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Citation for published version

Fordy, Allan P. and Xenitidis, Pavlos (2017) ZN graded discrete Lax pairs and Yang–Baxter maps. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science, 473 (2201). p. 20160946. ISSN 1364-5021.

DOI

https://doi.org/10.1098/rspa.2016.0946

Link to record in KAR

http://kar.kent.ac.uk/62291/

Document Version

Author's Accepted Manuscript

$\mathbb{Z}_N$ graded discrete Lax pairs and Yang-Baxter maps

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December 28, 2016

Abstract

We recently introduced a class of $\mathbb{Z}_N$ graded discrete Lax pairs and studied the associated discrete integrable systems (lattice equations). In this paper we introduce the corresponding Yang-Baxter maps. Many well known examples belong to this scheme for $N = 2$, so, for $N \geq 3$, our systems may be regarded as generalisations of these.

In particular, for each N we introduce a class of multi-component Yang-Baxter maps, which include H_{III}^B (of [6]), when $N = 2$, and that associated with the discrete modified Boussinesq equation, for $N = 3$. For $N \geq 5$ we introduce a new families of Yang-Baxter maps, which have no lower dimensional analogue. We also present new multi-component versions of the Yang-Baxter maps F_{IV} and F_V (given in the classification of [2]).

Keywords: Discrete integrable system, Lax pair, symmetry, Yang-Baxter map.

1 Introduction

The term "Yang-Baxter map" was introduced by Veselov [10] as an abbreviation for Drinfeld's notion of "set-theoretical solutions to the quantum Yang-Baxter equation". The basic ingredient is a map $R : \mathrm{X} \times \mathrm{X} \to \mathrm{X} \times \mathrm{X}$, where X is some algebraic variety. For the case $\mathrm{X} = \mathbb{CP}^1$, these were partially classified in [2, 6]. In [8] a symmetry approach was introduced to relate Yang-Baxter equations with $3D$ consistent equations on quad-graphs, which had been classified in [1]. Starting with any symmetry of an integrable equation on a quad-graph, the authors introduce invariant functions, which are then used to define a map. The Yang-Baxter relation was shown to be a *consequence* of $3D$ consistency. Multi-component Yang-Baxter maps are not yet classified, but several are known (see, for example, [9, 8, 7, 5, 3]).

We recently introduced a class of $\mathbb{Z}_N$ graded discrete Lax pairs and studied the associated discrete integrable systems [4]. Many well known examples belong to that scheme for $N = 2$, so, for $N \geq 3$, our systems may be regarded as generalisations of these. As mentioned above, the quad systems for $N = 2$ can be related to Yang-Baxter maps. In this paper we construct generalisations of these, associated with our generalised lattice equations.

In Section 2 we present the basic background theory of Yang-Baxter maps and their relationship to lattice equations on a quadrilateral lattice. In Section 3, we introduce the $\mathbb{Z}_N$-graded Lax pairs of [4] and derive the reduction to Yang-Baxter maps. We show that all such maps are equivalent to ones with "level structure" $(0, \delta; 0, \delta)$. For each N and δ, with $1 \leq \delta \leq \frac{N}{2}$, we present a Yang-Baxter map $R^{(\delta)}(a, b)$ with $2N - 2$ components (see Section 4). For $\delta = 1$, this

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includes the map H^B_{III} of [6], when $N = 2$, and the Yang-Baxter map associated with the discrete modified Boussinesq equation, for $N = 3$. The general map for $\delta = 1$ is known [5], but for $\delta \geq 2$ this is a new class of Yang-Baxter maps. In Section 5 we present a new multi-component generalisation of the Yang-Baxter maps F_{IV} and F_V (given in the classification of [2])

2 Basic Definitions

Let X be an algebraic variety. A *parametric Yang-Baxter map* $R(a, b)$, depending upon parameters (a, b), is a map

$$R(a, b) : \mathrm{X} \times \mathrm{X} \to \mathrm{X} \times \mathrm{X},$$

satisfying:

$$R_{23}(a_2, a_3) \circ R_{13}(a_1, a_3) \circ R_{12}(a_1, a_2) = R_{12}(a_1, a_2) \circ R_{13}(a_1, a_3) \circ R_{23}(a_2, a_3), \tag{2.1}$$

where $R_{ij}(a_i, a_j)$ is the map that acts as $R(a, b)$ on the i and j factor of $\mathrm{X}\times\mathrm{X}\times\mathrm{X}$, and identically on the other.

Definition 2.1 (Reversibility) *Let P be the involution given by $P(\mathbf{x}, \mathbf{y}; a, b) = (\mathbf{y}, \mathbf{x}; b, a)$. If $P \circ R(a, b)$ is also an involution, then the map $R(a, b)$ is said to be* reversible.

Remark 2.2 *An alternative way of writing this is that the map $P \circ R(a, b) \circ P$ is the inverse of $R(a, b)$.*

Lax pairs were defined for Yang-Baxter maps in [10, 9]. A matrix $L(\mathbf{x}, a)$, with $\mathbf{x} \in \mathrm{X}$, depending upon the YB parameter a and the spectral parameter λ is used to define the equation:

$$L(\mathbf{x}', a)L(\mathbf{y}', b) = L(\mathbf{y}, b)L(\mathbf{x}, a). \tag{2.2}$$

It was shown in [10] that if L satisfies this, then the map $(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}', \mathbf{y}')$ satisfies the parametric Yang-Baxter equation (2.1) and is *reversible*.

Definition 2.3 (The Companion Map) *The companion map $(\mathbf{x}, \mathbf{y}') \mapsto (\mathbf{x}', \mathbf{y})$ is obtained by solving equation (2.2) for the variables $(\mathbf{x}', \mathbf{y})$.*

2.1 Travelling Wave Reductions of a Lattice Equation

Suppose we have a square lattice with vertices labelled (m, n). At each vertex we have functions

$$\mathbf{u}_{m,n} = \left(u^{(0)}_{m,n}, \dots, u^{(N-1)}_{m,n}\right), \quad \mathbf{v}_{m,n} = \left(v^{(0)}_{m,n}, \dots, v^{(N-1)}_{m,n}\right),$$

and vector function $\Psi_{m,n}$, satisfying

$$\Psi_{m+1,n} = L(\mathbf{u}_{m,n}, a)\, \Psi_{m,n}, \quad \Psi_{m,n+1} = L(\mathbf{v}_{m,n}, b)\, \Psi_{m,n}, \tag{2.3}$$

with compatibility conditions

$$L(\mathbf{u}_{m,n+1}, a)L(\mathbf{v}_{m,n}, b) = L(\mathbf{v}_{m+1,n}, b)L(\mathbf{u}_{m,n}, a). \tag{2.4}$$

If we now consider the reduction

$$\mathbf{u}_{m,n} = \mathbf{x}_p, \quad \mathbf{v}_{m,n} = \mathbf{y}_{p+1}, \quad \text{where} \quad p = n - m, \tag{2.5}$$

then (2.4) reduces to (2.2), with $\mathbf{x} = \mathbf{x}_p$, $\mathbf{x}' = \mathbf{x}_{p+1}$, $\mathbf{y} = \mathbf{y}_p$, $\mathbf{y}' = \mathbf{y}_{p+1}$, with the map $(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}', \mathbf{y}')$ being Yang-Baxter.

Remark 2.4 *Notice that this does not rely on any underlying Lie point symmetry of the lattice equation. It is just a "travelling wave" solution of the lattice equation.*

3 $\mathbb{Z}_N$-Graded Lax Pairs

We now consider the specific discrete Lax pairs, which we introduced in [4]. Consider a pair of matrix equations of the form

$$\Psi_{m+1,n} = L_{m,n}\,\Psi_{m,n} \equiv \left(U_{m,n} + \lambda\,\Omega^{\ell_1}\right)\,\Psi_{m,n}, \tag{3.1a}$$

$$\Psi_{m,n+1} = M_{m,n}\,\Psi_{m,n} \equiv \left(V_{m,n} + \lambda\,\Omega^{\ell_2}\right)\,\Psi_{m,n}, \tag{3.1b}$$

where

$$U_{m,n} = \mathrm{diag}\left(u^{(0)}_{m,n},\dots,u^{(N-1)}_{m,n}\right)\Omega^{k_1}, \quad V_{m,n} = \mathrm{diag}\left(v^{(0)}_{m,n},\dots,v^{(N-1)}_{m,n}\right)\Omega^{k_2}, \tag{3.1c}$$

and

$$(\Omega)_{i,j} = \delta_{j-i,1} + \delta_{i-j,N-1}.$$

The matrix Ω defines a grading and the four matrices of (3.1) are said to be of respective levels k_i, ℓ_i, with $\ell_i \neq k_i$ (for each i). The Lax pair is characterised by the quadruple $(k_1,\ell_1;k_2,\ell_2)$, which we refer to as *the level structure* of system, and for consistency, we require

$$k_1 + \ell_2 \equiv k_2 + \ell_1 \ (\mathrm{mod} N). \tag{3.2}$$

Since matrices U, V and Ω are independent of λ, the compatibility condition of (3.1),

$$L_{m,n+1}M_{m,n} = M_{m+1,n}L_{m,n}, \tag{3.3}$$

splits into the system

$$U_{m,n+1}V_{m,n} = V_{m+1,n}U_{m,n}\,, \tag{3.4a}$$

$$U_{m,n+1}\Omega^{\ell_2} - \Omega^{\ell_2}U_{m,n} = V_{m+1,n}\Omega^{\ell_1} - \Omega^{\ell_1}V_{m,n}, \tag{3.4b}$$

which can be written explicitly as

$$u^{(i)}_{m,n+1}v^{(i+k_1)}_{m,n} = v^{(i)}_{m+1,n}u^{(i+k_2)}_{m,n}\,, \tag{3.5a}$$

$$u^{(i)}_{m,n+1} - u^{(i+\ell_2)}_{m,n} = v^{(i)}_{m+1,n} - v^{(i+\ell_1)}_{m,n}\,, \tag{3.5b}$$

or, in a solved form, as

$$u^{(i)}_{m,n+1} = \frac{u^{(i+\ell_2)}_{m,n} - v^{(i+\ell_1)}_{m,n}}{u^{(i+k_2)}_{m,n} - v^{(i+k_1)}_{m,n}}\,u^{(i+k_2)}_{m,n}\,, \quad v^{(i)}_{m+1,n} = \frac{u^{(i+\ell_2)}_{m,n} - v^{(i+\ell_1)}_{m,n}}{u^{(i+k_2)}_{m,n} - v^{(i+k_1)}_{m,n}}\,v^{(i+k_1)}_{m,n}, \tag{3.6}$$

assuming that $u^{(i)}_{m,n} \neq v^{(j)}_{m,n}$ for all i,j. In all the above formulae, i, j are taken $(\mathrm{mod} N)$.

It is easily seen that the quantities

$$a = \prod_{i=0}^{N-1} u^{(i)}_{m,n}, \quad b = \prod_{i=0}^{N-1} v^{(i)}_{m,n} \quad \text{satisfy} \quad \Delta_n(a) = \Delta_m(b) = 0, \tag{3.7}$$

where

$$\Delta_m = \mathcal{S}_m - 1, \quad \Delta_n = \mathcal{S}_n - 1, \quad \text{with} \quad \mathcal{S}_m f_{m,n} = f_{m+1,n}, \quad \mathcal{S}_n f_{m,n} = f_{m,n+1}.$$

3.1 Reduction to Yang-Baxter Maps

We can now employ the reduction (2.5), using (3.7) to replace the components $x_p^{(N-1)}$, $y_p^{(N-1)}$. This introduces parameters a, b into the Lax matrices. If we define

$$X_p = \operatorname{diag}\left(x_p^{(0)}, \dots, x_p^{(N-1)}\right), \quad Y_p = \operatorname{diag}\left(y_p^{(0)}, \dots, y_p^{(N-1)}\right), \tag{3.8}$$

then the compatibility condition (3.3) takes the form

$$(X_{p+1}\Omega^{k_1} + \lambda\Omega^{\ell_1})(Y_{p+1}\Omega^{k_2} + \lambda\Omega^{\ell_2}) = (Y_p\Omega^{k_2} + \lambda\Omega^{\ell_2})(X_p\Omega^{k_1} + \lambda\Omega^{\ell_1}), \tag{3.9}$$

and equations (3.5) take the form

$$x_{p+1}^{(i)} y_{p+1}^{(i+k_1)} = y_p^{(i)} x_p^{(i+k_2)}, \quad x_{p+1}^{(i)} + y_{p+1}^{(i+\ell_1)} = y_p^{(i)} + x_p^{(i+\ell_2)}. \tag{3.10}$$

We can write (3.9) as

$$(X_{p+1} + \lambda\Omega^{\delta})(\Omega^{k_1} Y_{p+1}\Omega^{-k_1} + \lambda\Omega^{\delta}) = (Y_p + \lambda\Omega^{\delta})(\Omega^{k_2} X_p \Omega^{-k_2} + \lambda\Omega^{\delta}), \tag{3.11}$$

where $0 < \delta \le N-1$, with $\delta \equiv \ell_i - k_i \ (\mathrm{mod} N)$. This allows us to reduce the general case with level structure $(k_1, \ell_1; k_2, \ell_2)$ to that with level structure $(0, \delta; 0, \delta)$. First, note that formula (3.11) can be written

$$(\bar{X}_{p+1} + \lambda\Omega^{\delta})(\bar{Y}_{p+1} + \lambda\Omega^{\delta}) = (\bar{Y}_p + \lambda\Omega^{\delta})(\bar{X}_p + \lambda\Omega^{\delta}), \tag{3.12}$$

where

$$\bar{X}_p = \operatorname{diag}\left(\bar{x}_p^{(0)}, \dots, \bar{x}_p^{(N-1)}\right), \quad \bar{Y}_p = \operatorname{diag}\left(\bar{y}_p^{(0)}, \dots, \bar{y}_p^{(N-1)}\right).$$

Comparing (3.12) and (3.11), we see that

$$\bar{x}_{p+1}^{(i)} = x_{p+1}^{(i)}, \quad \bar{y}_{p+1}^{(i)} = y_{p+1}^{(i+k_1)}, \quad \bar{x}_p^{(i)} = x_p^{(i+k_2)}, \quad \bar{y}_p^{(i)} = y_p^{(i)},$$

all taken $(\mathrm{mod} N)$. We see from (3.12) that the components $(\bar{x}_p^{(i)}, \bar{y}_p^{(i)})$ satisfy

$$\bar{x}_{p+1}^{(i)} \bar{y}_{p+1}^{(i)} = \bar{y}_p^{(i)} \bar{x}_p^{(i)}, \quad \bar{x}_{p+1}^{(i)} + \bar{y}_{p+1}^{(i+\delta)} = \bar{y}_p^{(i)} + \bar{x}_p^{(i+\delta)},$$

which are just (3.10) with $(k_i, \ell_i) = (0, \delta)$. We summarise these results in:

Proposition 3.1 *In the Yang-Baxter reduction, all systems with level structure* $(k_1, \ell_1; k_2, \ell_2)$*, for which* $\ell_i - k_i \equiv \delta \ (\mathrm{mod} N)$*, are equivalent (up to point transformation) to the system with level structure* $(0, \delta; 0, \delta)$*.*

4 The Yang-Baxter Map Corresponding to the Case $(0, \delta; 0, \delta)$

In this section we consider the Lax equations with level structure $(0, \delta; 0, \delta)$, with $0 < \delta \le N-1$. The resulting equations are *quadrirational*, with both the Yang-Baxter and companion maps being *birational*. We find that the Yang-Baxter maps corresponding to δ and $N - \delta$ are inverses to each other and that the companion map is periodic, with period N.

4.1 The Equations and Maps

With Lax matrices

$$L(\mathbf{x}, a) = X_p + \lambda\Omega^\delta, \quad L(\mathbf{y}, b) = Y_p + \lambda\Omega^\delta, \tag{4.1}$$

where X_p and Y_p are defined by (3.8), with

$$x_p^{(N-1)} = \frac{a}{\prod_{i=0}^{N-2} x_p^{(i)}}, \quad y_p^{(N-1)} = \frac{b}{\prod_{i=0}^{N-2} y_p^{(i)}}, \tag{4.2}$$

the Lax equation (2.2) implies

$$x_{p+1}^{(i)} y_{p+1}^{(i)} = y_p^{(i)} x_p^{(i)}, \quad x_{p+1}^{(i)} + y_{p+1}^{(i+\delta)} = y_p^{(i)} + x_p^{(i+\delta)}, \quad 0 \le i \le N-1. \tag{4.3}$$

Only the formulae with $0 \le i \le N-2$ are independent, but the full set is useful when discussing first integrals.

Remark 4.1 (Level structure $(\delta, 0; \delta, 0)$ vs $(0, \delta; 0, \delta)$) *Under the point transformation*

$$x_{p+1}^{(i)} = \tilde{x}_p^{(i+\delta)}, \quad x_p^{(i)} = \tilde{x}_{p+1}^{(i)}, \quad y_{p+1}^{(i)} = \tilde{y}_p^{(i)}, \quad y_p^{(i)} = \tilde{y}_{p+1}^{(i+\delta)},$$

equations (4.3) take the form

$$\tilde{x}_{p+1}^{(i)} \tilde{y}_{p+1}^{(i+\delta)} = \tilde{y}_p^{(i)} \tilde{x}_p^{(i+\delta)}, \quad \tilde{x}_{p+1}^{(i)} + \tilde{y}_{p+1}^{(i)} = \tilde{y}_p^{(i)} + \tilde{x}_p^{(i)}, \quad 0 \le i \le N-1,$$

which are just the equations for level structure $(\delta, 0; \delta, 0)$, so these structures are equivalent.

4.1.1 The Yang-Baxter map $R^{(\delta)}(a, b)$

Here we solve (4.3) for $(x_{p+1}^{(i)}, y_{p+1}^{(i)})$ as functions of $(x_p^{(i)}, y_p^{(i)})$ (with $0 \le i \le N-2$ and $x_p^{(N-1)}$, $y_p^{(N-1)}$ replaced by (4.2)). We write this map as $R^{(\delta)}(a, b)$, but when no ambiguity can arise, we suppress the parametric dependence by writing the map as $R^{(\delta)}$.

Notice that by shifting $i \mapsto i + N - \delta \equiv i - \delta \ (\mathrm{mod} N)$, the second part of equation (4.3) takes the form

$$x_{p+1}^{(i-\delta)} + y_{p+1}^{(i)} = y_p^{(i-\delta)} + x_p^{(i)},$$

which leads to:

Proposition 4.2 (Inverse Map) *The Yang-Baxter map $R^{(-\delta)}(a, b)$ is just the <u>inverse</u> of the map $R^{(\delta)}(a, b)$.*

This means that we only need to consider $\delta \le \frac{N}{2}$ and that, when $N = 2M$, the map $R^{(M)}(a, b)$ is an involution.

Proposition 4.3 (First Integrals) *The Yang-Baxter map $R^{(\delta)}(a, b)$ has the following N first integrals:*

$$x_p^{(i)} y_p^{(i)} = c_i, \quad 0 \le i \le N-2, \quad \sum_{i=0}^{N-1} (x_p^{(i+\delta)} + y_p^{(i)}) = c_{N-1}, \tag{4.4}$$

where, in the latter, $x_p^{(N-1)}$ and $y_p^{(N-1)}$ are replaced by (4.2).

The last of these integrals is obtained by summing the additive equations of (4.3).

4.1.2 The Companion Map $\varphi^{(\delta)}$

Here we solve (4.3) for $(x_{p+1}^{(i)}, y_p^{(i)})$ as functions of $(x_p^{(i)}, y_{p+1}^{(i)})$ (with $0 \le i \le N-2$ and $x_p^{(N-1)}$, $y_p^{(N-1)}$ replaced by (4.2)). Since p is no longer the evolution parameter, we relabel our variables as:

$$(x_p^{(i)}, y_{p+1}^{(i)}) = (x_q^{(i)}, y_q^{(i)}), \quad (x_{p+1}^{(i)}, y_p^{(i)}) = (x_{q+1}^{(i)}, y_{q+1}^{(i)}).$$

Remark 4.4 (A second travelling wave reduction) *This labelling follows directly from the travelling wave reduction*

$$\mathbf{u}_{m,n} = \mathbf{x}_q, \quad \mathbf{v}_{m,n} = \mathbf{y}_q, \quad \textit{where} \quad q = n + m$$

We can re-arrange the quadratic formulae in (4.3) (with this new labelling) to obtain $N-1$ first integrals:

$$\frac{x_q^{(i)}}{y_q^{(i)}} = c_i, \quad 0 \le i \le N-2. \tag{4.5}$$

We can also re-arrange the linear formulae of (4.3) to obtain

$$x_{q+1}^{(i)} - y_{q+1}^{(i)} = x_q^{(i+\delta)} - y_q^{(i+\delta)}, \quad 0 \le i \le N-1.$$

If we define

$$f(x, y) = x - y, \tag{4.6}$$

then

$$f(x_{q+1}^{(i)}, y_{q+1}^{(i)}) = f(x_q^{(i+\delta)}, y_q^{(i+\delta)}), \quad 0 \le i \le N-1. \tag{4.7}$$

We may use

$$\left(\frac{x_q^{(0)}}{y_q^{(0)}}, \dots, \frac{x_q^{(N-2)}}{y_q^{(N-2)}}, f\left(x_q^{(0)}, y_q^{(0)}\right), \dots, f\left(x_q^{(N-2)}, y_q^{(N-2)}\right) \right)$$

as coordinates and, in these coordinates, the map $\varphi^{(\delta)}$ just shifts the coordinates $f(x_q^{(i)}, y_q^{(i)})$ by δ, whilst leaving the coordinates $\frac{x_q^{(i)}}{y_q^{(i)}}$ fixed. This leads to the following:

Proposition 4.5 (Periodicity) *The map $\varphi^{(\delta)}$ is periodic with period N. When $(N, \delta) = 1$ this is the minimum period. Furthermore, we have that $\varphi^{(\delta)} = \varphi^{(1)} \circ \cdots \circ \varphi^{(1)}$ (the $\delta-$fold composition of $\varphi^{(1)}$).*

This statement is, of course, independent of coordinates.

Remark 4.6 ($(2N-2)$ first integrals) *Any cyclically symmetric function of $f(x_q^{(i)}, y_q^{(i)})$ is a first integral of the companion map, so it possesses $(2N-2)$ first integrals. The common level set is then finite, corresponding to the periodicity of the map.*

4.2 Examples of the map $R^{(\delta)}$

We can build hierarchies of Yang-Baxter maps for each δ. It follows from Proposition 4.2 that we only need to consider $\delta \le \frac{N}{2}$. However, as the value of N increases, so does the number of different maps $R^{(\delta)}$. We have:

Case $\delta = 1$: At $N = 2$, we only have the case $\delta = 1$, and $R^{(1)}$ is just the map H^B_{III} in the classification of scalar Yang-Baxter maps [6]. The map $R^{(1)}$ exists for all $N \geq 2$, which can therefore be considered as a multi-component generalisation of the scalar Yang-Baxter map H^B_{III}.

Case $\delta = 2$: For $N \geq 4$ we have the map $R^{(2)}$. When N is even, this map degenerates to lower dimensional maps (see the case $N = 4$ below), but when N is odd, we have a new sequence of Yang-Baxter maps which fully couple $2N - 2$ variables. The $8-$component case can be seen in the case $N = 5$ below.

Case $\delta = 3$: For $N \geq 6$ we have the map $R^{(3)}$, but again, this map degenerates to lower dimensional maps when N is a multiple of 3. The first fully coupled system is at $N = 7$.

Whilst the generalisation of $\delta = 1$ is already known [5], the maps $R^{(\delta)}$, for $\delta \geq 2$, are new classes of Yang-Baxter maps.

4.2.1 When $N = 2$

Here we only have the case $\delta = 1$, which leads to (with $x^{(0)} = x$, $y^{(0)} = y$, $x^{(1)} = a/x$, $y^{(1)} = b/y$)

$$x_{p+1} = y_p \left(\frac{a + xy}{b + xy}\right), \quad y_{p+1} = x_p \left(\frac{b + xy}{a + xy}\right), \tag{4.8}$$

which (up to a relabelling of parameters) is just the map H^B_{III} in the classification of scalar Yang-Baxter maps [6].

The existence of the two invariant functions (4.4) implies (the well known fact) that this map is an involution.

4.2.2 When $N = 3$

Here we have $\delta = 1$ and $\delta = 2$, but since $N - 1 = 2 \equiv -1 \,(\mathrm{mod} 3)$, the map $R^{(2)}$ is just the inverse of $R^{(1)}$. In this case $R^{(1)}$ takes the form:

$$x^{(i)}_{p+1} = y^{(i)}_p \frac{A^{(i)}}{A^{(i+1)}}, \qquad y^{(i)}_{p+1} = x^{(i)}_p \frac{A^{(i+1)}}{A^{(i)}}, \quad 0 \leq i \leq 1, \tag{4.9}$$

with upper indices taken (mod2) and where

$$A^{(0)} = a(x^{(1)}_p + y^{(0)}_p) + x^{(0)}_p x^{(1)}_p y^{(0)}_p y^{(1)}_p, \quad A^{(1)} = A^{(0)} + (b - a)x^{(1)}_p, \quad A^{(2)} = A^{(1)} + (b - a)y^{(0)}_p.$$

Remark 4.7 (Discrete Modified Boussinesq Equation) *This is equivalent to the Yang-Baxter map derived in [8], associated with the discrete modified Boussinesq equation (see equation (67a-b) of [8]). They are related by a simple point transformation:*

$$x^{(0)} \mapsto \frac{c_0}{x^1}, \quad x^{(1)} \mapsto c_0 x^2, \quad y^{(0)} \mapsto \frac{c_0\alpha_1}{\alpha_2 y^1}, \quad y^{(1)} \mapsto \frac{\alpha_1^2 y^2}{c_0^3}, \quad \textit{where} \quad c_0^4 = \frac{\alpha_1^3}{\alpha_2}.$$

4.2.3 When $N = 4$

For $\delta = 1$: We obtain the $6-$component version of (4.9).

For $\delta = 2$: Since $(N, \delta) = 2 \neq 1$, the map is reducible, with a 4−component subsystem:

$$
\begin{aligned}
x^{(0)}_{p+1} &= \frac{x^{(0)}_p(x^{(2)}_p + y^{(0)}_p)}{x^{(0)}_p + y^{(2)}_p}, \quad x^{(2)}_{p+1} = \frac{x^{(2)}_p(x^{(0)}_p + y^{(2)}_p)}{x^{(2)}_p + y^{(0)}_p}, \\
y^{(0)}_{p+1} &= \frac{y^{(0)}_p(x^{(0)}_p + y^{(2)}_p)}{x^{(2)}_p + y^{(0)}_p}, \quad y^{(2)}_{p+1} = \frac{y^{(2)}_p(x^{(2)}_p + y^{(0)}_p)}{x^{(0)}_p + y^{(2)}_p},
\end{aligned} \tag{4.10}
$$

in which the parameters (a, b) are absent.

The remaining pair of equations are a non-autonomous version of (4.8), with coefficients depending upon $(x^{(0)}_p, x^{(2)}_p, y^{(0)}_p, y^{(2)}_p)$:

$$
x^{(1)}_{p+1} = \frac{y^{(0)}_p y^{(2)}_p y^{(1)}_p \left(a + x^{(0)}_p x^{(2)}_p x^{(1)}_p y^{(1)}_p\right)}{x^{(0)}_p x^{(2)}_p \left(b + y^{(0)}_p y^{(2)}_p x^{(1)}_p y^{(1)}_p\right)}, \quad y^{(1)}_{p+1} = \frac{x^{(0)}_p x^{(2)}_p x^{(1)}_p \left(b + y^{(0)}_p y^{(2)}_p x^{(1)}_p y^{(1)}_p\right)}{x^{(0)}_p x^{(2)}_p \left(a + x^{(0)}_p x^{(2)}_p x^{(1)}_p y^{(1)}_p\right)}. \tag{4.11}
$$

Notice that this last pair could also be written

$$
x^{(1)}_{p+1} = \frac{x^{(1)}_p(x^{(3)}_p + y^{(1)}_p)}{x^{(1)}_p + y^{(3)}_p}, \quad y^{(1)}_{p+1} = \frac{y^{(1)}_p(x^{(1)}_p + y^{(3)}_p)}{x^{(3)}_p + y^{(1)}_p},
$$

which, with the constraint (4.2), explains the formulae in (4.11).

The 4−component system (4.10) has 4 independent first integrals

$$
I_1 = x^{(0)}_p y^{(0)}_p, \quad I_2 = x^{(2)}_p y^{(2)}_p, \quad I_3 = x^{(0)}_p x^{(2)}_p, \quad I_4 = x^{(0)}_p + x^{(2)}_p + y^{(0)}_p + y^{(2)}_p,
$$

so is periodic (and has period 2).

The remaining two equations (4.11) cannot be taken alone, but only as part of the 6−component system. This system has two more first integrals,

$$
I_5 = x^{(1)}_p y^{(1)}_p, \quad I_6 = x^{(1)}_p + \frac{a}{x^{(0)}_p x^{(1)}_p x^{(2)}_p} + y^{(1)}_p + \frac{b}{y^{(0)}_p y^{(1)}_p y^{(2)}_p},
$$

so is also periodic (of period 2). As commented after Proposition 4.2, this involutive property follows from $\delta = N - \delta$ for this case.

Remark 4.8 (Non-Coprime Case) *This decoupling, when $(N, \delta) \neq 1$, is a general feature.*

4.2.4 When $N = 5$

Here $\delta = 1$ and $\delta = 2$ give genuinely different maps.

For $\delta = 1$: The map $R^{(1)}$ takes the same form as (4.9):

$$
x^{(i)}_{p+1} = y^{(i)}_p \frac{A^{(i)}}{A^{(i+1)}}, \qquad y^{(i)}_{p+1} = x^{(i)}_p \frac{A^{(i+1)}}{A^{(i)}}, \quad 0 \leq i \leq 3, \tag{4.12}
$$

with upper indices taken (mod4) and where

$$
\begin{aligned}
A^{(0)} &= a(x^{(1)}_p x^{(2)}_p x^{(3)}_p + x^{(2)}_p x^{(3)}_p y^{(0)}_p + x^{(3)}_p y^{(0)}_p y^{(1)}_p + y^{(0)}_p y^{(1)}_p y^{(2)}_p) + \prod_{i=0}^{3} x^{(i)}_p y^{(i)}_p, \\
A^{(1)} &= A^{(0)} + (b - a) x^{(1)}_p x^{(2)}_p x^{(3)}_p, \quad A^{(2)} = A^{(1)} + (b - a) x^{(2)}_p x^{(3)}_p y^{(0)}_p, \\
A^{(3)} &= A^{(2)} + (b - a) x^{(3)}_p y^{(0)}_p y^{(1)}_p, \quad A^{(4)} = A^{(3)} + (b - a) y^{(0)}_p y^{(1)}_p y^{(2)}_p.
\end{aligned}
$$

For $\delta = 2$: The map $R^{(2)}$ takes the form:

$$x_{p+1}^{(0)} = y_p^{(0)} \frac{A^{(2)}}{A^{(3)}}, \quad x_{p+1}^{(1)} = y_p^{(1)} \frac{A^{(0)}}{A^{(1)}}, \quad x_{p+1}^{(2)} = y_p^{(2)} \frac{A^{(3)}}{A^{(4)}}, \quad x_{p+1}^{(3)} = y_p^{(3)} \frac{A^{(1)}}{A^{(2)}}, \tag{4.13}$$

and $y_{p+1}^{(i)} = \frac{x_p^{(i)} y_p^{(i)}}{x_{p+1}^{(i)}}$, with upper indices taken (mod4) and where

$$\begin{aligned} A^{(0)} &= a(x_p^{(3)} x_p^{(0)} x_p^{(2)} + x_p^{(0)} x_p^{(2)} y_p^{(1)} + x_p^{(2)} y_p^{(1)} y_p^{(3)} + y_p^{(1)} y_p^{(3)} y_p^{(0)}) + \prod_{i=0}^{3} x_p^{(i)} y_p^{(i)}, \\ A^{(1)} &= A^{(0)} + (b-a) x_p^{(3)} x_p^{(0)} x_p^{(2)}, \quad A^{(2)} = A^{(1)} + (b-a) x_p^{(0)} x_p^{(2)} y_p^{(1)}, \\ A^{(3)} &= A^{(2)} + (b-a) x_p^{(2)} y_p^{(1)} y_p^{(3)}, \quad A^{(4)} = A^{(3)} + (b-a) y_p^{(1)} y_p^{(3)} y_p^{(0)}. \end{aligned}$$

4.2.5 The Structure of the Formulae

The order of appearance of $A^{(i)}$ in (4.13) and the combination of variables appearing in the definition of $A^{(0)}$ is controlled by the following ordering of the variables $x_p^{(i)}, y_p^{(i)}$:

$$\{x^{(\delta-1)}, x^{(2\delta-1)}, \dots, x^{((N-1)\delta-1)}, y^{(\delta-1)}, y^{(2\delta-1)}, \dots, y^{((N-1)\delta-1)}\}.$$

When $(N, \delta) = 1$, the numbers $\{(m\delta - 1)\}_{m=1}^{N-1}$ form a permutation of the numbers $0, \dots, N-2$, so all the variables are included in this list. The formulae (4.13) are just

$$x_{p+1}^{(m\delta-1)} = y_p^{(m\delta-1)} \frac{A^{(m-1)}}{A^{(m)}}, \quad 1 \le m \le N-1. \tag{4.14}$$

The coefficient of the parameter a in function $A^{(0)}$ is constructed as follows: the first term is $\frac{\prod_{i=0}^{N-2} x^{(i)}}{x^{(\delta-1)}}$. We then repeatedly act by the permutation

$$x^{(\delta-1)} \to x^{(2\delta-1)} \to \cdots \to x^{((N-1)\delta-1)} \to y^{(\delta-1)} \to y^{(2\delta-1)} \to \cdots \to y^{((N-1)\delta-1)} \to x^{(\delta-1)},$$

for $(N-2)$ times, which ends with $\frac{\prod_{i=0}^{N-2} y^{(i)}}{y^{(N-1-\delta)}}$. The coefficient of a is then just the sum of these $(N-1)$ terms. The remaining term in $A^{(0)}$ is just $\prod_{i=0}^{N-2} x^{(i)} y^{(i)}$.

The functions $A^{(i)}$ are formed by successively changing the coefficient a to b at each of the terms in the above sum.

Example 4.9 (The case $N = 5$, $\delta = 2$) Here we have

$$x^{(1)} \to x^{(3)} \to x^{(0)} \to x^{(2)} \to y^{(1)} \to y^{(3)} \to y^{(0)} \to y^{(2)},$$

and

$$x_p^{(3)} x_p^{(0)} x_p^{(2)} \to x_p^{(0)} x_p^{(2)} y_p^{(1)} \to x_p^{(2)} y_p^{(1)} y_p^{(3)} \to y_p^{(1)} y_p^{(3)} y_p^{(0)},$$

giving the expression for $A^{(0)}$, given in the case of (4.13).

Example 4.10 (The case $N = 7$, $\delta = 3$) Here we have

$$x^{(2)} \to x^{(5)} \to x^{(1)} \to x^{(4)} \to x^{(0)} \to x^{(3)} \to y^{(2)} \to y^{(5)} \to y^{(1)} \to y^{(4)} \to y^{(0)} \to y^{(3)},$$

and

$$x_p^{(5)} x_p^{(1)} x_p^{(4)} x_p^{(0)} x_p^{(3)} \to x_p^{(1)} x_p^{(4)} x_p^{(0)} x_p^{(3)} y_p^{(2)} \to \cdots \to y_p^{(2)} y_p^{(5)} y_p^{(1)} y_p^{(4)} y_p^{(0)},$$

giving

$$A^{(0)} = a(x_p^{(5)}x_p^{(1)}x_p^{(4)}x_p^{(0)}x_p^{(3)} + x_p^{(1)}x_p^{(4)}x_p^{(0)}x_p^{(3)}y_p^{(2)} + \cdots + y_p^{(2)}y_p^{(5)}y_p^{(1)}y_p^{(4)}y_p^{(0)}) + \prod_{i=0}^{5} x_p^{(i)}y_p^{(i)}.$$

The remaining $A^{(i)}$ are then constructed by the above prescription and the map $R^{(3)}$, for $N = 7$ is given by (4.14), for $\delta = 3$.

4.3 The Quotient Potential Case and Symmetries

In [4] we introduced two potential forms of our equations (3.5). Here we briefly mention the "quotient potential", leaving the "additive potential" to Section 5.

Equations (3.5a) hold identically if we set

$$u_{m,n}^{(i)} = \alpha \frac{\phi_{m+1,n}^{(i)}}{\phi_{m,n}^{(i+k_1)}}, \quad v_{m,n}^{(i)} = \beta \frac{\phi_{m,n+1}^{(i)}}{\phi_{m,n}^{(i+k_2)}}, \tag{4.15}$$

where $a = \alpha^N$, $b = \beta^N$. Equations (3.5b) then take the form

$$\alpha \left(\frac{\phi_{m+1,n+1}^{(i)}}{\phi_{m,n+1}^{(i+k_1)}} - \frac{\phi_{m+1,n}^{(i+\ell_2)}}{\phi_{m,n}^{(i+\ell_2+k_1)}} \right) = \beta \left(\frac{\phi_{m+1,n+1}^{(i)}}{\phi_{m+1,n}^{(i+k_2)}} - \frac{\phi_{m,n+1}^{(i+\ell_1)}}{\phi_{m,n}^{(i+\ell_1+k_2)}} \right), \tag{4.16}$$

where indices are taken $(\bmod N)$.

These equations have a weighted scaling symmetry, whose invariants are given exactly by the formulae (4.15), leading us back to equations (3.5) and therefore to our previous Yang-Baxter maps.

5 The Additive Potential

Equations (3.5b) hold identically if we set

$$u_{m,n}^{(i)} = \chi_{m+1,n}^{(i)} - \chi_{m,n}^{(i+\ell_1)}, \quad v_{m,n}^{(i)} = \chi_{m,n+1}^{(i)} - \chi_{m,n}^{(i+\ell_2)}. \tag{5.1}$$

Equations (3.5a) then take the form

$$\frac{\left(\chi_{m+1,n+1}^{(i)} - \chi_{m,n+1}^{(i+\ell_1)}\right)}{\left(\chi_{m+1,n+1}^{(i)} - \chi_{m+1,n}^{(i+\ell_2)}\right)} = \frac{\left(\chi_{m+1,n}^{(i+k_2)} - \chi_{m,n}^{(i+k_2+\ell_1)}\right)}{\left(\chi_{m,n+1}^{(i+k_1)} - \chi_{m,n}^{(i+k_1+\ell_2)}\right)}, \tag{5.2}$$

and the first integrals (3.7) take the form

$$\prod_{i=0}^{N-1} \left(\chi_{m+1,n}^{(i)} - \chi_{m,n}^{(i+\ell_1)}\right) = a, \quad \prod_{i=0}^{N-1} \left(\chi_{m,n+1}^{(i)} - \chi_{m,n}^{(i+\ell_2)}\right) = b. \tag{5.3}$$

Remark 5.1 (Reduction) *It is not always possible to use these first integrals to explicitly reduce (5.2) to a system with $N-1$ components (eliminating $\chi_{m,n}^{(N-1)}$), and even when this is possible the spectral problem (3.1) cannot be written in terms of the reduced variables.*

In [4] we showed that it is possible to explicitly reduce the system with $(k_i, \ell_i) = (0, 1)$, which takes the form

$$\frac{\left(\chi^{(i)}_{m+1,n+1} - \chi^{(i+1)}_{m,n+1}\right)}{\left(\chi^{(i)}_{m+1,n+1} - \chi^{(i+1)}_{m+1,n}\right)} = \frac{\left(\chi^{(i)}_{m+1,n} - \chi^{(i+1)}_{m,n}\right)}{\left(\chi^{(i)}_{m,n+1} - \chi^{(i+1)}_{m,n}\right)}, \quad i = 0, \dots, N-3, \tag{5.4a}$$

$$\chi^{(N-2)}_{m+1,n+1} = \chi^{(0)}_{m,n} + \frac{1}{\chi^{(N-2)}_{m+1,n} - \chi^{(N-2)}_{m,n+1}} \left(\frac{a}{X} - \frac{b}{Y}\right), \tag{5.4b}$$

where $X = \prod_{j=0}^{N-3}(\chi^{(j)}_{m+1,n} - \chi^{(j+1)}_{m,n})$ and $Y = \prod_{j=0}^{N-3}(\chi^{(j)}_{m,n+1} - \chi^{(j+1)}_{m,n})$.

Remark 5.2 *This is a direct generalisation of equation $H1$ in the ABS classification [1].*

It is easy to see that the system (5.4) has the following pair of symmetry generators:

$$\mathbf{X}_t = \sum_{i=0}^{N-2} \omega^{m+n+i} \partial_{\chi^{(i)}_{m,n}}, \tag{5.5a}$$

$$\mathbf{X}_s = \sum_{i=0}^{N-2} \omega^{m+n+i} \chi^{(i)}_{m,n} \partial_{\chi^{(i)}_{m,n}}, \quad \omega \neq 1, \tag{5.5b}$$

where $\omega^N = 1$. It is therefore possible to write equations (5.4) in terms of the invariants of these symmetries. We can then reduce this form of the lattice equations to Yang-Baxter maps.

5.1 The Invariants of $\mathbf{X}_t$

It is straightforward to write a suitable "basis" for the invariants of $\mathbf{X}_t$. The formulae are more symmetric if we write "too many" invariants, which then satisfy some additional identities. We therefore define $4(N-1)$ invariants, satisfying $(N-1)$ identities. Furthermore, we make the reduction (2.5), so that we derive a map. Following [8], we denote these invariants by

$$x^{(i)} \equiv x^{(i)}_p, \quad y^{(i)} \equiv y^{(i)}_p, \quad u^{(i)} \equiv x^{(i)}_{p+1}, \quad v^{(i)} \equiv y^{(i)}_{p+1}, \quad \text{where } p = n - m, \tag{5.6}$$

corresponding to specific edges of the lattice square, as shown in Figure 1 and noting that the shifts $m \mapsto m-1$ and $n \mapsto n+1$ both correspond to $p \mapsto p+1$.

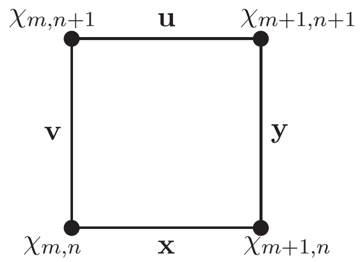

Figure 1: Invariants defined on edges

The $4(N-1)$ invariants:

$$x^{(i)} = \chi^{(i)}_{m+1,n} - \chi^{(i+1)}_{m,n}, \qquad i = 0,\dots,N-3, \quad x^{(N-2)} = \chi^{(N-2)}_{m+1,n} + \sum_{j=0}^{N-2} \chi^{(j)}_{m,n},$$

$$y^{(i)} = \chi^{(i)}_{m+1,n+1} - \chi^{(i+1)}_{m+1,n}, \quad i = 0,\dots,N-3, \quad y^{(N-2)} = \chi^{(N-2)}_{m+1,n+1} + \sum_{j=0}^{N-2} \chi^{(j)}_{m+1,n},$$

$$u^{(i)} = \chi^{(i)}_{m+1,n+1} - \chi^{(i+1)}_{m,n+1}, \quad i = 0,\dots,N-3, \quad u^{(N-2)} = \chi^{(N-2)}_{m+1,n+1} + \sum_{j=0}^{N-2} \chi^{(j)}_{m,n+1},$$

$$v^{(i)} = \chi^{(i)}_{m,n+1} - \chi^{(i+1)}_{m,n}, \qquad i = 0,\dots,N-3, \quad v^{(N-2)} = \chi^{(N-2)}_{m,n+1} + \sum_{j=0}^{N-2} \chi^{(j)}_{m,n},$$

satisfy $(N-1)$ identities:

$$x^{(i+1)} + y^{(i)} = u^{(i)} + v^{(i+1)}, \quad i = 0,\dots,N-3, \tag{5.7a}$$

$$y^{(N-2)} + \sum_{j=0}^{N-2} v^{(j)} = u^{(N-2)} + \sum_{j=0}^{N-2} x^{(j)}, \tag{5.7b}$$

and equations (5.4) take the form

$$u^{(i)} v^{(i)} = x^{(i)} y^{(i)}, \quad i = 0,\dots,N-3, \tag{5.7c}$$

$$u^{(N-2)} = \sum_{j=0}^{N-2} v^{(j)} + \frac{1}{x^{(N-2)} - v^{(N-2)}} \left(\frac{a}{\prod_{j=0}^{N-3} x^{(j)}} - \frac{b}{\prod_{j=0}^{N-3} v^{(j)}} \right). \tag{5.7d}$$

The Yang-Baxter map corresponds to the solution of equations (5.7) for $(u^{(i)}, v^{(i)})$. We do not have an explicit form of the solution in general, but for any given value of N, this can be found.

Remark 5.3 (The Case $N = 2$) *We already remarked that for $N = 2$ the lattice equation is just $H1$ in the ABS classification [1]. Using the symmetry $\mathbf{X}_t$, with $\omega = -1$ leads to the Yang-Baxter map*

$$u = y + \frac{a-b}{x-y}, \quad v = x + \frac{a-b}{x-y},$$

which is just F_V of the ABS classification of quadrirational maps [2] (the Adler map). Clearly, we may consider this whole family of maps as multi-component generalisations of F_V.

Example 5.4 (The Case $N = 3$) In this case, we find

$$\begin{aligned}
u^{(0)} &= y^{(0)} + \frac{(a-b)y^{(0)}}{b - x^{(0)}y^{(0)}(x^{(0)} + x^{(1)} - y^{(1)})}, \\
u^{(1)} &= y^{(1)} + \frac{(b-a)y^{(0)}}{b - x^{(0)}y^{(0)}(x^{(0)} + x^{(1)} - y^{(1)})} + \frac{(b-a)x^{(0)}}{a - x^{(0)}y^{(0)}(x^{(0)} + x^{(1)} - y^{(1)})}, \\
v^{(0)} &= x^{(0)} + \frac{(b-a)x^{(0)}}{a - x^{(0)}y^{(0)}(x^{(0)} + x^{(1)} - y^{(1)})}, \\
v^{(1)} &= x^{(1)} + \frac{(b-a)y^{(0)}}{b - x^{(0)}y^{(0)}(x^{(0)} + x^{(1)} - y^{(1)})}.
\end{aligned}$$

5.2 The Invariants of $\mathbf{X}_s$

Again we denote invariants as in (5.6) and Figure 1. The $4(N-1)$ invariants:

$$
\begin{aligned}
x^{(i)} &= \frac{\chi^{(i)}_{m+1,n}}{\chi^{(i+1)}_{m,n}}, \quad i=0,\dots,N-3, & x^{(N-2)} &= \chi^{(N-2)}_{m+1,n}\prod_{j=0}^{N-2}\chi^{(j)}_{m,n},\\
y^{(i)} &= \frac{\chi^{(i)}_{m+1,n+1}}{\chi^{(i+1)}_{m+1,n}}, \quad i=0,\dots,N-3, & y^{(N-2)} &= \chi^{(N-2)}_{m+1,n+1}\prod_{j=0}^{N-2}\chi^{(j)}_{m+1,n},\\
u^{(i)} &= \frac{\chi^{(i)}_{m+1,n+1}}{\chi^{(i+1)}_{m,n+1}}, \quad i=0,\dots,N-3, & u^{(N-2)} &= \chi^{(N-2)}_{m+1,n+1}\prod_{j=0}^{N-2}\chi^{(j)}_{m,n+1},\\
v^{(i)} &= \frac{\chi^{(i)}_{m,n+1}}{\chi^{(i+1)}_{m,n}}, \quad i=0,\dots,N-3, & v^{(N-2)} &= \chi^{(N-2)}_{m,n+1}\prod_{j=0}^{N-2}\chi^{(j)}_{m,n},
\end{aligned}
$$

satisfy $(N-1)$ identities:

$$
\begin{aligned}
u^{(i)}v^{(i+1)} &= x^{(i+1)}y^{(i)}, \quad i=0,\dots,N-3, && (5.8a)\\
u^{(N-2)}\prod_{j=0}^{N-2}x^{(j)} &= y^{(N-2)}\prod_{j=0}^{N-2}v^{(j)}, && (5.8b)
\end{aligned}
$$

and equations (5.4) take the form

$$
\begin{aligned}
u^{(i)}v^{(i+1)} &= \frac{(v^{(i)}-1)v^{(i+1)}-(x^{(i)}-1)x^{(i+1)}}{v^{(i)}-x^{(i)}}, \quad i=0,\dots,N-3, && (5.8c)\\
u^{(N-2)} &= \left(1+\frac{1}{x^{(N-2)}-v^{(N-2)}}\left(\frac{a}{X}-\frac{b}{Y}\right)\right)\prod_{j=0}^{N-2}v^{(j)}, && (5.8d)
\end{aligned}
$$

where $X=\prod_{j=0}^{N-3}(x^{(j)}-1)$, $Y=\prod_{j=0}^{N-3}(v^{(j)}-1)$.

Remark 5.5 (The Case $N=2$) *Again, since the lattice equation is just $H1$ in the ABS classification [1], the symmetry $\mathbf{X}_s$, with $\omega=-1$, leads to the Yang-Baxter map*

$$
u=y\left(1+\frac{a-b}{x-y}\right), \quad v=x\left(1+\frac{a-b}{x-y}\right),
$$

which is just F_{IV} of the ABS classification of quadrirational maps [2]. Clearly, we may consider this whole family of maps as multi-component generalisations of F_{IV}.

Example 5.6 (The Case $N=3$) In this case, we first define

$$
P_a = ax^{(0)}-(x^{(0)}-1)(y^{(0)}-1)(x^{(0)}x^{(1)}-y^{(1)}), \quad P_b = bx^{(0)}-(x^{(0)}-1)(y^{(0)}-1)(x^{(0)}x^{(1)}-y^{(1)}).
$$

We then have the map

$$
\begin{aligned}
u^{(0)} &= y^{(0)}\left(1-\frac{(a-b)x^{(0)}(y^{(0)}-1)}{(y^{(0)}-1)P_a-y^{(0)}P_b}\right),\\
u^{(1)} &= y^{(1)}\left(1-(a-b)\left(\frac{(x^{(0)}-1)y^{(0)}}{P_a}+\frac{(y^{(0)}-1)}{P_b}\right)\right),\\
v^{(0)} &= x^{(0)}\left(1-\frac{(a-b)(x^{(0)}-1)}{P_a}\right),\\
v^{(1)} &= x^{(1)}\left(1-\frac{(a-b)(y^{(0)}-1)x^{(0)}}{P_b}\right).
\end{aligned}
$$

Acknowledgements

PX acknowledges support from the EPSRC grant *Structure of partial difference equations with continuous symmetries and conservation laws*, EP/I038675/1, and an *Academic Development Fellowship* from the University of Leeds. We thank Jon Nimmo and Ralph Willox for bringing their paper [5] to our attention.

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