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INVERSES OF DISJOINTNESS PRESERVING OPERATORS IN FINITE DIMENSIONAL PRE-RIESZ SPACES

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Abstract. If $\mathbb{R}^n$ is partially ordered by a generating closed cone K, then $(\mathbb{R}^n, K)$ is a pre-Riesz space. We show for a disjointness preserving bijection T on $(\mathbb{R}^n, K)$ that the inverse of T is also disjointness preserving. We prove that for T there is $k \in \mathcal{P}(b)$ such that T^k is band preserving, where b is the number of bands in $(\mathbb{R}^n, K)$, and $\mathcal{P}(b)$ the set of orders of permutations on b symbols.

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Key words: Band, band preserving operator, disjointness preserving operator, d-isomorphism, finite dimensional, generating closed cone, pre-Riesz space.

1. Introduction. If X and Y are Banach lattices and $T\colon X \to Y$ is a disjointness preserving bijection, then in [8] and [9] it is shown that the inverse T^{-1} is also disjointness preserving. Various other conditions under which T^{-1} is disjointness preserving are given in [1] and [2], where it is also observed that for a disjointness preserving bijection between arbitrary vector lattices the inverse is not disjointness preserving, in general.

In [4] disjointness is introduced in the more general setting of partially ordered vector spaces. In a finite dimensional partially ordered vector space $(\mathbb{R}^n, K)$ with closed generating cone K, we will show that the inverse of a disjointness preserving operator is disjointness preserving.

In partially ordered vector spaces, bands are defined in [4] as sets that equal their double-disjoint complement. In [6] it is shown that the number b of bands in

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$(\mathbb{R}^n, K)$ does not exceed $\frac{1}{4}2^{2^n}$. Using this fact, for a disjointness preserving bijection $T\colon \mathbb{R}^n \to \mathbb{R}^n$ we will show that there is $k \in \mathbb{N}$ such that T^k is band preserving.

In the remainder of this section we fix our notation. Let X be a real vector space and let K be a *cone* in X, that is, K is a *wedge* ($x, y \in K$, $\lambda, \mu \geq 0$ imply $\lambda x + \mu y \in K$) and $K \cap (-K) = \{0\}$. In X a partial order is introduced by defining $x \leq y$ if and only if $y - x \in K$; we write (X, K) for a partially ordered vector space. We call (X, K) *Archimedean* if for every $x, y \in X$ with $nx \leq y$ for all $n \in \mathbb{N}$ one has that $x \leq 0$, and *directed* if $X = K - K$. Denote for a set $M \subseteq X$ the set of all upper bounds of M by

$$M^{\mathrm{u}} = \{x \in X;\ \forall m \in M\colon x \geq m\}.$$

For standard notations in the case that (X, K) is a vector lattice see [3].

By a subspace of a partially ordered vector space or a vector lattice we mean an arbitrary linear subspace with the inherited order. We do not require it to be a lattice or a sublattice. We call a subspace D of a partially ordered vector space Y *order dense* in Y if every $y \in Y$ is the greatest lower bound of the set $\{d \in D;\ y \leq d\}$ in Y, i.e.

$$y = \inf\{d \in D;\ y \leq d\}.$$

We continue by a notion which is closely related to the order dense embedding of a partially ordered vector space into a vector lattice. A partially ordered vector space X is called *pre-Riesz* if for every $x, y, z \in X$ the inclusion $\{x+y, x+z\}^{\mathrm{u}} \subseteq \{y, z\}^{\mathrm{u}}$ implies $x \in K$ [7, Definition 1.1(viii), Theorem 4.15]. Every pre-Riesz space is directed and every directed Archimedean partially ordered vector space is pre-Riesz [7]. Clearly, each vector lattice is pre-Riesz.

Recall that a linear map $i\colon X \to Y$, where X and Y are partially ordered vector spaces, is called *bipositive* if for every $x \in X$ one has $0 \leq x$ if and only if $0 \leq i(x)$. An embedding map is required to be linear and bipositive, which implies injectivity. For sets $L \subseteq X$ and $M \subseteq Y$ we denote $i[L] := \{i(x);\ x \in L\}$ and $[M]i := \{x \in X;\ i(x) \in M\}$.

Let X be a partially ordered vector space. The following statements are equivalent [7, Corollaries 4.9–11 and Theorems 3.5, 3.7, 4.13]:

(i) X is pre-Riesz.

(ii) There exist a vector lattice Y and a bipositive linear map $i\colon\ X \to Y$ such that $i[X]$ is order dense in Y.

(iii) There exist a vector lattice Y and a bipositive linear map $i\colon\ X \to Y$ such that $i[X]$ is order dense in Y and $i[X]$ generates Y as a vector lattice, i.e. for every $y \in Y$ there are $a_1, \ldots, a_m, b_1, \ldots, b_n \in i[X]$ such that

$$y = \bigvee_{i=1}^{m} a_i - \bigvee_{i=1}^{n} b_i.$$

A pair (Y, i) as in (ii) is called a *vector lattice cover* of X, a pair (Y, i) as in (iii) is called a *Riesz completion* of X. Since all spaces Y as in (iii) are isomorphic as vector lattices, we will speak of *the* Riesz completion of X and denote it by (X^ρ, i). If X is pre-Riesz and (Y, i) a vector lattice cover of X, then X^ρ is the vector lattice generated by $i[X]$.

Disjointness in a partially ordered vector space (X, K) is introduced in [4]. Two elements $x, y \in X$ are called *disjoint*, in symbols $x \perp y$, if

$$\{x + y, -x - y\}^{\mathrm{u}} = \{x - y, -x + y\}^{\mathrm{u}}.$$

If X is a vector lattice, then this notion of disjointness coincides with the usual one, see [3, Theorem 1.4(4)]. The *disjoint complement* of a subset $M \subseteq X$ is the set $M^{\mathrm{d}} = \{y \in X; \forall x \in M\colon\ y \perp x\}$. A subspace B of X is called a *band* if $B = B^{\mathrm{dd}}$. Note that if X is an Archimedean vector lattice, then this notion of a band coincides with the usual one (i.e. a band is defined to be an order closed ideal).

Let (X, K) be a pre-Riesz space and (Y, i) a vector lattice cover of X. The order denseness of $i[X]$ in Y implies that two elements in X are disjoint if and only if they are disjoint in Y [4, Proposition 2.1]. For a set $S \subseteq X$ this means

$$S^{\mathrm{d}} = [i[S]^{\mathrm{d}}]i. \tag{1}$$

This implies that disjoint complements in pre-Riesz spaces have similar properties as the ones in vector lattices, see [5, Theorem 5.10]. In particular, a disjoint complement is a band in X.

A linear operator $T\colon X \to X$ is called *disjointness preserving*, if for every $x, y \in X$ with $x \perp y$ one has that $Tx \perp Ty$. A linear bijection $T\colon X \to X$ is called a *d-isomorphism* if T is disjointness preserving and has a disjointness preserving inverse.

The paper is organized as follows. In Section 2 we consider disjointness preserving operators and d-isomorphisms in arbitrary pre-Riesz spaces and collect their properties in preparation for the main results in Section 3. In Section 3 we restrict to finite-dimensional pre-Riesz spaces. We prove that the inverse of a disjointness preserving bijection is also disjointness preserving (Theorem 3.4). Moreover, we show that a certain power of a disjointness preserving bijection is band preserving (Theorem 3.5).

2. On d-isomorphisms in pre-Riesz spaces. In the present section, let (X, K) be a pre-Riesz space and $T\colon X \to X$ a linear operator. Denote

$$\mathcal{B} := \{B \subseteq X;\ B \text{ is a band in } X\}$$

and define

$$\mathcal{T}\colon \mathcal{B} \to \mathcal{B}, \quad B \mapsto T[B]^{\mathrm{dd}}. \tag{2}$$

If $B_1, B_2 \in \mathcal{B}$ are such that $B_1 \subseteq B_2$, then $\mathcal{T}(B_1) \subseteq \mathcal{T}(B_2)$.

Proposition 2.1. *If T is disjointness preserving, then for $B_1, B_2 \in \mathcal{B}$ with $B_1 \perp B_2$ one has $\mathcal{T}(B_1) \perp \mathcal{T}(B_2)$.*

Proof. For every $B \in \mathcal{B}$ one has $\mathcal{T}(B) \subseteq \mathcal{T}(B^{\mathrm{d}})^{\mathrm{d}}$. Indeed, we show

$$T[B]^{\mathrm{dd}} \subseteq T\big[B^{\mathrm{d}}\big]^{\mathrm{d}} = T\big[B^{\mathrm{d}}\big]^{\mathrm{ddd}}. \tag{3}$$

Let $x \in T[B]^{\mathrm{dd}}$. Let $y \in T[B^{\mathrm{d}}]$ and $v \in B^{\mathrm{d}}$ be such that $Tv = y$. For $u \in B$ one has $u \perp v$ and, since T is disjointness preserving, $Tu \perp Tv$, hence $y \in T[B]^{\mathrm{d}}$. Therefore $x \perp y$, which yields (3).

As a consequence, one has $\mathcal{T}(B^{\mathrm{d}}) = \mathcal{T}(B^{\mathrm{d}})^{\mathrm{dd}} \subseteq \mathcal{T}(B)^{\mathrm{d}}$. For $B_1, B_2 \in \mathcal{B}$ with $B_1 \perp B_2$, from $B_2 \subseteq B_1^{\mathrm{d}}$ one obtains $\mathcal{T}(B_2) \subseteq \mathcal{T}(B_1^{\mathrm{d}}) \subseteq \mathcal{T}(B_1)^{\mathrm{d}}$, and we conclude $\mathcal{T}(B_2) \perp \mathcal{T}(B_1)$. □

Remark 2.2. If T is, in addition, injective, and B is a non-trivial band, then $\mathcal{T}(B)$ is a non-trivial band. Indeed, B contains an element $b \neq 0$, and since T is injective, one has $T(b) \neq T(0) = 0$. Then $T(b) \in T[B] \subseteq T[B]^{\mathrm{dd}} = \mathcal{T}(B)$. Similarly, since B^{d} contains a non-zero element c, one has $0 \neq T(c) \in T[B^{\mathrm{d}}] \subseteq T[B^{\mathrm{d}}]^{\mathrm{dd}} = \mathcal{T}(B^{\mathrm{d}}) \subseteq \mathcal{T}(B)^{\mathrm{d}}$, by Proposition 2.1.

Proposition 2.3. *If T is a d-isomorphism and $B \in \mathcal{B}$, then $\mathcal{T}(B^{\mathrm{d}}) = \mathcal{T}(B)^{\mathrm{d}}$.*

Proof. Let $B \in \mathcal{B}$. Since $B \perp B^{\mathrm{d}}$, by Proposition 2.1 one has $\mathcal{T}(B) \perp \mathcal{T}(B^{\mathrm{d}})$, hence $\mathcal{T}(B^{\mathrm{d}}) \subseteq \mathcal{T}(B)^{\mathrm{d}}$.

Next we show $T[B]^{\mathrm{d}} \subseteq T[B^{\mathrm{d}}]^{\mathrm{dd}}$. Let $x \in T[B]^{\mathrm{d}}$ and $v \in X$ be such that $Tv = x$. Let $y \in T[B^{\mathrm{d}}]^{\mathrm{d}}$. For $z \in B$ one has $x \perp Tz$ and, since T is a d-isomorphism, $v \perp z$. Therefore $v \in B^{\mathrm{d}}$, which implies $x = Tv \in T[B^{\mathrm{d}}]$. Consequently $y \perp x$, which yields $x \in T[B^{\mathrm{d}}]^{\mathrm{dd}}$.

It follows that $\mathcal{T}(B)^{\mathrm{d}} = T[B]^{\mathrm{ddd}} = T[B]^{\mathrm{d}} \subseteq T[B^{\mathrm{d}}]^{\mathrm{dd}} = \mathcal{T}(B^{\mathrm{d}})$. □

Proposition 2.4. *Let T be a d-isomorphism. Then T is band preserving if and only if $\mathcal{T}$ equals the identity.*

Proof. Let T be band preserving and $B \in \mathcal{B}$. Then $T[B] \subseteq B$, moreover $\mathcal{T}(B) = T[B]^{\mathrm{dd}} \subseteq B$. Similarly, $\mathcal{T}(B^{\mathrm{d}}) \subseteq B^{\mathrm{d}}$. By Proposition 2.3, $\mathcal{T}(B)^{\mathrm{d}} \subseteq B^{\mathrm{d}}$ and therefore $B = B^{\mathrm{dd}} \subseteq \mathcal{T}(B)^{\mathrm{dd}} = \mathcal{T}(B)$. We conclude $\mathcal{T}(B) = B$.

Conversely, let $\mathcal{T}$ be the identity and $B \in \mathcal{B}$. Then $T[B] \subseteq T[B]^{\mathrm{dd}} = \mathcal{T}(B) = B$, hence T is band preserving. □

Proposition 2.5. *If T is a d-isomorphism, then for every $k \in \mathbb{N}$ and every $B \in \mathcal{B}$ one has $\mathcal{T}^k(B) = \big(T^k[B]\big)^{\mathrm{dd}}$.*

Proof. We show $\mathcal{T}^2(B) = \big(T^2[B]\big)^{\mathrm{dd}}$, then the assertion follows by similar arguments. First observe that

$$\big(T^2[B]\big)^{\mathrm{dd}} = T[T[B]]^{\mathrm{dd}} \subseteq T\big[T[B]^{\mathrm{dd}}\big]^{\mathrm{dd}} = \mathcal{T}^2(B).$$

Similarly, $\big(T^2[B^{\mathrm{d}}]\big)^{\mathrm{dd}} \subseteq \mathcal{T}^2\big(B^{\mathrm{d}}\big)$. By Proposition 2.1, from $B \perp B^{\mathrm{d}}$ it follows that $\mathcal{T}^2(B) \perp \mathcal{T}^2\big(B^{\mathrm{d}}\big)$. Therefore

$$\mathcal{T}^2(B) \subseteq \mathcal{T}^2\big(B^{\mathrm{d}}\big)^{\mathrm{d}} \subseteq \big(T^2[B^{\mathrm{d}}]\big)^{\mathrm{ddd}} = \big(T^2[B^{\mathrm{d}}]\big)^{\mathrm{d}}.$$

It remains to show $\big(T^2[B^{\mathrm{d}}]\big)^{\mathrm{d}} \subseteq \big(T^2[B]\big)^{\mathrm{dd}}$. Indeed, let $x \in \big(T^2[B^{\mathrm{d}}]\big)^{\mathrm{d}}$. Let $y \in \big(T^2[B]\big)^{\mathrm{d}}$ and $v \in X$ be such that $T^2v = y$. For $z \in B$ one has $T^2z \perp y$, and, since T is a d-isomorphism, $z \perp v$. Therefore $v \in B^{\mathrm{d}}$, and hence $y = T^2v \in T^2[B^{\mathrm{d}}]$. Thus, $x \perp y$. □

If T is a d-isomorphism, then for $k \in \mathbb{N}$ the operator T^k is a d-isomorphism as well. Combining the Propositions 2.4 and 2.5, we obtain the following result.

PROPOSITION 2.6. *Let T be a d-isomorphism and $k \in \mathbb{N}$. Then T^k is band preserving if and only if $\mathcal{T}^k$ equals the identity.*

Now define

$$\mathcal{T}' \colon \mathcal{B} \to \mathcal{B}, \quad B \mapsto [B]T^{\mathrm{dd}}. \tag{4}$$

PROPOSITION 2.7. *If T is a d-isomorphism, then for every $B \in \mathcal{B}$ one has $\mathcal{T}'(\mathcal{T}(B)) = B$ and $\mathcal{T}(\mathcal{T}'(B)) = B$.*

Proof. We show the first equality. Given $B \in \mathcal{B}$, then

$$B \subseteq \mathcal{T}'(\mathcal{T}(B)) = \big[T[B]^{\mathrm{dd}}\big]T^{\mathrm{dd}}. \tag{5}$$

Indeed, let $x \in B$ and $y \in \big[T[B]^{\mathrm{dd}}\big]\, T^{\mathrm{d}}$. For $z \in T[B]^{\mathrm{dd}}$ and $v \in X$ with $Tv = z$, one has $v \perp y$. Since T is disjointness preserving, we obtain $Tv \perp Ty$, hence $z \perp Ty$. Thus, $Ty \in T[B]^{\mathrm{ddd}} = T[B]^{\mathrm{d}}$. In particular, $Tx \perp Ty$. Since T is a d-isomorphism, we get $x \perp y$, and hence $x \in \big[T[B]^{\mathrm{dd}}\big]\, T^{\mathrm{dd}}$. One obtains (5), and, moreover,

$$B^{\mathrm{d}} \subseteq \mathcal{T}'\big(\mathcal{T}\big(B^{\mathrm{d}}\big)\big). \tag{6}$$

By Proposition 2.1, $B \perp B^{\mathrm{d}}$ implies $\mathcal{T}(B) \perp \mathcal{T}(B^{\mathrm{d}})$. Applying Proposition 2.1 to the inverse of T, we get $\mathcal{T}'(\mathcal{T}(B)) \perp \mathcal{T}'(\mathcal{T}(B^{\mathrm{d}}))$. Together with (6), this boils down to

$$\mathcal{T}'(\mathcal{T}(B)) \subseteq \mathcal{T}'\big(\mathcal{T}\big(B^{\mathrm{d}}\big)\big)^{\mathrm{d}} \subseteq B^{\mathrm{dd}} = B.$$ □

3. Disjointness preserving bijections in finite dimensions. In this section, let K be a generating closed cone in $\mathbb{R}^n$. Note that $(\mathbb{R}^n, K)$ is then an Archimedean directed partially ordered vector space, and, hence, a pre-Riesz space, so that we can apply the results of the previous section. We intend to show that the inverse of a disjointness preserving bijection on $(\mathbb{R}^n, K)$ is disjointness preserving. For this purpose, we define

$$p \colon \mathcal{B} \to \mathbb{N}, \quad B \mapsto \dim(B) + \dim(B^{\mathrm{d}}),$$

and collect properties of p in the subsequent lemmas. The set $\mathcal{B}$ of bands in $(\mathbb{R}^n, K)$ is finite, see [6]. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a disjointness preserving linear bijection, and let $\mathcal{T}$ be defined as in (2).

Lemma 3.1. *(i) For every $B \in \mathcal{B}$ we have $p(\mathcal{T}(B)) \geq p(B)$.*

(ii) If $B \in \mathcal{B}$ is such that $p(\mathcal{T}(B)) = p(B)$, then $\mathcal{T}(B) = T[B]$ and $\mathcal{T}(B)^{\mathrm{d}} = T[B^{\mathrm{d}}]$.

Proof. (i) Clearly, $T[B] \subseteq T[B]^{\mathrm{dd}} = \mathcal{T}(B)$, and by Proposition 2.1

$$T[B^{\mathrm{d}}] \subseteq T[B^{\mathrm{d}}]^{\mathrm{dd}} = \mathcal{T}(B^{\mathrm{d}}) \subseteq \mathcal{T}(B)^{\mathrm{d}}.$$

Hence $\dim(\mathcal{T}(B)) \geq \dim(T[B]) = \dim(B)$ and $\dim(\mathcal{T}(B)^{\mathrm{d}}) \geq \dim(T[B^{\mathrm{d}}]) = \dim(B^{\mathrm{d}})$, therefore $p(\mathcal{T}(B)) \geq p(B)$.

(ii) Let B be a band such that $p(\mathcal{T}(B)) = p(B)$. Since $T[B]$ is a subspace of $\mathcal{T}(B)$, the assumption $\mathcal{T}(B) \neq T[B]$ implies $\dim(\mathcal{T}(B)) > \dim(T[B]) = \dim(B)$. Analogously, $T[B^{\mathrm{d}}]$ is a subspace of $\mathcal{T}(B)^{\mathrm{d}}$, and the assumption $\mathcal{T}(B)^{\mathrm{d}} \neq T[B^{\mathrm{d}}]$ implies $\dim(\mathcal{T}(B))^{\mathrm{d}} > \dim(T[B^{\mathrm{d}}]) = \dim(B^{\mathrm{d}})$. In both cases a contradiction to $p(\mathcal{T}(B)) = p(B)$ is obtained. □

Lemma 3.2. *(i) If $A, B \in \mathcal{B}$ are such that $\mathcal{T}(A) = T[B]$ and $\mathcal{T}(A)^{\mathrm{d}} = T[B^{\mathrm{d}}]$, then $A = B$.*

(ii) If $A, B \in \mathcal{B}$ are such that $\mathcal{T}(A) = \mathcal{T}(B)$ and $p(\mathcal{T}(B)) = p(B)$, then $A = B$.

Proof. (i) From $T[B] = \mathcal{T}(A) = T[A]^{\mathrm{dd}} \supseteq T[A]$ we obtain $B \supseteq A$. Similarly, as T is disjointness preserving, we get $T[B^{\mathrm{d}}] = \mathcal{T}(A)^{\mathrm{d}} = T[A]^{\mathrm{d}} \supseteq T[A^{\mathrm{d}}]$, hence $B^{\mathrm{d}} \supseteq A^{\mathrm{d}}$. As A and B are bands, it follows that $B = B^{\mathrm{dd}} \subseteq A^{\mathrm{dd}} = A$.

(ii) By Lemma 3.1 (ii), we obtain $\mathcal{T}(A) = \mathcal{T}(B) = T[B]$ and $\mathcal{T}(A)^{\mathrm{d}} = \mathcal{T}(B)^{\mathrm{d}} = T[B^{\mathrm{d}}]$. Now the statement follows from (i). □

Next we consider certain sets of non-trivial bands. For $k \in \mathbb{N}$ we denote

$$\mathcal{B}_k := \{B \in \mathcal{B} \setminus \{\mathbb{R}^n, \{0\}\} \colon p(B) = k\}\,.$$

Furthermore, let $m := \max\{p(B) \colon B \in \mathcal{B} \setminus \{\mathbb{R}^n, \{0\}\}\}\,.$

Lemma 3.3. *(i) For every $k \in \{2, \ldots, m\}$ one has $\mathcal{T}[\mathcal{B}_k] \subseteq \mathcal{B}_k$, and $\mathcal{T} \colon \mathcal{B}_k \to \mathcal{B}_k$ is a bijection.*

(ii) For every $B \in \mathcal{B}$ we have $T[B] \in \mathcal{B}$. For every $B \in \mathcal{B}$ there is $A \in \mathcal{B}$ such that $T[A] = B$ and $T[A^{\mathrm{d}}] = B^{\mathrm{d}}$.

Proof. We first consider the case $k = m$. Let $B \in \mathcal{B}_m$. By Remark 2.2, $\mathcal{T}(B)$ is a non-trivial band. From Lemma 3.1 (i) we obtain

$$m = p(B) \leq p(\mathcal{T}(B)) \leq m, \tag{7}$$

hence $\mathcal{T}(B) \in \mathcal{B}_m$. To show that $\mathcal{T}\colon \mathcal{B}_m \to \mathcal{B}_m$ is injective, let $A, B \in \mathcal{B}_m$ be such that $\mathcal{T}(A) = \mathcal{T}(B)$. By (7) and Lemma 3.2 (ii), we obtain $A = B$. As $\mathcal{B}_m$ is a finite set, it follows that $\mathcal{T}\colon \mathcal{B}_m \to \mathcal{B}_m$ is a bijection.

Next, assume that $\ell < m$ is such that for every $k \in \{\ell+1, \dots, m\}$ we have $\mathcal{T}[\mathcal{B}_k] \subseteq \mathcal{B}_k$, and $\mathcal{T}\colon \mathcal{B}_k \to \mathcal{B}_k$ is a bijection. Let $A \in \mathcal{B}_\ell$. Suppose that $\mathcal{T}(A) \notin \mathcal{B}_\ell$. Then, by Lemma 3.1 (i), there is $k \in \{\ell+1, \dots, m\}$ such that $\mathcal{T}(A) \in \mathcal{B}_k$. Since $\mathcal{T}\colon \mathcal{B}_k \to \mathcal{B}_k$ is a bijection, there exists $B \in \mathcal{B}_k$ such that $\mathcal{T}(B) = \mathcal{T}(A)$. Moreover, $p(\mathcal{T}(B)) = p(B) = k$, therefore Lemma 3.2 (ii) implies that $A = B$. This contradicts $p(A) = \ell < k = p(B)$. Hence $\mathcal{T}(A) \in \mathcal{B}_\ell$. We obtain $\mathcal{T}[\mathcal{B}_\ell] \subseteq \mathcal{B}_\ell$. If $A, B \in \mathcal{B}_\ell$ are such that $\mathcal{T}(A) = \mathcal{T}(B)$, then by the previous step we get $p(\mathcal{T}(B)) = \ell = p(B)$, such that Lemma 3.2 (ii) implies $A = B$. Thus, $\mathcal{T}\colon \mathcal{B}_\ell \to \mathcal{B}_\ell$ is injective, and, moreover, it is a bijection. By induction, the proof is complete.

(ii) For trivial bands the assertion is clear. Let $B \in \mathcal{B}$ be non-trivial, i.e. there exists $k \in \mathbb{N}$ such that $B \in \mathcal{B}_k$. By (i), we obtain $\mathcal{T}(B) \in \mathcal{B}_k$, i.e. $p(\mathcal{T}(B)) = p(B)$. Lemma 3.1 (ii) implies $\mathcal{T}(B) = T[B]$, hence $T[B] \in \mathcal{B}$. Moreover, by (i) there exists $A \in \mathcal{B}_k$ such that $\mathcal{T}(A) = B$. Then $p(\mathcal{T}(A)) = p(A)$, and Lemma 3.1 (ii) implies $\mathcal{T}(A) = T[A]$ and $\mathcal{T}(A)^{\mathrm{d}} = T[A^{\mathrm{d}}]$. Hence $B = T[A]$ and $B^{\mathrm{d}} = T[A^{\mathrm{d}}]$. □

As a consequence of the previous lemma, the main result on the inverses of disjointness preserving bijections in finite dimensions is obtained next.

THEOREM 3.4. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a disjointness preserving bijection. Then T is a d-isomorphism.*

Proof. We show that the inverse of T is disjointness preserving. Let $u, v \in \mathbb{R}^n$ be such that $u \perp v$, and let $x, y \in \mathbb{R}^n$ be such that $Tx = u$ and $Ty = v$. Let $B := \{u\}^{\mathrm{dd}}$, then $B \in \mathcal{B}$, $u \in B$ and $v \in B^{\mathrm{d}}$. By Lemma 3.3 (ii), there exists $A \in \mathcal{B}$ such that $B = T[A]$ and $B^{\mathrm{d}} = T[A^{\mathrm{d}}]$. Then $x \in A$ and $y \in A^{\mathrm{d}}$, hence $x \perp y$. □

The number b of bands in $(\mathbb{R}^n, K)$ is less or equal $\frac{1}{4}2^{2^n}$, see [6]. In the subsequent theorem, $\mathcal{P}(b)$ denotes the set of orders of permutations on b symbols.

THEOREM 3.5. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a disjointness preserving bijection. Then there is $k \in \mathcal{P}(b)$ such that T^k is band preserving.*

Proof. By Theorem 3.4, T is a d-isomorphism. Due to Proposition 2.7, the map $\mathcal{T}$ is a bijection, i.e. $\mathcal{T}$ is a permutation on the finite set $\mathcal{B}$. Hence there is $k \in \mathcal{P}(b)$ such that $\mathcal{T}^k$ is the identity. By Proposition 2.6, T^k is band preserving. □

Theorem 3.4 is a first instance of a theory on inverses of disjointness preserving operators on suitably normed pre-Riesz spaces. The finite dimensional spaces under consideration can be such that there are no non-trivial disjoint elements at all or such that there are even more disjoint elements than in a vector lattice of the same dimension. Apparently, that does not matter for Theorem 3.4. It is tempting to ask for similar results in an infinite dimensional setting. As in vector lattices, a general theory will at least either require more conditions on the operators or appropriate norms on the pre-Riesz space, similar to the Banach lattice case.

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