ON HYPERBOLIC KNOTS IN S^3 WITH EXCEPTIONAL SURGERIES AT MAXIMAL DISTANCE

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Abstract. Baker showed that 10 of the 12 classes of Berge knots are obtained by surgery on the minimally twisted 5–chain link. In this article we enumerate all hyperbolic knots in S^3 obtained by surgery on the minimally twisted 5–chain link that realise the maximal known distances between slopes corresponding to exceptional (lens, lens), (lens, toroidal) and (lens, Seifert fibred) pairs. In light of Baker's work, the classification in this paper conjecturally accounts for "most" hyperbolic knots in S^3 realising the maximal distance between these exceptional pairs. As a byproduct, we obtain that all examples that arise from the 5–chain link actually arise from the magic manifold. The classification highlights additional examples not mentioned in Martelli and Petronio's survey of the exceptional fillings on the magic manifold. Of particular interest, is an example of a knot with two lens space surgeries that is not obtained by filling the Berge manifold.

1. Introduction

Thurston's ground-breaking work in the 1970s showed that every non-trivial knot that is not a satellite is hyperbolic, and that non-hyperbolic surgeries on such knots are "exceptional". These deep and surprising results reinvented the field of hyperbolic geometry and knot theory. With the exception of S^3, given a non-hyperbolic manifold M the set of all cusped hyperbolic manifolds with M as a filling is unwieldy, and we shouldn't expect to be able to write down the set of all hyperbolic manifolds which have a lens space filling. However, in light of Thurston's work, it becomes reasonable to ask which hyperbolic knots in S^3 have a lens space surgery, or which hyperbolic knots have a toroidal filling that is "far" from the S^3 filling. This paper looks at hyperbolic knots in S^3 that have exceptional fillings that are "far" apart.

Let K be a knot in S^3 and consider its exterior $S^3 \setminus \nu(K)$ where $\nu(K)$ is a small open regular neighborhood of the knot. For a slope α (the isotopy class of an essential simple closed curve) on the boundary of the exterior of K, the closed manifold obtained from α-surgery (gluing a solid torus to the exterior of K by identifying the meridian to α) is denoted by $K(\alpha)$.

Suppose that K is hyperbolic, that is, its complement admits a Riemannian metric of constant sectional curvature -1 which is complete and of finite volume. Then Thurston's hyperbolic Dehn surgery theorem implies that all but finitely many slopes produce hyperbolic manifolds via surgery see [Th] and [BP]. The exceptional cases are called exceptional slopes and exceptional surgeries.

It is a consequence of the geometrization theorem that every exceptional surgery on a hyperbolic *link* is either S^3, a lens space, has an essential surface of non-negative Euler characteristic, or fibres over the sphere with three exceptional fibres. We now assign the following standard names to these classes of non-hyperbolic 3-manifolds following [G1]. We say that a manifold is of type D, A, S or T if it contains, respectively, an essential disc, annulus, sphere or torus, and of type S^H or T^H if it contains a Heegaard sphere or torus. Finally we denote by Z the type of small closed Seifert manifolds. Notice that $S^H = \{S^3\}$ and that T^H is the set of lens spaces (including $S^1 \times S^2$).

In the present paper, we are interested in hyperbolic manifolds X with a torus boundary component τ supporting a pair (α,β) of exceptional slopes whose associated surgeries lead respectively to manifolds of types C_1 and C_2, where $C_1, C_2 \in \{S^H, S, T^H, T, D, A, Z\}$, the set of exceptional type manifolds described

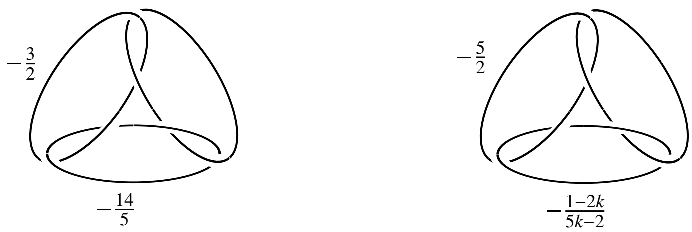

Figure 1: Surgery presentation for all (distinct) hyperbolic knots with two lens space fillings obtained by surgery on the minimally twisted 5–chain link.

above. We will summarize this situation by writing $(X, \tau; \alpha, \beta) \in (C_1, C_2)$. The distance (minimal geometric intersection) between two slopes α and β on a torus is denoted by $\Delta(\alpha, \beta)$. The *maximal distance between types of exceptional manifolds* C_1 and C_2 is defined as the max $\{\Delta(\alpha, \beta) \,|\, (X, \tau; \alpha, \beta) \in (C_1, C_2)\}$ and denoted by $\Delta(C_1, C_2)$.

Quite some energy has been devoted in the literature to the understanding of exceptional slopes on hyperbolic manifolds. In the case of hyperbolic knot exteriors there are strong restrictions on their exceptional surgeries or fillings. The S^H–filling is unique [GL1] and no knot exterior has a filling with an essential annulus or disc. Conjecturally no hyperbolic knot exterior has a reducible surgery [GAS]. So, there are nine possible exceptional pairs obtained by surgery on a hyperbolic knot in S^3, namely the (S^H, T^H), (S^H, T), (S^H, Z), (T^H, T^H), (T^H, T), (T^H, Z), (T, T), (T, Z) and (Z, Z) exceptional pairs.

The (S^H, T) pairs have been completely enumerated [GL2]. Examples of (S^H, Z) pairs have been constructed, see for example [Eud, Rou2]. The exceptional surgeries on the figure eight knot tell us that $\Delta(T^H, Z), \Delta(T, Z), \Delta(Z, Z) > 5$, and from [Ago] we know that there are only a finite number of examples realising these distances. The (S^H, T^H) pairs are conjecturally a subset of the Berge knots classified in [Be1]. It follows that, since the remaining three cases all involve a T^H surgery, an enumeration of the remaining three exceptional pairs is conjecturally an enumeration of a subset of Berge knots. Baker showed [Bak] that 10 of the 12 classes of Berge knots are obtained by surgery on the minimally twisted 5–chain link (5CL, see Figure 3). So, conjecturally, most of the hyperbolic knots realising (T^H, T^H), (T^H, T) and (T^H, Z) exceptional pairs of slopes are obtained by surgery on 5CL.

In this article we enumerate all hyperbolic knots obtained from surgery on the 5CL that realise the maximum known distance between the exceptional filling types. We completely classify the knots arising in this manner and having either two different lens space surgeries; a lens space surgery and a toroidal surgery at distance 3; or a lens space surgery and a small Seifert surgery at distance 2. In light of Baker's work, the classification in this article conjecturally accounts for most examples of hyperbolic knots with an exceptional pair of slopes at maximal distance. Our main result is the following:

Theorem 1.1. *Let K be a hyperbolic knot in S^3 obtained by surgery on the minimally twisted 5–chain link with two exceptional slopes α and β, and such that $K(\alpha)$ is a lens space.*

- *If $K(\beta)$ is a lens space, then K is found in Figure 1.*
- *If $K(\beta)$ is toroidal, then the distance between α and β is at most three and if the distance equals three, then K is found in Figure 2.*
- *If $K(\beta)$ fibres over the sphere with three exceptional fibres then the distance between α and β is at most two and if the distance equals two then K is found in Figure 2.*

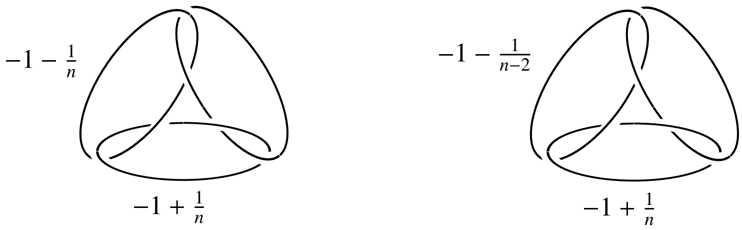

Figure 2: Surgery presentation for all (distinct) hyperbolic knots with a lens space and a toroidal filling at distance 3, or a lens space and a Seifert filling at distance 2, obtained by surgery on 5CL.

Given M, an orientable cusped hyperbolic 3–manifold and τ a fixed torus component of the boundary of its compactification, it is a consequence of [LM] that 8 is a universal upper bound for $\Delta(\alpha_1, \alpha_2)$ for each exceptional pair $(M, \tau; \alpha_1, \alpha_2)$. The celebrated Gordon-Luecke theorem [GL1] can be formulated by saying that $\Delta(S^H, S^H) = 0$, the Cabling conjecture by saying that $\Delta(S, S^H) = -\infty$ [GAS], the Berge conjecture implies that the Berge knots in [Be1] contain all exceptional pairs of type (S^H, T^H), and the theorem of [GL2] by saying that the knots realizing $\Delta(\alpha_1, \alpha_2) = \Delta(S^H, T)$ are precisely the Eudave-Muñoz knots.

It is natural to generalise these types of questions by asking whether we can find $\Delta(C_1, C_2)$ for each pair of classes $C_1, C_2 \in \{S^H, S, T^H, T, D, A, Z\}$, and whether we can enumerate all $(M, \tau; \alpha_1, \alpha_2)$ of type (C_1, C_2) with $\Delta(\alpha_1, \alpha_2) = \Delta(C_1, C_2)$. A great deal is known, see [GL3] or [G2] for an overview.

If a knot in S^3 is not a torus knot or a satellite knot then its exterior is a hyperbolic 3-manifold. We can consider all $(M_K, \tau; \alpha_1, \alpha_2)$ when M_K is the exterior of a knot K in S^3 and ask what is $\Delta(C_1, C_2)$ and which $(M_K, \tau; \alpha_1, \alpha_2)$ of type (C_1, C_2) have $\Delta(\alpha_1, \alpha_2) = \Delta(C_1, C_2)$ for this subclass of hyperbolic manifolds. Of course, this is the same as asking what is the greatest value of $\Delta(\alpha_2, \alpha_3)$ among exceptional triples $(M, \tau; \alpha_1, \alpha_2, \alpha_3)$ of type (S^H, C_1, C_2) and which $(M, \tau; \alpha_1, \alpha_2, \alpha_3)$ realise the maximum $\Delta(\alpha_2, \alpha_3)$. From this perspective, we enumerate in this article such (S^H, C_1, C_2) triples obtained from the minimally twisted 5–chain link.

In order to state some of the noteworthy remarks coming from the analysis done to establish Theorem 1.1 we need to introduce some more notation. The chain links that are ubiquitous throughout this paper are depicted in Figure 3. We keep the notation conventions of [MPR] and denote the minimally twisted 5–chain link by 5CL and its exterior by M_5; the 4–chain link is denoted by 4CL and its exterior is denoted by M_4. The minimally twisted 4–chain link M4CL, as well as its exterior F, will also appear extensively in the text. A (-1)–surgery on any component of 4CL gives a 3–chain link 3CL, whose exterior is denoted by M_3. We closely reference the tables from [MP] which give a classification of the exceptional surgeries on the mirror 3CL* shown in Figure 3. The exterior of this link is the "magic manifold" [GW] which we will denote by N.

The knots in Figures 1 and 2 are described by giving a filling instruction on two of the 3 boundary components of the magic manifold. The exceptional slopes on the knots $N(-\frac{3}{2}, -\frac{14}{5})$ and $N(-\frac{5}{2}, \frac{1-2k}{5k-2})$ from Figure 1 and the corresponding fillings are found in Theorem 3.1. The exceptional slopes on $N(-1 + \frac{1}{n}, -1 - \frac{1}{n})$ and $N(-1 + \frac{1}{n}, -1 - \frac{1}{n-2})$ from Figure 2 and the corresponding fillings are found in Theorem 4.1. Theorems 3.1 and 4.1 go further and show that the three families of knots and the isolated example shown in Figures 1 and 2 are all distinct knots.

There is a unique hyperbolic knot in a torus with two non-trivial surgeries [Be2]; the exterior of this knot is called the Berge manifold, which will appear frequently in the text. It can be obtained by filling one of the 3 boundary components of the magic manifold N. Indeed, the Berge manifold is $N(-\frac{5}{2})$. Cutting,

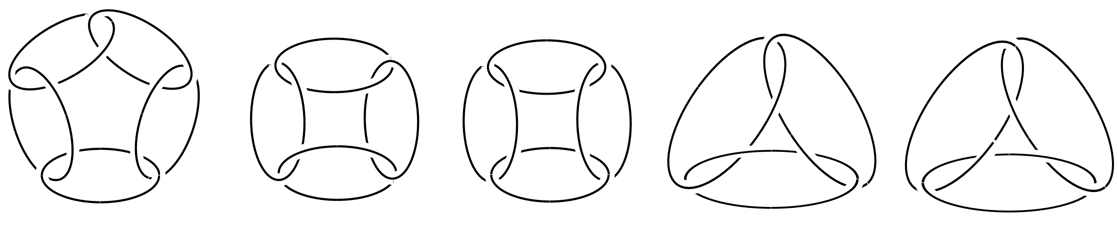

Figure 3: The minimally twisted 5–chain link 5CL, the 4–chain link 4CL, the minimally twisted 4–chain link M4CL and the 3–chain links 3CL and 3CL*. The exteriors of these links are respectively called M_5, M_4, F, M_3 and N.

twisting and filling the boundary of the torus yields an infinite family of inequivalent knots in S^3 with two lens space fillings. This family is precisely the set of $N(-\frac{5}{2}, \frac{1-2k}{5k-2})$ from Theorem 1.1. It should be highlighted that the example $N(-\frac{3}{2}, -\frac{14}{5})$ is not obtained by surgery on the Berge manifold (Theorem 3.1).

The article [BDH] contains a complete description of all surgeries on the 5CL with three cyclic fillings, which are fillings leading to type S^H or T^H manifolds. It is a more general question than our quest to find *knot exteriors* on the 5CL with two cyclic fillings and the techniques used in [BDH] to reduce the argument to an analysis of the fillings on the Magic manifold are different from ours. A translation of our results into the language of [BDH] follows. The family $\{N(-\frac{5}{2}, \frac{1-2k}{5k-2})\}$ is the family $\{B_{(2k-1)/(5k-2)}\} \subset \{B_{p/q}\}$ from [BDH], and the isolated example $N(-\frac{3}{2}, -\frac{14}{5})$ is $A_{2,3}$ from [BDH]. As a final remark, let us emphasize the fact that the family $N(-1 + \frac{1}{n}, -1 - \frac{1}{n})$ and its exceptional slopes and fillings are highlighted in [MP, Table 17], but the distinct family $N(-1 + \frac{1}{n}, -1 - \frac{1}{n-2})$ is not.

1.1. **Article structure.** The results in this article are obtained by a careful analysis of the classifications of exceptional sets of slopes on surgeries of the minimally twisted 5–chain link given in [MP] and [Rou2]. The work done there, translates the enumeration of exceptional pairs realising maximal distances into finding the solutions to a (long) list of elementary diophantine equations. The translation necessitates a table by table analysis of the work given in [MP] and [Rou2]. A collection of easy (but technical) lemmata in the Appendix facilitates the translation and reduces the amount of work needed. The proofs of the main results are littered with references to results in the Appendix, [MP], and [Rou2]. Most equalities and isomorphisms shall, for instance, be subscripted by a label that refers to the Appendix. Therefore, this article is best read with both articles and the Appendix in-hand.

Section 2 sets out the notation and conventions used throughout this article. Section 3 gives an enumeration of all exceptional (S^H, T^H, T^H) triples obtained by surgery on 5CL. Section 4 gives an enumeration of all exceptional (S^H, T^H, T) triples obtained by surgery on 5CL. Section 5 gives an enumeration of all exceptional (S^H, T^H, Z) triples obtained by surgery on 5CL. Sections 3-5 all proceed in the same way. The sections start with a precise statement about the enumeration of the exceptional triples. The results are established by first showing that all examples are obtained by surgery on 4CL, and then showing that all examples are obtained by surgery on 3CL. The final sections then enumerate all examples of exceptional triples obtained by surgery on 3CL.

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2. Notation and conventions

In this section we set out notation and conventions used throughout the article. We will use the conventions on surgery instructions set out in [Rou2] which we briefly outline. For more detailed descriptions, please refer to [Rou2]. Given an orientable compact 3-manifold X such that ∂X is a collection of tori, we use the term *slope* to indicate the isotopy class of a non-trivial unoriented essential simple closed curve on a component of ∂X. After fixing a choice of meridian and longitude on a boundary torus, a slope is naturally identified with an element in $\mathbb{Q} \cup \{\infty\}$. A *filling instruction* α (also denoted by $\mathcal{F}$) for X is a set consisting of either a slope or the empty set for each component of ∂X. The chain links have a rotational symmetry which allow us to unambiguously choose any component as the first component and order the remaining cyclically in the anticlockwise direction. The filling instructions shall then be identified with tuples of elements in $\mathbb{Q} \cup \{\infty\}$. The *filling* $X(\alpha)$ is the manifold obtained by gluing one solid torus to ∂X for each non-empty slope in α. The meridian of the solid torus is glued to the slope.

A very related concept to that of a filling is a *surgery* on a link $L \subset S^3$. By definition, a surgery on L is a filling of the exterior of L, $S^3 \setminus \nu(L)$, where $\nu(L)$ is an open regular neighborhood of L. By a *surgery instruction* for L we mean a filling instruction on the exterior of L.

In the present article we will be concerned with *exceptional fillings*. If the interior of X is hyperbolic but the interior of $X(\alpha)$ is not, we say that α is an *exceptional filling instruction* for X and $X(\alpha)$ is an exceptional filling. If the resulting manifold has a cyclic fundamental group, that is if it belongs to S^H or T^H, then we say that it is furthermore *cyclic*. We follow the notation used to describe the sets of exceptional slopes set out in [G2]. The set of exceptional slopes on a fixed toroidal boundary component τ of a hyperbolic 3-manifold X is denoted by $E_\tau(X)$, and the cardinality of $E_\tau(X)$ by $e_\tau(X)$. In our case τ will refer to the n^{th} component of the chain link with n components and is dropped throughout the article. A word of caution: when $\mathcal{F}$ is a filling instruction on M_5, we write the elements of $E(M_5(\mathcal{F}))$ with respect to the choice of bases on M_5 (*and not* $M_5(\mathcal{F})$!).

Beyond the exceptional pairs (C_1, C_2) explained in the introduction, we will work also with *exceptional* $(C_1, ..., C_n)$ *n-tuples*. By this we mean the following: if X is a hyperbolic 3-manifold and $\alpha_1, \dots, \alpha_n$ are exceptional slopes on a fixed toroidal boundary component of X, with $X(\alpha_i)$ a manifold of type C_i, then we say that $(X, \alpha_1, ..., \alpha_n)$ is an exceptional $(C_1, ..., C_n)$ n-tuple and write $(X, \alpha_1, ..., \alpha_n) \in (C_1, ..., C_n)$. There is a notion of equivalence among exceptional tuples. We will say that two exceptional n-tuples $(X_1, \alpha_1, ..., \alpha_n)$ and $(X_2, \beta_1, ..., \beta_n)$ are *equivalent* if there exists a homeomorphism $h : X_1 \to X_2$ with $X_2(h(\alpha_i)) = X_2(\beta_i)$. When two n-tuples $(X_1, \alpha_1, ..., \alpha_n)$, $(X_2, \beta_1, ..., \beta_n)$ are equivalent we write $(X_1, \alpha_1, ..., \alpha_n) \cong (X_2, \beta_1, ..., \beta_n)$.

We now recall the following important notion introduced in [Rou2]: given α, a filling instruction on a manifold X, we say that *α factors through a manifold Y* if there exists some filling instruction $\alpha' \subset \alpha$ such that $Y = X(\alpha')$.

To describe the exceptional fillings on the minimally twisted 5–chain link, we follow the standard choice of notation used to describe graph manifolds set out in [Rou2]. Very briefly, if G is an orientable surface with k boundary components and Σ is G minus n discs, we can construct homology bases $\{(\mu_i, \lambda_i)\}$ on $\partial(\Sigma \times S^1)$. For coprime pairs $\{(p_i, q_i)\}_{i=1}^n$ with $|p_i| \geq 2$ we get a Seifert manifold $(G, (p_1, q_1), \dots, (p_n, q_n))$ with fixed homology bases on its k boundary components. Given Seifert manifolds X and Y with boundary and orientable base surfaces as above and an element $B \in \mathrm{GL}_2(\mathbb{Z})$, we define $X \cup_B Y$ unambiguously to be the quotient manifold $X \cup_f Y$, with $f : T \to T'$ where T and T' are arbitrary boundary tori of X and Y, and f acting on homology by B with respect to the fixed bases. Similarly one can define $X/_B$ when X has at least two boundary components.

As is common in the litterature, we employ a somehow more flexible notation for lens spaces than the usual one. We will write $\mathrm{L}(2,q)$ for the real projective space, $\mathrm{L}(1,q)$ for the 3-sphere, $\mathrm{L}(0,q)$ for $S^2 \times S^1$ and $\mathrm{L}(p,q)$ for $\mathrm{L}(|p|,q')$ with $q \equiv q'$ modulo p and $0 < q' < |p|$, for any coprime p, q. Later in the paper, we will often be interested in understanding when $\mathrm{L}(x,y) = S^3$ where x and y shall have some complicated expression; as $\mathrm{L}(x,y) = S^3$ if and only if $|x| = 1$, we will often replace "y" with " $\star$ " to simplify matters.

Finally, throughout the text the symbols ε, ε_1, η, etc. will all denote ± 1, and k,n, etc. will denote integers.

3. (S^H, T^H, T^H) triples from 5CL

In this section, we enumerate all exceptional (S^H, T^H, T^H) triples obtained by surgery on the 5CL. Each one of these triples can be thought of as a knot in S^3 with two different lens space surgeries.

Theorem 3.1.
- *If* $\left(M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}), \alpha, \beta, \gamma\right) \in (S^H, T^H, T^H)$ *then it is equivalent to either* $\left(N(-\frac{3}{2}, -\frac{14}{5}), -2, -1, \infty\right)$ *or* $A_n := \left(N(-\frac{5}{2}, \frac{1-2n}{5n-2}), \infty, -2, -1\right)$ *for some* $n \in \mathbb{Z}$.
- *The sets of exceptional slopes and corresponding fillings of* $N(-\frac{3}{2}, -\frac{14}{5})$ *and of* $N(-\frac{5}{2}, \frac{1-2n}{5n-2})$*, for* $n \neq 0$*, are given in Table 1.*
- *All* $N(-\frac{5}{2}, \frac{1-2n}{5n-2})$ *are obtained by filling the Berge manifold; but none of the* A_n *is equivalent to* $\left(N(-\frac{3}{2}, -\frac{14}{5}), -2, -1, \infty\right)$.

Remark If $n = 0$ then $N(-\frac{5}{2}, \frac{1-2n}{5n-2})$ is the exterior of the $(-2, 3, 7)$ pretzel knot which has 7 exceptional slopes, see [MP, Table A.4] for details.

We prove Theorem 3.1 by first considering, in Section 3.1, all $(M_5(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, T^H)$ with $\mathcal{F}$ not factoring through M_4. This set will turn out to be empty and we proceed in Section 3.2 to investigate the $(M_4(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, T^H)$ with $\mathcal{F}$ not factoring through M_3. Again, there will be no such examples and we will finally consider in Section 3.3 the case $(M_3(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, T^H)$. We will produce a complete list of examples, the family A_n and the isolated example in the statement of Theorem 3.1. The fact that the examples we find are all different is an easy consequence of the results in [MP] and is shown at the end of Section 3.3. Throughout the argument, easy (but technical) lemmata from the Appendix are referenced.

3.1. **Hyperbolic knots with two lens surgeries arising from the 5–chain link.** In this section we prove that if $M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})$ is a hyperbolic knot exterior admitting two different lens space fillings then the instruction $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})$ factors through M_4. If $\left(M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}), \alpha, \beta, \gamma\right) \in (S^H, T^H, T^H)$ and $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})$ does not factor through M_4 then [Rou2, Theorem 4] tells that there are two different scenarios to consider: either there are 3 exceptional slopes, or there are more. We study separately these two cases, starting with the latter one.

3.1.1. *Case* $e\,(M_5(\mathcal{F})) > 3$. By application of [Rou2, Theorem 4], we know that the manifold $M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})$ is then equivalent to some $M_5(\mathcal{F})$ listed in [Rou2, Tables 6 – 11]. A careful inspection of the corresponding exceptional sets, given in [Rou2, Tables 14 – 20], shows that Tables 17 and 18 are the only ones where types S^H and T^H appear simultaneously. The only possible cases are hence $\mathcal{F} = (-2, \frac{p}{q}, 3, \frac{u}{v})$ [Rou2, Tables 17] and $\mathcal{F} = (-2, \frac{p}{q}, \frac{r}{s}, -2)$ [Rou2, Tables 18]. The exceptional slopes are then $\{-1, 0, 1, \infty\}$, but since

$n \in \mathbb{Z}\backslash\{0\}$,	$E(N(-\frac{5}{2}, \frac{1-2n}{5n-2})) = \{-3, -2, -\frac{3}{2}, -1, 0, \infty\}$
$\beta \in E(N(-\frac{5}{2}, \frac{1-2n}{5n-2}))$	$N(-\frac{5}{2}, \frac{1-2n}{5n-2})(\beta)$
$\beta = \infty$	S^3
$\beta = -3$	$(D, (2,1), (3,-2)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (2,1), (3n-1, 5n-2))$
$\beta = -2$	$L(18-49n, 7-19n)$
$\beta = -\frac{3}{2}$	$(D, (2,1), (3,1)) \bigcup_{\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}} (D, (2,1), (8n-3, 5n-2))$
$\beta = -1$	$L(49n-19, 31n-12)$
$\beta = 0$	$(D, (2,-1), (5n-2, 8n-3)) \bigcup_{\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}} (D, (2,1), (3,1))$

$E(N(-\frac{3}{2}, -\frac{14}{5})) = \{-3, -\frac{5}{2}, -2, -1, 0, \infty\}$	
$\beta \in E(N(-\frac{3}{2}, -\frac{14}{5}))$	$N(-\frac{3}{2}, -\frac{14}{5})(\beta)$
$\beta = \infty$	$L(32, -9)$
$\beta = -3$	$(S^2, (2,1), (3,2), (9,-5))$
$\beta = -\frac{5}{2}$	$(D, (2,1), (3,1)) \bigcup_{\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}} (D, (2,1), (4,-5))$
$\beta = -2$	S^3
$\beta = -1$	$L(31, 17)$
$\beta = 0$	$(D, (2,1), (5,-4)) \bigcup_{\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}} (D, (2,1), (3,1))$

Table 1: The exceptional slopes and corresponding fillings of hyperbolic knot exteriors in S^3 with two lens space fillings obtained by surgery on 5CL.

$\Delta(S^H, T^H) = \Delta(T^H, T^H) = 1$ [CGLS], 1 and -1 cannot yield simultaneously cyclic fillings, so 0 and ∞ are necessarily part of the exceptional triple.

If $\mathcal{F} = (-2, \frac{p}{q}, \frac{r}{s}, -2)$**:** 0 is the only possibility for the S^H–filling, and then either $\frac{p}{q}$ or $\frac{r}{s}$ is of the form $1 + \frac{1}{n}$; but ∞ is also a T^H–filling, so $|s| = |q| = 1$, that is $\frac{p}{q}, \frac{r}{s} \in \mathbb{Z}$. It follows that $1 + \frac{1}{n}$ is an integer, and the only possibilities are 0 or 2. But according Lemma A.4, if it is 0, then $M_5(\mathcal{F})$ is non-hyperbolic, and if it is 2, then it factors through M_4.

If $\mathcal{F} = (-2, \frac{p}{q}, 3, \frac{u}{v})$**:** On the one hand, 0 is a cyclic slope, it follows that either $\frac{u}{v} = 3$, $\frac{u}{v} = 3 + \frac{1}{k}$, $\frac{u}{v} = \frac{6n+7}{2n+3} = 3 - \frac{2}{2n+3}$ or $|(3+2n)u - (7+6n)v| = 1$ that is $\frac{u}{v} = 3 - \frac{2}{2n+3} + \frac{\varepsilon}{(2n+3)v}$. Moreover, in the last two cases, $\frac{p}{q} = 1 + \frac{1}{n}$ so we can assume that $n \neq -1, -2$ otherwise $M_5(\mathcal{F})$ would be non-hyperbolic or $\mathcal{F}$ would factor through M_4 because of Lemma A.4. We obtain hence the lower bound $\frac{u}{v} \geq 3 - \frac{2}{|2n+3|} - \frac{1}{|2n+3|} \geq 3 - \frac{3}{3} = 2$. On the other hand, ∞ is also a cyclic slope, so either $\frac{u}{v} = \frac{1}{3}$, $\frac{u}{v} = \frac{2k+1}{6k+1} = \frac{1}{3} + \frac{2}{3(6k+1)}$, $v - 3u = \varepsilon$ that is $\frac{u}{v} = \frac{1}{3} - \frac{\varepsilon}{3v}$, or $|(1+2k)v - (1+6k)u| = 1$ that is $\frac{u}{v} = \frac{1}{3} + \frac{2}{3(6k+1)} + \frac{\varepsilon}{(6k+1)v}$. It follows that we have the upper bound $\frac{u}{v} \leq \frac{1}{3} + \frac{2}{3} + 1 = 2$. In conclusion, $\frac{u}{v} = 2$ and $M_5(\mathcal{F})$ factors through M_4 because of Lemma A.4.

3.1.2. *Case* $e(M_5(\mathcal{F})) = 3$. By application of [Rou2, Theorem 4], we know that the exceptional set of slopes is $\{0, 1, \infty\}$. Moreover, we have

$$M_5(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g}{h})(\infty) \underset{(28)}{=} F(-\tfrac{a}{b}, \tfrac{f}{e}, \tfrac{d}{c}, -\tfrac{g}{h}). \tag{1}$$

Recall that if one of $a,b,c,d,e,f,g,h = 0$ then $M_5(\frac{a}{b},\frac{c}{d},\frac{e}{f},\frac{g}{h})$ is non-hyperbolic by Lemma A.4. The enumeration of closed fillings of F is found in [Rou2, Table 4]. We will use (1) to translate instructions on M_5 to instructions on F and carefully consider the entries from [Rou2, Table 4]. In the analysis, T4.n will denote the n^{th} line of this table.

Considering the $T^H \cup S^H$–fillings of F listed in [Rou2, Table 4], and in view of Lemma A.12, we learn that, up to a D_4 permutation of slopes, T4.2–T4.5 is a complete list of necessary and sufficient conditions for $M_5(\frac{a}{b},\frac{c}{d},\frac{e}{f},\frac{g}{h})(\infty) \in S^H \cup T^H$. The lines T4.2 and T4.3, which correspond to $\frac{p}{q} = 0$, can be ignored since by (1) and Lemma A.4 they yield a non hyperbolic filling.

The entry T4.4 tells us that if $M_5(\frac{a}{b},\frac{c}{d},\frac{e}{f},\frac{g}{h})(\infty) = F(-\frac{a}{b},\frac{f}{e},\frac{d}{c},-\frac{g}{h}) \in S^H \cup T^H$, then, taking into consideration the action of D_4 on F, one of the following conditions necessarily holds:

$$(i)\ \tfrac{a}{b}=\tfrac{1}{n}\ \&\ \tfrac{e}{f}=k \qquad (ii)\ \tfrac{c}{d}=n\ \&\ \tfrac{g}{h}=\tfrac{1}{k} \qquad (iii)\ \tfrac{a}{b}=\tfrac{1}{n}\ \&\ \tfrac{g}{h}=\tfrac{1}{k} \qquad (iv)\ \tfrac{c}{d}=n\ \&\ \tfrac{e}{f}=k.$$

These conditions can all be identified using Lemma A.3. In fact, to identify for example case (i) with case (iii) it suffices to remark that $\left(M_5(\frac{a}{b},\frac{c}{d},\frac{e}{f},\frac{g}{h}),\infty\right) \underset{(9)^3\circ(11)}{\cong} \left(M_5(\frac{a}{a-b},\frac{d-c}{d},\frac{h}{g},\frac{f}{e}),\infty\right)$. In a similar way, case (i) can be identified with case (ii) using (9) and (10), and case (i) can be identified with case (iv) using (18).

The entry T4.5 tells us that if $M_5(\frac{a}{b},\frac{c}{d},\frac{e}{f},\frac{g}{h})(\infty) = F(-\frac{a}{b},\frac{f}{e},\frac{d}{c},-\frac{g}{h}) \in S^H \cup T^H$, then one of the following conditions necessarily holds:

$$(i)\ \tfrac{a}{b}=\tfrac{1}{n}\ \&\ \tfrac{c}{d}=\tfrac{\varepsilon+nk}{k} \quad (ii)\ \tfrac{a}{b}=\tfrac{k}{\varepsilon+nk}\ \&\ \tfrac{c}{d}=n \quad (iii)\ \tfrac{g}{h}=\tfrac{1}{n}\ \&\ \tfrac{e}{f}=\tfrac{\varepsilon+nk}{k} \quad (iv)\ \tfrac{g}{h}=\tfrac{k}{\varepsilon+nk}\ \&\ \tfrac{e}{f}=n$$

where $\varepsilon = \pm 1$. Then Case (i) is identified with Case (iv) using (18). Moreover, Case (i) is identified with Case (iii), and Case (ii) is identified with Case (iv) using $(9)^4\circ(10)$.

Therefore, any $M_5(\frac{a}{b},\frac{c}{d},\frac{e}{f},\frac{g}{h})$ with $\left(M_5(\frac{a}{b},\frac{c}{d},\frac{e}{f},\frac{g}{h}),\alpha,\beta,\gamma\right) \in (S^H,T^H,T^H)$ and $\{\alpha,\beta,\gamma\} = \{0,1,\infty\}$ is equivalent to one of:

$$M_5(\tfrac{1}{n},\tfrac{c}{d},k,\tfrac{g}{h}) \quad \text{(Family 1)} \qquad \text{or} \qquad M_5(\tfrac{1}{n},\tfrac{\varepsilon+nk}{k},\tfrac{e}{f},\tfrac{g}{h}) \quad \text{(Family 2)}.$$

We first consider the examples from Family 1. We know that both 0 and 1 correspond to S^H or T^H–slopes. Examining the 1–slope we obtain

$$M_5(\tfrac{1}{n},\tfrac{c}{d},k,\tfrac{g}{h})(1) \underset{(29)}{=} F(\tfrac{1-n}{n},\tfrac{c}{d},k,\tfrac{g-h}{h}) \underset{\text{Lemma } A.11}{=} \big(D,(1{-}n,n),(k,1)\big)\bigcup\nolimits_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(D,(c,d),(g{-}h,h)\big)$$

which has an essential torus unless $0,\pm1 \in \{1-n,k,c,g-h\}$ by Lemma A.2. Since we are interested in hyperbolic manifolds and instructions not factoring through M_4, we can use Lemma A.4 to rule out the possibilities $1-n, k \in \{0,\pm1\}$ and $c, g-h = 0$. We are then left with the cases $c = \pm1$ and $g = h \pm 1$.

Case $c = \pm 1$: Turning now our attention to the slope 0 and writing $\frac{c}{d} = \frac{1}{m}$, it holds

$$M_5(\tfrac{1}{n},\tfrac{1}{m},k,\tfrac{g}{h})(0) \underset{(30)}{=} F(\tfrac{n}{n-1},1-m,-\tfrac{h}{g},k-1),$$

which, by Lemmata A.2 and A.11, has an essential torus unless we are in the case $0,\pm1 \in \{n,1-m,-h,k-1\}$. This time Lemma A.4 leaves us with the necessary condition $h = \pm1$, which translates to $\frac{g}{h} \in \mathbb{Z}$. Combining the two necessary conditions and writing $\frac{g}{h} = l$, we learn that

$$M_5(\tfrac{1}{n},\tfrac{1}{m},k,l)(1) \underset{(29)}{=} F(\tfrac{1-n}{n},\tfrac{1}{m},k,l-1) \underset{(27)}{=} \big(D,(1{-}n,n),(k,1)\big)\bigcup\nolimits_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(D,(1,m),(l{-}1,1)\big)$$

$$\underset{(6)}{=} \big(S^2,(1{-}n,n),(k,1),(1{+}ml{-}m,1{-}l)\big)$$

which is in $S^H \cup T^H$ only if $\pm1 \in \{1+m(l-1), 1-n, k\}$ by Lemma A.2. Lemma A.4 is used to rule out any of these cases occurring. For instance, if $1+m(l-1) = -1$, then $m(1-l) = 2$ and

$m \in \{\pm 1, \pm 2\}$; moreover, if $m = -2$, then $1 - l = -1$ and $l = 2$, meaning that it factors through M_4 by Lemma A.4.

Case $g = h \pm 1$**:** As before, turning now our attention to the slope 0 and writing $\frac{g}{h} = 1 + \frac{1}{m}$, we have

$$M_5(\tfrac{1}{n}, \tfrac{c}{d}, k, \tfrac{m+1}{m})(0) \underset{(30)}{=} F(\tfrac{n}{n-1}, \tfrac{c-d}{c}, -\tfrac{m}{m+1}, k-1),$$

which, unless $0, \pm 1 \in \{n, c-d, m, k-1\}$, will have an essential torus by Lemmata A.2 and A.11. Just as in the preceding case, we can use Lemma A.4 to conclude that the only possibility is $c - d = \pm 1$, which is equivalent to $\frac{c}{d} = 1 + \frac{1}{l}$. Combining the necessary conditions, we obtain that

$$M_5(\tfrac{1}{n}, \tfrac{l+1}{l}, k, \tfrac{m+1}{m})(1) \underset{(29)}{=} F(\tfrac{1-n}{n}, \tfrac{l+1}{l}, k, \tfrac{1}{m}) \underset{(27)}{=} (D, (1-n, n), (k, 1)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (l+1, l), (1, m))$$
$$\underset{(6)}{=} (S^2, (1-n, n), (k, 1), (l+ml+m, -l-1))$$

which is in $S^H \cup T^H$ only if $\pm 1 \in \{l + ml + m, 1 - n, k\}$ by Lemma A.2. Lemma A.4 is then used to rule out any of these cases. For instance, if $l + ml + m = 1$, then $m(1 + l) = 1 - l$; if $l = -1$ then $\frac{l+1}{l} = 0$, otherwise $\frac{2}{1+l} - 1 = m \in \mathbb{Z}$ so $1 + l \in \{\pm 1, \pm 2\}$, that is $\frac{1+l}{l} \in \{\frac{2}{3}, \frac{1}{2}, \infty, 2\}$; moreover, if $\frac{1+l}{l} = \frac{2}{3}$, then $m = -2$ and $\frac{m+1}{m} = \frac{1}{2}$, meaning that it factors through M_4 by Lemma A.4. The remaining three cases for $\frac{1+l}{l}$ are directly excluded by Lemma A.4.

We consider now the examples from Family 2. The analysis follows verbatim the steps considered in the study of Family 1. We have assumed that both 0 and 1 correspond to S^H or T^H–slopes. The manifold $M_5(\frac{1}{n}, \frac{\varepsilon+kn}{k}, \frac{e}{f}, \frac{g}{h})(1) \underset{(29)}{=} F(\frac{1-n}{n}, \frac{\varepsilon+kn}{k}, \frac{e}{f}, \frac{g-h}{h})$ has an essential torus unless $0, \pm 1 \in \{1 - n, \varepsilon + kn, e, g - h\}$ by Lemmata A.2 and A.11. Lemma A.4 implies that $1 - n, \varepsilon + kn \notin 0, \pm 1$ and $e, g - h \neq 0$. We are thus left with the possibilities $e = \pm 1$ and $g - h = \pm 1$.

Case $e = \pm 1$**:** Writing $\frac{e}{f} = \frac{1}{m}$ we have

$$M_5(\tfrac{1}{n}, \tfrac{\varepsilon+kn}{k}, \tfrac{1}{m}, \tfrac{g}{h})(0) \underset{(30)}{=} F(\tfrac{n}{n-1}, \tfrac{\varepsilon+(n-1)k}{\varepsilon+kn}, -\tfrac{h}{g}, \tfrac{1-m}{m})$$

which has again an essential torus unless $0, \pm 1 \in \{n, \varepsilon + (n-1)k, h, 1 - m\}$. Lemma A.4 leaves us with the necessary condition $h = \pm 1$. Indeed, among the other cases, the worst situation is $\varepsilon + (n-1)k = -\varepsilon$, but then $(n-1)k = -2\varepsilon$ and $n - 1 \in \{\pm 1, \pm 2\}$. The first three cases can directly be ruled out by Lemma A.4, and the assumption $n - 1 = 2$ implies that $k = -\varepsilon$ and hence that $\frac{\varepsilon+kn}{k} = 2$, meaning that it factors through M_4 by Lemma A.4. We can hence set $\frac{g}{h} = l$. This gives

$$M_5(\tfrac{1}{n}, \tfrac{\varepsilon+kn}{k}, \tfrac{1}{m}, l)(1) \underset{(29)}{=} F(\tfrac{1-n}{n}, \tfrac{\varepsilon+kn}{k}, \tfrac{1}{m}, l-1) \underset{(27)}{=} (D, (1-n, n), (1, m)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (\varepsilon+kn, k), (l-1, 1))$$
$$\underset{(6)}{=} (S^2, (n+m(1-n), n-1), (l-1, 1), (\varepsilon+nk, k))$$

which is in $S^H \cup T^H$ only when $\pm 1 \in \{n + m(1 - n), l - 1, \varepsilon + nk\}$ by Lemma A.2. These cases are all discounted using Lemma A.4.

Case $g = h \pm 1$**:** Turning now our attention to the slope 0 and writing $\frac{g}{h} = 1 + \frac{1}{m}$, we get

$$M_5(\tfrac{1}{n}, \tfrac{\varepsilon+kn}{k}, \tfrac{e}{f}, \tfrac{m+1}{m})(0) \underset{(30)}{=} F(\tfrac{n}{n-1}, \tfrac{\varepsilon+(n-1)k}{\varepsilon+kn}, -\tfrac{m}{m+1}, \tfrac{e-f}{f})$$

which, unless $0, \pm 1 \in \{n, \varepsilon + (n-1)k, m, e - f\}$, will have an essential torus by Lemmata A.2 and A.11. Once again we use Lemma A.4 to conclude that the only possibility is $e - f = \pm 1$, which can be reformulated as $\frac{e}{f} = \frac{l+1}{l}$. Combining the necessary conditions, we obtain

$$M_5(\tfrac{1}{n}, \tfrac{\varepsilon+kn}{k}, \tfrac{l+1}{l}, \tfrac{m+1}{m})(1) \underset{(29)}{=} F(\tfrac{1-n}{n}, \tfrac{\varepsilon+kn}{k}, \tfrac{l+1}{l}, \tfrac{1}{m}) \underset{(27)}{=} (D, (1-n, n), (l+1, l)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (\varepsilon+kn, k), (1, m))$$

$$\underset{(6)}{=} \big(S^2, (1-n, n), (l+1, l), (k+m(\varepsilon+kn), -kn-\varepsilon)\big)$$

which is in $S^H \cup T^H$ only if $\pm 1 \in \{k+m(\varepsilon+kn), 1-n, l+1\}$ by Lemma A.2. Lemma A.4 is used to directly rule out the $\pm 1 \in \{1-n, l+1\}$ cases. Now, if $k+m(\varepsilon+kn) = \eta$, then $m = -\frac{1}{n} + \frac{\eta}{\varepsilon+kn} + \frac{\varepsilon}{n(\varepsilon+kn)}$. We can assume that $n, \varepsilon + kn \notin \{0, \pm 1\}$ otherwise Lemma A.4 would apply. It follows that $m \in [-\frac{3}{2}, \frac{3}{2}]$ and hence that $m \in \{0, \pm 1\}$ which, again, can be ruled out because of Lemma A.4.

We conclude that if $(M_5(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, T^H)$ then $\mathcal{F}$ factors through M_4. This completes Section 3.1.

3.2. **Hyperbolic knots with two lens surgeries arising from the 4–chain link.** In this section we prove that if $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ is hyperbolic with three fillings in $S^H \cup T^H$ then the instruction $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ factors through M_3. From [Rou2, Theorem 5] and a careful inspection of [Rou2, Tables 12, 21, 22] we deduce that if the triple $\big(M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}), \alpha, \beta, \gamma\big)$ is in (S^H, T^H, T^H), then $e\big(M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})\big) = 4$ and $\{\alpha, \beta, \gamma\} \subset \{0, 1, 2, \infty\}$. Since $\Delta(S^H, T^H) = \Delta(T^H, T^H) = 1$ [CGLS], it follows that either $\{\alpha, \beta, \gamma\} = \{1, 2, \infty\}$ or $\{\alpha, \beta, \gamma\} = \{0, 1, \infty\}$. But one can observe that

$$\begin{aligned} M_4(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g}{h}) &\underset{\text{Lemma A.5}}{\cong} M_5(\tfrac{a}{b}, \tfrac{c-d}{d}, -1, \tfrac{e-f}{f}, \tfrac{g}{h}) \underset{(19)\circ(9)^2}{\cong} M_5(\tfrac{c-2d}{c-d}, \tfrac{b}{b-a}, -1, \tfrac{h}{h-g}, \tfrac{e-2f}{e-f}) \\ &\underset{\text{Lemma A.5}}{\cong} M_4(\tfrac{c-2d}{c-d}, \tfrac{2b-a}{b-a}, \tfrac{2h-g}{h-g}, \tfrac{e-2f}{e-f}) \underset{\text{Lemma A.7}}{\cong} M_4(\tfrac{e-2f}{e-f}, \tfrac{c-2d}{c-d}, \tfrac{2b-a}{b-a}, \tfrac{2h-g}{h-g}). \end{aligned}$$

It follows that $\big(M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}), 0, 1, \infty\big) \cong \big(M_4(\frac{e-2f}{e-f}, \frac{c-2d}{c-d}, \frac{2b-a}{b-a}), 2, \infty, 1\big)$. It is hence sufficient to study the case $\{\alpha, \beta, \gamma\} = \{1, 2, \infty\}$. We examine now the necessary conditions on the filling instruction $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ imposed from 1,2 and ∞ being $S^H \cup T^H$–slopes.

3.2.1. *Necessary conditions from $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})(2)$.* We have

$$M_4(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f})(2) \underset{(26)}{=} \big(D, (a-b, b), (e-f, f)\big) \bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} \big(D, (c, d), (2, -1)\big)$$

which is in $S^H \cup T^H$ only if $0, \pm 1 \in \{a-b, e-f, c\}$ by Lemma A.2. If $a-b=0$, $e-f=0$, $c=0$ then $\frac{a}{b}=1$, $\frac{e}{f}=1$, $\frac{c}{d}=0$ respectively, which are all excluded by Lemma A.6 since we are only interested in the hyperbolic case. We continue with a case by case analysis:

Case $|a-b| = 1$**:** Up to a simultaneous change of signs for a and b, we may assume that $a-b=1$. This gives us, again by Lemma A.2,

$$M_4(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f})(2) \underset{(26)}{=} \big(D, (1, b), (e-f, f)\big) \bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} \big(D, (c, d), (2, -1)\big) \underset{(6)}{=} \big(S^2, (c, d), (2, -1), (f+b(e-f), f-e)\big)$$

which is in $S^H \cup T^H$ only if $\pm 1 \in \{c, f+b(e-f)\}$. Up to changing the signs of c and d or of e and f, we may hence assume that either $c=1$ or $b(f-e) = 1+f$.

Case $|e-f| = 1$**:** Lemma A.7 tells us that $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})(2) = M_4(\frac{e}{f}, \frac{c}{d}, \frac{a}{b})(2)$. So, any example found in this case is contained in the case $|a-b| = 1$;

Case $|c = 1|$**:** Up to a simultaneous change of signs for c and d, we may assume that $c = 1$. We get

$$\begin{aligned} M_4(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f})(2) &\underset{(26)}{=} \big(D, (a-b, b), (e-f, f)\big) \bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} \big(D, (1, d), (2, -1)\big) \\ &\underset{(6)}{=} \big(S^2, (e-f, f), (a-b, b), (1-2d, 2)\big) \end{aligned}$$

which is in $S^H \cup T^H$ only when $\pm 1 \in \{a-b, e-f, 1-2d\}$ by Lemma A.2. If $1-2d = \pm 1$, then $d \in \{0, 1\}$, that is $\frac{c}{d} \in \{1, \infty\}$, which is excluded by Lemma A.6. So either $|a-b| = 1$ or $|e-f|$ equals 1, and we are left with one of the previous cases.

To summarise, if $M_4(\frac{a}{b},\frac{c}{d},\frac{e}{f})(2)\in S^H\cup T^H$, then one of the following sets of conditions holds:

$$\boxed{\begin{array}{ll} a-b=1 & (C_2^0)\\ c=1 & (C_2') \end{array}}\,(C_2^1)\quad\text{or}\quad \boxed{\begin{array}{ll} a-b=1 & (C_2^0)\\ b(f-e)=1+f & (C_2'') \end{array}}\,(C_2^2).$$

3.2.2. *Necessary conditions from $M_4(\frac{a}{b},\frac{c}{d},\frac{e}{f})(1)$.* We have

$$M_4(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f})(1)\underset{(25)}{=}(S^2,(a-2b,b),(c-d,c),(e-2f,f))$$

which is in $S^H\cup T^H$ only if $\pm1\in\{a-2b,c-d,e-2f\}$ by Lemma A.2. So, one of the following conditions necessarily holds:

$$\boxed{a-2b=\varepsilon_1}\,(C_1^1)\quad\text{or}\quad\boxed{c-d=\varepsilon_1}\,(C_1^2)\quad\text{or}\quad\boxed{e-2f=\varepsilon_1}\,(C_1^3).$$

3.2.3. *Necessary conditions from $M_4(\frac{a}{b},\frac{c}{d},\frac{e}{f})(\infty)$.* We have

$$M_4(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f})(\infty)\underset{(23)}{=}(S^2,(a,b),(d,-c),(e,f))$$

which is in $S^H\cup T^H$ only when $\pm1\in\{a,d,e\}$ by Lemma A.2. So, one of the following conditions necessarily holds:

$$\boxed{a=\varepsilon_\infty}\,(C_\infty^1)\quad\text{or}\quad\boxed{d=\varepsilon_\infty}\,(C_\infty^2)\quad\text{or}\quad\boxed{e=\varepsilon_\infty}\,(C_\infty^3).$$

3.2.4. *Enumeration of $M_4(\frac{a}{b},\frac{c}{d},\frac{e}{f})$ satisfying the necessary conditions.* We have shown that if the triple $(M_4(\mathcal{F}),\alpha,\beta,\gamma)$ is in (S^H,T^H,T^H) then $\mathcal{F}$ is equivalent to a filling instruction $(\frac{a}{b},\frac{c}{d},\frac{e}{f})$ satisfying one of C_2^1 or C_2^2, one of C_1^1, C_1^2 or C_1^3 and one of C_∞^1, C_∞^2 or C_∞^3. We will now show that any such $(\frac{a}{b},\frac{c}{d},\frac{e}{f})$ must factor through M_3. First, we begin by emphasizing a few incompatibilities between the above conditions.

$C_2^0+C_1^1$: substituting $a-b=1$ into $a-2b=\varepsilon_1$ gives $\frac{a}{b}=1+\frac{1}{1-\varepsilon_1}\in\{\frac{3}{2},\infty\}$, which is excluded by Lemma A.6.

$C_2^0+C_\infty^1$: substituting $a=\varepsilon_\infty$ into $a-b=1$ gives $\frac{a}{b}=\frac{\varepsilon_\infty}{\varepsilon_\infty-1}=1+\frac{1}{\varepsilon_\infty-1}\in\{\frac{1}{2},\infty\}$ which is excluded by Lemma A.6.

$C_2'+C_1^2$: substituting $c=1$ into $c-d=\varepsilon$ gives $\frac{c}{d}=\frac{1}{1-\varepsilon_1}\in\{\frac{1}{2},\infty\}$, which is excluded by Lemma A.6.

$C_2'+C_\infty^2$: this gives $\frac{c}{d}=\pm1$, which is excluded by Lemma A.6.

$C_2^2+C_1^3$: C_1^3 implies $e-f=f+\varepsilon_1$ which we substitute into $b(f-e)=1+f$ to get $-b(f+\varepsilon_1)=1+f$. If $f=-\varepsilon_1$ then $\frac{e}{f}=-1$ which is excluded by Lemma A.6, and otherwise $-b=1+\frac{1-\varepsilon_1}{f+\varepsilon_1}$. If $\varepsilon_1=1$, then $b=-1$, $a=0$ and $\frac{a}{b}=0$ which are excluded by Lemma A.6. If $\varepsilon_1=-1$ then $\frac{2}{f-1}=-b-1$ is an integer, so $f-1$ divides 2 and $f\in\{-1,0,1,2,3\}$. If $f=3$, then $b=-2$, $a=-1$ and $\frac{a}{b}=\frac{1}{2}$ which is excluded by Lemma A.6. Otherwise, $\frac{e}{f}=2-\frac{1}{f}\in\{1,\frac{3}{2},3,\infty\}$ which is excluded by Lemma A.6;

$C_2^2+C_\infty^3$: if $e=\varepsilon_\infty$ then $f\neq\pm1$ by Lemma A.6. Substituting $e=\varepsilon_\infty$ into $b(f-e)=1+f$ gives $b=1+\frac{1+\varepsilon_\infty}{f-\varepsilon_\infty}$. If $\varepsilon_\infty=-1$, then $b=1$, $a=2$ and $\frac{a}{b}=2$ which is excluded by Lemma A.6. If $\varepsilon_\infty=1$ then $\frac{2}{f-1}=b-1$ is an integer, and $f-1$ divides 2 which implies $f\in\{-1,0,1,2,3\}$. If $f=3$, then $b=2$, $a=3$ and $\frac{a}{b}=\frac{3}{2}$ which is excluded by Lemma A.6. Otherwise, $\frac{e}{f}=\frac{1}{f}\in\{-1,\frac{1}{2},1,\infty\}$ which is excluded by Lemma A.6.

$C_1^2+C_\infty^2$: we have $c=\varepsilon_\infty+\varepsilon_1$ and hence $\frac{c}{d}=\frac{\varepsilon_\infty+\varepsilon}{\varepsilon_\infty}=1+\varepsilon_1\varepsilon_\infty\in\{0,2\}$, which is excluded by Lemma A.6.

$C_1^3 + C_\infty^3$: we have $2f = \varepsilon_\infty - \varepsilon_1 \Rightarrow f \in \{0, \pm 1\}$, so $\frac{e}{f} \in \{\pm 1, \infty\}$ which is excluded by Lemma A.6.

We now observe that the above analysis is enough to conclude:

- C_2^0 necessarily holds. The above analysis implies that C_1^1 or C_∞^1 do not hold.
- If C_2^1 holds then, because of C_2', neither C_1^2 or C_∞^2 hold. It follows that both C_1^3 and C_∞^3 hold, but they can't hold simultaneously. So C_2^2 holds.
- If C_2^2 holds then neither C_1^3 or C_∞^3 can hold. It follows that both C_1^2 and C_∞^2 hold, but they can't hold simultaneously.

We conclude that, as announced, if $(M_4(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, T^H)$ then $\mathcal{F}$ factors through M_3.

3.3. **Hyperbolic knots with two lens surgeries arising from the 3–chain link.** We now enumerate all the hyperbolic knots with two lens space surgeries obtained by surgery on the 3–chain link. We prove the following result:

Proposition 3.2. *If* $\left(M_3(\frac{a}{b}, \frac{c}{d}), \alpha, \beta, \gamma\right) \in (S^H, T^H, T^H)$ *then it is equivalent to* $\left(N(-\frac{5}{2}, \frac{1-2k}{5k-2}), \infty, -2, -1\right)$, *for some* $k \in \mathbb{Z}$, *or to* $\left(N(-\frac{3}{2}, -\frac{14}{5}), -2, -1, \infty\right)$.

The enumeration of all (S^H, T^H, T^H) triples obtained by surgery on 3CL comes from [MP, Theorem 1.3] and a careful examination of [MP, Tables 2–3]. It should be noted that the classification of exceptional fillings on the exterior of the 3-chain link in [MP] is performed on the exterior of the mirror image 3CL*. The exterior of 3CL* is denoted by N, and, of course, $M_3(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}) = N(-\frac{a}{b}, -\frac{c}{d}, -\frac{e}{f})$. For the sake of clarity when referencing to tables, we will adopt the convention in [MP].

First, we note that [MP, Table 4] involves no fillings of the form $L(\star, \star)$ so we restrict our attention to [MP, Tables 2–3]. In Table 3, there are some entries where $N(\frac{p}{q}, \frac{r}{s}, \frac{t}{u}) = L(\star, \star)$; however, in each case, slopes -1 or 0 are involved so, if $\left(N(\frac{a}{b}, \frac{c}{d}), \alpha, \beta, \gamma\right)$ was a (S^H, T^H, T^H) triple which had some entries in Table 3, then β or γ would be the -1 or 0 slope, otherwise $N(\frac{a}{b}, \frac{c}{d})$ would not be hyperbolic, and then there would be no value for α to carry an S^H–surgery. It follows then from [MP, Theorem 1.3] that, up to equivalency, if $\left(N(\frac{a}{b}, \frac{c}{d}), \alpha, \beta, \gamma\right) \in (S^H, T^H, T^H)$ then we can assume that at least one of the slopes is -3, -2, -1, 0 or ∞ and that, because of Lemma A.10, $\alpha, \beta, \gamma \in \{-3, -2, -1, 0, \infty\}$, otherwise $M_3(\frac{a}{b}, \frac{c}{d})$ would not be hyperbolic. In particular the S^H–slope α is in $\{-3, -2, -1, 0, \infty\}$. We now examine each case individually.

3.3.1. *Case 0 is an S^H–slope.* We see directly from [MP, Table 2] that if $N(\frac{r}{s}, \frac{t}{u})(0) = L(\star, \star)$ then $\frac{r}{s} = n$, $\frac{t}{u} = -4 - n + \frac{1}{m}$ and $N(\frac{r}{s}, \frac{t}{u})(0) = L(6m-1, 2m-1)$. So, if $N(\frac{r}{s}, \frac{t}{u})(0) = S^3$ then $m = 0$ and $\frac{t}{u} = \infty$, which is discarded by Lemma A.10.

3.3.2. *Case -1 is an S^H–slope.* We see directly from [MP, Table 2] that if $N(\frac{r}{s}, \frac{t}{u})(-1) = L(\star, \star)$ then $\frac{r}{s} = -3 + \frac{1}{n}$, and $N(\frac{r}{s}, \frac{t}{u})(-1) = L(2n(t+3u)-t-u, \star)$. If $L(2n(t+3u)-t-u, \star) = S^3$ then $2n(t+3u) - t - u = \pm 1$. By changing the signs of both t and u, we may assume w.l.o.g. that

$$2n(t + 3u) - t - u = 1. \tag{2}$$

Moreover, we know by [CGLS] that $\Delta(S^H, T^H) = 1$ and since $\Delta(-3, -1) = 2$, it follows that $\beta, \gamma \in \{-2, 0, \infty\}$. But, by [CGLS], we also know that $\Delta(T^H, T^H) = 1$, so the only pairs of possibilities for the T^H–slopes are $\{-2, \infty\}$ and $\{0, \infty\}$. From [MP, Theorem 1.3] we know that $N(\frac{r}{s}, \frac{t}{u})(\infty)$ is always a lens space.

We will now use (see [MP, Table 2]) to further refine the constraints, $\frac{r}{s} = -3 + \frac{1}{n}$ and (2), that we have found from imposing -1 to be a S^3–slope. This time we will analyse the restrictions we obtain by considering 0 and -2 to be lens space slopes and through this analysis we will enumerate all (S^H, T^H, T^H) triples. We will denote the new parameters with primes.

Case 0 is a T^H–slope: Either $\frac{r}{s} = -3 + \frac{1}{n} = n'$ or $\frac{r}{s} = -3 + \frac{1}{n} = -4 - n' + \frac{1}{m'}$.

Case $\frac{r}{s} = -3 + \frac{1}{n} = n'$: Then $n' = -2$ or -4, and $\frac{t}{u} = -4 - n' + \frac{1}{m'}$. The case $n' = -2$ is excluded by Lemma A.10. The case $n' = -4$ implies that $n = -1$ and that $\frac{t}{u} = \frac{1}{m'}$, that is $u = m't$. From (2), we have then $-t(3 + 7m') = 1$ which cannot hold.

Case $\frac{r}{s} = -3 + \frac{1}{n} = -4 - n' + \frac{1}{m'}$: Then $n'+1 = \frac{1}{m'} - \frac{1}{n} \in [-2, 2] \cup \{\infty\}$ so $n' \in \{-3, -2, -1, 0, 1, \infty\}$. But $\frac{t}{u} = n'$ and, by Lemma A.10, we know that $n' = 1$ otherwise $N(\frac{r}{s}, \frac{t}{u})$ would not be hyperbolic. It follows that $n = -1$ and substituting this information in (2) we obtain $-10u = 1$ which cannot hold.

Case -2 is a T^H–slope: Either $\frac{r}{s} = -3 + \frac{1}{n} = -2 + \frac{1}{n'}$ or $-2 + \frac{1}{n'} = \frac{t}{u}$.

Case $\frac{r}{s} = -2 + \frac{1}{n'} = -3 + \frac{1}{n}$: Then $n = 2$ and by (2), $\left(N(-\frac{5}{2}, \frac{t}{u}), -1, -2, \infty\right)$ is a (S^H, T^H, T^H) triple whenever $3t + 11u = 1$. Namely, for $t = 4 - 11k$ and $u = 3k - 1$ with any $k \in \mathbb{Z}$. That is $\boxed{\left(N(-\frac{5}{2}, \frac{4-11k}{3k-1}), -1, -2, \infty\right)}$ are (S^H, T^H, T^H) triples for every $k \in \mathbb{Z}$.

Case $-2 + \frac{1}{n'} = \frac{t}{u}$: Then $\frac{t}{u} = \frac{1-2n'}{n'}$ so (2) becomes $2n(1 + n') + n' \in \{0, 2\}$.

If $2n(1 + n') + n' = 0$, then $2n = \frac{1}{1+n'} - 1 \in [-2, 0]$ so $(n, n') \in \{(-1, -2), (0, 0)\}$. The first case leads to the (S^H, T^H, T^H) triple $\boxed{\left(N(-4, -\frac{5}{2}), -1, -2, \infty\right)}$ whereas the second is discarded by Lemma A.10 since $\frac{r}{s} = \frac{t}{u} = \infty$.

If $2n(1 + n') + n' = 2$, then $\frac{3}{2n+1} = n' + 1 \in \mathbb{Z}$. It follows that $n \in \{-2, -1, 0, 1\}$. For $n \in \{0, 1\}$, we have $\frac{r}{s} = -3 + \frac{1}{n} \in \{-2, \infty\}$ so the associated space is non-hyperbolic by Lemma A.10. For $n = -2$ and -1 we find that $\boxed{\left(N(-\frac{7}{2}, -\frac{5}{2}), -1, -2, \infty\right)}$ and $\boxed{\left(N(-4, -\frac{9}{4}), -1, -2, \infty\right)}$ are (S^H, T^H, T^H) triples.

3.3.3. *Case -2 is an S^H–slope.* We see directly from [MP, Table 2] that if $N(\frac{r}{s}, \frac{t}{u})(-2) = L(\star, \star)$ with $N(\frac{r}{s}, \frac{t}{u})$ hyperbolic then $\frac{r}{s} = -2 + \frac{1}{n}$, and $N(\frac{r}{s}, \frac{t}{u})(-2) = L((3n(t+2u)-2t-u, \star)$. So, up to simultaneously reversing the signs of t and u, we may assume w.l.o.g. that

$$3n(t + 2u) - 2t - u = 1. \tag{3}$$

As in the previous section, since $\Delta(S^H, T^H) = \Delta(T^H, T^H) = 1$ the only possible pairs of T^H–slopes are $\{-3, \infty\}$ and $\{-1, \infty\}$. We know that the ∞–filling is always a lens space by [MP, Theorem 1.3]. We now enumerate the new conditions arising from -1 or -3 being T^H–slopes. We will denote the new parameters with primes.

Case -1 is a T^H–slope: From [MP, Table 2], either $-2 + \frac{1}{n} = -3 + \frac{1}{n'}$ or $\frac{t}{u} = -3 + \frac{1}{n'}$.

Case $-2 + \frac{1}{n} = -3 + \frac{1}{n'}$: Then $n = -2$ and we find that $\left(N(-\frac{5}{2}, \frac{t}{u}), -1, -2, \infty\right)$ is a (S^H, T^H, T^H) triple whenever $8t + 13u + 1 = 0$, that is for $t = 13k - 5$ and $u = 3 - 8k$ with any $k \in \mathbb{Z}$. So $\boxed{\left(N(-\frac{5}{2}, \frac{13k-5}{3-8k}), -2, -1, \infty\right)}$ is a (S^H, T^H, T^H) triple for every $k \in \mathbb{Z}$.

Case $-3 + \frac{1}{n'} = \frac{t}{u}$: In this case $\frac{t}{u} = \frac{1-3n'}{n'}$ and (3) becomes $3n(1 - n') + 5n' \in \{1, 3\}$.

If $3n(1 - n') + 5n' = 1$ then $\frac{4}{5-3n} = 1 - n' \in \mathbb{Z}$. It follows that $n \in \{1, 2, 3\}$. For $n \in \{2, 3\}$ we find that $\boxed{\left(N(-\frac{3}{2}, -\frac{14}{5}), -2, -1, \infty\right)}$ and $\boxed{\left(N(-\frac{5}{3}, -\frac{5}{2}), -2, -1, \infty\right)}$ are (S^H, T^H, T^H) triples.

If $3n(1 - n') + 5n' = 3$ then $\frac{2}{5-3n} = 1 - n' \in \mathbb{Z}$. It follows that $n \in \{1, 2\}$. For $n = 1$, we have $\frac{r}{s} = -1$ which makes $N(\frac{r}{s}, \frac{t}{u})$ non-hyperbolic by Lemma A.10. For $n = 2$ we find that $\boxed{\left(N(-\frac{3}{2}, -\frac{8}{3}), -2, -1, \infty\right)}$ is a (S^H, T^H, T^H) triple.

Case -3 is a T^H–slope: If $\frac{r}{s}$ or $\frac{t}{u}$ is -2 then $N(\frac{r}{s},\frac{t}{u})$ is non-hyperbolic by Lemma A.10. So, from [MP, Table 2], $-2+\frac{1}{n} = -1+\frac{1}{n'}$ making $n = 2$ and $\frac{t}{u} = -1+\frac{1}{m'} = \frac{1-m'}{m'}$. Using (3), we obtain $m' \in \{-\frac{5}{7}, -\frac{3}{7}\}$ which is not an integer.

3.3.4. *Case -3 is an S^H–slope.* From [MP, Table 2], if $N(\frac{r}{s},\frac{t}{u})(-3) = L(\star,\star)$ then either $\frac{t}{u} = -2$, which is excluded by Lemma A.10, or $\frac{r}{s} = -1+\frac{1}{n}$ and $\frac{t}{u} = -1+\frac{1}{m}$. In the latter case we have $N(-1+\frac{1}{n}, -1+\frac{1}{m})(-3) = L((2n+1)(2m+1)-4,\star) = S^3$ if and only if $(2n+1)(2m+1)-4 = \pm1$; that is $(2n+1)(2m+1) = 3$ or 5. Since both 3 and 5 are primes, it follows that either $2n+1$ or $2m+1$ is ±1. By symmetry, we may assume that $2n+1 = \pm1$, making $n = -1$ or 0 which are both excluded by Lemma A.10.

3.3.5. *Case ∞ is an S^H–slope.* From [MP, Theorem 1.3], $N(\frac{r}{s},\frac{t}{u})(\infty) = L(tr-us,\star)$. So, ∞ is an S^H–slope if and only if

$$tr - us = \pm1. \tag{4}$$

As before, we have $\Delta(S^H,T^H) = \Delta(T^H,T^H) = 1$ so the only possible pairs of T^H slopes are $\{-3,-2\}$, $\{-2,-1\}$ or $\{-1,0\}$. Each T^H–slope imposes conditions on $\frac{r}{s}$, $\frac{t}{u}$. We will use primes on the parameters to denote the conditions imposed from the smallest T^H–slope and double primes on the parameters coming from the conditions on the second T^H–slope.

Case 0 is a T^H–slope: From [MP, Table 2] we have $\frac{r}{s} = n'$ and $\frac{t}{u} = -4-n'+\frac{1}{m'} = \frac{1-m'(n'+4)}{m'}$. Equation (4) becomes then $(1-m'(n'+4))n' = m' \pm 1$. According to Lemma A.16, we have $n' \in \{-5,-4,-3,-2,-1,0,1\}$. For $N(\frac{r}{s},\frac{t}{u})$ to be hyperbolic, n' cannot be in $\{-3,-2,-1,0\}$ because of Lemma A.10; we are hence left with cases $(n',m') \in \{(-5,-1),(-4,-5),(-4,-3),(1,0)\}$. If $(n',m') = (-5,-1)$ then $\frac{t}{u} = 0$, and if $(n',m') = (1,0)$ then $\frac{t}{u} = \infty$. So, these cases are both excluded by Lemma A.10. The other two cases $\boxed{\big(N(-4,-\frac{1}{5}),\infty,-1,0,\big)}$ and $\boxed{\big(N(-4,-\frac{1}{3}),\infty,-1,0\big)}$ are indeed (S^H,T^H,T^H) triples.

Case -2 and -1 are the T^H–slopes: In this case, either $-2+\frac{1}{n'} = -3+\frac{1}{n''}$ or, up to symmetry, $\left(\frac{r}{s},\frac{t}{u}\right) = \left(\frac{1-2n'}{n'},\frac{1-3n''}{n''}\right)$.

Case $-2+\frac{1}{n'} = -3+\frac{1}{n''}$: Then $n' = -2$ and up to symmetry, we may assume that $\frac{r}{s} = -\frac{5}{2}$. Up to a simultaneous change of sign for t and u, equation (4) becomes $5t+2u = 1$ and this leads to $\boxed{\big(N(-\frac{5}{2},\frac{1-2k}{5k-2}),\infty,-2,-1\big)}$ which is indeed a (S^H,T^H,T^H) triple for every $k \in \mathbb{Z}$.

Case $\left(\frac{r}{s},\frac{t}{u}\right) = \left(\frac{1-2n'}{n'},\frac{1-3n''}{n''}\right)$: Equation (4) becomes $2n'+3n''-5n'n'' \in \{0,2\}$.
If $2n'+3n'' = 5n'n''$, then $n' = \frac{3n''}{5n''-2} \in \mathbb{Z}$. If $n'' \geq 0$ then the condition $5n''-2 \leq 3n''$ implies that $n'' \leq 1$. If $n'' \leq 0$ then the condition $3n'' \leq 5n''-2$ implies that $n'' \geq 1$. It follows that either $n'' = 0$, and then $\frac{t}{u} = \infty$, or $n'' = 1$, and then $\frac{t}{u} = -2$. Both cases are excluded by Lemma A.10.
If $2n'+3n'' = 2+5n'n''$, then $n' = \frac{3n''-2}{5n''-2} \in \mathbb{Z}$. If $n'' < 0$ then $0 < n' < 1$, and if $n'' \geq \frac{2}{3}$ then $0 < n' < 1$. So $n'' = 0$ and $\frac{t}{u} = \infty$ which is excluded by Lemma A.10.

Case -3 is a T^H–slope: From [MP, Table 2] we have $\frac{r}{s} = -2$, which is excluded by Lemma A.10, or $\frac{r}{s} = \frac{1-n'}{n'}$ and $\frac{t}{u} = \frac{1-m'}{m'}$. In the latter case (4) becomes $n'+m' = 0$ or 2. Using $\Delta(T^H,T^H) = 1$, if -3 is a T^H–slope then -2 is the only possible second T^H. But then, according to [MP, Table 2], we have $-1+\frac{1}{n'} = -2+\frac{1}{n''}$ and hence$n' = -2$. Subbing this value into (4) with $\frac{t}{u} = \frac{1-m'}{m'}$ we find that either $m' = 2$ or 4. This leads to $\boxed{\big(N(-\frac{3}{2},-\frac{1}{2}),\infty,-3,-2\big)}$ and $\boxed{\big(N(-\frac{3}{2},-\frac{3}{4}),\infty,-3,-2\big)}$ which are actually (S^H,T^H,T^H) triples.

3.3.6. *Identifying cases.* In the above analysis we have proved that the only (S^H, T^H, T^H) triples of the form $\left(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma\right)$ are:

(i) $A_n := \left(N(-\frac{5}{2}, \frac{1-2n}{5n-2}), \infty, -2, -1\right)$ for $n \in \mathbb{Z}$;
(ii) $A'_n := \left(N(-\frac{5}{2}, \frac{4-11n}{3n-1}), -1, -2, \infty\right)$ for $n \in \mathbb{Z}$;
(iii) $A''_n := \left(N(-\frac{5}{2}, \frac{13n-5}{3-8n}), -2, -1, \infty\right)$ for $n \in \mathbb{Z}$;
(iv) $\left(N(-4, -\frac{5}{2}), -1, -2, \infty\right)$;
(v) $\left(N(-\frac{7}{2}, -\frac{5}{2}), -1, -2, \infty\right)$;
(vi) $\left(N(-4, -\frac{9}{4}), -1, -2, \infty\right)$;
(vii) $\left(N(-\frac{3}{2}, -\frac{14}{5}), -2, -1, \infty\right)$;
(viii) $\left(N(-\frac{5}{3}, -\frac{5}{2}), -2, -1, \infty\right)$;
(ix) $\left(N(-\frac{3}{2}, -\frac{8}{3}), -2, -1, \infty\right)$;
(x) $\left(N(-4, -\frac{1}{5}), \infty, -1, 0\right)$;
(xi) $\left(N(-4, -\frac{1}{3}), \infty, -1, 0\right)$;
(xii) $\left(N(-\frac{3}{2}, -\frac{1}{2}), \infty, -3, -2\right)$;
(xiii) $\left(N(-\frac{3}{2}, -\frac{3}{4}), \infty, -3, -2\right)$.

In this list there are many repetitions. Indeed, using the first equality in [MP, Theorem 1.5], we obtain that cases (iv) and (ix), cases (vi) and (vii), cases (x) and (xiii), and cases (xi) and (xii) are pairwise isomorphic. Using the third equality in [MP, Theorem 1.5], we obtain that cases (vii) and (xiii) on the one hand, and cases (ix) and (xii) on the other hand, are pairwise isomorphic. Moreover, using the second equality in [MP, Theorem 1.5], we see that $A''_n \cong A'_n \cong A_n$ for every $n \in \mathbb{Z}$. Finally, it can be noted that, up to Lemma A.8, case (iv) is A'_0, case (iv) is A'_1 and case (viii) is A''_0. Summing up, all cases are either isomorphic to case (vii) or to A_n for some $n \in \mathbb{Z}$.

3.3.7. *Distinctness of examples.* The Berge manifold is the unique hyperbolic knot exterior in a solid torus T with three distinct solid torus fillings [Gab]. The Berge manifold is equal to $N(-\frac{5}{2})$ [MP]. By filling along a $\frac{1}{n}$–slope on ∂T we obtain a family of hyperbolic knot exteriors with two lens space fillings. As our enumeration of (S^H, T^H, T^H) triples obtained by surgery on 5CL is exhaustive, the family of (S^H, T^H, T^H) triples obtained by filling along a boundary component of the Berge manifold is $\left\{\left(N(-\frac{5}{2}, \frac{1-2n}{5n-2}), \infty, -2, -1\right)\right\}$.

By considering the sets of exceptional fillings, we will now show that $N(-\frac{3}{2}, -\frac{14}{5}) \neq N(-\frac{5}{2}, \frac{1-2n}{5n-2})$ for any n. Using [MP, Tables 2–3] we can write down the set of exceptional slopes and fillings of $N(-\frac{5}{2}, \frac{1-2n}{5n-2})$ and $N(-\frac{3}{2}, -\frac{14}{5})$; the result is shown in Table 1. We immediately observe that $N(-\frac{5}{2}, \frac{1-2n}{5n-2})$ has three distinct toroidal fillings, and that $N(-\frac{3}{2}, -\frac{14}{5})$ has only two toroidal filling. This shows $N(-\frac{3}{2}, -\frac{14}{5}) \neq N(-\frac{5}{2}, \frac{1-2n}{5n-2})$ for any $n \in \mathbb{Z}$.

4. (S^H, T^H, T) TRIPLES

In this section, we enumerate all (S^H, T^H, T) triples obtained by surgery on the 5CL and realizing the maximal distance. We know, from [Rou2, Theorem 1], that if $(M_5(\mathcal{F}), \beta, \gamma) \in (T^H, T)$, then $\Delta(\beta, \gamma) \leq 3$.

Theorem 4.1.

- *If* $(M_5(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, T)$ *with* $\Delta(\beta, \gamma) = 3$*, then it is equivalent to either the triple* $B_n := \left(N(-1 + \frac{1}{n}, -1 - \frac{1}{n}), \infty, -3, 0\right)$ *for some integer* $n \geq 2$*, or to* $C_n := \left(N(-1 + \frac{1}{n}, -1 - \frac{1}{n-2}), \infty, -3, 0\right)$ *for some integer* $n \geq 4$.
- *For* $n > 2$*,* $E(B_n) = \{-3, -2, -1, 0, \infty\}$ *and the exceptional fillings are given in Table 2. For* $n = 2$*,* B_n *is the exterior of the pretzel knot* $(-2, 3, 7)$ *and* $e(B_n) = 7$.
- *For* $n \geq 4$*,* $E(C_n) = \{-3, -2, -1, 0, \infty\}$ *and the exceptional fillings are given in Table 2.*
- *None of the* B_n *is equivalent to a* C_k.

In Section 4.1, we show that if $(M_5(\mathcal{F}), \alpha, \beta, \gamma)$ or $(M_4(\mathcal{F}), \alpha, \beta, \gamma)$ is in (S^H, T^H, T) with $\Delta(\beta, \gamma) = 3$, then $\mathcal{F}$ factors through M_3. If $M_3(\mathcal{F})$ is hyperbolic then, by [MP, Corollay A.6], we know that either $e(M_3(\mathcal{F})) > 5$, and then it appears in [MP, Tables A.2–A.9], or $e(M_3(\mathcal{F})) = 5$; Sections 4.2–4.3 investigate the exceptional triples arising from M_3 in these two cases. Note that [MP] classifies the exceptional

$n > 2, \quad E(N(-1 + \frac{1}{n}, -1 - \frac{1}{n})) = \{-3, -2, -1, 0, \infty\}$	
$\beta \in E(N(-1 + \frac{1}{n}, -1 - \frac{1}{n}))$	$N(-1 + \frac{1}{n}, -1 - \frac{1}{n}))(\beta)$
$\beta = \infty$	S^3
$\beta = -3$	$L(4n^2+3, 2n^2+n+2)$
$\beta = -2$	$(S^2, (3, 2), (1+n, n), (1-n, n))$
$\beta = -1$	$(S^2, (2, 1), (1+2n, -n), (1-2n, n))$
$\beta = 0$	$(D, (n, 1+n), (n, n-1)) \bigcup_{\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}} (D, (2, 1), (3, 1))$
$n \geq 4, \quad E(N(-1 + \frac{1}{n}, -1 - \frac{1}{n-2})) = \{-3, -2, -1, 0, \infty\}$	
$\beta \in E(N(-1 + \frac{1}{n}, -1 - \frac{1}{n-2}))$	$N(-1 + \frac{1}{n}, -1 - \frac{1}{n-2}))(\beta)$
$\beta = \infty$	S^3
$\beta = -3$	$L(4n^2+8n-1, 2n^2-3n)$
$\beta = -2$	$(S^2, (1+n, n), (3-n, n-2), (3, 2))$
$\beta = -1$	$(S^2, (2, 1), (1+2n, -n), (5-2n, n-2))$
$\beta = 0$	$(D, (n, 1+n), (2-n, 2-n))3 - 3n \bigcup_{\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}} (D, (2, 1), (3, 1))$

Table 2: The sets of exceptional slopes and fillings of all knot exteriors obtained by surgery on the minimally twisted 5-chain link realising $\Delta(T^H, T) = 3$ or $\Delta(T^H, Z) = 2$.

filling instructions and fillings on N, the exterior of the mirror image of 3CL. Of course N and M_3 are homeomorphic but, for the instructions, the slopes differ by a sign change; namely, $M_3(\alpha_1, \alpha_2, \alpha_3) = N(-\alpha_1, -\alpha_2, -\alpha_3)$. For the sake of clarity, as we work with the Tables in [MP], we will use the filling instructions on N. Finally, Section 4.4 concludes the proof by comparing the different families thus obtained.

4.1. **Triples from M_5 and M_4.** A complete enumeration of $E(M_5(\mathcal{F}))$, for $\mathcal{F}$ not factoring through M_4, is given in [Rou2, Theorem 4]. If $E(M_5(\mathcal{F})) = \{0, 1, \infty\}$, then all slopes are at distance 1. Moreover, a careful inspection of [Rou2, Tables 6 –11] shows that only Table 6 has exceptional slopes at distance 3, but then [Rou2, Table 14] shows that none of these examples contains an S^H–slope. Any exceptional triple $(M_5(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, T)$ with $\Delta(\beta, \gamma) = 3$ has $\mathcal{F}$ factoring through M_4.

Similarly, [Rou2, Theorem 5] gives a complete enumeration of $E(M_4(\mathcal{F}))$ for $\mathcal{F}$ not factoring through M_3. Again, if $E(M_5(\mathcal{F})) = \{0, 1, 2, \infty\}$ then all exceptional slopes are at distance at most 2, and [Rou2, Tables 21–22] shows that, otherwise, there is no example containing simultaneously S^H, T^H and T slopes. Any triple $(M_4(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, T)$ with $\Delta(\beta, \gamma) = 3$ must have $\mathcal{F}$ factoring through M_3.

4.2. **Exceptional triples from $M_3(\mathcal{F})$ with $e(M_3(\mathcal{F})) > 5$.** We recall that, for the sake of clarity, we use here filling instructions on N, and that they actually differ in sign from the filling instructions on M_3.

Any filling instruction $\mathcal{F}$ on N consisting of two slopes and such that $e(N(\mathcal{F})) > 5$ can be found in [MP, Tables A.2-A.9]. The tables A.2, A.3, A.4 and A.9 each contain a finite list of $N(\mathcal{F})$. The remaining tables consist of four infinite families. We proceed to examine each of these tables in our quest for examples.

4.2.1. *Examples arising from* [MP, Tables A.2–A.4 and A.9]. The only hyperbolic knots, *i.e.* $N(\mathcal{F})$ with an S^H–filling, listed are $N(1, 2)$—also known as the Figure-8 knot—in Table A.2 and $N(-4, -\frac{1}{3})$—the $(-2, 3, 7)$ pretzel knot—in Table A.4. The former has no lens space filling while the latter gives a unique

(S^H, T^H, T) triple with $\Delta(T^H, T) = 3$. So, from Tables A.2–A.4 and A.9 the only example we get is $\boxed{\left(N(-4,-\frac{1}{3}),\infty,0,-3\right) \in (S^H, T^H, T).}$

4.2.2. *Examples arising from* [MP, Table A.5]. This table enumerates $N(\mathcal{F}) = N(1, \frac{r}{s})$ cases with exceptional $\frac{p}{q} \in E(N(\mathcal{F})) = \{-3,-2,-1,0,1,\infty\}$. By a direct inspection, we see that a S^H–filling can arise in only two ways: either $\frac{p}{q} = \infty$ and $\frac{r}{s} = -5 + \frac{1}{n}$ with $n = 0$ — but then $\frac{r}{s} = \infty$ so $\mathcal{F}$ contains ∞, which is discarded by Lemma A.10 — or $\frac{p}{q} = \infty$ with $r - s = \pm 1$. In the latter case, up to a simultaneous change of sign for r and s, we can even assume w.l.o.g. that $r = s + 1$. Moreover, if $N(1, \frac{r}{s})(\alpha)$ is toroidal then α is either -3 or 1. We study both cases separately.

If -3 is a toroidal slope on $N(1, \frac{r}{s})$: Then $\frac{p}{q} = 0$ is the only case satisfying $N(1, \frac{r}{s})(\frac{p}{q}) \in T^H$ and $\Delta(\frac{p}{q}, -3) = 3$. But moreover, $\frac{r}{s}$ would be equal to $-5 + \frac{1}{n}$ from [MP, Table A.5] which is not compatible with the relation $r = s + 1$, otherwise we would obtain $6 = \frac{1}{n} - \frac{1}{s} \in [-2, 2] \cup \{\infty\}$.

If 1 is a toroidal slope on $N(1, \frac{r}{s})$): Then $\frac{p}{q} = -2$ is the only case satisfying $N(1, \frac{r}{s})$ hyperbolic, $N(1, \frac{r}{s})(\frac{p}{q}) \in T^H$ and $\Delta(\frac{p}{q}, 1) = 3$. But moreover, $\frac{r}{s}$ would be equal to $-2 + \frac{1}{n}$ which is not compatible with the relation $r = s + 1$, otherwise we would obtain $3 = \frac{1}{n} - \frac{1}{s} \in [-2, 2] \cup \{\infty\}$.

4.2.3. *Examples arising from* [MP, Table A.6]. This table enumerates $N(\mathcal{F}) = N(-\frac{3}{2}, \frac{r}{s})$ cases with exceptional $\frac{p}{q} \in E(N(\mathcal{F})) = \{-3, -\frac{5}{2}, -2, -1, 0, \infty\}$. By a direct inspection, we see that the possible S^H–slopes are -3, -2, -1 and ∞. Examining each possible case individually we find:

If $\frac{p}{q} = \infty$ is a S^H–slope: Then $\Delta(\frac{p}{q}, \alpha) \le 2$ for all $\alpha \in E(N(\mathcal{F}))$.

If $\frac{p}{q} = -3$ is a S^H–slope: Then $\frac{r}{s} = -1 + \frac{1}{n}$ and $6n + 7 = \pm 1$. The only possibility is $n = -1$ but then $\frac{r}{s} = -2$ and $N(\mathcal{F})$ is non-hyperbolic by Lemma A.10.

If $\frac{p}{q} = -2$ is a S^H–slope: Then $4r + 11s = \pm 1$, and up to a simultaneous change of sign for r and s, we may even assume that $4r + 11s = 1$, or equivalently that $\frac{r}{s} = \frac{1}{4s} - \frac{11}{4}$. The distance 3 pairs of slopes from $E(N(-\frac{3}{2}, \frac{r}{s}))$ are $\{-3, 0\}$ and $\{-\frac{5}{2}, -1\}$. In the first case, -3 must be the lens space surgery and $\frac{r}{s}$ is forced to be $-1 + \frac{1}{n}$; then $-1 + \frac{1}{n} = \frac{r}{s} = \frac{1}{4s} - \frac{11}{4}$ from where we arrive to the contradiction $\frac{7}{4} = \frac{1}{4s} - \frac{1}{n} \le \frac{5}{4}$. In the second case, -1 must be the lens space surgery and $\frac{r}{s}$ is forced to be $-3 + \frac{1}{n}$; then $-3 + \frac{1}{n} = \frac{r}{s} = \frac{1}{4s} - \frac{11}{4}$, that is $4s - n = ns$. According to Lemma A.15 we have then $(n, s) \in \{(0, 0), (3, 3), (5, -5), (8, -2), (6, -3), (2, 1)\}$.

Case $(n, s) = (0, 0)$: Then $\frac{r}{s} = -3 + \frac{1}{n} = \infty$ and the space is non-hyperbolic by Lemma A.10.

Case $(n, s) = (3, 3)$: Then $\frac{r}{s} = -3 + \frac{1}{n} = -\frac{8}{3}$ and this is excluded from Table A.6.

Case $(n, s) = (5, -5)$: We obtain then $\boxed{\left(N(-\frac{3}{2}, -\frac{14}{5}), -2, -1, -\frac{5}{2}\right)}$ which is indeed a (S^H, T^H, T) triple with $\Delta(T^H, T) = 3$.

Case $(n, s) = (8, -2)$: then $r = \frac{1}{4} - \frac{11s}{4} = \frac{23}{4} \notin \mathbb{Z}$, which is a contradiction.

Case $(n, s) = (6, -3)$: then $r = \frac{34}{4} \notin \mathbb{Z}$, which is a contradiction.

Case $(n, s) = (2, 1)$: then $r = -\frac{10}{4} \notin \mathbb{Z}$, which is a contradiction.

If $\frac{p}{q} = -1$ is a S^H–slope: Then $\frac{r}{s} = -3 + \frac{1}{n}$ and $6n + 1 = \pm 1$. The only possibility is $n = 0$ but then $\frac{r}{s} = \infty$ and $N(\mathcal{F})$ is non-hyperbolic by Lemma A.10.

4.2.4. *Examples arising from* [MP, Table A.7]. This table enumerates $N(\mathcal{F}) = N(-\frac{5}{2}, \frac{r}{s})$ cases with exceptional $\frac{p}{q} \in E(N(\mathcal{F})) = \{-3, -2, -\frac{3}{2}, -1, 0, \infty\}$. By a direct inspection, we see that the possible T^H–slopes are -2, -1 and ∞. But none of these slopes are at distance 3 from any other slope in $E(N(\mathcal{F}))$.

4.2.5. *Examples arising from* [MP, Table A.8]. This table enumerates $N(\mathcal{F}) = N(-\frac{1}{2}, \frac{r}{s})$ cases with exceptional $\frac{p}{q} \in E(N(\mathcal{F})) = \{-4, -3, -2, -1, 0, \infty\}$. By a direct inspection, we see that the possible S^H–slopes are -3, -1 and ∞. However, if $\frac{p}{q} \in \{-3, -1\}$ corresponds to an S^H–filling, then $\frac{r}{s}$ is either $-1 + \frac{1}{n}$ or $-3 + \frac{1}{n}$

with $n = 0$, that is $\frac{r}{s} = \infty$, which makes $N(\mathcal{F})$ non-hyperbolic by Lemma A.10. We can hence assume that the S^H–slope is ∞ and, up to a simultabeous change of sign for r and s, that $r + 2s = 1$, that is $\frac{r}{s} = -2 + \frac{1}{s}$. Now, the only pairs of slopes at distance 3 in $E(N(\mathcal{F}))$ are $(-4, -1)$ and $(-3, 0)$, and Table A.8 tells us that neither -4 or 0 can correspond to a lens space filling. On the other hand, if $\frac{p}{q} \in \{-3, -1\}$ corresponds to an T^H–filling, then $\frac{r}{s}$ is either $-1 + \frac{1}{n}$ or $-3 + \frac{1}{n}$. But since $\frac{r}{s} = -2 + \frac{1}{s}$, it follows that $\frac{r}{s}$ is either $-\frac{5}{2}$ or $-\frac{3}{2}$, which are both excluded from Table A.8.

4.3. **Exceptional triples arising from $N(\mathcal{F})$ with $e(N(\mathcal{F})) = 5$.** The same arguments presented at the beginning of Section 3.3 reduce the study of the cases coming from [MP, Theorem 1.3 and Tables 2–4] to just Table 2 and Theorem 1.3; namely to the hyperbolic $N(\frac{r}{s}, \frac{t}{u})$ with $E(N(\frac{r}{s}, \frac{t}{u})) = \{-3, -2, -1, 0, \infty\}$. Such $N(\frac{r}{s}, \frac{t}{u})(\frac{p}{q})$ can be toroidal only when $\frac{p}{q} = -3$ or 0.

4.3.1. *Case $\frac{p}{q} = -3$ is the T–filling.* In this case, [MP, Table 2] gives us the conditions that $\frac{r}{s}, \frac{t}{u} \neq -1 - \frac{1}{n}$. We also require the lens space slope to be at distance 3 from the toroidal slope so $\frac{p}{q} = 0$ should be the lens space slope and this implies that $\{\frac{r}{s}, \frac{t}{u}\} = \{n, -4 - n + \frac{1}{m}\}$. Up to symmetry, we may hence assume that $\frac{r}{s} = n$ and $\frac{t}{u} = -4 - n + \frac{1}{m}$. The possible S^H–slopes are now -1, -2 and ∞.

Case $\frac{p}{q} = -1$ is the S^H–slope: Then either $\frac{r}{s} = -3 + \frac{1}{n'} = n$ or $\frac{t}{u} = -3 + \frac{1}{n'} = -4 - n + \frac{1}{m}$.

If $\frac{r}{s} = -3 + \frac{1}{n'} = n$, then $\frac{1}{n'} = 3 + n \in \mathbb{Z}$ so $3 + n = \pm 1$ and $n \in \{-4, -2\}$. For $N(\mathcal{F})$ to be hyperbolic, n is necessarily -4 because of Lemma A.10, so $\frac{t}{u} = \frac{1}{m}$. However, from [MP, Table 2] we get then $N(\mathcal{F})(-1) = N(-4, \frac{1}{m})(-1) = L(-3 - 7m, \star) \neq S^3$.

If $\frac{t}{u} = -3 + \frac{1}{n'} = -4 - n + \frac{1}{m}$, then $1 + n = \frac{1}{m} - \frac{1}{n'} \in [-2, 2]$ so $n \in \{-3, -2, -1, 0, 1\}$. Because of Lemma A.10, $N(\mathcal{F})$ is hyperbolic only when $n = 1$. But then $n' = -1$ and $\frac{t}{u} = -4$ so, again from [MP, Table 2], $N(\mathcal{F})(-1) = N(1, -4)(-1) = L(-10, \star) \neq S^3$.

Case $\frac{p}{q} = -2$ is the S^H–slope: Then either $\frac{r}{s} = -2 + \frac{1}{n'} = n$ or $\frac{t}{u} = -2 + \frac{1}{n'} = -4 - n + \frac{1}{m}$.

If $\frac{r}{s} = -2 + \frac{1}{n'} = n$, then $\frac{1}{n'} = 2 + n \in \mathbb{Z}$ so $n = -3$ or $n = -1$. In both cases $N(\mathcal{F})$ is non-hyperbolic by Lemma A.10.

If $\frac{t}{u} = -2 + \frac{1}{n'} = -4 - n + \frac{1}{m}$, then $2 + n = \frac{1}{m} - \frac{1}{n'} \in [-2, 2]$, so $n \in \{-4, -3, -2, -1, 0\}$. Because of Lemma A.10, $N(\mathcal{F})$ is hyperbolic only when $n = -4$. But then $n' = 1$ and $\frac{t}{u} = -2 + \frac{1}{n'} = -1$, which makes $N(\mathcal{F})$ non-hyperbolic by Lemma A.10.

Case $\frac{p}{q} = \infty$ is the S^H–slope: Then from [MP, Theorem 1.3] we know

$$N(\mathcal{F})(\infty) = N\left(n, \frac{1 - m(n+4)}{m}\right)(\infty) = L\Big((1 - m(n+4))n - m, \star\Big).$$

For $N(\mathcal{F})(\infty)$ to be S^3, it is hence required that $(1 - m(n+4))n = m \pm 1$. By Lemma A.16 it follows then that $n \in \{-5, -4, -3, -2, -1, 0, 1\}$. For $N(\mathcal{F}) = N(n, \frac{1-m(n+4)}{m})$ to be hyperbolic, n cannot be in $\{-3, -2, -1, 0\}$ because of Lemma A.10, so we are left with the cases $(n, m) \in \{(-5, -1), (-4, -5), (-4, -3), (1, 0)\}$. The cases $(n, m) = (-5, -1), (1, 0)$ yield again a non-hyperbolic $N(\mathcal{F})$, while $(n, m) = (-4, -5), (-4, -3)$ give the (S^H, T^H, T) triples $\boxed{\left(N(-4, -\frac{1}{5}), \infty, 0, -3\right)}$ and $\boxed{\left(N(-4, -\frac{1}{3}), \infty, 0, -3\right)}$ with $\Delta(T^H, T) = 3$.

4.3.2. *Case $\frac{p}{q} = 0$ is a T–filling.* To have a T^H–slope at distance 3 from the toroidal slope, we need that $N(\frac{r}{s}, \frac{t}{u})(-3) \in T^H$. According to [MP, Table 2], it follows that either $\frac{r}{s} = -2$, but then $N(\mathcal{F})$ is non hyperbolic because of Lemma A.10, or $\frac{r}{s} = -1 + \frac{1}{n}$ and $\frac{t}{u} = -1 + \frac{1}{m}$. The S^H–slope is then one of -1, -2 or ∞.

Case $\frac{p}{q} = -2$ is the S^H–slope: Then, up to symmetry, $\frac{r}{s} = -2 + \frac{1}{n'} = -1 + \frac{1}{n}$ so $n' = 2$. Since $\frac{t}{u} = \frac{1-m}{m}$, [MP, Table 2] tells us that $N(\mathcal{F})(-2) = L(4 + 7m, \star) \neq S^3$.

Case $\frac{p}{q} = -1$ is the S^H–slope: Then, up to symmetry, $\frac{r}{s} = -3 + \frac{1}{n'} = -1 + \frac{1}{n}$ so $\frac{r}{s} = -2$ which makes $N(\mathcal{F})$ non-hyperbolic by Lemma A.10.

Case $\frac{p}{q} = \infty$ is the S^H–slope: Then from [MP, Theorem 1.3], we know

$$N(\mathcal{F})(\infty) = N\left(\frac{1-n}{n}, \frac{1-m}{m}\right)(\infty) = L(1-n-m, \star).$$

For $N(\mathcal{F})(\infty)$ to be S^3, it is hence required that $1 - n - m = \pm 1$, or equivalently that $n + m \in \{0, 2\}$. The first case leads to $\boxed{B_n := \left(N(-1+\frac{1}{n}, -1-\frac{1}{n}), \infty, -3, 0\right)}$ for $n \in \mathbb{Z} \setminus \{0, \pm 1\}$ and the second to $\boxed{C_n := \left(N(-1+\frac{1}{n}, -1-\frac{1}{n-2}), \infty, -3, 0\right)}$ for $n \in \mathbb{Z} \setminus \{0, \pm 1, 2, 3\}$. Both families are indeed (S^H, T^H, T) triples with $\Delta(T^H, T) = 3$.

4.4. **Conclusion.** Along Sections 4.1 – 4.3, we have proved that the only (S^H, T^H, T) triples obtained by surgery on the 5CL and that realize $\Delta(T^H, T) = 3$ are:

- $B_n = N\left(-1+\frac{1}{n}, -1-\frac{1}{n}\right)(\infty, -3, 0)$ for $n \in \mathbb{Z} \setminus \{0, \pm 1\}$;
- $C_n = N\left(-1+\frac{1}{n}, -1-\frac{1}{n-2}\right)(\infty, -3, 0)$ for $n \in \mathbb{Z} \setminus \{0, \pm 1, 2, 3\}$;
- $N\left(-\frac{3}{2}, -\frac{14}{5}\right)\left(-2, -1, -\frac{5}{2}\right)$;
- $N\left(-4, -\frac{1}{3}\right)(\infty, 0, -3)$;
- $N\left(-4, -\frac{1}{5}\right)(\infty, 0, -3)$.

The last three isolated cases can be seen to be redundant using [MP, Theorem 1.5]. It follows indeed from the first equality that $\left(N(-4, -\frac{1}{3}), \infty, 0, -3\right) \cong B_{-2}$ and $\left(N(-4, -\frac{1}{5}), \infty, 0, -3\right) \cong C_{-2}$, and from the third equality that $\left(N\left(-\frac{3}{2}, -\frac{14}{5}\right), -2, -1, -\frac{5}{2}\right) \cong C_{-2}$. This completes the proof that every (S^H, T^H, T) triple with $\Delta(T^H, T) = 3$ is equivalent to some B_n or C_n. But besides, $B_{-n} \cong B_n$ and $C_{-n} \cong C_{n+2}$ for all n because of Lemma A.8.

We now show that these two families are distinct. This is done by comparing their exceptional fillings. Using [MP, Theorem 1.3] and [MP, Table 2] we can indeed write down the exceptional slopes and fillings of both $N(-1+\frac{1}{n}, -1-\frac{1}{n})$ and $N(-1+\frac{1}{n}, -1-\frac{1}{n-2})$; the result is shown in Table 2. We note that both B_n and C_n have a unique lens space filling, namely $B_n(-3) = L(4n^2+3, 2n^2+n+2)$ and $C_n(-3) = L(4n^2+8n-1, 2n^2-3n)$. If $B_n(-3) = C_k(-3)$ for some $n, k \in \mathbb{Z}$, then the order of their fundamental groups should be equal. But it is well known that $\pi_1(L(p, q))$ is the cyclic group of order p (see for example [Rol, Exercise 9.B.5]); so it would follow that $3 + 4n^2 = 4k^2 + 8k - 1 \Leftrightarrow 4(n-k)(k+n) = 2(4k-1)$, and this would imply that $2 \mid 4k-1$, which is a contradiction. Hence, $\{B_n\} \cap \{C_n\} = \emptyset$ and the proof of Theorem 4.1 is complete.

Remark B_2 is the exterior of the $(-2, 3, 7)$ pretzel knot. In this case, $e(B_2) = 7$ and the exceptional slopes and fillings can be found in [MP, Table A.2]. In the second family, $E(C_{-2}) = \{-3, -\frac{5}{2}, -2, -1, 0, \infty\}$; $C_{-2}(\alpha)$ is found in Table 2 for $\alpha \in E(C_{-2}) \setminus \{-\frac{5}{2}\}$, and $C_{-2}(-\frac{5}{2}) = (D, (2,1), (3,1)) \bigcup_{\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}} (D, (2,1), (5,4))$.

5. (S^H, T^H, Z) TRIPLES

In this section, we enumerate all (S^H, T^H, Z) triples obtained by surgery on the 5CL and realizing the maximal distance. It shall turn out that all such triples are obtain by surgery on the 3CL.

Theorem 5.1.

- *If $\left(M_5(\frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{x}{y}), \alpha, \beta, \gamma\right) \in (S^H, T^H, Z)$ then $\Delta(\beta, \gamma) \leq 2$.*
- *If $\left(M_5(\frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{x}{y}), \alpha, \beta, \gamma\right) \in (S^H, T^H, Z)$ with $\Delta(\beta, \gamma) = 2$ then it is equivalent to either $B'_n := \left(N(-1+\frac{1}{n}, -1-\frac{1}{n}), \infty, -3, -1\right)$ for some integer $n \geq 2$, or $C'_n := \left(N(-1+\frac{1}{n}, -1-\frac{1}{n-2}), \infty, -3, -1\right)$ for some integer $n \geq 4$.*

Remark Note that the knot exteriors in Theorem 5.1 are the same as the knot exteriors in Theorem 4.1. Therefore, we know that these examples are distinct and that the exceptional slopes and fillings are given in Table 2.

The proof shall proceed in three steps. In Section 5.1, we show that if $(M_5(\mathcal{F}),\alpha,\beta,\gamma)\in(S^H,T^H,Z)$ and $\Delta(\beta,\gamma)\geq 2$ then $\mathcal{F}$ factors through M_4. Then, in Section 5.2, we show that if $(M_4(\mathcal{F}),\alpha,\beta,\gamma)\in(S^H,T^H,Z)$ and $\Delta(\beta,\gamma)\geq 2$ then $\mathcal{F}$ factors through M_3. Finally, in Section 5.3 we show that if $(N(\mathcal{F}),\alpha,\beta,\gamma)\in(S^H,T^H,Z)$ then $\Delta(\beta,\gamma)\leq 2$ and all triples with $\Delta(\beta,\gamma)=2$ are then enumerated.

5.1. (S^H,T^H,Z) **triples from** M_5**.** If $(M_5(\mathcal{F}),\alpha,\beta,\gamma)\in(S^H,T^H,Z)$ and $\mathcal{F}$ does not factor through M_4 then, from [Rou2, Theorem 4], we have that either $E(M_5(\mathcal{F}))=\{0,1,\infty\}$, or $\mathcal{F}$ is equivalent to one of the surgery instructions in [Rou2, Tables 14 – 20]. But clearly, if $E(M_5(\mathcal{F}))=\{0,1,\infty\}$ then all exceptional slopes are at distance 1. Any (S^H,T^H,Z) triple realising $\Delta(T^H,Z)\geq 2$ should hence be found in [Rou2, Tables 14 – 20]. In these tables, the only simultaneous occurence of S^H, T^H and Z–slopes are in Table 17, with $\mathcal{F}=(-2,\frac{p}{q},3,\frac{u}{v})$, and in Table 18, with $\mathcal{F}=(-2,\frac{p}{q},\frac{r}{s},-2)$. In both cases, the exceptional slopes are 0, ±1 and ∞, so as a consequence $\Delta(\beta,\gamma)\leq 2$. Moreover, the only possibility for (Z,T^H)–slopes to realise $\Delta(\beta,\gamma)=2$ is that $\{\beta,\gamma\}=\{\pm1\}$; the S^H–slope is then either 0 or ∞. We proceed now with a case by case analysis.

Case $\mathcal{F}=(-2,\frac{p}{q},3,\frac{u}{v})$**:** By applying $(9)^{-2}\circ(16)$, we may assume that 0 corresponds to the S^H–slope. It follows then from [Rou2, Table 17] that either $\frac{p}{q}=1+\frac{1}{n}$ and $|(3+2n)u-(7+6n)v|=1$, or $\frac{u}{v}=3+\frac{1}{k}$ and $|(3+2k)p-(1+2k)q|=1$. Moreover, from -1 being a T^H or Z–slope, we also know that either $|p|=1$ or $|u+v|=1$. These conditions shall be shown to be incompatible.

Case $\frac{p}{q}=1+\frac{1}{n}$ **and** $|(3+2n)u-(7+6n)v|=1$**:**

If $|p|=1$ then, up to reversing both p and q, we may assume that $p=1$ so that $-1+\frac{1}{1-q}=n\in\mathbb{Z}$, that is $q\in\{0,2\}$. But then $\frac{p}{q}\in\{\frac{1}{2},\infty\}$, which is discarded by Lemma A.4.

If $|u+v|=1$ then, up to reversing both u and v, we may assume that $u=1-v$. Subbing this into $|(3+2n)u-(7+6n)v|=1$ and solving for v in terms of n, we obtain that v is either $\frac{2+n}{5+4n}$ or $\frac{1+n}{5+4n}$. But $v\in\mathbb{Z}$, so in the first case, n shall be -1 or -2, that is $\frac{u}{v}\in\{0,\infty\}$; and in the second case, n shall be -1, that is $\frac{u}{v}=\infty$. All these cases are discarded by Lemma A.4.

Case $\frac{u}{v}=3+\frac{1}{k}$ **and** $|(3+2k)p-(1+2k)q|=1$**:**

If $|u+v|=1$, then $1=|u+v|=|(3k+1)+k|$ meaning that $k=0$ and that $\frac{u}{v}=\infty$ which is excluded by Lemma A.4.

If $|p|=1$ then, up to reversing both p and q, we may assume that $p=1$ and subbing this in $|(3+2k)p-(1+2k)q|=1$ we obtain that q is either $1+\frac{1}{1+2k}$ or $1+\frac{3}{1+2k}$. But $q\in\mathbb{Z}$, so $q\in\{0,\pm2,4\}$. If $q\in\{0,2\}$, then $\frac{p}{q}\in\{\frac{1}{2},\infty\}$ and this is discarded by Lemma A.4. If $q\in\{-2,4\}$, then $k\in\{-1,0\}$ and $\frac{u}{v}\in\{2,\infty\}$ which is also discarded by Lemma A.4.

Case $\mathcal{F}=(-2,\frac{p}{q},\frac{r}{s},-2)$**:** From [Rou2, Table 18] we see that for -1 to be a type Z or T^H–slope we need $|q|=1$ or $|s|=1$. By applying $(9)^{-1}\circ(10)$, we may assume that $|q|=1$. But from the same table, we also see that for 1 to be a type Z or T^H–slope we need $|p|=1$ or $|r|=1$. Since $|q|=1$, the case $|p|=1$ is discarded by Lemma A.4. It follows hence that $|r|=1$. But now, [Rou2, Table 18] also tells that the only possible S^H–slope is 0, and that it requires either $\frac{p}{q}=1+\frac{1}{n}$ or $\frac{r}{s}=1+\frac{1}{n}$. But, since $|q|=1$, the first condition implies that $\frac{p}{q}\in\{0,2\}$ and, since $|r|=1$, the second condition implies that $\frac{r}{s}=\frac{1}{2}$; all these are discarded by Lemma A.4.

5.2. (S^H,T^H,Z) **triples from** M_4**.** If $(M_4(\mathcal{F}),\alpha,\beta,\gamma)\in(S^H,T^H,Z)$ with $\mathcal{F}$ not factoring through M_3, then according to [Rou2, Theorem 5], either $E(M_4(\mathcal{F}))=\{0,1,2,\infty\}$ or $\mathcal{F}$ is equivalent to a filling instruction

$\infty \rightsquigarrow S^H$	$0 \rightsquigarrow T^H$	$2 \rightsquigarrow Z$
$a = \pm 1$	$\frac{a}{b} = n$	$\frac{a}{b} = 1 + \frac{1}{p}$
$d = \pm 1$	$\frac{c}{d} = 2 + \frac{1}{k}$	$\frac{e}{f} = 1 + \frac{1}{p}$
$e = \pm 1$		$\frac{c}{d} = \frac{1}{p}$

Table 3: Necessary conditions for $(\infty, 0, 2)$ to be a (S^H, T^H, Z) triple.

listed in [Rou2, Tables 21 – 22]. But in these tables, S^H and T^H–slopes never occur simultaneously. It follows that $E(M_4(\mathcal{F})) = \{0, 1, 2, \infty\}$. In particular, $\Delta(\beta, \gamma) \leq 2$ and if $\Delta(\beta, \gamma) = 2$ then $\{\beta, \gamma\} = \{0, 2\}$ and $\alpha \in \{1, \infty\}$. But one can observe that, on one hand,

$$M_4(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g}{h}) \underset{\text{Lemma A.5}}{\cong} M_5(\tfrac{a}{b}, \tfrac{c-d}{d}, -1, \tfrac{e-f}{f}, \tfrac{g}{h}) \underset{(13)}{\cong} M_5(\tfrac{g}{g-h}, \tfrac{b-a}{b}, -1, \tfrac{d}{c-d}, \tfrac{2f-e}{f})$$
$$\underset{\text{Lemma A.5}}{\cong} M_4(\tfrac{g}{g-h}, \tfrac{2b-a}{b}, \tfrac{c}{c-d}, \tfrac{2f-e}{f}) \underset{\text{Lemma A.7}}{\cong} M_4(\tfrac{2b-a}{b}, \tfrac{c}{c-d}, \tfrac{2f-e}{f}, \tfrac{g}{g-h}),$$

and that, on the other hand,

$$M_4(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g}{h}) \underset{\text{Lemma A.5}}{\cong} M_5(\tfrac{a}{b}, \tfrac{c-d}{d}, -1, \tfrac{e-f}{f}, \tfrac{g}{h}) \underset{(9)^2\circ(11)}{\cong} M_5(\tfrac{2d-c}{d}, \tfrac{f}{e-f}, -1, \tfrac{h-g}{h}, \tfrac{a}{a-b})$$
$$\underset{\text{Lemma A.5}}{\cong} M_4(\tfrac{2d-c}{d}, \tfrac{e}{e-f}, \tfrac{2h-g}{h}, \tfrac{a}{a-b}) \underset{\text{Lemma A.7}}{\cong} M_4(\tfrac{a}{a-b}, \tfrac{2d-c}{d}, \tfrac{e}{e-f}, \tfrac{2h-g}{h}).$$

Consequently, it follows then directly that $\left(M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}), 1, 0, 2\right) \cong \left(M_4(\frac{e-2f}{e-f}, \frac{c-2d}{c-d}, \frac{2b-a}{b-a}), \infty, 0, 2\right)$ and that $\left(M_4(\frac{p}{q}, \frac{r}{s}, \frac{u}{v}), \infty, 0, 2\right) \cong \left(M_4(\frac{p}{p-q}, \frac{2s-r}{s}, \frac{u}{u-v}), \infty, 2, 0\right)$. Up to these equivalencies, we can hence assume that ∞ is the S^H–slope, 0 is the T^H–slope and 2 the Z–slope. We set the filling instruction on M_4 to be $\mathcal{F} = (\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$.

Since $M_4(\mathcal{F})(\infty) = S^3$, we know by (23) and Lemma A.2, that one of $|a|$, $|d|$ or $|e|$ is 1.

Since $M_4(\mathcal{F})(0)$ is T^H, we know by (24) and Lemma A.2, that one of b, f or $c - 2d$ is in $\{0, \pm 1\}$. But if one of them is 0, then one of $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ is in $\{2, \infty\}$ and $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ is non-hyperbolic by Lemma A.6. We conclude that one of $|b|$, $|f|$ or $|c - 2d|$ is 1, that is either $\frac{a}{b} = n$, $\frac{e}{f} = n$ or $\frac{c}{d} = 2 + \frac{1}{k}$. Using Lemma A.7, we may even assume that either $\frac{a}{b} = n$ or $\frac{c}{d} = 2 + \frac{1}{k}$.

Since $M_4(\mathcal{F})(2) \in Z$, we know by (26) and Lemma A.2, that one of $a - b$, c or $e - f$ is in $\{0, \pm 1\}$. But if one of them is 0, then one of $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ is in $\{0, 1\}$ and $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ is non-hyperbolic by Lemma A.6. We conclude that one of $|a - b|$, $|c|$ or $|e - f|$ is 1, that is either $\frac{a}{b} = 1 + \frac{1}{p}$, $\frac{e}{f} = 1 + \frac{1}{p}$ or $\frac{c}{d} = \frac{1}{p}$.

Collecting the necessary conditions for $(\infty, 0, 2)$ to be a (S^H, T^H, Z) triple found above, we see that at least one condition from each column in Table 3 must be fulfilled.

Case $\frac{c}{d} = 2 + \frac{1}{k}$: Then $\frac{c}{d} = \frac{2k+1}{k}$ and identity (24) implies that

$$M_4(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f})(0) \underset{(24)}{\cong} (D, (f, -e), (b, 2b-a)) \bigcup\nolimits_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (2, 1), (1, k))$$
$$\underset{(6)}{\cong} (S^2, (f, -e), (b, 2b-a), (2k+1, -2)).$$

But since $M_4(\mathcal{F})(0)$ is a lens space, and according to Lemma A.2, it follows that either b, f or $2k + 1$ is ± 1.

If $2k + 1 = \pm 1$, then $k \in \{-1, 0\}$ and $\frac{c}{d} = 2 + \frac{1}{k} \in \{1, \infty\}$, which is ruled out by Lemma A.6.

If $|b|$ or $|f|$ is 1, then up to Lemma A.7, we may even assume that $|b| = 1$. But now, we can claim that $a, d \neq \pm 1$, otherwise $\frac{a}{b}$ would be ± 1, or $\frac{c}{d} = 2 + \frac{1}{k}$ would be 1 or 3, and those are discarded by Lemma A.6. Looking at the first column of Table 3, we conclude hence that $e = \pm 1$. Looking now at the third column of Table 3, we see that either $\frac{a}{b} = 1 + \frac{1}{p}$, but then condition $b = \pm 1$ implies that $\frac{a}{b} \in \{0, 2\}$; or $\frac{e}{f} = 1 + \frac{1}{p}$, but then condition $e = \pm 1$ implies that $\frac{e}{f} = \frac{1}{2}$, or $\frac{c}{d} = \frac{1}{p}$, but then condition $\frac{c}{d} = 2 + \frac{1}{k}$ implies that $\frac{c}{d} = 1$. All those are discarded by Lemma A.6.

Case $\frac{a}{b} = n$: Then identity (24) implies that

$$M_4(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f})(0) \underset{(24)}{=} (D, (f, -e), (1, 2-n)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (2, 1), (c-2d, d)) \underset{(6)}{=} (S^2, (2, 1), (c-2d, d), (f(2-n)-e, -f)).$$

But since $M_4(\mathcal{F})(0)$ is a lens space, and according to Lemma A.2, it follows that either $c - 2d$ or $e + f(n-2)$ is ± 1.

If $|c - 2d| = 1$, then $\frac{c}{d} = 2 + \frac{1}{k}$ and we are left to the previous case.

If $e + f(n-2) = \pm 1$, that is $\frac{e}{f} = 2 - n + \frac{1}{k}$, then $\frac{a}{b} \neq 1 + \frac{1}{p}$, otherwise $\frac{a}{b} = n$ would be 0 or 2 and this is discarded by Lemma A.6. But $\frac{e}{f}$ is also distinct from $1 + \frac{1}{p}$. Indeed, $1 + \frac{1}{p} = 2 - n + \frac{1}{k}$ would imply that $(n, k) \in \{(1, p), (0, 2), (2, 2)\}$ and then either $\frac{a}{b} = n = 1$ or $\frac{e}{f} \in \{\frac{1}{2}, \frac{3}{2}\}$, both are discarded by Lemma A.6. Looking at the third column of Table 3, we conclude hence that $\frac{c}{d} = \frac{1}{p}$. Looking now at the first column of Table 3, we see that either $a = \pm 1$, but then condition $\frac{a}{b} = n$ implies that $\frac{a}{b} = \pm 1$; or $d = \pm 1$, but then condition $\frac{c}{d} = \frac{1}{p}$ implies that $\frac{e}{f} = \pm 1$, or $e = \pm 1$, but then condition $\frac{e}{f} = 2 - n + \frac{1}{k}$ implies that $\frac{a}{b} = n \in \{1, 2, 3\}$. All those are discarded by Lemma A.6.

This ends the proof that if $(M_5(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ and $\Delta(\beta, \gamma) \geq 2$ then $\mathcal{F}$ factors through M_3.

5.3. (S^H, T^H, Z) triples from M_3.

Proposition 5.2. *If $\left(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma\right) \in (S^H, T^H, Z)$ then $\Delta(\beta, \gamma) \leq 2$.*

Proof. If $N(\frac{r}{s}, \frac{t}{u})$ is hyperbolic, then [MP, Corollary A.6] tells us that either $e(N(\frac{r}{s}, \frac{t}{u})) = 5$ or $N(\frac{r}{s}, \frac{t}{u})$ is found in [MP, Tables A.2 – A.9]. Moreover, if $e(N(\frac{r}{s}, \frac{t}{u})) = 5$ then, it is a consequence of [MP, Theorem 1.3] and Lemma A.10 that $E(N(\frac{r}{s}, \frac{t}{u})) = \{\infty, -3, -2, -1, 0\}$. Since we just want to dismiss pairs of exceptional slopes at distance greater than two, we only have to consider the case $\{\beta, \gamma\} = \{0, -3\}$.

Case $\{\beta, \gamma\} = \{0, -3\}$: In this case, we can see in [MP, Table 2] that if $N(\frac{r}{s}, \frac{t}{u})$ is hyperbolic with $N(\frac{r}{s}, \frac{t}{u})(-3) \in T^H$ then either $\frac{r}{s} = -2$, but this is dismissed by Lemma A.10, or $\frac{r}{s} = -1 + \frac{1}{n}$ and $\frac{t}{u} = -1 + \frac{1}{m}$. But then $N(\frac{r}{s}, \frac{t}{u})(0) \in Z$ and [MP, Table 2] tells us that one of $\frac{r}{s} = -1 + \frac{1}{n}$ or $\frac{t}{u} = -1 + \frac{1}{m}$ is an integer, so that one of $\frac{r}{s}$ and $\frac{t}{u}$ is in $\{-2, 0\}$, which is forbidden by Lemma A.10.

On the other hand, if $N(\frac{r}{s}, \frac{t}{u})(-3) \in Z$, then the same table tells us[1] that, up to Lemma A.8, $\frac{r}{s} = -1 + \frac{1}{n}$. But then $N(\frac{r}{s}, \frac{t}{u})(0) \in T^H$ and [MP, Table 2] tells us that $\{\frac{r}{s}, \frac{t}{u}\} = \{k, -4 - k + \frac{1}{m}\}$. The case $\frac{r}{s} = -1 + \frac{1}{n} = k$ would imply $\frac{r}{s} \in \{-2, 0\}$ and is hence dismissed by Lemma A.10; so we can assume that $\frac{r}{s} = -1 + \frac{1}{n} = -4 - k + \frac{1}{m}$ and $\frac{t}{u} = k$. As $n = \pm 1$ would make $N(\frac{r}{s}, \frac{t}{u})$ non-hyperbolic because of Lemma A.10, we have $3 + k = \frac{1}{m} - \frac{1}{n} \in \{0, \pm 1\}$, that is $k \in \{-4, -3, -2\}$. But since $\frac{t}{u} = k$, cases $k = -3$ and $k = -2$ are dismissed by Lemma A.10. If $k = -4$ then $\frac{t}{u} = -4$ and $\frac{r}{s} = -1 + \frac{1}{n} = \frac{1}{m}$; it follows that $\frac{r}{s} = -\frac{1}{2}$, but $N(\frac{r}{s}, \frac{t}{u}) = N(-4, -\frac{1}{2})$ is non-hyperbolic, see [MP, Table 1].

Case $N(\frac{r}{s}, \frac{t}{u})$ is found in [MP, Tables A.2 – A.9] **and $\{\beta, \gamma\} \neq \{0, -3\}$:** It is immediately clear that the only (S^H, T^H, Z) triple in Tables A.2–A.4 and Table A.9 is the triple obtained from the $(-2, 3, 7)$ pretzel knot, and in this case $\Delta(\beta, \gamma) = 2$.

If $\left(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma\right) \in (S^H, T^H, Z)$ with $\Delta(\beta, \gamma) > 2$ is found in [MP, Table A.5] then $E(N(\frac{r}{s}, \frac{t}{u})) = \{-3, -2, -1, 0, 1, \infty\}$ so that $1 \in \{\beta, \gamma\}$. However, this table tells us that $N(\frac{r}{s}, \frac{t}{u})(1)$ is never in $T^H \cup Z$.

[1] A word of caution: as one can read in the arXiv preprint of this article, the relation between the entries on the $\frac{r}{s}$ column and the $\frac{t}{u}$ column in [MP, Table 2] is more intricate than what a reader might appreciate in the published version, and the conditions $\frac{r}{s} = -1 + \frac{1}{n}$ and $\frac{r}{s} \neq -2$ are actually shared by lines 4 and 5.

If $\left(N(\frac{r}{s},\frac{t}{u}),\alpha,\beta,\gamma\right) \in (S^H, T^H, Z)$ with $\Delta(\beta,\gamma) > 2$ is found in [MP, Table A.6] then $E(N(\frac{r}{s},\frac{t}{u})) = \{-3,-\frac{5}{2},-2,-1,0,\infty\}$ and $-\frac{5}{2} \in \{\beta,\gamma\}$. This table also tells us that $N(\frac{r}{s},\frac{t}{u})(-\frac{5}{2}) \notin T^H$ and that $N(\frac{r}{s},\frac{t}{u})(-\frac{5}{2}) \in Z$ only when $\frac{r}{s} = -2+\frac{1}{n}$. Moreover, if $\Delta(\beta,-\frac{5}{2}) > 2$ then $\beta \in \{-1,0\}$; but 0 is not a T^H–slope and if -1 is, then $\frac{r}{s} = -2+\frac{1}{n} = -3+\frac{1}{k}$, that is $\frac{r}{s} = -\frac{5}{2}$, which is actually excluded from this table.

If $\left(N(\frac{r}{s},\frac{t}{u}),\alpha,\beta,\gamma\right) \in (S^H, T^H, Z)$ with $\Delta(\beta,\gamma) > 2$ is found in [MP, Table A.7] then $E(N(\frac{r}{s},\frac{t}{u})) = \{-3,-2,-\frac{3}{2},-1,0,\infty\}$ and $-\frac{3}{2} \in \{\beta,\gamma\}$. Moreover this table tells us $N(\frac{r}{s},\frac{t}{u})(-\frac{3}{2}) \notin T^H$ so that $\gamma = -\frac{3}{2}$; but if $\Delta(\beta,-\frac{3}{2}) > 2$ then $\beta \in \{-3,0\}$ and neither of them is a T^H–slope.

Finally, if $\left(N(\frac{r}{s},\frac{t}{u}),\alpha,\beta,\gamma\right) \in (S^H, T^H, Z)$ with $\Delta(\beta,\gamma) > 2$ is found in [MP, Table A.8] then $E(N(\frac{r}{s},\frac{t}{u})) = \{-4,-3,-2,-1,0,\infty\}$ and $-4 \in \{\beta,\gamma\}$. This table also tells us that $N(\frac{r}{s},\frac{t}{u})(-4) \notin T^H$ and that $N(\frac{r}{s},\frac{t}{u})(-4) \in Z$ only when $\frac{r}{s} \in \mathbb{Z}$. Moreover, if $\Delta(\beta,-4) > 2$ then $\beta \in \{-1,0\}$; but 0 is not a T^H–slope, and if -1 is, then $\frac{r}{s} = -3+\frac{1}{n}$ and, since $\frac{r}{s} \in \mathbb{Z}$, $\frac{r}{s} \in \{-4,-2\}$, which are actually excluded from this table.

□

Proposition 5.3. *If* $\left(N(\frac{r}{s},\frac{t}{u}),\alpha,\beta,\gamma\right) \in (S^H, T^H, Z)$ *and* $\Delta(\beta,\gamma) = 2$ *then it is equivalent to either* $B'_n := \left(N(-1+\frac{1}{n},-1-\frac{1}{n}),\infty,-3,-1\right)$ *with* $n \in \mathbb{Z}\setminus\{0,\pm1\}$, *or to* $C'_n := \left(N(-1+\frac{1}{n},-1-\frac{1}{n-2}),\infty,-3,-1\right)$ *with* $n \in \mathbb{Z}\setminus\{0,\pm1,2,3\}$.

Proof. By the same discussion that the one which begins the proof of Proposition 5.2, we know that either $E(N(\frac{r}{s},\frac{t}{u})) = \{0,-1,-2,-3,\infty\}$ or $\{0,-1,-2,-3,\infty\} \subsetneq E(N(\frac{r}{s},\frac{t}{u}))$ and, in the latter case, $N(\frac{r}{s},\frac{t}{u})$ and $E(N(\frac{r}{s},\frac{t}{u}))$ are found in [MP, Tables A.2 – A.9].

Case $\{\alpha,\beta,\gamma\} \not\subset \{0,-1,-2,-3,\infty\}$**:** In this case, $N(\frac{r}{s},\frac{t}{u})$, $E(N(\frac{r}{s},\frac{t}{u}))$ are found in [MP, Tables A.2 – A.9]. It is immediately clear that the only $\left(N(\frac{r}{s},\frac{t}{u}),\alpha,\beta,\gamma\right) \in (S^H, T^H, Z)$ in [MP, Tables A.2 – A.4 and Table A.9] is the $(-2,3,7)$ pretzel knot exterior $\boxed{\left(N(-4,-\frac{1}{3}),\infty,0,-2\right)}$.

If $\left(N(\frac{r}{s},\frac{t}{u}),\alpha,\beta,\gamma\right) \in (S^H, T^H, Z)$ is found in [MP, Table A.5] then the exceptional set $E(N(\frac{r}{s},\frac{t}{u}))$ is $\{-3,-2,-1,-0,1,\infty\}$ so that $1 \in \{\alpha,\beta,\gamma\}$; but in this table $N(1,\frac{r}{s})(1) \notin T^H \cup S^H \cup Z$.

If $\left(N(\frac{r}{s},\frac{t}{u}),\alpha,\beta,\gamma\right) \in (S^H, T^H, Z)$ is found in [MP, Table A.6] then the exceptional set $E(N(\frac{r}{s},\frac{t}{u}))$ is $\{-3,-\frac{5}{2},-2,-1,-0,\infty\}$ so that $-\frac{5}{2} \in \{\alpha,\beta,\gamma\}$. Moreover this table tells us that $N(\frac{r}{s},-\frac{3}{2})(-\frac{5}{2})$ is in $S^H \cup T^H \cup Z$ only if $\frac{r}{s} = -2+\frac{1}{n}$, in which case $N(-2+\frac{1}{n},-\frac{3}{2})(-\frac{5}{2}) \in Z$. But we have $\Delta(\beta,-\frac{5}{2}) = 2$ for $\beta \in E(N(\frac{r}{s},\frac{t}{u}))$ only when $\beta = \infty$. We can also see from Table A.6 that the only possible S^H–slopes on hyperbolic $N(\frac{r}{s},-\frac{3}{2})$ are ∞ which is already the T^H–slope in our case, -3 but then $\frac{r}{s} = -2$ and this is discarded by Lemma A.10, -2, and -1 but then $\frac{r}{s} = \infty$ and this is discarded by Lemma A.10. We are hence left with $\alpha = -2$ and $|4r+11s| = 1$. But $\frac{r}{s} = \frac{1-2n}{n}$, so $|4r+11s| = 1$ if and only if $n = -1$. It would follow that $\frac{r}{s} = -3$ and this is excluded by Lemma A.10.

If $\left(N(\frac{r}{s},\frac{t}{u}),\alpha,\beta,\gamma\right) \in (S^H, T^H, Z)$ is found in [MP, Table A.7] then the exceptional set $E(N(\frac{r}{s},\frac{t}{u}))$ is $\{-3,-2,-\frac{3}{2},-1,0,\infty\}$ so that $-\frac{3}{2} \in \{\alpha,\beta,\gamma\}$. This table also tells us that $N(\frac{r}{s},-\frac{5}{2})(-\frac{3}{2})$ is in $S^H \cup T^H \cup Z$ only if $\frac{r}{s} = -2+\frac{1}{n}$, in which case $N(-2+\frac{1}{n},-\frac{5}{2})(-\frac{3}{2}) \in Z$. Moreover, the only possible S^H–slopes found in this table are ∞, -2 and -1. More precisely, one can read that

- $N\left(-2+\frac{1}{n},-\frac{3}{2}\right)(\infty) = L(5-8n,\star)$ which is S^H if and only if $|5-8n| = 1$, but this has no integer solution;
- $N\left(-2+\frac{1}{n},-\frac{3}{2}\right)(-2) = L(8-3n,\star)$ which is S^H if and only if $|8-3n| = 1$, that is when $n = 3$, but then $\frac{r}{s} = -\frac{5}{3}$ and we get the $(-2,3,7)$ pretzel knot which is excluded from [MP, Table A.7];
- $N\left(-2+\frac{1}{n},-\frac{3}{2}\right)(-1) = L(3+5n,\star)$ which is S^H if and only if $|3+5n| = 1$, but this has no integer solution.

If $\left(N(\frac{r}{s},\frac{t}{u}),\alpha,\beta,\gamma\right) \in (S^H, T^H, Z)$ is found in [MP, Table A.8] then the exceptional set $E(N(\frac{r}{s},\frac{t}{u}))$ is $\{-4,-3,-2,-1,0,\infty\}$ so that $-4 \in \{\alpha,\beta,\gamma\}$. This table also tells us that $N(\frac{r}{s},-\frac{1}{2})(-4)$ is in $S^H \cup T^H \cup Z$ only if $\frac{r}{s} = n$, in which case $N(n,-\frac{1}{2})(-4) \in Z$. Moreover, the only possible S^H–slope are ∞, -3, -2 and -1, but in the last three cases $\frac{r}{s} = \infty$ and this is discarded by Lemma A.10. Finally, $N\left(n,-\frac{1}{2}\right)(\infty) = L(n+2,\star)$ is S^H if and only if $n = -3$, which makes $N\left(n,-\frac{1}{2}\right)$ non-hyperbolic by Lemma A.10.

Case $\{\alpha,\beta,\gamma\} \subset \{0,-1,-2,-3,\infty\}$**:** All examples can be constructed from [MP, Table 2]. However, as noted in the footnote on page 22, we warn the reader that Table 2, as given in the published version, misses some separating lines. We recommend hence to look at the arXiv version.

We first show that we may assume that ∞ corresponds to the S^H–slope. Indeed, we know by [CGLS] that the distance between the S^H and the T^H–slope is 1, and we are looking to (S^H, T^H, Z) triple realizing $\Delta(T^H, Z) = 2$. So, if ∞ is not an S^H–slope, then the triple (α,β,γ) belongs to $\{(-3,-2,0),(-2,-3,-1),(-2,-1,-3),(-1,-2,0),(-1,0,-2),(0,-1,-3)\}$.

Case $(\alpha,\beta,\gamma) = (-3,-2,0)$**:** Then for -3 to be a S^H–slope, we need that either $\frac{r}{s} = -2$, but this is discarded by Lemma A.10, or $\frac{r}{s} = -1 + \frac{1}{n}$ and $\frac{t}{u} = -1 + \frac{1}{m}$. But since 0 is a Z–slope, one of $\frac{r}{s}$ or $\frac{t}{u}$ has to be in $\mathbb{Z}$ and hence equal to -2 or 0. This is again forbidden by Lemma A.10.

Cases $(\alpha,\beta,\gamma) = (-2,-3,-1)$ **and** $(\alpha,\beta,\gamma) = (-2,-1,-3)$**:** Using Lemma A.8 and Identity (1.3) in [MP, Proposition 1.5], we obtain that any such example is going to be equivalent to $\left(N(-\frac{3}{2},\frac{2t+5u}{t+2u}),\infty,-\frac{2\beta+5}{\beta+2},-\frac{2\gamma+5}{\gamma+2}\right)$, where ∞ is the S^H–slope. This identifies the triple $(\alpha,\beta,\gamma) = (-2,-3,-1)$ with $(\alpha,\beta,\gamma) = (\infty,-1,-3)$ and the triple $(\alpha,\beta,\gamma) = (-2,-1,-3)$ with $(\alpha,\beta,\gamma) = (\infty,-3,-1)$ which are considered later.

Case $(\alpha,\beta,\gamma) = (-1,-2,0)$**:** Then, for -2 to be a T^H–slope, we need that $\frac{r}{s} = -2+\frac{1}{n}$. Moreover, for -1 to be a S^H–slope, we need that either $\frac{r}{s}$ or $\frac{t}{u}$ is of the form $-3+\frac{1}{k}$.

If $\frac{r}{s} = -3+\frac{1}{k} = -2+\frac{1}{n}$, then $\frac{r}{s} = -\frac{5}{2}$ so for 0 to a Z–slope, $\frac{t}{u}$ must be an integer l. In this case, $N(-\frac{5}{2},l)(-1) = L(3l+11,\star)$ is S^3 if and only if $l = -4$; and indeed $\boxed{\left(N(-4,-\frac{5}{2}),-1,-2,0\right)}$ is a $\in (S^H, T^H, Z)$ triple.

If $\frac{t}{u} = -3+\frac{1}{k}$, then for 0 to a Z–slope, $\frac{r}{s} = -2+\frac{1}{n}$ or $\frac{t}{u} = -3+\frac{1}{k}$ must be an integer. But the values -3, -2 and -1 are all discarded by Lemma A.10, so the only remaining case is $\frac{t}{u} = -4$. In this case, $N(-4,-2+\frac{1}{n})(-1) = L(-3-n,\star)$ is S^3 if and only if $n \in \{-2,-4\}$; and indeed $\left(N(-\frac{5}{2},-4),-1,-2,0\right)$ (already listed) and $\boxed{\left(N(-4,-\frac{9}{4}),-1,-2,0\right)}$ are (S^H, T^H, Z) triples.

Case $(\alpha,\beta,\gamma) = (-1,0,-2)$**:** Then, for -1 to be a S^H–slope, we need that $\frac{r}{s} = -3+\frac{1}{n}$. Moreover, for 0 to be a T^H–slope, we need that $\{\frac{r}{s},\frac{t}{u}\} = \{k,-4-k+\frac{1}{m}\}$.

If $\frac{r}{s} = k = -3+\frac{1}{n}$, then either $\frac{r}{s} = -2$, but this is discarded by Lemma A.10, or $\frac{r}{s} = -4$ and then $\frac{t}{u} = \frac{1}{m}$. In this case $N(-4,\frac{1}{m})(-1) = L(-3-7m,\star) \neq S^3$.

If $\frac{r}{s} = -4-k+\frac{1}{m} = -3+\frac{1}{n}$ then $k = -1+\frac{1}{m}-\frac{1}{n}$ so that $\frac{t}{u} = k \in \{-3,-2,-1,0,1\}$. But the values -3, -2, -1 and 0 are all discarded by Lemma A.10, so the only remaining case is $\frac{t}{u} = 1$, implying that $n = -1$ so that $\frac{r}{s} = -4$. In this case, $N(-4,1)(-1) = L(-10,\star) \neq S^3$.

Case $(\alpha,\beta,\gamma) = (0,-1,-3)$**:** Then, for 0 to be a S^H–slope, we need that $\{\frac{r}{s},\frac{t}{u}\} = \{n,-4-n+\frac{1}{m}\}$ and moreover that $m = 0$. It follows that either $\frac{r}{s}$ or $\frac{t}{u}$ is ∞, which is discarded by Lemma A.10.

We can now assume that ∞ is the S^H–slope, that is $\alpha = \infty$ and

$$|rt - su| = 1. \tag{5}$$

Moreover, $\{\beta,\gamma\} \subset \{-3,-2,-1,0\}$, so for $\Delta(\beta,\gamma) = 2$ to hold, the only possibilities are $(\beta,\gamma) \in \{(-3,-1),(-1,3),(-2,0),(0,-2)\}$.

Case $(\alpha,\beta,\gamma) = (\infty,-3,-1)$**:** Then for -3 to be a T^H–slope, we need that either $\frac{r}{s} = -2$, but this is discarded by Lemma A.10, or $\frac{r}{s} = -1 + \frac{1}{n}$ and $\frac{t}{u} = -1 + \frac{1}{m}$. Condition (5) becomes then $(1-n)(1-m)-nm = \pm 1$, that is $m = 1 \pm 1 - n$. This leads to $\boxed{\big(N(-1+\frac{1}{n},-1-\frac{1}{n}),\infty,-3,-1\big)}$ and $\boxed{\big(N(-1+\frac{1}{n},-1-\frac{1}{n-2}),\infty,-3,-1\big)}$ which are indeed families of (S^H,T^H,Z) triples.

Case $(\alpha,\beta,\gamma) = (\infty,-1,-3)$**:** Then for -1 to be a T^H–slope, we need that $\frac{r}{s} = -3+\frac{1}{n}$. Moreover, for -3 to be a Z–slope, we need that either $\frac{r}{s}$ or $\frac{t}{u}$ is of the form $-1+\frac{1}{k}$ and this cannot be $\frac{r}{s}$ otherwise we would have $\frac{r}{s} = -2$, which is discarded by Lemma A.10. We have then $(\frac{r}{s},\frac{t}{u}) = (-3+\frac{1}{n},-1+\frac{1}{m})$ and condition (5) becomes $(1-3n)(1-m)-nm = \pm 1$, that is $m = \frac{3n}{2n-1}$ or $\frac{3n-2}{2n-1}$. This implies that $n \in \{-1,0,1,2\}$ in the former case and that $n \in \{0,1\}$ in the latter. Cases $n \in \{0,1\}$ make $\frac{r}{s} \in \{-2,\infty\}$ and are excluded by Lemma A.10. If $n = -1$ then $m = 1$ and $\frac{t}{u} = 0$ which is also excluded by Lemma A.10. If $n = 2$ then $m = 2$, $\frac{r}{s} = -\frac{5}{2}$ and $\frac{t}{u} = -\frac{1}{2}$; and indeed $\boxed{\big(N(-\frac{5}{2},-\frac{1}{2}),\infty,-1,-3\big)}$ is a (S^H,T^H,Z) triple.

Case $(\alpha,\beta,\gamma) = (\infty,-2,0)$**:** Then for -2 to be a T^H–slope, we need that $\frac{r}{s} = -2+\frac{1}{n}$. Moreover, for 0 to be a Z–slope, we need that either $\frac{r}{s}$ or $\frac{t}{u}$ is an integer k and this cannot be $\frac{r}{s}$ otherwise we would have $\frac{r}{s} \in \{-3,-1\}$, which is discarded by Lemma A.10. We have then $(\frac{r}{s},\frac{t}{u}) = (-2+\frac{1}{n},k)$ and condition (5) becomes $k(1-2n)-n = \pm 1$, that is $k = \frac{n+1}{1-2n}$ or $\frac{n-1}{1-2n}$. This implies that $(n,k) \in \{(-1,0),(0,-1),(0,1),(1,-2),(1,0),(2,-1)\}$, and in all cases we have either $n \in \{-1,0,1\}$ or $k = -1$, that is either $\frac{r}{s} \in \{-3,-1,\infty\}$ or $\frac{t}{u} = -1$. All are discarded by Lemma A.10.

Case $(\alpha,\beta,\gamma) = (\infty,0,-2)$**:** Then for 0 to be a T^H–slope, we need that $\frac{r}{s} = n$ and $\frac{t}{u} = -4-n+\frac{1}{m}$. Condition (5) becomes $n(1-4m-nm) = m \pm 1$ whose solutions are given Lemma A.16. First, we can exclude all solution with $\frac{r}{s} = n \in \{-3,-2,-1,0\}$ which are discarded by Lemma A.10. We also exclude $(n,k) = (-5,-1)$ and $(n,k) = (1,0)$ which give $\frac{t}{u} \in \{0,\infty\}$, also discarded by Lemma A.10. We are then left with $\big(N(-4,-\frac{1}{3}),\infty,0,-2\big)$ (already listed) and $\boxed{\big(N(-4,-\frac{1}{5}),\infty,0,-2\big)}$ which are indeed (S^H,T^H,Z) triples.

In the above analysis, we have proved that every $\big(N(\frac{r}{s},\frac{t}{u}),\alpha,\beta,\gamma\big)$ which is a (S^H,T^H,Z) triple with $\Delta(\beta,\gamma) = 2$ is equivalent to one of

(i) $B'_n := \big(N(-1+\frac{1}{n},-1-\frac{1}{n}),\infty,-3,-1\big)$ for $n \in \mathbb{Z} \setminus \{0,\pm 1\}$;
(ii) $C'_n := \big(N(-1+\frac{1}{n},-1-\frac{1}{n-2}),\infty,-3,-1\big)$ for $n \in \mathbb{Z} \setminus \{0,\pm 1,2,3\}$;

(iii) $\big(N(-4,-\frac{1}{3}),\infty,0,-2)\big)$;
(iv) $\big(N(-4,-\frac{1}{5}),\infty,0,-2)\big)$;
(v) $\big(N(-4,-\frac{5}{2}),-1,-2,0)\big)$;
(vi) $\big(N(-4,-\frac{9}{4}),-1,-2,0)\big)$;
(vii) $\big(N(-\frac{5}{2},-\frac{1}{2}),\infty,-1,-3)\big)$.

But now, using Identitiy (1.1) in [MP, Proposition 1.5], we obtain that cases (iii), (iv), (v) and (vi) are respectivelu equivalent to B'_{-2}, C'_{-2}, $\big(N(-\frac{3}{2},-\frac{8}{3}),-2,-1,-3)\big)$ and $\big(N(-\frac{3}{2},-\frac{14}{5}),-2,-1,-3)\big)$; and that, using Identitiy (1.3) in [MP, Proposition 1.5], that the latter two are equivalent to, respectively, B'_2 and C'_2. Finally, using Lemma A.8 and Identitiy (1.4) in [MP, Proposition 1.5], we obtain that case (vii) is equivalent to B'_2. Moreover, $B'_{-n} \cong B'_n$ and $C'_{-n} \cong C_{n+2}$ for every $n \geq 2$ because of Lemma A.8, and we already observed in Section 4.4 that the two families are distinct. This completes the proof. □

Appendix A. Facts used liberally throughout this article

The classification in this article comes from a careful consideration of the tables found in [MP] and [Rou2]. Often, cases considered in the enumeration are identified and/or discounted using technical results, most of which are found in [MP] and [Rou2]. To keep this article as self-contained as possible we list the technical lemmata that are used in this article.

A.1. **Identities between graph manifolds.** The following lemma consists of a list of identities between graph manifolds which are found in both [Rou2] and [MP]. Details can be found in [FM].

Lemma A.1. *The following identities on graph manifolds hold:*

$$(D,(1,b),(c,d))\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}(D,(e,f),(g,h)) = (D,(e,f),(g,h))\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}(D,(1,b),(c,d)) \tag{6}$$

$$= (S^2,(e,f),(g,h),(d+bc,-c))$$

$$(S^2,(a,b),(c,d),(0,1)) = L(a,b)\#L(c,d) \tag{7}$$

$$(S^2,(a,b),(c,d),(1,e)) = L(a(d+ce)+bc,\star) \tag{8}$$

The following obvious lemma is used throughout the article.

Lemma A.2.

- *If* $(D,(a,b),(c,d))\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}(D,(e,f),(g,h))$ *is a Seifert space, a lens space or* S^3*, then one of* $|a|$, $|c|$, $|e|$ *or* $|g|$ *is less than or equal to 1.*
- *If* $(S^2,(a,b),(c,d),(e,f))$ *is a lens space or* S^3*, then one of* $|a|$, $|c|$ *or* $|e|$ *is equal to 1.*

A.2. **Concerning surgery instruction on 5CL.**

Lemma A.3 ([Rou2, Lemma 2.2])**.** *The action of* $\mathrm{Aut}(M_5)$ *on surgery instructions on* 5CL *is generated by* (9)–(21)*. Moreover, for* $11 \le n \le 21$ *each* (n) *corresponds to the action of a distinct element of* $\mathrm{Aut}(M_5)/G$ *where* G *is the subgroup generated by the elements* (9)–(10) *corresponding to the generators of the link symmetry group of* 5CL.

$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) \longmapsto (\alpha_5,\alpha_1,\alpha_2,\alpha_3,\alpha_4) \tag{9}$$

$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) \longmapsto (\alpha_5,\alpha_4,\alpha_3,\alpha_2,\alpha_1) \tag{10}$$

$$(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}) \longmapsto (\tfrac{f}{e},\tfrac{j-i}{j},\tfrac{a}{a-b},\tfrac{d-c}{d},\tfrac{h}{g}) \tag{11}$$

$$(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}) \longmapsto (\tfrac{b}{b-a},\tfrac{i-j}{i},\tfrac{e-f}{e},\tfrac{d}{d-c},\tfrac{g}{h}) \tag{12}$$

$$(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}) \longmapsto (\tfrac{i}{i-j},\tfrac{b-a}{b},\tfrac{f}{e},\tfrac{d}{c},\tfrac{h-g}{h}) \tag{13}$$

$$(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}) \longmapsto (\tfrac{j}{j-i},\tfrac{e}{f},\tfrac{b}{b-a},\tfrac{c-d}{c},\tfrac{g-h}{g}) \tag{14}$$

$$(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}) \longmapsto (\tfrac{a}{a-b},\tfrac{e}{e-f},\tfrac{i}{i-j},\tfrac{c}{c-d},\tfrac{g}{g-h}) \tag{15}$$

$$(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}) \longmapsto (\tfrac{h}{g},\tfrac{j}{i},\tfrac{f-e}{f},\tfrac{c}{c-d},\tfrac{b-a}{b}) \tag{16}$$

$$(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}) \longmapsto (\tfrac{h}{h-g},\tfrac{a}{b},\tfrac{f}{f-e},\tfrac{c-d}{c},\tfrac{i-j}{i}) \tag{17}$$

$$(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}) \longmapsto (\tfrac{g}{g-h},\tfrac{f-e}{f},\tfrac{b}{a},\tfrac{d}{c},\tfrac{j-i}{j}) \tag{18}$$

$$(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}) \longmapsto (\tfrac{g-h}{g},\tfrac{f}{f-e},\tfrac{i}{j},\tfrac{d}{d-c},\tfrac{a-b}{a}) \tag{19}$$

$$(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}) \longmapsto (\tfrac{h-g}{h},\tfrac{b}{a},\tfrac{j}{i},\tfrac{d-c}{d},\tfrac{e}{e-f}) \tag{20}$$

$$(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}) \longmapsto (\tfrac{a-b}{a},\tfrac{e-f}{e},\tfrac{h}{h-g},\tfrac{c}{d},\tfrac{j}{j-i}). \tag{21}$$

Lemma A.4 ([Rou2, Theorem 4 and Eq. (70)]). *The following statements hold:*

- *If* $0,1,\infty \in \{\frac{a}{b},\frac{c}{d},\frac{e}{f},\frac{g}{h},\frac{i}{j}\}$ *then* $M_5(\frac{a}{b},\frac{c}{d},\frac{e}{f},\frac{g}{h},\frac{i}{j})$ *is non-hyperbolic.*
- *If* $-1,\frac{1}{2},2 \in \{\frac{a}{b},\frac{c}{d},\frac{e}{f},\frac{g}{h},\frac{i}{j}\}$ *then* $(\frac{a}{b},\frac{c}{d},\frac{e}{f},\frac{g}{h},\frac{i}{j})$ *factors through* M_4.

As highlighted in [MPR]:

Lemma A.5. *The following identity holds:*

$$M_5(\tfrac{a}{b},\tfrac{c}{d},-1,\tfrac{e}{f},\tfrac{g}{h}) = M_4(\tfrac{a}{b},\tfrac{c+d}{d},\tfrac{e+f}{f},\tfrac{g}{h}). \tag{22}$$

A.3. **Concerning surgery instructions on 4CL.** From [Rou2] we have the following identities:

$$M_4(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f})(\infty) = (S^2,(a,b),(d,-c),(e,f)), \tag{23}$$

$$M_4(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f})(0) = (D,(f,-e),(b,2b-a))\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}(D,(2,1),(c-2d,d)), \tag{24}$$

$$M_4(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f})(1) = (S^2,(a-2b,b),(c-d,c),(e-2f,f)), \tag{25}$$

$$M_4(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f})(2) = (D,(a-b,b),(e-f,f))\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}(D,(c,d),(2,-1)). \tag{26}$$

Lemma A.6 ([Rou2, Theorem 5 and Eq. (69)]). *The following statements hold:*

- *If* $(\frac{a}{b},\frac{c}{d},\frac{e}{f},\frac{g}{h})$ *is an instruction on* 4CL *and one of the slopes is in* $\{0,1,2,\infty\}$ *then* $M_4(\frac{a}{b},\frac{c}{d},\frac{e}{f},\frac{g}{h})$ *is non-hyperbolic.*
- *If* $(\frac{a}{b},\frac{c}{d},\frac{e}{f},\frac{g}{h})$ *is an instruction on* 4CL *and one of the slopes is in* $\{-1,\frac{1}{2},\frac{3}{2},3\}$ *then* $(\frac{a}{b},\frac{c}{d},\frac{e}{f},\frac{g}{h})$ *factors through* M_3. *In particular,* $M_4(\frac{a}{b},-1,\frac{c}{d},\frac{e}{f}) = M_3(\frac{a}{b}+1,\frac{c}{d}+1,\frac{e}{f})$

Lemma A.7. *For a filling instruction* $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ *on* M_4 *we have* $M_4(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = M_4(\alpha_{\sigma(1)},\alpha_{\sigma(2)},\alpha_{\sigma(3)},\alpha_{\sigma(4)})$ *for every* $\sigma \in D_4$.

A.4. **Concerning surgery instructions on 3CL.**

Lemma A.8. *If* $\sigma \in S_3$ *and* $(\alpha_1,\alpha_2,\alpha_3)$ *is a filling instruction on* N *then*

$$N(\alpha_1,\alpha_2,\alpha_3) = N(\alpha_{\sigma(1)},\alpha_{\sigma(2)},\alpha_{\sigma(3)}).$$

Lemma A.9. *For all filling instructions it holds* $M_3(\frac{a}{b},\frac{c}{d},\frac{e}{f}) = N(-\frac{a}{b},-\frac{c}{d},-\frac{e}{f})$.

Lemma A.10 ([MP, Theorem 1.2]). *If* $(\frac{a}{b},\frac{c}{d})$ *is an instruction on* N *and one of the slopes is* $\{0,-1,-2,-3,\infty\}$ *then* $N(\frac{a}{b},\frac{c}{d})$ *is non-hyperbolic.*

A.5. **Concerning surgery instructions on M4CL.** [Rou2, Proposition 2.1] gives us a complete enumeration of the Dehn fillings on F, the exterior of the minimally twisted 4 chain link. We have:

Lemma A.11. *For slopes* $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$, $\frac{g}{h}$ *on* M4CL *the following identity holds:*

$$F(\tfrac{a}{b}, \tfrac{e}{f}, \tfrac{c}{d}, \tfrac{g}{h}) = \big(D, (a, b), (c, d)\big) \bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} \big(D, (e, f), (g, h)\big) \tag{27}$$

Lemma A.12. *For a filling instruction* $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ *on* F *we have* $F(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = F(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)}, \alpha_{\sigma(4)})$ *for every* $\sigma \in D_4$.

In fact, “most" exceptional fillings of M_5 are obtained by filling F (c.f. [Rou2, Proposition 3.1]).

Lemma A.13. *The following identities hold:*

$$M_5(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g}{h})(\infty) = F(-\tfrac{a}{b}, \tfrac{f}{e}, \tfrac{d}{c}, -\tfrac{g}{h}) \tag{28}$$

$$M_5(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g}{h})(1) = F(\tfrac{a-b}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g-h}{h}) \tag{29}$$

$$M_5(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g}{h})(0) = F(\tfrac{b}{b-a}, \tfrac{c-d}{c}, -\tfrac{h}{g}, \tfrac{e-f}{f}) \tag{30}$$

Consequently,

Lemma A.14. *The following identities hold:*

$$F(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g}{h}) = M_5(-\tfrac{a}{b}, \tfrac{f}{e}, \tfrac{d}{c}, -\tfrac{g}{h})(\infty) \tag{31}$$

$$F(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g}{h}) = M_5(\tfrac{a+b}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g+h}{h})(1) \tag{32}$$

$$F(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g}{h}) = M_5(\tfrac{a-b}{a}, \tfrac{d}{d-c}, -\tfrac{h}{g}, \tfrac{f+e}{f})(0) \tag{33}$$

A.6. **Some elementary diophantine equations.**

Lemma A.15. *For* $(n, s) \in \mathbb{Z}^2$*, we have*

- $s - n = ns \implies (n, s) \in \{(0, 0), (2, -2)\}$ *;*
- $2s - n = ns \implies ((n, s) \in \{(0, 0), (1, 1), (3, -3), (4, -2)\}$ *;*
- $4s - n = ns \implies ((n, s) \in \{(0, 0), (3, 3), (5, -5), (8, -2), (6, -3), (2, 1)\}$ *;*
- $s - n = 3ns \implies (n, s) \in \{(0, 0)\}$ *;*
- $2s - n = 3ns \implies ((n, s) \in \{(0, 0), (1, -1)\}$ *;*
- $4s - n = 3ns \implies ((n, s) \in \{(0, 0), (1, 1), (2, -1)\}$ *;*
- $8s - n = 3ns \implies (n, s) \in \{(0, 0), (3, -3), (2, 1), (4, -1)\}$ *;*
- $5s - n = 3ns \implies ((n, s) \in \{(0, 0), (2, -2)\}$ *;*
- $s - n = -5ns \implies (n, s) \in \{(0, 0)\}$ *;*
- $2s - n = -5ns \implies (n, s) \in \{(0, 0)\}$ *;*
- $4s - n = -5ns \implies (n, s) \in \{(0, 0), (-1, 1)\}$ *;*
- $8s - n = -5ns \implies (n, s) \in \{(0, 0), (-2, 1)\}$ *:*
- $3s - n = -5ns \implies (n, s) \in \{(0, 0)\}$.

Proof. Here, we consider equations of the form $\alpha s - n = \beta ns$ for some $\alpha,\beta \in \mathbb{Z}$. They are solved by induction on the number of prime factor of α.

Indeed, we first note that $s \mid n$ and $n \mid \alpha s$.

- If actually $n \mid s$, then $s = \pm n$ and n satisfies either $(\alpha - 1)n = \beta n^2$ or $(\alpha + 1)n = \beta n^2$. It follows that $(n, s) = (0, 0)$, or $\left(\frac{\alpha-1}{\beta}, \frac{\alpha-1}{\beta}\right)$ if $\frac{\alpha-1}{\beta} \in \mathbb{Z}$, or $\left(\frac{\alpha+1}{\beta}, -\frac{\alpha+1}{\beta}\right)$ if $\frac{\alpha+1}{\beta} \in \mathbb{Z}$.
- If $n \nmid s$ then $n = kn'$ with some prime divisor of α, but then $\frac{\alpha}{k}n' - s = \beta n' s$ and by induction, we know all such (n'_0, s_0) and each of them leads to a solution (kn'_0, s_0).

□

Lemma A.16. *If m, n are integers such that $(1 - m(n+4))n = m \pm 1$ then*

$$(n, m) \in \{(-5,-1), (-4,-3), (-4,-5), (-3,1), (-3,2), (-2,1), (-1,0), (-1,1), (0,-1), (0,1), (1,0)\}.$$

Proof. Then $m(1 + n(n+4)) = n \pm 1$. So either $m = 0$ or $(1 + n(n+4)) \mid n \pm 1$.

Case $m = 0$**:** then $n = \pm 1$.

Case $m(1 + n(n+4)) \mid n+1$**:** then $1 + n(n+4) \leq |n+1|$.

If $n + 1 \geq 0$**:** then $n(n+3) \leq 0$ so $n \in \{-3,-2,-1,0\}$. Only -1 and 0 satisfy $n+1 \geq 0$, leading to solutions $(n,m) \in \{(-1,0),(0,1)\}$.

If $n + 1 \leq 0$**:** then $n^2 + 5n + 2 \leq 0$ so $n \in \left[\frac{-5-\sqrt{17}}{2}, \frac{-5+\sqrt{17}}{2}\right] \cap \mathbb{Z} = \{-4,-3,-2,-1\}$. If $n = -2$, then $m = \frac{1}{3} \notin \mathbb{Z}$. Other cases lead to solutions $(n,m) \in \{(-4,-3),(-3,1),(-1,0)\}$.

Case $m(1 + n(n+4)) \mid n-1$**:** $1 + n(n+4) \leq |n-1|$.

If $n - 1 \geq 0$**:** then $(n+1)(n+2) \leq 0$ so $n \in \{-2,-1\}$ and doesn't satisfy $n - 1 \geq 0$.

If $n - 1 \leq 0$**:** then $n(n+5) \leq 0$ so $n \in \{-5,-4,-3,-2,-1,0\}$ leading to solutions $(n,m) \in \{(-5,-1),(-4,-5),(-3,2),(-2,1),(-1,1),(0,-1)\}$.

□

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