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p-Sidon Sets and a Uniform Property

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Let G be a compact abelian group with dual group Γ. Denote by $L^p(E)$ and $M(E)$ the usual spaces of Haar-measurable functions and bounded regular Borel measures, respectively, which are supported on the subset E of G or Γ. The Haar measure on G is normalized and its dual is the Haar measure on Γ. Let $\hat{\varphi}$ denote the Fourier or Fourier-Stieltjes transform of the function or measure φ. A subset $E \subset \Gamma$ is said to be *p-Sidon* for some $1 \leqq p < 2$ (not interesting for $p \geqq 2$) if there is an $\alpha > 0$ such that $||\hat{\varphi}||_p \leqq \alpha \, ||\varphi||_\infty$ for all trigonometric polynomials φ on G with supp $\hat{\varphi} \subset E$. This is equivalent to the dual statement: E is p-Sidon if and only if $L^q(E) \subset M(G)\hat{}\,|_E$, where $1/p + 1/q = 1$ and "$|_E$" denotes restriction to E. Hereafter p and q will *always* be as above.

The concept of a p-Sidon set was independently introduced in [2, 4, and 5] as a natural generalization of the classical Sidon sets (*i.e.*, 1-Sidon sets). In each of these articles, the various equivalent definitions for p-Sidon sets are given. They correspond to the classical equivalent definitions of a Sidon set as presented in [8, Theorem 5.7.3]. In [5], Hahn extends a theorem of J. P. Kahane to give the best known necessary conditions for a set to be p-Sidon when $\Gamma = \mathbf{Z}$, the integers. Edwards and Ross present the most comprehensive treatment of the subject in [4]. It is there that the first non Sidon p-Sidon set is constructed via an extremely ingenious application of the tensor algebraic techniques of Varapoulos. Their methods are extended in [6] to prove that the classes of all $2n/(n + 1)$-Sidon sets are distinct for $n = 1, 2, \cdots$. One will also find in [6] all known non Sidon p-Sidon sets to date (except for unions with finite sets). For a somewhat more skillful application of the Varapoulos techniques to this problem, we refer the reader to [1].

In this paper, we adapt an idea of Rider [7] in defining the class of uniformizable p-Sidon sets. The class is, by design, closed under finite unions. Of course, its members are p-Sidon sets. Our main result is that Sidon sets are uniformizable p-Sidon sets for all p. Its proof is a variant of Drury's famous technique which resembles most closely the approach found in [3]. As a corollary we prove that the union of a Sidon set with any p-Sidon set is again p-Sidon, thus enabling one to exhibit many new non Sidon p-Sidon sets. We conclude with a slight extension of the results in [6], using an argument similar to the one presented there, and a list of open questions.

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In what follows, $L^p(\Gamma)_E$ denotes the $L^p(\Gamma)$ functions supported on $E \subset \Gamma$ and I_E denotes the characteristic function of E. We begin with a useful technical result of a standard type.

Lemma 1. *$E \subset \Gamma$ is a p-Sidon set if and only if there exist $\beta > 0$ and $0 < \delta < 1$ such that for each $\varphi \in L^q(\Gamma)_E$ there is a $\mu \in M(G)$ satisfying*

(i) $\|\mu\| \leqq \beta \|\varphi\|_q$; *and*

(ii) $\|\hat{\mu} I_E - \varphi\|_q < \delta \|\varphi\|_q$.

Proof. Suppose E is a p-Sidon set. Define the relation $\sim$ on $M(G)$ by $\mu \sim \nu$ if $\hat{\mu} - \hat{\nu} \equiv 0$ on E. Let $M(G)/\sim$ denote the usual Banach quotient space. By definition $L^q(\Gamma)_E$ naturally embeds in $M(G)/\sim$. Moreover, the uniqueness of the Fourier-Stieltjes transform yields that the graph of this map is closed; hence (i). Of course, (ii) holds for any $\delta > 0$.

For the converse, let $\varphi \in L^q(\Gamma)_E$. Then (i) and (ii) yields inductively a sequence $\{\mu_n\} \subset M(G)$ with μ_1 satisfying $\|\mu_1\| \leqq \beta \|\varphi\|_q$ and $\|\hat{\mu}_1 I_E - \varphi\|_q \leqq \delta \|\varphi\|_q$ and continuing

$$\|\mu_n\| \leqq \beta \delta^{n-1} \|\varphi\|_q$$

and

$$\left\| \hat{\mu}_n I_E - \left(\varphi - \sum_0^{n-1} \hat{\mu}_k I_E\right) \right\|_q \leqq \delta^n \|\varphi\|_q .$$

Since $\sum^\infty \|\mu_k\| \leqq \beta \|\varphi\|_q (1 - \delta)^{-1}$, the sum $\mu = \sum^\infty \mu_k$ converges in $M(G)$; clearly $\hat{\mu} = \varphi$ on E. □

Definition. $E \subset \Gamma$ is a *uniformizable* p-Sidon set if for each $\delta > 0$ there exists a $\beta > 0$ such that for any $\varphi \in L^q(\Gamma)_E$ there is a $\mu \in M(G)$ satisfying

(i) $\|\mu\| \leqq \beta \|\varphi\|_q$; and

(ii) $\|\hat{\mu} - \varphi\|_q \leqq \delta \|\varphi\|_q$.

Denote by $\mathfrak{U}_p$ the class of all uniformizable p-Sidon sets on Γ.

It is clear that each element of $\mathfrak{U}_p$ is p-Sidon. The full strength of the definition is summed up in the following theorem.

Theorem 1. *$E \in \mathfrak{U}_p$ if and only if for each $\delta > 0$ there exists a $\beta > 0$ such that for any $\varphi \in L^q(\Gamma)_E$ there is a $\mu \in M(G)$ satisfying*

(i) $\|\mu\| \leqq \beta \|\varphi\|_q$;

(ii) $\hat{\mu} \equiv \varphi$ *on E; and*

(iii) $\left(\sum_{\gamma \notin E} |\hat{\mu}(\gamma)|^q\right)^{1/q} \leqq \delta \|\varphi\|_q$.

Proof. Suppose $E \in \mathfrak{U}_p$. Let $\varphi \in L^q(\Gamma)_E$ and choose any $0 < \delta_0 < 1$. Set $\delta = \delta_0/2$. According to the definition of $\mathfrak{U}_p$, there is a $\beta > 0$ and a $\mu_1 \in M(G)$ such that $\|\mu_1\| \leqq \beta \|\varphi\|_q$ and $\|\hat{\mu}_1 - \varphi\|_q \leqq \delta \|\varphi\|_q$. Apply the definition again to $\varphi - \hat{\mu}_1 I_E$ with the same δ and continue in this manner. This gives rise to a sequence $\{\mu_n\} \subset M(G)$ as in Lemma 1. Thus

$$\mu = \sum^{\infty} \mu_n \in M(G), \qquad \|\mu\| \leqq \beta \, \|\varphi\|_q \, (1 - \delta)^{-1},$$

and $\hat{\mu} \equiv \varphi$ on E. But this time

$$\left\|\hat{\mu}_n - \left(\varphi - \sum_1^{n-1} \hat{\mu}_k\right)\right\|_q \leqq \left\|\hat{\mu}_n - \left(\varphi - \sum_1^{n-1} \hat{\mu}_k I_E\right)\right\|_q + \left\|\sum_1^{n-1} \hat{\mu}_k(1 - I_E)\right\|$$

$$\leqq \delta^n \, \|\varphi\|_q + \sum_1^{n-1} \delta^k \, \|\varphi\|_q < \delta_0 \, \|\varphi\|_q \, .$$

In particular, (iii) is valid. Of course, (i)–(iii) are sufficient to imply $E \in \mathfrak{U}_p$. □

Our next theorem is rather trivial at this point, but worth mentioning.

Theorem 2. *$\mathfrak{U}_p$ is closed under finite unions for $1 \leqq p < 2$.*

Proof. Suppose $E_1, E_2 \in \mathfrak{U}_p$ and set $E = E_1 \cup E_2$. Since subsets of elements in $\mathfrak{U}_p$ are also in $\mathfrak{U}_p$, we can assume that $E_1 \cap E_2 = \emptyset$. Let $\varphi \in L^q(\Gamma)_E$, let $\varphi_i = \varphi I_{E_i}$ for $i = 1, 2$, and choose any $\delta_0 > 0$. By definition there exist $\beta > 0$ and measures μ_1, μ_2 such that $\delta = \delta_0/2$, β, μ_i, φ_i satisfy (i) and (ii) of the definition for a $\mathfrak{U}_p$ set. Thus $\|\mu_1 + \mu_2\| \leqq 2\beta \, \|\varphi\|_q$ and

$$\|\hat{\mu}_1 + \hat{\mu}_2 - \varphi\|_q \leqq \|\hat{\mu}_1 - \varphi_1\|_q + \|\hat{\mu}_2 - \varphi_2\|_q < \delta_0 \, \|\varphi\|_q \, . \quad \square$$

Remark. This author had originally announced [Notices Amer. Math. Soc. **21** (1974), A-163] a somewhat different definition for $\mathfrak{U}_p$. Specifically, replace "for each $\delta > 0$" by "for some $0 < \delta < 1$" in Theorem 1. Under this change, Theorem 2 would read "the union of any two elements of $\mathfrak{U}_p$ is p-Sidon." The formally stronger definition that we are now using seems to better reflect the structure of p-Sidon sets.

We now turn to the question of existence of nontrivial uniformizable p-Sidon sets. Fortunately, Drury's theorem implies that $\mathfrak{U}_1$ consists of all Sidon sets. But this yields no information about $\mathfrak{U}_p$ for $p \neq 1$. In fact, the relationship between $\mathfrak{U}_p$ and $\mathfrak{U}_r$ for $1 \leqq p \neq r < 2$ is not at all clear. Our next theorem sheds some light on the matter by showing $\mathfrak{U}_1 \subset \mathfrak{U}_p$. The key is the observation that $\mathfrak{U}_p$ contains all dissociate sets for $1 \leqq p < 2$. For, Drury's techniques allow us in this context to essentially consider any Sidon set as a dissociate set. We emphasize that many of the techniques used in our next proof parallel those of [3]. A subset E of an abelian group Λ is *dissociate* if the only solutions to $\sum \delta_\gamma \gamma = 0$ (finite sum) with $\gamma \in E$ and $\delta_\gamma \in \{-2, -1, 0, 1, 2\}$ are $\delta_\gamma = 0$ for all γ. As is custom, we denote by $B(\Gamma)$ the space $M(G)^\wedge$ with the norm, $\|\hat{\mu}\|_B \equiv \|\mu\|$.

Theorem 3. *Sidon sets are uniformizable p-Sidon sets for all p.*

Proof. Let $E \subset \Gamma$ be a Sidon set. Following Drury [3], fix a positive integer n and let $\gamma_1, \cdots, \gamma_n \in E$ be any choice of n distinct nonzero elements. Let Λ be the discrete abelian group generated by $F \equiv \{\gamma_1, \cdots, \gamma_n\}$ over, say, $\mathbf{Z}$ mod (3) where $\gamma_1, \cdots, \gamma_n$ are simply considered as n independent symbols. That is $\Lambda \cong (\mathbf{Z} \bmod (3))^n$. The dual H of Λ is isomorphic to Λ but it can also be realized as the set of all maps $h : F \to T_3$ where T_3 is the set of 3rd roots of unity. The group operation, represented by $+$, is just pointwise multiplication. We insist that H have Haar measure 1. Then the dual Haar measure on Λ is simply the counting measure.

Consider first the group $\Gamma \times H$ which has dual $G \times \Lambda$. Since E is 1-Sidon, there exists an $\alpha > 0$ such that for each $h \in H$ there is a $\mu_h \in M(G)$ satisfying $\|\mu_h\| \leqq \alpha$ and $\hat{\mu}_h \equiv h$ on F. Set $g(\gamma, h) = \hat{\mu}_h(\gamma)$. Then $g(\gamma_i, \cdot)$ is a character on H. Together with the properties of μ_h, this yields

(1′) $\quad g(\cdot, h) \in B(\Gamma) \quad \text{with} \quad \|g(\cdot, h)\|_B \leqq \alpha \quad \text{for all} \quad h \in H$

and

(2′) $\quad g(\gamma, \cdot) \in B(H) \quad \text{with} \quad \|g(\gamma, \cdot)\|_B = 1 \quad \text{for all} \quad \gamma \in F.$

We adjust these two statements as follows. Define the function

$$r(\gamma, \cdot) \equiv g(\gamma, \cdot) \overset{*}{_H} g(\gamma, o) \quad \text{(convolution over } H).$$

Since $\|g(\gamma, \cdot)\|_\infty \leqq \alpha$, it follows that $\|g(\gamma, \cdot)\|_2 \leqq \alpha$; hence $\|r(\gamma, \cdot)\|_B \leqq \alpha^2$ for all $\gamma \in \Gamma$. Since $r(\cdot, h)$ is a convex linear combination of products of the $\hat{\mu}_\ell$, $\ell \in H$, it follows that $r(\cdot, h) \in B(\Gamma)$ and $\|r(\cdot, h)\|_B \leqq \alpha^2$ for $h \in H$. That is,

(1) $\|r(\cdot, h)\|_B \leqq \alpha^2$ for all $h \in H$;

(2) $\|r(\gamma, \cdot)\|_B \leqq \alpha^2$ for all $\gamma \in \Gamma$; and

(3) $r(\gamma, h) = h(\gamma)$ on F for all $h \in H$;

where (3) is immediate from the definition of r.

At this point we fix a real-valued $\varphi \in L^q(\Gamma)_F$ with $\|\varphi\|_q = 1$. Let $0 < \epsilon \leqq 1$ and set $x_j = (\gamma_j, \gamma_j) \in \Gamma \times \Lambda$ for $1 \leqq j \leqq n$. Define the Riesz polynomials P_e and P_0 on $G \times H$ by

$$P_e(z) = \prod_{j=1}^{n} [1 + \epsilon/2\varphi(\gamma_j)(x_j(z) + \overline{x_j(z)})]$$

and

$$P_0(z) = \prod_{j=1}^{n} [1 + \epsilon/2i\varphi(\gamma_j)(x_j(z) - \overline{x_j(z)})].$$

Since these functions are nonnegative $\|P_e\|_1 = \hat{P}_e(0)$ and $\|P_0\|_1 = \hat{P}_0(0)$. Their formal expansions can be described in the following terms. Set $\Omega = \{-1, 0, 1\}^n$, let $\delta = (\delta_1, \cdots, \delta_n)$ be a generic point of Ω, and adopt the convention $0^0 = 1$. Then, using the additive group notation, we have

$$P_e(z) = \sum_{\delta \in \Omega} \left[\prod_{j=1}^{n} (\epsilon/2\varphi(\gamma_j))^{|\delta_j|} \right] (\delta_1 x_1 + \cdots + \delta_n x_n)(z)$$

and

$$P_0(z) = \sum_{\delta \in \Omega} \left[\prod_{j=1}^{n} (\delta_j \epsilon/2i\varphi(\gamma_j))^{|\delta_j|} \right] (\delta_1 x_1 + \cdots + \delta_n x_n)(z).$$

Note that by definition of Λ the set $\{x_1, \cdots, x_n\}$ is dissociate; hence distinct $\delta \in \Omega$ give distinct characters $\delta_1 x_1 + \cdots + \delta_n x_n$ on $G \times H$. In particular, $\|P_\epsilon\|_1 = \|P_0\|_1 = 1$. Moreover, $\hat{P}_\epsilon$, $\hat{P}_0$ are supported on points of the form

$$y = \sum^n \delta_j x_j \quad \text{with} \quad \hat{P}_\epsilon(y) = \prod^n (\epsilon/2\varphi(\gamma_j))^{|\delta_j|}$$

and

$$\hat{P}_0(y) = \prod^n (\delta_j \epsilon/2i\varphi(\gamma_j))^{|\delta_j|}.$$

Also note, $\hat{P}_0(\pm x_j) = \pm\epsilon/2i\varphi(\gamma_j)$.

For a continuous P on $G \times H$, denote its transform with respect to the jth variable by $\hat{P}^j$ ($j = 1, 2$). It follows that $(\hat{P}^1)\hat{}^2 = \hat{P}$ and that $\|\hat{P}^1(\gamma, \cdot)\|_1 \leqq \|P_1\|$. In particular, the functions

$$s_\epsilon(\gamma) = (P_\epsilon(\gamma, \cdot) - 1)\hat{}^1 \underset{H}{*} r(\gamma, \cdot)(0)$$

and

$$s_0(\gamma) = (iP_0(\gamma, \cdot) - i)\hat{}^1 \underset{H}{*} r(\gamma, \cdot)(0)$$

are convex linear combinations of $B(\Gamma)$ functions with norm bounded by $2\alpha^2$. Thus

$$s \equiv s_\epsilon + s_0 \in B(\Gamma) \quad \text{and} \quad \|s\|_B \leqq 4\alpha^2. \tag{4}$$

Moreover, since $r(\gamma_j, h) = h(\gamma_j)$ for $1 \leqq j \leqq n$,

$$s(\gamma_j) = \hat{P}_\epsilon(x_j) + i\hat{P}_0(x_j) = \epsilon\varphi(\gamma_j). \tag{5}$$

We now want to estimate $\|s - \epsilon\varphi\|_q$. To this end, denote the Dirac point measure at $0 \in \Lambda$ by δ_0. Then applying Parseval's formula (relative to H) to the definition of $s(\gamma)$ yields

$$\begin{aligned} |s(\gamma)| &= \left| \int_H [P_\epsilon(\gamma, h) + iP_0(\gamma, h) - (1 + i)]\hat{}^1 r(\gamma, -h)\, dh \right| \\ &= \left| \int_\Lambda [\hat{P}_\epsilon(\gamma, \lambda) + i\hat{P}_0(\gamma, \lambda) - (1 + i)\delta_0]\hat{r}^2(\gamma, \lambda)\, d\lambda \right| \\ &\leqq \|\hat{P}_\epsilon(\gamma, \cdot) + i\hat{P}_0(\gamma, \cdot) - (1 + i)\delta_0\|_\infty \alpha^2 \quad \text{for all} \quad \gamma \in \Gamma \end{aligned}$$

by (2). Set $R = \hat{P}_\epsilon + i\hat{P}_0 - (1 + i)\delta_0$. The preceding inequalities and (5) yield

$$\|s - \epsilon\varphi\|_q^q = \sum_{\gamma \notin F} |s(\gamma)|^q \leqq \alpha^{2q} \sum_{\substack{x \in \Gamma \times \Lambda \\ x \neq x_j, 1 \leqq j \leqq n}} |R(x)|^q = \alpha^{2q}[\|R\|_q^q - \epsilon^q \|\varphi\|_q^q]. \tag{6}$$

To estimate $\|R\|_q$, partition Ω by the equivalence relation $\delta \sim \sigma$ if and only if $|\delta_j| = |\sigma_j|$ for $1 \leqq j \leqq n$. Call this partition $\mathcal{E}$. Given $u \in \mathcal{E}$ and any $\delta \in u$, define

$$|u| = \sum^n |\delta_j| \quad \text{and} \quad A_u = \prod^n |\varphi(\gamma)_j|^{|\delta_j|}.$$

Both symbols are well defined. Let $z = (x_1, \cdots, x_n)$. Then the expansions obtained earlier for P_e and P_0 yield

$$|R(\delta \cdot z)| = (\epsilon/2)^{|u|} A_u \, |1 + \beta_\delta| \quad \text{for} \quad \delta \in u \in \mathcal{E},$$

where $\delta \cdot z$ denotes the usual vector inner product and $\beta_\delta \in \{\pm 1, \pm i\}$. It is important to note that $R(x_j) = \epsilon\varphi(\gamma_j)$ and $R(0) = R(-x_j) = 0$ for $1 \leqq j \leqq n$. Since the cardinality of each $u \in \mathcal{E}$ is $2^{|u|}$, it follows that

$$\sum_{\delta \in u} |R(\delta \cdot z)|^q \leqq 2^{|u|}(\epsilon/2)^{|u|q} A_u^q 2^q \quad \text{if} \quad |u| > 1,$$

$$\sum_{\delta \in u} |R(\delta \cdot z)|^q = (\epsilon/2)^{|u|q} A_u^q 2^q \quad \text{if} \quad |u| = 1,$$

and

$$\sum_{\delta \in u} |R(\delta \cdot z)|^q = 0 \quad \text{if} \quad |u| = 0.$$

Thus

$$\begin{aligned} ||R||_q^q &= \sum_{u \in \mathcal{E}} \sum_{\delta \in u} |R(\delta \cdot z)|^q \\ &\leqq \sum_{u \in \mathcal{E}} 2^{|u|}(\epsilon/2)^{|u|q} A_u^q 2^q - \sum_{|u|=1} (\epsilon/2)^q A_u^q 2^q - 2^q \\ &= 2^q\Big(\sum_{u \in \mathcal{E}} 2^{|u|}(\epsilon/2)^{|u|q} A_u^q - (\epsilon/2)^q - 1\Big), \end{aligned}$$

where the second line of the inequality reflects, via subtraction, the differences between the cases $|u| > 1$, $|u| = 1$, and $|u| = 0$. We have also used

$$\sum_{|u|=1} (A_u)^q = (||\varphi||_q)^q = 1.$$

This can be further simplified with the aid of the equation

$$\sum_{u \in \mathcal{E}} 2^{|u|}(\epsilon/2)^{|u|q} A_u^q = \prod_{j=1}^{n} (1 + 2 \, |\epsilon/2\varphi(\gamma_j)|^q)$$

and the inequality

$$\begin{aligned} \ln \prod_{j=1}^{n} (1 + 2 \, |\epsilon/2\varphi(\gamma_j)|^q) &= \sum_{j=1}^{n} \ln (1 + 2 \, |\epsilon/2\varphi(\gamma_j)|^q) \\ &\leqq \sum_{j=1}^{n} 2 \, |\epsilon/2\varphi(\gamma_j)|^q = 2(\epsilon/2)^q \, ||\varphi||_q^q = 2(\epsilon/2)^q. \end{aligned}$$

In fact a slight computation yields

$$||R||_q^q \leqq 2^q[\exp(2(\epsilon/2)^q) - 1 - (\epsilon/2)^q].$$

Together with (6), this yields

$$\begin{aligned} ||s - \epsilon\varphi||_q &\leqq 2\alpha^2[\exp(2(\epsilon/2)^q) - (1 + 2(\epsilon/2)^q)]^{1/q} \\ &\leqq \alpha^2\epsilon^2. \end{aligned}$$

Now apply (4) and (5). We conclude: (i) $\epsilon^{-1}s \in B(\Gamma)$ and $\|\epsilon^{-1}s\|_B \leqq 4\epsilon^{-1}\alpha^2$; (ii) $\epsilon^{-1}s = \varphi$ on F; (iii) $\|\epsilon^{-1}s - \varphi\|_q \leqq \epsilon\alpha^2$. In particular, given any $\psi \in L^q(\Gamma)_F$ we can apply (i)–(iii) to its normalized real and imaginary parts. It follows that there is a $\mu \in M(G)$ satisfying

$$\text{(7)} \qquad \begin{aligned} &\text{(a) } \|\mu\| \leqq 8\epsilon^{-1}\alpha^2\, \|\psi\|_q\,, \\ &\text{(b) } \hat{\mu} = \psi \quad \text{on} \quad F, \text{ and} \\ &\text{(c) } \|\hat{\mu} - \psi\|_q \leqq 2\,\varepsilon\,\alpha^2\, \|\psi\|_q\,. \end{aligned}$$

The argument extends from finite sets F to E via a standard weak* compactness argument. □

We can now describe a large variety of new p-Sidon sets. Just consider the sets in [6] together with the following corollary.

Corollary. *Suppose $S \subset \Gamma$ is Sidon and $E \subset \Gamma$ is p-Sidon. Then $S \cup E$ is p-Sidon.*

Proof. We can assume $S \cap E = \varnothing$. The p-Sidon property and Theorem 3 imply that there exists $\beta > 0$ such that for any $\varphi \in L^q(\Gamma)_{S\cup E}$ there are measures μ, μ_1, $\mu_2 \in M(G)$ satisfying
(1) $\|\mu\| \leqq \beta$, $\hat{\mu} = 1$ on S, $|\hat{\mu}| < 1/4$ off S;
(2) $\|\mu_1\| \leqq \beta\, \|\varphi I_S\|_q$, $\hat{\mu}_1 = \varphi$ on S, $\|\hat{\mu}_1 - \varphi I_S\|_q \leqq 1/4\, \|\varphi I_S\|_q$;
(3) $\|\mu_2\| \leqq \beta\, \|\varphi I_E\|_q$, $\hat{\mu}_2 = \varphi$ on E.
Set $\hat{\nu} = (1 - \hat{\mu})\hat{\mu}_2 + \hat{\mu}_1$. Then $\nu \in M(G)$ and $\|\nu\| \leqq (1 + \beta)2\beta\, \|\varphi\|_q$. Moreover

$$\|\hat{\nu} I_S - \varphi I_S\|_q = 0$$

and

$$\|\hat{\nu} I_E - \varphi I_E\|_q = \|-\hat{\mu}\hat{\mu}_2 I_E + \hat{\mu}_1 I_E\|_q \leqq \tfrac{1}{2}\, \|\varphi\|_q\,.$$

Thus

$$\|\hat{\nu} I_{S\cup E} - \varphi\|_q \leqq \tfrac{1}{2}\, \|\varphi\|_q\,.$$

Now apply Lemma 1. □

Our last result exhibits some additional p-Sidon sets as an extension to the result in [6]. We outline much of the proof and refer the reader to [6] for the details. By $\pm A \pm B$ we mean $\{\delta a + \delta' b : \delta, \delta' \in \{-1, 1\}$ and $a \in A$, $b \in B\}$.

Theorem 4. *Suppose $A_1, \cdots, A_n$ are mutually disjoint infinite subsets of Γ whose union is dissociate. Then $E = \pm A_1 \pm A_2 \pm \cdots \pm A_n$ is p-Sidon if and only if $p \geqq 2n/(n + 1)$.*

Proof. Lemma 1 in [6] implies that $p \geqq 2n/(n + 1)$ if E is p-Sidon. Thus we need only prove that E is $p \equiv 2n/(n + 1)$-Sidon. To begin note that the 2^n sets of the form $E_\beta = \sum \beta_j A_j$ where $\beta = (\beta_1, \cdots, \beta_n) \in \{-1, 1\}^n$ are mutually disjoint since $\cup A_j$ is dissociate. Choose any β and a $\varphi \in L^q(\Gamma)_{E_\beta}$. We shall show that there is a $\mu_\beta \in M(G)$ such that $\hat{\mu}_\beta = \varphi$ on E_β while $\hat{\mu}_\beta \equiv 0$ on E_α for $\alpha \neq \beta$. The theorem then follows by considering sums of the form $\sum \mu_\beta$. It is sufficient to restrict our attention to real-valued φ and to $\beta \equiv (-1, 1, \cdots, 1) \in \{-1, 1\}^n$. Fix such a φ. As argued in [6], it follows that $\varphi \in C(A_1) \hat{\otimes} \cdots \hat{\otimes} C(A_n)$; hence we need only prove the following fact concerning basic tensor elements: there exists a constant $K > 0$ such that for any choice of real-valued functions $\varphi_1, \cdots, \varphi_n$ on $A_1, \cdots, A_n$, respectively, there is a $\mu \in M(G)$ with $\|\mu\| \leqq K \|\varphi_1\|_\infty \cdots \|\varphi_n\|_\infty$ satisfying $\hat{\mu} = 0$ on E_α for $\alpha \neq \beta$ and

$$\hat{\mu}(-\gamma_1 + \gamma_2 + \cdots + \gamma_n) = \varphi_1(\gamma_1) \cdots \varphi_n(\gamma_n)$$

on $E_\beta = -A_1 + A_2 + \cdots + A_n$.

To this end, assume for the moment that each A_j is finite and fix a choice of $\varphi_1, \cdots, \varphi_n$. We consider the Riesz polynomials

$$p_j(x) = \prod_{\gamma \in A_j} [1 + (2 \|\varphi_j\|_\infty)^{-1} \varphi_j(\gamma)(\gamma(x) + \overline{\gamma(x)})], \qquad 1 \leqq j \leqq n,$$

$$q_1(x) = \prod_{\gamma \in A_1} [1 + (2i \|\varphi_1\|_\infty)^{-1} \varphi_1(\gamma)(-\gamma(x) + \overline{\gamma(x)})],$$

and

$$q_j(x) = \prod_{\gamma \in A_j} [1 + (2i \|\varphi_j\|_\infty)^{-1} \varphi_j(\gamma)(\gamma(x) - \overline{\gamma(x)})], \qquad 2 \leqq j \leqq n.$$

The discussion of such polynomials in Theorem 3 implies that $\|p_j\|_1 = \|q_j\|_1 = 1$ and that $\hat{p}_j(\pm\gamma) = \varphi_j(\gamma)/(2\|\varphi_j\|_\infty)$, $\hat{q}_1(\pm\gamma) = \mp\varphi_1(\gamma)/(2i\|\varphi_1\|_\infty)$, and $q_j(\pm\gamma) = \pm\varphi_j(\gamma)/(2i \|\varphi_j\|_\infty) (j \neq 1)$, for all γ in the corresponding A_j, $1 \leqq j \leqq n$. In particular, the polynomials

$$P_j = (p_j - 1) \|\varphi_j\|_\infty, \qquad Q_j = (q_j - 1)i \|\varphi_j\|_\infty$$

and

$$R = \prod_{j=1}^{n} (P_j + Q_j)$$

satisfy

(1) $(P_j + Q_j)\hat{}(0) = 0$,

(2) $(P_1 + Q_1)\hat{}(\gamma) = 0$ and $(P_1 + Q_1)\hat{}(-\gamma) = \varphi_1(\gamma)$ for $\gamma \in A_1$,

(3) $(P_j + Q_j)\hat{}(\gamma) = \varphi_j(\gamma)$ and $(P_j + Q_j)\hat{}(-\gamma) = 0$ for $\gamma \in A_j$, $2 \leqq j \leqq n$, and

(4) $$\|R\|_1 \leqq 2^{2n} \prod_{j=1}^{n} \|\varphi_j\|_\infty .$$

Here (1)–(3) are immediate from the definitions and the fact that $\cup A_j$ is dissociate. To see (4) observe that R is the sum of 2^n terms, each of which has precisely n factors consisting of some combination of P_j's and Q_j's—each appearing only once. Since $||P_j||_1$, $||Q_j||_1 \leqq 2\ ||\varphi_j||_\infty$, it follows that each of those terms has L^1-norm bounded by $2^n \prod_j ||\varphi_j||_\infty$; whence (4). Again we use the dissociate property of $\cup A_j$, this time in conjunction with (1)–(3) to conclude

$$\hat{R}(-\gamma_1 + \gamma_2 + \cdots + \gamma_n) = \varphi_1(\gamma_1) \cdots \varphi_n(\gamma_n) \quad \text{for} \quad \gamma_j \in A_j$$

and

$$\hat{R} = 0 \quad \text{on} \quad E_\alpha \quad \text{for} \quad \alpha \neq \beta. \tag{5}$$

In light of (4), a weak* compactness argument extends (5) to infinite A_j for some $R \in M(G)$. □

Open questions.

1. Are all p-Sidon sets uniformizable r-Sidon sets for some $1 \neq p \leqq r < 2$? Indeed, do there exists uniformizable p-Sidon sets which are not Sidon sets? To be specific, let $A = \{3^{2n}\}_1^\infty$ and $B = \{3^{2n+1}\}_1^\infty$. Is $A + B$ a uniformizable p-Sidon set?

2. Is the union of two p-Sidon sets ($p \neq 1$) an r-Sidon set for some $p \leqq r < 2$? This is open even if one of the sets is assumed to be a uniformizable p-Sidon set.

3. There is a form of the Kahane and Salem necessary condition for Sidon sets for p-Sidon subsets of $\mathbf{Z}$ (see [5]). It extends immediately to any discrete Γ for which every $\gamma \neq 0$ has infinite order and actually improves somewhat for other discrete Γ's. The condition appears fairly tight. But what about sufficient conditions? For Sidon sets we at least have the Stečkin type conditions (see [7] or [8, Section 5.7.5]). For p-Sidon sets ($p \neq 1$) the best result so far in this direction is our Theorem 4. Is there some analogue to the Stečkin condition for p-Sidon sets?

4. Let S_p be the class of all p-Sidon subsets of Γ. It is immediate that $S_p \subset S_r$ if $p \leqq r$. Moreover, if $p_n = 2n/(n+1)$, then [6] tells us that $S_{p_n} \not\subseteq S_{p_{n+1}}$. If $1 \leqq p \neq r < 2$ must it follow that $S_p \neq S_r$?

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