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A note on graphs and rational balls

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Abstract In this short note we study some particular graphs associated to small Seifert spaces and Montesinos links. The study of these graphs leads to new examples of Seifert manifolds bounding rational homology balls and Montesinos links bounding smoothly and properly embedded surfaces (possibly not orientable) in the 4 ball with Euler characteristic equal to 1.

Keywords Seifert manifolds · Rational homology balls · Slice-ribbon conjecture

Mathematics Subject Classification 57M25

1 Introduction

The general question of determining whether or not a given 3–manifold bounds a rational homology ball was probably first raised by Andrew Casson and it is collected in Kirby's problem list [3, Problem 4.5]. In the case we are studying here, the answer to this question has direct implications in another problem in low dimensional topology, namely the sliceness of Montesinos knots. Indeed, the double cover of S^3 branched over a Montesinos knot is a Seifert space and it is well known that if a knot is slice, then its double branched cover is the boundary of a rational homology ball [2, Lemma 17.2].

Several partial results have appeared in the literature concerning the sliceness of Montesinos knots [1,5–7] among others. The ideas in the present article stem from Lisca's pioneer work on the sliceness of two bridge knots (a particular type of Montesinos knots). To every

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To Professor Maria Teresa Lozano on the occasion of her 70th birthday.

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two bridge knot Lisca associates a linear weighted graph Γ and a quantity $I(\Gamma)$ (see the next section for the precise definition) which can be read off from the graph and plays an essential role in the proofs. It turns out that if a two bridge knot is slice, then its double branched cover is the boundary of a plumbed manifold on a linear graph with $I(\Gamma) \in \{-1, -2, -3\}$. In [5] this study was extended to more general Montesinos knots and graphs with a single trivalent vertex and with the property $I(\Gamma) < -1$.

The main result of this note is a partial answer to the case $I(\Gamma) = -1$. There are two different families to consider. First, we deal with the case in which the Montesinos link with $I(\Gamma) = -1$ has two complementary legs. A thorough study of these links was carried out in [4] and Proposition 3.2 builds on the results therein. Then, we deal with the family of Montesinos links with $I(\Gamma) = -1$ and no complementary legs, which has no intersection with the results in [4]. We show that all knots corresponding to the graphs in Fig. 2 are slice. Our main results are summarized in the following theorem. The more technical and precise statements can be found in Sect. 3.

Theorem 1.1 *Consider a three legged Montesinos link whose double branched cover is the boundary of a negative definite 4-manifold obtained by plumbing according to a graph* Γ *with* $I(\Gamma) = -1$.

- *If the Montesinos link has two complementary legs, then it bounds a properly embedded surface* $\Sigma \subset B^4$ *such that* $\chi(\Sigma) = 1$ *if and only if it belongs to the list in Proposition 3.2.*
- *All the Montesinos links corresponding to the graphs in Fig. 2 bound properly embedded surfaces* $\Sigma \subset B^4$ *such that* $\chi(\Sigma) = 1$.

We can rephrase the above theorem in terms of Seifert spaces bounding rational homology balls.

Theorem 1.2 *Let* Y *be a small Seifert space bounding a negative definite 4-manifold obtained by plumbing according to a graph* Γ *with* $I(\Gamma) = -1$.

- *If the Seifert space has two complementary legs, then it is the boundary of a rational homology ball if and only if it belongs to the list in Proposition 3.2.*
- *All the Seifert spaces corresponding to the graphs in Fig. 2 bound rational homology balls.*

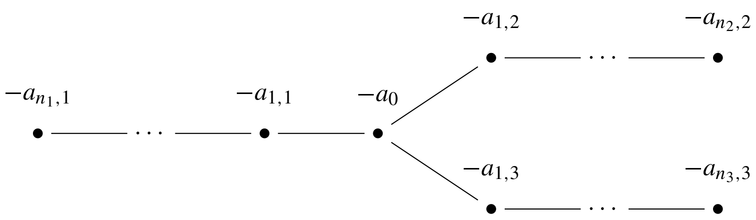
1.1 Organization of the paper

In Sect. 2 we introduce the strict minimum regarding Seifert manifolds and Montesinos links and in Sect. 3 we carry out the proofs of the theorems in the introduction.

2 Notation and conventions

2.1 Graphs and Seifert spaces

It is well known that, at least with one of the two orientations, a Seifert fibered rational homology sphere Y with three exceptional fibers is the boundary of a negative definite 4-manifold obtained by plumbing according to a three legged graph [10]:



where $a_{j,i} \geq 2$ and $a_0 \geq 1$. The Seifert invariants $Y = Y(b; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ can be read off from the graph as follows:

$$\frac{\alpha_i}{\beta_i} = [a_{1,i}, \ldots, a_{n_i,i}] := a_{1,i} - \cfrac{1}{a_{2,i} - \cfrac{1}{\ddots \, a_{n_i-1,i} - \cfrac{1}{a_{n_i,i}}}} \quad \text{and} \quad a_0 = -b.$$

If we are given a weighted graph P, like for example the one above, we will denote by M_P the 4-manifold obtained by plumbing according to P and by Y_P its boundary. The incidence matrix of the graph, which represents the intersection pairing on $H_2(M_P; \mathbb{Z})$ with respect to the natural basis given by the plumbing description, will be denoted by Q_P. The number of vertices in P coincides with $b_2(M_P) =: n$ and we will call $(\mathbb{Z}^n, Q_P)$ the intersection lattice associated to P. Let $(\mathbb{Z}^n, -\mathrm{Id})$ be the standard negative diagonal lattice. If the intersection lattice $(\mathbb{Z}^n, Q_P)$ admits a finite index extension $(\mathbb{Z}^n, -\mathrm{Id})$, we will write $P \subseteq \mathbb{Z}^n$.

The vertices of P are indexed by elements of the set $J := \{(s, \alpha) |\, s \in \{0, 1, \ldots, n_\alpha\},\ \alpha \in \{1, 2, 3\}\}$. Here, α labels the legs of the graph and $L_\alpha := \{v_{i,\alpha} \in P \,|\, i = 1, 2, \ldots, n_\alpha\}$ is the set of vertices of the α-leg. The *string* associated to the leg L_α is the n_α-tuple of integers $(a_{1,\alpha}, \ldots, a_{n_\alpha,\alpha})$, where $a_{i,\alpha} := -Q_P(v_{i,\alpha}, v_{i,\alpha}) \geq 0$. As a shortcut in notation we will write

$$(\ldots, 2^{[t]}, \ldots) := \left(\ldots, \underbrace{2, \ldots, 2}_{t}, \ldots \right).$$

The three legs are connected to a common central vertex, which we denote indistinctly by $v_0 = v_{0,1} = v_{0,2} = v_{0,3}$ (notice that, with our notation, v_0 does not belong to any leg).

A dichotomy which will play an essential role in the next section is given by the existence of *complementary legs* in a Seifert space. These are pairs of legs in the graph P associated to Seifert invariants with the property $\frac{\beta_i}{\alpha_i} + \frac{\beta_j}{\alpha_j} = 1$. If the legs L_2 and L_3 are complementary, it is well known then that the strings of integers $(a_{1,2}, \ldots, a_{n_2,2})$ and $(a_{1,3}, \ldots, a_{n_3,3})$ are related to one another by Riemenschneider's point rule [11]. A notion that will appear in the text and is related to complementary legs is that of the *dual* of a string of integers $(a_1, \ldots, a_n)$, $a_i \geq 2$. It refers to the unique string $(b_1, \ldots, b_m)$, $b_i \geq 2$, such that

$$\frac{1}{[a_1, \ldots, a_n]} + \frac{1}{[b_1, \ldots, b_m]} = 1.$$

Notice that the dual of a string of integers can be obtained via Riemenschneider's point rule, which guarantees its existence and uniqueness.

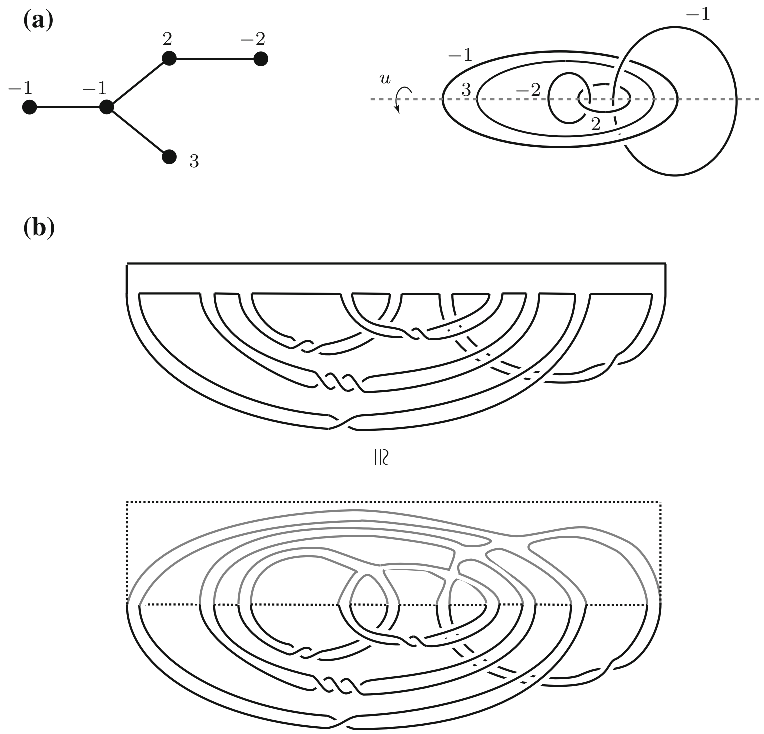


Fig. 1 Part **a** shows a 3 legged plumbing graph and its associated Kirby diagram as a strongly invertible link. The bottom picture shows the branch surface in B^4 which is the image of $\mathrm{Fix}(u)$. This surface is homeomorphic to the result of plumbing bands according to the initial graph: the gray lines retract onto the sides of the rectangle

Given a weighted graph P, the following quantity will be assumed to be equal to -1 in all the results in this note:

$$I(P) := a_0 - 3 + \sum_{(s,\alpha)\in J, s\neq 0} (a_{s,\alpha} - 3).$$

A property of this quantity is that given a *linear* graph P and the graph P^*, whose associated string is dual to that of P, it holds $I(P) + I(P^*) = -2$.

2.2 Montesinos links

Seifert spaces admit an involution which presents them as double covers of S^3 with branch set a link. This link can be easily recovered from any plumbing graph P which provides a surgery presentation for the Seifert manifold Y_P. As shown in Fig. 1 the Kirby diagram obtained from P is a strongly invertible link. The involution on Y_P, which is a restriction of an involution on M_P, yields as quotient S^3 and the branch set of the involution on M_P consists of a collection of twisted bands plumbed together. The twists correspond to the weights in the graph and the plumbing to the edges. The *Montesinos link* is by definition the boundary of the surface obtained by plumbing bands according to P [8]. An example is shown in Fig. 1.

Given a three legged graph P, there is a unique Montesinos link ML_P which can be obtained by the above procedure. If we start with a linear graph P', then the same procedure yields a 2-bridge link and in this case $Y_{P'}$ is a lens space.

3 Proofs

Theorems 1.1 and 1.2 are direct consequences of the results in this section. The statement concerning Montesinos links/Seifert spaces with complementary legs is related to Proposition 3.2 and Corollary 3.3. The idea of the proof is the following: if a Montesinos link in S^3 is the boundary of a smoothly and properly embedded surface Σ in B^4 such that $\chi(\Sigma) = 1$, then, its double branched cover, a Seifert space, is the boundary of a rational homology ball. On the other hand, every Seifert space is the boundary of a negative definite 4-dimensional manifold obtained by plumbing according to a star-shaped graph. By glueing the plumbing to the rational homology ball, we obtain a closed, smooth, negative definite 4-manifold, which then, by Donaldson's theorem, has a standard intersection form. It follows that there exists an embedding of the lattice associated to the plumbing into the negative standard lattice of the same rank. Proposition 3.2 studies the consequences of the existence of this embedding building on previous work [4,7]. Specially relevant will be the following classification due to Lisca [7, Lemmas 7.1 and 7.2].

Lemma 3.1 (Lisca) *Given a* linear *weighted graph P with associated string $(a_1, \ldots, a_n)$ and an intersection lattice admitting an embedding into $(\mathbb{Z}^n, -\mathrm{Id})$, the following holds:*

1. *If $I(P) = -3$, then $(a_1, \ldots, a_n) = (b_k, \ldots, b_1, 2, c_1, \ldots, c_\ell)$ where $k, \ell \geq 1$, the k-tuple of integers $b_1, \ldots, b_k \geq 2$ is arbitrary and $c_1, \ldots, c_\ell$ are obtained from $b_1, \ldots, b_k$ using Riemenschneider's point rule.*
2. *If $I(P) = -2$ either $(a_1, \ldots, a_n)$ or $(a_n, \ldots, a_1)$ has one of the following forms:*

   (a) $(2^{[t]}, 3, 2+s, 2+t, 3, 2^{[s]})$, $\quad s, t \geq 0$,
   (b) $(2^{[t]}, 3+s, 2, 2+t, 3, 2^{[s]})$, $\quad s, t \geq 0$,
   (c) $(b_k, \ldots, b_1 + 1, 2, 2, 1 + c_1, \ldots, c_\ell)$, *for arbitrary integers $b_1, \ldots, b_k \geq 2$ and $c_1, \ldots, c_\ell$ obtained from $b_1, \ldots, b_k$ using Riemenschneider's point rule.*

The next proposition establishes a complete list of three legged graphs with $I = -1$, two complementary legs and such that the corresponding Seifert manifolds bound rational homology balls. The proof relies on the above described existence of an embedding of the lattice associated to the graph P into the standard negative lattice of the same rank.

Proposition 3.2 *Let $P \subseteq \mathbb{Z}^n$ be a weighted graph with a trivalent vertex v_0, two complementary legs, L_2 and L_3, $I(P) = -1$ and such that Y_P bounds a rational homology ball. Then, the numbers $\{a_0, a_{1,1}, \ldots, a_{n_3,3}\}$ satisfy:*

- *The strings associated to the complementary legs, namely $(a_{1,2}, \ldots, a_{n_2,2})$ and $(a_{1,3}, \ldots, a_{n_3,3})$, are related to each other by Riemenschneider's point rule.*
- *The linear set $L_1 \cup \{v_0\}$ has an associated string $(a_{n_1,1}, \ldots, a_{1,1}, a_0)$ which is obtained by one of the two following ways:*

  1. *either from the dual string associated to a linear set with $I = -2$ by adding 1 to the final vertex that plays the role of central vertex in P. The complete list of the possible $(a_{n_1,1}, \ldots, a_{1,1}, a_0)$ is the following.*
     $(t+2, 3, 2^{[s-1]}, 3, 2^{[t-1]}, 3, s+3)$ $\qquad \forall s, t \geq 1$,

$(2, 3, 2^{[s-1]}, 4, s+3)$ $\forall s \geq 1,$
$(s+2, 4, 2^{[s-1]}, 3, 3)$ $\forall s \geq 1,$
$(t+2, 2^{[s]}, 4, 2^{[t-1]}, 3, s+3)$ $\forall s \geq 0, \ \forall t \geq 1,$
$(s+2, 3, 2^{[t-1]}, 4, 2^{[s]}, t+3)$ $\forall s \geq 0, \ \forall t \geq 1,$
$(b_k, b_{k-1}, \ldots, b_1, 5, c_1, \ldots, c_{\ell-1}, c_\ell+1)$, *for arbitrary integers* $b_1, \ldots, b_k \geq 2$ *and for* $c_1, \ldots, c_\ell$ *obtained from* $b_1, \ldots, b_k$ *using Riemenschneider's point rule.*

2. *or from the dual string associated to a linear set with* $I = -3$ *by adding a final 2 and augmenting of 1 the value of its adjacent number in the string. The final 2 will play the role of central vertex in* P.

Proof Since Y_P has two complementary legs and bounds a rational homology ball, it belongs to the list $\mathscr{L}$ in [4, Section 3.1]. It follows that the linear set $L_1 \cup \{v_0\}$ is of the form $(a_{n_1,1}, \ldots, a_{r,1}, 2^{[k]})$ with $a_{r,1} > 2$, $k + r = n_1 \geq 1$ and $k \geq 0$. By [4, Proposition 3.1] the lens space with associated string $(a_{n_1,1}, \ldots, a_{r,1}, 2^{[k-1]}, 1)$, if $k \geq 1$, or $(a_{n_1,1}, \ldots, a_{r,1} - 1)$, if $k = 0$, is rational homology cobordant to Y_P and thus bounds a rational homology ball.

A straightforward computation gives $I(P) = I(L_1 \cup \{v_0\}) - 2$ and therefore by our assumptions $I(L_1 \cup \{v_0\}) = 1$. If $k = 0$ then $(a_{n_1,1}, \ldots, a_{r,1} - 1)$ defines a lens space with $I = 0$ which bounds a rational homology ball. It follows by work of Lisca [7] that this string is necessarily dual to a string with $I = -2$ in Remark 3.1. Computing the dual strings (and their reversals) and adding one to the final vector that plays the role of central vertex in P we obtain the list of strings in statement (1) of the theorem.

If $k > 0$ then we claim that $k = 1$. Indeed, if $k \geq 2$ and $I(L_1 \cup \{v_0\}) = 1$ we have that $(a_{n_1,1}, \ldots, a_{r,1} - 1)$ bounds a rational homology ball (since it is cobordant to $(a_{n_1,1}, \ldots, a_{r,1}, 2^{[k-1]}, 1)$) while satisfying $I((a_{n_1,1}, \ldots, a_{r,1} - 1)) \geq 2$, which contradicts the results in [7]. We are only left with the case $k = 1$ and $(a_{n_1,1}, \ldots, a_{r,1} - 1)$ bounding a rational homology ball while satisfying $I((a_{n_1,1}, \ldots, a_{r,1} - 1)) = 1$. We know then that this string is dual to one in the family with $I = -3$ in Remark 3.1 and statement (2) in the theorem follows. □

Corollary 3.3 *A Montesinos knot with two complementary legs and* $I = -1$ *is slice if and only if it is of the form described in the statement of Proposition 3.2.*

Proof The double cover of a slice Montesinos knot with two complementary legs and $I = -1$ is a Seifert fibered space which bounds a rational homology ball and satisfies the hypothesis of Proposition 3.2. The converse holds by [4, Proposition 3.5]. □

In the following proposition we deal with Montesinos links (or small Seifert fibered spaces) with $I = -1$ and no complementary legs. We present 8 infinite families of Seifert spaces and describe explicitly, via Kirby calculus, how they bound rational homology balls. This result can be directly translated using the correspondence in Fig. 1 into a description of ribbon surfaces of Euler characteristic one bounded by the corresponding Montesinos links.

Proposition 3.4 *The graphs* P *in Fig. 2 have no complementary legs, satisfy* $I(P) = -1$ *and the associated Montesinos links bound a ribbon surface,* $\Sigma \subset S^3$*, possibly not orientable, such that* $\chi(\Sigma) = 1$. *This implies in particular that the associated Seifert fibered spaces bound rational homology balls.*

Proof The families $(a1)$ to (d) are represented schematically in Fig. 3 while families $(e1)$ and $(e2)$ are treated in Fig. 5. A black square on a circle represents a possible linear plumbing linked to it. The thicker circle represents an added 2-handle. We perform Kirby calculus on

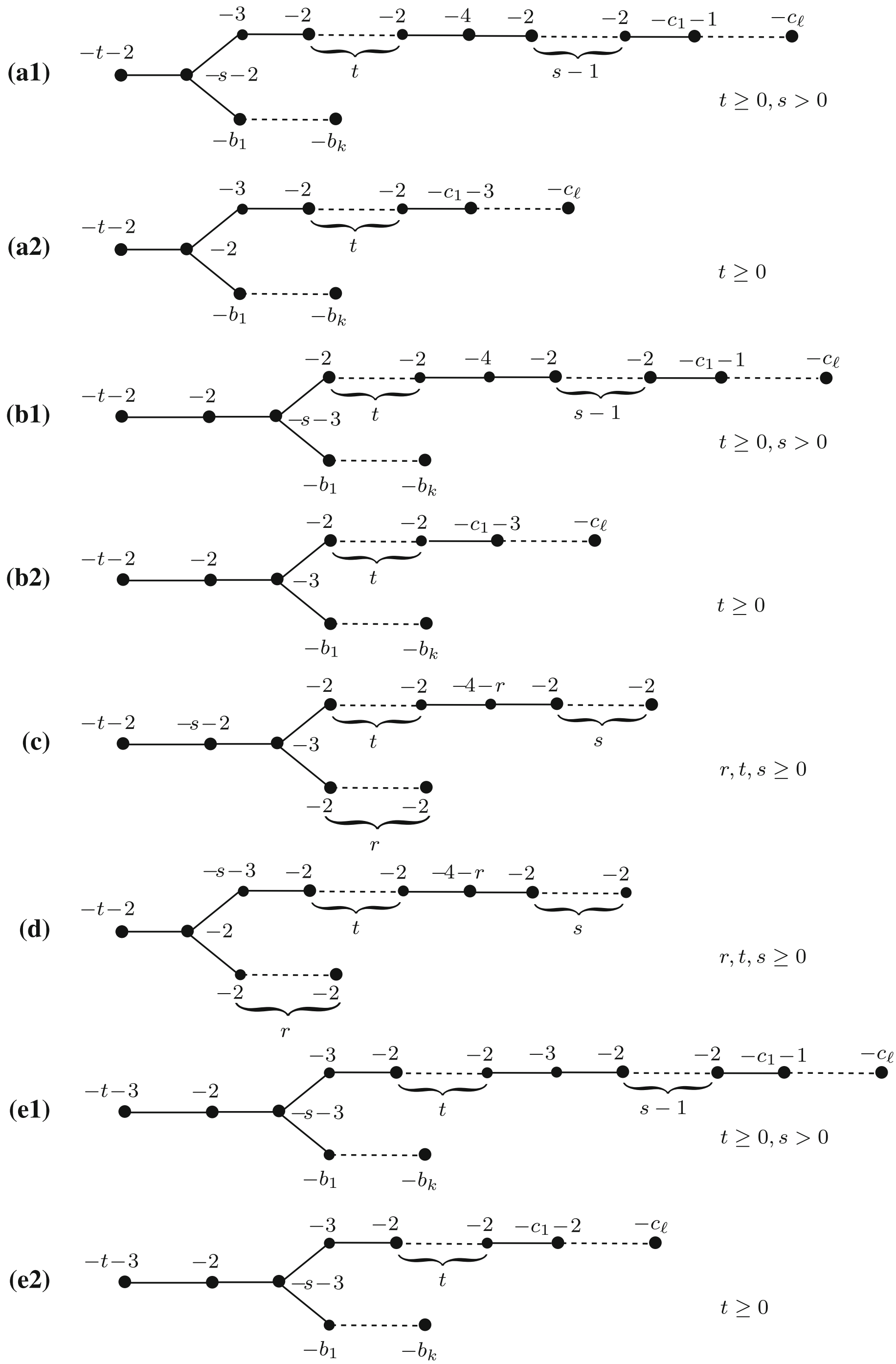


Fig. 2 Notice that, family ($a2$) [resp. ($b2$), ($e2$)] corresponds to family ($a1$) [resp. ($b1$), ($e1$)] with $s = 0$

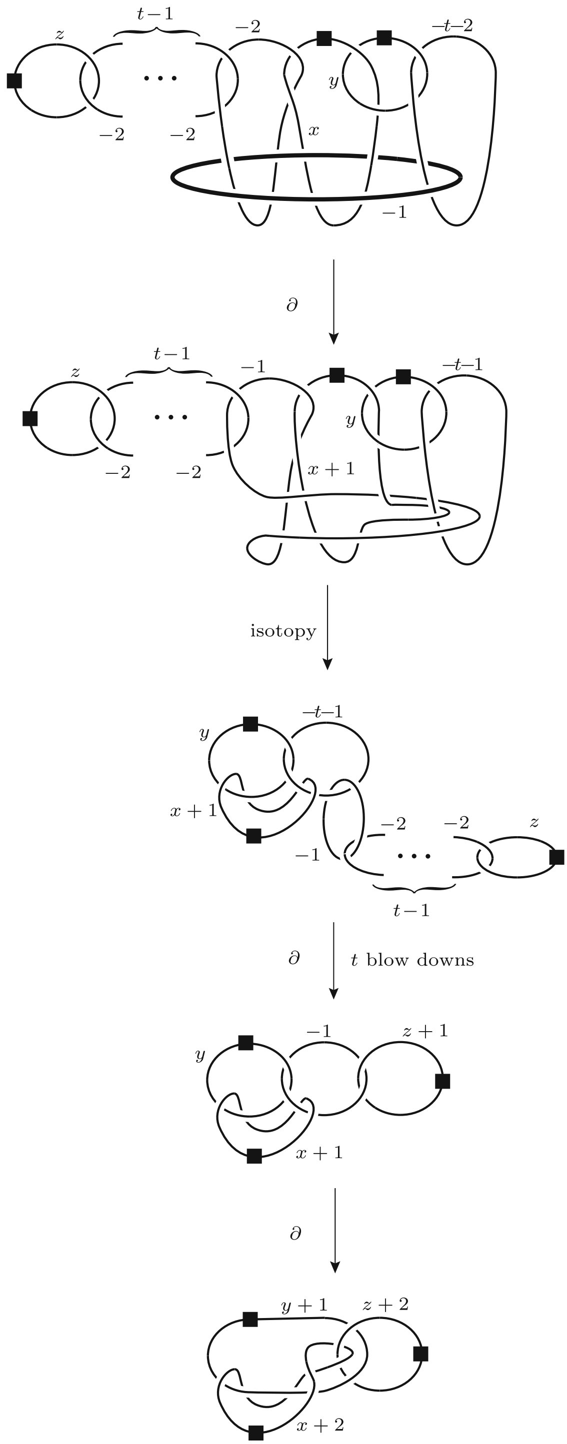


Fig. 3 This diagram shows how to add a 2-handle with framing −1 to the families (*a*1) to (*d*) of Fig. 2, in order to obtain a 4-manifold with boundary $S^1 \times S^2$. The framings x, y and z, and the linear plumbings represented by black squares differ in the six families. The arrows in the diagram represent either blow downs or isotopies

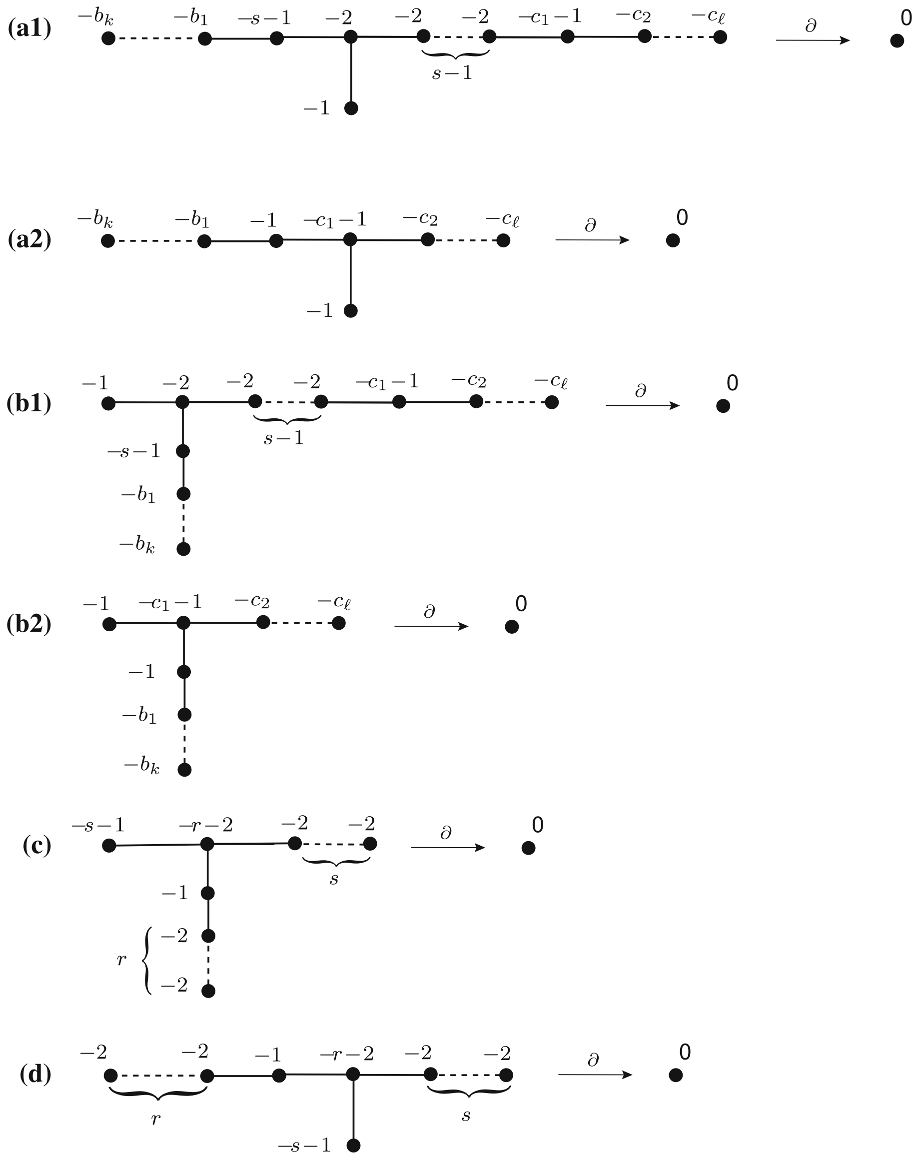


Fig. 4 Graphs $(a1)$ to (d) are obtained from the last diagram of Fig. 3 by substituting x, y, z and the black squares with the data from the corresponding graphs in Fig. 2

these general figures and then recover the different families by substituting x, y, z and the black squares with the corresponding framings and linear plumbings respectively. This is done in Fig. 4, where each graph is obtained substituting in the last diagram of Fig. 3, which is star-shaped, the data of the corresponding family. Every graph in Fig. 4 has a vertex with weight -1. Blowing down this -1 we obtain a new graph with a new vertex with weight -1. In each family this blowing down operation can be iterated until we are left with a graph

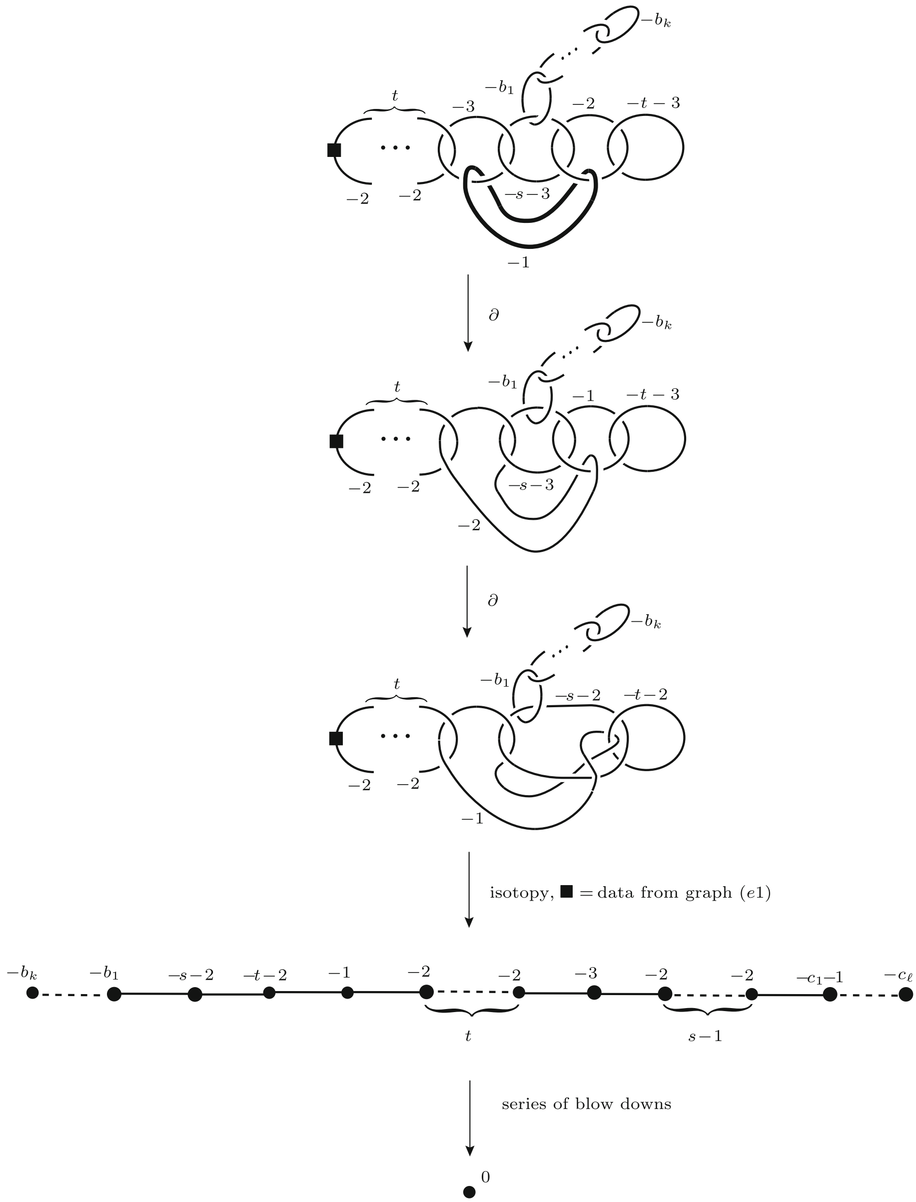


Fig. 5 This diagram shows how to add a 2-handle with framing -1 (the thicker one) to the families ($e1$) and ($e2$) of Fig. 2, in order to obtain a 4-manifold with boundary $S^1 \times S^2$. The vertical arrows in the diagram represent either blow downs or isotopies. The linear graph is obtained substituting ■ with the data from family ($e1$)

with only one vertex of weight 0, which represents the 4-manifold $D^2 \times S^2$ with boundary $S^1 \times S^2$.

The first diagram of Figs. 3 and 5 is a strongly invertible link in S^3 with respect to the involution u as shown in Fig. 6. Let us call M' the 4-manifold with boundary $S^1 \times S^2$

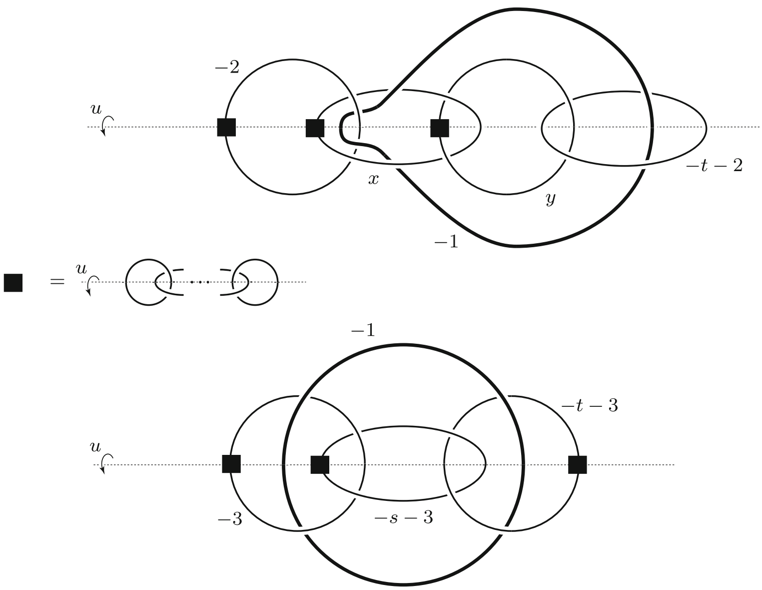


Fig. 6 Strongly invertible link representation of the first links in Figs. 3 and 5

obtained from M_P by adding the 4-dimensional 2-handle of Figs. 3 and 5. It follows from [9] that M_P is the double cover of D^4 branched along a surface B_P, which consists of bands plumbed according to the graph P. In turn, M' is the double cover of D^4 branched over the surface B', which is B_P with an additional band. The addition of this band corresponds to a ribbon move on the Montesinos link $\mathrm{ML}_P = \partial B_P$. This ribbon move necessarily leads to two unlinked unknots, since $\partial M' = S^1 \times S^2$ and whenever $S^1 \times S^2$ double branch covers S^3, the branch set is the unlink of two unknotted components. By elementary facts on the classification of surfaces with boundary it follows that, if ML_P is a knot, then it bounds a ribbon disc in S^3; if ML_P is a 2-component link, it bounds a ribbon surface in S^3 which is the disjoint union of a disc and a Möbius band; finally, if ML_P is a 3-component link, then it bounds a ribbon surface in S^3 which is the disjoint union of a disc and an annulus. □

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