THE CELLULAR SECOND FUNDAMENTAL THEOREM OF INVARIANT THEORY FOR CLASSICAL GROUPS

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Abstract. We construct explicit integral bases for the kernels and the images of diagram algebras (including the symmetric groups, orthogonal and symplectic Brauer algebras) acting on tensor space. We do this by providing an axiomatic framework for studying quotients of diagram algebras.

Contents

Introduction

Schur–Weyl duality relates the classical matrix groups $\mathsf{GL}(V)$, $\mathsf{SL}(V)$, $\mathsf{O}(V)$, or $\mathsf{Sp}(V)$, where V is a finite dimensional vector space, with certain quotients of diagram algebras – symmetric group algebras, Brauer algebras or walled Brauer algebras – via mutually centralizing actions on tensor space. The surjectivity of the map from the diagram algebra to the centralizer algebra of the matrix group is equivalent to the first fundamental theorem of invariant theory. Any effective description of the kernel of the map is a form of the second fundamental theorem (SFT) of invariant theory.

This paper studies the centralizer algebras and the second fundamental theorem from the point of view of cellularity [19]. We construct integral cellular bases for the centralizer algebras, and simultaneously bases of the kernel of the map from the diagram algebras to the centralizer algebras.

There are two remarkable cellular bases of the Iwahori Hecke algebras of finite type A – the Kazhdan–Lusztig bases [27, 19] and the Murphy bases [34]. Each has its own merits. The Kazhdan–Lusztig bases encode a great deal of representation theory and have a deep relation to geometry. The Murphy bases are simpler and more explicit; they encode the restriction of cell-modules along the tower of Hecke algebras; they are related to the seminormal bases by a dominance triangular transformation and consequently the Jucys–Murphy elements act on the Murphy bases by dominance triangular matrices. Relationships between the two types of bases are investigated in [15]. As evidence of the enduring utility of the Murphy bases, we mention that they were used in [22] to construct graded cellular bases of the Hecke algebras.

The Kazhdan–Lusztig bases have been generalized in [40] to the Brauer centralizer algebras and to many other examples using the theory of dual canonical bases of quantum groups from [30]. In this paper we concentrate on generalizing the Murphy bases. In previous work [11], we have already generalized the Murphy bases to the Brauer diagram algebras (and to other diagram algebras related to the Jones basic construction). In this paper we extend this analysis to encompass centralizer algebras for the classical groups. The bases we obtain, for the diagram algebras and for the centralizer algebras, share all the properties of original Murphy bases mentioned above.

We are using the phrase "centralizer algebra" as a shorthand for the image of the diagram algebra (for example, the Brauer diagram algebra) acting on tensor space, over an arbitrary field or over the integers. In fact, these algebras are generically centralizer algebras in the classical sense, see Theorems 7.1 and 8.1.

In order to produce Murphy bases of centralizer algebras, we first develop a quotient construction for cellularity of towers of diagram algebras. We then apply this construction to the integral versions of the Brauer algebras acting on orthogonal or symplectic tensor space. The construction involves modifying the Murphy type basis of the tower of diagram algebras constructed following [11] in such a way that the modified basis splits into a basis of the kernel of the map Φ from the diagram algebra to endomorphisms of tensor space, and a subset which maps onto a cellular basis of the image of Φ.[1] This construction thus provides simultaneously an integral cellular basis of the centralizer algebra, and a version of the SFT, namely an explicit description of the kernel of Φ. Moreover, it is evident from the construction that $\ker(\Phi)$ is generated as an ideal by certain "diagrammatic minors" or "diagrammatic Pfaffians", so we also recover the version of the SFT from [13]. The combinatorics underlying our construction is the same as that in [38], namely the cellular basis of the centralizer algebra is indexed by pairs of "permissible paths" on the generic branching diagram for the tower of diagram algebras. The cell modules of the integral centralizer algebras are in general proper quotients of certain cell modules of the integral diagram algebra.

All of these results are compatible with reduction from $\mathbb{Z}$ to a field of arbitrary characteristic (except that characteristic 2 is excluded in the orthogonal case). For a symplectic or orthogonal bilinear form on a finite dimensional vector space V over a field $\Bbbk$, and for Φ the corresponding map from the Brauer diagram algebra to $\mathrm{End}(V^{\otimes r})$, our bases of $\mathrm{im}(\Phi)$ and of $\ker(\Phi)$ are independent of the field, of the characteristic, and of the choice of the bilinear form. They depend only on the dimension of V. It follows from our results that for a fixed field $\Bbbk$ and fixed $\dim(V)$, and for fixed symmetry type of the form (symplectic or orthogonal) the Brauer

[1] Integral bases of the Brauer diagram algebras with a similar splitting property were constructed in [9, 21].

centralizer algebra im(Φ) is independent, up to isomorphism, of the choice of the form. For example, if the field is the real numbers, and the form is symmetric, the Brauer centralizer algebra is independent, up to isomorphism, of the signature of the form.

We also explain in our context the well-known phenomenon that the seminormal representations of centralizer algebras of the classical groups are truncations of the seminormal representations of the corresponding diagram algebras.

We wish to remark upon our emphasis on working over the integers. As noted in [14], cellularity "provides a systematic framework for studying the representation theory of non-semisimple algebras which are deformations of semisimple ones." Typically, a "cellular algebra" A is actually a family of algebras A_S defined over various ground rings S, and typically there is a generic ground ring R such that: each instance A_S of A is a specialization of A_R, i.e. $A_S = A_R \otimes_R S$; with $\mathbb{F}$ the field of fractions of R, $A_{\mathbb{F}}$ is semisimple; and if k is any field, the cell modules and cellular basis of A_k are obtained by specializing those of A_R, and the simple A_k modules appear as heads of (some of) the cell modules. This point of view was not stressed in the original papers [19, 34], but is a sort of folk wisdom. In our applications, the integers are the generic ground ring for the centralizer algebras; it is not altogether obvious, but it follows from our results that the centralizer algebras over fields of prime characteristic are specializations of the integral versions; see Sections 7 and 8 for precise statements.

In the orthogonal and symplectic cases, our bases are new. In the general linear case, our result is equivalent to [20] for tensor space and [37, 39] for mixed-tensor space, respectively. A completely different and very general approach to proving the existence of abstract cellular bases of centralizer algebras of quantum groups over a field has been developed in [1].

Our method should apply to other examples as well. The case of the BMW algebra acting on symplectic tensor space should be straightforward, using the q–analogue of the diagrammatic Pfaffians obtained in [23]. The case of the BMW algebra acting on orthogonal tensor space could be more challenging as the appropriate q–analogues of the diagrammatic minors are not yet available.

Outline. The paper is structured as follows. In Section 1 we recall the necessary background material on diagram algebras and their branching graphs; this is taken from [4, 11, 17, 18, 19]. In Section 2, we introduce an axiomatic framework for cellularity of a sequence of quotients of a sequence of diagram algebras. This culminates in Theorem 2.7, which contains the main result on cellular bases of quotient algebras as well as an abstract "second fundamental theorem" — that is, a description of the kernel of the quotient map.

In Section 4 we treat the Murphy basis of the symmetric group algebras, and a dual version, twisted by the automorphism $s_i \mapsto -s_i$ of the symmetric group. Section 5 treats the Murphy and dual Murphy bases of Brauer algebras. Finally, in Sections 7 and 8 we apply our abstract theory to the main examples of interest in this paper, namely to the Brauer algebra acting on symplectic or orthogonal tensor space.

There are four appendices in the arXiv version of this paper. In Appendix A we review results of Härterich [20] regarding the action of the symmetric group and the Hecke algebra on ordinary tensor space. In Appendices B and C we construct Murpy bases of the walled Brauer algebras and of their quotients acting on mixed tensor space, following the techniques used in the main text. In Appendix D we review results on diagrammatic minors and Pfaffians which are needed for our treatment of the SFT.

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1. Diagram algebras

For the remainder of the paper, we shall let R be an integral domain with field of fractions $\mathbb{F}$. In this section, we shall define diagram algebras and recall the construction of their Murphy bases in terms of "up" and "down" branching factors, following [11]. As in [4], we emphasize crucial factorization and compatibility relations between the "up" and "down" branching factors.

1.1. **Cellular algebras.** We first recall the definition of a cellular algebra, as in [19].

Definition 1.1. Let R be an integral domain and let A be a unital algebra over R. A cell datum for A is a tuple $(A, *, \widehat{A}, \trianglerighteq, \mathrm{Std}(\cdot), \mathscr{A})$ where:

(1) $* : A \to A$ is an algebra involution, that is, an R–linear anti–automorphism of A such that $(x^*)^* = x$ for $x \in A$.
(2) $(\widehat{A}, \trianglerighteq)$ is a finite partially ordered set, and for each $\lambda \in \widehat{A}$, $\mathrm{Std}(\lambda)$ is a finite indexing set.
(3) The set

$$\mathscr{A} = \left\{ c^{\lambda}_{\mathsf{st}} \;\middle|\; \lambda \in \widehat{A} \text{ and } \mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda) \right\},$$

is an R–basis for A.

Let $A^{\rhd\lambda}$ denote the R–module with basis

$$\left\{ c^{\mu}_{\mathsf{st}} \;\mid \mu \rhd \lambda \text{ and } \mathsf{s}, \mathsf{t} \in \mathrm{Std}(\mu) \right\}.$$

(4) The following two conditions hold for the basis $\mathscr{A}$.
(a) Given $\lambda \in \widehat{A}$, $\mathsf{t} \in \mathrm{Std}(\lambda)$, and $a \in A$, there exist coefficients $r(a; \mathsf{t}, \mathsf{v}) \in R$, for $\mathsf{v} \in \mathrm{Std}(\lambda)$, such that, for all $\mathsf{s} \in \mathrm{Std}(\lambda)$,

$$c^{\lambda}_{\mathsf{st}} a \equiv \sum_{\mathsf{v} \in \mathrm{Std}(\lambda)} r(a; \mathsf{t}, \mathsf{v}) c^{\lambda}_{\mathsf{sv}} \mod A^{\rhd\lambda}, \tag{1.1}$$

(b) If $\lambda \in \widehat{A}$ and $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda)$, then $(c^{\lambda}_{\mathsf{st}})^* \equiv (c^{\lambda}_{\mathsf{ts}}) \mod A^{\rhd\lambda}$.

A is called a cellular algebra if it has a cell datum. The basis $\mathscr{A}$ is called a cellular basis of A.

If A is a cellular algebra over R, and $R \to S$ is a homomorphism of integral domains, then the specialization $A^S = A \otimes_R S$ is a cellular algebra over S, with cellular basis

$$\mathscr{A}^S = \{ c^{\lambda}_{\mathsf{st}} \otimes 1_S \mid \lambda \in \widehat{A}, \text{ and } \mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda) \}.$$

In particular, $A^{\mathbb{F}}$ is a cellular algebra. Since the map $a \mapsto a \otimes 1_F$ is injective, we regard A as contained in $A^{\mathbb{F}}$ and we identify $a \in A$ with $a \otimes 1_F \in A^{\mathbb{F}}$.

An order ideal $\Gamma \subset \widehat{A}$ is a subset with the property that if $\lambda \in \Gamma$ and $\mu \trianglerighteq \lambda$, then $\mu \in \Gamma$. It follows from the axioms of a cellular algebra that for any order ideal Γ in $\widehat{A}$,

$$A^{\Gamma} = \mathrm{Span} \left\{ c^{\lambda}_{\mathsf{st}} \;\middle|\; \lambda \in \Gamma \text{ and } \mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda) \right\}$$

is an involution–invariant two sided ideal of A. In particular $A^{\rhd\lambda}$, defined above, and

$$A^{\trianglerighteq\lambda} = \mathrm{Span} \left\{ c^{\mu}_{\mathsf{st}} \;\middle|\; \mu \trianglerighteq \lambda \text{ and } \mathsf{s}, \mathsf{t} \in \mathrm{Std}(\mu) \; \right\}$$

are involution–invariant two sided ideals.

Definition 1.2. Let A be a cellular algebra over R and $\lambda \in \hat{A}$. The cell module $\Delta(\lambda)$ is the right A–module defined as follows. As an R–module, $\Delta(\lambda)$ is free with basis indexed by $\mathrm{Std}(\lambda)$, say $\{c^{\lambda}_{\mathsf{t}} \mid \mathsf{t} \in \mathrm{Std}(\lambda)\}$. The right A–action is given by

$$c^{\lambda}_{\mathsf{t}} a = \sum_{\mathsf{v} \in \hat{A}^{\lambda}} r(a; \mathsf{t}, \mathsf{v}) c^{\lambda}_{\mathsf{v}},$$

where the coefficients $r(a; \mathsf{t}, \mathsf{v})$ are those of Equation (1.1).

Thus, for any $\mathsf{s} \in \mathrm{Std}(\lambda)$, a model for the cell module $\Delta(\lambda)$ is given by

$$\mathrm{Span}\{c^\lambda_{\mathsf{st}} + A^{\rhd\lambda} \mid \mathsf{t} \in \mathrm{Std}(\lambda)\} \subseteq A^{\trianglerighteq\lambda}/A^{\rhd\lambda}.$$

When we need to emphasize the algebra or the ground ring, we may write $\Delta_A(\lambda)$ or $\Delta^R(\lambda)$. Note that whenever $R \to S$ is a homomorphism of integral domains, $\Delta^S(\lambda) = \Delta(\lambda) \otimes_R S$ is the cell module for A^S corresponding to λ.

If A is an R–algebra with involution $*$, then $*$ induces functors $M \to M^*$ interchanging left and right A–modules, and taking A–A bimodules to A–A bimodules. We identify M^{**} with M via $x^{**} \mapsto x$ and for modules ${}_AM$ and N_A we have $(M \otimes_R N)^* \cong N^* \otimes_R M^*$, as A–A bimodules, with the isomorphism determined by $(m \otimes n)^* \mapsto n^* \otimes m^*$. For a right A–module M_A, using both of these isomorphisms, we identify $(M^* \otimes M)^*$ with $M^* \otimes M^{**} = M^* \otimes M$, via $(x^* \otimes y)^* \mapsto y^* \otimes x$. Now we apply these observations with A a cellular algebra and $\Delta(\lambda)$ a cell module. The assignment

$$\alpha_\lambda : c^\lambda_{\mathsf{st}} + A^{\rhd\lambda} \mapsto (c^\lambda_{\mathsf{s}})^* \otimes (c^\lambda_{\mathsf{t}})$$

determines an A–A bimodule isomorphism from $A^{\trianglerighteq\lambda}/\,\mathrm{A}^{\rhd\lambda}$ to $(\Delta(\lambda))^* \otimes_R \Delta(\lambda)$. Moreover, we have $* \circ \alpha_\lambda = \alpha_\lambda \circ *$, which reflects the cellular algebra axiom $(c^\lambda_{\mathsf{st}})^* \equiv c^\lambda_{\mathsf{ts}} \mod A^{\rhd\lambda}$.

A certain bilinear form on the cell modules plays an essential role in the theory of cellular algebras. Let A be a cellular algebra over R and let $\lambda \in \widehat{A}$. The cell module $\Delta(\lambda)$ can be regarded as an $A/A^{\rhd\lambda}$ module. For $x, y, z \in \Delta(\lambda)$, it follows from the definition of the cell module and the map α_λ that $x\alpha_\lambda^{-1}(y^* \otimes z) \in Rz$. Define $\langle x, y\rangle$ by

$$x\alpha_\lambda^{-1}(y^* \otimes z) = \langle x, y\rangle z. \tag{1.2}$$

Then $\langle x, y\rangle$ is R-linear in each variable and we have $\langle xa, y\rangle = \langle x, ya^*\rangle$ for $x, y \in \Delta(\lambda)$ and $a \in A$. Note that

$$c^\lambda_{\mathsf{st}} c^\lambda_{\mathsf{uv}} = \langle c^\lambda_{\mathsf{t}}, c^\lambda_{\mathsf{u}}\rangle c^\lambda_{\mathsf{sv}},$$

which is the customary definition of the bilinear form.

Definition 1.3 ([16])**.** A cellular algebra, A, is said to be cyclic cellular if every cell module is cyclic as an A-module.

If A is cyclic cellular, $\lambda \in \widehat{A}$, and $\delta(\lambda)$ is a generator of the cell module $\Delta(\lambda)$, let m_λ be a lift in $A^{\trianglerighteq\lambda}$ of $\alpha_\lambda^{-1}(\delta(\lambda)^* \otimes \delta(\lambda))$, i.e. $\alpha_\lambda^{-1}(\delta(\lambda)^* \otimes \delta(\lambda)) = m_\lambda + A^{\rhd\lambda}$.

Lemma 1.4. *The element m_λ has the following properties:*

(1) $m_\lambda \equiv m_\lambda^* \mod A^{\rhd\lambda}$.
(2) $A^{\trianglerighteq\lambda} = Am_\lambda A + A^{\rhd\lambda}$.
(3) $(m_\lambda A + A^{\rhd\lambda})/A^{\rhd\lambda} \cong \Delta(\lambda)$*, as right A–modules.*

Proof. Lemma 2.5 in [16]. □

We will call the elements m_λ cell generators; in examples of interest to us, they are given explicitly and satisfy $m_\lambda^* = m_\lambda$.

We will need the following elementary lemma regarding specializations of algebras.

Lemma 1.5. *Let R be a commutative ring with identity, A an R–algebra, and M an A–module Let $\tau : R \to S$ be a unital ring homomorphism. Note that $M \otimes_R S$ is an $A \otimes_R S$ module. Let $\varphi : A \to \mathrm{End}_R(M)$ be the homomorphism corresponding to the A–module structure of M, and $\varphi_S : A \otimes_R S \to \mathrm{End}_S(M \otimes_R S)$ the homomorphism corresponding to the $A \otimes_R S$–module structure of $M \otimes_R S$. Then there exists an R–algebra homomorphism $\theta : \varphi(A) \to \varphi_S(A \otimes_R S)$,* *making the following diagram commute:*

$$\begin{array}{ccc} A & \xrightarrow{\quad\varphi\quad} & \varphi(A) \\ \Big\downarrow{\scriptstyle \otimes 1_S} & & \Big\downarrow{\scriptstyle \theta} \\ A \otimes_R S & \xrightarrow[\varphi_S]{\quad\quad} & \varphi_S(A \otimes_R S) \end{array} \quad . \tag{1.3}$$

Proof. Note that φ_S is defined by $\varphi_S(a \otimes 1_S)(m \otimes 1_S) = \varphi(a)(m) \otimes 1_S$. Define $\theta(\varphi(a)) = \varphi_S(a \otimes 1_S)$. This is well defined because if $a \in \ker(\varphi)$, then $a \otimes 1_S \in \ker \varphi_S$. $\square$

Remark 1.6. In case $R \subset S$ are fields, the map θ in (1.3) is injective, because $\theta(\varphi(a))(m \otimes 1_S) = \varphi(a)(m) \otimes 1_S$. If $\theta(\varphi(a)) = 0$, then $\varphi(a)(m) = 0$ for all $m \in M$, so $\varphi(a) = 0$.

1.2. **Sequences of diagram algebras.** Here and in the remainder of the paper, we will consider an increasing sequence $(A_r)_{r \geqslant 0}$ of cellular algebras over an integral domain R with field of fractions $\mathbb{F}$. We assume that all the inclusions $A_r \hookrightarrow A_{r+1}$ are unital and that the involutions are consistent; that is the involution on A_{r+1}, restricted to A_r, agrees with the involution on A_r. We will establish a list of assumptions (D1)–(D6). For convenience, we call an increasing sequence of cellular algebras satisfying these assumptions a sequence of diagram algebras.

Let $(\widehat{A}_r, \trianglerighteq)$ denote the partially ordered set in the cell datum for A_r. For $\lambda \in \widehat{A}_r$, let $\Delta_r(\lambda)$ denote the corresponding cell module. If S is an integral domain with a unital homomorphism $R \to S$, write $A_r^S = A_r \otimes_R S$ and $\Delta_r^S(\lambda)$ for $\Delta_r(\lambda) \otimes_R S$. In particular, write $A_r^{\mathbb{F}} = A_r \otimes_R \mathbb{F}$ and $\Delta_r^{\mathbb{F}}(\lambda)$ for $\Delta_r(\lambda) \otimes_R \mathbb{F}$.

Definition 1.7. Let A be a cellular algebra over R. If M is a right A–module, a cell-filtration of M is a filtration by right A–modules

$$\{0\} = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r = M,$$

such that $M_i/M_{i-1} \cong \Delta(\lambda^{(i)})$ for some $\lambda^{(i)} \in \widehat{A}$. We say that the filtration is order preserving if $\lambda^{(i)} \rhd \lambda^{(i+1)}$ in $\widehat{A}$ for all $i \geqslant 1$.

Definition 1.8. Let $A \subseteq B$ be a unital inclusion of cellular algebras over an integral domain R (with consistent involutions).

(1) Say the inclusion is restriction–coherent if for every $\mu \in \widehat{B}$, the restricted module $\mathrm{Res}^B_A(\Delta_B(\mu))$ has an order preserving cell-filtration (as an A–module).
(2) Say the inclusion is induction–coherent if for every $\lambda \in \widehat{A}$, the induced module $\mathrm{Ind}^B_A(\Delta_A(\lambda))$ has an order preserving cell-filtration (as a B–module).

Definition 1.9 ([17, 18])**.** Let $(A_r)_{r \geqslant 0}$ be an increasing sequence of cellular algebras over an integral domain R. We say the tower is restriction–coherent if each inclusion $A_r \subseteq A_{r+1}$ is restriction coherent, and induction–coherent if each inclusion is induction coherent. We say the tower is coherent if it is both restriction– and induction–coherent.

Remark 1.10. We have changed the terminology from [17, 18, 11], as the weaker notion of coherence, in which the order preserving requirement is omitted, plays no role here.

We now list the first of our assumptions for a sequence of diagram algebras:

(D1) $A_0 = R$.
(D2) The algebras A_r are cyclic cellular for all $r \geqslant 0$.

For all r and for all $\lambda \in \widehat{A}_r$, fix once and for all a bimodule isomorphism $\alpha_\lambda : A_r^{\trianglerighteq\lambda}/A_r^{\rhd\lambda} \to (\Delta_r(\lambda))^* \otimes_R \Delta_r(\lambda)$, a generator $\delta_r(\lambda)$ of the cyclic A_r–module $\Delta_r(\lambda)$, and a cell generator $m_\lambda \in A_r^{\trianglerighteq\lambda}$ satisfying $\alpha_\lambda(m_\lambda + A_r^{\rhd\lambda}) = (\delta_r(\lambda))^* \otimes \delta_r(\lambda)$, as in the discussion preceding Lemma 1.4. We require the following mild assumption on the cell generators.

(D3) The cell generators satisfy $m_\lambda = m_\lambda^*$.

Our list of assumptions continues as follows:

(D4) $A_r^{\mathbb{F}}$ is split semisimple for all $r \geqslant 0$.

(D5) The sequence of algebras $(A_r)_{r\geqslant 0}$ is restriction–coherent.

As discussed in [11, Section 3], under the assumptions (D1)–(D5) above, there exists a well-defined multiplicity–free branching diagram $\widehat{A}$ associated with the sequence $(A_r)_{r\geqslant 0}$. The branching diagram is an infinite, graded, directed graph with vertices $\widehat{A}_r$ at level r and edges determined as follows. If $\lambda \in \widehat{A}_{r-1}$ and $\mu \in \widehat{A}_r$, there is an edge $\lambda \to \mu$ in $\widehat{A}$ if and only if $\Delta_{r-1}(\lambda)$ appears as a subquotient of an order preserving cell filtration of $\mathrm{Res}^{A_r}_{A_{r-1}}(\Delta_r(\mu))$. In fact, $\lambda \to \mu$ if and only if the simple $A_{r-1}^{\mathbb{F}}$–module $\Delta_{r-1}^{\mathbb{F}}(\lambda)$ is a direct summand of the restriction of $\Delta_r^{\mathbb{F}}(\mu)$ to $A_{r-1}^{\mathbb{F}}$. Note that $\widehat{A}_0$ is a singleton; we denote its unique element by $\varnothing$. We can choose $\Delta_0(\varnothing) = R$, $\delta_0(\varnothing) = 1$, and $m_\varnothing = 1$.

Definition 1.11. Given $\nu \in \widehat{A}_r$, we define a standard tableau of shape ν to be a directed path t on the branching diagram $\widehat{A}$ from $\varnothing \in \widehat{A}_0$ to ν,

$$\mathsf{t} = (\varnothing = \mathsf{t}(0) \to \mathsf{t}(1) \to \mathsf{t}(2) \to \cdots \to \mathsf{t}(r-1) \to \mathsf{t}(r) = \nu). \tag{1.4}$$

We let $\mathrm{Std}_r(\nu)$ denote the set of all such paths and we set $\mathrm{Std}_r = \cup_{\nu\in\widehat{A}_r}\mathrm{Std}_r(\nu)$.

Given an algebra satisfying axioms (D1) to (D5) it is shown in [11, Section 3] that there exist certain "down–branching factors" $d_{\lambda\to\mu} \in A_r$, for $\lambda \in \widehat{A}_{r-1}$ and $\mu \in \widehat{A}_r$ with $\lambda \to \mu$ in $\widehat{A}$, related to the cell filtration of $\mathrm{Res}^{A_r}_{A_{r-1}}(\Delta_r(\mu))$. Given a path $\mathsf{t} \in \mathrm{Std}_r(\nu)$ as in (1.4) define the ordered product d_t of branching factors by

$$d_\mathsf{t} = d_{\mathsf{t}(r-1)\to\mathsf{t}(r)}d_{\mathsf{t}(r-2)\to\mathsf{t}(r-1)}\cdots d_{\mathsf{t}(0)\to\mathsf{t}(1)}. \tag{1.5}$$

We say two cellular bases of an algebra A with involution are equivalent if they determine the same two sided ideals $A^{\trianglerighteq\lambda}$ and isomorphic cell modules.

Theorem 1.12 ([11], Section 3)**.** *Let $(A_r)_{r\geqslant 0}$ be a sequence of algebras satisfying assumptions (D1)–(D5).*

(1) *Let $\lambda \in \widehat{A}_r$. The set $\{m_\lambda d_\mathsf{t} + A_r^{\triangleright\lambda} \mid \mathsf{t} \in \mathrm{Std}_r(\lambda)\}$ is a basis of the cell module $\Delta_r(\lambda)$.*

(2) *The set $\{d_\mathsf{s}^* m_\lambda d_\mathsf{t} \mid \lambda \in \widehat{A}_r$ and $\mathsf{s},\mathsf{t} \in \mathrm{Std}_r(\lambda)\}$ is a cellular basis of A_r, equivalent to the original cellular basis.*

(3) *For a fixed $\lambda \in \widehat{A}_r$, we let $\mu(1) \triangleright \mu(2) \triangleright \cdots \triangleright \mu(s)$ be a listing of the $\mu \in \widehat{A}_{r-1}$ such that $\mu \to \lambda$. Let*

$$M_j = \mathrm{Span}_R\left\{m_\lambda d_\mathsf{t} + A_r^{\triangleright\lambda} \,\middle|\, \mathsf{t} \in \mathrm{Std}_r(\lambda), \mathsf{t}(k-1) \trianglerighteq \mu(j)\right\}.$$

Then

$$(0) \subset M_1 \subset \cdots \subset M_s = \Delta_r(\lambda)$$

is a filtration of $\Delta_r(\lambda)$ by A_{r-1}-submodules, and $M_j/M_{j-1} \cong \Delta_{r-1}(\mu(j))$.

We will now continue with our list of assumed properties of the sequence of algebra $(A_r)_{r\geqslant 0}$ with one final key axiom.

(D6) There exist "up–branching factors" $u_{\lambda\to\mu} \in A_r^R$ for $\lambda \in \widehat{A}_{r-1}$ and $\mu \in \widehat{A}_r$ satisfying the compatibility relations

$$m_\mu d_{\lambda\to\mu} = (u_{\lambda\to\mu})^* m_\lambda. \tag{1.6}$$

Example 1.13. It is shown in [11] that the Hecke algebras of type A, the symmetric group algebras, the Brauer algebras, the Birman–Wenzl–Murakami algebras, the partition algebras, and the Jones–Temperley–Lieb algebras all are examples of sequences of algebras satisfying properties (D1)–(D6). In Appendix B in the arXiv version of this paper, we show that one can extract single sequences from the double sequence of walled Brauer algebras, so that properties

(D1)–(D6) are satisfied. In each case the ground ring R can be taken to be the generic ground ring for the class of algebras. For example, for the Hecke algebras, this is $\mathbb{Z}[\boldsymbol{q}, \boldsymbol{q}^{-1}]$, and for the Brauer algebras it is $\mathbb{Z}[\boldsymbol{\delta}]$, where $\boldsymbol{q}$ and $\boldsymbol{\delta}$ are indeterminants.

Remark 1.14. In all the examples listed above, the branching factors $d_{\lambda\to\mu}$ and $u_{\lambda\to\mu}$ and the cell generators m_λ are determined explicitly. For the symmetric group algebras and the Hecke algebras of finite type A, the branching factors can be chosen so that the basis $\{d_{\mathsf{s}}^* m_\lambda d_{\mathsf{t}}\}$ coincides with Murphy's cellular basis or its dual version, see Section 4. In all of these examples, u-branching factors are related to cell filtrations of induced cell modules; see [11] for details. However, for the purposes of this paper it is enough to know that the u-branching coefficients exist and are explicitly determined.

Definition 1.15. We write $m_{\mathsf{st}}^\lambda = d_{\mathsf{s}}^* m_\lambda d_{\mathsf{t}}$. Also write $m_{\mathsf{t}} = m_\lambda d_{\mathsf{t}} + A_r^{\rhd\lambda} \in \Delta_r(\lambda)$. We refer to the cellular basis $\{m_{\mathsf{st}}^\lambda \mid \lambda \in \widehat{A}_r \text{ and } \mathsf{s}, \mathsf{t} \in \mathrm{Std}_r(\lambda)\}$ as the Murphy cellular basis of A_r and $\{m_{\mathsf{t}}^\lambda \mid \mathsf{t} \in \mathrm{Std}_r(\lambda)\}$ as the Murphy basis of the cell module $\Delta_r(\lambda)$.

Remark 1.16. (Remark on notation for branching factors) Let $\lambda \in \widehat{A}_{r-1}$ and $\mu \in \widehat{A}_r$ with $\lambda \to \mu \in \widehat{A}$. In situations where it seems helpful to emphasize the level on the branching diagram, we will write, for example, $d_{\lambda\to\mu}^{(r)}$ instead of $d_{\lambda\to\mu}$. See for instance, Theorem 1.24.

Definition 1.17. Given $0 \leqslant s \leqslant r$ and $\lambda \in \widehat{A}_s$, $\nu \in \widehat{A}_r$, we define a skew standard tableau of shape $\nu \setminus \lambda$ and degree $r - s$ to be a directed path t on the branching diagram $\widehat{A}$ from λ to μ,

$$\lambda = \mathsf{t}(s) \to \mathsf{t}(s+1) \to \mathsf{t}(s+2) \to \cdots \to \mathsf{t}(r-1) \to \mathsf{t}(r) = \nu. \tag{1.7}$$

We let $\mathrm{Std}_{s,r}(\nu \setminus \lambda)$ denote the set of all such paths with given λ and ν. Given $0 \leqslant s \leqslant r$, we set $\mathrm{Std}_{s,r} = \cup_{\lambda\in\widehat{A}_s, \nu\in\widehat{A}_r} \mathrm{Std}_{s,r}(\nu \setminus \lambda)$.

Given two paths $\mathsf{s} \in \mathrm{Std}_{q,s}(\mu \setminus \lambda)$ and $\mathsf{t} \in \mathrm{Std}_{s,r}(\nu \setminus \mu)$ such that the final point of s is the initial point of t, define $\mathsf{s} \circ \mathsf{t} \in \mathrm{Std}_{q,r}(\nu \setminus \lambda)$ to be the obvious path obtained by concatenation.

Remark 1.18. Given a path $\mathsf{t} \in \mathrm{Std}_{s,r}(\nu \setminus \lambda)$ as in (1.7) define

$$d_{\mathsf{t}} = d_{\mathsf{t}(r-1)\to\mathsf{t}(r)} d_{\mathsf{t}(r-2)\to\mathsf{t}(r-1)} \cdots d_{\mathsf{t}(s)\to\mathsf{t}(s+1)},$$

and

$$u_{\mathsf{t}} = u_{\mathsf{t}(s)\to\mathsf{t}(s+1)} \cdots u_{\mathsf{t}(r-2)\to\mathsf{t}(r-1)} u_{\mathsf{t}(r-1)\to\mathsf{t}(r)}.$$

Then it follows from the compatibility relation (1.6) and induction on $r - s$ that

$$u_{\mathsf{t}}^* m_\lambda = m_\nu d_{\mathsf{t}}. \tag{1.8}$$

Because $m_\varnothing$ can be chosen to be 1, this gives in particular for $\mathsf{t} \in \mathrm{Std}_r(\nu)$,

$$u_{\mathsf{t}}^* = m_\nu d_{\mathsf{t}}. \tag{1.9}$$

Therefore the cellular basis $\{m_{\mathsf{st}}^\nu\}$ can also be written in the apparently asymmetric form

$$m_{\mathsf{st}}^\nu = d_{\mathsf{s}}^* m_\nu d_{\mathsf{t}} = d_{\mathsf{s}}^* u_{\mathsf{t}}^*.$$

Using the symmetry of the cellular basis $(m_{\mathsf{st}}^\nu)^* = m_{\mathsf{ts}}^\nu$ (which follows from the assumption (D3)), we also get

$$m_{\mathsf{st}}^\nu = u_{\mathsf{s}} d_{\mathsf{t}}.$$

Using (1.9), we have the following form for the basis $\{m_{\mathsf{t}}^\lambda \mid \mathsf{t} \in \mathrm{Std}_r(\nu)\}$ of the cell module $\Delta_r(\nu)$:

$$m_{\mathsf{t}}^\lambda = u_{\mathsf{t}}^* + A_r^{\rhd\nu}. \tag{1.10}$$

Now, for any $0 \leqslant q \leqslant s \leqslant r$, let $\mathsf{t}_{[q,s]}$ denote the truncated path,

$$\mathsf{t}(q) \to \mathsf{t}(q+1) \to \mathsf{t}(q+2) \to \cdots \to \mathsf{t}(s-1) \to \mathsf{t}(s).$$

The representative u_{t}^* of m_{t} has the remarkable property that for any $0 \leqslant s \leqslant r$,

$$u_{\mathsf{t}}^* = u_{\mathsf{t}_{[s,r]}}^* u_{\mathsf{t}_{[0,s]}}^*, \tag{1.11}$$

and

$$u_{\mathsf{t}_{[0,s]}}^* = m_{\mathsf{t}(s)} d_{\mathsf{t}_{[0,s]}} \in m_{\mathsf{t}(s)} A_s \subseteq A_s^{\trianglerighteq \mathsf{t}(s)}. \tag{1.12}$$

Here, (1.11) follows directly from the definition of u_{t}, while (1.12) comes from applying (1.9) to $u_{\mathsf{t}_{[0,s]}}^*$ in place of u_{t}^*. The compatibility relations (1.8) together with the factorizability (1.11) of representatives u_{t}^* of the Murphy basis play a crucial role in this paper. In our view, these are the distinguishing properties of the Murphy bases of diagram algebras, and even in the original context of the Hecke algebras [34] these properties provide new insight.

1.3. **Seminormal bases, dominance triangularity, and restriction of cell modules.** We have explored certain consequences of our standing assumptions (D1)–(D6) in an companion paper [4]. We recall some of the results of that paper that will be applied here.

One can define analogues of seminormal bases in the algebras and the cell modules defined over the field of fractions $\mathbb{F}$, as follows. Let z_r^λ denote the minimal central idempotent in $A_r^{\mathbb{F}}$ corresponding to the minimal two sided ideal labeled by $\lambda \in \widehat{A}_r$. For $r \geqslant s$ and for a path $\mathsf{t} \in \mathrm{Std}_{s,r}(\nu \setminus \lambda)$ as in (1.7), define

$$F_{\mathsf{t}} = \prod_{s \leqslant j \leqslant r} z_j^{\mathsf{t}(j)}.$$

The factors are mutually commuting so the order of the factors does not have to be specified. In particular the set of F_{t} for $\mathsf{t} \in \mathrm{Std}_r(\nu)$ and $\nu \in \widehat{A}_r$, is a family of mutually orthogonal minimal idempotents, with $\sum_{\mathsf{t} \in \mathrm{Std}_r(\nu)} F_{\mathsf{t}} = z_r^\nu$. The collection of idempotents F_{t} (for $r \geqslant 1$, $\nu \in \hat{A}_r$, and $\mathsf{t} \in \mathrm{Std}_r(\nu)$) is called the family of *Gelfand-Zeitlin* idempotents for the tower $(A_r)_{r \geqslant 0}$. The family is characterized in [18, Lemma 3.10].

Define $f_{\mathsf{t}} = m_{\mathsf{t}} F_{\mathsf{t}}$ in $\Delta_r^{\mathbb{F}}(\nu)$ and $F_{\mathsf{st}} = F_{\mathsf{s}} m_{\mathsf{st}}^\lambda F_{\mathsf{t}}$, for $\nu \in \widehat{A}_r$ and $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_r(\nu)$. These are analogues for diagram algebras of the seminormal bases of the Hecke algebras of the symmetric groups. This construction, and its relation to other constructions of seminormal bases, is discussed in detail in [4].

The following two partial orders on standard tableaux play an important role in the theory of diagram algebras.

Definition 1.19 (Dominance order for paths)**.** For $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_{s,r}$, define $\mathsf{s} \trianglerighteq \mathsf{t}$ if $\mathsf{s}(j) \trianglerighteq \mathsf{t}(j)$ for all $s \leqslant j \leqslant r$.

This is evidently a partial order, which we call the dominance order. In particular the dominance order is defined on Std_r and on $\mathrm{Std}_r(\nu)$ for $\nu \in \widehat{A}_r$. The corresponding strict partial order is denoted $\mathsf{s} \triangleright \mathsf{t}$ if $\mathsf{s} \neq \mathsf{t}$ and $\mathsf{s} \trianglerighteq \mathsf{t}$.

Definition 1.20 (Reverse lexicographic order for paths)**.** For $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_{s,r}$, define $\mathsf{s} \succcurlyeq \mathsf{t}$ if $\mathsf{s} = \mathsf{t}$ or if for the last index j such that $\mathsf{s}(j) \neq \mathsf{t}(j)$, we have $\mathsf{s}(j) \triangleright \mathsf{t}(j)$.

This is also a partial order on paths. The corresponding strict partial order is denoted $\mathsf{s} \succ \mathsf{t}$ if $\mathsf{s} \neq \mathsf{t}$ and $\mathsf{s} \succcurlyeq \mathsf{t}$. Evidently $\mathsf{s} \triangleright \mathsf{t}$ implies $\mathsf{s} \succ \mathsf{t}$.

We now review several results from [4]. The most useful technical result is that the Murphy bases and the seminormal bases of the cell modules are related by a dominance–unitriangular transformation.

Theorem 1.21 ([4], Theorem 3.3)**.** *Fix $\lambda \in \widehat{A}_r$. For all $\mathsf{t} \in \mathrm{Std}_r(\lambda)$, there exist coefficients $r_{\mathsf{s}}, r'_{\mathsf{s}} \in \mathbb{F}$ such that*

$$m_{\mathsf{t}}^\lambda = f_{\mathsf{t}}^\lambda + \sum_{\substack{\mathsf{s} \in \mathrm{Std}_r(\lambda) \\ \mathsf{s} \triangleright \mathsf{t}}} r_{\mathsf{s}} f_{\mathsf{s}}^\lambda \qquad f_{\mathsf{t}}^\lambda = m_{\mathsf{t}}^\lambda + \sum_{\substack{\mathsf{s} \in \mathrm{Std}_r(\lambda) \\ \mathsf{s} \triangleright \mathsf{t}}} r'_{\mathsf{s}} m_{\mathsf{s}}^\lambda.$$

Corollary 1.22 ([4], Corollary 3.4). *For $r \geqslant 0$, we have that*

(1) $\{f_{\mathsf{t}}^{\lambda} \mid \mathsf{t} \in \mathrm{Std}_r(\lambda)\}$ *is a basis of* $\Delta_r^{\mathbb{F}}(\lambda)$ *for all* $\lambda \in \widehat{A}_r$.
(2) $\{F_{\mathsf{st}}^{\lambda} \mid \lambda \in \widehat{A}_r$ *and* $\mathsf{s}, \mathsf{t} \in \widehat{A}_r\}$ *is a cellular basis of* $A_r^{\mathbb{F}}$.

Proposition 1.23 ([4], Proposition 3.9). *Let $1 \leqslant s < r$, $\nu \in \widehat{A}_r$, $\lambda \in \widehat{A}_s$ and $\mathsf{t} \in \mathrm{Std}_{s,r}(\nu \setminus \lambda)$. Let $x \in m_\lambda A_s$ and write*

$$x = \sum_{\mathsf{s} \in \mathrm{Std}_s(\lambda)} \alpha_{\mathsf{s}} u_{\mathsf{s}}^* + y,$$

with $y \in A_s^{\rhd\lambda}$. Then there exist coefficients $r_{\mathsf{z}} \in R$, such that

$$u_{\mathsf{t}}^* x \equiv \sum_{\mathsf{s} \in \mathrm{Std}_s(\lambda)} \alpha_{\mathsf{s}} u_{\mathsf{t}}^* u_{\mathsf{s}}^* + \sum_{\mathsf{z}} r_{\mathsf{z}} u_{\mathsf{z}}^* \qquad \mathrm{mod}\ A_r^{\rhd\nu},$$

where the sum is over $\mathsf{z} \in \mathrm{Std}_r(\nu)$ such that $\mathsf{z}_{[s,r]} \rhd \mathsf{t}$ and $\mathsf{z}(s) \rhd \lambda$.

Finally we mention, without going into details, the relation of the assumptions (D1)–(D6) to Jucys–Murphy elements. Assume that $(A_r^S)_{r\geqslant 0}$ is a tower of algebras satisfying assumptions (D1)–(D6) and assume in addition that the tower has Jucys–Murphy elements, in the sense of [18]. This assumption holds for Hecke algebras of type A, the symmetric group algebras, the Brauer algebras, the Birman–Wenzl–Murakami algebras, the partition algebras, and the Jones–Temperley–Lieb algebras. We will see in Sections 7 and 8 that it also holds for the Brauer centralizer algebras acting on symplectic and orthogonal tensor spaces. It is shown in [4] that the Jucys–Murphy elements act diagonally on the seminormal bases and dominance unitriangularly on the Murphy bases, generalizing a result of Murphy [33, Theorem 4.6] for the Hecke algebras.

1.4. **Cellularity and the Jones basic construction.** In this section, we recall the framework of [17, 18, 11]. This framework allows one to lift the cellular structure from a coherent sequence $(H_r)_{r\geqslant 0}$ of cyclic cellular algebras to a second sequence $(A_r)_{r\geqslant 0}$, related to the first sequence by "Jones basic constructions". Most importantly, we will recall how the branching factors and cell generators for the tower $(A_r)_{r\geqslant 0}$ can be explicitly constructed from those of the tower $(H_r)_{r\geqslant 0}$.

The list of assumptions regarding the two sequence of algebras, from [11, Section 5], is the following: $(H_r)_{r\geqslant 0}$ and $(A_r)_{r\geqslant 0}$ are both sequences of algebras over an integral domain R with field of fractions $\mathbb{F}$. The inclusions are unital, and both sequences of algebras have consistent algebra involutions $*$. Moreover:

(J1) $A_0 = H_0 = R$ and $A_1 = H_1$ (as algebras with involution).
(J2) There is a $\delta \in S$ and for each $r \geqslant 2$, there is an element $e_{r-1} \in A_r$ satisfying $e_{r-1}^* = e_{r-1}$ and $e_{r-1}^2 = \delta e_{r-1}$. For $r \geqslant 2$, $e_{r-1} e_r e_{r-1} = e_{r-1}$ and $e_r e_{r-1} e_r = e_r$.
(J3) For $r \geqslant 2$, $A_r/(A_r e_{r-1} A_r) \cong H_r$ as algebras with involution.
(J4) For $r \geqslant 1$, e_r commutes with A_{r-1} and $e_r A_r e_r \subseteq A_{r-1} e_r$.
(J5) For $r \geqslant 1$, $A_{r+1} e_r = A_r e_r$, and the map $x \mapsto x e_r$ is injective from A_r to $A_r e_r$.
(J6) For $r \geqslant 2$, $e_{r-1} A_r e_{r-1} A_r = e_{r-1} A_r$.
(J7) For all r, $A_r^{\mathbb{F}} := A_r \otimes_R \mathbb{F}$ is split semisimple.
(J8) $(H_r)_{r\geqslant 0}$ is a coherent tower of cyclic cellular algebras.

The conclusion ([11, Theorem 5.5]) is that $(A_r)_{r\geqslant 0}$ is a coherent tower of cyclic cellular algebras over R (in particular the tower $(A_r)_{r\geqslant 0}$ satisfies conditions (D1), (D2), (D4), and (D5). We let $(\widehat{H}_r, \trianglerighteq)$ denote the partially ordered set in the cell datum for H_r. Then the partially ordered set in the cell datum for A_r is

$$\widehat{A}_r = \{(\lambda, l) \mid 0 \leqslant l \leqslant \lfloor r/2 \rfloor \text{ and } \lambda \in \widehat{H}_{r-2l}\},$$

with partial order $(\lambda, l) \trianglerighteq (\mu, m)$ if $l > m$ or if $l = m$ and $\lambda \trianglerighteq \mu$. The branching diagram for the tower $(A_r)_{r\geqslant 0}$ is $\widehat{A} = \bigsqcup_{r\geqslant 0} \widehat{A}_r$ with the branching rule $(\lambda, l) \to (\mu, m)$ if $l = m$ and $\lambda \to \mu$

in $\widehat{H}$ or if $m = l+1$ and $\mu \to \lambda$ in $\widehat{H}$. We call this the branching diagram obtained by reflections from $\widehat{H}$.

We will now explain how the branching factors and cell generators for the tower $(A_r)_{r\geqslant 0}$ can be explicitly constructed from those of the tower $(H_r)_{r\geqslant 0}$. For $r \geqslant 2$, let

$$e_{r-1}^{(l)} = \begin{cases} 1 & \text{if } l = 0 \\ \underbrace{e_{r-2l+1}e_{r-2l+3}\cdots e_{r-1}}_{l \text{ factors}} & \text{if } l = 1, \dots, \lfloor r/2 \rfloor, \text{ and} \\ 0 & \text{if } l > \lfloor r/2 \rfloor. \end{cases} \tag{1.13}$$

Let $d_{\lambda\to\mu}$ and $u_{\lambda\to\mu}$ denote down– and up–branching factors, and let m_λ denote cell generators for $(H_r)_{r\geqslant 0}$. Let $\bar{d}_{\lambda\to\mu}$, $\bar{u}_{\lambda\to\mu}$, and $\bar{m}_\lambda$ denote liftings of these elements in the algebras A_r. Then we have the following two results:

Theorem 1.24 ([11], Theorem 5.7)**.** *The branching factors for the tower $(A_r)_{r\geqslant 0}$ can be chosen to satisfy:*

(1) $d^{(r+1)}_{(\lambda,l)\to(\mu,l)} = \bar{d}^{(r+1-2l)}_{\lambda\to\mu}\, e^{(l)}_{r-1}$.
(2) $u^{(r+1)}_{(\lambda,l)\to(\mu,l)} = \bar{u}^{(r+1-2l)}_{\lambda\to\mu}\, e^{(l)}_{r}$.
(3) $d^{(r+1)}_{(\lambda,l)\to(\mu,l+1)} = \bar{u}^{(r-2l)}_{\mu\to\lambda}\, e^{(l)}_{r-1}$.
(4) $u^{(r+1)}_{(\lambda,l)\to(\mu,l+1)} = \bar{d}^{(r-2l)}_{\mu\to\lambda}\, e^{(l+1)}_{r}$.

Lemma 1.25 ([11], Section 5.5)**.** *For $(\lambda, l) \in \widehat{A}_r$, the cell generator $m_{(\lambda,l)}$ in $A_r^{\trianglerighteq(\lambda,l)}$ can be chosen as $m_{(\lambda,l)} = \bar{m}_\lambda e^{(l-1)}_{r-1}$.*

Remark 1.26. Although these results involve unspecified liftings of elements from H_r to A_r, in the examples, the liftings are chosen explicitly. Moreover, the cell generators m_λ in H_r and $m_{(\lambda,l)}$ in A_r are chosen to be $*$–invariant, so that the tower (A_r) satisfies axiom (D3). Furthermore, in the examples, the branching factors and cell generators in the algebras H_r satisfy the compatibility relation (D6), and their liftings can be chosen to satisfy these relations as well. It then follows from Theorem 1.24 and Lemma 1.25 that the branching factors and cell generators in the algebras A_r also satisfy the compatibility relations (D6).

Now the tower $(A_r)_{r\geqslant 0}$ in particular satisfies the conditions (D1)–(D6) over R, so each A_r has a Murphy type cellular basis obtained by the prescription of Theorem 1.12, using ordered product of d–branching factors along paths on $\widehat{A}$.

Remark 1.27. For the standard examples of diagram algebras, for example the Brauer algebras, all this works not over the generic ground ring $R = \mathbb{Z}[\boldsymbol{\delta}]$, but only over $R[\boldsymbol{\delta}^{-1}]$. However, the branching factors and cell generators obtained from Theorem 1.24 and Lemma 1.25 do lie in the algebras over the generic ground ring. Furthermore, one can check that the transition matrix between the diagram basis of the algebras and the Murphy type cellular basis is invertible over the generic ground ring; this step is case–by–case and somewhat *ad hoc*. It follows that the tower of algebras $(A_r^R)_{r\geqslant 0}$ over the generic ground ring satisfies all of the conditions (D1)–(D6). This is explained in detail in [11, Sections 5 and 6].

2. A framework for cellularity of quotient algebras

As explained in the introduction, cellularity does not pass to quotients in general, but nevertheless we intend to show that cellularity does pass to the quotients of certain abstract diagram or tangle algebras acting on tensor space. In this section, we will develop an axiomatic framework for this phenomenon. In the remainder of the paper, this framework will be applied to Brauer's centralizer algebras acting on orthogonal or symplectic tensor space. In Appendix C in the arXiv version of this paper we show that the walled Brauer algebras acting on mixed tensor space can be treated in an identical fashion.

2.1. **A setting for quotient towers.** We consider a tower of cellular algebras $(A_r)_{r\geqslant 0}$ over an integral domain R satisfying the properties (D1)–(D6) of Section 1.2. In particular, for each r, we have the cellular basis

$$\{d_{\mathsf{s}}^* m_\lambda d_{\mathsf{t}} \mid \lambda \in \widehat{A}_r \text{ and } \mathsf{s},\mathsf{t} \in \mathrm{Std}_r(\lambda)\}$$

of A_r from Theorem 1.12, and we write $m_{\mathsf{st}}^\lambda = d_{\mathsf{s}}^* m_\lambda d_{\mathsf{t}}$.

Suppose that S is an integral domain with field of fractions $\mathbb{K}$ and that $\pi : R \to S$ is a surjective ring homomorphism. We consider the specialization $A_r^S = A_r \otimes_R S$ of the algebras A_r. Let $(Q_r^{\mathbb{K}})_{r\geqslant 0}$ be a tower of unital algebras over $\mathbb{K}$, with common identity, and with surjective homomorphisms $\phi_r : A_r^{\mathbb{K}} \to Q_r^{\mathbb{K}}$. We denote $\phi_r(A_r^S) \subseteq Q_r^{\mathbb{K}}$ by Q_r^S.

We suppose that the homomorphisms are consistent with the inclusions of algebras, $\phi_{r+1} \circ \iota = \iota \circ \phi_r$, where ι denotes both the inclusions $\iota : A_r^{\mathbb{K}} \to A_{r+1}^{\mathbb{K}}$ and $\iota : Q_r^{\mathbb{K}} \to Q_{r+1}^{\mathbb{K}}$. In particular, this implies that $\ker(\phi_r) \subseteq \ker(\phi_{r+1})$. Because of this, we will usually just write ϕ instead of writing ϕ_r.

Definition 2.1. We say that $(Q_r^S)_{r\geqslant 0}$ is a quotient tower of $(A_r^S)_{r\geqslant 0}$ if the following axioms hold.

(Q1) There is a distinguished subset $\widehat{A}_{r,\mathrm{perm}}$ of "permissible" points in $\widehat{A}_r$. The point $\varnothing \in \widehat{A}_0$ is permissible, and for each r and permissible μ in $\widehat{A}_r$, there exists at least one permissible ν in $\widehat{A}_{r+1}$ with $\mu \to \nu$ in $\widehat{A}$, and (for $r \geqslant 1$) at least one permissible λ in $\widehat{A}_{r-1}$ with $\lambda \to \mu$ in $\widehat{A}$.

A path $\mathsf{t} \in \mathrm{Std}_r(\nu)$ will be called permissible if $\mathsf{t}(k)$ is permissible for all $0 \leqslant k \leqslant r$. Write $\mathrm{Std}_{r,\mathrm{perm}}(\nu)$ for the set of permissible paths in $\mathrm{Std}_r(\nu)$.

(Q2) If $\mathsf{t} \in \mathrm{Std}_r(\nu)$ is not permissible, let $1 \leqslant k \leqslant r$ be the first index such that $\mu = \mathsf{t}(k)$ is not permissible. Then there exist elements $\mathfrak{b}_\mu$ and $\mathfrak{b}'_\mu$ in A_k^S such that
  (a) $m_\mu = \mathfrak{b}_\mu - \mathfrak{b}'_\mu$.
  (b) $\mathfrak{b}_\mu \in \ker(\phi)$.
  (c) $\mathfrak{b}'_\mu \in m_\mu A_k^{\mathbb{K}} \cap (A_k^S)^{\rhd\mu}$.

(Q3) With $\overline{\mathbb{K}}$ the algebraic closure of $\mathbb{K}$, we have

$$\dim_{\overline{\mathbb{K}}}(Q_r^{\overline{\mathbb{K}}}) = \sum_{\nu\in\widehat{A}_{r,\mathrm{perm}}} (\sharp\mathrm{Std}_{r,\mathrm{perm}}(\nu))^2.$$

Remark 2.2. (Some notation and terminology) Let $\mathfrak{p} = \ker(\pi)$, a prime ideal in R, and let $R_{\mathfrak{p}} \subset \mathbb{F}$ be the localization of R at $\mathfrak{p}$. Thus $R_{\mathfrak{p}}$ is a local ring with unique maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$ and residue field $\mathbb{K}$, and $\pi : R \to S$ extends to a surjective ring homomorphism $\pi : R_{\mathfrak{p}} \to \mathbb{K}$. We have surjective evaluation maps, also denoted π from $A_s^{R_{\mathfrak{p}}}$ to $A_s^{\mathbb{K}}$ given by $\pi(\sum \alpha_{\mathsf{uv}}^\alpha m_{\mathsf{uv}}^\alpha) = \sum \pi(\alpha_{\mathsf{uv}}^\alpha) m_{\mathsf{uv}}^\alpha$. and from $\Delta_{A_s}^{R_{\mathfrak{p}}}(\lambda)$ to $\Delta_{A_s}^{\mathbb{K}}(\lambda)$ given by $\pi(\sum_{\mathsf{t}} \alpha_{\mathsf{t}} m_{\mathsf{t}}^\lambda) = \sum \pi(\alpha_{\mathsf{t}}) m_{\mathsf{t}}^\lambda$. We often refer to $R_{\mathfrak{p}}$, or $A_s^{R_{\mathfrak{p}}}$, or $\Delta_{A_s}^{R_{\mathfrak{p}}}(\lambda)$ as the set of *evaluable* elements (in $\mathbb{F}$, or $A_s^{\mathbb{F}}$, or $\Delta_{A_s}^{\mathbb{F}}$, respectively).

2.2. **Cellular bases of quotient towers.** We are now going to show that under the assumptions (Q1)–(Q3), the quotient algebras Q_r^S are cellular algebras with a cellular basis $\{\phi(d_{\mathsf{s}}^* m_\lambda d_{\mathsf{t}}) \mid \lambda \in \widehat{A}_{r,\mathrm{perm}} \text{ and } \mathsf{s},\mathsf{t} \in \mathrm{Std}_{r,\mathrm{perm}}(\lambda)\}$. Furthermore, we will produce a cellular basis $\{\widetilde{m}_{\mathsf{st}}^\lambda\}$ of A_r^S, equivalent to the cellular basis $\{m_{\mathsf{st}}^\lambda\}$, with the properties that $\widetilde{m}_{\mathsf{st}}^\lambda = m_{\mathsf{st}}^\lambda$ in case both s and t are permissible, and $\widetilde{m}_{\mathsf{st}}^\lambda \in \ker(\phi)$ otherwise. In particular, the set of $\widetilde{m}_{\mathsf{st}}^\lambda$ such that at least one of s and t is not permissible constitutes an S–basis of $\ker(\phi)$.

Lemma 2.3. *Assume as in the discussion above that S is an integral domain with field of fractions $\mathbb{K}$ and that $\pi : R \to S$ is a surjective ring homomorphism. Let $0 \leqslant s < r$, $\mu \in \widehat{A}_s$, and*

$x \in m_\mu A_s^{\mathbb{K}} \cap (A_s^S)^{\rhd\mu}$. *Let* $\lambda \in \widehat{A}_r$ *and suppose* $\mathsf{t} \in \mathrm{Std}_{s,r}(\lambda \setminus \mu)$. *Then there exist coefficients* $\alpha_{\mathsf{z}} \in S$ *such that*

$$u_{\mathsf{t}}^* x \equiv \sum_{\mathsf{z}} \alpha_{\mathsf{z}} m_\lambda d_{\mathsf{z}} \qquad \mathrm{mod}\ ((A_r^S)^{\rhd\lambda} \cap m_\lambda A_r^S),$$

where the sum is over $\mathsf{z} \in \mathrm{Std}_r(\lambda)$ *with* $\mathsf{z}_{[s,r]} \rhd \mathsf{t}$ *and* $\mathsf{z}(s) \rhd \mu$.

Proof. We will apply Proposition 1.23, but we cannot do so directly. Recall the notation from Remark 2.2. By hypothesis, $x = m_\mu \beta$, where $\beta \in A_s^{\mathbb{K}}$ and $x \in (A_s^S)^{\rhd\mu}$. Lift β to an element $\beta_0 \in A_s^{R_{\mathfrak{p}}}$ and let $x_0 = m_\mu \beta_0$. Since $x_0 \in m_\mu A_s^{R_{\mathfrak{p}}}$, we can write

$$x_0 \equiv \sum_{\mathsf{v} \in \mathrm{Std}_s(\mu)} r_{\mathsf{v}} m_\mu d_{\mathsf{v}} \qquad \mathrm{mod}\ (A_s^{R_{\mathfrak{p}}})^{\rhd\mu}.$$

Since $\pi(x_0) = x \in (A_s^S)^{\rhd\mu}$ it follows that $\pi(r_{\mathsf{v}}) = 0$ for all $\mathsf{v} \in \mathrm{Std}_s(\mu)$. Now we can apply Proposition 1.23 to x_0, with R replaced by $R_{\mathfrak{p}}$, which gives us

$$u_{\mathsf{t}}^* x_0 \equiv \sum_{\mathsf{v} \in \mathrm{Std}_s(\mu)} r_{\mathsf{v}} m_\lambda d_{\mathsf{t}} d_{\mathsf{v}} + \sum_{\substack{\mathsf{z} \in \mathrm{Std}_r(\lambda) \\ \mathsf{z}_{[s,r]} \rhd \mathsf{t} \\ \mathsf{z}(s) \rhd \mu}} r'_{\mathsf{z}} m_\lambda d_{\mathsf{z}} \qquad \mathrm{mod}\ (A_r^{R_{\mathfrak{p}}})^{\rhd\lambda},$$

Applying the evaluation map π and recalling that $\pi(r_{\mathsf{v}}) = 0$ gives

$$u_{\mathsf{t}}^* x = \sum_{\substack{\mathsf{z} \in \mathrm{Std}_r(\lambda) \\ \mathsf{z}_{[s,r]} \rhd \mathsf{t} \\ \mathsf{z}(s) \rhd \mu}} \alpha_{\mathsf{z}} m_\lambda d_{\mathsf{z}} + z \tag{2.1}$$

where $\alpha_{\mathsf{z}} \in \mathbb{K}$ and $z \in (A_r^{\mathbb{K}})^{\rhd\lambda}$. But since $u_{\mathsf{t}}^* x \in A_r^S$, we must have $\alpha_{\mathsf{z}} \in S$ and $z \in (A_r^S)^{\rhd\lambda}$. Finally, since $u_{\mathsf{t}}^* x \in m_\lambda A_r^S$, it follows from (2.1) that $z \in m_\lambda A_r^S$. □

Lemma 2.4. *Assume* (Q1)–(Q3). *Let* $\lambda \in \widehat{A}_r$ *and let* $\mathsf{t} \in \mathrm{Std}_r(\lambda)$. *If* t *is not permissible, then there exist coefficients* $r_{\mathsf{v}} \in S$ *such that*

$$m_\lambda d_{\mathsf{t}} = \sum_{\substack{\mathsf{v} \in \mathrm{Std}_{r,\mathrm{perm}}(\lambda) \\ \mathsf{v} \succ \mathsf{t}}} r_{\mathsf{v}} m_\lambda d_{\mathsf{v}} + x_1 + x_2,$$

where $x_1 \in \ker(\phi)$, *and* $x_2 \in (A_r^S)^{\rhd\lambda} \cap m_\lambda A_r^S$. *Hence for all for all* $\mathsf{s} \in \mathrm{Std}_r(\lambda)$,

$$m_{\mathsf{st}}^\lambda \equiv \sum_{\substack{\mathsf{v} \in \mathrm{Std}_{r,\mathrm{perm}}(\nu) \\ \mathsf{v} \succ \mathsf{t}}} r_{\mathsf{v}} m_{\mathsf{sv}}^\lambda \qquad \mathrm{mod}\ ((A_r^S)^{\rhd\lambda} + \ker(\phi)).$$

Proof. Since t is not permissible, by assumption (Q2) there exists $0 \leqslant k \leqslant r$ such that $\mu = \mathsf{t}(k)$ satisfies the following: there are elements $\mathfrak{b}_\mu$ and $\mathfrak{b}'_\mu$ in A_k^S such that $m_\mu = \mathfrak{b}_\mu - \mathfrak{b}'_\mu$, $\mathfrak{b}_\mu \in \ker(\phi)$, and

$$\mathfrak{b}'_\mu \in m_\mu A_k^{\mathbb{K}} \cap (A_k^S)^{\rhd\mu}. \tag{2.2}$$

Write $\mathsf{t}_1 = \mathsf{t}_{[0,k]}$ and $\mathsf{t}_2 = \mathsf{t}_{[k,r]}$ Using the branching compatibility relation (1.8),

$$m_\lambda d_{\mathsf{t}} = m_\lambda d_{\mathsf{t}_2} d_{\mathsf{t}_1} = u_{\mathsf{t}_2}^* m_\mu d_{\mathsf{t}_1} = u_{\mathsf{t}_2}^* \mathfrak{b}_\mu d_{\mathsf{t}_1} - u_{\mathsf{t}_2}^* \mathfrak{b}'_\mu d_{\mathsf{t}_1} \tag{2.3}$$

The first term $u_{\mathsf{t}_2}^* \mathfrak{b}_\mu d_{\mathsf{t}_1}$ in (2.3) lies in $\ker(\phi)$. Recall $\mathfrak{b}'_\mu d_{\mathsf{t}_1} \in m_\mu A_k^{\mathbb{K}} \cap (A_k^S)^{\rhd\mu}$ by (2.2), and so we can apply Lemma 2.3 to conclude that the second term $-u_{\mathsf{t}_2}^* \mathfrak{b}'_\mu d_{\mathsf{t}_1}$ in (2.3) satisfies

$$-u_{\mathsf{t}_2}^* \mathfrak{b}'_\mu d_{\mathsf{t}_1} \equiv \sum_{\substack{\mathsf{v} \in \mathrm{Std}_r(\lambda) \\ \mathsf{v}_{[k,r]} \rhd \mathsf{t}_2}} \alpha_{\mathsf{v}} m_\lambda d_{\mathsf{v}} \qquad \mathrm{mod}\ ((A_r^S)^{\rhd\lambda} \cap m_\lambda A_r^S), \tag{2.4}$$

where $\alpha_{\mathsf{v}} \in S$. Note that the condition $\mathsf{v}_{[k,r]} \rhd \mathsf{t}_2$ implies that $\mathsf{v} \succ \mathsf{t}$. This gives us

$$m_\lambda d_{\mathsf{t}} \equiv \sum_{\substack{\mathsf{v}\in \mathrm{Std}_r(\lambda)\\ \mathsf{v}\succ\mathsf{t}}} \alpha_{\mathsf{v}} m_\lambda d_{\mathsf{v}} \qquad \mathrm{mod}\ (\ker(\phi) + (A_r^S)^{\rhd\lambda} \cap m_\lambda A_r^S).$$

By induction on the ordering, $\succ$, on $\mathrm{Std}_r(\nu)$ we obtain

$$m_\lambda d_{\mathsf{t}} \equiv \sum_{\substack{\mathsf{v}\in \mathrm{Std}_{r,\mathrm{perm}}(\lambda)\\ \mathsf{v}\succ\mathsf{t}}} r_{\mathsf{v}} m_\lambda d_{\mathsf{v}} \qquad \mathrm{mod}\ (\ker(\phi) + (A_r^S)^{\rhd\lambda} \cap m_\lambda A_r^S),$$

where now the sum is over permissible paths only. This gives the first assertion in the statement of the lemma. Finally, multiplying on the left by d_{s}^* yields the second statement. □

We are now going to produce the cellular basis $\{\widetilde{m}_{\mathsf{st}}^\lambda\}$ of A_r, equivalent to the original cellular basis $\{m_{\mathsf{st}}^\lambda\}$ with the properties that $\widetilde{m}_{\mathsf{st}}^\lambda = m_{\mathsf{st}}^\lambda$ in case both s and t are permissible, and $\widetilde{m}_{\mathsf{st}}^\lambda \in \ker(\phi)$ otherwise.

Let $\mathsf{t} \in \mathrm{Std}_r(\lambda)$ be a non–permissible path. Let $1 \leqslant k \leqslant r$ be the first index such that $\mu = \mathsf{t}(k)$ is not permissible. It follows from (Q2) that the element $\mathfrak{b}_\mu$ is in $\ker(\phi) \cap m_\mu A_k^{\mathbb{K}}$, so there exists a $\beta_\mu \in A_k^{\mathbb{K}}$ with $\mathfrak{b}_\mu = m_\mu \beta_\mu$. Let $\mathsf{t}_1 = \mathsf{t}_{[0,k]}$ and $\mathsf{t}_2 = \mathsf{t}_{[k,r]}$. Following the proof of Lemma 2.4, and using in particular (2.3) and (2.4), we get

$$m_\lambda d_{\mathsf{t}} \equiv u_{\mathsf{t}_2}^* \mathfrak{b}_\mu d_{\mathsf{t}_1} + \sum_{\substack{\mathsf{v}\in \mathrm{Std}_r(\lambda)\\ \mathsf{v}\succ\mathsf{t}}} \alpha_{\mathsf{v}} m_\lambda d_{\mathsf{v}} \qquad \mathrm{mod}\ (A_r^S)^{\rhd\lambda}, \tag{2.5}$$

for $\alpha_{\mathsf{v}} \in S$. Since $\mathfrak{b}_\mu = m_\mu\beta_\mu$, we have $u_{\mathsf{t}_2}^* \mathfrak{b}_\mu d_{\mathsf{t}_1} = m_\lambda d_{\mathsf{t}_2} \beta_\mu d_{\mathsf{t}_1}$, using (1.8). Substitute this into (2.5) and transpose to get

$$m_\lambda d_{\mathsf{t}_2} \beta_\mu d_{\mathsf{t}_1} \equiv m_\lambda d_{\mathsf{t}} - \sum_{\substack{\mathsf{v}\in \mathrm{Std}_r(\lambda)\\ \mathsf{v}\succ\mathsf{t}}} \alpha_{\mathsf{v}} m_\lambda d_{\mathsf{v}} \qquad \mathrm{mod}\ (A_r^S)^{\rhd\lambda}. \tag{2.6}$$

Note that the left hand expression is in $\ker(\phi)$. For any non-permissible path t, we define $a_{\mathsf{t}} = d_{\mathsf{t}_2}\beta_\mu d_{\mathsf{t}_1}$ to be the element which we arrived at in (2.6). Although a_{t} is *a priori* in $A_r^{\mathbb{K}}$, (2.6) shows that $m_\lambda a_{\mathsf{t}} \in m_\lambda A_r^S$. Passing to the cell module $\Delta_r(\lambda)$, we have

$$m_\lambda a_{\mathsf{t}} + (A_r^S)^{\rhd\lambda} = m_{\mathsf{t}}^\lambda - \sum_{\substack{\mathsf{v}\in \mathrm{Std}_r(\lambda)\\ \mathsf{v}\succ\mathsf{t}}} \alpha_{\mathsf{v}} m_{\mathsf{v}}^\lambda. \tag{2.7}$$

For $\mathsf{t} \in \mathrm{Std}_r(\lambda)$ permissible, define $a_{\mathsf{t}} = d_{\mathsf{t}}$. For any $\mathsf{u}, \mathsf{v} \in \mathrm{Std}_r(\lambda)$, permissible or not, define $\widetilde{m}_{\mathsf{v}}^\lambda = m_\lambda a_{\mathsf{v}} + (A_r^S)^{\rhd\lambda}$, and $\widetilde{m}_{\mathsf{uv}}^\lambda = a_{\mathsf{u}}^* m_\lambda a_{\mathsf{v}}$. We remark that in all examples, the elements $\mathfrak{b}_\mu$, $\mathfrak{b}'_\mu$, and β_μ will be explicitly described as elementary sums of Brauer-type diagrams.

Theorem 2.5. *Assume (Q1)–(Q3). The set*

$$\mathbb{B}_r = \left\{ \widetilde{m}_{\mathsf{st}}^\lambda \;\middle|\; \widetilde{m}_{\mathsf{st}}^\lambda := a_{\mathsf{s}}^* m_\lambda a_{\mathsf{t}}, \lambda \in \widehat{A}_r \text{ and } \mathsf{s}, \mathsf{t} \in \mathrm{Std}_r(\lambda) \right\}$$

is a cellular basis of A_r^S equivalent to the original cellular basis. It has the property that $\widetilde{m}_{\mathsf{st}}^\lambda = m_{\mathsf{st}}^\lambda$ if both s and t are permissible and $\widetilde{m}_{\mathsf{st}}^\lambda \in \ker(\phi)$ otherwise.

Proof. Equation (2.7) shows that $\{\widetilde{m}_{\mathsf{v}}^\lambda \mid \mathsf{v} \in \mathrm{Std}_r(\lambda)\}$ is related to the S–basis $\{m_{\mathsf{v}}^\lambda \mid \mathsf{v} \in \mathrm{Std}_r(\lambda)\}$ of the cell module $\Delta_r^S(\lambda)$ by a unitriangular transformation with coefficients in S, and therefore $\{\widetilde{m}_{\mathsf{v}}^\lambda \mid \mathsf{v} \in \mathrm{Std}_r(\lambda)\}$ is also an S–basis of the cell module.

For u and v arbitrary elements of $\mathrm{Std}_r(\lambda)$ we have $\alpha_\lambda(\widetilde{m}_{\mathsf{uv}}^\lambda + (A_r^S)^{\rhd\lambda}) = (\widetilde{m}_{\mathsf{u}}^\lambda)^* \otimes \widetilde{m}_{\mathsf{v}}^\lambda$. It follows from [16, Lemma 2.3] that $\{\widetilde{m}_{\mathsf{uv}}^\lambda \mid \lambda \in \widehat{A}_r, \mathsf{u}, \mathsf{v} \in \mathrm{Std}_r(\lambda)\}$ is a cellular basis of A_r^S equivalent to the original cellular basis $\{m_{\mathsf{uv}}^\lambda\}$.

It is evident from the construction that $\widetilde{m}_{\mathsf{st}}^\lambda = m_{\mathsf{st}}^\lambda$ if both s and t are permissible and $\widetilde{m}_{\mathsf{st}}^\lambda \in \ker(\phi)$ otherwise. □

Definition 2.6. Call $\mu \in \widehat{A}_s$ a marginal point if μ is not permissible and there exists a path $\mathsf{t} \in \mathrm{Std}_s(\mu)$ such that $\mathsf{t}(k)$ is permissible for all $k < s$.

Theorem 2.7. *Assume (Q1)–(Q3). Then*

(1) $\ker(\phi_r)$ *is globally invariant under the involution* $*$. *Hence one can define an algebra involution on* $Q_r^S = \phi(A_r^S)$ *by* $(\phi(a))^* = \phi(a^*)$.

(2) *The algebra* $Q_r^S = \phi(A_r^S)$ *is a cellular algebra over* S *with cellular basis*

$$\mathbb{A}_r = \left\{ \phi(m_{\mathsf{st}}^\lambda) \;\middle|\; \lambda \in \widehat{A}_{r,\mathrm{perm}} \text{ and } \mathsf{s}, \mathsf{t} \in \mathrm{Std}_{r,\mathrm{perm}}(\lambda) \right\}.$$

More precisely, the cell datum is the following: the involution * *on* Q_r^S *defined in part (1); the partially ordered set* $(\widehat{A}_{r,\mathrm{perm}}, \trianglerighteq)$ *of permissible points in* $\widehat{A}_r$*; for each* $\lambda \in \widehat{A}_{r,\mathrm{perm}}$*, the index set* $\mathrm{Std}_{r,\mathrm{perm}}(\lambda)$ *of permissible paths of shape* λ*; and finally the basis* $\mathbb{A}_r$.

(3) *The set*

$$\kappa_r = \{\widetilde{m}_{\mathsf{st}}^\lambda \mid \lambda \in \widehat{A}_r \text{ and } \mathsf{s} \text{ or } \mathsf{t} \text{ is not permissible}\}$$

is an S*–basis of* $\ker(\phi_r)$.

(4) $\ker(\phi_r)$ *is the ideal* $\mathfrak{I}_r$ *in* A_r^S *generated by the set of* $\mathfrak{b}_\mu$*, where* μ *is a marginal point of* $\widehat{A}_s$ *for some* $0 < s \leqslant r$.

Proof. Since $\mathbb{B}_r$ is a basis of A_r^S, by Theorem 2.5, it follows that $\phi(\mathbb{B}_r)$ spans $Q_r^S = \phi(A_r^S)$ over S. But $\phi(\widetilde{m}_{\mathsf{st}}^\lambda) = \phi(m_{\mathsf{st}}^\lambda)$ if both s and t are permissible, and $\phi(\widetilde{m}_{\mathsf{st}}^\lambda) = 0$ otherwise. It follows that $\mathbb{A}_r$ spans Q_r^S over S, hence spans $Q_r^{\overline{\mathbb{K}}}$ over $\overline{\mathbb{K}}$. Since by assumption (Q3), $\dim_{\overline{\mathbb{K}}}(Q_r^{\overline{\mathbb{K}}}) = \sharp(\mathbb{A}_r)$, it follows that $\mathbb{A}_r$ is linearly independent over $\overline{\mathbb{K}}$. Thus $\mathbb{A}_r$ is an S–basis of Q_r^S.

The S–span of κ_r is contained in $\ker(\phi_r)$ by Theorem 2.5. On the other hand, it follows from the linear independence of $\mathbb{A}_r$ that $\ker(\phi_r)$ has trivial intersection with the S–span of $\{\widetilde{m}_{\mathsf{st}}^\lambda \mid \lambda \in \widehat{A}_r \text{ and } \mathsf{s}, \mathsf{t} \in \mathrm{Std}_{r,\mathrm{perm}}(\lambda)\}$. It follows from this that κ_r spans, and hence is a basis of, $\ker(\phi_r)$.

The cellular basis $\mathbb{B}_r$ of A_r^S satisfies $(\widetilde{m}_{\mathsf{st}}^\lambda)^* = \widetilde{m}_{\mathsf{ts}}^\lambda$, and it follows that $\ker(\phi_r)$, namely the S-span of κ_r, is globally invariant under $*$. Hence one can define an algebra involution on $Q_r^S = \phi(A_r^S)$ by $(\phi(a))^* = \phi(a^*)$.

So far, we have proved points (1) and (3), and shown that $\mathbb{A}_r$ is an S–basis of Q_r^S. Next we check that $\mathbb{A}_r$ is a cellular basis of Q_r^S, by appealing to Theorem 2.5. For $\lambda \in \widehat{A}_{r,\mathrm{perm}}$ and $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_{r,\mathrm{perm}}(\lambda)$, and for $a \in A_r^S$, we have by cellularity of A_r^S with respect to the basis $\mathbb{B}_r$,

$$\widetilde{m}_{\mathsf{st}}^\lambda a \equiv \sum_{\mathsf{v} \in \mathrm{Std}_r(\lambda)} r_{\mathsf{v}} \widetilde{m}_{\mathsf{sv}}^\lambda \qquad \mathrm{mod}\ (A_r^S)^{\rhd\lambda},$$

where the coefficients are in S and independent of s, and the sum goes over all $\mathsf{v} \in \mathrm{Std}_r(\lambda)$. When we apply ϕ, only those terms with permissible v survive:

$$\phi(m_{\mathsf{st}}^\lambda)\phi(a) \equiv \sum_{\mathsf{v} \in \mathrm{Std}_{r,\mathrm{perm}}(\lambda)} r_{\mathsf{v}} \phi(m_{\mathsf{sv}}^\lambda) \qquad \mathrm{mod}\ \phi((A_r^S)^{\rhd\lambda}).$$

Again by Theorem 2.5, we have $\phi((A_r^S)^{\rhd\lambda}) = (Q_r^S)^{\rhd\lambda}$. This verifies the multiplication axiom for a cellular basis. The involution axiom is easily verified using part (1), namely $\phi(m_{\mathsf{st}}^\lambda)^* = \phi((m_{\mathsf{st}}^\lambda)^*) = \phi(m_{\mathsf{ts}}^\lambda)$. This completes the proof of part (2).

It remains to check part (4). By construction of the basis $\mathbb{B}_r$, we have that

$$\kappa_r \subseteq \mathfrak{I}_r \subseteq \ker(\phi_r).$$

Therefore it follows from part (3) that $\ker(\phi_r) = \mathfrak{I}_r$. □

Since Q_r^S is a quotient of A_r^S, in particular its cell modules are A_r^S–modules. We observe that the cell modules of Q_r^S are quotients of cell modules of A_r^S, when regarded as A_r^S–modules.

Corollary 2.8. *Assume (Q1)–(Q3). Then*

(1) *For λ a permissible point in $\widehat{A}_r$, $\kappa(\lambda) = \mathrm{Span}_S\{\widetilde{m}^\lambda_{\mathsf{t}} \mid \mathsf{t}$ is not permissible$\}$ is an A^S_r–submodule of the cell module $\Delta^S_{A_r}(\lambda)$, and $\Delta^S_{Q_r}(\lambda) \cong \Delta^S_{A_r}(\lambda)/\kappa(\lambda)$ as A_r–modules.*

(2) *Assume $Q^{\mathbb{K}}_r$ is split semisimple and λ is a permissible point in $\widehat{A}_r$. Then $\kappa(\lambda) \otimes_S K = \mathrm{Span}_{\mathbb{K}}\{\widetilde{m}^\lambda_{\mathsf{t}} \mid \mathsf{t}$ is not permissible$\}$ is the radical of $\Delta^{\mathbb{K}}_{A_r}(\lambda)$.*

Proof. By Theorem 2.5, we have $\phi((A^S_r)^{\trianglerighteq\lambda}) = (Q^S_r)^{\trianglerighteq\lambda}$ and $\phi((A^S_r)^{\triangleright\lambda}) = (Q^S_r)^{\triangleright\lambda}$, so ϕ induces an A^S_r–A^S_r bimodule homomorphism $(A^S_r)^{\trianglerighteq\lambda}/(A^S_r)^{\triangleright\lambda} \to (Q^S_r)^{\trianglerighteq\lambda}/(Q^S_r)^{\triangleright\lambda}$. For a fixed permissible $\mathsf{s} \in \mathrm{Std}_{r,\mathrm{perm}}(\lambda)$,

$$\widetilde{m}^\lambda_{\mathsf{st}} + (A^S_r)^{\triangleright\lambda} \mapsto \phi(\widetilde{m}^\lambda_{\mathsf{st}}) + (Q^S_r)^{\triangleright\lambda}$$

defines a right A_r–module homomorphism from $\Delta^S_{A_r}(\lambda)$ to $\Delta^S_{Q_r}(\lambda)$, with kernel $\kappa(\lambda)$. It follows that $\Delta^{\mathbb{K}}_{Q_r}(\lambda) \cong \Delta^{\mathbb{K}}_{A_r}(\lambda)/(\kappa(\lambda) \otimes_S \mathbb{K})$. If $Q^{\mathbb{K}}_r$ is split semisimple, then its cell modules are simple, so $\Delta^{\mathbb{K}}_{Q_r}(\lambda)$ is the simple head of $\Delta^{\mathbb{K}}_{A_r}(\lambda)$ and $\kappa(\lambda) \otimes_S K$ is the radical. □

Let us review what we have accomplished here, with a view towards our applications in Sections 7 and 8. Suppose we have a tower $(A^R_r)_{r\geqslant 0}$ of diagram algebras, satisfying axioms (D1) to (D6) of Section 1.2. and specializations A^S_r together with maps $\phi_r : A^S_r \to Q^S_r$ which satisfy the conditions of Definition 2.1. Then we can produce all of the following:

(1) A modified Murphy basis $\{\widetilde{m}^\lambda_{\mathsf{st}}\}$ of each of the algebras A^S_r which is equivalent to the basis $\{m^\lambda_{\mathsf{st}}\}$ of Theorem 1.12. If s and t are both permissible, then $\widetilde{m}^\lambda_{\mathsf{st}} = m^\lambda_{\mathsf{st}}$. However, if either s or t is impermissible, then $\widetilde{m}^\lambda_{\mathsf{st}}$ belongs to the kernel of ϕ_r.
(2) An S-linear basis of $\ker(\phi_r)$ consisting of those $\widetilde{m}^\lambda_{\mathsf{st}}$ with at least one of s or t not permissible.
(3) A (small) generating set for $\ker(\phi_r)$ as an ideal in A^S_r.
(4) A cellular basis of Q^S_r consisting of the image under ϕ_r of Murphy basis elements m^λ_{st} such that both s and t are permissible.

In the applications, points (2) and (3) of this list are two different versions of a second fundamental theorem of invariant theory, while point (4) shows that the classical centralizer algebras – Brauer's centralizer algebras on orthogonal or symplectic tensor space, or the image of the walled Brauer algebras on mixed tensor space – are cellular algebras over the integers.

3. Supplements on quotient towers

In this section, we provide some supplementary material on quotient towers. This material is not strictly needed to appreciate the applications in the subsequent sections, so it could be safely skipped on the first reading.

3.1. **Quotient towers are themselves towers of diagram algebras.** In this section, we show that the tower $(Q^S_r)_{r\geqslant 0}$ is restriction coherent, and that the d–branching factors associated to restrictions of cell modules in this tower are just those obtained by applying ϕ to the d–branching factors of the tower $(A^S_r)_{r\geqslant 0}$. It follows that the tower $(Q^S_r)_{r\geqslant 0}$ satisfies all of the axioms (D1) to (D6) of Section 1.2, with the possible exception of axiom (D4). If we assume that the quotient algebras $Q^{\mathbb{K}}_r$ are split semisimple – and this will be valid in our applications – then all the consequences of (D1) to (D6) are available to us; see Section 1.3 and [4].

First we demonstrate that the tower of cellular algebras $(Q^S_r)_{r\geqslant 0}$ is restriction coherent. We write Q_k for Q^S_k. Write $\widehat{Q}_r$ for $\widehat{A}_{r,\mathrm{perm}}$. We have the branching diagram $\widehat{Q} = \bigsqcup_r \widehat{Q}_r$, with the branching rule $\lambda \to \mu$ for $\lambda \in \widehat{Q}_{r-1}$ and $\mu \in \widehat{Q}_r$ if and only if $\lambda \to \mu$ in $\widehat{A}$. For $\nu \in \widehat{Q}_r$, the set of $\mu \in \widehat{Q}_{r-1}$ such that $\mu \to \nu$ is totally ordered, because it is a subset of the set of $\mu \in \widehat{A}_{r-1}$ such that $\mu \to \nu$. For $\nu \in \widehat{Q}_r$ let $\Delta_{Q_r}(\nu)$ denote the corresponding cell module of Q_r,

$$\Delta_{Q_r}(\nu) = \mathrm{Span}_R \left\{ \phi(m_\nu d_{\mathsf{t}}) + Q^{\triangleright\nu}_r \;\middle|\; \mathsf{t} \in \mathrm{Std}_{r,\mathrm{perm}}(\nu) \right\}.$$

Lemma 3.1. *Let $r \geqslant 1$, let $\nu \in \widehat{Q}_r$ and $\mu \in \widehat{Q}_{r-1}$ with $\mu \to \nu$. Let $u = u_{\mu\to\nu}$. Let $x \in \phi(m_\mu)Q_{r-1} \cap Q_{r-1}^{\rhd\mu}$. Then*

$$\phi(u^*)x \equiv \sum_{\mathsf{z}} r_{\mathsf{z}}\phi(m_\nu d_{\mathsf{z}}) \mod Q_r^{\rhd\nu},$$

where the sum is over $\mathsf{z} \in \mathrm{Std}_{r,\mathrm{perm}}(\nu)$ such that $\mathsf{z}(r-1) \rhd \mu$.

Proof. Since $x \in \phi(m_\mu)Q_{r-1}$, there exists $b \in A_{r-1}^S$ such that $x = \phi(m_\mu b)$. Using Lemma 2.4, we can write

$$m_\mu b = \sum_{\mathsf{s}\in\mathrm{Std}_{r-1,\mathrm{perm}}(\mu)} r_{\mathsf{s}} m_\mu d_{\mathsf{s}} + y_1 + y_2,$$

where $y_1 \in \ker(\phi)$ and $y_2 \in (A_{r-1}^S)^{\rhd\mu} \cap m_\mu A_{r-1}^S$. But since $x = \phi(m_\mu b) \in Q_{r-1}^{\rhd\mu}$, all the coefficients r_s are zero and

$$m_\mu b = y_1 + y_2$$

Now apply Lemma 2.3 to y_2,

$$u^* m_\mu b = u^* y_1 + \sum_{\mathsf{v}} \alpha_{\mathsf{v}} m_\nu d_{\mathsf{v}} + y_3,$$

where the sum is over $\mathsf{v} \in \mathrm{Std}_r(\nu)$ such that $\mathsf{v}(r-1) \rhd \mu$, and $y_3 \in (A_r^S)^{\rhd\nu}$. The v appearing in the sum may not be permissible, but we can apply Lemma 2.4 to any term $m_\nu d_{\mathsf{v}}$ with v not permissible, to replace it with a linear combination of terms $m_\nu d_{\mathsf{z}}$, modulo $(\ker(\phi) + (A_r^S)^{\rhd\nu})$, where $\mathsf{z} \in \mathrm{Std}_r(\nu)$ is permissible and satisfies $\mathsf{z} \succ \mathsf{v}$. But if $\mathsf{z} \succ \mathsf{v}$, then $\mathsf{z}(r-1) \trianglerighteq \mathsf{v}(r-1) \rhd \mu$. ☐

Corollary 3.2. *Let $r \geqslant 1$, $\nu \in \widehat{Q}_r$, $\mathsf{t} \in \mathrm{Std}_{r,\mathrm{perm}}(\nu)$, and $a \in Q_{r-1}$. Write $\mu = \mathsf{t}(r-1)$ and $\mathsf{t}' = \mathsf{t}_{[0,r-1]}$. Suppose*

$$\phi(m_\mu d_{\mathsf{t}'})a \equiv \sum_{\mathsf{s}\in\mathrm{Std}_{r-1,\mathrm{perm}}(\mu)} r_{\mathsf{s}}\phi(m_\mu d_{\mathsf{s}}) \mod Q_{r-1}^{\rhd\mu}.$$

Then

$$\phi(m_\nu d_{\mathsf{t}})a \equiv \sum_{\mathsf{s}\in\mathrm{Std}_{r-1,\mathrm{perm}}(\mu)} r_{\mathsf{s}}\phi(m_\nu d_{\mu\to\nu} d_{\mathsf{s}}) + \sum_{\mathsf{z}} r_{\mathsf{z}}\phi(m_\nu d_{\mathsf{z}}) \mod Q_r^{\rhd\nu},$$

where the sum is over $\mathsf{z} \in \mathrm{Std}_{r,\mathrm{perm}}(\nu)$ such that $\mathsf{z}(r-1) \rhd \mu$.

Proof. Write

$$\phi(m_\mu d_{\mathsf{t}'})a \equiv \sum_{\mathsf{s}\in\mathrm{Std}_{r-1,\mathrm{perm}}(\mu)} r_{\mathsf{s}}\phi(m_\mu d_{\mathsf{s}}) + y,$$

where $y \in Q_{r-1}^{\rhd\mu} \cap \phi(m_\mu)Q_{r-1}$. Multiply both sides on the left by $\phi(u^*)$, where $u = u_{\mu\to\nu}$, and apply Lemma 3.1 to $\phi(u^*)y$. ☐

Proposition 3.3. *Let $r \geqslant 1$, let $\nu \in \widehat{Q}_r$, and let $\mu(1) \rhd \mu(2) \rhd \cdots \rhd \mu(s)$ be the list of $\mu \in \widehat{Q}_{r-1}$ such that $\mu \to \nu$. Define*

$$M_j = \mathrm{Span}_R \left\{ \phi(m_\nu d_{\mathsf{t}}) + Q_r^{\rhd\nu} \;\middle|\; \mathsf{t} \in \mathrm{Std}_{r,\mathrm{perm}}(\nu), \mathsf{t}(r-1) \trianglerighteq \mu(j) \right\}.$$

and $M_0 = (0)$. Then

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_s = \Delta_{Q_r}(\nu)$$

is a filtration of $\Delta_{Q_r}(\nu)$ by Q_{r-1} submodules, and $M_j/M_{j-1} \cong \Delta_{Q_{r-1}}(\mu(j))$.

Proof. Immediate from Corollary 3.2. ☐

Corollary 3.4. *The branching factors associated to the filtrations in Proposition 3.3 are $\phi(d_{\mu\to\nu})$ for $\mu \in \widehat{Q}_{r-1}$ and $\nu \in \widehat{Q}_r$ with $\mu \to \nu$.*

Proof. The isomorphism $M_j/M_{j-1} \to \Delta_{Q_{r-1}}(\mu(j))$ is

$$\phi(m_\nu d_{\mu(j)\to\nu} d_{\mathsf{s}}) + M_{j-1} + Q_r^{\rhd\nu} \mapsto \phi(m_{\mu(j)} d_{\mathsf{s}}) + Q_{r-1}^{\rhd\mu(j)}.$$

□

Proposition 3.5. *Assume (Q1)–(Q3), and that $Q_r^{\mathbb{K}}$ is split semisimple for all r. It follows that the tower $(Q_r^S)_{r\geqslant 0}$ satisfy axioms (D1)–(D6) of Section 1.2.*

Proof. One only has to observe that the branching coefficients satisfy

$$\phi(m_\nu)\phi(d_{\mu\to\nu}) = \phi(u_{\mu\to\nu})^*\phi(m_\mu).$$

□

Remark 3.6. The assumption that $Q_r^{\mathbb{K}}$ is split semisimple for all r implies that $\Delta^S_{Q_{r-1}}(\mu)$ appears as a subquotient in a cell filtration of $\mathrm{Res}(\Delta^S_{Q_r}(\nu))$ if and only if the simple $Q^{\mathbb{K}}_{r-1}$–module $\Delta^K_{Q_{r-1}}(\mu)$ is a direct summand of $\mathrm{Res}(\Delta^K_{Q_r}(\nu))$, if and only if $\mu \to \nu$ in $\widehat{Q}$. See [17, Lemma 2.2].

3.2. **Jucys–Murphy elements in quotient towers.** Consider a sequence of cellular algebras $(A_r)_{r\geqslant 0}$ over an integral domain R with field of fractions $\mathbb{F}$, satisfying the properties (D1)–(D6) of Section 1.2. In [4, Definition 4.1], following [18], one defines a sequence $\{L_i\}_{i\geqslant 1}$ of *additive Jucys–Murphy elements* by the two conditions:

(JM1) For $r \geqslant 1$, $L_r \in A_r^R$, $L_r = L_r^*$, and L_r commutes pointwise with A^R_{r-1}.

(JM2) For all $r \geqslant 1$ and $\lambda \in \widehat{A}_r$, there exists $d(\lambda) \in R$ such that $L_1 + \cdots + L_r$ acts as the scalar $d(\lambda)$ on the cell module $\Delta_r^R(\lambda)$.

Using that $A_r^{\mathbb{F}}$ is split semisimple for all r, condition (JM2) is equivalent to

(JM3) For all $r \geqslant 1$, $L_1 + \cdots + L_r$ is in the center of A_r^R.

For each edge $\lambda \to \mu$ in the branching diagram $\hat{A}$ for $(A_r^{\mathbb{F}})_{r\geqslant 0}$, write $\kappa(\lambda\to\mu) = d(\mu) - d(\lambda) \in R$, where by convention $d(\varnothing) = 0$. For a path $\mathsf{t} \in \mathrm{Std}_r$ and $i \leqslant r$, write $\kappa_{\mathsf{t}}(i)$ for $\kappa(\mathsf{t}(i-1) \to \mathsf{t}(i))$. The elements $\kappa(\lambda\to\mu)$ or $\kappa_{\mathsf{t}}(i)$ are called contents since they generalize the contents of standard tableaux in the theory of the symmetric group. It is shown in [4, Section 4], strengthening results of [18], that

(JM4) $f_{\mathsf{t}}^\lambda L_i = \kappa_{\mathsf{t}}(i) f_{\mathsf{t}}$ for all paths $\mathsf{t} \in \mathrm{Std}_r(\lambda)$ and for $i \leqslant r$, and

(JM5) $m_{\mathsf{t}}^\lambda L_i = \kappa_{\mathsf{t}}(i) m_{\mathsf{t}}^\lambda + \sum_{\mathsf{s}\rhd\mathsf{t}} r_{\mathsf{s}} m_{\mathsf{s}}^\lambda$, for some coefficients $r_{\mathsf{s}} \in R$.

Condition (JM5) is an instance of Mathas's abstraction of Jucys–Murphy elements from [31]. We note that conditions (JM4) and (JM5) do not depend on Mathas's separation condition being satisfied, as in [31, Section 3].

Suppose we are given an additive sequence of JM elements. Then conditions (JM1), (JM2), (JM3), and (JM5) remain valid in any specialization $A_r^S = A_r^R \otimes_R S$, where L_i is replaced by $L_i \otimes 1_S$ and $d(\lambda)$ and $\kappa(\lambda \to \mu)$ by their images in S, so in particular every specialization has JM elements in the sense of Mathas.

Now, finally, suppose the hypotheses (Q1)–(Q3) are satisfied and that the quotient algebras $Q_r^{\mathbb{K}}$ are split semisimple, so that the quotient tower $(Q_r^S)_{r\geqslant 0}$ is a sequence of diagram algebras satisfying the properties (D1)–(D6). Clearly, the defining conditions (JM1) and (JM2) for JM elements are satisfied, with L_i replaced by $\phi(L_i \otimes 1_S)$, and (JM3) follows. (Of course, versions of (JM4) and (JM5) must hold as well, but this is not very useful in this generality as we cannot relate the seminormal bases of the quotient tower with that of the original tower. This defect is removed in Section 3.3.)

This discussion holds just as well for *multiplicative Jucys–Murphy elements*, see [4, Definition 4.3] and [18].

3.3. **Seminormal bases of quotient towers.** In this section we examine seminormal bases and seminormal representations in quotient towers. We work in the following setting: we assume the setting of Section 2.1, in particular that (Q1)–(Q3) are satisfied; in particular, recall the notation and terminology from Remark 2.2. We assume this existence of additive or multiplicative JM elements for the tower $(A_r^R)_{r\geqslant 0}$, as in Section 3.2; in particular, conditions (JM4) and (JM5) of Section 3.2 hold. We assume, moreover, that the separation condition of Mathas holds; that is if s and t are distinct paths in Std_r, then there exists $i \leqslant r$ such that $\kappa_{\mathsf{t}}(i) \neq \kappa_{\mathsf{s}}(i)$. We assume that the quotient algebras $Q_r^{\mathbb{K}}$ are split semisimple, so that the quotient tower $(Q_r^S)_{r\geqslant 0}$ is a tower of diagram algebras satisfying (D1)–(D6). As a tower of diagram algebras, $(Q_r^S)_{r\geqslant 0}$ has its own seminormal bases. Finally, we assume the following condition, which will allow us to connect these seminormal bases to those of the original tower:

(SN) Whenever $\mathsf{t} \in \mathrm{Std}_{r,\mathrm{perm}}$, it follows that F_{t} is evaluable.

It follows from (SN) that if $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_r(\lambda)$ are permissible, then f_{t}^{λ} and $F_{\mathsf{st}}^{\lambda}$ are evaluable, and also that $\langle f_{\mathsf{s}}^{\lambda}, f_{\mathsf{t}}^{\lambda}\rangle \in R_{\mathfrak{p}}$.

We are interested in verifying these assumptions, and in particular condition (SN), for quotients of diagram algebras acting on tensor spaces. The following lemma provides a sufficient condition for (SN) to hold. Associate to each path $\mathsf{t} \in \mathrm{Std}_r$ its *content* sequence $\kappa_{\mathsf{t}}(i)$ for $1 \leqslant i \leqslant r$ and its *residue* sequence $r_{\mathsf{t}}(i) = \pi(\kappa_{\mathsf{t}}(i))$. Say two paths s, t are *residue equivalent*, and write $\mathsf{t} \approx \mathsf{s}$, if they have the same residue sequences.

Lemma 3.7. *Suppose that each permissible path in* Std_r *is residue equivalent only to itself. Then* (SN) *holds.*

Proof. This follows from [31, Lemma 4.2]. It is also easy to prove directly using the following recursive formula for the idempotents F_{t}. For $\mathsf{t} \in \mathrm{Std}_{k+1}$, let t' denote the truncation $\mathsf{t}' = \mathsf{t}_{[0,k]}$. Then

$$F_{\mathsf{t}} = F_{\mathsf{t}'} \prod_{\substack{\mathsf{s}\neq\mathsf{t} \\ \mathsf{s}'=\mathsf{t}'}} \frac{L_r - \kappa_{\mathsf{s}}(r)}{\kappa_{\mathsf{t}}(r) - \kappa_{\mathsf{s}}(r)}.$$

□

Remark 3.8. We will use the hypothesis of Lemma 3.7 in the following equivalent form: For all $r \geqslant 1$ and for all $\mathsf{t}, \mathsf{s} \in \mathrm{Std}_r$ with $\mathsf{s}' = \mathsf{t}'$, if at least one of s, t is permissible, then $\pi(\kappa_t(r)) \neq \pi(\kappa_s(r))$.

Example 3.9. In the case of the symmetric group algebras acting on ordinary or mixed tensor space, no specialization of the ground ring is involved, so condition (SN) is vacuous. The sufficient condition of Lemma 3.7 is verified for the Brauer algebras acting on symplectic tensor space in Section 7.3, and for the walled Brauer algebras in Appendix C.4 in the arXiv version of this paper. It is verified for the partition algebras, with an appropriate permissibility condition, in [3, Section 6]. For the Brauer algebra acting on orthogonal tensor space, the sufficient condition of Lemma 3.7 holds for odd integers values, but fails for even integer values; in the latter case a more subtle argument is needed in order to verify (SN); for this, see [10].

Lemma 3.10. *Let* $a \in A_r^R$ *and let* $a(\mathsf{s}, \mathsf{t})$ *denote the matrix coefficients of* a *with respect to the seminormal basis* $\{f_{\mathsf{t}}^{\lambda}\}$ *of* $\Delta_{A_r}^{\mathbb{F}}(\lambda)$,

$$f_{\mathsf{t}}^{\lambda} a = \sum a(\mathsf{s}, \mathsf{t}) f_{\mathsf{s}}^{\lambda}.$$

If s, t *are permissible paths in* $\mathrm{Std}_r(\lambda)$*, then* $a(\mathsf{s}, \mathsf{t}) \in R_{\mathfrak{p}}$.

Proof. We have $f_{\mathsf{t}}^{\lambda} a F_{\mathsf{s}} = a(\mathsf{s}, \mathsf{t}) f_{\mathsf{s}}^{\lambda} = a(\mathsf{s}, \mathsf{t}) m_{\mathsf{s}}^{\lambda} + \sum_{\mathsf{v} \rhd \mathsf{s}} \gamma_{\mathsf{v}} m_{\mathsf{v}}^{\lambda}$, using Theorem 1.21. By assumption (SN), the element on the left side of the equation is evaluable, and hence the coefficients on the right side lie in $R_{\mathfrak{p}}$. □

We will now construct a cellular basis $\{h^\lambda_{\mathsf{st}} \mid \lambda \in \hat{A}_r \text{ and } \mathsf{s}, \mathsf{t} \in \mathrm{Std}_r(\lambda)\}$ with the properties that

(1) If both s, t are permissible paths, then $h^\lambda_{\mathsf{st}} = \pi(F^\lambda_{\mathsf{st}})$.
(2) If at least one of s, t is not permissible, then $h^\lambda_{\mathsf{st}} \in \ker(\phi)$.

If $\mathsf{t} \in \mathrm{Std}_r$ is permissible, define $[\mathsf{t}] = \{\mathsf{t}\}$. If t is not permissible, then let $[\mathsf{t}]$ denote the set of paths $\mathsf{s} \in \mathrm{Std}_r$ such that $\mathsf{s} \approx \mathsf{t}$. By assumption (SN) and [31, Lemma 4.2], $F_{[\mathsf{t}]} := \sum_{\mathsf{s}\in[\mathsf{t}]} F_{\mathsf{s}}$ is an evaluable idempotent. For any $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_r(\lambda)$, permissible or not, define $h^\lambda_{\mathsf{t}} = \widetilde{m}^\lambda_{\mathsf{t}}\pi(F_{[\mathsf{t}]})$, and $h^\lambda_{\mathsf{st}} = \pi(F_{[\mathsf{s}]})\widetilde{m}^\lambda_{\mathsf{st}}\pi(F_{[\mathsf{t}]})$.

Lemma 3.11. *For $\lambda \in \hat{A}_r$ and $\mathsf{t} \in \mathrm{Std}_r(\lambda)$ there exist coefficients $\beta_{\mathsf{s}} \in \mathbb{K}$ such that*

$$h^\lambda_{\mathsf{t}} = m^\lambda_{\mathsf{t}} + \sum_{\mathsf{s}\succ\mathsf{t}} \beta_{\mathsf{s}} m^\lambda_{\mathsf{s}}. \tag{3.1}$$

Proof. We know from Equation (2.7) that the basis elements $\widetilde{m}^\lambda_{\mathsf{t}}$ of $\Delta^S_{A_r}(\lambda)$ are related to the basis elements m^λ_{t} by a unitriangular transformation with coefficients in S,

$$\widetilde{m}^\lambda_{\mathsf{t}} = m^\lambda_{\mathsf{t}} + \sum_{\mathsf{s}\succ\mathsf{t}} \alpha_{\mathsf{s}} m^\lambda_{\mathsf{s}}.$$

Lift this relation to $\Delta^R_{A_r}(\lambda)$, defining $\widetilde{m}^\lambda_{\mathsf{t}}$ by $\widetilde{m}^\lambda_{\mathsf{t}} = m^\lambda_{\mathsf{t}} + \sum_{\mathsf{s}\succ\mathsf{t}} \alpha'_{\mathsf{s}} m^\lambda_{\mathsf{s}}$, where $\alpha'_{\mathsf{s}} \in R$ and $\pi(\alpha'_{\mathsf{s}}) = \alpha_{\mathsf{s}}$. Apply Theorem 1.21 to get $\widetilde{m}^\lambda_{\mathsf{t}} = f^\lambda_{\mathsf{t}} + \sum_{\mathsf{s}\succ\mathsf{t}} \gamma_{\mathsf{s}} f^\lambda_{\mathsf{s}}$, where the coefficients are now in $\mathbb{F}$. Applying $F_{[\mathsf{t}]}$ yields $\widetilde{m}^\lambda_{\mathsf{t}} F_{[\mathsf{t}]} = f^\lambda_{\mathsf{t}} + \sum_{\mathsf{s}\succ\mathsf{t},\mathsf{s}\in[\mathsf{t}]} \gamma_{\mathsf{s}} f^\lambda_{\mathsf{s}}$, and using Theorem 1.21 again produces

$$\widetilde{m}^\lambda_{\mathsf{t}} F_{[\mathsf{t}]} = m^\lambda_{\mathsf{t}} + \sum_{\mathsf{s}\succ\mathsf{t}} \beta'_{\mathsf{s}} m^\lambda_{\mathsf{s}}.$$

The coefficients are *a priori* in $\mathbb{F}$, but since $\widetilde{m}^\lambda_{\mathsf{t}} F_{[\mathsf{t}]}$ is evaluable, the coefficients are evaluable. Finally applying π yields (3.1), with $\beta_{\mathsf{s}} = \pi(\beta'_{\mathsf{s}})$. □

Corollary 3.12.

(1) *For $\lambda \in \hat{A}_r$, $\{h^\lambda_{\mathsf{t}} \mid \mathsf{t} \in \mathrm{Std}_r(\lambda)\}$ is a basis of the cell module $\Delta^{\mathbb{K}}_{A_r}(\lambda)$ and $\{h^\lambda_{\mathsf{st}} \mid \lambda \in \hat{A}_r$ and $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_r(\lambda)\}$ is a cellular basis of $A^{\mathbb{K}}_r$ equivalent to the Murphy basis.*
(2) *If both s, t are permissible paths, then $h^\lambda_{\mathsf{st}} = \pi(F^\lambda_{\mathsf{st}})$. If at least one of s, t is not permissible, then $h^\lambda_{\mathsf{st}} \in \ker(\phi)$.*

Proof. The first assertion follows from Lemma 3.11 by familiar argument, compare Theorem 2.5. The second assertion is evident from the construction and properties of $\{\widetilde{m}^\lambda_{\mathsf{st}}\}$. □

When $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_r(\lambda)$ are permissible, define $\bar{F}^\lambda_{\mathsf{st}} = \phi \circ \pi(F^\lambda_{\mathsf{st}}) = \phi(h^\lambda_{\mathsf{st}})$ and $\bar{F}_{\mathsf{t}} = \phi \circ \pi(F_{\mathsf{t}})$.

Corollary 3.13.

(1) *The set $\{\bar{F}^\lambda_{\mathsf{st}} \mid \lambda$ is permissible and $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_{r,\mathrm{perm}}(\lambda)\}$ is a basis of $Q^{\mathbb{K}}_r$.*
(2) *The set $\{h^\lambda_{\mathsf{st}} \mid \lambda \in \widehat{A}_r$ and s or t is not permissible$\}$ is a $\mathbb{K}$–basis of $\ker(\phi_r)$.*

Proof. Adapt the proof of Theorem 2.7 parts (2) and (3). □

In Corollary 2.8, for $\lambda \in \hat{A}_r$ permissible, we identified $\Delta^{\mathbb{K}}_{Q_r}(\lambda)$ with the simple head of $\Delta^{\mathbb{K}}_{A_r}(\lambda)$, and we showed that the radical of $\Delta^{\mathbb{K}}_{A_r}(\lambda)$ is $\kappa^{\mathbb{K}}(\lambda) := \mathrm{Span}_{\mathbb{K}}\{\widetilde{m}^\lambda_{\mathsf{t}} \mid \mathsf{t} \text{ is not permissible}\}$. Overloading notation, let us write ϕ for the quotient map $\phi : \Delta^{\mathbb{K}}_{A_r}(\lambda) \to \Delta^{\mathbb{K}}_{A_r}(\lambda)/\kappa^{\mathbb{K}}(\lambda)$.

For $\mathsf{t} \in \mathrm{Std}_r(\lambda)$ permissible, write $\bar{f}^\lambda_{\mathsf{t}} = \phi(h^\lambda_{\mathsf{t}}) = \phi(\widetilde{m}^\lambda_{\mathsf{t}}\pi(F_{\mathsf{t}}))$.

Corollary 3.14.

(1) *For $\lambda \in \hat{A}_r$ permissible, $\{\bar{f}^\lambda_{\mathsf{t}} \mid \mathsf{t}$ is permissible$\}$ is a basis of $\Delta^{\mathbb{K}}_{Q_r}(\lambda)$.*
(2) *The set $\{h^\lambda_{\mathsf{t}} \mid \mathsf{t}$ is not permissible$\}$ is a basis of $\mathrm{rad}(\Delta^{\mathbb{K}}_{A_r}(\lambda))$.*

Proof. Write Δ for $\Delta^{\mathbb{K}}_{A_r}(\lambda)$. For t not permissible, we have $h^\lambda_{\mathsf{t}} = \widetilde{m}^\lambda_{\mathsf{t}}\pi(F_{[\mathsf{t}]}) \in \mathrm{rad}(\Delta)$ because $\widetilde{m}^\lambda_{\mathsf{t}} \in \mathrm{rad}(\Delta)$ by Corollary 2.8, and $\mathrm{rad}(\Delta)$ is a submodule. The two conclusions follow by a dimension argument as in the proof of Theorem 2.7 parts (2) and (3). □

The following result says that the seminormal representations of the quotient algebras $Q^{\mathbb{K}}_r$ are obtained as truncations of the seminormal representations of the diagram algebras $A^{\mathbb{F}}_r$. The application of this result to the Brauer algebras and their quotients acting on orthogonal or symplectic tensor space recovers a well-known phenomenon, which is implicit in [35, 28, 36], and explicit in [6, Theorem 5.4.3]. See also [10].

Theorem 3.15.

(1) *The family of idempotents $\bar{F}_{\mathsf{t}} = \phi \circ \pi(F_{\mathsf{t}})$, where $r \geqslant 1$ and $\mathsf{t} \in \mathrm{Std}_r$ is permissible, is the family of Gelfand-Zeitlin idempotents for the tower $Q^{\mathbb{K}}_r$.*

In the following statements, let λ and μ be permissible points in $\hat{A}_r$ for some r, and let $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_{r,\mathrm{perm}}(\lambda)$, and $\mathsf{u}, \mathsf{v} \in \mathrm{Std}_{r,\mathrm{perm}}(\mu)$.

(2) $\bar{f}^\lambda_{\mathsf{t}}\bar{F}^\mu_{\mathsf{uv}} = \delta_{\lambda,\mu}\delta_{\mathsf{t},\mathsf{u}}\pi(\langle f^\lambda_{\mathsf{t}}, f^\lambda_{\mathsf{t}}\rangle)\bar{f}^\lambda_{\mathsf{v}}$, *and* $\bar{F}^\lambda_{\mathsf{st}}\bar{F}^\mu_{\mathsf{uv}} = \delta_{\lambda,\mu}\delta_{\mathsf{t},\mathsf{u}}\pi(\langle f^\lambda_{\mathsf{t}}, f^\lambda_{\mathsf{t}}\rangle)\bar{F}^\lambda_{\mathsf{sv}}$.

(3) $\pi(\langle f^\lambda_{\mathsf{t}}, f^\lambda_{\mathsf{s}}\rangle) \neq 0$ *if and only if* $\mathsf{t} = \mathsf{s}$.

(4) $\pi(\langle f^\lambda_{\mathsf{t}}, f^\lambda_{\mathsf{t}}\rangle)^{-1}\bar{F}^\lambda_{\mathsf{tt}} = \bar{F}_{\mathsf{t}}$

(5) *The set of elements $\bar{E}^\lambda_{\mathsf{st}} = \pi(\langle f^\lambda_{\mathsf{s}}, f^\lambda_{\mathsf{s}}\rangle)^{-1}\bar{F}^\lambda_{\mathsf{st}}$ for $\lambda \in \widehat{A}_r$ and $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_{r,\mathrm{perm}}(\lambda)$ is a complete family of matrix units with $\bar{E}^\lambda_{\mathsf{st}}\bar{E}^\mu_{\mathsf{uv}} = \delta_{\lambda,\mu}\delta_{\mathsf{t},\mathsf{u}}\bar{E}^\lambda_{\mathsf{sv}}$, and $\bar{E}^\lambda_{\mathsf{ss}} = \bar{F}_{\mathsf{s}}$.*

(6) *Suppose that $a \in A^R_r$ has matrix coefficients $a(\mathsf{s}, \mathsf{t})$ with respect to the seminormal basis $\{f^\lambda_{\mathsf{t}}\}$ of $\Delta^{\mathbb{F}}_{A_r}(\lambda)$,*

$$f^\lambda_{\mathsf{t}} a = \sum_{\mathsf{s} \in \mathrm{Std}_r(\lambda)} a(\mathsf{s}, \mathsf{t}) f^\lambda_{\mathsf{s}}.$$

Then for $\mathsf{t} \in \mathrm{Std}_r(\lambda)$ permissible, we have

$$\bar{f}^\lambda_{\mathsf{t}}\phi \circ \pi(a) = \sum_{\mathsf{s} \in \mathrm{Std}_{r,\mathrm{perm}}(\lambda)} \pi(a(\mathsf{s}, \mathsf{t}))\bar{f}^\lambda_{\mathsf{s}}. \tag{3.2}$$

Proof. We remark that if t is permissible, then statement (2) follows from the definitions and the corresponding properties of the elements f^λ_{t} in $\Delta^{\mathbb{F}}_{A_r}(\lambda)$ and F^μ_{uv} in $A^{\mathbb{F}}_r$, see [4, Lemma 3.8]. (This follows because $[\mathsf{t}] = \{\mathsf{t}\}$ by definition, when t is permissible.) We already know from [4, Lemma 3.8] that $\langle f^\lambda_{\mathsf{t}}, f^\lambda_{\mathsf{s}}\rangle = 0$ if $\mathsf{s} \neq \mathsf{t}$ and also that for all t, $F^\lambda_{\mathsf{tt}} = \langle f^\lambda_{\mathsf{t}}, f^\lambda_{\mathsf{t}}\rangle F_{\mathsf{t}}$. It follows that $\bar{F}^\lambda_{\mathsf{tt}} = \pi(\langle f^\lambda_{\mathsf{t}}, f^\lambda_{\mathsf{t}}\rangle)\bar{F}_{\mathsf{t}}$. If for some t, $\pi(\langle f^\lambda_{\mathsf{t}}, f^\lambda_{\mathsf{t}}\rangle) = 0$, then $\bar{F}^\lambda_{\mathsf{tt}} = 0$, contradicting Corollary 3.13. This proves points (3) and (4) and point (5) also follows from the previous statements.

For $\mathsf{t} \in \mathrm{Std}_{r,\mathrm{perm}}(\lambda)$ and $\mathsf{v} \in \mathrm{Std}_{r,\mathrm{perm}}(\mu)$, we have $\bar{f}^\mu_{\mathsf{v}}\bar{F}_{\mathsf{t}} = \delta_{\lambda,\mu}\delta_{\mathsf{v},\mathsf{t}}\bar{f}^\mu_{\mathsf{v}}$, and it follows that $\sum_{\mathsf{t} \in \mathrm{Std}_{r,\mathrm{perm}}(\lambda)} \bar{F}_{\mathsf{t}}$ is the minimal central idempotent in $Q^{\mathbb{K}}_r$ corresponding to the simple module $\Delta^{\mathbb{K}}_{Q_r}(\lambda)$. Let $s \leqslant r$, $\mathsf{s} \in \mathrm{Std}_{s,\mathrm{perm}}$, $\mathsf{t} \in \mathrm{Std}_{r,\mathrm{perm}}$. We have $F_{\mathsf{s}}F_{\mathsf{t}} = \delta_{\mathsf{s},\mathsf{t}_{[0,s]}}F_{\mathsf{t}}$, in $A^{\mathbb{F}}_r$ from the definition of the Gelfand-Zeitlin idempotents. But then the corresponding property $\bar{F}_{\mathsf{s}}\bar{F}_{\mathsf{t}} = \delta_{\mathsf{s},\mathsf{t}_{[0,s]}}\bar{F}_{\mathsf{t}}$ holds in $Q^{\mathbb{K}}_r$. By [18, Lemma 3.10], these properties characterize the family of Gelfand-Zeitlin idempotents, so point (1) holds.

For point (6), suppose that $a \in A^R_r$ and that a has matrix coefficients $a(\mathsf{s}, \mathsf{t})$ with respect to the seminormal basis $\{f^\lambda_{\mathsf{t}}\}$ of $\Delta^{\mathbb{F}}_{A_r}(\lambda)$. Then when t and s are both permissible elements of $\mathrm{Std}_r(\lambda)$, we have $m^\lambda_{\mathsf{t}}F_{\mathsf{t}}aF_{\mathsf{s}} = a(\mathsf{s}, \mathsf{t})m^\lambda_{\mathsf{s}}F_{\mathsf{s}}$. As this equality involves evaluable elements, we can apply $\phi \circ \pi$ to get $\bar{f}^\lambda_{\mathsf{t}}\phi \circ \pi(a)\bar{F}_{\mathsf{s}} = \pi(a(\mathsf{s}, \mathsf{t}))\bar{f}^\lambda_{\mathsf{s}}$. Now sum over s and use that $\sum_{\mathsf{s} \in \mathrm{Std}_{r,\mathrm{perm}}(\lambda)} \bar{F}_{\mathsf{s}}$ acts as the identity on the cell module $\Delta^{\mathbb{K}}_{Q_r}(\lambda)$. This yields (3.2). □

4. The Murphy and dual Murphy bases of the symmetric groups

A *partition* λ of r, denoted $\lambda \vdash r$, is defined to be a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of non-negative integers such that the sum $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ equals

r. Let $\widehat{\mathfrak{S}}_r$ denote the set of all partitions of r. With a partition, λ, is associated its *Young diagram*, which is the set of nodes

$$[\lambda] = \left\{(i,j) \in \mathbb{Z}^2_{>0} \mid j \leqslant \lambda_i\right\}.$$

We identify partitions with their Young diagrams. There is a unique partition of size zero, which we denote $\varnothing$. We let λ' denote the *conjugate partition* obtained by flipping the Young diagram $[\lambda]$ through the diagonal. Let $\lambda, \mu \in \widehat{\mathfrak{S}}_r$, we say that λ dominates μ and write $\lambda \trianglerighteq \mu$ if, for all $1 \leqslant k \leqslant r$, we have

$$\sum_{i=1}^{k} \lambda_i \geqslant \sum_{i=1}^{k} \mu_i.$$

Define column dominance order, denoted by $\trianglerighteq_{\text{col}}$, by $\lambda \trianglerighteq_{\text{col}} \mu$ if and only if $\lambda' \trianglerighteq \mu'$. It is known that column dominance order is actually the opposite order to dominance order.

Young's graph or lattice, $\widehat{\mathfrak{S}}$, is the branching diagram with vertices $\widehat{\mathfrak{S}}_r$ on level r and a directed edge $\lambda \to \mu$ if μ is obtained from λ by adding one box. We define a standard tableau of shape λ to be a directed path on $\widehat{\mathfrak{S}}$ from $\varnothing$ to λ. (Such paths are easily identified with the usual picture of standard tableaux as fillings of the Young diagram of λ with the numbers 1 through r, so that the entries are increasing in rows and columns.) For $\lambda \vdash r$, denote the set of standard tableaux of shape λ by $\mathrm{Std}_r(\lambda)$. If $\mathsf{s} \in \mathrm{Std}_r(\lambda)$ is the path

$$\varnothing = \mathsf{s}(0) \to \mathsf{s}(1) \to \mathsf{s}(2) \to \cdots \to \mathsf{s}(r) = \lambda,$$

then the conjugate standard tableaux $\mathsf{s}' \in \mathrm{Std}_r(\lambda')$ is the path

$$\varnothing = \mathsf{s}(0) \to \mathsf{s}(1)' \to \mathsf{s}(2)' \to \cdots \to \mathsf{s}(r)' = \lambda'.$$

For any ring R, and for all $r \geqslant 0$, the group algebra $R\mathfrak{S}_r$ has an algebra involution determined by $w^* = w^{-1}$ and an automorphism $\#$ determined by $w^{\#} = \mathrm{sign}(w)w$ for $w \in \mathfrak{S}_r$. The involution $*$, the automorphism $\#$, and the inclusions $R\mathfrak{S}_r \hookrightarrow R\mathfrak{S}_{r+1}$ are mutually commuting. Let $s_1, \ldots, s_{r-1}$ be the usual generators of the symmetric group $\mathfrak{S}_r$, $s_i = (i, i+1)$. If $1 \leqslant a \leqslant i$, define

$$s_{a,i} = s_a s_{a+1} \cdots s_{i-1} = (i, i-1, \ldots, a) \tag{4.1}$$

and $s_{i,a} = s^*_{a,i}$. Therefore $s_{a,a} = 1$, the identity in the symmetric group. We let

$$x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} w \qquad y_\lambda = \sum_{w \in \mathfrak{S}_{\lambda'}} \mathrm{sign}(w) w \tag{4.2}$$

where $\mathfrak{S}_\lambda = \mathfrak{S}_{\{1,2,\ldots,\lambda_1\}} \times \mathfrak{S}_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \ldots$ is the Young subgroup labeled by λ and $\mathfrak{S}_{\lambda'}$ is the Young subgroup labeled by λ'. Given $\mu \vdash i-1$ and $\lambda \vdash i$ with $\lambda = \mu \cup \{j, \lambda_j\}$, we set $a = \sum_{r=1}^{j} \lambda_r$ and let $b = \sum_{r=1}^{\lambda_j} \lambda'_r$. We define the branching factors as follows:

$$d_{\mu \to \lambda} = s_{a,i} \quad u_{\mu \to \lambda} = s_{i,a} \sum_{r=0}^{\mu_j} s_{a,a-r} \tag{4.3}$$

and (conjugating and applying the automorphism $\#$) we obtain the dual branching factors

$$b_{\mu \to \lambda} = (-1)^{b-i} s_{b,i} \quad v_{\mu \to \lambda} = s_{i,b} \sum_{r=0}^{j} (-1)^{r+b-i} s_{b,b-r}. \tag{4.4}$$

For $\lambda \in \widehat{\mathfrak{S}}_r$ and $\mathsf{t} \in \mathrm{Std}_r(\lambda)$ let d_t be the ordered product of the d–branching factors along t and let b_t be the ordered product of b–branching factors along t, i.e. $b_\mathsf{t} = (d_{\mathsf{t}'})^{\#}$. Given $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda)$ we let

$$x_{\mathsf{st}} = d^*_\mathsf{s} x_\lambda d_\mathsf{t} \qquad y_{\mathsf{st}} = b^*_\mathsf{s} y_\lambda b_\mathsf{t}.$$

Theorem 4.1 ([11, 34])**.** *The algebra $R\mathfrak{S}_r$ has cellular bases*

$$\mathcal{X} = \{x_{\mathsf{st}} \mid \mathsf{s}, \mathsf{t} \in \mathrm{Std}_r(\lambda) \text{ for } \lambda \in \widehat{\mathfrak{S}}_r\} \qquad \mathcal{Y} = \{y_{\mathsf{st}} \mid \mathsf{s}, \mathsf{t} \in \mathrm{Std}_r(\lambda) \text{ for } \lambda \in \widehat{\mathfrak{S}}_r\}$$

with the involution $*$ and the posets $(\widehat{\mathfrak{S}}_r, \trianglerighteq)$ and $(\widehat{\mathfrak{S}}_r, \trianglerighteq_{\mathrm{col}})$ respectively. These bases are the Murphy and dual-Murphy bases defined in [34]*.*

Proof. It is shown in [11, Corollary 4.8] that $\mathcal{X}$ coincides with the Murphy cellular basis defined in [34, Theorem 4.17]. Since $y_{\mathsf{st}} = (x_{\mathsf{s}'\mathsf{t}'})^{\#}$, it follows that $\mathcal{Y}$ is also a cellular basis; the basis $\mathcal{Y}$ appears in [34] in a subsidiary role. □

It is shown in [11, Section 4 and Appendix A], following [34, 26, 25, 8, 32], that the sequence of symmetric group algebras $(\mathbb{Z}\mathfrak{S}_r)_{r\geqslant 0}$, endowed with the Murphy cellular bases, satisfies axioms (D1)–(D6) of Section 1.2. In fact, the sequence is induction coherent, and the u–branching coefficients are those derived from cell filtrations of induced cell modules. The cell generators are the elements x_λ. The branching diagram associated to the sequence is Young's lattice $\widehat{\mathfrak{S}}$. The d– and u–branching factors satisfy the compatibility relations (1.6), by [11, Appendix A]. The corresponding results for the dual-Murphy basis follow by conjugating and applying the automorphism $\#$.

5. The Murphy and dual Murphy bases of the Brauer algebra

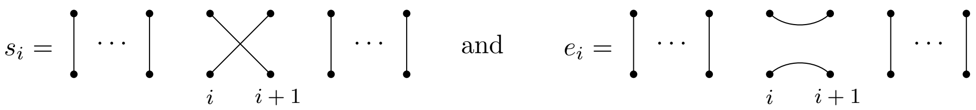
In this section we recall the definition of the Brauer algebra and the construction of its Murphy and dual Murphy bases. In subsequent sections, we will require the Murphy basis for examining Brauer algebras acting on symplectic tensor space, whereas we require the dual Murphy basis for examining Brauer algebras acting on orthogonal tensor space.

An r–strand Brauer diagram is a figure consisting of r points on the top edge, and another r on the bottom edge of a rectangle $\mathcal{R}$ together with r curves in $\mathcal{R}$ connecting the $2r$ points in pairs, with two such diagrams being identified if they induce the same matching of the $2r$ points. We call the distinguished points vertices and the curves strands. A strand is vertical if it connects a top vertex with a bottom vertex and horizontal otherwise. We label the top vertices by $\mathbf{1}, \dots, \boldsymbol{r}$ and the bottom vertices by $\overline{\mathbf{1}}, \dots, \overline{\boldsymbol{r}}$ from left to right.

Let S be an integral domain with a distinguished element $\delta \in S$. As an S-module, the r–strand Brauer algebra $B_r(S;\delta)$ is the free S–module with basis the set of r–strand Brauer diagrams. The product ab of two Brauer diagrams is defined as follows: stack a over b to obtain a figure $a * b$ consisting of a Brauer diagram c together with some number j of closed loops. Then ab is defined to be $\delta^j c$. The product on $B_r(S;\delta)$ is the bilinear extension of the product of diagrams.

The Brauer algebra $B_r(S;\delta)$ has an S–linear involution $*$ defined on diagrams by reflection in a horizontal line. The r–strand algebra embeds in the $r+1$–strand algebra by the map defined on diagrams by adding an additional top vertex $\boldsymbol{r+1}$ and an additional bottom vertex $\overline{\boldsymbol{r+1}}$ on the right, and connecting the new pair of vertices by a vertical strand.

The r–strand Brauer algebra is generated as a unital algebra by the following Brauer diagrams:



(5.1)

We have $e_i^2 = \delta e_i$, $e_i^* = e_i$ and $s_i^* = s_i$. An r–strand Brauer diagram with only vertical strands can be identified with a permutation in $\mathfrak{S}_r$, and the product of such diagrams agrees with composition of permutations. The linear span of the permutation diagrams is thus a subalgebra of $B_r(S;\delta)$ isomorphic to $S\mathfrak{S}_r$. This subalgebra is generated by the diagrams s_i in (5.1).

The linear span of r–strand Brauer diagrams with at least one horizontal strand is an ideal J_r of $B_r(S;\delta)$, and J_r is generated as an ideal by any of the elements e_i. The quotient $B_r(S;\delta)/J_r$ is also isomorphic to the symmetric group algebra, as algebras with involution.

The rank of a Brauer diagram is the number of its vertical strands; the corank is 1/2 the number of horizontal strands.

Denote the generic ground ring $\mathbb{Z}[\boldsymbol{\delta}]$ for the Brauer algebras by R, and let $R' = \mathbb{Z}[\boldsymbol{\delta}, \boldsymbol{\delta}^{-1}]$. It was shown in [11, Section 6.3] that the pair of towers of algebras $(B_r(R';\boldsymbol{\delta}))_{r\geqslant 0}$ and $(R'\mathfrak{S}_r)_{r\geqslant 0}$ satisfy (J1)–(J8) of Section 1.4, where $R'\mathfrak{S}_r$ is endowed with the Murphy cellular basis. But the same is true if $R'\mathfrak{S}_r$ is endowed instead with the dual Murphy cellular basis. Following through the work outlined in Section 1.4, based on the Murphy basis or the dual Murphy basis of the symmetric group algebras, one obtains two different cellular bases on the Brauer algebras $B_r(R;\boldsymbol{\delta})$ over the generic ground ring R, which we also call the Murphy and dual Murphy cellular bases. The tower of Brauer algebras over R, with either cellular structure, is restriction–coherent, with branching diagram $\widehat{B}$ obtained by reflections from Young's lattice. Explicitly, the branching rule is as follows: if $(\lambda, l) \in \widehat{B}_r$ and $(\mu, m) \in \widehat{B}_{r+1}$, then $(\lambda, l) \to (\mu, m)$ if and only if the Young diagram μ is obtained from the Young diagram λ by either adding or removing one box. The partial order $\trianglerighteq$ on $\widehat{B}_r$ for the Murphy cell datum is: $(\lambda, l) \trianglerighteq (\mu, m)$ if $l > m$ or if $l = m$ and $\lambda \trianglerighteq \mu$ The partial order $\trianglerighteq_{\text{col}}$ on $\widehat{B}_r$ for the dual Murphy cell datum is analogous, but with dominance order replaced with column dominance order.

The branching factors and cell generators in the Brauer algebras, computed using Theorem 1.24 and Lemma 1.25, involve liftings of elements of the symmetric group algebras to the Brauer algebras; for any element $x \in R\mathfrak{S}_r$, we lift x to the "same" element in the span of permutation diagrams in $B_r(r;\boldsymbol{\delta})$. Thus, for $(\lambda, l) \in \widehat{B}_r$, define

$$x_{(\lambda,l)} = x_\lambda e_{r-1}^{(l)} \quad \text{and} \quad y_{(\lambda,l)} = y_\lambda e_{r-1}^{(l)}. \tag{5.2}$$

These are the cell generators for the Murphy and dual Murphy cellular structures on $B_r(R;\boldsymbol{\delta})$. The branching factors $d_{(\lambda,l)\to(\mu,m)}$ and $u_{(\lambda,l)\to(\mu,m)}$ for the Murphy basis are obtained using the formulas of Theorem 1.24 from the d– and u–branching factors of the symmetric group algebras; and similarly the $b_{(\lambda,l)\to(\mu,m)}$ and $v_{(\lambda,l)\to(\mu,m)}$ branching factors for the dual Murphy basis are obtained from the b– and v–branching factors of the symmetric group algebras. These satisfy the compatibility relations

$$x_{(\mu,m)} d_{(\lambda,l)\to(\mu,m)} = u^*_{(\lambda,l)\to(\mu,m)} x_{(\lambda,l)} \qquad y_{(\mu,m)} b_{(\lambda,l)\to(\mu,m)} = v^*_{(\lambda,l)\to(\mu,m)} y_{(\lambda,l)}. \tag{5.3}$$

The compatibility relation is established in [11, Appendix A] for the Murphy basis, and the argument holds just as well for the dual Murphy basis. For $(\lambda, l) \in \widehat{B}_r$ and $\mathsf{t} \in \mathrm{Std}_r(\lambda, l)$ let d_{t} be the ordered product of the d–branching factors along t, and b_{t} the ordered product of the b–branching factors along t. For $(\lambda, l) \in \widehat{B}_r$ and $\mathsf{s}, \mathsf{t} \in \mathrm{Std}_k(\lambda, l)$, define

$$x^{(\lambda,l)}_{\mathsf{st}} = d^*_{\mathsf{s}} x_{(\lambda,l)} d_{\mathsf{t}} \qquad y^{(\lambda,l)}_{\mathsf{st}} = b^*_{\mathsf{s}} y_{(\lambda,l)} b_{\mathsf{t}}.$$

We have that

$$\{x^{(\lambda,l)}_{\mathsf{st}} \mid (\lambda, l) \in \widehat{B}_r \text{ and } \mathsf{s}, \mathsf{t} \in \mathrm{Std}_k(\lambda, l)\} \tag{5.4}$$

$$\{y^{(\lambda,l)}_{\mathsf{st}} \mid (\lambda, l) \in \widehat{B}_r \text{ and } \mathsf{s}, \mathsf{t} \in \mathrm{Std}_k(\lambda, l)\} \tag{5.5}$$

are the Murphy and dual Murphy cellular bases of $B_r(R;\boldsymbol{\delta})$.

Theorem 5.1. *The sequence of Brauer algebras $(B_r(R;\delta))_{r\geqslant 0}$ over the generic ground ring $R = \mathbb{Z}[\boldsymbol{\delta}]$ with either the Murphy cellular bases (5.4) or the dual Murphy cellular bases (5.5), satisfies (D1)–(D6) of Section 1.2. The data entering into the definition of the Murphy bases and dual Murphy bases are explicitly determined using equation (5.2) and the formulas of Theorem 1.24.*

Proof. For both the Murphy and dual Murphy bases, conditions (D1), (D2), (D4), and (D5) follow from the general framework of [11], as outlined in Section 1.4. Condition (D3) follows from (5.2) and condition (D6) from (5.3). ☐

We will require the following lemma in Sections 7 and 8.

Lemma 5.2. *If D is an r–strand Brauer diagram of corank $\geqslant m+1$, then for all $\mu \vdash r-2m$, D is an element of the ideal $B_r(R;\boldsymbol{\delta})^{\rhd(\mu,m)}$ defined using the Murphy basis. Likewise, D is an element of the ideal $B_r(R;\boldsymbol{\delta})^{\rhd_{\mathrm{col}}(\mu,m)}$ defined using the dual Murphy basis.*

Proof. It follows from the computation of the transition matrix between the Murphy basis and the diagram basis in [11, Section 6.2.3] that for all $(\lambda,l)\in\widehat{B}_r$ and all $\mathsf{s},\mathsf{t}\in\mathrm{Std}_r(\lambda,l)$, the Murphy basis element $x^{(\lambda,l)}_{\mathsf{st}}$ is an integer linear combination of Brauer diagrams with corank l. Thus the transition matrix is block diagonal, and the inverse transition matrix is also block diagonal. Hence if the corank of D is $l>m$, then D is a linear combination of Murphy basis elements $x^{(\lambda,l)}_{\mathsf{st}}$ with $\lambda\vdash r-2l$ and $\mathsf{s},\mathsf{t}\in\mathrm{Std}_r(\lambda,l)$. It follows that $D\in B_r(R;\boldsymbol{\delta})^{\rhd(\mu,m)}$ for any $\mu\vdash r-2m$. The same argument holds for the dual Murphy basis. ☐

6. Bilinear forms and the action of the Brauer algebra on tensor space

Let V be a finite dimensional vector space over a field $\Bbbk$ with a non–degenerate bilinear form denoted $[\ ,\ ]$. For the moment we make no assumption about the symmetry of the bilinear form. The non–degenerate form induces an isomorphism $\eta: V\to V^*$, defined by $\eta(v)(w)=[w,v]$ and hence a linear isomorphism $A: V\otimes V\to \mathrm{End}(V)$ defined by $A(v\otimes w)(x)=\eta(v)(x)\,w=[x,v]\,w$. We write $x\cdot(v\otimes w)=A(v\otimes w)(x)=[x,v]\,w$, and also $(v\otimes w)\cdot x=v[w,x]$.

For all $r\geqslant 1$, extend the bilinear form to $V^{\otimes r}$ by $[x_1\otimes x_2\cdots x_r, y_1\otimes y_2\cdots y_r]=\prod_i[x_i,y_i]$. Then this is also an non–degenerate bilinear form so induces isomorphisms $\eta_r: V^{\otimes r}\to (V^*)^{\otimes r}$ and $A_r: V^{\otimes 2r}\to\mathrm{End}(V^{\otimes r})$. We will generally just write η and A instead of η_r and A_r. In the following, let G denote the group of linear transformations of V preserving the bilinear form.

Lemma 6.1. *Let $\{v_i\}$ and $\{v_i^*\}$ be dual bases of V with respect to the bilinear form, i.e. bases such that $[v_i,v_j^*]=\delta_{i,j}$, and let $\omega=\sum_i v_i^*\otimes v_i$. Then:*

(1) *For all $x\in V$, $x\cdot\omega=\omega\cdot x=x$. In particular, $\omega=A^{-1}(\mathrm{id}_V)$ and ω is independent of the choice of the dual bases.*
(2) *For all $x,y\in V$, $[x\otimes y,\omega]=[y,x]$.*
(3) *ω is G–invariant.*

Proof. For any j, $v_j\cdot\omega=\sum_i[v_j,v_i^*]v_i=v_j$. Similarly, $\omega\cdot v_j^*=v_j^*$. Hence, for all x, $x\cdot\omega=\omega\cdot x=x$. We have $[x\otimes y,\omega]=\sum_i[x,v_i^*][y,v_i]=[y,x\cdot\omega]=[y,x]$. For the G–invariance of ω, note that for $g\in G$, $g\cdot\omega=\sum_i gv_i^*\otimes gv_i=\omega$, because $\{gv_i\}$ and $\{gv_i^*\}$ is another pair of dual bases. ☐

For the remainder of this section, we assume that the bilinear form $[\ ,\ ]$ is either symmetric or skew–symmetric. Note that in both cases the bilinear form induced on $V\otimes V$ is symmetric. Because the bilinear form on $V^{\otimes r}$ is non–degenerate, $\mathrm{End}(V^{\otimes r})$ has a $\Bbbk$–linear algebra involution $*$ defined by $[T^*(x),y]=[x,T(y)]$, for $T\in\mathrm{End}(V^{\otimes r})$ and $x,y\in V^{\otimes r}$. (The involution property depends on the bilinear form being either symmetric or skew–symmetric.)

Define $E,S\in\mathrm{End}(V\otimes V)$ by $(x\otimes y)E=[x,y]\omega$, and $(x\otimes y)S=y\otimes x$. These will be used to define a right action of Brauer algebras on tensor powers of V.

Lemma 6.2. *Write $\epsilon=1$ if the bilinear form $[\ ,\ ]$ on V is symmetric and $\epsilon=-1$ if the bilinear form is skew symmetric.*

(1) *$ES=SE=\epsilon E$*
(2) *$E^2=(\epsilon\dim V)E$.*
(3) *$E=E^*$ and $S=S^*$ in $\mathrm{End}(V\otimes V)$.*

(4) *E and S commute with the action of G on $V \otimes V$.*

Proof. These statements follow from straightforward computations. The proof of the last statement, on G–invariance, uses Lemma 6.1, part (3). $\square$

For $r \geqslant 1$ and for $1 \leqslant i \leqslant r-1$ define E_i and S_i in $\mathrm{End}(V^{\otimes r})$ to be E and S acting in the i–th and $i+1$–st tensor places.

Proposition 6.3 (Brauer, [5])**.** *Let V be a finite dimensional vector space over $\Bbbk$ with a non–degenerate symmetric or skew–symmetric bilinear form $[\ ,\ ]$, and let G be the group of linear transformations of V preserving the bilinear form. Then for $r \geqslant 1$, $e_i \mapsto E_i$ and $s_i \mapsto \epsilon S_i$ determines a homomorphism*

$$\Phi_r : B_r(\Bbbk; \epsilon \dim V) \longrightarrow \mathrm{End}_G(V^{\otimes r}).$$

Proof. Brauer works over the complex numbers, but his argument in [5, page 869] is equally valid over any field. Alternatively, one can use the presentation of the Brauer algebra, see [2, Proposition 2.7] for example, and verify that the images of the generators s_i, e_i satisfy the defining relations. $\square$

Note that the symmetric group (contained in the Brauer algebra) acts on $V^{\otimes r}$ by place permutations if the bilinear form $[\ ,\ ]$ on V is symmetric and by signed place permutations if the bilinear form is skew–symmetric.

We have $\Phi_{r+1} \circ \iota = \iota \circ \Phi_r$, where we have used ι to denote both the embedding of B_r into B_{r+1} and the embedding of $\mathrm{End}(V^{\otimes r})$ into $\mathrm{End}(V^{\otimes(r+1)})$, namely $\iota : T \mapsto T \otimes \mathrm{id}_V$. In particular, this implies $\ker(\Phi_r) \subseteq \ker(\Phi_{r+1})$. Because of this, we will sometimes just write Φ instead of Φ_r.

Lemma 6.4. *The homomorphism Φ_r respects the involutions on $B_r(\Bbbk; \epsilon \dim V)$ and $\mathrm{End}(V^{\otimes r})$, i.e. $\Phi_r(a^*) = \Phi_r(a)^*$. Consequently, the image $\mathrm{im}(\Phi_r)$ is an algebra with involution, and $\ker(\Phi_r)$ is a $*$–invariant ideal in $B_r(\Bbbk; \epsilon \dim V)$.*

Proof. Follows from Lemma 6.2, part (3). $\square$

7. The Brauer algebra on symplectic tensor space

Let V be a $2N$–dimensional vector space over a field $\Bbbk$ with a symplectic form $\langle\ ,\ \rangle$, i.e., a non–degenerate, alternating (and thus skew–symmetric) bilinear form. One can show that V has a Darboux basis, i.e. a basis $\{v_i\}_{1 \leqslant i \leqslant 2N}$ such that the dual basis $\{v_i^*\}$ with respect to the symplectic form is $v_i^* = v_{2N+1-i}$ if $1 \leqslant i \leqslant N$ and $v_i^* = -v_{2N+1-i}$ if $N+1 \leqslant i \leqslant 2N$. Hence, one can assume without loss of generality that $V = \Bbbk^{2N}$ with the standard symplectic form

$$\langle x, y \rangle = \sum_{i=1}^{N} (x_i y_{2N+1-i} - y_i x_{2N+1-i}).$$

For $r \geqslant 1$, let $\Phi_r : B_r(\Bbbk; -2N) \to \mathrm{End}(V^{\otimes r})$ be the homomorphism defined as in Section 6 using the symplectic form. When required for clarity, we write $V_\Bbbk$ for $V = \Bbbk^{2N}$ and $\Phi_{r,\Bbbk}$ for Φ_r. The image $\mathrm{im}(\Phi_{r,\Bbbk})$ is known as the (symplectic) Brauer centralizer algebra.

Theorem 7.1 ([7])**.** *Let $\Lambda : \Bbbk\,\mathsf{Sp}(V) \to \mathrm{End}(V^{\otimes r})$ denote the homomorphism corresponding to the diagonal action of the symplectic group $\mathsf{Sp}(V)$ on $V^{\otimes r}$.*

(1) *If $\Bbbk$ is a quadratically closed infinite field, then $\mathrm{im}(\Phi_r) = \mathrm{End}_{\mathsf{Sp}(V)}(V^{\otimes r})$ and $\mathrm{im}(\Lambda) = \mathrm{End}_{B_r(\Bbbk;-2N)}(V^{\otimes r})$.*
(2) *The dimension of $\mathrm{im}(\Phi_r)$ is independent of the field and of the characteristic, for infinite fields $\Bbbk$.*

Remark 7.2. The special case when $\Bbbk$ is the field of complex numbers is due to Brauer [5]. The statement of part (1) in [7] is more general, allowing general infinite fields at the cost of replacing the symplectic group with the symplectic similitude group.

Let $\Phi_{r,\mathbb{Z}}$ denote the restriction of $\Phi_{r,\mathbb{C}}$ to $B_r(\mathbb{Z};-2N)$; the image $\mathrm{im}(\Phi_{r,\mathbb{Z}})$ is the $\mathbb{Z}$-subalgebra of $\mathrm{End}(V_{\mathbb{C}}^{\otimes r})$ generated by E_i and S_i for $1 \leqslant i \leqslant r-1$. Let $V_{\mathbb{Z}}$ be the $\mathbb{Z}$-span of the standard basis $\{e_i \mid 1 \leqslant i \leqslant 2N\}$. Thus $V_{\mathbb{Z}}^{\otimes r} \subset V_{\mathbb{C}}^{\otimes r}$ and $\mathrm{End}_{\mathbb{Z}}(V_{\mathbb{Z}}^{\otimes r}) \subset \mathrm{End}(V_{\mathbb{C}}^{\otimes r})$. Since E_i and S_i leave $V_{\mathbb{Z}}^{\otimes r}$ invariant, we can also regard $\mathrm{im}(\Phi_{r,\mathbb{Z}})$ as a $\mathbb{Z}$-subalgebra of $\mathrm{End}_{\mathbb{Z}}(V_{\mathbb{Z}}^{\otimes r})$.

For any $\Bbbk$, $B_r(\Bbbk;-2N) \cong B_r(\mathbb{Z};-2N) \otimes_{\mathbb{Z}} \Bbbk$ and $V_{\Bbbk} \cong V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$. For $a \in B_r(\mathbb{Z};-2N)$ and $w \in V_{\mathbb{Z}}$, we have $\Phi_{r,\Bbbk}(a \otimes 1_{\Bbbk})(w \otimes 1_{\Bbbk}) = \Phi_{r,\mathbb{Z}}(a)(w) \otimes 1_{\Bbbk}$. Therefore, we are in the situation of Lemma 1.5, and there exists a map $\theta : \mathrm{im}(\Phi_{r,\mathbb{Z}}) \to \mathrm{im}(\Phi_{r,\Bbbk})$ making the diagram commute:

$$\begin{array}{ccc} B_r(\mathbb{Z};-2N) & \xrightarrow{\Phi_{r,\mathbb{Z}}} & \mathrm{im}(\Phi_{r,\mathbb{Z}}) \\ \downarrow{\scriptstyle \otimes 1_{\Bbbk}} & & \downarrow{\scriptstyle \theta} \\ B_r(\Bbbk;-2N) & \xrightarrow[\Phi_{r,\Bbbk}]{} & \mathrm{im}(\Phi_{r,\Bbbk}) \end{array} \quad . \tag{7.1}$$

7.1. **Murphy basis over the integers.**

Definition 7.3. Write $A_r^{\mathrm{s}}(**) = \Phi(B_r(**;-2N))$, where $**$ stands for $\mathbb{C}$, $\mathbb{Q}$, or $\mathbb{Z}$. Thus $A_r^{\mathrm{s}}(\mathbb{Z})$ is the $\mathbb{Z}$–algebra generated by $E_i = \Phi(e_i)$ and $S_i = -\Phi(s_i)$. (The superscript "s" in this notation stands for "symplectic".)

Let $R = \mathbb{Z}[\boldsymbol{\delta}]$. Endow $B_r(R;\boldsymbol{\delta})$ with the Murphy cellular structure described in Section 5 with the Murphy type basis

$$\left\{ x_{\mathsf{st}}^{(\lambda,l)} \;\middle|\; (\lambda,l) \in \widehat{B}_r \text{ and } \mathsf{s},\mathsf{t} \in \mathrm{Std}_r(\lambda,l) \right\}.$$

By Theorem 5.1, the tower $(B_r(R;\boldsymbol{\delta}))_{r\geqslant 0}$ satisfies the assumptions (D1)–(D6) of Section 1.2.

We want to show that the maps $\Phi_{r,\mathbb{Z}} : B_r(\mathbb{Z};-2N) \to A_r^{\mathrm{s}}(\mathbb{Z})$ satisfy the assumptions (Q1)–(Q3) of Section 2. It will follow that the integral Brauer centralizer algebras $A_r^{\mathrm{s}}(\mathbb{Z})$ are cellular over the integers.

First we need the appropriate notion of permissibility for points in $\widehat{B}_r$ and for paths on $\widehat{B}$.

Definition 7.4. A $(-2N)$–permissible partition λ is a partition such that $\lambda_1 \leqslant N$. We say that an element $(\lambda,l) \in \widehat{B}_r$ is $(-2N)$–permissible if λ is $(-2N)$–permissible. We let $\widehat{B}^{\mathrm{s}}_{r,\mathrm{perm}} \subseteq \widehat{B}_r$ denote the subset of $(-2N)$–permissible points.

A path $\mathsf{t} \in \mathrm{Std}_r(\lambda,l)$ is $(-2N)$–permissible if $\mathsf{t}(k)$ is $(-2N)$–permissible for all $0 \leqslant k \leqslant r$. We let $\mathrm{Std}^{\mathrm{s}}_{r,\mathrm{perm}}(\lambda,l) \subseteq \mathrm{Std}_r(\lambda,l)$ denote the subset of $(-2N)$–permissible paths.

Note that this set of permissible points satisfies condition (Q1).

For any ring U and any $\delta \in U$, and any natural numbers r, s, there is an injective U–algebra homomorphism $B_r(U;\delta) \otimes B_s(U;\delta) \to B_{r+s}(U;\delta)$ defined on the basis of Brauer diagrams by placing diagrams side by side. We also write $x \otimes y$ for the image of $x \otimes y$ in $B_{r+s}(U;\delta)$.

Definition 7.5. Define $\mathfrak{b}_r \in B_r(\mathbb{Z};-2N)$ to be the sum of all Brauer diagrams on r strands and $\mathfrak{b}'_r$ to be the sum of all Brauer diagrams on r strands of corank $\geqslant 1$. For $\lambda = (\lambda_1,\ldots,\lambda_s)$ a partition of r, write

$$\mathfrak{b}_\lambda = \mathfrak{b}_{\lambda_1} \otimes x_{(\lambda_2,\ldots,\lambda_s)} \quad \text{and} \quad \mathfrak{b}'_\lambda = \mathfrak{b}'_{\lambda_1} \otimes x_{(\lambda_2,\ldots,\lambda_s)}. \tag{7.2}$$

For $(\lambda,l) \in \widehat{B}_r$, write

$$\mathfrak{b}_{(\lambda,l)} = \mathfrak{b}_\lambda e^{(l)}_{r-1} \quad \text{and} \quad \mathfrak{b}'_{(\lambda,l)} = \mathfrak{b}'_\lambda e^{(l)}_{r-1}. \tag{7.3}$$

Remark 7.6. Thus, for all $r \geqslant 0$ and for all $(\lambda,l) \in \widehat{B}_r$,

$$x_{(\lambda,l)} = \mathfrak{b}_{(\lambda,l)} - \mathfrak{b}'_{(\lambda,l)}. \tag{7.4}$$

Lemma 7.7. *There exists $\beta'_r \in B_r(\mathbb{Q};-2N)$ such that $\mathfrak{b}'_r = x_{(r)}\beta'_r$.*

Proof. $\mathfrak{S}_r$ acts on the left on the set of Brauer diagrams on r strands with corank $\geqslant 1$. Choose a representative of each orbit. Then

$$\mathfrak{b}'_r = x_{(r)} \sum_x \frac{1}{|\mathrm{Stab}(x)|} x,$$

where the sum is over orbit representatives x and $|\mathrm{Stab}(x)|$ is the cardinality of the stabilizer of x in $\mathfrak{S}_r$. □

It follows that for all $r \geqslant 0$ and for all $(\lambda, l) \in \widehat{B}_r$,

$$\mathfrak{b}'_{(\lambda,l)} = x_{(\lambda,l)} \beta'_{\lambda_1}. \tag{7.5}$$

Fix $r \geqslant 1$. The multilinear functionals on V^{2r} of the form $(w_1, \dots, w_{2r}) \mapsto \prod \langle w_i, w_j \rangle$, where each w_i occurs exactly once, are evidently Sp(V)–invariant. Moreover, there are some obvious relations among such functionals. If we take $r = N+1$, then for any choice of $(w_1, \dots, w_{2r})$, the $2r$–by–$2r$ skew-symmetric matrix $(\langle w_i, w_j \rangle)$ is singular and therefore the Pfaffian of this matrix is zero, which provides such a relation. These elementary observations are preliminary to the first and second fundamental theorems of invariant theory for the symplectic groups, see [41, Section 6.1]. The following proposition depends on the second of these observations.

Proposition 7.8. *The element $\mathfrak{b}_{N+1}$ is in $\ker(\Phi)$. Hence if $r \geqslant N+1$ and $(\lambda, l) \in \widehat{B}_r$ with $\lambda_1 = N+1$, then $\mathfrak{b}_{(\lambda,l)} \in \ker(\Phi)$.*

Proof. Set $r = N+1$. There exist linear isomorphisms $A : V^{\otimes 2r} \to \mathrm{End}(V^{\otimes r})$ and $\eta : V^{\otimes 2r} \to (V^{\otimes 2r})^*$. The proof consists of showing that $(\eta \circ A^{-1} \circ \Phi(\mathfrak{b}_{N+1}))(w_1 \otimes \cdots \otimes w_{2r})$ is up to a sign the Pfaffian of the singular matrix $(\langle w_i, w_j \rangle)$. Hence $\mathfrak{b}_{N+1} \in \ker(\Phi)$. This is explained in [12, Section 3.3]. We have also provided an exposition in Appendix D in the arXiv version of this paper. Another proof can be found in [21, Proposition 4.6]. □

We can now verify axiom (Q2). Let $\mathsf{t} \in \mathrm{Std}_r(\lambda, l)$ be a path which is not $(-2N)$-permissible. Let $k \leqslant r$ be the first index such that $\mathsf{t}(k) = (\mu, m)$ satisfies $\mu_1 = N+1$. It follows from Proposition 7.8 that

$$\mathfrak{b}_{(\mu,m)} \in \ker(\Phi_r). \tag{7.6}$$

By (7.5) and (5.2), we have that

$$\mathfrak{b}'_{(\mu,m)} = x_{(\mu,m)} \beta'_{\mu_1} = x_\mu \beta'_{\mu_1} e^{(m)}_{k-1}$$

is a linear combination of Brauer diagrams of corank at least $m+1$, and therefore $\mathfrak{b}'_{(\mu,m)} \in B_r(\mathbb{Z}; -2N)^{\rhd(\mu,m)}$, using Lemma 5.2. Hence

$$\mathfrak{b}'_{(\mu,m)} \in x_{(\mu,m)} B_r(\mathbb{Q}; -2N) \cap B_r(\mathbb{Z}; -2N)^{\rhd(\mu,m)} \tag{7.7}$$

as required. Taken together, (7.4)–(7.7) show that axiom (Q2) holds.

Finally, it is shown in [38, Theorem 3.4, Corollary 3.5] that

$$\dim_{\mathbb{C}}(A^{\mathrm{s}}_r(\mathbb{C})) = \sum_{(\lambda,l) \in \widehat{B}^{\mathrm{s}}_{r,\mathrm{perm}}} (\sharp \mathrm{Std}^{\mathrm{s}}_{r,\mathrm{perm}}(\lambda, l))^2.$$

Thus axiom (Q3) holds.

Since assumptions (Q1)–(Q3) of Section 2 are satisfied, we can produce a modified Murphy basis of $B_r(\mathbb{Z}; -2N)$,

$$\left\{ \widetilde{x}^{(\lambda,l)}_{\mathsf{st}} \;\middle|\; (\lambda, l) \in \widehat{B}_r \text{ and } \mathsf{s}, \mathsf{t} \in \mathrm{Std}_k(\lambda, l) \right\},$$

following the procedure described before Theorem 2.5. The following theorem gives a cellular basis for Brauer's centralizer algebra acting on symplectic tensor space, valid over the integers. It also gives two descriptions of the kernel of the map $\Phi_{r,\mathbb{Z}} : B_r(\mathbb{Z}; -2N) \to \mathrm{End}(V^{\otimes r})$, one by providing a basis of $\ker(\Phi_{r,\mathbb{Z}})$ over the integers, and the other by describing the kernel as the ideal generated by a single element. Each of these statements is a form of the second fundamental theorem of invariant theory for the symplectic groups.

Theorem 7.9. *The integral (symplectic) Brauer centralizer algebra $A_r^{\mathrm{s}}(\mathbb{Z})$ is a cellular algebra over $\mathbb{Z}$ with basis*

$$\mathbb{A}_r^{\mathrm{s}}(\mathbb{Z}) = \left\{ \Phi_{r,\mathbb{Z}}(\widetilde{x}_{\mathsf{st}}^{(\lambda,l)}) \mid (\lambda,l) \in \widehat{B}^{\mathrm{s}}_{r,\mathrm{perm}} \text{ and } \mathsf{s},\mathsf{t} \in \mathrm{Std}^{\mathrm{s}}_{r,\mathrm{perm}}(\lambda,l) \right\}$$

with the involution $*$ determined by $E_i^* = E_i$ and $S_i^* = S_i$ and the partially ordered set $(\widehat{B}^{\mathrm{s}}_{r,\mathrm{perm}}, \trianglerighteq)$. The ideal $\ker(\Phi_{r,\mathbb{Z}}) \subseteq B_r(\mathbb{Z};-2N)$ has $\mathbb{Z}$-basis

$$\kappa_r = \left\{ \widetilde{x}_{\mathsf{st}}^{(\lambda,l)} \mid (\lambda,l) \in \widehat{B}_r \text{ and } \mathsf{s} \text{ or } \mathsf{t} \text{ not } (-2N)\text{–permissible} \right\}.$$

Moreover, for $r > N$, $\ker(\Phi_{r,\mathbb{Z}})$ is the ideal generated by the single element $\mathfrak{b}_{N+1} \in B_r(\mathbb{Z};-2N)$. For $r \leqslant N$, $\ker(\Phi_{r,\mathbb{Z}}) = 0$.

Proof. The construction of the cellular basis of $A_r^{\mathrm{s}}(\mathbb{Z})$ and the basis of $\ker(\Phi_r)$ follows immediately from Theorem 2.7 since (Q1)–(Q3) have been verified.

Now, if $r \leqslant N$ then $\ker(\Phi_{r,\mathbb{Z}}) = 0$, since all paths on $\widehat{B}$ of length $\leqslant N$ are $(-2N)$–permissible. For $r > N$, the kernel is the ideal generated by all the $\mathfrak{b}_{(\mu,m)}$ such that (μ,m) is a marginal point in $\widehat{B}_k$ for some $0 < k \leqslant r$, using Theorem 2.7. But the marginal points are all of the form (μ,m) for some μ with $\mu_1 = N+1$. Now, by (7.2) and (7.3) we have that

$$\mathfrak{b}_{(\mu,m)} = \mathfrak{b}_\mu e_{r-1}^{(m)} = \mathfrak{b}_{N+1} \otimes x_{(\lambda_2,\ldots,\lambda_s)} e_{k-1}^{(m)}$$

and so we are done. □

Remark 7.10. Applying Corollary 2.8, for every $r \geqslant 0$ and for every permissible point $(\lambda,l) \in \widehat{B}_r$, the cell module $\Delta^{\mathbb{C}}_{A_r^{\mathrm{s}}}((\lambda,l))$, regarded as a $B_r(\mathbb{C};-2N)$–module, is the simple head of the cell module $\Delta_{B_r(\mathbb{C};-2N)}((\lambda,l))$. Thus the cell module $\Delta^{\mathbb{Z}}_{A_r^{\mathrm{s}}}((\lambda,l))$ is an integral form of the simple $B_r(\mathbb{C};-2N)$–module $L_{B_r(\mathbb{C};-2N)}((\lambda,l))$. In this way, all the simple $B_r(\mathbb{C};-2N)$–modules that factor through the representation on symplectic tensor space are provided with integral forms.

7.2. **Murphy basis over an arbitrary field.** We return to the general situation described at the beginning of Section 7: V is a vector space of dimension $2N$ over an arbitrary field $\Bbbk$, with a symplectic form, and for $r \geqslant 1$, $\Phi_{r,\Bbbk} : B_r(\Bbbk;-2N) \to \mathrm{End}(V^{\otimes r})$ is Brauer's homomorphism. We assume without loss of generality that $V = \Bbbk^{2N}$, with the standard symplectic form. We have the commutative diagram (7.1).

For the modified Murphy basis $\{\widetilde{x}_{\mathsf{st}}^{(\lambda,l)}\}$ of $B_r(\mathbb{Z};-2N)$, we also write $\widetilde{x}_{\mathsf{st}}^{(\lambda,l)}$ instead of $\widetilde{x}_{\mathsf{st}}^{(\lambda,l)} \otimes 1_\Bbbk$ for the corresponding basis element of $B_r(\Bbbk;-2N)$.

Theorem 7.11. *Let V be a $2N$–dimensional vector space over a field $\Bbbk$. Assume that V has a symplectic form, and let $\Phi_{r,\Bbbk} : B_r(\Bbbk;-2N) \to \mathrm{End}(V^{\otimes r})$ be Brauer's homomorphism defined using the symplectic form. Write $A_r^{\mathrm{s}}(\Bbbk)$ for the Brauer centralizer algebra $\mathrm{im}(\Phi_{r,\Bbbk})$.*

The Brauer centralizer algebra $A_r^{\mathrm{s}}(\Bbbk)$ is a cellular algebra over $\Bbbk$ with basis

$$\mathbb{A}_r^{\mathrm{s}}(\Bbbk) = \left\{ \Phi_{r,\Bbbk}(\widetilde{x}_{\mathsf{st}}^{(\lambda,l)}) \mid (\lambda,l) \in \widehat{B}^{\mathrm{s}}_{r,\mathrm{perm}} \text{ and } \mathsf{s},\mathsf{t} \in \mathrm{Std}^{\mathrm{s}}_{r,\mathrm{perm}}(\lambda,l) \right\}$$

with the involution $*$ determined by $E_i^* = E_i$ and $S_i^* = S_i$ and the partially ordered set $(\widehat{B}^{\mathrm{s}}_{r,\mathrm{perm}}, \trianglerighteq)$. The ideal $\ker(\Phi_{r,\Bbbk}) \subseteq B_r(\Bbbk;-2N)$ has basis

$$\kappa_r = \left\{ \widetilde{x}_{\mathsf{st}}^{(\lambda,l)} \mid (\lambda,l) \in \widehat{B}_r \text{ and } \mathsf{s} \text{ or } \mathsf{t} \text{ not } (-2N)\text{–permissible} \right\}.$$

Moreover, $\ker(\Phi_{r,\Bbbk})$ is the ideal generated by the single element $\mathfrak{b}_{N+1} \in B_r(\Bbbk;-2N)$ for $r > N$ (and is zero for $r \leqslant N$).

Proof. Refer to the commutative diagram (7.1). If s or t is not $(-2N)$–permissible, then $\Phi_{r,\Bbbk}(\widetilde{x}_{\mathsf{st}}^{(\lambda,l)}) = \theta \circ \Phi_{r,\mathbb{Z}}(\widetilde{x}_{\mathsf{st}}^{(\lambda,l)}) = 0$. Thus $\kappa_r \subset \ker(\Phi_{r,\Bbbk})$. It follows from this that $\mathbb{A}_r^{\mathsf{s}}(\Bbbk)$ spans $A_r^{\mathsf{s}}(\Bbbk)$. Once we have established that $\mathbb{A}_r^{\mathsf{s}}(\Bbbk)$ is linearly independent, the argument of Theorem 2.7 shows that κ_r is a basis of $\ker(\Phi_{r,\Bbbk})$, and that $\mathbb{A}_r^{\mathsf{s}}(\Bbbk)$ is a cellular basis of the Brauer centralizer algebra $A_r^{\mathsf{s}}(\Bbbk)$.

Suppose first that $\Bbbk$ is an infinite field. By Theorem 7.1 part (2), the dimension of $\ker(\Phi_{r,\Bbbk})$ and the dimension of $A_r^{\mathsf{s}}(\Bbbk)$ are independent of the (infinite) field $\Bbbk$ and of the characteristic. Therefore $\mathbb{A}_r^{\mathsf{s}}(\Bbbk)$ is a basis of $A_r^{\mathsf{s}}(\Bbbk)$.

Now consider the case that $\Bbbk$ is any field; let $\overline{\Bbbk}$ be the algebraic closure of $\Bbbk$. Applying Lemma 1.5 again, we have a commutative diagram

$$\begin{array}{ccc} B_r(\Bbbk; -2N) & \xrightarrow{\Phi_{r,\Bbbk}} & A_r^{\mathsf{s}}(\Bbbk) \\ \Big\downarrow \otimes 1_{\overline{\Bbbk}} & & \Big\downarrow \eta \\ B_r(\overline{\Bbbk}; -2N) & \xrightarrow[\Phi_{r,\overline{\Bbbk}}]{} & A_r^{\mathsf{s}}(\overline{\Bbbk}) \end{array} .$$

We conclude that $\mathbb{A}_r^{\mathsf{s}}(\Bbbk)$ is linearly independent over $\Bbbk$, since $\eta(\mathbb{A}_r^{\mathsf{s}}(\Bbbk)) = \mathbb{A}_r^{\mathsf{s}}(\overline{\Bbbk})$ is linearly independent over $\overline{\Bbbk}$. As noted above, it now follows that $\mathbb{A}_r^{\mathsf{s}}(\Bbbk)$ is a cellular basis of $A_r^{\mathsf{s}}(\Bbbk)$, and κ_r is a basis of $\ker(\Phi_{r,\Bbbk})$. For the final statement, it suffices to show that elements of κ_r are in the ideal generated by $\mathfrak{b}_{N+1}$ and this follows from Theorem 7.9. □

Remark 7.12. Theorem 7.11 extends the constancy of dimension of Brauer's centralizer algebras $A_r^{\mathsf{s}}(\Bbbk)$ and of $\ker(\Phi_{r,\Bbbk})$, from Theorem 7.1 part (2), to all fields $\Bbbk$.

We also have the following corollary.

Corollary 7.13. *Adopt the hypothesis of Theorem 7.11. The Brauer centralizer algebra $A_r^{\mathsf{s}}(\Bbbk)$ acting on $V^{\otimes r}$ is the specialization of the integral Brauer centralizer algebra $A_r^{\mathsf{s}}(\mathbb{Z})$, i.e. $A_r^{\mathsf{s}}(\Bbbk) \cong A_r^{\mathsf{s}}(\mathbb{Z}) \otimes_{\mathbb{Z}} \Bbbk$.*

Proof. In general, if A is and R–algebra which is free as an R–module with basis $\{b_i\}$ and structure constants $r_{ij}^k \in R$, then any specialization $A^S = A \otimes_R S$ is characterized by being free as an S–module with basis $\{b_i \otimes 1_S\}$ and structure constants $r_{ij}^k \otimes 1_S$. Now $A_r^{\mathsf{s}}(\mathbb{Z})$ has $\mathbb{Z}$–basis $\{\Phi_{r,\mathbb{Z}}(\widetilde{x}_{\mathsf{st}}^{(\lambda,l)}) \mid (\lambda, l), \mathsf{s}, \mathsf{t} \text{ permissible}\}$; and if

$$\widetilde{x}_{\mathsf{st}}^{(\lambda,l)} \widetilde{x}_{\mathsf{uv}}^{(\mu,m)} = \sum r(\nu, n, \alpha, \beta)\ \widetilde{x}_{\alpha,\beta}^{(\nu,n)}$$

in $B_r(\mathbb{Z}; -2N)$, where the sum runs over all (ν, n) and α, β, then

$$\Phi_{r,\mathbb{Z}}(\widetilde{x}_{\mathsf{st}}^{(\lambda,l)}) \Phi_{r,\mathbb{Z}}(\widetilde{x}_{\mathsf{uv}}^{(\mu,m)}) = \sum{}' \ r(\nu, n, \alpha, \beta)\ \Phi_{r,\mathbb{Z}}(\widetilde{x}_{\alpha,\beta}^{(\nu,n)}),$$

where now the sum is restricted to permissible (ν, n) and α, β.

$A_r^{\mathsf{s}}(\Bbbk)$ has $\Bbbk$–basis $\{\Phi_{r,\Bbbk}(\widetilde{x}_{\mathsf{st}}^{(\lambda,l)}) \mid (\lambda, l), \mathsf{s}, \mathsf{t} \text{ permissible}\}$. Moreover, in $B_r(\Bbbk; -2N)$, we have

$$\widetilde{x}_{\mathsf{st}}^{(\lambda,l)} \widetilde{x}_{\mathsf{uv}}^{(\mu,m)} = \sum (r(\nu, n, \alpha, \beta) \otimes 1_{\Bbbk})\ \widetilde{x}_{\alpha,\beta}^{(\nu,n)},$$

and in $A_r^{\mathsf{s}}(\Bbbk)$,

$$\Phi_{r,\Bbbk}(\widetilde{x}_{\mathsf{st}}^{(\lambda,l)}) \Phi_{r,\Bbbk}(\widetilde{x}_{\mathsf{uv}}^{(\mu,m)}) = \sum{}' \ (r(\nu, n, \alpha, \beta) \otimes 1_{\Bbbk})\ \Phi_{r,\Bbbk}(\widetilde{x}_{\alpha,\beta}^{(\nu,n)}),$$

with the sum again restricted to permissible (ν, n) and α, β. This shows that $A_r^{\mathsf{s}}(\Bbbk)$ is a specialization of $A_r(\mathbb{Z})$, as required. □

7.3. **Jucys–Murphy elements and seminormal representations.** We verify that the setting of Section 3.3 applies to the Brauer algebras, their specializations $B_r(\mathbb{Z}; -2N)$, and the quotients of these specializations acting on symplectic tensor space.

We have $R = \mathbb{Z}[\boldsymbol{\delta}]$, the generic ground ring, and the quotient map $\pi : \mathbb{Z}[\boldsymbol{\delta}] \to \mathbb{Z}$ determined by $\boldsymbol{\delta} \mapsto -2N$. The kernel of this map is the prime ideal $\mathfrak{p} = (\boldsymbol{\delta} + 2N)$. The subring of evaluable elements in $\mathbb{F} = \mathbb{Q}(\boldsymbol{\delta})$ is $R_\mathfrak{p}$. The subring of evaluable elements in $B_r(\mathbb{F}; \boldsymbol{\delta})$ is $B_r(R_\mathfrak{p}; \boldsymbol{\delta})$; c.f. Remark 2.2. The sequence of Brauer algebras with the Murphy cellular basis satisfies (D1)–(D6) according to Theorem 5.1. We have verified in Section 7.1 that the quotient axioms (Q1)–(Q3) are satisfied by the maps Φ_r from $B_r(\mathbb{Z}; -2N)$ to endomorphisms of symplectic tensor space. Jucys–Murphy elements for the Brauer algebras were defined by Nazaorv [35]. It is shown in [18] that these are an additive family of JM elements, with contents

$$\kappa((\lambda, l) \to (\mu, m)) = \begin{cases} c(a), & \text{if } \mu = \lambda \cup \{a\}, \\ 1 - \boldsymbol{\delta} - c(a), & \text{if } \mu = \lambda \setminus \{a\}. \end{cases} \tag{7.8}$$

where the content $c(a)$ of a node a of a Young diagram is the column index of a minus the row index of a. It is easy to check that the separation condition is satisfied. It remains to check condition (SN).

Lemma 7.14. *Let t be an $-2N$–permissible path in Std_r. Then F_t is evaluable.*

Proof. We apply Lemma 3.7 and Remark 3.8. Let s, t be two paths of length $r \geqslant 1$ with $\mathsf{s}' = \mathsf{t}'$ and with at least one of the paths $-2N$ permissible. We have to show that $\kappa_\mathsf{t}(r) - \kappa_\mathsf{s}(r) \not\equiv 0 \mod (\boldsymbol{\delta} + 2N)$. In order to reach a contradiction, assume $\kappa_\mathsf{t}(r) - \kappa_\mathsf{s}(r) \equiv 0 \mod (\boldsymbol{\delta} + 2N)$. This can only happen if one of the two edges $\mathsf{t}(r-1) \to \mathsf{t}(r)$ and $\mathsf{t}(r-1) \to \mathsf{s}(r)$ involves adding a node to $\mathsf{t}(r-1)$ and the other involves removing a node. Assume wlog that $\mathsf{t}(r-1) = (\lambda, l)$, $\mathsf{t}(r) = (\lambda \cup \{\alpha\}, l)$ for an addable node α of λ, and $\mathsf{s}(r) = (\lambda \setminus \{\beta\}, l+1)$, for a removable node β of λ. Our condition is then $c(\alpha) + c(\beta) = 1 + 2N$. But since $\lambda_1 \leqslant N$, we have $c(a), c(\beta) \leqslant N$ and $c(a) + c(\beta) \leqslant 2N$. □

8. The Brauer algebra on orthogonal tensor space

Let V be an N–dimensional vector space over a field $\Bbbk$ with $\mathrm{char}(\Bbbk) \neq 2$, with a non–degenerate, symmetric bilinear form (,); we will call such forms orthogonal forms for brevity. For $r \geqslant 1$, let $\Psi_r : B_r(\Bbbk; N) \to \mathrm{End}(V^{\otimes r})$ be Brauer's homomorphism defined in Section 6 using the orthogonal form. The image $\mathrm{im}(\Psi_r)$ is known as the (orthogonal) Brauer centralizer algebra.

Over general fields, the classification of orthogonal forms is complicated. However, if the field is quadratically closed, then one can easily show that V has an orthonormal basis, or alternatively a basis $\{v_i\}_{1 \leqslant i \leqslant N}$ whose dual basis $\{v_i^*\}$ with respect to the orthogonal form is $v_i^* = v_{N+1-i}$. Therefore, we can assume (when $\Bbbk$ is quadratically closed) that $V = \Bbbk^N$ with the standard orthogonal form $(x, y) = \sum_i x_i y_{N+1-i}$.

Theorem 8.1 ([9])**.** *Let $\Lambda : \Bbbk\,\mathsf{O}(V) \to \mathrm{End}(V^{\otimes r})$ denote the homomorphism corresponding to the diagonal action of the orthogonal group $\mathsf{O}(V)$ on $V^{\otimes r}$.*

(1) *If $\Bbbk$ is a quadratically closed infinite field, with $\mathrm{char}(\Bbbk) \neq 2$, then $\mathrm{im}(\Psi_r) = \mathrm{End}_{\mathsf{O}(V)}(V^{\otimes r})$ and $\mathrm{im}(\Lambda) = \mathrm{End}_{B_r(\Bbbk;N)}(V^{\otimes r})$.*
(2) *The dimension of $\mathrm{im}(\Psi_r)$ is independent of the field and of the characteristic, for infinite quadratically closed fields $\Bbbk$ with $\mathrm{char}(\Bbbk) \neq 2$.*

The special case of this theorem when $\Bbbk = \mathbb{C}$ is due to Brauer [5]. Let us continue to assume, for now, that $\Bbbk$ is quadratically closed and that $V = \Bbbk^N$ with the standard orthogonal form. When we need to emphasize the field we write $V_\Bbbk$ for V and $\Psi_{r,\Bbbk}$ for Ψ_r.

Let $\Psi_{r,\mathbb{Z}}$ denote the restriction of $\Psi_{r,\mathbb{C}}$ to $B_r(\mathbb{Z}; N)$; the image $\mathrm{im}(\Psi_{r,\mathbb{Z}})$ is the $\mathbb{Z}$-subalgebra of $\mathrm{End}(V_\mathbb{C}^{\otimes r})$ generated by E_i and S_i for $1 \leqslant i \leqslant r-1$. Let $V_\mathbb{Z}$ be the $\mathbb{Z}$-span of the standard basis $\{e_i \mid 1 \leqslant i \leqslant 2N\}$. Thus $V_{\mathbb{Z}}^{\otimes r} \subset V_{\mathbb{C}}^{\otimes r}$ and $\mathrm{End}_{\mathbb{Z}}(V_{\mathbb{Z}}^{\otimes r}) \subset \mathrm{End}(V_{\mathbb{C}}^{\otimes r})$. Since E_i and S_i leave $V_{\mathbb{Z}}^{\otimes r}$ invariant, we can also regard $\mathrm{im}(\Psi_{r,\mathbb{Z}})$ as a $\mathbb{Z}$-subalgebra of $\mathrm{End}_{\mathbb{Z}}(V_{\mathbb{Z}}^{\otimes r})$.

For any (quadratically closed) $\Bbbk$, $B_r(\Bbbk; N) \cong B_r(\mathbb{Z}; N) \otimes_{\mathbb{Z}} \Bbbk$ and $V_{\Bbbk} \cong V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$. For $a \in B_r(\mathbb{Z}; N)$ and $w \in V_{\mathbb{Z}}$, we have $\Psi_{r,\Bbbk}(a \otimes 1_{\Bbbk})(w \otimes 1_{\Bbbk}) = \Psi_{r,\mathbb{Z}}(a)(w) \otimes 1_{\Bbbk}$. Therefore, we are in the situation of Lemma 1.5, and there exists a map $\theta : \mathrm{im}(\Psi_{r,\mathbb{Z}}) \to \mathrm{im}(\Psi_{r,\Bbbk})$ making the diagram commute:

$$\begin{array}{ccc} B_r(\mathbb{Z}; N) & \xrightarrow{\Psi_{r,\mathbb{Z}}} & \mathrm{im}(\Psi_{r,\mathbb{Z}}) \\ \big\downarrow{\scriptstyle \otimes 1_{\Bbbk}} & & \big\downarrow{\scriptstyle \theta} \\ B_r(\Bbbk; N) & \xrightarrow[\Psi_{r,\Bbbk}]{} & \mathrm{im}(\Psi_{r,\Bbbk}) \end{array} . \tag{8.1}$$

8.1. **Murphy basis over the integers.**

Definition 8.2. Write $A_r^{\mathrm{o}}(**) = \Psi_r(B_r(**; N))$, where $**$ stands for $\mathbb{C}$, $\mathbb{Q}$, or $\mathbb{Z}$. Thus $A_r^{\mathrm{o}}(\mathbb{Z})$ is the $\mathbb{Z}$–algebra generated by $E_i = \Psi_r(e_i)$ and $S_i = \Psi_r(s_i)$. (The superscript "o" in this notation stands for "orthogonal".)

Let $R = \mathbb{Z}[\boldsymbol{\delta}]$. Endow $B_r(R; \boldsymbol{\delta})$ with the dual Murphy cellular structure described in Section 5, with cellular basis

$$\left\{ y_{\mathsf{st}}^{(\lambda,l)} \;\middle|\; (\lambda, l) \in \widehat{B}_r \text{ and } \mathsf{s}, \mathsf{t} \in \mathrm{Std}_r(\lambda, l) \right\}.$$

By Theorem 5.1, the tower $(B_r(R; \boldsymbol{\delta}))_{r \geqslant 0}$ satisfies the assumptions (D1)–(D6) of Section 1.2.

We want to show that the maps $\Psi_{r,\mathbb{Z}} : B_r(\mathbb{Z}; N) \to A_r^{\mathrm{o}}(\mathbb{Z})$ satisfy the assumptions (Q1)–(Q3) of Section 2. It will follow that Brauer's centralizer algebras $A_r^{\mathrm{o}}(\mathbb{Z})$ are cellular over the integers.

First we define the appropriate permissible points in $\widehat{B}_r$ and permissible paths in $\widehat{B}$.

Definition 8.3. An N–permissible partition λ is a partition such that $\lambda'_1 + \lambda'_2 \leqslant N$. We say that an element $(\lambda, l) \in \widehat{B}_r$ is N–permissible if λ is N–permissible. We let $\widehat{B}^{\mathrm{o}}_{r,\mathrm{perm}} \subseteq \widehat{B}_r$ denote the subset of N–permissible points.

A path $\mathsf{t} \in \mathrm{Std}_r(\lambda, l)$ is N–permissible if $\mathsf{t}(k)$ is N–permissible for all $0 \leqslant k \leqslant r$. We let $\mathrm{Std}^{\mathrm{o}}_{r,\mathrm{perm}}(\lambda, l) \subseteq \mathrm{Std}_r(\lambda, l)$ denote the subset of N–permissible paths.

Note that this set of permissible points satisfies condition (Q1).

We will require the notion of a walled Brauer diagram. Consider Brauer diagrams with $a + b$ strands. Divide the top vertices into a left cluster of a vertices and a right cluster of b vertices, and similarly for the bottom vertices. The (a, b)–walled Brauer diagrams are those in which no vertical strand connects a left vertex and a right vertex, and every horizontal strand connects a left vertex and a right vertex. For any ground ring U and loop parameter δ, the U–span of (a, b)–walled Brauer diagrams is a unital involution–invariant subalgebra of $B_{a+b}(U; \delta)$, called the walled Brauer algebra, and denoted $B_{a,b}(U; \delta)$. (One imagines a wall dividing the left and right vertices; thus the terminology "walled Brauer diagram" and "walled Brauer algebra".)

Definition 8.4. If d is an (a, b)-walled Brauer diagram, then the diagram obtained by exchanging the top and bottom vertices to the right of the wall is a permutation diagram. Define the sign of d, denoted $\mathrm{sign}(d)$ to be the sign of the permutation diagram.

Example 8.5. For $a, b \in \mathbb{N}$, let $e_{a,b}$ be the (a, b)–walled Brauer diagram with horizontal edges $\{\mathbf{1}, \boldsymbol{a} + \boldsymbol{b}\}$ and $\{\overline{\mathbf{1}}, \overline{\boldsymbol{a} + \boldsymbol{b}}\}$ and vertical edges $\{\boldsymbol{j}, \overline{\boldsymbol{j}}\}$ for $j \neq 1, a + b$. Then the permutation corresponding to $e_{a,b}$ is the transposition $(1, a + b)$, and hence $\mathrm{sign}(e_{a,b}) = -1$.

Definition 8.6. Let $a, b \geqslant 0$. We define elements $\mathfrak{d}_{a,b}$ and $\mathfrak{d}'_{a,b}$ in $B_{a,b} \subseteq B_{a+b}$.

(1) Let $\mathfrak{d}_{a,b} = \sum_d \mathrm{sign}(d)\, d$, where the sum is over all (a, b)-walled Brauer diagrams.
(2) Let $\mathfrak{d}'_{a,b} = \sum_d \mathrm{sign}(d)\, d$, where now the sum is over all (a, b)-walled Brauer diagrams of corank $\geqslant 1$.

Definition 8.7. For a composition λ , the row antisymmetrizer of λ is $A_\lambda = \sum_{\pi \in \mathfrak{S}_\lambda} \operatorname{sign}(\pi)\, \pi$. In case λ is a partition, we have $A_\lambda = y_{\lambda'}$, where λ' is the conjugate partition.

Lemma 8.8. *For $a, b \geqslant 0$, there exists an element $\beta'_{a,b} \in B_{a,b}(\mathbb{Z}; N) \subseteq B_{a+b}(\mathbb{Z}; N)$ such that $\mathfrak{d}'_{a,b} = A_{(a,b)} \beta'_{a,b}$.*

Proof. $\mathfrak{S}_a \times \mathfrak{S}_b$ acts freely (by multiplication on the left) on the set $D'_{a,b}$ of (a, b)-walled Brauer diagrams with corank $\geqslant 1$, and

$$\operatorname{sign}(w\, d) = \operatorname{sign}(w) \operatorname{sign}(d),$$

for $w \in \mathfrak{S}_a \times \mathfrak{S}_b$ and $d \in D'_{a,b}$. Choose a representative of each orbit of the action. Then

$$\mathfrak{d}'_{a,b} = A_{(a,b)} \sum_x \operatorname{sign}(x) x,$$

where the sum is over the chosen orbit representatives. □

Given λ a Young diagram with more than two columns, we vertically slice λ into two parts after the second column. The left and right segments of the sliced partition are then defined as follows,

$$\lambda^L = (\lambda'_1, \lambda'_2)' \quad \lambda^R = (\lambda'_3, \lambda'_4, \dots)'.$$

Definition 8.9.

(1) If λ is a Young diagram with at most two columns, define $\mathfrak{d}_\lambda = \mathfrak{d}_{\lambda'_1, \lambda'_2}$ and $\mathfrak{d}'_\lambda = \mathfrak{d}'_{\lambda'_1, \lambda'_2}$.

(2) For λ a Young diagram, with more than 2 columns we write

$$\mathfrak{d}_\lambda = \mathfrak{d}_{\lambda^L} \otimes y_{\lambda^R} \quad \text{and} \quad \mathfrak{d}'_\lambda = \mathfrak{d}'_{\lambda^L} \otimes y_{\lambda^R}. \tag{8.2}$$

(3) For $(\lambda, l) \in \widehat{B}_r$, write

$$\mathfrak{d}_{(\lambda,l)} = \mathfrak{d}_\lambda e^{(l)}_{r-1} \quad \text{and} \quad \mathfrak{d}'_{(\lambda,l)} = \mathfrak{d}'_\lambda e^{(l)}_{r-1}. \tag{8.3}$$

Remark 8.10. It is immediate that for all r and for all $(\lambda, l) \in \widehat{B}_r$,

$$y_{(\lambda,l)} = \mathfrak{d}_{(\lambda,l)} - \mathfrak{d}'_{(\lambda,l)}. \tag{8.4}$$

Moreover, it follows from Lemma 8.8 that

$$\mathfrak{d}'_{(\lambda,l)} = y_{(\lambda,l)} \beta', \tag{8.5}$$

where $\beta' = \beta'_{\lambda'_1, \lambda'_2}$.

Fix $r \geqslant 1$. The multilinear functionals on V^{2r} of the form $(w_1, \dots, w_{2r}) \mapsto \prod (w_i, w_j)$, where each w_i occurs exactly once, are evidently $\mathrm{O}(V)$–invariant. Moreover, there are some evident relations among such functionals, stemming from the following observation. If we take $r = N + 1$ and fix disjoint sets S, S' of size $N + 1$ with $S \cup S' = \{1, 2, \dots, 2N + 2\}$, then $(w_1, \dots, w_{2r}) \mapsto \det((w_i, w_j))_{i \in S, j \in S'}$ is zero, because the matrix $((w_i, w_j))_{i \in S, j \in S'}$ is singular. These elementary observations are preliminary to the first and second fundamental theorems of invariant theory for the orthogonal groups. See the preamble to [41, Theorem 2.17.A]. The following proposition depends on the second of these observations.

Proposition 8.11. *If $a + b = N + 1$, then $\mathfrak{d}_{a,b} \in \ker(\Psi)$. Hence if $r \geqslant N + 1$ and $(\lambda, l) \in \widehat{B}_r$ with $\lambda'_1 + \lambda'_2 = N + 1$, then $\mathfrak{d}_{(\lambda,l)} \in \ker(\Psi)$.*

Proof. Set $r = N + 1$. There exist linear isomorphisms $A : V^{\otimes 2r} \to \operatorname{End}(V^{\otimes r})$ and $\eta : V^{\otimes 2r} \to (V^{\otimes 2r})^*$. The proof consists of showing that $\eta \circ A^{-1} \circ \Psi(\mathfrak{d}_{a,b})$ is a functional of the sort described above, $(\eta \circ A^{-1} \circ \Psi(\mathfrak{d}_{a,b}))(w_1 \otimes \cdots \otimes w_{2r}) = \det((w_i, w_j))_{i \in S, j \in S'}$, for suitable choice of S, S'. Hence $\mathfrak{d}_{a,b} \in \ker(\Psi)$. This is explained in [12, Section 3.3] or [29, Lemma 3.3]. We have also provided an exposition in Appendix D in the arXiv version of this paper. □

We can now verify axiom (Q2). Let $\mathsf{t} \in \mathrm{Std}_r(\lambda, l)$ be a path which is not N–permissible. Let $k \leqslant r$ be the first index such that $\mathsf{t}(k) = (\mu, m)$ satisfies $\mu'_1 + \mu'_2 = N + 1$. It follows from Proposition 8.11 that

$$\mathfrak{d}_{(\mu,m)} \in \ker(\Psi). \tag{8.6}$$

By (8.5) and (5.2), we have that

$$\mathfrak{d}'_{(\mu,m)} = y_{(\mu,m)}\beta' = y_\mu \beta' e_{k-1}^{(m)},$$

where $\beta' = \beta'_{\mu'_1,\mu'_2}$. This exhibits $\mathfrak{d}_{(\mu,m)}$ as an element of $y_{(\mu,m)}B_r(\mathbb{Z}; N)$. Moreover, it shows that $\mathfrak{d}'_{(\mu,m)}$ is a linear combination of Brauer diagrams of corank at least $m + 1$, and therefore $\mathfrak{d}'_{(\mu,m)} \in B_r(\mathbb{Z}; N)^{\rhd_{\mathrm{col}}(\mu,m)}$, using Lemma 5.2. Hence

$$\mathfrak{d}'_{(\mu,m)} \in y_{(\mu,m)}B_r(\mathbb{Z}; N) \cap B_r(\mathbb{Z}; N)^{\rhd_{\mathrm{col}}(\mu,m)}. \tag{8.7}$$

Equations (8.4)–(8.7) show that axiom (Q2) holds. It is shown in [38, Theorem 3.4, Corollary 3.5] that

$$\dim_{\mathbb{C}}(A^{\mathrm{o}}_r(\mathbb{C})) = \sum_{(\lambda,l)\in \widehat{B}^{\mathrm{o}}_{r,\mathrm{perm}}} (\sharp\mathrm{Std}^{\mathrm{o}}_{r,\mathrm{perm}}(\lambda, l))^2.$$

Thus axiom (Q3) holds.

Since assumptions (Q1)–(Q3) of Section 2 are satisfied, we can produce a modified dual Murphy basis of $B_r(\mathbb{Z}; N)$,

$$\left\{ \widetilde{y}^{(\lambda,l)}_{\mathsf{st}} \;\middle|\; (\lambda, l) \in \widehat{B}_r \text{ and } \mathsf{s}, \mathsf{t} \in \mathrm{Std}_r(\lambda, l) \right\},$$

following the procedure described before Theorem 2.5. The following theorem gives a cellular basis for Brauer's centralizer algebra acting on orthogonal tensor space, valid over the integers. It also gives two descriptions of the kernel of the map $\Psi_r : B_r(\mathbb{Z}; N) \to \mathrm{End}(V^{\otimes r})$, one by providing a basis of $\ker(\Psi_r)$ over the integers, and the other by describing the kernel as the ideal generated by a small set of elements. Each of these statements is a form of the second fundamental theorem of invariant theory for the orthogonal groups.

Theorem 8.12. *The integral (orthogonal) Brauer centralizer algebra $A^{\mathrm{o}}_r(\mathbb{Z})$ is a cellular algebra over $\mathbb{Z}$ with basis*

$$\mathbb{A}^{\mathrm{o}}_r = \left\{ \Psi(\widetilde{y}^{(\lambda,l)}_{\mathsf{st}}) \mid (\lambda, l) \in \widehat{B}^{\mathrm{o}}_{r,\mathrm{perm}} \text{ and } \mathsf{s}, \mathsf{t} \in \mathrm{Std}^{\mathrm{o}}_{r,\mathrm{perm}}(\lambda, l) \right\},$$

with the involution $*$ determined by $E_i^* = E_i$ and $S_i^* = S_i$ and the partially ordered set $(\widehat{B}^{\mathrm{o}}_{r,\mathrm{perm}}, \trianglerighteq_{\mathrm{col}})$. The ideal $\ker(\Psi_r) \subseteq B_r(\mathbb{Z}; N)$ has $\mathbb{Z}$-basis

$$\kappa_r = \left\{ \widetilde{y}^{(\lambda,l)}_{\mathsf{st}} \mid (\lambda, l) \in \widehat{B}_{r,\mathrm{perm}} \text{ and } \mathsf{s} \text{ or } \mathsf{t} \text{ is not } N\text{–permissible} \right\}.$$

Moreover, for $r > N$, $\ker(\Psi_r)$ is the ideal generated by the set $\{\mathfrak{d}_{a,b} \mid a+b = N+1\}$. For $r \leqslant N$, $\ker(\Psi_r) = 0$.

Proof. The construction of the cellular basis of $A^{\mathrm{o}}_r(\mathbb{Z})$ and of the basis of $\ker(\Psi_r)$ follows immediately from Theorem 2.7 since (Q1)–(Q3) have been verified.

Now, if $r \leqslant N$ then $\ker(\Psi_r) = 0$ since all paths on $\widehat{B}$ of length $\leqslant N$ are N–permissible. For $r > N$, the kernel is the ideal generated by all the $\mathfrak{d}_{(\mu,m)}$ such that (μ, m) is a marginal point in $\widehat{B}_k$ for some $0 < k \leqslant r$, using Theorem 2.7. But the marginal points are all of the form (μ, m) for some μ with $\mu'_1 + \mu'_2 = N + 1$. Now by (8.2) and (8.3),

$$\mathfrak{d}_{(\mu,m)} = \mathfrak{d}_\mu e^{(m)}_{k-1} = (\mathfrak{d}_{\mu^L} \otimes y_{\mu^R})e^{(m)}_{k-1},$$

and so the result follows. □

Remark 8.13. As in Remark 7.10, our construction provides an integral form of the simple $B_r(\mathbb{C}; N)$-modules labeled by permissible partitions.

8.2. **Murphy basis over an arbitrary field.** We return to the general situation described at the beginning of Section 8: V is a vector space of dimension N over an arbitrary field $\Bbbk$, with $\mathrm{char}(\Bbbk) \neq 2$, with an orthogonal form, and for $r \geqslant 1$, $\Psi_{r,\Bbbk} : B_r(\Bbbk; N) \to \mathrm{End}(V^{\otimes r})$ is Brauer's homomorphism. The map $\Psi_{r,\Bbbk}$ actually depends upon the particular orthogonal form, and, in contrast to the symplectic case, we may not assume in this generality that we are dealing with the standard orthogonal form on $\Bbbk^N$.

For the modified Murphy basis $\{\widetilde{y}_{\mathsf{st}}^{(\lambda,l)}\}$ of $B_r(\mathbb{Z}; N)$, we also write $\widetilde{y}_{\mathsf{st}}^{(\lambda,l)}$ instead of $\widetilde{y}_{\mathsf{st}}^{(\lambda,l)} \otimes 1_\Bbbk$ for the corresponding basis element of $B_r(\Bbbk; N) \cong B_r(\mathbb{Z}; N) \otimes \Bbbk$.

Theorem 8.14. *Let V be a vector space of dimension N over a field $\Bbbk$ with $\mathrm{char}(\Bbbk) \neq 2$. Assume V has an orthogonal form $(\ ,\ )$, and let $\Psi_{r,\Bbbk} : B_r(\Bbbk; N) \to \mathrm{End}(V^{\otimes r})$ be Brauer's homomorphism defined using the orthogonal form. Write $A_r^{\mathrm{o}}(\Bbbk)$ for the Brauer centralizer algebra $\mathrm{im}(\Psi_{r,\Bbbk})$.*

The algebra $A_r^{\mathrm{o}}(\Bbbk)$ is a cellular algebra over $\Bbbk$ with basis

$$\mathbb{A}_r^{\mathrm{o}}(\Bbbk) = \left\{ \Psi_{r,\Bbbk}(\widetilde{y}_{\mathsf{st}}^{(\lambda,l)}) \mid (\lambda, l) \in \widehat{B}_{r,\mathrm{perm}}^{\mathrm{o}} \text{ and } \mathsf{s}, \mathsf{t} \in \mathrm{Std}_{r,\mathrm{perm}}^{\mathrm{o}}(\lambda, l) \right\}.$$

with the involution $*$ determined by $E_i^* = E_i$ and $S_i^* = S_i$ and the partially ordered set $(\widehat{B}_{r,\mathrm{perm}}^{\mathrm{o}}, \trianglerighteq_{\mathrm{col}})$. The ideal $\ker(\Psi_{r,\Bbbk}) \subseteq B_r(\Bbbk; N)$ has basis

$$\kappa_r = \left\{ \widetilde{y}_{\mathsf{st}}^{(\lambda,l)} \mid (\lambda, l) \in \widehat{B}_r \text{ and } \mathsf{s} \text{ or } \mathsf{t} \text{ not } N\text{–permissible} \right\}.$$

Moreover, for $r > N$, $\ker(\Psi_{r,\Bbbk})$ is the ideal generated by the set $\{\mathfrak{d}_{a,b} \mid a + b = N + 1\}$. For $r \leqslant N$, $\ker(\Psi_{r,\Bbbk}) = 0$.

Proof. Assume first that $\Bbbk$ is infinite and quadratically closed. In this case, we may assume that $V = \Bbbk^N$ with the standard orthogonal form, and moreover, we have the commutative diagram (8.1). Now we can argue exactly as in the proof of Theorem 7.11, using the constancy of dimension of the orthogonal Brauer centralizer algebra from Theorem 8.1, to obtain the desired conclusions.

Now consider the general case. Let $\overline{\Bbbk}$ be the algebraic closure of $\Bbbk$. Extend the orthogonal form to $V_{\overline{\Bbbk}} = V \otimes_\Bbbk \overline{\Bbbk}$ by $(v \otimes 1_{\overline{\Bbbk}}, w \otimes 1_{\overline{\Bbbk}}) = (v, w) \in \Bbbk \subset \overline{\Bbbk}$. Let $\Psi_{r,\overline{\Bbbk}} : B_r(\overline{\Bbbk}; N) \to \mathrm{End}(V_{\overline{\Bbbk}}^{\otimes r})$ be the corresponding Brauer homomorphism. It is easy to check that we are again in the situation of Lemma 1.5, and we have a commutative diagram:

$$\begin{array}{ccc} B_r(\Bbbk; N) & \xrightarrow{\Psi_{r,\Bbbk}} & A_r^{\mathrm{o}}(\Bbbk) \\ \downarrow{\scriptstyle \otimes 1_{\overline{\Bbbk}}} & & \downarrow{\scriptstyle \eta} \\ B_r(\overline{\Bbbk}; N) & \xrightarrow[\Psi_{r,\overline{\Bbbk}}]{} & A_r^{\mathrm{o}}(\overline{\Bbbk}) \end{array} \quad . \tag{8.8}$$

Moreover, from the first paragraph of the proof, we know that $\mathbb{A}_r^{\mathrm{o}}(\overline{\Bbbk})$ is a $\overline{\Bbbk}$–basis of $A_r^{\mathrm{o}}(\overline{\Bbbk})$. The map η in (8.8) is injective, by Remark 1.6. If $y \in \kappa_r \subset B_r(\Bbbk; N)$, then $0 = \Psi_{r,\overline{\Bbbk}}(y \otimes 1_{\overline{\Bbbk}}) = \eta(\Psi_{r,\Bbbk}(y))$. Since η is injective, it follows that $\kappa_r \subseteq \ker(\Psi_{r,\Bbbk})$, and from this it follows that $\mathbb{A}_r^{\mathrm{o}}(\Bbbk)$ spans $A_r^{\mathrm{o}}(\Bbbk)$. But $\mathbb{A}_r^{\mathrm{o}}(\Bbbk)$ is linearly independent over $\Bbbk$, because $\eta(\mathbb{A}_r^{\mathrm{o}}(\Bbbk)) = \mathbb{A}_r^{\mathrm{o}}(\overline{\Bbbk})$ is linearly independent over $\overline{\Bbbk}$. Now we conclude as in the proof of Theorem 2.7 that κ_r is a basis of $\ker(\Psi_{r,\Bbbk})$ and that $\mathbb{A}_r^{\mathrm{o}}(\Bbbk)$ is a cellular basis of $A_r^{\mathrm{o}}(\Bbbk)$. To finish, it suffices to observe that κ_r is contained in the ideal generated by the elements $\mathfrak{d}_{a,b}$, and this follows from Theorem 8.12. ☐

Remark 8.15. Over fields of characteristic not equal to 2, it is shown in [29, 24] that $\ker(\Psi_{r,\Bbbk})$ is actually generated by the single element $\mathfrak{d}_{\lceil N/2 \rceil, \lfloor N/2 \rfloor}$.

Remark 8.16. As in the symplectic case, Theorem 8.14 extends the constancy of dimension of the Brauer centralizer algebras $A_r^{\mathrm{o}}(\Bbbk)$ and of $\ker(\Psi_{r,\Bbbk})$ from Theorem 8.1 to all fields $\Bbbk$ of characteristic different from 2, and, moreover, to all orthogonal forms on a finite dimensional $\Bbbk$–vector space.

We observe that the isomorphism type of the Brauer centralizer algebra $A_r^{\mathrm{o}}(\Bbbk)$ is independent of the choice of the orthogonal bilinear form; for example, when $\Bbbk = \mathbb{R}$, the field of real numbers, the isomorphism type of $A_r^{\mathrm{o}}(\mathbb{R})$ does not depend on the signature of the form:

Corollary 8.17. *Adopt the hypotheses of Theorem 8.14. The Brauer centralizer algebra $A_r^{\mathrm{o}}(\Bbbk)$ acting on $V^{\otimes r}$ is the specialization of the integral Brauer centralizer algebra $A_r^{\mathrm{o}}(\mathbb{Z})$, i.e. $A_r^{\mathrm{o}}(\Bbbk) \cong A_r^{\mathrm{o}}(\mathbb{Z}) \otimes_{\mathbb{Z}} \Bbbk$. Consequently, the isomorphism type of $A_r^{\mathrm{o}}(\Bbbk)$ is independent of the choice of the orthogonal form on V.*

Proof. The proof is exactly the same as that of Corollary 7.13. The final statement holds since $A_r^{\mathrm{o}}(\mathbb{Z}) \otimes_{\mathbb{Z}} \Bbbk$ doesn't depend on the choice of the bilinear form. □

8.3. **Jucys–Murphy elements and seminormal representations.** The discussion in Section 7.3 before Lemma 7.14 carries over to the orthogonal case with small changes. However, the condition of Lemma 3.7 fails for even integer values of the parameter, so a different argument is needed to verify condition (SN). This is done in [10].

Example 8.18. Let $N = 2M$ be a positive even integer. Let $\lambda = (M, M-1)'$, $\mu^+ = (M+1, M-1)'$ and $\mu^- = (M-1, M-1)'$. Then all three Young diagrams are $2M$–permissible, and $\kappa((\lambda, 0) \to (\mu^+, 0)) \equiv \kappa((\lambda, 0) \to (\mu^-, 1)) \mod (\boldsymbol{\delta} - 2M)$.

References

1. Henning Haahr Andersen, Catharina Stroppel, and Daniel Tubbenhauer, *Cellular structures using U_q-tilting modules*, Pacific J. Math. **292** (2018), no. 1, 21–59. MR 3708257
2. S. Ariki, A. Mathas, and H. Rui, *Cyclotomic Nazarov-Wenzl algebras*, Nagoya Math. J. **182** (2006), 47–134.
3. C. Bowman, M. De Visscher, and J. Enyang, *Simple modules for the partition algebra and monotone convergence of Kronecker coefficients*, Int. Math. Res. Not. IMRN (to appear), doi:10.1093/imrn/rnx095.
4. Christopher Bowman, John Enyang, and Frederick Goodman, *Diagram algebras, dominance triangularity and skew cell modules*, J. Aust. Math. Soc. **104** (2018), no. 1, 13–36. MR 3745412
5. R. Brauer, *On algebras which are connected with the semisimple continuous groups*, Ann. of Math. (2) **38** (1937), no. 4, 857–872.
6. M. De Visscher and P. P. Martin, *On Brauer algebra simple modules over the complex field*, Trans. Amer. Math. Soc. **369** (2017), no. 3, 1579–1609.
7. R. Dipper, S. Doty, and J. Hu, *Brauer algebras, symplectic Schur algebras and Schur-Weyl duality*, Trans. Amer. Math. Soc. **360** (2008), no. 1, 189–213 (electronic).
8. R. Dipper and G. James, *Representations of Hecke algebras of general linear groups*, Proc. London Math. Soc. (3) **52** (1986), no. 1, 20–52.
9. S. Doty and J. Hu, *Schur-Weyl duality for orthogonal groups*, Proc. Lond. Math. Soc. (3) **98** (2009), no. 3, 679–713.
10. J. Enyang and F. M. Goodman, *The seminormal representations of the Brauer algebras*, in preparation.
11. ———, *Cellular Bases for Algebras with a Jones Basic Construction*, Algebr. Represent. Theory **20** (2017), no. 1, 71–121.
12. F. Gavarini, *A Brauer algebra-theoretic proof of Littlewood's restriction rules*, J. Algebra **212** (1999), no. 1, 240–271.
13. ———, *On the radical of Brauer algebras*, Math. Z. **260** (2008), no. 3, 673–697.
14. M. Geck, *Hecke algebras of finite type are cellular*, Invent. Math. **169** (2007), no. 3, 501–517.
15. Meinolf Geck, *Kazhdan-Lusztig cells and the Murphy basis*, Proc. London Math. Soc. (3) **93** (2006), no. 3, 635–665. MR 2266962
16. T. Geetha and F. M. Goodman, *Cellularity of wreath product algebras and A-Brauer algebras*, J. Algebra **389** (2013), 151–190.
17. F. M. Goodman and J. Graber, *Cellularity and the Jones basic construction*, Adv. in Appl. Math. **46** (2011), no. 1-4, 312–362.
18. ———, *On cellular algebras with Jucys Murphy elements*, J. Algebra **330** (2011), 147–176.
19. J. J. Graham and G. I. Lehrer, *Cellular algebras*, Invent. Math. **123** (1996), no. 1, 1–34.
20. M. Härterich, *Murphy bases of generalized Temperley-Lieb algebras*, Arch. Math. (Basel) **72** (1999), no. 5, 337–345.
21. J. Hu, *Specht filtrations and tensor spaces for the Brauer algebra*, J. Algebraic Combin. **28** (2008), no. 2, 281–312.

22. J. Hu and A. Mathas, *Graded cellular bases for the cyclotomic Khovanov-Lauda-Rouquier algebras of type A*, Adv. Math. **225** (2010), no. 2, 598–642.
23. J. Hu and Z. Xiao, *On tensor spaces for Birman-Murakami-Wenzl algebras*, J. Algebra **324** (2010), no. 10, 2893–2922.
24. ———, *On a theorem of Lehrer and Zhang*, Doc. Math. **17** (2012), 245–270.
25. G. D. James, *The representation theory of the symmetric groups*, Lecture Notes in Mathematics, vol. 682, Springer, Berlin, 1978.
26. Thomas Jost, *Morita equivalence for blocks of Hecke algebras of symmetric groups*, J. Algebra **194** (1997), no. 1, 201–223.
27. David Kazhdan and George Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), no. 2, 165–184. MR 560412
28. R. Leduc and A. Ram, *A ribbon Hopf algebra approach to the irreducible representations of centralizer algebras: the Brauer, Birman-Wenzl, and type A Iwahori-Hecke algebras*, Adv. Math. **125** (1997), no. 1, 1–94.
29. G. Lehrer and R. Zhang, *The second fundamental theorem of invariant theory for the orthogonal group*, Ann. of Math. (2) **176** (2012), no. 3, 2031–2054.
30. G. Lusztig, *Introduction to quantum groups*, Progress in Mathematics, vol. 110, Birkhäuser Boston, Inc., Boston, MA, 1993.
31. A. Mathas, *Seminormal forms and Gram determinants for cellular algebras*, J. Reine Angew. Math. **619** (2008), 141–173, With an appendix by Marcos Soriano.
32. ———, *A Specht filtration of an induced Specht module*, J. Algebra **322** (2009), no. 3, 893–902.
33. G. E. Murphy, *On the representation theory of the symmetric groups and associated Hecke algebras*, J. Algebra **152** (1992), no. 2, 492–513.
34. ———, *The representations of Hecke algebras of type A_n*, J. Algebra **173** (1995), no. 1, 97–121.
35. M. Nazarov, *Young's orthogonal form for Brauer's centralizer algebra*, J. Algebra **182** (1996), no. 3, 664–693.
36. A. Ram and Hans Wenzl, *Matrix units for centralizer algebras*, J. Algebra **145** (1992), no. 2, 378–395.
37. F. Stoll and M. Werth, *A cell filtration of mixed tensor space*, Math. Z. **282** (2016), no. 3-4, 769–798.
38. Hans Wenzl, *On the structure of Brauer's centralizer algebras*, Ann. of Math. (2) **128** (1988), no. 1, 173–193.
39. M. Werth, *Murphy bases for endomorphism rings of tensor space*, Ph.D. thesis, Universität Stuttgart, 2015.
40. B. W. Westbury, *Invariant tensors and cellular categories*, J. Algebra **321** (2009), no. 11, 3563–3567.
41. H. Weyl, *The Classical Groups. Their Invariants and Representations*, Princeton University Press, Princeton, N.J., 1939.

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