UNDECIDABLE PROBLEMS ON REPRESENTABILITY AS BINARY RELATIONS.

ROBIN HIRSCH AND MARCEL JACKSON

Abstract. In this article we establish the undecidability of representability and of finite representability as algebras of binary relations in a wide range of signatures. In particular, representability and finite representability are undecidable for Boolean monoids and lattice ordered monoids, while representability is undecidable for Jónsson's relation algebra. We also establish a number of undecidability results for representability as algebras of injective functions.

§1. Introduction. The application of abstract algebra in logic and computer science rests heavily on the abstract characterisation of algebras of binary relations. One wants a simple set of axioms that precisely characterises some particular class of algebras of relations of interest—a familiar example might be that the group axioms completely describe algebras of permutations under the operations of composition and inverse. Ideally the axiomatisation will be finite (as it is for groups), however failing that, a reasonable test for being simple might be that the axioms are algorithmically verifiable on finite algebras. Equivalently, when presented with a finite general algebraic structure (henceforth, an *algebra*) it is desirable that there be an algorithm to recognise when it is isomorphic to an actual algebra of relations.

The most famous algebra of relations is that initiated by Tarski in the 1940s [59]. He proposed a finite set of axioms for the algebra of binary relations endowed with the Boolean operations of union, intersection, complementation, composition, converse and the identity. Tarski's proposed axiomatisation (here referred to as *relation algebra*) was shown to be incomplete by Lyndon [38, 39] who also produced a complete set of axioms, but Lyndon's axioms are infinite and complicated. In 1964, Monk [44] showed that no finite axiomatisation is possible. An enormous amount of further work has been done in this area since, including books such as Andréka, Monk and Németi [5], Hirsch and Hodkinson [22], Madarász and Crvenković [40] and Maddux [41]. The surveys by Schein [53] or by Mikulas [42] for example detail many of the results concerning axiomatisations. Finally, Hirsch and Hodkinson [21] showed that representability

2000 *Mathematics Subject Classification.* Primary: 03G15, 20M30. Secondary: 20F10, 08A99.

Key words and phrases. Relation algebra, binary relations, representability, Boolean monoid, lattice ordered monoid, Brandt semigroup.

is undecidable for relation algebras: there is no algorithm to decide if a finite algebra in Tarski's signature is representable as an algebra of binary relations.

While Tarski's relation algebras continue to hold active interest, these dramatically negative results have led to increased attention toward weaker signatures. In the present article we show that the behaviour witnessed for the full Tarski signature continues to hold in some surprisingly weak signatures, including signatures formed from constructions currently investigated in the modeling of computer programs. We do not identify a formal boundary between decidability of representability and undecidability of representability for finite algebras, however a very slight weakening of some of the weaker signatures covered by our results changes this decision problem from undecidable to decidable.

We use three main methods. The first consists of quite straightforward arguments based on the existing result of Hirsch and Hodkinson [21]. For some signatures $\mathscr{F}$ it is relatively straightforward to show that representability of a relation algebra is equivalent to representability of its $\mathscr{F}$-reduct. Using such observations we show that the result of Hirsch and Hodkinson already gives undecidability of representability in several cases: in particular any signature containing the lattice-ordered monoid signature $\{\cdot, +, ;, 1'\}$ (see Theorem 6.1 below).

The second method is probably the most substantial innovation of the article. We develop a Boolean monoid interpretation of the uniform word problem for groups and use it to show undecidability of representability for a wide range of converse-free signatures: signatures expressible using the Boolean monoid operations $\{+, \cdot, -, ;, 0, 1, 1'\}$. By using the undecidability of the uniform word problem for finite groups, the method also yields corresponding undecidability results for *finite* representability: representability as an algebra of binary relations on a finite base set. The corresponding problem for the full Tarski signature remains open [22, Problem 18.18]; indeed to the best of our knowledge, the current undecidability of finite representability results are the first known for reducts of the Tarksi relation algebra signature. Amongst the problems we show undecidable using this method are the following.

1. For converse free signatures $\mathscr{F}$ capable of expressing either of the signatures $\{x \cdot y + x \cdot z, ;, 1'\}$ or $\{(x + y) \cdot (x + z), ;, 1'\}$ (so in particular, for signatures $\{+, \cdot, -, ;, 0, 1, 1'\}$, $\{+, \cdot, ;, 1'\}$, $\{\Rightarrow, ;, 1'\}$ and $\{\backslash, ;, 1'\}$):
   - the problems of deciding representability and of deciding finite representability for finite $\mathscr{F}$-algebras;
   - the problems of deciding if a finite $\mathscr{F}$-algebra is a subreduct of a (not necessarily representable) relation algebra or of a finite relation algebra (Section 6).
2. Problems concerning "constrained" representability and finite representability of semigroups and ordered semigroups (Section 6). For example, we show that the following class is not recursive: the class of triples (S, I, R) where S is a finite semigroup, $I, R \subseteq S$ and S can be faithfully represented as a semigroup of binary relations in such a way that the elements of I are injective relations and the elements of R are reflexive.
3. Problems concerning the representability of enriched semigroups as semigroups of injective partial functions (Section 7). In particular, if $\mathscr{F}$ is a

signature under which injective partial functions are closed and $\mathscr{F}$ contains composition as well as the "equal domains" operator $\sqcap$ and the "equal ranges" operator $\sqcup$, then representability and finite representability of $\mathscr{F}$-algebras as injective functions is undecidable.

We also present concrete examples of finite unrepresentable algebras and of finite algebras that are representable but not finitely representable (or are subreducts of relation algebras, but not of any finite relation algebra); see Example 5.3 for example.

This method owes much to an idea initially developed by Hall, Kublanovsky, Margolis, Sapir and Trotter in [17], where problems relating to the embeddability of semigroups in Brandt semigroups were shown to be undecidable. Essentially, we show how such techniques can be applied to the complex algebra (the powerset algebra) of a Brandt groupoid. The results are also related to a result of Gould and Kambites [16] which showed that the class of ample semigroup subreducts of an inverse semigroup is nonrecursive. Their result (whose proof is also at least loosely related to [17]) can be rephrased as the undecidability of representability as a semigroup of injective partial maps with unary operations of domain and range (a specific instance of a result covered by item (3) above); see Jackson and Stokes [29, §6].

These new results all concern converse-free signatures. To finish the article we revisit the construction of Hirsch and Hodkinson to prove one further result in the theme of the article. We show that any reduct of Tarski's signature containing $\{\cdot, ;, ^{\smile}\}$ has undecidability of representability (Theorem 8.1). The signature $\{\cdot, ;, ^{\smile}\}$ was considered in 1959 by Jonsson [33]; he obtained a complete axiomatisation that is somewhat more transparent than that of Lyndon's for the full relation algebra signature (there is only one technical axiom schema). The present result shows that satisfaction of this axiom schema is not an algorithmically verifiable property for finite algebras. An analogous result (Corollary 8.2) is deduced for *allegories* in the sense of Freyd and Scedrov [13].

With the exception of the results in item (3) we also observe that the results continue to hold if the unary operation of reflexive transitive closure * is included (Theorem 6.7). At present, signatures at least as rich as $\{+, ;, ^*, 0, 1'\}$ are commonly encountered in algebraic models of program logics, such as propositional dynamic logic (see Desharnais, Möller and Struth [10] for one example). At the same time, there have been recent successes in incorporating both converse and intersection into dynamic logic (Göller, Lohrey and Lutz [15] for example). The present results show that one cannot hope for representation theorems for algebraic models of such logics.

1.1. Structure of the article. We begin in Section 2 with some basic definitions, facts and linking relationships between Boolean monoids, relation algebras and Brandt semigroups. The section finishes with a more detailed overview of the structure of the proof: Subsection 2.4. In Section 3 we revisit the partial group constructions used to encode the uniform word problem for groups. This section also constructs some small examples used later in the article, including a small pattern that cannot appear within the table of any finite group, yet can appear in the table of a larger group. This preliminary section is essentially a specifically tailored presentation of the ideas found in Evans [11] and Hall et al. [17]. The main construction is presented in Section 4, with the final details of the main proof completed in Section 5. The main proof is developed in the language of Boolean monoids, though it ties representability (or finite representability) to embeddability of a related partial group into a group (or finite group). The full array of results for subreducts of the Tarski signatures are developed in Section 6, by establishing various different intermediate conditions between the Boolean monoid signature and the partial group embedding. Results from item (3) in the list above require a slight adaptation of the construction, and this is given in Section 7. Finally, in Section 8 we revisit the Hirsch and Hodkinson construction and show how to make it work using only the Jónsson signature.

§2. Preliminaries: Boolean monoids, relation algebras and complex algebras. In this paper we have various binary operations. We will let $*$ denote a partial binary operation on a set (for example, in a partial algebra) and ; will denote a normal additive binary operator in a Boolean algebra with operators (for example, in a Boolean monoid or relation algebra). In context, we occasionally use $*$ as a total operation in structures embedding partial structures (in particular, groups and Brandt semigroups), and ; in structures arising as reducts of a Boolean algebra with operators (such as ordered semigroups). In one instance only, we consider the symbol $*$ in superscript to denote the unary operation of reflexive transitive closure. We frequently identify the structure with its domain, for example, $x \in \mathbf{A}$ means that x belongs to the domain of the algebra $\mathbf{A}$.

2.1. Boolean monoids. A *Boolean monoid* $\mathbf{M} = \langle M, 0, 1, +, -, ;, 1' \rangle$ is an algebraic structure on the set M, where 0, 1, $1'$ are constants, $-$ is a unary operation and $+$, ; are binary operations satisfying the following properties.

Boolean Algebra: $\langle M, 0, 1, +, - \rangle$ is a Boolean algebra with top element 1 and bottom element 0.

Monoid: $\langle M, ;, 1' \rangle$ is a monoid. That is ; is associative, and $1'$ is a two sided identity for ;.

Operator: ; is additive and normal. That is, $x ; (y + z) = x ; y + x ; z$ and $(x + y) ; z = x ; z + y ; z$ (additive) and $0 ; x = x ; 0 = 0$ (normal).

We use standard abbreviations $a \cdot b = -(-a + -b)$ and $a \leq b$ for $a + b = b$. We write $a < b$ if $a \leq b$ and $b \not\leq a$. We also adopt the usual bracketing conventions regarding associativity and let ; take precedent over $+$ and $\cdot$, so that (for example) $a + b ; c \cdot -c = a + ((b ; c) \cdot (-c))$. We avoid treating $-$ as a binary relation.

Define the following two term operations in the language of Boolean monoids:

$$D(x) = x ; 1 \cdot 1' \text{ and } R(x) = 1 ; x \cdot 1'.$$

A Boolean monoid $\mathbf{M}$ is said to be *normal* if it satisfies the equations

$$D(x) ; x = x = x ; R(x). \tag{2.1}$$

A concrete example of a Boolean monoid is $(\mathcal{S}, \varnothing, 1, \cup, \setminus, ;, \mathrm{id}_X)$, where $\mathcal{S} \subseteq \wp(X \times X)$ is a set of binary relations over some domain X containing the empty relation $\varnothing$, some biggest relation 1, the identity id_X over X, closed under union, complement relative to 1 and ordinary composition of binary relations, ;. In a concrete example of this type, it follows from the boolean axioms that $1 \geq 1'$ so the biggest relation 1 has to be reflexive and since $1 \geq 1;1 \geq 1;1' = 1$, 1 must also be transitive. Conversely, given any set X and reflexive, transitive binary relation θ over X, we may construct an algebra $\mathrm{Bm}(\theta) := \langle \wp(\theta), \varnothing, \theta, \cup, \setminus, ;, \mathrm{id}_X \rangle$ on $\wp(\theta)$ (so that the top element is θ and the bottom element is $\varnothing$), letting ; be composition of relations. The reader will verify that this is a Boolean monoid. In general we say that a Boolean monoid $\mathbf{M}$ is *representable* if $\mathbf{M}$ is isomorphic to a sub-Boolean monoid of $\mathrm{Bm}(\theta)$ for some reflexive, transitive relation θ. If θ is a relation over a finite base set X, then $\mathbf{M}$ is said to be *finitely representable*. A Boolean monoid has a *square representation* if it embeds into $\mathrm{Bm}(X \times X)$ for some set X.

The *representation problem* for finite relation algebras is the problem of deciding if a finite relation algebra is representable. (Here we restrict to finite relation algebras to be sure that we have a well-defined decision problem.) The representation problem was shown to be undecidable by the first author and Hodkinson [21]. The *finite representation problem* is the problem of deciding if a finite relation algebra is representable over a finite base set. The decidability of this problem is still open.

When the relation θ is symmetric, it is easy to see that $\mathrm{Bm}(\theta)$ is a normal Boolean monoid. In Lemma 2.2 below we show that the converse holds. The main step of the proof is the next lemma which is also important later.

Let the *symmetric interior* of a relation r be the relation $r^\circ := \{(x, y) \mid (x, y) \in r \text{ and } (y, x) \in r\}$ (it is the complement of the symmetric closure of the complement of r). If θ is a reflexive transitive relation, then θ° is easily seen to be the largest equivalence relation contained within θ.

LEMMA 2.1. *Let $\mathbf{M} = \langle M, 0, 1, +, -, ;, 1' \rangle$ be a normal Boolean monoid. Say that there is a reflexive, transitive relation θ on a set X and an injective map $\phi : M \to \mathrm{Bm}(\theta)$ preserving $\{\cdot, ;, 1'\}$. Then the map $\phi^\circ : a \mapsto a^\phi \cap \theta^\circ$ is an injective map $\phi^\circ : M \to \mathrm{Bm}(\theta^\circ)$ preserving $\{\cdot, ;, 1'\}$ and with 1^{ϕ° equal to the equivalence relation θ. Moreover, if ϕ preserves some subset of $\{0, +, -\}$, then so does ϕ°.*

PROOF. Now ϕ° clearly preserves $\cdot$, as well as any subset of $\{0, +, -\}$ if ϕ does. Also, $(1')^{\phi^\circ} = \mathrm{id}_X = (1')^\phi$. Note also that if $e \leq 1'$, then $e^{\phi^\circ} = e^\phi$. For preservation of composition, first consider $(x, y) \in (a\,; b)^{\phi^\circ}$ (for some $a, b \in M$). Then $(y, x) \in 1^\phi$ and there is z such that $(x, z) \in a^\phi$ and $(z, y) \in b^\phi$. So $(y, z) \in 1^\phi ; a^\phi \subseteq 1^\phi$ and similarly, $(z, x) \in 1^\phi$. Hence $(x, z) \in a^{\phi^\circ}$ and $(z, y) \in b^{\phi^\circ}$ showing that $(x, y) \in a^{\phi^\circ} ; b^{\phi^\circ}$. The reverse containment is easier and left to the reader.

Finally, we must verify injectivity of ϕ°. Let $a \neq b$ without loss suppose $b \not\leq a$, so $b \cdot (-a) \neq 0$, and by normality, $D(b \cdot (-a)) \neq 0$. Hence $(D(b \cdot (-a)))^{\phi^\circ} = (D(b \cdot (-a)))^\phi \neq 0^\phi$. However $(D(a \cdot (-a)))^{\phi^\circ} = (D(0))^{\phi^\circ} = 0^{\phi^\circ} = 0^\phi$. Hence $a^{\phi^\circ} \neq b^{\phi^\circ}$ as required. ⊣

LEMMA 2.2. *Let θ be a reflexive transitive relation. Then $\mathrm{Bm}(\theta)$ is a normal Boolean monoid if and only if θ is symmetric.*

Proof. The if direction was observed above. Now say that $\mathrm{Bm}(\theta)$ is a normal Boolean monoid, and note that as $\theta^\circ \subseteq \theta$ we have that $\theta^\circ \in \mathrm{Bm}(\theta)$. Let $\iota : \mathrm{Bm}(\theta) \to \mathrm{Bm}(\theta)$ be the identity isomorphism. Lemma 2.1 shows that ι° is an injective homomorphism identifying θ and θ°, showing that $\theta = \theta^\circ$ is symmetric. ⊣

The following lemma is easy, and its proof is omitted (cf. [35, Theorem 4.28]).

Lemma 2.3. *Let θ be an equivalence relation on a set X. Then* $\mathrm{Bm}(\theta)$ *is a subdirect product of* $\{\mathrm{Bm}(x/\theta \times x/\theta) \mid x \in X\}$.

Let $t(x_1, \dots , x_n)$ be a term in the Boolean monoid signature, $\mathbf{M}$ be a normal Boolean monoid and θ be an equivalence relation over X. If $\phi : M \to \wp(\theta)$, then we say that t is *preserved* by ϕ if $(t^{\mathbf{M}}(m_1, \dots , m_n))^\phi = t^{\mathrm{Bm}(\theta)}(m_1^\phi, \dots , m_n^\phi)$ for every $m_1, \dots , m_n$ in M.

Lemma 2.4. *Let $\mathbf{M}$ be a normal Boolean monoid. If there is an injective map $\phi : M \to \wp(X \times X)$ for some set X preserving either $\{x \cdot y + x \cdot z, ;, 1'\}$ or $\{(x + y) \cdot (x + z), ;, 1'\}$, then $\mathbf{M}$ is representable. Moreover, $\mathbf{M}$ is finitely representable if X is finite.*

Proof. Whichever of the two alternative sets of operators is preserved by ϕ, note that $\cdot$ and $+$ must also be preserved: in the first case we have $x \cdot y = x\cdot y+x\cdot y,\ x+y = 1\cdot x+1\cdot y$ and in the second case $x\cdot y = (0+x)\cdot(0+y),\ x+y = (x + y)\cdot(x + y)$. Now as $0 \le 1'$, it is represented as a restriction of the diagonal map. Let Z be the domain of 0 and $Y := X\backslash Z$. Define $\phi' : M \to \wp(Y \times Y)$ by $m \mapsto x^\phi \cap (Y \times Y)$. As $0 \cdot m = 0$ and $m ; 0 = 0 ; m = 0$ for every $m \in M$, it is easily seen that ϕ' is injective and preserves $\{\cdot, +, ;, 1'\}$ as well as correctly representing 0 as the empty relation. Finally, complementation is preserved relative to 1 because the complement of $m^{\phi'}$ under 1^ϕ is the unique element a for which $m^{\phi'} \cap a = \varnothing$ and $m^{\phi'} \cup a = 1^{\phi'}$ (and $(-m)^{\phi'}$ has these properties). ⊣

2.2. Relation algebras. A *relation algebra* $\mathbf{A} = \langle A, 0, 1, +, -, 1', {}^\smile, ;\rangle$ is an algebraic structure on the set A, where ${}^\smile$ is a unary operation and the following properties hold.

Boolean Monoid: $\langle A, 0, 1, +, -, 1', ;\rangle$ is a Boolean monoid.
Operator: ${}^\smile$ is additive and normal (i.e. $0^\smile = 0$ and $(x + y)^\smile = x^\smile + y^\smile$).
Involution: $(x^\smile)^\smile = x$ and $(x; y)^\smile = y^\smile; x^\smile$.
Triangle Law: $x^\smile; (-(x; y)) \le -y$.

Consider a set X and an equivalence relation θ on X. A motivating example of a relation algebra $\mathrm{Rel}(\theta) := \langle\wp(\theta), 0, 1, +, -, 1', {}^\smile, ;\rangle$ can be defined on $\wp(\theta)$ by adjoining the operation of relational converse to the Boolean monoid $\mathrm{Bm}(\theta)$: so $(x, y) \in r^\smile$ if and only if $(y, x) \in r$. (Note that the assumed choice of θ as an equivalence relation is essentially forced by the fact that $\mathrm{Bm}(\theta)$ is a Boolean monoid and $1 = 1^\smile$.) A *relation algebra representation* ϕ is an isomorphism from a relation algebra $\mathbf{R}$ to a subalgebra of $\langle\wp(\theta), \varnothing, \theta, \cup, -_\theta, \mathrm{id}_\theta, {}^\smile, ;\rangle$ for some equivalence relation θ (and where $-_\theta$ is the unary operation of complementation in θ). More generally, for any signature $\mathscr{F}$ consisting of term operations in the relation algebra signature, an *$\mathscr{F}$-representation* is an isomorphism from an $\mathscr{F}$-algebra to an algebra of binary sub-relations of some binary relation θ, where each operator in $\mathscr{F}$ is defined using $\langle\varnothing, \theta, \cup, -_\theta, \mathrm{id}_\theta, {}^{\smile}, ;\rangle$, using its term definition. We may say that an algebra is $\mathscr{F}$-representable in this case.

The *representation problem* for relation algebras is the problem of deciding if a finite relation algebra is representable. The representation problem was shown to be undecidable by the first author and Hodkinson [21]. The *finite representation problem* is the problem of deciding if a finite relation algebra is representable over a finite base set. The decidability of this problem is still open.

In this article we also consider the *subreduct problem* for a class K of relation algebras. Fix a signature $\mathscr{F}$ consisting of term operations in the relation algebra signature. The $\mathscr{F}$-subreduct problem for K asks if a given finite algebra of the same type as $\mathscr{F}$ is a $\mathscr{F}$ subreduct of some relation algebra in K. When K consists of representable relation algebras the subreduct problem for K can be shown to be the same as the $\mathscr{F}$-representability problem. In this article we also consider the case where K is the class of all relation algebras, and the class of all finite relation algebras.

The following lemmas are consequences of the relation algebra axioms (e.g., [22, lemma 3.12] proves Lemma 2.5 which can be used to prove Lemma 2.6, use [35, theorem 4.15] to prove Lemma 2.7).

LEMMA 2.5. *Let* $\mathbf{R}$ *be a relation algebra and* $a, b, c \in R$.

$$a\,;b\cdot c = 0 \iff b\,;c^{\smile}\cdot a^{\smile} = 0 \iff c^{\smile}\,;a\cdot b^{\smile} = 0$$

LEMMA 2.6. *Let* $\mathbf{R}$ *be a relation algebra and* $e \leq 1' \in R$. *Then*

$$e^{\smile} = e \text{ and } e\,;e = e.$$

Let $a \in R$. *Then* $a\,;a^{\smile}\cdot 1' = a\,;1\cdot 1'$ $(= D(a))$ *and* $D(a);a = a$.

LEMMA 2.7. *Let* $\mathbf{R}$ *be a relation algebra and consider two elements* $z, r \in R$ *with the properties that* $r\,;r = r^{\smile} = r$, $r\cdot z = z\cdot r = r\,;z = z\,;r = z\,;z = z^{\smile} = z$. *Let* S *be the set* $\{s \in R \mid z \leq s \leq r\}$, *which is a* $\{+,\cdot,;,{}^{\smile}\}$ *subreduct of* $\mathbf{R}$. *Define* $1'_S = 1'\cdot r$ *and* $-_S a := r\cdot(-a)$. *Then* $\mathbf{S} = \langle S, z, r, +, -_S, {}^{\smile}, ;, 1'_S\rangle$ *is a relation algebra.*

In the following lemma (and elsewhere) the square brackets are used to give two separate series of equivalent conditions: those when the bracketed statements are removed, and those where the brackets (but not the statements they bridge) are removed.

LEMMA 2.8. *Let* $\mathbf{M}$ *be a simple, normal Boolean monoid. The following are equivalent.*

1. $\mathbf{M}$ *is representable* [*over a finite base set*].
2. $\mathbf{M}$ *has a square representation* [*over a finite base set*].
3. $\mathbf{M}$ *is a* $\{0, 1, +, -, ;, 1'\}$*-subreduct of a simple representable relation algebra* [*representable over a finite base set*].
4. $\mathbf{M}$ *is a subreduct of a representable relation algebra* [*representable over a finite base set*].

PROOF. For 1 implies 2, use Lemma 2.1 to obtain a representation into $\mathrm{Bm}(\theta)$ for some equivalence relation θ on a set X. Then, use Lemma 2.3 and the fact that $\mathbf{M}$ is (isomorphic to) a simple subalgebra of $\mathrm{Bm}(\theta)$ to obtain a representation of $\mathbf{M}$ into $\mathrm{Bm}(Y \times Y)$, where Y is one of the blocks of θ (note that Y is finite if X is). Now 1 is represented as $Y \times Y$, a square representation.

For 2 implies 3, use the fact that $\mathrm{Bm}(Y \times Y)$ is the $\{0, 1, +, -, ;, 1'\}$ reduct of $\mathrm{Rel}(Y \times Y)$, which is simple.

Implications 3 implies 4 and 4 implies 1 are trivial. $\dashv$

Lemma 2.9. *Let $\mathbf{R}$ be a relation algebra and consider an injective function ϕ from R to $\wp(X \times X)$ for some set X. If ϕ preserves either $\{x \cdot y + x \cdot z, ;, 1'\}$ or $\{(x + y) \cdot (x + z), ;, 1'\}$, then $\mathbf{R}$ is a representable relation algebra.*

Proof. By Lemma 2.6, the Boolean monoid reduct of $\mathbf{R}$ is normal, and by Lemma 2.4 it is representable. By Lemma 2.1, there is a representation $\psi : \mathbf{R} \to \mathrm{Bm}(\theta)$ for the Boolean monoid reduct of $\mathbf{R}$ in which the top element is represented as the equivalence relation θ. It remains to show that $\smile$ can be correctly represented.

Consider any $r \in R$ and some pair $(x, y) \in r^\psi$, and note that $(y, x) \in 1^\psi$. We use the law $a ; (-(a^\smile)) \leq -1'$, which is an instance of the triangle axiom. This implies that $(x, x) \notin (r ; (-(r^\smile)))^\psi = r^\psi ; (-(r^\smile))^\psi$. Hence $(y, x) \notin (-(r^\smile))^\psi$, showing that $(y, x) \in (r^\smile)^\psi$ as required. $\dashv$

We can also extend Lemma 2.4 to subreducts of relation algebras. The proof is essentially the same, except that Lemma 2.7 is used.

Lemma 2.10. *Let $\mathbf{M}$ be a normal Boolean monoid, $\mathbf{R}$ be a relation algebra and consider an injective function ϕ from M to R. If ϕ preserves either $\{x \cdot y + x \cdot z, ;, 1'\}$ or $\{(x + y) \cdot (x + z), ;, 1'\}$, then $\mathbf{M}$ is a subreduct of a relation algebra. Moreover, if $\mathbf{R}$ is finite, then $\mathbf{M}$ is a subreduct of a finite relation algebra.*

Definition 2.11. *Define a unary relation $\mathtt{i}$ in the language of relation algebras by*

$$x \in \mathtt{i} \Leftrightarrow x ; x^\smile \leq 1' \ \ \& \ \ x^\smile ; x \leq 1'.$$

Define the diversity element $0'$ *to be* $-1'$.

It is easy to see that in $\mathrm{Rel}(\theta)$, the relation $\mathtt{i}$ consists of precisely those relations that are injective partial maps (here called injective functions), while on $\mathrm{Rel}(X \times X)$, the diversity element is the relation of inequality $\neq$ on X. The next lemma is crucial in our main proof and shows that on a relation algebra, the unary relation $\mathtt{i}$ coincides with the solution set of a one variable formula in the language of Boolean monoids (in fact, in the signature $\{\cdot, ;, 0'\}$).

Lemma 2.12. *Let $\mathbf{R}$ be a relation algebra. Then $a \in \mathtt{i}^{\mathbf{R}}$ if and only if*

$$0' ; a \cdot a = 0 = a ; 0' \cdot a.$$

Proof. First note that $a \in \mathtt{i}^{\mathbf{R}}$ if and only if $a ; a^\smile \cdot 0' = 0 = a^\smile ; a \cdot 0$. Now note that Lemma 2.5 shows that for any $a \in R$, we have $a ; a^\smile \cdot 0' = 0$ if and only if $(0')^\smile ; a \cdot a = 0$. Dually, $a ; 0' \cdot a = 0$ is equivalent to $a^\smile ; a \cdot 0' = 0$ $\dashv$

Lemma 2.12 shows that we can unambiguously refer to the relation $\mathtt{i}$ in a normal Boolean monoid too.

2.3. Brandt groupoids and complex algebras. In this article we take, for each cardinal λ and each group $\mathbf{G}$, the *Brandt groupoid* $\mathsf{Br}_\lambda(\mathbf{G})$ (of *dimension* λ) to be the partial algebra on the set $\lambda \times G \times \lambda$ with a single partial operation $*$ of binary multiplication, given by $(i,g,j)*(j,h,k) = (i,gh,k)$ for every $i,j,k < \lambda$ and $g \in G$. Note that the set of *idempotents* of $\mathsf{Br}_\lambda(\mathbf{G})$ is $\{(i,e,i) : i < \lambda\}$, where e is the group identity.

The *complex algebra* $\mathfrak{C}(\mathsf{Br}_\lambda(\mathbf{G}))$ of the Brandt groupoid $\mathsf{Br}_\lambda(\mathbf{G})$ is the normal Boolean monoid on the powerset $\wp(\mathsf{Br}_\lambda(\mathbf{G}))$, where $0, 1, +, -$ take their usual set theoretic interpretations and composition ; is defined pointwise

$$A\,;B := \{a * b \mid a \in A, b \in B \text{ and } a \cdot b \text{ is defined}\}.$$

The algebra $\mathfrak{C}(\mathsf{Br}_\lambda(\mathbf{G}))$ is a complete atomic normal Boolean monoid and is representable. Indeed, $\mathfrak{C}(\mathsf{Br}_\lambda(\mathbf{G}))$ is isomorphic to a subalgebra of $\mathrm{Bm}((\lambda \times \mathbf{G}) \times (\lambda \times \mathbf{G}))$ by identifying the atom $\{(i,g,j)\}$ with the injective function from domain $\{i\} \times G$ to range $\{j\} \times G$ defined by $(i,h) \mapsto (j,hg)$ (and extending the representation by complete additivity of $\mathfrak{C}(\mathsf{Br}_\lambda(\mathbf{G}))$). Note that $\mathfrak{C}(\mathsf{Br}_\lambda(\mathbf{G}))$ is finite iff λ and $\mathbf{G}$ are finite.

A *Brandt semigroup* is a semigroup obtained from a Brandt groupoid by adjoining a new element, 0, and making the multiplication totally defined by letting any undefined products take the value 0. Standard notation for the semigroup formed from $\mathsf{Br}_\lambda(\mathbf{G})$ in this way is $\mathsf{B}_\lambda(\mathbf{G})$. Brandt semigroups play a ubiquitous role in the study of ideals of semigroups; see any semigroup text (Howie [26] for example). The following proposition details an equivalent abstract characterisation, and follows easily from the Rees-Suskevich Theorem; see [26, Chapter 3].

Proposition 2.13. *A semigroup $\mathbf{S} = \langle S, ;\rangle$ with a multiplicative zero element 0 is isomorphic to $\mathsf{B}_\lambda(\mathbf{G})$ for some group $\mathbf{G}$ if and only if the following three properties hold.*

0-simplicity: *Whenever $a, b \in S\backslash\{0\}$ then there are elements $c_1, c_2, d_1, d_2 \in S \cup \{\Lambda\}$, where Λ is an additional identity element, such that $c_1\,; a\,; d_1 = b$ and $c_2\,; b\,; d_2 = a$ (that is, $\mathbf{S}$ is 0-simple).*
Primitivity: *Every product of distinct idempotent elements is 0.*
Dimension λ: *The cardinality of the set of non-zero idempotents is λ.*

A Brandt groupoid also admits a natural inverse: $(i,g,j)^{-1} := (j,g^{-1},i)$. This extends to an operation of converse on the complex algebra $\mathfrak{C}(\mathsf{Br}_\lambda(\mathbf{G}))$ by $S^{\smile} = \{s^{-1} \mid s \in S\}$.

Proposition 2.14 ([35, theorem 5.8]). *Let $\mathcal{A}$ be a relation algebra. The following are equivalent.*

- *$\mathcal{A} \subseteq \mathfrak{C}(\mathsf{Br}_\lambda(\mathbf{G}))$, for some Brandt groupoid $\mathsf{Br}_\lambda(\mathbf{G})$,*
- *$\mathcal{A}$ is representable.*

The following lemma provides a way of identifying Brandt semigroups in the $\{;\}$-reduct of a relation algebra and is used in our main proof.

Lemma 2.15. *Let $\mathbf{R}$ be a relation algebra and say that there are elements $a, b \in \mathbf{R}$ with the following properties*:

- $a, b \in \mathtt{i}^{\mathbf{R}}$;
- $R(a) = D(b)$;
- $D(a)$, $R(a)$ *and* $R(b)$ *are pairwise disjoint (that is,* $D(a) \cdot R(a) = D(a) \cdot R(b) = R(a) \cdot R(b) = 0$*),*

let e_0, e_1 *and* e_2 *denote the elements* $D(a)$, $R(a)$ *and* $D(b)$ *respectively. Then the set*

$$B = \{x \in R \mid x = 0 \text{ or } (\exists i, j \in 3)\, x \,;\, x^{\smile} = e_i \ \& \ x^{\smile} \,;\, x = e_j\}$$

is a Brandt semigroup of dimension 3 *when given the operation* ;.

Proof. The 0-simplicity condition follows easily from the fact that $\smile$ is an involution, that $a, b \in \mathtt{I}^{\mathbf{R}}$ and the fact that $x \,;\, x^{\smile} = e_i$ and $x^{\smile} \,;\, x = e_j$ imply $e_i \,;\, x = x = x \,;\, e_j$. The primitivity condition will follow from the assumptions on the idempotent elements e_i, provided we can verify that there are no other nonzero idempotent elements. Say that x is a nonzero idempotent element. Let $i, j \in 3$ be such that $x;x^{\smile} = e_i \ \& \ x^{\smile};x = e_j$. As $x = x;x = x;e_j;e_i;x$ we have that $e_j = e_i$, or equivalently, that $i = j$. Then $x = x;e_i = x;x;x^{\smile} = x;x^{\smile} = e_i$. So e_1, e_2, e_3 are the only nonzero idempotents as required, showing that $\langle B, ;\rangle$ is a Brandt semigroup of dimension 3. ⊣

2.4. Overview of main proof. Hall et al. [17] showed that the class of finite semigroups that can be embedded into a Brandt semigroup (or into a finite Brandt semigroup) is nonrecursive. The goal is to find a construction of a Boolean monoid that incorporates certain elements with similar behaviour to the construction in [17]. In particular, these key elements will be contained in the unary relation $\mathtt{i}$ via Lemma 2.12. Lemma 2.8 relates representability for normal Boolean monoids to embeddability into relation algebras and then Lemma 2.15 will show that the key elements embed in an appropriate way into a Brandt semigroup.

§3. Preliminaries: Partial groups and undecidability.

Before presenting the main construction, we recall some useful concepts that are used in the proof and give the intuition behind the proof.

3.1. Partial groups. In this article we conceive of a *partial group* as a system $\langle A, *, e\rangle$, where A is a set, $* : A \times A \to A$ is a partial binary operation and $e \in A$ has the property that $e*x = x*e = x$ whenever $*$ is defined at a tuple containing x. (Usually one requires that $e * x = x * e = x$ for all x but this minor difference has no impact on our proofs and simplifies later definitions.) Here and below we interpret the truth of equalities in partial algebras to mean that both sides are defined and equal.

Definition 3.1. *Let* $\mathbf{A} = \langle A, *, e\rangle$ *be a partial group. We say that* $\mathbf{A}$ *is a* square *partial group if there is a subset* $\sqrt{A}$ *of* A *containing* e *and the following properties hold.*

1. $a * b$ *is defined if and only if* $a, b \in \sqrt{A}$.
2. $\sqrt{A} * \sqrt{A} = A$; *that is, for every* $c \in A$ *there are* $a, b \in \sqrt{A}$ *such that* $a * b = c$.

Note that if $\sqrt{A}$ exists then it is completely determined by $*$, and that as $e \in A$, we have $e * a = a * e = a$ whenever $a \in \sqrt{A}$. In [17], Hall et al. introduced a notion of an *extension* $\mathbf{A}$ *of rank* k *of a partial group* $\mathbf{B}$, which has been widely used in a number of subsequent undecidability results in semigroup theory. The notion of a square partial group essentially corresponds to the rank 2 case, but captures the information required in our proof more succinctly.

Every group $\mathbf{G}$ is a square partial group, where $\sqrt{G} = G$ and e is chosen as the identity element. The following example gives a more typical form for a square partial group and forms a useful counterexample later.

EXAMPLE 3.2. *The following table defines the structure of a square partial group* $\mathbf{F}$ *on the set* $F = \{a, b, c, e\}$*, with* $\sqrt{F} = \{a, b, e\}$*:*

$*$	e	a	b
e	e	a	b
a	a	b	c
b	b	c	a

A homomorphism from a partial group $\langle A, *, e\rangle$ to a group $\langle G, \star, 1\rangle$ is a function $\phi : A \to G$ satisfying $(a * b)^\phi = a^\phi \star b^\phi$ whenever $a * b$ is defined. The partial group $\mathbf{A}$ *embeds* into a group $\mathbf{G}$ if there is an injective homomorphism from $\mathbf{A}$ to $\mathbf{G}$.

A *cancellative* partial group is a partial group satisfying the cancellation laws

$$(x * y = x * z \to y = z) \ \& \ (x * y = z * y \to x = z)\,. \tag{3.1}$$

Thus the cancellativity property for partial groups simply ensures that the partial multiplication table has no repeats in any row nor in any column. Cancellativity is obviously a necessary condition for a partial group to be embeddable in a group. It is routine to verify that all 2-element and 3-element cancellative square partial groups embed into finite groups. Thus the following example from Jackson [27] is of minimal size.

EXAMPLE 3.3. *The 4-element square partial group* $\mathbf{F}$ *of Example 3.2 is cancellative but does not embed into any group.*

PROOF. Without loss of generality, we may assume (for contradiction) that the table for $\mathbf{F}$ is a restriction of a multiplication table of some group $\mathbf{G}$. Then associativity of $\mathbf{G}$ forces $a*e = a = b*b = (a*a)*(a*a) = a*(a*(a*a)) = a*c$, and then cancellativity of $\mathbf{G}$ contradicts $e \neq c$. $\dashv$

3.2. Uniform word problems, partial groups and homotopy embeddings. Consider a group presentation $\langle A \mid R\rangle$. Let us say that an interpretation of $\langle A \mid R\rangle$ in a group $\mathbf{G}$ is a map ϕ from the free group on A into $\mathbf{G}$ such that each word $r \in R$ has $r^\phi = 1$. Now fix a class of groups K; in the present article we will be interested in the case where K is the class of all groups, or where K is the class of all finite groups.

- The *uniform word problem* (abbreviated to uwp) for K takes as an instance a finite group presentation $\langle A \mid R\rangle$ and a group word w in the alphabet A. This is a YES instance provided every interpretation ϕ of $\langle A \mid R\rangle$ has $w^\phi = 1$.

- The *partial group embedding problem* for K takes as an instance a finite partial group $\mathbf{A}$. This is a YES instance provided that there is a group $\mathbf{G} \in K$ and an injective map $\phi : A \to G$ preserving all products defined in $\mathbf{A}$.

Trevor Evans [11] showed that the partial algebra embedding problem for K is decidable if and only if the uwp for K is decidable. This proof is easily adapted to show that the problem of deciding embeddability of finite square partial groups into K is undecidable. In fact we have the following lemma (Lemma 1.1 of [31]), which uses the extension of Evans' argument to square partial groups and the fact that every class of groups containing the class of finite groups has undecidable uwp (this is essential due to Slobodskoi [56], see Kharlampovic and Sapir [36, §7.4.3]).

LEMMA 3.4. *Let $\mathscr{G}_{\text{fin}}$ denote the class of finite groups and $\mathscr{G}$ denote the class of all groups. If $\mathscr{G}_{\text{fin}} \subseteq \mathscr{H} \subseteq \mathscr{G}$, then the class of finite cancellative square partial groups embeddable into a group from $\mathscr{H}$ is not recursive.*

We direct the reader to [31] for a fuller discussion of the proof of this lemma. The following example (essentially along the lines of the example suggested by Evans in [11]) contains the essence of the idea and serves as a useful counterexample later.

EXAMPLE 3.5. *1. The table in Figure 1 defines the multiplication of a* 13*-element partial group* $\mathbf{T}$ *on the set* $\{1, 2, \ldots, 13\}$ *that is embeddable in a group but not in any finite group.*
2. There is a square partial group $\mathbf{Q}$ *with at most* 61 *elements and with* $|\sqrt{Q}| =$ 9 *that is embeddable in a group but not in any finite group.*

PROOF. We consider Higman's finitely presented group $\mathbf{H}$ (see [19]) with group presentation $\langle a, b, c, d : ab = bba, bc = ccb, cd = ddc, da = aad \rangle$. For consistency, we let e denote the identity element of $\mathbf{H}$. The key property established in [19] is that $\mathbf{H}$ has no nontrivial finite quotients.

Now consider the 9-element set $S = \{e, a, b, c, d, aa, bb, cc, dd\}$ and let $T = \{e, a, aa, b, bb, c, cc, d, dd, ab, bc, cd, da\}$. Consider the partial group on T with multiplication $x * y$ defined provided that $x, y \in S$, and

$$x * y := \begin{cases} xy & \text{if } xy \in T, \\ ab & \text{if } x = bb \text{ and } y = a, \\ bc & \text{if } x = cc \text{ and } y = b, \\ cd & \text{if } x = dd \text{ and } y = c, \\ da & \text{if } x = aa \text{ and } y = d. \end{cases}$$

All other products are undefined. The definition of this partial multiplication is simply chosen to be enough to determine the relations in the presentation of $\mathbf{H}$. The table for $\mathbf{T}$ is given in Figure 1, with the numbers 1–12 taking the role of the elements $e, a, aa, b, bb, c, cc, d, dd, ab, bc, cd, da$.

It is routine to verify that each element of T is distinct in $\mathbf{H}$; we omit the details. Also, each of the defined products in $\mathbf{T}$ agree with those in $\mathbf{H}$. So the table of the partial group $\mathbf{T}$ is a restriction of the (infinite) Cayley table for $\mathbf{H}$.

$*$	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	3		10					
3	3							13	
4	4			5		11			
5	5	10							
6	6					7		12	
7	7			11					
8	8	13						9	
9	9					12			

FIGURE 1. A partial group embeddable in a group but not into any finite group.

However any group **G** embedding **T** has distinct elements a, b, c, d satisfying the relations in the presentation of **H**; it follows that there is a nontrivial homomorphism from **H** into **G**, which is therefore infinite, as **H** has no nontrivial finite quotients.

The partial group **T** is not a square partial group. To make a square partial group **Q** with the desired properties (and with $\surd$ equal to the 9-element set S), one needs to complete the entries of the table in Figure 1, possibly adding new elements. There are only finitely many ways to do this: there are only 52 unfilled entries in the table whence at most 52 extra elements are required, and they can be arranged in only a finite number of ways. (The number 52 may be reduced to at most 48, taking into account the fact that in Higman's group $a(bb) = (bb)(ab)$, and the three other symmetric equalities.) One of these "square completions" coincides (up to isomorphism) with a restriction of the multiplication table for **H** (whence is the desired cancellative square partial group **Q** embeddable in **H**, and with at most $48+13 = 61$ elements), but none can embed into a finite group, as they all embed **T**. ⊣

Consider a partial group $\mathbf{A} = \langle A, * \rangle$. If $\mathbf{B} = \langle B, \times \rangle$ is an algebra with a total binary operation, then a *homotopy* from **A** to **B** is a triple of partial functions $(\alpha_1, \alpha_2, \alpha_3)$ from A into B, where the domain of α_1 is $\{a \in A \mid (\exists b \in A)\, a*b \text{ is defined}\}$, and dually for α_2 and where α_3 is total, such that $a^{\alpha_1} \times b^{\alpha_2} = (a*b)^{\alpha_3}$ (all a, b in the domain of α_1, α_2, respectively). The homotopy $(\alpha_1, \alpha_2, \alpha_3)$ is a *homotopy embedding* if each of the contributing functions are injective. The notion of homotopy as described was introduced by Albert in [1] in the context of quasigroups and latin squares. The following lemma is essentially the argument in [17].

LEMMA 3.6. (*Hall et al.* [17], *after Albert* [1].) *A partial group* $\mathbf{A} = \langle A, *, e \rangle$ *embeds into a group* **G** *if and only if it homotopy embeds into* **G**.

PROOF. (Sketch.) The left to right implication is trivial, while the reverse implication follows because if $(\alpha_1, \alpha_2, \alpha_3)$ is a homotopy embedding into a group **G**, then the map $a \mapsto a^{\alpha_3}(e^{\alpha_2})^{-1}(e^{\alpha_1})^{-1}$ is an injective homomorphism (see [31, Lemma 1.2] for details). ⊣

The following example is of some interest from the perspective of Latin squares, as Lemma 3.6 shows that we can drop the row and column labels from the table in Figure 1 and obtain a partial Latin square with similar properties to the corresponding partial group.

EXAMPLE 3.7. *The content of the table in Figure 1 gives a pattern that does not appear in any Latin square isotopic to the multiplication table of any finite group but that does appear in the multiplication table of an infinite group.*

An interesting combinatorial problem is to find the smallest number of entries such a partial Latin square may have. A careful analysis of the proof of [31, Lemma 1.2], shows that in the partial table of Figure 1 we do not need all of the entries resulting from products with the element 1 (5 entries may be dropped from the existing 29; again, we omit details).

§4. The main construction. Throughout this section we consider a fixed finite, cancellative, square partial group $\mathbf{A}$. We define a finite (whence complete atomic) Boolean monoid $\mathbf{M}(\mathbf{A})$.

The atoms (with respect to the Boolean algebra reduct of $\mathbf{M}(\mathbf{A})$) are

$$\{\mathsf{e}_{ii} \mid i \in 3\} \cup \{\mathsf{w}_{ij} \mid i, j \in 3\} \cup \{\mathsf{a}_{01}, \mathsf{a}_{12} \mid a \in \sqrt{A}\} \cup \{\mathsf{b}_{02} \mid b \in A\}.$$

The remaining elements of $\mathbf{M}(\mathbf{A})$ are arbitrary sums of these atoms, so that $\mathbf{M}(\mathbf{A})$ has 2^n elements, for $n = 3+9+2\times|\sqrt{A}|+|A|$. This determines the Boolean reduct of $\mathbf{M}(\mathbf{A})$; it remains to define the identity element and composition. The identity is $1' := \mathsf{e}_{11} + \mathsf{e}_{22} + \mathsf{e}_{33}$. The elements w_{ij} for $i, j < 3$ are called *white*.

In order to define composition it is convenient to introduce some notation. We let 1_{ij} denote the sum of all atoms with subscript ij. Thus

$$1_{ij} := \begin{cases} \mathsf{w}_{ij} & \text{if } i > j, \\ \mathsf{w}_{ii} + \mathsf{e}_{ii} & \text{if } i = j, \\ \mathsf{w}_{ij} + \sum_{a \in \sqrt{A}} \mathsf{a}_{ij} & \text{if } j - i = 1, \\ \mathsf{w}_{ij} + \sum_{b \in A} \mathsf{a}_{ij} & \text{if } j - i = 2. \end{cases}$$

Let 0 denote the empty sum of atoms, $1 := \sum_{i,j \in 3} 1_{ij}$ and $A_{ij} := 1_{ij} - \mathsf{w}_{ij}$. In general, we let x_{ij}, y_{ij}, z_{ij} be variables taking values amongst the atoms beneath 1_{ij}.

Composition will defined on the atoms, then extended to arbitrary sums of atoms using additivity:

$$\sum S \,;\, \sum T := \sum_{s \in S, t \in T} s \,;\, t.$$

Composition is defined on the atoms by the following 6 items.

($1_{;}$) $x_{ij} \,;\, y_{j'k} = 0$ if $j \neq j'$.

($2_{;}$) $\mathsf{e}_{ii} \,;\, x_{ij} = x_{ij} \,;\, \mathsf{e}_{jj} = x_{ij}$.

($3_{;}$) $\mathsf{a}_{01} \,;\, \mathsf{b}_{12} = (a * b)_{02}$, for $a, b \in \sqrt{A}$.

(Note that by additivity and (3) we now have (for any $i, j, k \in 3$ with $i < j$) that $\mathsf{a}_{ij} \,;\, A_{jk} = \sum\{(a * b)_{ik} \mid \mathsf{b}_{jk} \in A_{jk}\}$. The composition $A_{ij} \,;\, a_{jk}$ is similarly forced when $j < k$. This enables the next part of the definition.)

($4_;$) If $i < j$, then $\mathsf{a}_{ij} ; \mathsf{w}_{jk} = 1_{ik} - \mathsf{a}_{ij} ; A_{jk}$ for $a \in \sqrt{A}$ if $j - i = 1$ and $a \in A$ if $j - i = 2$.

($5_;$) Dually, if $j < k$, then $\mathsf{w}_{ij} ; \mathsf{a}_{jk} = 1_{ik} - A_{ij} ; \mathsf{a}_{jk}$ for $a \in \sqrt{A}$ if $k - j = 1$ and $a \in A$ if $k - j = 2$.

($6_;$) $\mathsf{w}_{ij} ; \mathsf{w}_{jk} = 1_{ik}$.

Thus we arrive at an algebraic structure $\mathbf{M} = \mathbf{M}(\mathbf{A})$ of the same signature as a Boolean monoid. Moreover the reduct to $\{+, \cdot, -, 0, 1\}$ is a Boolean algebra, the element $1'$ is a multiplicative identity element (by (2)), and the composition is additive by definition. Note that part (3) of the definition of composition ensures that there is a homotopy embedding of $\mathbf{A}$ into the ;-reduct of $\mathbf{M}(\mathbf{A})$ given by $\alpha_1(a) = \mathsf{a}_{01}$, $\alpha_2(a) = \mathsf{a}_{12}$ and $\alpha_3(a) = \mathsf{a}_{02}$.

We have not yet used the cancellativity condition 3.1 for $\mathbf{A}$. This is used in the proof of the following lemma to guarantee that composition on $\mathbf{M}(\mathbf{A})$ is associative.

Lemma 4.1. $\mathbf{M}(\mathbf{A})$ *is a finite, simple, normal Boolean monoid with* $3 + 9 + 2 \times |\sqrt{A}| + |A|$ *distinct atoms.*

Proof. It remains to check associativity, the normal law and simplicity. For associativity, it suffices to show $(x; y); z = x; (y; z)$ for any three atoms $x, y, z \in \mathsf{At}(\mathbf{M})$. If the subscripts do not match then both sides will be zero, by definition ($1_;$). Now suppose the three atoms are x_{ij}, y_{jk}, z_{kl}, for some $i, j, k, l < 3$. We have to check that

$$(x_{ij} ; y_{jk}) ; z_{kl} = x_{ij} ; (y_{jk} ; z_{kl}). \tag{4.1}$$

The case where $i = j$ and $x_{ij} = \mathsf{e}_{ii}$ is $(\mathsf{e}_{ii} ; y_{ik}) ; z_{kl} = y_{ik} ; z_{kl} = \mathsf{e}_{ii} ; (y_{ik} ; z_{kl})$. Similarly if $j = k$ and $y_{jk} = \mathsf{e}_{jj}$ or $k = l$ and $z_{kl} = \mathsf{e}_{kk}$ then (4.1) holds. Now suppose that none of x_{ij}, y_{jk}, z_{kl} is an identity atom. If $x_{ij} = \mathsf{a}_{ij}$ for some $a \in A$ then we must have $i < j$ (we cannot have $i = j$ since we have discounted the case where x_{ij} is an identity atom) and a similar argument applies to y_{jk}, z_{kl}. Since we cannot have $i < j < k < l < 3$, at least one of x_{ij}, y_{jk}, z_{kl} is white. If two of x_{ij}, y_{jk}, z_{kl} are white then it is easy to check that each side of (4.1) equals 1_{il}.

We are left with the three cases where exactly one of x_{ij}, y_{jk}, z_{kl} is white and none of them is below the identity. The first case is where $x_{ij} = \mathsf{w}_{ij},\ y_{jk} = \mathsf{b}_{jk},\ z_{kl} = \mathsf{c}_{kl}$, where $b, c \in \sqrt{A}$ and $j < k < l$, hence $j = 0,\ k = 1,\ l = 2$. If $i > 0$ then $A_{i0} = 0$ so

$$\begin{aligned}
(\mathsf{w}_{i0} ; \mathsf{b}_{01}) ; \mathsf{c}_{12} = (1_{i1} - A_{i0} ; \mathsf{b}_{01}) ; \mathsf{c}_{12} = 1_{i1} ; \mathsf{c}_{12} = 1_{i2} &\qquad \text{(by (5))} \\
= 1_{i2} - A_{i0} ; (b * c)_{02} &\qquad (A_{i0} = 0) \\
= \mathsf{w}_{i0} ; (b * c)_{02} &\qquad \text{(by (5))} \\
= \mathsf{w}_{i0} ; (\mathsf{b}_{01} ; \mathsf{c}_{12}) &\qquad \text{(by (3))}.
\end{aligned}$$

If $i = 0$ then $A_{00} = \mathsf{e}_{00}$ so the left hand side of (4.1) is

$$
\begin{aligned}
(\mathsf{w}_{00} ; \mathsf{b}_{01}) ; \mathsf{c}_{12} &= (1_{01} - A_{00} ; \mathsf{b}_{01}) ; \mathsf{c}_{12} && \text{(by (5))}\\
&= (1_{01} - \mathsf{b}_{01}) ; \mathsf{c}_{12} && (A_{00} = \mathsf{e}_{00})\\
&= (\mathsf{w}_{01} + (A_{01} - \mathsf{b}_{01})) ; \mathsf{c}_{12} && (1_{01} = \mathsf{w}_{01} + A_{01})\\
&= \mathsf{w}_{01} ; \mathsf{c}_{12} + \sum_{b' \neq b,\ b' \in \sqrt{(A)}} (b'_{01} ; \mathsf{c}_{12}) && \text{(additivity)}\\
&= \mathsf{w}_{01} ; \mathsf{c}_{12} + \sum_{b' \neq b,\ b' \in \sqrt{(A)}} (b' * c)_{02} && \text{(by (3))}\\
&= 1_{02} - (b * c)_{02} && \text{(by (3.1))}\\
&= 1_{02} - \mathsf{e}_{00} ; (b * c)_{02}\\
&= \mathsf{w}_{00} ; (\mathsf{b}_{01} ; \mathsf{c}_{12}) && \text{(by (5), } A_{00} = \mathsf{e}_{00}),
\end{aligned}
$$

the right hand side of (4.1), as required. The cases $y_{jk} = \mathsf{w}_{jk}$ and $z_{kl} = \mathsf{w}_{kl}$ are similar. This proves that **M** is associative, hence a boolean monoid.

Now to verify that **M** is normal. For $i, j < 3$ and any $x_{ij} \in M$ we have $1_{ii} \geq x_{ij} ; 1_{ji} \geq \mathsf{e}_{ii}$ (see definition of ; on **M**), hence $D(x_{ij}) = \mathsf{e}_{ii}$ and similarly $R(x_{ij}) = \mathsf{e}_{jj}$. Therefore $D(x_{ij}) ; x_{ij} = \mathsf{e}_{ii} ; x_{ij} = x_{ij} = x_{ij} ; \mathsf{e}_{jj} = x_{ij} ; R(x_{ij})$, so **M** is normal.

Finally, **M** is simple because every nontrivial congruence on a Boolean algebra identifies some nonzero element a with 0. But then as $1 ; a ; 1 = 1$ and $1 ; 0 ; 1 = 0$, every nontrivial congruence identifies the top element 1 with the bottom element 0. ⊣

The next lemma motivates the construction of $\mathbf{M}(\mathbf{A})$ from **A** and is crucial in proving the main results.

Lemma 4.2. *If* **A** *is embeddable in a group* **G** *then* $\mathbf{M}(\mathbf{A})$ *is isomorphic to a subalgebra of the complex algebra of the Brandt groupoid* $\mathsf{Br}_3(\mathbf{H})$ *for some group either equal to* **G** *or to* $\mathbf{G} \times \mathbf{G} \times \mathbf{G}$.

Proof. Say that **A** is embeddable in the group **G** (we take the embedding to be the inclusion map for simplicity). Without loss of generality, we may assume that $|G| > 2|A|$ (otherwise just replace **G** by $\mathbf{G} \times \mathbf{G} \times \mathbf{G}$) and that the multiplication in **A** is a restriction of that of **G**. The identity element of **G** is e. We now define a bijection between the atoms of $\mathbf{M}(\mathbf{A})$ and the elements of a partition of $\mathsf{Br}_3(\mathbf{G})$ and verify that the identity and composition operation are preserved.

The bijection, ι, is as follows.

- $\mathsf{a}_{ij} \mapsto \{(i, a, j)\}$,
- $\mathsf{e}_{ii} \mapsto \{(i, e, i)\}$,
- if $i > j$, then $\mathsf{w}_{ij} \mapsto \{i\} \times G \times \{j\}$,
- if $i = j$, then $\mathsf{w}_{ij} \mapsto \{i\} \times (G \backslash \{e\}) \times \{j\}$,
- if $j - i = 1$, then $\mathsf{w}_{ij} \mapsto \{i\} \times (G \backslash \sqrt{A}) \times \{j\}$,
- if $j - i = 2$, then $\mathsf{w}_{ij} \mapsto \{i\} \times (G \backslash A) \times \{j\}$.

It is clear that ι maps atoms of **M** to the elements of a partition of the underlying set of $\mathsf{Br}_3(\mathbf{G})$, hence can be extended to a Boolean algebra isomorphism from

$\mathbf{M}$ to the Boolean algebra of subsets of $\mathsf{Br}_3(\mathbf{G})$ generated by the blocks of the partition. Note that $\iota(1_{ij}) = \{i\} \times G \times \{j\}$. It is also trivial that $\iota(1') = \{(0, e, 0), (1, e, 1), (2, e, 2)\}$ of $\mathfrak{C}(\mathsf{Br}_3(\mathbf{G}))$, showing that $1'$ is preserved as a nullary. Thus it only remains to verify that composition is preserved.

Let x_{ij} and y_{jk} be atoms of $\mathbf{M}$. There are six parts to the definition of composition; cases ($1_;$)–($3_;$) are clearly preserved under the proposed embedding. Preservation of composition in cases ($4_;$) and ($5_;$) follows immediately from the easily verified group theoretic fact that for $g \in G$ and $P \subseteq G$, we have $g * P = G\backslash\ (g * (G\backslash P))$.

For the final case ($6_;$) of the definition of composition, note that $\iota(\mathsf{w}_{ij}) * \iota(\mathsf{w}_{jk}) \subseteq \{i\} \times G \times \{k\} = \iota(1_{ik})$ trivially. Conversely, let $(i, g, k) \in \iota(1_{ik})$. Since $|G| > 2|A|$, there exists $h \in G$ such that neither $g * h$ nor h^{-1} is contained in A. Then $(i, g * h, j) \in (\{i\} \times G\backslash A \times \{j\}) \subseteq \iota(\mathsf{w}_{ij})$ and $(j, h^{-1}, k) \in (\{j\} \times G\backslash A \times \{k\}) \subseteq \iota(\mathsf{w}_{jk})$, so that $(i, gh, j) * (j, h^{-1}, k) = (i, g, k) \in \iota(\mathsf{w}_{ij}) * \iota(\mathsf{w}_{jk})$, showing that $\iota(1_{ik}) \subseteq \iota(\mathsf{w}_{ij}) * \iota(\mathsf{w}_{jk})$ also. This completes the final check for preservation of ; and completes the proof. ⊣

§5. Undecidability of representability. Our main results will follow from embellishments of the following proposition. (Recall that square bracketed statements are a separate series of equivalences.)

Proposition 5.1. *Let* $\mathbf{A}$ *be a finite, cancellative, square partial group. The following are equivalent*:

1. $\mathbf{M}(\mathbf{A})$ *is representable* [*over a finite base set*];
2. $\mathbf{M}(\mathbf{A})$ *embeds into* $\mathrm{Bm}(X \times X)$ *for some* [*finite*] *set* X;
3. $\mathbf{M}(\mathbf{A})$ *is a subreduct of a* [*finitely*] *representable relation algebra*;
4. $\mathbf{M}(\mathbf{A})$ *is a subreduct of a* [*finite*] *relation algebra*;
5. $\mathbf{A}$ *is embeddable in a* [*finite*] *group.*

Proof. Implication (1) implies (2) follows from Lemma 2.8 and the fact that $\mathbf{M}(\mathbf{A})$ is simple and normal (Lemma 4.1). Implications (2) implies (3) and (3) implies (4) are trivial. Implication (5) implies (1) follows immediately from Lemma 4.2 and the fact that $\mathfrak{C}(\mathsf{Br}_3(\mathbf{H}))$ has a square representation (over a finite base set if and only if the group $\mathbf{H}$ is finite).

Now we prove the implication (4) implies (5). Assume that $\mathbf{M} = \mathbf{M}(\mathbf{A})$ is a sub-Boolean monoid of the Boolean monoid reduct of a relation algebra $\mathbf{R}$. We are going to show that $\mathbf{A}$ embeds into a subgroup of the ;-reduct of $\mathbf{R}$. (Whence if $\mathbf{R}$ is finite, then $\mathbf{A}$ embeds into a finite group.)

Let S denote the set

$$\{\mathsf{a}_{ij} \mid i, j \in 3, a \in \sqrt{A},\ \text{and}\ j - i = 1\} \cup \{\mathsf{b}_{02} \mid b \in A\}.$$

Claim 1. $S \subseteq \mathtt{i}^{\mathbf{M}}$.

Proof. For any $x_{ij} \in S$, we have from ($4_;$) that

$$(1_{ii} - \mathsf{e}_{ii}) \mathbin{;} x_{ij} = \mathsf{w}_{ii} \mathbin{;} x_{ij} = 1_{ij} - A_{ii} \mathbin{;} x_{ij} = 1_{ij} - \mathsf{e}_{ii} x_{ij} = 1_{ij} - x_{ij} \tag{5.1}$$

Suppose, for contradiction, that $x_{ij} \notin \mathtt{i}^{\mathbf{M}}$. Then by lemma 2.12 either $0' ; x_{ij} \geq x_{ij}$ or $x_{ij} ; 0' \geq x_{ij}$. In the former case, $(1_{ii} - \mathsf{e}_{ii}) ; x_{ij} \geq x_{ij}$, contradicting (5.1), and the latter case is also impossible, similarly. Hence $S \subseteq \mathtt{i}^{\mathbf{R}}$. ⊣

Now consider the subset

$$B := \{x \in R \mid x = 0 \text{ or } (\exists i, j \in 3)\, x ; x^{\smile} = \mathsf{e}_{ii} \;\;\&\;\; x ; x^{\smile} = \mathsf{e}_{jj}\}.$$

Claim 1 shows that $S \subseteq B$. As $\mathsf{e}_{ii} ; \mathsf{e}_{jj} \neq 0$ if and only if $i = j$, Lemma 2.15 shows that $\langle B, ;\rangle$ is a Brandt semigroup of dimension 3 (finite, if R is finite). Say, $\langle B, ;\rangle \cong \mathsf{B}_3(\mathbf{G})$ for some group $\mathbf{G}$ (finite if $\mathbf{R}$ is a finite relation algebra).

The remaining part of the argument is along similar lines to the general arguments in [17]; see in particular [17, Lemma 2.3] and [31, Proposition 3.1][1]. Let $\iota : \langle B, ;\rangle \to \mathsf{B}_3(\mathbf{G})$ be the isomorphism. The three nonzero idempotent elements of $\mathsf{B}_3(\mathbf{G})$ are $(0, e, 0)$, $(1, e, 1)$ and $(2, e, 2)$ and without loss of generality we may assume that $\mathsf{e}_{ii}^{\iota} = (i, e, i)$. Then by $(1_;)$, for all $x_{ij} \in S$ we have that $x_{ij}^{\iota} \in \{i\} \times G \times \{j\}$. This in turn enables the definition of three injective maps from A into $\mathbf{G}$ as follows. For $a \in \sqrt{A}$, define a^{α_1} to be the group coordinate of a_{01}^{ι} and a^{α_2} to be the group coordinate of a_{12}^{ι}. For $a \in A$ define a^{α_3} to be the group coordinate of a_{02}^{ι}. Because $|G| \geq |A|$, the maps α_1 and α_2 can be extended from domain $\sqrt{A}$ to injective maps from A into $\mathbf{G}$ (in an arbitrary fashion). Let us assume that we have made one such extension. We now show that $(\alpha_1, \alpha_2, \alpha_3)$ form a homotopy embedding of $\mathbf{A}$ into $\mathbf{G}$. Certainly all are injective. Now say that $a * b$ is defined in $\mathbf{A}$. So $a, b \in \sqrt{A}$ and $\mathsf{a}_{01}, \mathsf{b}_{12}$ are contained in S. Let $g, h \in G$ be the values of a^{α_1} and b^{α_2} respectively, so that $a^{\alpha_1} b^{\alpha_2} = gh$. Now $(a * b)^{\alpha_3}$ is the group coordinate of $(a * b)_{02}$. We have $(a * b)_{02}^{\iota} = \mathsf{a}_{01}^{\iota} \mathsf{b}_{12}^{\iota} = (0, g, 1) ; (1, h, 2) = (0, gh, 2)$. So $(a * b)_3^{\alpha} = gh$ as required. By Lemma 3.6 we have that $\mathbf{A}$ embeds into $\mathbf{G}$. This completes the proof of the implication (4) implies (5), whence the proof. ⊣

This proposition and Lemma 3.4 give the following corollary.

Corollary 5.2. *Let K be any class of finite Boolean monoids containing the class of finite Boolean monoids that have a finite square representation over a finite base set, and contained within the class of finite Boolean monoids that are subreducts of relation algebras. Then K is not recursive.*

Example 5.3.
1. *Let $\mathbf{A}$ be the partial group in Example 3.2. Then $\mathbf{M}(\mathbf{A})$ is an example of a normal Boolean monoid that is not a subreduct of any relation algebra.*
2. *Now let $\mathbf{A}$ be the square cancellative partial group shown to exist in Example 3.5. Then $\mathbf{M}(\mathbf{A})$ is an example of a finite Boolean monoid that is a subreduct of a relation algebra, but not of any finite relation algebra.*

§6. Undecidability of representability for reducts. In fact the proofs of the previous section continue to hold in a much restricted signature. The most common signatures considered from the perspective of "reducts of relation algebra" are subsets of $\{0, 1, +, \cdot, -, \leq, \smile, ;, 1'\}$. More generally though, one may

[1]There is an error in the proof of part (b) of [17, Lemma 2.3]. The corrected details can be found in [31].

fix any term t in the language of relation algebras, and consider the term reduct signatures containing t given the status of a fundamental operation (we've already done this in the case of $\cdot$ and the nullary term $0'$). Important examples of term functions are the operation of set subtraction $x\backslash y := x \cdot (-y)$ and its dual operation of implication $x \Rightarrow y := y + -x$, as well as the derived unary operations D and R considered in the present article.

Lemmas 2.4 and 2.10 immediately give us some undecidability results for some of these reducts, though we note that these will be subsumed by the results of Section 8 to follow.

THEOREM 6.1. *1. The representation problem is undecidable for any reduct of the relation algebra signature capable of expressing either* $\{x\cdot y+x\cdot z, ;, 1'\}$ *or* $\{(x+y)\cdot(x+z), ;, 1'\}$.
2. The finite representation problem is undecidable for any reduct of the Boolean monoid signature capable of expressing $\{x\cdot y+x\cdot z, ;, 1'\}$ *or* $\{(x+y)\cdot(x+z), ;, 1'\}$.
3. The subreduct problems for relation algebras and for finite relation algebras are undecidable for any reduct of the Boolean monoid signature capable of expressing $\{x\cdot y+x\cdot z, ;, 1'\}$ *or* $\{(x+y)\cdot(x+z), ;, 1'\}$.

PROOF. (1) Consider some reduct signature $\mathscr{F}$ expressing either $\{x\cdot y+x\cdot z, ;, 1'\}$ or $\{(x+y)\cdot(x+z), ;, 1'\}$. Let $\mathbf{R}$ be a relation algebra and let $\mathbf{R}^{\mathscr{F}}$ denote the $\mathscr{F}$-reduct of $\mathbf{R}$. If $\mathbf{R}$ is representable, then so is $\mathbf{R}^{\mathscr{F}}$. But by Lemma 2.10, if $\mathbf{R}^{\mathscr{F}}$ is representable then so is $\mathbf{R}$. The claim follows because representability is undecidable for finite relation algebras [21].

(2) has an almost identical proof using Lemma 2.4 and Corollary 5.2, while (3) follows similarly using Lemma 2.10 in place of Lemma 2.4. $\dashv$

We give three examples covered by Theorem 6.1.

EXAMPLE 6.2. *Representability and finite representability are undecidable for each of the signatures* $\{\cdot, +, ;, 1'\}$, $\{\backslash, ;, 1'\}$ *and* $\{\Rightarrow, ;, 1'\}$ (*where* $\backslash$ *denotes set subtraction, the term function* $x\backslash y := x\cdot -y$ *and* $\Rightarrow$ *is the term function of implication:* $x \Rightarrow y := -x+y$). *In each case, the subreduct problem is undecidable for relation algebras and for finite relation algebras.*

PROOF. For example, it is routine to verify that term function $x\cdot y+x\cdot z$ can be written in terms of $\backslash$ via $x\cdot y+x\cdot z = x\backslash[(x\backslash y)\backslash z]$. $\dashv$

Let $\sqsupset$ denote the binary relation of domain equivalence; in other words, if $r, s \in \wp(X\times X)$ are binary relations then $r \sqsupset s$ when $\{x\in X \mid (\exists y\in X)\,(x,y)\in r\} = \{x\in X \mid \exists y\in X\,(x,y)\in s\}$. An abstraction of the relation $\sqsupset$ can be defined on any normal Boolean monoid by $x \sqsupset y$ if $D(x) = D(y)$. (We will use the same notation for the relation-theoretic and the algebraic $\sqsupset$, as they coincide in the algebras of relations we encounter.) Dually, we can define $\sqcup$ to be the relation of range equivalence, which can also be defined abstractly using R on a Boolean monoid.

Recall that a map $\phi : \mathbf{A} \to \mathbf{B}$ between structures whose signature contains some n-ary relation r is said to *preserve* r if $(a_1,\dots,a_n)\in r^{\mathbf{A}}$ implies $(a_1^{\phi},\dots,a_n^{\phi})\in r^{\mathbf{B}}$. The map ϕ is an r-*embedding* if it is an injective r-preserving function such that $(a_1^{\phi},\dots,a_n^{\phi})\in r^{\mathbf{B}}$ implies $(a_1,\dots,a_n)\in r^{\mathbf{A}}$.

Now we look more closely at the proof of Proposition 5.1, and find a some further conditions that can be added to the list of equivalent properties. The main proof essentially hinges on the ability to define

- the relation $\mathtt{i}$ of Lemma 2.12,
- composition,
- the relation of domain equivalence and range equivalence (used in Lemma 2.15).

This is clarified in the proof of the next result.

PROPOSITION 6.3. *The following are equivalent for a square cancellative partial group* $\mathbf{A}$.

1. *There is a* $\{;, \mathtt{i}, \sqcap, \sqcup\}$*-embedding of* $\mathbf{M}(\mathbf{A})$ *into* $\mathrm{Rel}(X \times X)$ *for some set* X.
2. *There is a* $\{;\}$*-embedding of* $\mathbf{M}(\mathbf{A})$ *into the* $\{;\}$*-reduct of the representable relation algebra* $\mathrm{Rel}(X \times X)$ *that preserves the relations* $\mathtt{i}$, $\sqcap$ *and* $\sqcup$.
3. $\mathbf{A}$ *embeds into a group* $\mathbf{G}$.

Moreover, the set X *can be chosen to be finite if and only if the group* $\mathbf{G}$ *can be chosen to be finite.*

PROOF. (1) implies (2) is trivial, while (3) implies (1) is trivial once given Lemma 4.2. To prove (2) implies (3), assume that ϕ is a faithful representation of the semigroup $\langle M(\mathbf{A}), ;\rangle$ into $\langle \wp(X \times X), ;\rangle$ which additionally preserves the relations $\{\mathtt{i}, \sqcap, \sqcup\}$. The proof is now essentially the same as the proof of (4) implies (5) in Proposition 5.1, using the concrete relation algebra $\mathrm{Rel}(X \times X)$ instead of the abstract relation algebra $\mathbf{R}$. $\dashv$

As an example, consider the following "constrained representation problem" for semigroups, which we denote by $\mathsf{Rep}(;, \mathtt{i}, \mathtt{r})$.
INSTANCE: a finite semigroup $\mathbf{S}$ with two subsets I and R of S.
QUESTION: can $\mathbf{S}$ be faithfully represented as a semigroup of binary relations in such a way that the elements of I are injective functions and the elements in R are reflexive relations?

We let $\mathsf{Rep}_{\mathrm{fin}}(;, \mathtt{i}, \mathtt{r})$ denote the same problem, except that the semigroup representation is required to be over a finite base set.

THEOREM 6.4. *The problems* $\mathsf{Rep}(;, \mathtt{i}, \mathtt{r})$ *and* $\mathsf{Rep}_{\mathrm{fin}}(;, \mathtt{i}, \mathtt{r})$ *are undecidable.*

PROOF. Let $\mathbf{S}$ be the $\{;\}$-reduct of $\mathbf{M}(\mathbf{A})$, and choose I to be the relation $\mathtt{i}$ as defined in $\mathbf{M}(\mathbf{A})$ and let R be the set $\{m \mid m \geq 1'\}$. Now notice that if r is a reflexive relation and a, b are relations with $a ; r = b ; r$, then $a \sqcap b$, and dually $r ; a = r ; b$ implies $a \sqcup b$.

If $\mathbf{A}$ embeds under some map ϕ into a group (or a finite group), then $\mathbf{S}$ constitutes a YES instance of $\mathsf{Rep}(;, \mathtt{i}, \mathtt{r})$. Conversely though, say that $\mathbf{S}$ can be faithfully represented (as a semigroup) with the desired constraints on I and R being preserved. Thus, trivially, the relation $\mathtt{i}$ on $\mathbf{M}(\mathbf{A})$ is preserved. Now say that $x \sqcap y$ in $\mathbf{M}(\mathbf{A})$. So $x ; 1 = y ; 1$. Now $1 \in R$, so 1^ϕ is reflexive. Hence $x^\phi ; 1^\phi = y^\phi ; 1^\phi$, from which it easily follows that $D(x^\phi) = D(y^\phi)$. That is $x^\phi \sqcap y^\phi$. Similarly $\sqcup$ is preserved. By Proposition 6.3, the partial group $\mathbf{A}$ embeds into a group (finite, if $\mathbf{S}$ was represented over a finite base set). $\dashv$

A more restrictive formulation of the problem $\mathsf{Rep}(;,\mathtt{i},\mathtt{r})$ is to require that the semigroup $\mathbf{S}$ have a representation that is a $\{\mathtt{i},\mathtt{r}\}$-embedding (rather than just be $\{\mathtt{i},\mathtt{r}\}$-preserving). It is clear from the proof and Proposition 6.3 that this formulation is also undecidable.

Further variations are to replace the "reflexive relation" constraint in the problem $\mathsf{Rep}(;,\mathtt{i},\mathtt{r})$ by other constraints. For example, one may use "full relations"[2], "equivalence relations", or "symmetric relations", amongst others (for symmetric relation, one needs to make use of the fact that $1 ; a ; 1 = 1$ for all $a \neq 0$. Alternatively one may simply require that some designated element (namely, the element 1 of $\mathbf{M}(\mathbf{A})$) be represented as the universal relation: essentially the square representation problem for $\{1, ;, \mathtt{i}\}$. In each such case there is a corresponding undecidability result.

A similar result can be obtained for ordered semigroups.

THEOREM 6.5. *There is no algorithm to decide if an arbitrary finite ordered semigroup $\langle S, \leq\rangle$ with distinguished subset I can be faithfully represented as an ordered semigroup of binary relations in such a way that the elements of I are injective functions. The same is true if one requires representability over a finite base set.*

PROOF. Let $\mathbf{S}$ be the $\{;\}$-reduct of $\mathbf{M}(\mathbf{A})$, and define $\leq$ to be the boolean ordering on $\mathbf{M}(\mathbf{A})$ and $I := \mathtt{i}^{\mathbf{M}(\mathbf{A})}$. If $\mathbf{A}$ embeds into a group then $\mathbf{S}$ is a YES instance of the decision problem. For the converse, let $\phi : S \to \wp(X \times X)$ be an injective map preserving ;, embedding $\leq$ and with $s \in I$ implies $s^\phi \in \mathtt{i}$. Now, $1' \in I$ and is ;-idempotent, which forces $(1')^\phi$ to be a restriction of the identity map on X. Also, as $1'$ is a ;-identity element for $\mathbf{S}$ we may further assume that $(1')^\phi$ is the identity map on X (otherwise, we may represent $\mathbf{S}$ over the domain of $(1')^\phi$). Now follow the proof of Theorem 6.4 using the fact that the reflexive elements are exactly those ordered above $1'$. ⊣

The restriction of this problem to the case where I is empty corresponds to the problem of deciding representability of ordered semigroups. Zareckii [62] proved that every ordered semigroup is isomorphic to an ordered semigroup of binary relations (over a finite base set, if the semigroup is finite). At the other extreme, the case where I is forced to be all of S corresponds to the problem of deciding representability of ordered semigroups as ordered semigroups of injective functions. Schein [48] (see also [52]) showed that an ordered semigroup is isomorphic to an ordered semigroup of injective functions (over a finite base set if the semigroup is finite) if and only if it satisfies $xv \geq z \ \& \ uv \geq z \ \& \ uy \geq z \to xy \geq z$.

We finish this section with a sample of further undecidability results using $\mathbf{M}(\mathbf{A})$ (many other variations can be found as well).

THEOREM 6.6. *The following problems are undecidable.*

1. *Given a finite $\{\cdot\}$-semilattice ordered semigroup $\mathbf{S} = \langle S, ;, \cdot\rangle$ with distinguished element $0' \in S$, decide if $\langle S, ;, \cdot, 0'\rangle$ is isomorphic to a $\cap$-ordered semigroup of binary relations [over a finite domain] in which $0'$ is represented as the diversity element.*

[2]Recall that a binary relation a is said to be *full* if it $D(a) = R(a) = \mathrm{id}$.

2. *Given an ordered monoid* $\mathbf{S} = \langle S, ;, \leq, 1' \rangle$ *and a binary operation* $-$ *on* S, *decide if* $\langle S, ;, -, \leq, 1' \rangle$ *is isomorphic to an ordered monoid of binary relations* (*on the base set* X *say*) *in which* $-$ *is preserved as complementation in* $X \times X$.

Proof. We give only a sketch in each case. Also, in each case we start by taking $\mathbf{S}$ to be the suitable reduct of $\mathbf{M}(\mathbf{A})$. Thus, if $\mathbf{A}$ is embeddable into a group (or finite group), then the corresponding construction of $\mathbf{S}$ will be a YES instance of the decision problem (because $\mathbf{M}(\mathbf{A})$ has a square representation as a Boolean monoid). Thus in each case it suffices to assume that $\mathbf{S}$ has been represented in the desired fashion and then deduce that $\mathbf{M}(\mathbf{A})$ is representable as a Boolean monoid (whence $\mathbf{A}$ is embeddable in a group, or even a finite group).

(1) The main argument will require that the element 0 of $\mathbf{M}(\mathbf{A})$ is correctly represented as the empty relation, so we show this first. Let ϕ denote the representation of $\mathbf{S}$ into $\mathrm{Bm}(X \times X)$ (that is, an injective map from $\mathbf{M}(\mathbf{A})$ into $\wp(X \times X)$ preserving $\{\cdot, ;, 0'\}$). Assume, for a contradiction that there are $x, y \in X$ with $(x, y) \in 0^\phi$. Now as $0 \cdot 0' = 0$, we have that $0^\phi \cap (0')^\phi = 0^\phi$, and as $0'$ is the relation of inequality on X, we have that $x \neq y$. Hence $(y, x) \in (0')^\phi$ showing that $(x, x) \in 0^\phi ; (0')^\phi = 0^\phi \subseteq (0')^\phi$, contradicting our assumption that $0'$ is represented as the diversity relation. Hence $0^\phi = \varnothing$.

Lemma 2.12 shows that relation $\mathtt{i}$ can be defined using only $\cdot, ;, 0', 0$, while if $0'$ is correctly represented, then $1 = 0' ; 0'$ also is represented correctly. Hence $\sqcap$ and $\sqcup$ are preserved because in $\mathbf{M}(\mathbf{A})$ we have $x \sqcap y$ if and only if $x ; 1 = y ; 1$ (and dually for $\sqcup$).

(2) Note that as $0'$ is the complement of $1'$ it will be correctly represented. Now we may define $\mathtt{i}$ by adjusting the property in Lemma 2.12 as follows: $x \in \mathtt{i}$ if and only if $-x \geq x ; 0'$ & $-x \geq 0' ; x$. Now we are in the situation of Theorem 6.5. ⊣

The first undecidability result in this theorem provides a quite a sharp boundary between decidability and undecidability because Bredikhin and Schein [8] showed that every semilattice ordered semigroup is isomorphic to an $\cap$-ordered semigroup of binary relations. Later, Bredikhin [7] gave finite axiomatisations for the representable algebras in the signature $\{\cdot, ;, 1\}$ and $\{\cdot, ;, 0, 1\}$ (where 1 is to be represented as the universal relation). A problem related to item (2) is raised in Schein's [53], where a characterisation of ordered semigroups of binary relations with complement is sought.

Many applications of algebras of relations incorporate the unary operation of reflexive transitive closure; see Pratt [45] or Möller and Struth [43] as a small sample. We use the notation r^* to represent the reflexive transitive closure of the relation r. Unlike for other operations considered in this article, the reflexive transitive closure of a relation r is not definable from r in the first order theory of binary relations, though it can be written as an infinite union:

$$r^* = \bigcup_{i \in \omega} r^i, \tag{6.1}$$

where $r^0 := 1'$ and $r^{i+1} = r ; r^i$. From the perspective of axiomatisation, this undefinability opens the door to extreme complexity: for example the universal

Horn theory of algebras of relations in the signature $\{+, 0, ;, 1', {}^*\}$ is Π^1_1-complete, hence not recursively enumerable (see Hardin and Kozen [18]). However the issue of complexity of representability takes as its instance a finite algebra, and in such cases the infinite union of Equation 6.1 can be expressed as a finite union, because the ;-reduct will be a periodic semigroup (so that $r^n = r^m$ for some $n \neq m$). In the case of the construction $\mathbf{M}(\mathbf{A})$ for example, it is routine to verify that each element x satisfies $x\,;x\,;x = x\,;x$, whence x^* can be expressed as $1' + x + x\,;x$. A similar statement holds for the tiling algebras of Hirsch and Hodkinson, or in fact any finite construction whose signature can be extended to include $\{+, ;, 1'\}$ without affecting representability. Because of this we can state the following meta-theorem.

THEOREM 6.7. *Results in the present article that concern undecidability of representability or of finite representability for signatures $\mathscr{F}$ as algebras of arbitrary binary relations can be extended to corresponding results for the signature $\mathscr{F} \cup \{{}^*\}$.*

Expressed differently, Theorem 6.7 applies to all undecidability results of the article except for those of the next section (where only very specific kinds of binary relations are allowed).

§7. Representability problems for algebras of injective partial functions. We let $\mathcal{I}(X)$ denote the set of all injective partial maps on the set X. Problems concerning the representation of various kinds of enriched semigroups as algebras of injective partial functions have been widely studied. Probably the most important of the variations that have been considered is the class of inverse semigroups, which correspond to the signature $\{\circ, {}^{-1}\}$. However many other signatures have been considered. Even the plain semigroup theoretic signature $\{;\}$ (with or without nullaries for $\varnothing$ and Δ) is interesting and nontrivial. This was first characterised by Schein [49] who showed it to be a nonfinitely axiomatisable quasivariety (see [55] for discussion of the history, and Jackson and Volkov [32] for a very elementary proof of the nonfinite basis property). The signature $\{;, \leq\}$ (with or without constants for 0 and $1'$) was encountered above (after Theorem 6.5) and gives rise to a finitely based quasivariety [48]. Now define relations $\ulcorner$ and $\llcorner$ on $\mathcal{I}(X)$ (or indeed, on $\wp(X \times X)$) by $f \ulcorner g$ if the domain of f is a subset of the domain of g. Define $\llcorner$ similarly in terms of range. (Note for example, that $f \sqcap g$ if and only if $f \ulcorner g$ and $g \ulcorner f$.) In Schein [52], results are surveyed for signatures that include ; with various combinations of $\ulcorner$, $\llcorner$ and $\leq$. In [54], Schein characterises semigroups of injective functions with set subtraction $\setminus$; they are a finitely based variety. The elements $0, 1$ can also be represented correctly (see Stokes [57] for example). Other operations that have been considered are

- intersection $\cdot$,
- the unary operations of domain D and range R (used extensively in the present article),
- the unary operation of *domain complement* P given by $P(f) = \mathrm{id} \cap (-D(f))$,
- the unary operation of *range complement* Q given by $Q(f) = \mathrm{id} \cap (-R(f))$,

- the binary operation $\ltimes$ of *first restrictive multiplication*, given by $f \ltimes g = D(f)g$, and
- the binary operation $\ltimes$ of *second restrictive multiplication*, given by $f \rtimes g = fR(g)$.

For example, the class of unary semigroups representable as semigroups of injective functions with domain are precisely the class of *right type A semigroups* (now called *left ample semigroups*) in the sense of Fountain [12] (the R case corresponds to right ample semigroups). The operations P and Q arise in the study of modal operators. Indeed algebras of relations in the signature $\{;, P\}$ essentially correspond to semigroups of modal operators (while Q corresponds to backward modal operators; see [10] and [43] for example). Restricting to injective functions corresponds to restricting the allowed modal systems. The operations $\ltimes$ and $\rtimes$ are considered by Schein in [50] for example[3].

Aside from $0, 1, P, Q, \backslash, \cdot$, all of the operations and relations just discussed can be defined using inverse and composition: for example, $D(x) = x \,;\, x^{-1}$. Inverse semigroups endowed with most of the combinations of $0, 1, P, Q, \backslash, \cdot$ have been characterised in the literature discussed above and all are finitely based varieties.

Let us divide the above operations and relations into three groupings as follows.

- Domain constructions: $\{P, D, \ltimes, \ulcorner, \sqcap\}$.
- Range constructions: $\{Q, R, \rtimes, \llcorner, \sqcup\}$.
- Other: $\{;, \backslash, \cdot, \leq, 0, 1\}$.

Lemma 7.1. *The relation $\sqcap$ is definable by a conjunction of atomic formulas in any signature containing a domain construction and ;. The relation $\sqcup$ is definable by a conjunction of atomic formulas in any signature containing a range construction and ;.*

Proof. This is obvious. For example $x \sqcap y$ if and only if $x \ulcorner y \;\&\; y \ulcorner x$ if and only if $x \ltimes y = y \;\&\; y \ltimes x = x$, and so on. In fact it is easy to see that in the presence of ;, P defines D, which defines $\ulcorner$ which defines $\sqcap$. $\dashv$

The combination $\{\subseteq, \ulcorner, \llcorner\}$ is mentioned as unsolved in [51] (see page 42), while the combination $\{\ulcorner, \llcorner\}$ is noted as unsolved in Schein [52]. In fact these and other articles reveal that most of the combinations (of the operations and relations above) that are *not* covered by Theorem 7.2 (that is, where not both domain- and range-related constructions are included) have been characterised.

The signatures between $\{;\}$ and $\{;, \leq, 0, 1'\}$ follow from Schein [48], while those under $\{;, \cdot, \leq, 0, 1'\}$ can be found in Garvackii [14]. Signatures between $\{;, \backslash\}$ and $\{;, \backslash, \cdot, \leq, 0, 1'\}$ were characterised by Schein [54]. Semigroups of injective functions with D were mentioned above (the left ample semigroups), while the signatures $\{;, P\}$ and $\{;, P, \backslash\}$ (possibly with some combination of 0, $1'$) have been characterised in a forthcoming article by the second author and Stokes [30].

Gould and Kambites [16] showed that representability and finite representability as injective functions are undecidable in the signature $\{;, D, R\}$ (representability is not explicitly addressed in [16]; the easy connecting details are

[3]Schein uses the notation $\rhd$ instead of $\ltimes$, however this notation has a different meaning in commonly encountered relation algebra literature.

discussed in [29, Theorem 6.2]). The main result of this section is an extension of this result.

THEOREM 7.2. *Let $\mathscr{F}$ be any combination of the above operations and relations that contains composition, at least one domain construction and at least one range construction. Then the representability and finite representability problems for $\mathscr{F}$-algebras of injective functions is undecidable.*

PROOF. We again base our arguments around $\mathbf{M}(\mathbf{A})$, however this time we cannot use it directly, as even the $\{;\}$-reduct of $\mathbf{M}(\mathbf{A})$ is not representable as a semigroup of injective functions. Instead we consider the subalgebra of $\mathbf{M}(\mathbf{A})$ generated by the elements

$$S(\mathbf{A}) := \{\mathsf{a}_{01}, \mathsf{a}_{12} \mid a \in \sqrt{A}\} \cup \{\mathsf{b}_{02} \mid b \in A\}$$

using only the operations and relations of $\mathscr{F}$. Let us denote this subreduct of $\mathbf{M}(\mathbf{A})$ by $\mathbf{S}_{\mathscr{F}}(\mathbf{A})$. Observe that if $\mathscr{F} \subseteq \{;, \cdot, 0, \leq, \ltimes, \rtimes, \sqcap, \sqcup, \ulcorner, \llcorner\}$, then $\mathbf{S}_{\mathscr{F}}(\mathbf{A})$ consists of the just the elements of $S(\mathbf{A})$ and 0 and is a 3-nilpotent semigroup (the composition of any three elements equals 0).

Now, if $\mathbf{A}$ embeds into a group (or finite group), then $\mathbf{M}(\mathbf{A})$ is isomorphic to a sub-Boolean monoid of $\mathrm{Bm}(X \times X)$ for some set X (or finite set, respectively). Moreover, the elements of $S(\mathbf{A})$ are represented as injective functions, as they are in the $\mathtt{i}$ relation on $\mathbf{M}(\mathbf{A})$. Now, as injective functions are closed under the operations of $\mathscr{F}$,it follows that this representation is an $\mathscr{F}$-embedding of $\mathbf{S}_{\mathscr{F}}(\mathbf{A})$.

Conversely, say that $\mathbf{S}_{\mathscr{F}}(\mathbf{A})$ is isomorphic to a substructure of $\mathcal{I}(X)$ with the usual interpretations of $\mathscr{F}$. Lemma 7.1 shows that without loss of generality we may add $\sqcap$ and $\sqcup$ to the signature $\mathscr{F}$. Consider now $\mathrm{Rel}(X \times X)$. We have a representation of $\mathbf{S}_{\mathscr{F}}(\mathbf{A})$ into the reduct of $\mathrm{Rel}(X \times X)$ to the operations in $\mathscr{F}$, in which all of the elements of $S_{\mathscr{F}}(\mathbf{A})$ have been represented inside the relation $\mathtt{i}$ as defined in $\mathrm{Rel}(X \times X)$ (as this is exactly the set $\mathcal{I}(X)$). Also for $i < j \leq 2$ and $k < \ell \leq 2$ we have that $\mathsf{a}_{ij} \sqcap \mathsf{b}_{k\ell}$ if and only if $i = k$, and similarly, $\mathsf{a}_{ij} \sqcup \mathsf{b}_{k\ell}$ if and only if $j = \ell$. If we consider any $a, b \in \sqrt{A}$ then the two elements a_{01} and b_{12} are easily seen to satisfy the conditions of Lemma 2.15. Hence we can again follow the argument given in Proposition 5.1 to show that $\mathbf{A}$ embeds into a $\{;\}$-subgroup of $\mathrm{Rel}(X \times X)$. ⊣

We mention that Theorem 6.7 applies only vacuously to Theorem 7.2, as the signatures here do not include union.

§8. Starting from tilings.

The methods developed in the present article are unable to cover signatures involving the operation $^\smile$. On the other hand, the construction used by Hirsch and Hodkinson [21] to prove the undecidability of representability for Tarski relation algebras makes essential use of converse. We now revisit that construction to establish the following result (the proof covers the remainder of the section).

THEOREM 8.1. *Let $\mathscr{F}$ be a relation algebra reduct containing $\{\cdot, ^\smile, ;\}$. Then the representability problem is undecidable for $\mathscr{F}$.*

The signatures $\{\cdot, ^\smile, ;\}$ and $\{\cdot, ^\smile, ;, 1'\}$ were studied and characterised by Jónsson in [33] and are sometimes referred to as *relation algebras in the sense of Jónsson*. Undecidability of finite representability for these signatures remains open.

Theorem 8.1 also presents something close to a boundary between decidability and undecidability of representability: Bredikhin [6] showed that the class of representable algebras in the signature $\{;, ^\smile, D, R, \leq\}$ is finitely axiomatisable, whence has decidable membership.

The notion of an *allegory* was introduced by Freyd and Scedrov [13] to provide a category-theoretic model of relations in a signature essentially the same as that of Jónsson: along with the usual objects and morphisms of a category, there is a binary operation of intersection and a unary operation of converse. The motivating case is the allegory $\mathrm{Rel}(\mathscr{S})$, where the objects are sets, morphisms are binary relations between sets, and intersection and converse have their usual interpretation. The following is an immediate corollary of Theorem 8.1 and the fact that an algebra of relations in the signature $\{\cdot, ^\smile, ;, 1'\}$ forms a one-object allegory.

Corollary 8.2. *There is no algorithm to decide if a finite one-object allegory embeds into the allegory* $\mathrm{Rel}(\mathscr{S})$.

Representability of one-object allegories is considered in [13, II.2.158].

8.1. Deterministic tiling problem. The argument of [21] uses an undecidable tiling problem. Our revisit of this argument will require a slight refinement of the standard tiling undecidability results.

Let us consider a *set of tiles* to be a family τ of symbols (which we will think of as representing squares of unit edge length and refer to as "tiles", with upper, lower, left and right edges) and two binary relations $\sim_L$ and $\sim_U$ on τ: these corresponding to the allowed neighbourings of tiles; thus we think of $s \sim_L t$ as indicating that tile s can be placed to the left of tile t, while $s \sim_U t$ indicates that tile s can be placed underneath (below) t. A natural geometric consistency condition that can be imposed is

$$(s_1 \sim_L t_1 \ \ \& \ \ s_2 \sim_L t_1 \ \ \& \ \ s_2 \sim_L t_2) \to s_1 \sim_L t_2$$

and analogously for $\sim_U$. These conditions will hold for the tiling instances we construct.

Let us interpret the relations $\sim_L$ and $\sim_U$ on the set $\mathbb{Z} \times \mathbb{Z}$ by letting $(i, j) \sim_L (i+1, j)$ and $(i, j) \sim_U (i, j+1)$. A *tiling of the plane* is a map γ from $\mathbb{Z} \times \mathbb{Z}$ to τ preserving the two relations $\sim_L$ and $\sim_U$. A *partial tiling* is a homomorphism γ from an induced substructure of $\langle \mathbb{Z} \times \mathbb{Z}, \sim_L, \sim_U \rangle$ into τ.

An instance of the *tiling completion problem* takes as its instance a set of tiles τ including one distinguished tile[4] t. This instance is a YES instance provided that there is a tiling γ with $\gamma(0, 0) = t$. The undecidability of the tiling completion problem is well known and has a quite transparent proof based on interpreting the undecidability of the halting problem for Turing machine programs started on blank tapes. The argument in Robinson [47, §4] is one example.

[4]More generally the problem starts with some finite partial tiling by τ, but in the present case we want the single distinguished tile.

We now describe a restriction of this problem.

Definition 8.3. *A set of tiles τ with distinguished tile t is* deterministic *if there is a sequence of triples $(u_0, v_0, t_0), (u_1, v_1, t_1), \ldots$ in $\mathbb{Z} \times \mathbb{Z} \times \tau$ with the following properties.*

- $t_0 = t$, $u_0 = v_0 = 0$ *(tile t is placed at $(0, 0)$).*
- $\{(u_i, v_i) : i < \omega\} = \mathbb{Z} \times \mathbb{Z}$ *(the sequence of coordinates covers the plane).*
- *For each $i > 0$,* if *the sequence $(u_0, v_0, t_0), (u_1, v_1, t_1), \ldots, (u_i, v_i, t_i)$ is a partial tiling, then either*
  (a) *there is no tile in τ that can be placed at position (u_{i+1}, v_{i+1}) such that $\{(u_j, v_j, t_j) : j \leq i+1\}$ is a partial tiling (in this case the value of t_{i+1} is unrestricted), or*
  (b) *there is a unique tile t' that can be placed at (u_{i+1}, v_{i+1}) such that $\{(u_j, v_j) : j \leq i+1\}$ is a partial tiling, and $t_{i+1} = t'$.*

Note that if τ tiles the plane and is deterministic then the described sequence forces the tiling to be unique. Note also that we do not need to know the tiling sequence in advance, just that it exists. We show that there is a reduction of the halting problem to the tiling completion problem in which the constructed tiling instances are provably deterministic.

Theorem 8.4. *The deterministic tiling problem is not decidable.*

Proof. We prove the theorem by reducing the *Empty Input Non-Halting Problem* (EINHP) to the deterministic tiling problem. Let $T = (Q, \Sigma, \delta, q_0, F)$ be an instance of EINHP, so Q is a finite set of states, Σ is a finite alphabet, $\delta : ((Q \setminus F) \times \Sigma) \to (Q \times \Sigma \times \{L, R\})$ is the transition function, $q_0 \in Q$ is the start state and $F \subseteq Q$ is the set of final states. T is a yes-instance of EINHP if T never enters a final state when started in q_0 on a blank, two-way infinite tape, it is a no-instance if eventually T enters a final state.

We reduce T to an instance (τ, t) of the determinstic tiling problem, where the set of tiles τ is defined to be

$$(Q \times \Sigma \times H \times V) \ \cup \ \{B\}$$

where $H = \{\Rightarrow, \rightarrow, \perp, \leftarrow, \Leftarrow\}$, $V = \{\mathsf{Init}, \mathsf{Later}\}$ and $t = (q_0, \mathsf{blank}, \perp, \mathsf{Init})$. It may be helpful to think of the tile (q, s, h, v) as representing a cell in a configuration of T where the current state is q, s is in the current cell, h is $\Rightarrow, \rightarrow, \perp, \leftarrow, \Leftarrow$, respectively, means that the tape head is at least two to the right, immediately to the right, at the same position, immediately to the left and at least two to the left of the current cell and v is Init in the initial configuration but $v = \mathsf{Later}$ in all subsequent configurations. The tile B will be used to cover cells in the lower half plane. If $q \in F$ then there are no tiles that can be placed above (q, s, h, v).

To formalise this, we must define the adjacencies.

Horizontal adjacency:

(H0) If $x, y \in \tau$ and $x \sim_L y$ then either $x = y = B$ or $B \notin \{x, y\}$.
Furthermore, if $(q, s, h, v) \sim_L (q', s', h', v')$ then
(H1) $q = q'$ and $v = v'$,
(H2) $(h, h') \in \{(\Rightarrow, \Rightarrow), (\Rightarrow, \rightarrow), (\rightarrow, \perp), (\perp, \leftarrow), (\leftarrow, \Leftarrow), (\Leftarrow, \Leftarrow)\}$, and
(H3) if v is Init then $s = s' = \mathsf{blank}$.

Vertical adjacency:

(V0) If $x \sim_U y$ then $x = B \iff y = B$ or $\exists q', s', h'$ such that $y = (q', s', h', \mathsf{Init})$.

Furthermore, if $(q, s, h, v) \sim_U (q', s', h', v')$ then

(V1) $v' = \mathsf{Later}$,

(V2) If $h \neq \perp$ then $s = s'$,

(V3) If $h = \perp$ then either (a) $\delta(q, s) = (q', s', R)$ and $h' = \rightarrow$ or (b) $\delta(q, s) = (q', s', L)$ and $h' = \leftarrow$.

This defines the reduction.

We must show that (τ, t) is deterministic. Formally, this should involve the specification of a sequence $((u_0, v_0, t_0), (u_1, v_1, t_1), \dots)$ as above. However, we do not specify such a sequence here, instead we sketch how to determine each tile in the plane and leave out the details of the exact order that this should be done.

First observe that if $t = (q_0, \mathsf{blank}, \perp, \mathsf{Init})$ is placed at $(0, 0)$ then by (V0) only B may be placed at $(0, -1)$ and it is easily seen by repeated use of (H0) and (V0) that only B can be placed at any position in the lower half plane. Furthermore, by (H1), (H2) and (H3), the only tile that can be placed at $(1, 0)$ is $(q_0, \mathsf{blank}, \leftarrow, \mathsf{Init})$ and the only possible tile that can be placed at $(x, 0)$ for $x > 1$ is $(q_0, \mathsf{blank}, \Leftarrow, \mathsf{Init})$. Similarly, tiles placed at $(x, 0)$ for $x < 0$ are determined, using (H1), (H2) and (H3).

Suppose now, for some $y \geq 0$, that we have determined a tile $(q, s_0, \perp, v)$ at (x, y) (some $x \in \mathbb{Z}$) and we have also determined tiles at $(x-1, y), (x+1, y)$. By (H1) and (H2) the tile at $(x-1, y)$ must be $(q, s_{-1}, \rightarrow, v)$ (some s_{-1}) and the tile at $(x+1, y)$ must be $(q, s_1, \leftarrow, v)$ (some s_1). If $q \in F$ then by (V3) there is no tile that can be placed at $(x, y+1)$. If $\delta(q, s_0) = (q', s'_0, R)$ then the tile at $(x, y+1)$ is determined to be $(q', s'_0, \rightarrow, \mathsf{Later})$ by (V1) and (V3(a)), and in this case the tiles at $(x-1, y+1), (x+1, y+1)$ are forced to be $(q', s_{-1}, \Rightarrow, \mathsf{Later}), (q', s_1, \perp, \mathsf{Later})$, respectively, by (H0), (H1), (H2) and (V2). Similarly, if $\delta(q, t) = (q', t', L)$ then the tiles at $(x-1, y+1), (x+1, y+1)$ are determined. And provided there is a tile, say $(q', s_2, \Leftarrow, v)$, at $(x+2, y)$, the tile at $(x+2, y+1)$ is determined (if $\delta(q, s_0) = (q', s'_0, R)$ then it must be $(q', s_2, \leftarrow, \mathsf{Later})$ and if $\delta(q, s_0) = (q', s'_0, L)$ it must be $(q', s_2, \Leftarrow, \mathsf{Later})$) by (H0), (H1), (H2) and (V2). In this way we can tile as much as we like of row $y + 1$.

This shows that (τ, t) is an instance of the deterministic tiling problem. To see that the reduction is correct, observe that in any tiling, for fixed $y \in \mathbb{Z}$, the tiles at $((x, y) : x \in \mathbb{Z})$ form the configuration of the Turing machine T after y steps. Let q be the state of this configuration. We cannot have $q \in F$ as there is no tile that could be placed above such a tile, hence T never halts. Conversely, if T never halts then the tiling procedure above can be continued to give a tiling of the entire plane. This shows that the reduction is correct. Since EINHP is not recursive, the theorem follows. $\dashv$

8.2. The Tiling Algebra. The following tiling algebra appeared in [21], or see [22, chapter 18]. Given an instance (τ, T^0) of the tiling problem we construct a finite relation algebra $\mathsf{RA}(\tau, T^0)$.

The Atoms. If (τ, T^0) is a tiling instance with k tiles including a specified tile $T^0 \in \tau$ then $\mathsf{RA}(\tau, T^0)$ has $2k+28$ atoms. The elements of $\mathsf{RA}(\tau, T^0)$ are then arbitrary sums of these atoms, so $|\mathsf{RA}(\tau, T^0)| = 2^{2k+28}$. The atoms are

start	end	Atoms
0	0	e_0, w_0
0	1	$g_{01}, u_{01}, v_{01}, w_{01}$
0	2	$g_{02}, u_{02}, v_{02}, w_{02}$
1	1	$e_1, +1_1, -1_1, w_1$
2	2	$e_2, +1_2, -1_2, w_2$
1	2	$T_{12}\,(T \in \tau),\ w_{12}$

plus the converses of the $(0-1), (0-2)$ and $(1-2)$ atoms. If $i, j < 3$, $i \neq j$, and a_{ij} is any $(i$–$j)$ atom, we write a_{ji} for $\breve{a}_{ij}$. Thus, the converse of g_{02} is g_{20}. We consider some of the atoms to be coloured: the atoms g_{01}, g_{10}, g_{02}, and g_{20} are green, and the atoms $w_0, w_1, w_2, w_{01}, w_{02}, w_{12}$ and their converses are white.
The Atom Structure. To define the relation algebra $\mathsf{RA}(\tau, T^0)$ it remains to define the operations of converse and composition on the atoms. The operations on arbitrary elements are then defined by distribution over disjunction: see [38]. The identity is $e_0+e_1+e_2$. For converse, we have already defined the converse of atoms with distinct subscripts. All the rest are self-converse except the following: the converse of $+1_1$ is -1_1 and the converse of $+1_2$ is -1_2, and vice versa.

Now we define composition. We do this by listing the inconsistent triangles (a, b, c) of atoms. This is defined to mean that $a\,;b \cdot c = 0$. Recall that the *Peircean transforms* of the triangle (a, b, c) are (a, b, c), $(b, \breve{c}, \breve{a})$, $(\breve{c}, a, \breve{b})$, $(\breve{b}, \breve{a}, \breve{c})$, $(c, \breve{b}, a)$, $(b, \breve{c}, \breve{a})$. By Lemma 2.5, it follows from the inconsistency of (a, b, c) that its Peircean transforms must also be inconsistent. The following triangles, plus all Peircean transforms of them, are defined to be inconsistent. Firstly, any triangle where the indices do not match is inconsistent: e.g., (x_{ij}, y_{kl}, a) and (x_j, y_{kl}, a) are inconsistent if $j \neq k$, for any atom a. Secondly, a triangle (e_i, x, y) is inconsistent unless $x = y$. Thirdly, the following are all inconsistent:

(8.1) (g_{10}, g_{02}, w_{12})

(8.2) $(T_{12}, T'_{21}, +1_1)$ any $T, T' \in \tau$, unless $T \sim_L T'$

(8.3) (u_{10}, g_{02}, T_{12}) any $T \in \tau \setminus \{T^0\}$

(8.4) $(v_{10}, g_{01}, \pm 1_1)$

There are three dual rules for inconsistent triangles, obtained from rules 8.2, 8.3 and 8.4 by swapping the subscripts 1 and 2 throughout and replacing $\sim_L$ by $\sim_U$, respectively. We will refer to these inconsistent triangles by "rules 8.1 to 8.4". We make no use of the atoms u_{01}, v_{01} in the following, nor of inconsistency rules 8.3, 8.4. They are used in [21] to prove that a (τ, T^0)-tiling implies the representability of $\mathsf{RA}(\tau, T^0)$, but fortunately we can do without them here.

All other triangles are defined to be consistent. This suffices to define composition.

THEOREM 8.5. [21] *Let (τ, T^0) be an instance of the tiling completion problem. Then $\mathsf{RA}(\tau, T^0)$ is a representable relation algebra if and only if (τ, T^0) is a yes instance of the tiling completion problem.*

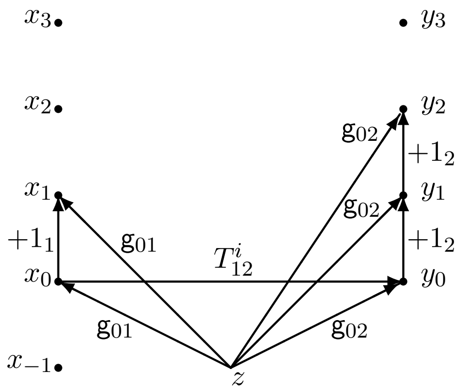

FIGURE 2. Representation implies Tiling

THEOREM 8.6. *Let* (τ, T^0) *be a deterministic instance of the tiling completion problem. Then* (τ, T^0) *is a yes instance if and only if there is a* $\{\cdot, \smile, ;\}$-*representation of* $\mathsf{RA}(\tau, T^0)$.

PROOF. If (τ, T^0) is a yes instance of the deterministic tiling problem then it is also a yes instance of the ordinary tiling problem so, by the theorem, $\mathsf{RA}(\tau, T^0)$ is a representable relation algebra and certainly its reduct to $\{\cdot, \smile, ;\}$ is representable.

Conversely, suppose there is a $\{\cdot, \smile, ;\}$-representation ϕ of $\mathsf{RA}(\tau, T^0)$. We have to prove that τ can tile the plane, with T^0 at $(0,0)$. Let

$$((u_0, v_0, t^0), (u_1, v_1, t^1), \dots)$$

be the sequence required to show that (τ, T^0) is deterministic, note that $t^0 = T^0$.

Consider the element $T^0_{12} \in \mathsf{At}(\mathsf{RA}(\tau, T^0))$. As $T^0_{12} \neq 0 \cdot T^0_{12} = 0$ and ϕ respects $\cdot$ we have $(T^0_{12})^\phi \neq 0^\phi \cap (T^0_{12})^\phi = 0^\phi$ so $0^\phi \subsetneq (T^0_{12})^\phi$ and since ϕ is faithful there is a pair

$$(x_0, y_0) \in (T^0_{12})^\phi \backslash 0^\phi. \tag{8.5}$$

Note that ϕ need not respect 0, so it is possible that $0^\phi \neq \varnothing$. As ϕ preserves converse we have $(y_0, x_0) \in (T^0_{21})^\phi$. Now $(\mathsf{g}_{10}, \mathsf{g}_{02}, T^0_{12})$ is not forbidden, so $\mathsf{g}_{10} ; \mathsf{g}_{02} \geq T^0_{12}$, showing that $(x_0, y_0) \in (\mathsf{g}_{10} ; \mathsf{g}_{02})^\phi = \mathsf{g}^\phi_{10} ; \mathsf{g}^\phi_{02}$. Therefore there is a point z such that $(x_0, z) \in \mathsf{g}^\phi_{10}$ and $(z, y_0) \in \mathsf{g}^\phi_{02}$. See Figure 2.

The triple $(+1_1, \mathsf{g}_{10}, \mathsf{g}_{10})$ is not forbidden, so $+1_1 ; \mathsf{g}_{10} \geq \mathsf{g}_{10}$. As $(x_0, z) \in \mathsf{g}^\phi_{10}$, there is a point x_1 such that $(x_0, x_1) \in +1^\phi_1$ and $(x_1, z) \in \mathsf{g}^\phi_{10}$. Continuing in this way we may find, for each $i \in \mathbb{N}$, points $x_2, x_3, \dots$ such that $(x_i, x_{i+1}) \in +1^\phi_1$ and $(x_i, z) \in \mathsf{g}^\phi_{10}$. Also, as $-1_1 ; \mathsf{g}_{10} \geq \mathsf{g}_{10}$ there is x_{-1} such that $(x_0, x_{-1}) \in (-1)^\phi$ and $(x_{-1}, z) \in \mathsf{g}^\phi_{10}$. Continuing in this way we can find for each $i \in \mathbb{N}$ points $x_{-1}, x_{-2}, \dots$ such that

$$(x_{-(i+1)}, x_{-i}) \in (+1)^\phi \text{ and } (x_{-i}, z) \in \mathsf{g}^\phi_{10}. \tag{8.6}$$

Similarly, using $\mathsf{g}_{02} ; (-1_2) \geq \mathsf{g}_{02}$ and $\mathsf{g}_{02} ; (+1_2) \geq \mathsf{g}_{02}$, there are points $y_1, y_2, \dots$ and $y_{-1}, y_{-2}, \dots$ such that $(z, y_i) \in \mathsf{g}^\phi_{02}$ and $(y_i, y_{i+1}) \in (+1_2)^\phi$ and $(z, y_i) \in \mathsf{g}^\phi_{02}$ for $i \in \mathbb{Z}$.

Claim 0. For every $i, j \in \mathbb{Z}$ the pair (x_i, y_j) is not contained in 0^ϕ.

PROOF. If (x_i, y_j) were contained in 0^ϕ then as $(x_k, z) \in \mathsf{g}_{10}^\phi$ and $(z, y_k) \in \mathsf{g}_{02}^\phi$ for every k we would have

$$x_0 \ \mathsf{g}_{10}^\phi \ z \ \mathsf{g}_{01}^\phi \ x_i \ 0^\phi \ y_j \ \mathsf{g}_{20}^\phi \ z \ \mathsf{g}_{02}^\phi \ y_0,$$

showing that $(x_0, y_0) \in (\mathsf{g}_{10} ; \mathsf{g}_{01} ; 0 ; \mathsf{g}_{20} ; \mathsf{g}_{02})^\phi = 0^\phi$ (and contradicting the choice of $(x_0, y_0) \notin 0^\phi$). ⊣

Note also that as $\mathsf{g}_{10} ; \mathsf{g}_{02} = \sum_{T \in \tau} T_{12}$ by (8.1) we have

$$(x_{u_i}, y_{v_j}) \in \big(\sum_{T \in \tau} T_{12}\big)^\phi. \tag{8.7}$$

Now we consider the edges (x_i, y_j) in the order $((x_{u_i}, y_{v_i}) \mid i \in \mathbb{N})$ (recall that $\{(u_i, v_i) \mid i \in \mathbb{N}\}$ covers the plane). We now prove by induction that for each $k \in \mathbb{N}$

- $(x_{u_k}, y_{v_k}) \in (t_{12}^k)^\phi$,
- $(x_{u_i}, y_{v_i}, t_{12}^i : i \leq k)$ is a partial tiling.

The case $k = 0$ is already established (8.5). Now let $k > 0$ and assume that the claim is true for all $i < k$. The goal is to use the determinism of (τ, T^0) to refine property (8.7) and replace $\sum_{T \in \tau} T_{12}$ by the single element t_{12}^k and prove that $(x_{u_k}, y_{v_k}) \in (t_{12}^k)^\phi$.

Claim 1. Suppose $(x_{u_k}, y_{v_k}) \sim_L (x_{u_i}, y_{v_i})$ (some $i < k$), i.e. $x_{u_k} = x_{u_i} - 1$, $y_{v_k} = y_{v_i}$, and $T \in \tau$ is a tile where $T \not\sim_L t^i$. Then $(x_{u_k}, y_{v_k}) \in (-T)^\phi$.

PROOF. The induction hypothesis gives $(x_{u_i}, y_{v_i}) \in (t_{12}^i)^\phi$, while $(x_{u_i - 1}, x_{u_i}) \in (+1_1)^\phi$ by (8.6). Hence $(x_{u_i - 1}, y_{v_i}) \in (+1_1 ; t_{12}^i)^\phi$. Since $T \not\sim_L t^i$, by (8.2), $+1_1 ; t_{12}^i \cdot T = 0$ so $+1_1 ; t_{12}^i \leq -T$. It follows that $(x_{u_k}, y_{v_k}) = (x_{u_i - 1}, y_{v_i}) \in (-T)^\phi$, proving the claim. ⊣

Similar claims can be proved for other adjacencies. Now, since (τ, T^0) is deterministic and $((u_i, v_i, t^i) : i < k)$ is a partial tiling, if $T \in \tau \setminus t^k$ then placing T at (u_k, v_k) violates an adjacency with the partial tiling $((u_i, v_i, t^i) : i < k)$. By claim 1, for each $T \in \tau \setminus \{t^k\}$ we have $(x_{u_k}, y_{v_k}) \in (-T_{12})^\phi$, so by (8.7), $(x_{u_k}, y_{v_k}) \in \bigcap_{T \in \tau \setminus \{t_{12}^k\}} (-T)^\phi \cap (\sum_{T \in \tau} T)^\phi = (t_{12}^k)^\phi$. Furthermore, it must be consistent with all the adjacencies to place t^k at (u_k, v_k), else $(x_{u_k}, y_{v_k}) \in (-t_{12}^k)^\phi$, by claim 1, which would imply that $(x_{u_k}, y_{v_k}) \in (t_{12}^k \cdot - t_{12}^k)^\phi = 0^\phi$, contrary to claim 0. Thus $(x_{u_k}, y_{v_k}) \in (t_{12}^k)^\phi$ and $((u_i, v_i, t^i) : i \leq k)$ is a partial tiling, completing the induction step.

By induction, $((u_i, v_i, t^i) : i < \omega)$ is a tiling of the whole plane, hence (τ, T^0) is a yes-instance of the deterministic tiling problem. ⊣

Proof of Theorem 8.1 Let $\mathscr{F}$ be an **RA** subsignature containing $\{\cdot, \smile, ;\}$. By Theorems 8.5 and 8.6, the deterministic tiling completion problem reduces to the representability problem for finite $\mathcal{F}$-algebras. By Theorem 8.4, the deterministic tiling completion problem is undecidable, hence membership of $\mathbf{R}(\mathscr{F})$ is undecidable, for finite $\mathscr{F}$-algebras.

Signature $\mathscr{F}$	Fin. ax.	Dec. Eq. Th.	Dec. Rep. Fin. Alg.
$\mathscr{F} = \mathbf{RA}$	$\times$[44]	$\times$[60]	$\times$[21]
$\mathscr{F} = \mathbf{BA}$	✓[58]	✓	✓[58]
$\mathscr{F} \subseteq \{0, 1, +, -, \leq, 1', ^{\smile}\}$	✓[53]	✓[46]	✓
$\mathscr{F} \subseteq \{0, 1, +, \cdot, \leq, 1', ^{\smile}, ;\}$	d	✓[2]	d
$\mathscr{F} \supseteq \{\cdot, ^{\smile}, ;\}$	$\times$[24]	d	$\times$Thm. 8.1
$\mathscr{F} \supseteq \{+, \cdot, ;\}$	$\times$[4]	d	?
$\mathscr{F} \supseteq \{+, \cdot, 1', ;\}$	$\times$	d	$\times$Thm. 6.1
$\mathscr{F} = \{+, ;\}$	$\times$[3]	✓	?
$\mathscr{F} = \{\cdot, 1', ;\}$	$\times$[23]	✓	?
$\mathscr{F} = \{\leq, ;\}$	✓[53]	✓	✓[53]
$\mathscr{F} = \{\leq, 1', ;\}$	$\times$[20]	✓	?

FIGURE 3. Finite axiomatisability of $\mathbf{R}(\mathscr{F})$, decidability of the equational theory of $\mathbf{R}(\mathscr{F})$ and decidability of representability for finite $\mathscr{F}$-algebras. ✓ means "yes", $\times$ mean "no" and d means that the result depends on the choice of $\mathscr{F}$.

§9. Conclusion. Ever since Lyndon proved that Tarski's axiomatisation of relation algebras was not complete over the class of representable relation algebras [38], the subject has been dogged by negative results; the results of the current paper can be thought of as the next in a long line. **RRA** cannot be defined by any finite set of formulas [44]. Any sound and complete equational axiomatisation of **RRA** must involve infinitely many variables [34], it must include infinitely many non-Sahlqvist equations [61] indeed it must include infinitely many non-canonical equations [25]. Tarski showed that the equational theory of relation algebra, although by definition finitely axiomatisable, is not decidable. Membership of the class of finite representable relation algebras is known to be undecidable [21].

Given all these negative results, various researchers investigated reducts of **RA** in the hope of finding algebras with better logical and computational behaviour. Let $\mathscr{F}$ be a set of Boolean or non-Boolean operators, definable by relation algebra operators. $\mathbf{R}(\mathscr{F})$ denotes the class of all $\mathscr{F}$-algebras that are representable as fields of binary relations, respecting each operator in $\mathscr{F}$. Figure 3 summarises some of the key discoveries, focussing on the finite axiomatisability of $\mathbf{R}(\mathscr{F})$, the decidability of the equational theory of $\mathbf{R}(\mathscr{F})$ and the decidability of membership of $\mathbf{R}(\mathscr{F})$ for finite $\mathscr{F}$-algebras. From the figure, the main positive result is that the equational theory of $\mathbf{R}(\mathscr{F})$ is decidable if $\mathscr{F}$ is a *positive* reduct (definable from $\{0, 1, +, ., 1', ^{\smile}, ;\}$) or if each operator in $\mathscr{F}$ can be defined by $\{0, 1, +, -, 1', ^{\smile}\}$ (i.e. without composition). The results for finite axiomatisability of $\mathbf{R}(\mathscr{F})$ are disappointing: $\mathbf{R}(\mathscr{F})$ is finitely axiomatisable if $\mathscr{F}$ can be defined by $\{0, 1, +, -, 1', ^{\smile}\}$, but the only other known finitely axiomatisable representation class is for $\mathscr{F} = \{\leq, ;\}$. Similarly, the results for decidability of membership of $\mathbf{R}(\mathscr{F})$ for finite $\mathscr{F}$-algebras are mostly negative.

For various signatures $\mathscr{F}$ without converse (for example, the signature of Boolean monoids) we have shown that membership of $\mathbf{R}(\mathscr{F})$ is not decidable for finite $\mathscr{F}$-algebras, but we have also shown that finite representability is undecidable for finite $\mathscr{F}$-algebras. As far as we know, this is the first case where finite representability has been shown to be undecidable for subsignatures of **RA**. Aside from the question marks in Figure 3, one key problem remains open: is the class of finitely representable relation algebras decidable?

REFERENCES

[1] A.A. Albert, Quasigroups I, Trans. Amer. Math. Soc. 54 (1943), 507–519.

[2] H. Andréka and D. Bredikhin. The equational theory of union-free algebras of relations. Algebra Universalis 33 (1995), 516–532.

[3] H. Andréka. On the 'union-relation composition' reducts of relation algebras. Abstracts Amer. Math. Soc., 10(2):174, 1989. Full, unpublished manuscript called 'On the representation problem of distributive semilattice-ordered semigroups', Mathematical Institute, Hungarian Academy of Sciences, Budapest.

[4] H. Andréka. Representation of distributive lattice-ordered semigroups with binary relations. Algebra Universalis 28 (1991), 12–25.

[5] H. Andréka, D. Monk and I. Németi (eds.), Algebraic Logic. Colloq. Math. Soc. J. Bolyai 54, North Holland, Amsterdam, 1991.

[6] D.A. Bredikhin, An abstract characterization of some classes of algebras of binary relations, in *Algebra and number theory*, No. 2, pp. 3–19. Kabardino-Balkarsk. Gos. Univ., Nalchik, 1977. [In Russian.]

[7] D.A. Bredikhin, Reducts of Tarski relation algebras, Algebra and Logic 37 (1998), 1–8.

[8] D.A. Bredikhin and B.M. Schein, Representations of ordered semigroups and lattices by binary relations, Colloq. Math. 39 (1978), 1–12.

[9] S. Comer. A remark on representable positive cylindric algebras. Algebra Universalis, 28 (1991), 150–151.

[10] J. Desharnais, B. Möller and G. Struth, Kleene algebra with domain, ACM Trans. Comput. Log. 7 (2006), 798–833.

[11] T. Evans, Embeddability and the word problem, J. London Math. Soc. 28 (1953), 76–80.

[12] J. Fountain, Free right type A semigroups, Glasgow Math. J. 33 (1991), 135–148.

[13] P. Freyd and A. Scedrov, Categories, Allegories, Mathematical Library Vol 39., North Holland, 1990.

[14] V.S. Garvac'kii, $\cap$-semigroups of transformations, in Theory of Semigroups and its Applications 2 (Izdat. Saratov. Uni., Saratov) (1971), 2–13. [In Russian.]

[15] S. Göller, M. Lohrey and C. Lutz, PDL with intersection and converse: satisfiability and infinite-state model checking, J. Symbolic Logic 74 (2009), 279–314.

[16] V. Gould and M. Kambites, Faithful functors from cancellative categories to cancellative monoids with an application to abundant semigroups, Internat. J. Algebra Comput. 15 (2005), 683–698.

[17] T.E. Hall, S.I. Kublanovsky, S. Margolis, M.V. Sapir, and P.G. Trotter, Decidable and undecidable problems related to finite 0-simple semigroups, J. Pure Appl. Algebra. 119 (1997), 75–96.

[18] C. Hardin and D. Kozen, The complexity of the Horn theory of REL, Technical Report TR2003-1896, Computer Science Department, Cornell University, May 2003.

[19] G. Higman, A finitely generated infinite simple group, J. London Math. Soc. 26 (1951), 61–64.

[20] R. Hirsch, The class of representable ordered monoids has a recursively enumerable, universal axiomatisation but it is not finitely axiomatisable, Log. J. IGPL, 13 (2005), 159–171.

[21] R. Hirsch and I. Hodkinson, Representability is not decidable for finite relation algebras, Trans. Amer. Math. Soc. 353 (2001), 1403–1425.

[22] R. Hirsch and I. Hodkinson, Relation algebras by games. Studies in Logic and the Foundations of Mathematics, Volume 147, North Holland. Elsevier Science, Amsterdam, 2002.

[23] R. Hirsch and Sz. Mikulás, Representable semilattice-ordered monoids, Algebra Universalis 57 (2007), 333–370.

[24] I. Hodkinson and S. Mikulás. A non-finite axiomatizability result for a reduct of RRA. Algebra Universalis, 43 (2000), 127–156.

[25] I. Hodkinson and Y. Venema. Canonical varieties with no canonical axiomatisation. Trans. Amer. Math. Soc., 357 (2005), 4579–4605.

[26] J.M. Howie, *Fundamentals of semigroup theory* Oxford University Press, New York, 2nd ed., 1995.

[27] M. Jackson, Some undecidable embedding problems for finite semigroups, Proc. Edinburgh Math. Soc. (II) 42 (1998), 113–125.

[28] M. Jackson and T. Stokes, An invitation to C-semigroups, Semigroup Forum 62 (2001), 279–310.

[29] M. Jackson and T. Stokes, The algebra of partial maps with domain and range: extending Schein's representation, Comm. Algebra 37 (2009), 2845–2870.

[30] M. Jackson and T. Stokes, Modal restriction semigroups: towards an algebra of functions, Internat. J. Algebra Comput. 21 (2011), 1053–1095.

[31] M. Jackson and M.V. Volkov, Undecidable problems for completely 0-simple semigroups, J. Pure Appl. Algebra 213 (2009), 1961–1978.

[32] M. Jackson and M.V. Volkov, Relatively inherently nonfinitely q-based semigroups, Trans. Amer. Math. Soc. 361 (2009), 2181–2206.

[33] B. Jónsson, Representation of modular lattices and relation algebras, Trans. Amer. Math. Soc. 92 (1959), 449–464.

[34] B Jónsson. The theory of binary relations, in H. Andréka, J. Monk, and I. Németi, editors, Algebraic logic, volume 54 of Colloq. Math. Soc. J. Bolyai, pages 245–292. North-Holland, Amsterdam, 1991.

[35] B. Jónsson and A. Tarski, Boolean algebras with operators. Part II, Amer. J. Math. 74, (1952), 127–162.

[36] O. Kharlampovich and M. Sapir, Algorithmic problems in varieties, Internat. J. Algebra Comput. 5 (1995), 379–602.

[37] A. Kurucz. Decision problems in algebraic logic, PhD thesis, Hungarian Academy of Sciences, Budapest, 1997. Available at '`http://www.math-inst.hu/algebraic-logic/kuagthes.dvi`'.

[38] R. Lyndon, The representation of relational algebras, Ann. Math. 51 (1950), 707–729.

[39] R. Lyndon, The representation of relation algebras II, Ann. Math. 63 (1956), 294–307.

[40] R.Sz. Madarász and S. Crvenković, Relacione algebre, Matematički Institut, Belgrade, 1992. [Serbo-Croatian.]

[41] R.D. Maddux, Relation Algebras. Studies in Logic and the Foundations of Mathematics, Volume 150, North Holland. Elsevier Science, Amsterdam, 2006.

[42] Sz. Mikulás, Axiomatizability of algebras of binary relations. Classical and new paradigms of computation and their complexity hierarchies, 187–205, Trends Log. Stud. Log. Libr., 23, Kluwer Acad. Publ., Dordrecht, 2004

[43] B. Möller and G. Struth, Algebras of modal operators and partial correctness, *Theoret. Comput. Sci.* 351 (2006), 221–239.

[44] D. Monk, On representable relation algebras, Michigan Math. J. 11 (1964), 207–210.

[45] V. Pratt, Dynamic algebras as a well-behaved fragment of relation algebras, in C.H. Bergman, R.D. Maddux, D.L. Pigozzi (Eds.), Algebraic Logic and Universal Algebra in Computer Science, LNCS 425, Springer-Verlag, 1990, pp. 77–110.

[46] I. Németi. Decidability of relation algebras with weakened associativity. Proc. Amer. Math. Soc., 100(2) (1987), 340–344.

[47] R.M. Robinson, Undecidability and nonperiodicity for tilings of the plane, Inventiones math. 12 (1971), 177–209.

[48] B.M. Schein, The representation of ordered semigroups, Mat. Sb. (N.S.) 65 (107) (1964), 188–197. [In Russian.]

[49] B.M. Schein, On certain classes of semigroups of binary relations, Sibirsk. Mat. Zh. 6 (1965), 616–635. [Russian; English translation in Amer. Math. Soc. Trans. Series 2. Vol. 139. (1988), pp. 117–137.]

[50] B.M. Schein,Restrictively multiplicative algebras of transformations, Izv. Vysš. Učebn. Zaved. Matematika 1970 no. 4 95 91–102. [In Russian.]

[51] B.M. Schein, Relation algebras and function semigroups, Semigroup Forum 1 (1970), 1–62.

[52] B.M. Schein, Lectures on transformation semigroups, Idzat. Saratov Univ., Saratov, 1970. [Russian; English translation in Translations of the Amer. Math. Soc. (2) 113 (1979), 123–181.]

[53] B.M. Schein, Representation of reducts of Tarski relation algebras, Coll. Math. Soc. János Bolyai 54 (1991), 379–394.

[54] B.M. Schein, Difference semigroups, Comm. Algebra 20 (1992), 2153–2169.

[55] B.M. Schein, Subsemigroups of inverse semigroups, Matematiche 51 (1996), 205–227.

[56] A. Slobodskoĭ, Undecidability of the universal theory of finite groups, Algebra i Logika 20 (1981) 207–230, 251 [Russian; English translation in Algebra and Logic 20 (1981), 139–156 (1982)].

[57] T. Stokes, Eq-monoids, Acta Sci. Math. (Szeged) 72 (2006), 481–506.

[58] M. Stone. The theory of representations for Boolean algebras. Trans. Amer. Math. Soc., 40 (1936), 37–111.

[59] A. Tarski, On the calculus of relations, J. Symbolic Logic 6 (1941), 73–89.

[60] A. Tarski and S. Givant. A formalization of set theory without variables. Number 41 in Colloquium Publications. Amer. Math. Soc., Providence, Rhode Island, 1987.

[61] Y. Venema. Atom structures and Sahlqvist equations. Algebra Universalis, 38 (1997), 185–199.

[62] K.A. Zareckiĭ, The representation of ordered semigroups by binary relations, Izv. Vyss. Ucebn. Zaved. Matematika 1959 1959 no. 6 (13), 48–50.

DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY COLLEGE LONDON
LONDON WC1E 6BT
U.K.
E-mail: R.Hirsch@cs.ucl.ac.uk

DEPARTMENT OF MATHEMATICS AND STATISTICS
LA TROBE UNIVERSITY
VICTORIA 3086
AUSTRALIA
E-mail: M.G.Jackson@latrobe.edu.au