic sets $\Gamma_{h\ell}$ whose elements are permuted by the finite group, $\mathfrak{S}_{h\ell} \leqslant \widehat{\mathfrak{S}_{h\ell}}$, and such that the stabilizer of any given point $\lambda \in \Gamma_{h\ell}$ is the trivial subgroup.

DEFINITION 10.22. We say that $\gamma \in \mathbb{E}^{\circledcirc}_{h,\ell}$ is a *maximal vertex* if for every $\alpha \in \Phi$ there exists some $m \in \mathbb{Z}$ such that $\gamma \in \mathbb{E}_{\alpha,me}$. By assumption, the parts are γ are all distinct and therefore there exists $\sigma \in \mathfrak{S}_{h\ell}$ such that

$$\langle \gamma, \varepsilon_{\sigma(h\ell)} \rangle < \cdots < \langle \gamma, \varepsilon_{\sigma(2)} \rangle < \langle \gamma, \varepsilon_{\sigma(1)} \rangle$$

We let $\mathfrak{S}_{h\ell} \leqslant \widehat{\mathfrak{S}_{h\ell}}$ denote the group generated by the reflections which fix $\gamma \in \mathscr{P}^{\ell}_n(h)$. We let $\alpha \in \mathscr{P}^{\ell}_n$ denote any multipartition such that

$$\langle \alpha, \varepsilon_{\sigma(h\ell)} \rangle < \cdots < \langle \alpha, \varepsilon_{\sigma(2)} \rangle < \langle \alpha, \varepsilon_{\sigma(1)} \rangle < e.$$

We let $\Gamma_{h\ell}$ denote the set $\Gamma_{h\ell} = \mathfrak{S}_{h\ell} \cdot (\gamma + \alpha)$.

There are $(h\ell)!$ distinct elements of $\Gamma_{h\ell}$ and the group $\mathfrak{S}_{h\ell}$ acts faithfully by permuting the elements of $\Gamma_{h\ell}$. We have chosen α in such a way that $\gamma + w\alpha$ is e-regular and $\ell(\gamma + w\alpha) = \ell(\gamma + \alpha) - \ell(w)$ for any $w \in \mathfrak{S}_{h\ell}$.

EXAMPLE 10.23. For $\ell = 1$, the point γ is a translate of the Steinberg point $(e-1)\rho$. The corresponding set $\Gamma_{h\ell}$ is given by an e-regular set of points 'around a Steinberg vertex'.

THEOREM 10.24. *For $\lambda, \mu \in \Gamma_{h\ell}$ such that $\lambda \uparrow \mu$, we have that*

$$\dim_t(\mathrm{Hom}_{A_h(n,\theta,\kappa)}(\Delta(\mu), \Delta(\lambda))) = t^{\ell(\mu)-\ell(\lambda)} + \cdots$$

where the remaining terms are all of strictly smaller degree.

Proof. Given $\lambda, \mu \in \Gamma_{h,\ell}$ we have that $\ell_\alpha(\mu) - \ell_\alpha(\lambda) = 0, 1$ for any $\alpha \in \Phi$. Therefore any $\mathsf{s} \in \mathrm{Path}(\lambda, \mathsf{t}^\mu)$ is obtained from t^μ by applying a sequence of reflections from $\mathfrak{S}_{h\ell}$ (and *not* any of the parallel translates of these reflections). Now, any hyperplane labelled by such a reflection belongs to $\mathbb{E}^{\circledcirc}_{h,\ell}$ and so this path is dominant, as required. The result follows by Theorem 10.11. □

10.3. Nonparabolic affine behaviour. One of the features of the previous two sections (and the classical results that these sections generalize) is that they relate Weyl and Specht modules which are labelled by points which are 'close together' in the alcove geometry. We now consider a family of homomorphisms from alcoves which are 'as far apart as possible'. These points are related by permuting like-labelled columns between distinct components.

DEFINITION 10.25. We say that $\gamma \in \mathbb{E}^{\circledcirc}_{h,\ell}$ is an *affine vertex* if $\gamma^{(i)}_{t+1} < \gamma^{(j)}_{t-1}$ for all $1 \leqslant i, j \leqslant \ell\}$ for some fixed $1 \leqslant t \leqslant h$. We let $\mathfrak{S}_\ell$ denote the subgroup generated by the reflections

$$\langle s_{\varepsilon_{hi+t}-\varepsilon_{hj+t},me} \mid 1 \leqslant i < j \leqslant \ell, m \in \mathbb{Z}\rangle \tag{10.14}$$

and we set

$$\Gamma_t = \{\lambda \in \mathbb{E}^{\circledcirc}_{h,\ell} \mid \lambda \in \mathfrak{S}_\ell \cdot \gamma\}.$$

THEOREM 10.26. *Let $1 \leqslant t \leqslant h$. For $\lambda, \mu \in \Gamma_t$ such that $\lambda \uparrow \mu$, we have that*

$$\dim_t(\mathrm{Hom}_{A_h(n,\theta,\kappa)}(\Delta(\mu), \Delta(\lambda))) = t^{\ell(\mu)-\ell(\lambda)} + \cdots$$

where the remaining terms are all of strictly smaller degree.

Proof. For $h = t = 1$ the fact that Γ_t is generic is clear. For more general $h, t \in \mathbb{Z}_{>0}$, the result follows by translating the paths within the geometry. □

REMARK 10.27. For $\ell = 1$, we have that $\mathfrak{S}_\ell$ is the trivial group and these sets are clearly trivial. For any $\ell > 1$, we will *always* obtain many infinite chains (as $n \to \infty$) of homomorphisms of the form in Theorem 10.26.

EXAMPLE 10.28. We let $\kappa = (0, 3)$ and $e = 7$. Given $\gamma = ((3, 2, 1^9), (3^2, 2^{12})) \in \mathbb{E}^{\circledcirc}_{h,\ell}$, we have that

$$\Gamma_2 = \begin{Bmatrix} ((3, 2, 1^9), (3^2, 2^{12})), & ((3, 2^2, 1^8), (3^2, 2^{11}, 1)), \\ ((3, 2^9, 1), (3^2, 2^4, 1^8)), & ((3, 2^8, 1^2), (3^2, 2^5, 1^7)) \end{Bmatrix}.$$

We picture these bipartitions as in Figure 6; we depict the final three by α, β and δ, respectively. Notice that the third column (of any given component) is much shorter than the first column (of any given component).

These multipartitions belong to a single line in the 6-dimensional space $\mathbb{E}^{\circledcirc}_{3,2}$. Projecting down onto this line, we obtain the set of four points depicted in Figure 7. We have illustrated the direction of the two affine vertex homomorphisms with arrows. Notice that they correspond to a reflection through a wall of the alcove containing the origin.

Finally, we note that this example can easily be generalized to an infinite sequence (as $n \to \infty$) of homomorphisms whose piecewise composition is nonzero (by verifying the conditions of Corollary 10.12 for these paths). The first four homomorphisms of such a chain are depicted in Figure 8. This makes clear the fact that these homomorphisms are from points 'as far apart as possible' in the geometry.

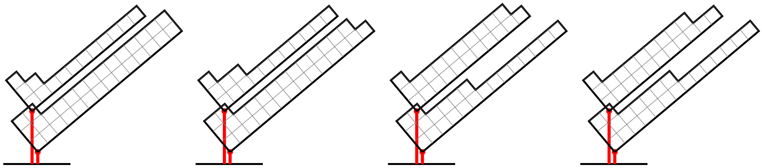


Figure 6. The set of bipartitions Γ_2 as in Example 10.28.

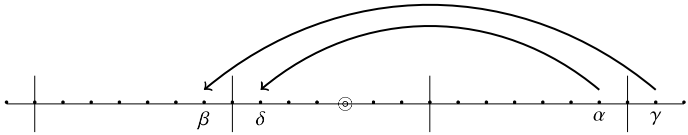


Figure 7. Some simple homomorphisms from affine vertices. This is the projection of $\mathbb{E}^{\circledcirc}_{3,2}$ onto the line $\mathbb{R}\{\varepsilon_2, \varepsilon_5\}/(\varepsilon_2 + \varepsilon_5)$.

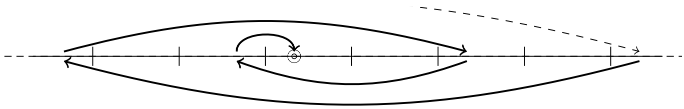


Figure 8. A simple infinite chain of homomorphisms arising from an affine vertex.

EXAMPLE 10.29. An example of affine vertices in a more complicated geometry is given in Figure 10.

REMARK 10.30. For the blob algebra, $\mathcal{Q}_{1,2,n}(\kappa)$, over $\mathbb{C}$, the homomorphisms of this subsection were first constructed in [**MW00**, (9.1) Theorem]. Moreover, these homomorphisms in conjunction with those of Remark 10.21 provide all homomorphisms between Weyl modules for the blob algebra over $\mathbb{C}$. Both families of homomorphisms continue to play an important role (and to be defined) over fields of positive characteristic [**CGM03**, Section 6].

11. Decomposition numbers over fields of large and infinite characteristic

In this section we shall give a combinatorial method for calculating certain parabolic affine Kazhdan–Lusztig polynomials in terms of the degrees of our paths in our alcove geometry. Over $\mathbb{C}$, this allows us to compute the decomposition matrix of these algebras (thus generalizing earlier work of [**BCS17**]). However, our main interest (as in the previous chapters) will be in what can be said over fields of positive characteristic. We begin by reviewing the necessary Kazhdan–Lusztig theory, in the spirit (and notation) of [Soe97].

Let $(\mathcal{W}, \mathcal{S})$ be a Coxeter system, and $\mathcal{S}_f \subset \mathcal{S}$. Then there is an associated parabolic subgroup $\mathcal{W}_f < \mathcal{W}$ generated by the set $\mathcal{S}_f$. Deodhar [Deo87] showed how to associate to the pair $(\mathcal{W}_f, \mathcal{W})$ certain parabolic Kazhdan–Lusztig polynomials. In particular, let $\mathcal{W}^f$ be a set of minimal length representatives of right cosets of $\mathcal{W}_f$ in $\mathcal{W}$ (with respect to the usual Coxeter length function). Then for any pair of elements $x, y \in \mathcal{W}^f$, there is an associated (inverse) parabolic Kazhdan–Lusztig polynomial $n_{x,y} \in \mathbb{Z}[t]$. We are interested in the special case where $\mathcal{W}$ is the affine Weyl group of type $\widehat{A}_{h\ell-1}$ (or type $A_{h\ell-1}$ if $e = \infty$) and the parabolic is the Weyl group of type

$$\underbrace{A_{h-1} \times A_{h-1} \times \cdots \times A_{h-1}}_{\ell \text{ copies}}.$$

As we have already seen, we can define a geometry associated with this choice of $\mathcal{W}$ on $\mathbb{E}_{h,\ell}$. We call the connected components of the complement of the union of the various reflecting hyperplanes the alcoves of this geometry. As described in [Soe97, Section 4] there is a natural bijection between $\mathcal{W}$ and the set of alcoves $\mathcal{A}$.

Soergel goes on to consider a certain set $\mathcal{A}^+ \subset \mathcal{A}$ which is precisely the set of alcoves lying in some fixed choice of fundamental domain for the finite Weyl group of type $A_{h\ell-1}$ inside $\widehat{A}_{h\ell-1}$. Then there is an induced bijection from the set of right cosets of this finite Weyl group inside the affine Weyl group to the alcoves in $\mathcal{A}^+$. However, we can instead consider the finite parabolic $\mathcal{W}_f$ inside $\mathcal{W}$ as defining our choice of fundamental domain $\mathcal{A}^+$, and then identify $\mathcal{W}^f$ with $\mathcal{A}^+$. This choice of $\mathcal{A}^+$ consists precisely of those alcoves contained in the dominant Weyl chamber $\mathbb{E}^{\circledcirc}_{h,\ell}$ defined in Section 6.2.

Choosing a fixed weight λ in an alcove, there is for each alcove in $\mathcal{A}$ an associated weight $w \cdot \lambda$ in that alcove, and in this way we can identify pairs of alcoves with pairs of weights in a given $\mathcal{W}$-orbit. Via these various identifications we can now define for any pair of weights λ and μ in the same $\mathcal{W}$-orbit a parabolic Kazhdan–Lusztig polynomial $n_{\lambda\mu} \in \mathbb{Z}[t]$.

In [GW01], Goodman and Wenzl have shown how to associate a parabolic Kazhdan–Lusztig polynomial to any pair of dominant weights, not just those lying in the interior of an alcove. Further, they give an algorithm for determining these polynomials in terms of certain piecewise linear paths in the geometry. As in [Soe97], their results are all stated for the case of a finite Weyl group considered as a parabolic in the associated affine Weyl group, but by making the same modifications as described above it is straightforward to see that this procedure extends to pairs of weights in the dominant Weyl chamber $\mathbb{E}^{\circledcirc}_{h,\ell}$ for our choice of $\mathcal{W}$ and $\mathcal{W}_f$.

We can now re-express the main result from [GW01] in terms of the paths which we have defined in Section 6.3.

THEOREM 11.1. *Let* $\lambda, \mu \in \mathbb{E}_{h,\ell}^{\circledcirc}$ *with* $\lambda \in \mathcal{W} \cdot \mu$ *and let* t^{μ} *be any admissible path of degree* n *from* $\circledcirc$ *to* μ. *The associated polynomial*

$$N_{\lambda\mathsf{t}^{\mu}} = \sum_{\mathsf{s} \in \mathrm{Path}_n^+(\lambda, \mathsf{t}^{\mu})} t^{\deg(\mathsf{s})} \tag{11.1}$$

is an element of $\mathbb{Z}_{\geqslant 0}[t, t^{-1}]$. *We can rewrite the polynomials* $N_{\lambda\mathsf{t}^{\mu}}$ *in the form*

$$N_{\lambda\mathsf{t}^{\mu}} = \sum_{\lambda \uparrow \nu \uparrow \mu} n_{\lambda\nu} \underline{N}_{\nu\mu}$$

for some $n_{\lambda\nu} \in \mathbb{Z}_{\geqslant 0}[t]$, *and* $\underline{N}_{\nu\mu} \in \mathbb{Z}_{\geqslant 0}[t + t^{-1}]$; *this expression is unique. Further, we have* $n_{\lambda\nu}$ *is the (inverse) parabolic affine Kazhdan–Lusztig polynomial associated to the pair* $(\mathcal{W}_f, \mathcal{W})$.

Proof. All that remains to be verified is that our paths are examples of the piecewise linear paths considered in [GW01], and that the associated degree functions for the two definitions coincide. That our paths are examples of their piecewise linear paths is obvious.

It remains to show that [GW01, Equation (2.4)] agrees with Definition 6.13. Recall from Definition 6.13, that the degree of a path $\mathsf{s}(k)$ is obtained from that of the path $\mathsf{s}(k-1)$ by counting hyperplanes with alternating signs; it is simple to see rephrase this alternating sign in terms of the length function used in [GW01]. Setting Λ in [GW01, Equation (2.4)] to be equal to the segment $(\mathsf{s}(k-1), \mathsf{s}(k))$ in the path s, we obtain the desired equality. □

11.1. Decomposition numbers over the complex field. We now use the results above to determine the decomposition numbers for our algebras in characteristic zero.

PROPOSITION 11.2. *If* $R = \mathbb{C}$, *then:*

(i) $\dim_t(\Delta_\mu(\lambda)) = N_{\lambda\mathsf{t}^{\mu}} \in \mathbb{Z}_{\geqslant 0}[t, t^{-1}]$ *and* $\dim_t(L_\mu(\lambda)) \in \mathbb{Z}_{\geqslant 0}[t + t^{-1}]$;

(ii) *if* $\dim_t(\Delta_\mu(\lambda)) = 0$, *then* $d_{\lambda\mu}(t) = 0$;

(iii) *we have* $\dim_t(\Delta_\mu(\mu)) = \dim_t(L_\mu(\mu)) = 1$;

(iv) *if* $\mathrm{Path}(\lambda, \mathsf{t}^{\mu}) = \emptyset$, *then* $\dim_t(\Delta_\mu(\lambda)) = 0$;

(v) *if* $\mathrm{Path}(\lambda, \mathfrak{t}^\mu) = \emptyset$, *then* $\dim_t(L_\mu(\lambda)) = 0$;

(vi) *we have that*

$$\dim_t(\Delta_\mu(\lambda)) = \sum_{\substack{\nu \neq \mu \\ \lambda \uparrow \nu \uparrow \mu}} d_{\lambda\nu}(t) \dim_t(L_\mu(\nu)) + d_{\lambda\mu}(t).$$

Proof. Part (i) is clear by Proposition 2.2 and (11.1). Part (iii) is a restatement of the condition that $\mathfrak{t}^\mu$ is the only path in $\mathrm{Path}(\mu, \mathfrak{t}^\mu)$. A necessary condition for $\dim_t(\mathrm{Hom}(P(\mu), \Delta(\lambda)) \neq 0$ is that $\Delta_\mu(\lambda) \neq 0$, therefore (ii) follows. Part (iv) is by definition, and part (v) follows from the cellular structure. Finally, (vi) follows from (i), (iii), (v) and Theorem 4.7. □

THEOREM 11.3. *Let $R = \mathbb{C}$. The graded decomposition numbers of $A_h(n, \theta, \kappa)$ are given by*

$$d_{\lambda\mu}(t) = n_{\lambda\mu}.$$

Proof. By Proposition 11.2(ii), we may assume $\mathrm{Path}(\lambda, \mathfrak{t}^\mu) \neq \emptyset$. We now calculate $d_{\lambda\mu}(t)$ and $\dim_t(L_\mu(\lambda))$ by induction on the length ordering. Induction begins when $\ell(\mu) - \ell(\lambda) = 0$, hence $\mu = \lambda$, and we have $d_{\mu\mu}(t) = 1$ by Proposition 11.2(iii) and $\dim_t(L_\mu(\mu)) = 1$.

Let $\ell(\mu) - \ell(\lambda) \geqslant 1$. By induction on the length ordering, we know $d_{\lambda\nu}(t)$ and $\dim_t(L_\mu(\nu))$ for points $\nu \in \Lambda_n$ such that $\lambda \uparrow \nu \uparrow \mu$. By Proposition 11.2(vi) we have

$$\dim_t(L_\mu(\lambda)) + d_{\lambda\mu}(t) = \dim_t(\Delta_\mu(\lambda)) - \sum_{\substack{\nu \neq \mu,\, \nu \neq \lambda \\ \lambda \uparrow \nu \uparrow \mu}} d_{\lambda\nu}(t) \dim_t(L_\mu(\nu)).$$

By induction and Proposition 11.2(i), we have that

$$\dim_t(L_\mu(\lambda)) + d_{\lambda\mu}(t) = N_{\lambda\mathfrak{t}^\mu} - \sum_{\substack{\nu \neq \mu,\, \nu \neq \lambda \\ \lambda \uparrow \nu \uparrow \mu}} n_{\lambda\nu} \underline{N}_{\nu\mu}.$$

Recall that $\underline{N}_{\nu\mu} = \dim_t(L_\mu(\nu)) \in \mathbb{Z}_{\geqslant 0}[t + t^{-1}]$ and $n_{\lambda\nu} = d_{\lambda\mu}(t) \in t\mathbb{Z}_{\geqslant 0}[t]$. Therefore there is a unique solution to the equality (see [**KN10**, Section 4.1: Basic Algorithm]). By Theorem 11.1 this solution is given by $d_{\lambda\mu}(t) = n_{\lambda\mu}$. □

11.2. Decomposition numbers over fields of large characteristic. As we have made clear throughout the paper, our principal interest is in fields of positive characteristic. Indeed, we informally defined our algebra $\mathcal{Q}_{\ell,h,n}(\kappa)$ as the *largest quotient of $H_n(\kappa)$ for which a generalized Lusztig conjecture could possibly hold* for fields of positive characteristic.

DEFINITION 11.4. We say that $\lambda \in \mathbb{E}^+_{h,\ell}$ belongs to *the first ep-alcove* if $\ell_\alpha(\lambda) < p$ for all $\alpha \in \Phi$.

CONJECTURE 11.5. *Let $e > h\ell$, $\kappa \in I^\ell$ be an h-admissible multicharge, and $\Bbbk$ be field of characteristic $p \gg h\ell$. The decomposition numbers of $A_h(n, \theta, \kappa)$ are given by*

$$d_{\lambda\mu}(t) = n_{\lambda\mu}(t)$$

for $\lambda, \mu \in \mathscr{P}^\ell_n(h)$ in the first ep-alcove and $n_{\lambda\mu}(t)$ the associated affine parabolic Kazhdan–Lusztig polynomial of type $A_{h-1} \times A_{h-1} \times \cdots \times A_{h-1} \subseteq \widehat{A}_{\ell h-1}$.

For $e = \infty$ we make a stronger conjecture concerning the algebras $A(n, \theta, \kappa)$ in which the statement of Conjecture 11.5 simplifies in two ways. The simpler geometry controlling the $e = \infty$ case means that there is no notion of a 'first *ep*-alcove'. Second, the subalgebra isomorphic to the Hecke algebra of the symmetric group is semisimple; therefore we need not restrict the characteristic of the field according to the number of columns of $\lambda, \mu \in \mathscr{P}^\ell_n$.

CONJECTURE 11.6. *Let $e > n$ and $\Bbbk$ be field of characteristic $p \gg \ell$. Suppose that $\kappa \in I^\ell$ has no repeated entries. The decomposition numbers of $A(n, \theta, \kappa)$ are given by*

$$d^{A(n,\theta,\kappa)}_{\lambda\mu}(t) = n_{\lambda\mu}(t)$$

for $\lambda, \mu \in \mathscr{P}^\ell_n$ where $n_{\lambda\mu}(t)$ is the associated Kazhdan–Lusztig polynomials of type $A_{n-1} \times A_{n-1} \times \cdots \times A_{n-1} \subseteq A_{n\ell-1}$.

REMARK 11.7. Notice that the Kazhdan–Lusztig polynomials in the latter conjecture are nonaffine (see Remark 6.23) and that the parabolic is determined by products of A_{n-1} (rather than A_{h-1}) and that it covers *all* decomposition numbers of the diagrammatic Cherednik algebra. Finally, recall that our assumption on $\kappa \in I^\ell$ is less strict than in Conjecture 11.6 is weaker than in Conjecture 11.5.

REMARK 11.8. For $e = \infty$, the unique graded decomposition matrix of $H_n(\kappa)$ appears as the (in general proper) submatrix of that of $A(n, \theta, \kappa)$ whose columns are labelled by so-called FLOTW multipartitions.

Conjecture 11.5 is the first conjectural attempt to describe an infinite family of decomposition numbers of (quiver) Hecke algebras for $e \in \mathbb{Z}_{>0}$. A more optimistic version of Conjecture 11.6 appeared in [**KR11**, Conjecture 7.3] (without the assumptions on $\kappa \in I^\ell$ or $p \gg \ell$) but was later disproven in [**Wil14**, Section 4.2]. This counterexample is for $\ell = 5$ and $p < \ell$ and does not seem surprising from the point of view of this paper (indeed we posed Conjecture 11.6 in its current form before being informed about the results of [**KR11**, **Wil14**] by Liron Speyer).

EXAMPLE 11.9. In Theorem 10.19, we saw that the above conjecture holds for any λ, μ in a maximal finite parabolic orbit with $\Bbbk$ arbitrary.

EXAMPLE 11.10. Let $\ell = 2$ and $h = 1$ and $e > 2$ and $\Bbbk$ be arbitrary. In this case $\mathcal{Q}_{h,\ell,n}(\kappa)$ is isomorphic to the blob algebra. The conjecture follows from [**CGM03**, Section 8].

EXAMPLE 11.11. If $\ell = 1$, then $\mathcal{Q}_{h,\ell,n}(\kappa)$ is isomorphic to the generalized Temperley–Lieb algebra of Härterich [**H99**, Section 1]. The result therefore follows by [**RW16**, Theorem 1.9].

EXAMPLE 11.12. If $\ell = 2$ and $e > n$, then $A(n, \theta, \kappa)$ is positively graded. Therefore any simple module is 1-dimensional. Therefore the decomposition numbers of $A_h(n, \theta, \kappa)$ are independent of the characteristic of the field and the result follows. An identical proof of this is given in [**HM15**, Appendix B] (in fact, the quiver Schur algebra of Hu and Mathas is isomorphic to our algebra $A(n, \theta, \kappa)$ in this case) and an earlier proof of this result for $e = \infty$ was given in [**BS11**].

EXAMPLE 11.13. Consider $A(n, \theta, \kappa)$ with $\ell = 3$ and $e = \infty$. This algebra is nonnegatively graded. For any $\lambda, \mu \in \mathscr{P}_n^3$ there is at most one path, s say, of degree zero in $\mathrm{Path}(\lambda, \mathsf{t}^\mu)$. For example if $\kappa = (0, 1, 2)$ then $\mathrm{Path}((2), \varnothing, (2)), (1), (1), (2))$ has two elements; one of degree 0 and one of degree 2. Therefore it suffices to check that $C_\mathsf{s}^* C_\mathsf{s} = C_{\mathsf{t}_\lambda^\mu}$ in order to conclude that $L(\lambda)$ is 2-dimensional with basis $\{C_\mathsf{s}, C_{\mathsf{t}^\lambda}\}$. Having done so, one can conclude that the decomposition numbers are independent of the characteristic of the field. We leave this as an exercise for the reader.

12. Alcove geometries in $\mathbb{R}^2$

We now discuss the algebras $\mathcal{Q}_{\ell,h,n}(\kappa)$ which are controlled by geometries which can be visualized within the plane $\mathbb{R}^2$. We encounter all the usual Lie theoretic geometries, as well as new geometries which do not arise in the representation theory of (affine) Lie algebras and related objects.

There are a total of five distinct affine geometries that we can picture via an embedding into $\mathbb{R}^2$.

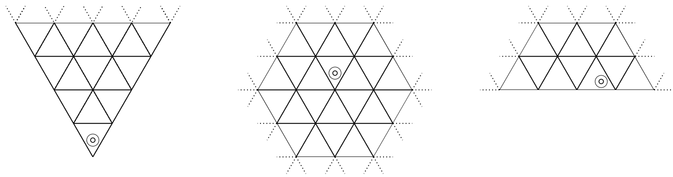


Figure 9. Geometries of type $A_2 \subseteq \widehat{A}_2$, $\widehat{A}_2$, and $A_1 \subseteq \widehat{A}_2$, respectively. These tile one sixth of $\mathbb{R}^2$, the whole of $\mathbb{R}^2$, and one half of $\mathbb{R}^2$, respectively.

The first two of these are of type $A_1 \subseteq \widehat{A}_1$ for $h = 2$ and $\ell = 1$ and of type $\widehat{A}_1$ for $\ell = 2$ and $h = 1$. In the former (respectively latter) case, we visualize $\mathbb{E}^{\circledcirc}_{h,\ell}$ as half of the real line (respectively the whole real line). These geometries already appear in classical Lie theory. The former (respectively latter) controls the representation theory of the Lie algebra $\mathfrak{sl}_2$ (respectively the Kac–Moody algebra $\widehat{\mathfrak{sl}}_2$).

There are three further affine geometries which can be visualized in $\mathbb{R}^2$ (depicted in Figure 9).

The first is that of type $A_2 \subseteq \widehat{A}_2$ which controls the representation theory of $\mathcal{Q}_{1,3,n}(\kappa)$. The algebra $\mathcal{Q}_{1,3,n}(\kappa)$ is the Ringel dual of the image of $U_q(\mathfrak{sl}_3)$ in $\mathrm{End}((\Bbbk^3)^{\otimes n})$. Obviously, this is the same as the geometry controls the representation theory of the Lie algebra $\mathfrak{sl}_3$. The dominant region $\mathbb{E}^{\circledcirc}_{h,\ell}$ is one sixth of the plane $\mathbb{R}^2$.

The second is that of type $\widehat{A}_2$ which controls the representation theory of $\mathcal{Q}_{3,1,n}(\kappa)$. This is the same as the geometry controls the representation theory of the Kac–Moody algebra $\widehat{\mathfrak{sl}}_3$. The dominant region $\mathbb{E}^{\circledcirc}_{h,\ell}$ is the entirety of the plane $\mathbb{R}^2$. Third, we encounter a geometry of type $A_1 \subseteq \widehat{A}_2$ which controls a portion of the representation theory of $\mathcal{Q}_{2,2,n}(\kappa)$; namely the subcategory of representations whose simple constituents are labelled by points in the space

$$\mathbb{E}^{\circledcirc}_{h,\ell} \cap \mathbb{R}_{\geqslant 0}\{\varepsilon_1, \varepsilon_3, \varepsilon_4\} \quad \text{or} \quad \mathbb{E}^{\circledcirc}_{h,\ell} \cap \mathbb{R}_{\geqslant 0}\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}.$$

The corresponding quotient algebra can be explicitly constructed as in Definition 5.2, but this is unnecessary as all the results are identical. This geometry does not appear in classical Lie theory. In either case, the dominant region is one half of the plane $\mathbb{R}^2$.

We now consider an example in this new geometry (of type $A_1 \subseteq \widehat{A}_2$) which illustrates both the local generic behaviour of Section 10.2 and the nonlocal generic behaviour of Section 10.3.

EXAMPLE 12.1. We let $e = 5$, $h = \ell = 2$, $n = 13$, and $\kappa = (0, 2) \in (\mathbb{Z}/5\mathbb{Z})^2$. We let $\Bbbk$ be an arbitrary field. Thus we are considering a chunk of the modular representation theory of the Hecke algebra of type B. We consider the linkage class containing the element $\alpha = ((1^{11}), (1^2))$. We consider the closed subset

$$\{\xi = ((\xi_1^{(1)}, \xi_2^{(1)}), (\xi_1^{(2)}, \xi_2^{(2)})) \in \mathscr{P}_{13}^2(2) \mid \xi \trianglerighteq \alpha \text{ and } \xi_2^{(1)} = 0\} \subset \mathscr{P}_{13}^2(2)$$

which can be embedded in $\mathbb{E}_{2,2}^{\circledcirc} \cap \mathbb{Z}\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ as depicted in Figure 10. This subalgebra of $A_2(13, \theta, (0, 2))$ has 15 simple modules. The decomposition matrix of this algebra (which appears as a submatrix of $A_2(13, \theta, (0, 2))$) is as follows:

$$\begin{array}{c|cccccc|cccccc|ccc}
 & \alpha & \beta & \beta' & \gamma & \gamma' & \delta & \lambda & \mu & \mu' & \nu & \nu' & \pi & \rho & \tau & \sigma \\
\hline
\alpha & & & & & & & & & & & & & & & \\
\beta & & & & & & & & & & & & & & & \\
\beta' & & & & M & & & & & & & & & & & \\
\gamma & & & & & & & & & & & & & & & \\
\gamma' & & & & & & & & & & & & & & & \\
\delta & & & & & & & & & & & & & & & \\
\hline
\lambda & & & & & & & & & & & & & & & \\
\mu & & & & & & & & & & & & & & & \\
\mu' & & & & t^1 M & & & & & & M & & & & & \\
\nu & & & & & & & & & & & & & & & \\
\nu' & & & & & & & & & & & & & & & \\
\pi & & & & & & & & & & & & & & & \\
\hline
\rho & 0 & t^2 & t^2 & t & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 1 & & \\
\tau & t^2 & t^3 & t & t^2 & 0 & t & 0 & 0 & t^2 & 0 & t & 0 & t & 1 & \\
\sigma & t^3 & 0 & t^2 & t^3 & t & t^2 & 0 & 0 & 0 & 0 & t^2 & t & 0 & t & 1
\end{array}$$

where the matrix M records the (nonparabolic) Kazhdan–Lusztig polynomials of type A_2. This is given as follows,

$$M = \begin{pmatrix} 1 & & & & & \\ t & 1 & & & & \\ t & 0 & 1 & & & \\ t^2 & t & t & 1 & & \\ t^2 & t & t & 0 & 1 & \\ t^3 & t^2 & t^2 & t & t & 1 \end{pmatrix}.$$

All the zero entries in the above can be deduced via Theorem 9.1.

The set of points $\{\alpha, \beta, \beta', \gamma, \gamma', \delta, \lambda, \mu, \mu', \nu, \nu', \tau\}$ is a generic set. One can deduce that M appears twice along the diagonal using the 'Steinberg-like' results in Section 10.2. That $t^1 M$ appears as a submatrix follows by combining the 'maximal nonparabolic' results of Section 10.3 (which gives the diagonal entries of $t^1 M$) with the 'Steinberg-like' results in Section 10.2 (which give the off-diagonal entries).

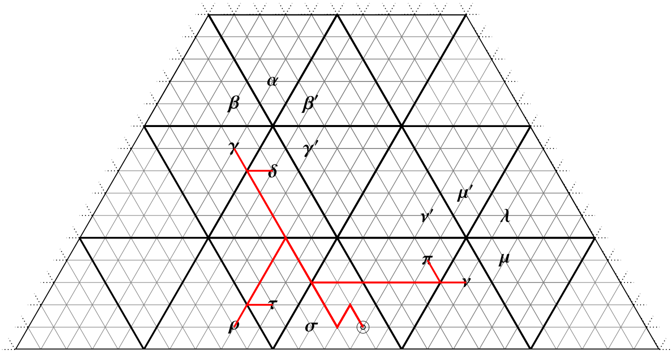


Figure 10. The six elements of $\mathrm{Path}^+(-, \mathsf{t}^\gamma)$ in the geometry of type $A_1 \subseteq \widehat{A}_2$. Each of the pairs (α, λ), (β, μ), (γ, ν), and (δ, π) are related by an affine vertex reflection (as are their primed versions) and so these pairs are generic sets as in Section 10.3. The pairs (μ', ρ), (ν', τ), (π, σ) are all generic sets as in Section 10.3.

We now consider the final three rows of the decomposition matrix. The nonzero entries in the final nine columns can all be calculated using the generic results of previous sections. In particular, the final three columns are given by the Kazhdan–Lusztig polynomials of type $A_2 \subseteq \widehat{A}_2$.

Finally, we consider the decomposition numbers in the final three rows intersected with the first six columns. These are the most interesting decomposition numbers in our matrix as they cannot be calculated using the generic results or Theorem 9.1. These can be calculated using the same considerations as in Example 10.7.

REMARK 12.2. Note that the generic results only give lower bounds on decomposition numbers in M and $t^1 M$ in Example 12.1. One must also verify that the paths of degree zero $\mathsf{s} \sim \mathsf{t}^\alpha$ and $\mathsf{t} \sim \mathsf{t}^\lambda$ do correspond to weight spaces in certain simple modules.

Acknowledgements

The authors would like to thank the referee for their helpful comments. They are also grateful for the financial support received from the Royal Commission for the Exhibition of 1851 and EPSRC grant EP/L01078X/1.

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