View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by EKE Repository of Publications

Annales Mathematicae et Informaticae
45 (2015) pp. 61–68
http://ami.ektf.hu

On geometric Hermite arcs*

Imre Juhász

Department of Descriptive Geometry, University of Miskolc, Hungary
agtji@uni-miskolc.hu

Submitted June 3, 2015 — Accepted September 27, 2015

Abstract

A geometric Hermite arc is a cubic curve in the plane that is specified by its endpoints along with unit tangent vectors and signed curvatures at them. This problem has already been solved by means of numerical procedures. Based on projective geometric considerations, we deduce the problem to finding the base points of a pencil of conics, that reduces the original quartic problem to a cubic one that easier can exactly be solved. A simple solvability criterion is also provided.

Keywords: Hermite arc, geometric constraint, pencil of conincs

MSC: 65D17, 68U07

1. Introduction

In Computer Aided Geometric Design curves are often specified by means of some constraints that the required curve has to fulfill, instead of by those data that are necessary for a certain representation form (e.g. Bézier or B-spline). A classical example is the Hermite arc, where endpoints are given along with the tangent vector (i.e. the first derivative of the curve with respect to the parameter) at them from which data a cubic arc has to be determined. This task always has a unique solution.

A slight modification of the previous problem is the so called geometric Hermite arc, where endpoints of a cubic arc in plane with unit tangent vectors and signed

*This research was carried out in the framework of the Center of Excellence of Mechatronics and Logistics at the University of Miskolc.

curvatures at them are given. Unlike in the classical Hermite arc, a solution to this problem does not always exist.

This problem pops up in [1] in connection with equidistants of plane cubic spline curves. In [2] G^2 cubic plane interpolating splines are constructed with geometric Hermite arcs, moreover a thorough analysis and a criterion for solvability is provided. In [3] G^2 end conditions are discussed for C^2 planar cubic interpolating splines (Ferguson splines). [4] studies transition curves between circular and conic arcs meeting G^2 continuity requirements, while [5] provides G^2 transition curves between two circles. There are several generalizations of the problem. [6] generalizes the problem to rational cubics and [7] to interpolating spline surfaces, while [8] restricts the solution to Pythagorean-hodograph cubics.

No matter in what way we seek the solution to the original problem, it always ends in a system of quadratic equations in two variables. In all the cited publications this system is solved numerically. Equations of the system are quite special that gives the hope of a simple exact solution. In this contribution we provide such a solution based on projective geometric considerations.

2. Bézier points of the arc

Let us denote the endpoints by $\mathbf{p}_0$ and $\mathbf{p}_1$, the signed curvatures at them by κ_0 and κ_1, and the unit tangent vectors by $\mathbf{t}_0$ and $\mathbf{t}_1$.We want to produce the Bézier representation

$$\mathbf{b}(u) = \sum_{i=0}^{3} B_i^3(u)\,\mathbf{b}_i,\ \ u \in [0,1]$$

of the cubic Hermite arc, where $B_i^3(u)$ denotes the ith cubic Bernstein polynomial. Its control points $\{\mathbf{b}_i\}_{i=0}^{3}$ can be expressed in the form

$$\begin{aligned}
\mathbf{b}_0 &= \mathbf{p}_0, \\
\mathbf{b}_3 &= \mathbf{p}_1, \\
\mathbf{b}_1 &= \mathbf{p}_0 + l_0\mathbf{t}_0, && (2.1)\\
\mathbf{b}_2 &= \mathbf{p}_1 - l_1\mathbf{t}_1, && (2.2)
\end{aligned}$$

in which positive lengths l_0 and l_1 are unknown.

The signed curvature of a cubic planar Bézier curve is of the form

$$\kappa(u) = \frac{\dot{\mathbf{b}}(u) \wedge \ddot{\mathbf{b}}(u)}{\left|\dot{\mathbf{b}}(u)\right|^3},$$

where $\dot{\mathbf{b}}(u) \wedge \ddot{\mathbf{b}}(u)$ is the third component of the cross product of the vectors $\dot{\mathbf{b}}(u)$ and $\ddot{\mathbf{b}}(u)$. Its value at the first and last points are

$$\kappa_0 := \kappa(0) = \frac{2(\mathbf{b}_1 - \mathbf{b}_0) \wedge (\mathbf{b}_2 - \mathbf{b}_1)}{3l_0^3}$$

$$= \frac{2A(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2)}{3l_0^3}$$

and

$$\kappa_1 := \kappa(1) = \frac{2(\mathbf{b}_2 - \mathbf{b}_1) \wedge (\mathbf{b}_3 - \mathbf{b}_2)}{3l_1^3}$$
$$= \frac{2A(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)}{3l_1^3}$$

respectively, where $A(\mathbf{a}, \mathbf{b}, \mathbf{c})$ stands for the signed area of the triangle determined by the sequence of vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Denoting the signed distance of the control point $\mathbf{b}_2$ from the directed straight line $\mathbf{b}_0, \mathbf{b}_1$ by d_0 (cf. Fig. 1) we obtain for the signed area

$$A(\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2) = \frac{l_0 d_0}{2}$$

and for the signed curvature

$$\kappa_0 = \frac{2}{3} \frac{d_0}{l_0^2}. \tag{2.3}$$

Analogously, the signed curvature κ_1 is of the form

$$\kappa_1 = \frac{2}{3} \frac{d_1}{l_1^2}, \tag{2.4}$$

where d_1 is the signed distance of the control point $\mathbf{b}_1$ from the directed line $\mathbf{b}_2, \mathbf{b}_3$.

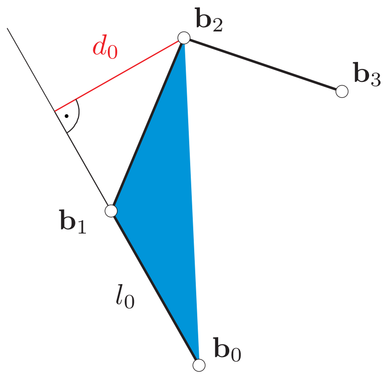

Figure 1: The signed curvature at the first point of a Bézier curve is proportional to the signed area of the triangle with vertices $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2$

2.1. Parallel tangent vectors

If $\mathbf{t}_0 \| \mathbf{t}_1$ then Eqs. (2.1) and (2.2) imply equality $d = |d_0| = |d_1|$, where d is the distance between the parallel lines determined by point and direction vector pairs $\mathbf{p}_0, \mathbf{t}_0$ and $\mathbf{p}_1, \mathbf{t}_1$.

If $d = 0$ then on the basis of Eqs. (2.3) and (2.4) it is obvious that a solution to the problem exists if and only if $\kappa_0 = \kappa_1 = 0$, in which case the number of solutions is infinite since both l_0 and l_1 can be considered as free parameters. The resulted curve is always a straight line segment, i.e. the cubic curve degenerates.

Assumption $d \neq 0$ implies $\kappa_0 \neq 0$ and $\kappa_1 \neq 0$, and for any such a pair of signed curvatures there is the unique solution

$$l_0 = \sqrt{\frac{2}{3}\frac{d}{\kappa_0}} \text{ and } l_1 = \sqrt{\frac{2}{3}\frac{d}{\kappa_1}}$$

that can be obtained by Eqs. (2.3) and (2.4).

2.2. Generic case

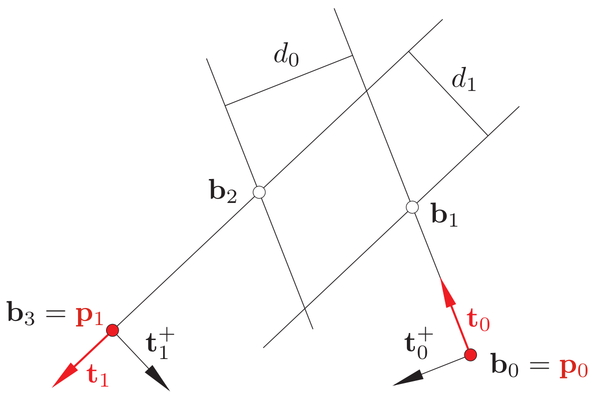

Figure 2: Determination of Bézier points

Hereafter we assume that $\mathbf{t}_0 \nparallel \mathbf{t}_1$. Using the notations of Fig. 2, distance d_1 is of the form

$$\begin{aligned} d_1 &= (\mathbf{b}_1 - \mathbf{b}_3) \cdot \mathbf{t}_1^+ = (\mathbf{p}_0 + l_0\mathbf{t}_0 - \mathbf{p}_1) \cdot \mathbf{t}_1^+ \\ &= l_0\mathbf{t}_0 \cdot \mathbf{t}_1^+ - (\mathbf{p}_1 - \mathbf{p}_0) \cdot \mathbf{t}_1^+ \end{aligned}$$

and the distance d_0 is

$$\begin{aligned} d_0 &= (\mathbf{b}_2 - \mathbf{b}_0) \cdot \mathbf{t}_0^+ = (\mathbf{p}_1 - l_1\mathbf{t}_1 - \mathbf{p}_0) \cdot \mathbf{t}_0^+ \\ &= (\mathbf{p}_1 - \mathbf{p}_0) \cdot \mathbf{t}_0^+ - l_1\mathbf{t}_1 \cdot \mathbf{t}_0^+. \end{aligned}$$

Introducing notations $a = \mathbf{t}_0 \cdot \mathbf{t}_1^+ = -\mathbf{t}_1 \cdot \mathbf{t}_0^+$, $b = (\mathbf{p}_1 - \mathbf{p}_0) \cdot \mathbf{t}_1^+$ and $c = (\mathbf{p}_1 - \mathbf{p}_0) \cdot \mathbf{t}_0^+$, where $\mathbf{t}_i^+$ denotes the positive normal vector of $\mathbf{t}_i$, i.e. $\mathbf{t}_i$ is rotated through $\pi/2$ in counterclockwise direction, we obtain equalities

$$d_0 = al_1 + c, \tag{2.5}$$

$$d_1 = al_0 - b. \tag{2.6}$$

Eqs. (2.3) and (2.5) yield equation

$$3\kappa_0 l_0^2 - 2al_1 - 2c = 0 \tag{2.7}$$

and Eqs. (2.4) and (2.6)

$$3\kappa_1 l_1^2 - 2al_0 + 2b = 0 \tag{2.8}$$

for the unknown distances l_0 and l_1. Therefore, the solution of the geometric Hermite interpolation is reduced to the solution of the system of quadratic equations (2.7), (2.8) of unknowns l_0 and l_1. This system does not always has a solution, or if it has, the the obtained values of l_0 and l_1 are not necessarily positive.

Before we solve the generic case, we have a look at those special cases when one or both of the curvatures vanish. If, e.g. $\kappa_0 = 0$ and $\kappa_1 \neq 0$ then the problem is reduced to a linear one, the solution is

$$\begin{aligned} l_0 &= \frac{3}{2}\kappa_1 \frac{c^2}{a^3} + \frac{b}{a}, \\ l_1 &= -\frac{c}{a} \end{aligned}$$

and control points $\mathbf{b}_0, \mathbf{b}_1$ and $\mathbf{b}_2$ become collinear, since $\kappa_0 = 0$ implies $d_0 = 0$. This case is illustrated in Fig. 3.

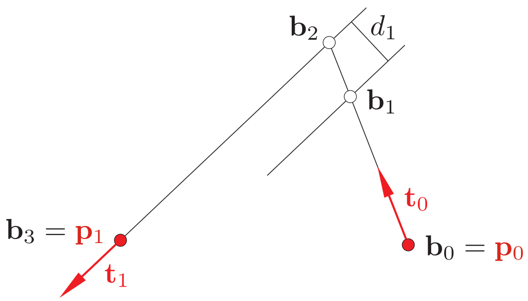

Figure 3: The case $\kappa_0 = 0$ and $\kappa_1 \neq 0$

If both curvatures vanish, then $\kappa_0 = \kappa_1 = 0$ involves $d_0 = d_1 = 0$ thus control points $\mathbf{b}_1$ and $\mathbf{b}_2$ coincide and are the point of intersection of the two tangent lines that must exist due to the assumption $\mathbf{t}_0 \nparallel \mathbf{t}_1$ (cf. Fig. 4).

3. An exact solution of the quadratic system

Hereafter we assume that $\mathbf{t}_0 \nparallel \mathbf{t}_1$ and $\kappa_0 \neq 0$, $\kappa_1 \neq 0$. In this case equalities (2.7) and (2.8) describe parabolas in the coordinate system (l_0, l_1) the axis of which are

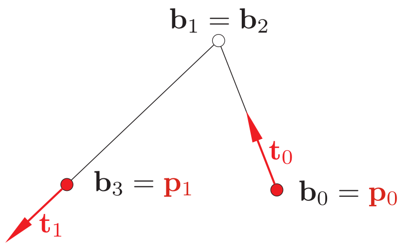

Figure 4: The case $\kappa_0 = \kappa_1 = 0$

perpendicular. Using homogeneous coordinates, the matrices of these parabolas are

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & -a \\ 0 & 3\kappa_1 & 0 \\ -a & 0 & 2b \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 3\kappa_0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & -a & -2c \end{bmatrix}.$$

To solve the system means to find points of intersection of these parabolas. These two parabolas establish a pencil of conics, the elements of which are of the form

$$\mathbf{C} = \alpha\mathbf{A} + \beta\mathbf{B},\ \alpha, \beta \in \mathbb{R},\ |\alpha| + |\beta| \neq 0. \tag{3.1}$$

To find points of intersection of the parabolas is equivalent to determine the base points of the pencil. One of the parameters can be eliminated from matrix (3.1), since matrices $\mathbf{C}$ and $\gamma\mathbf{C}$, $(0 \neq \gamma \in \mathbb{R})$ determine the same conic, therefore we will study the matrix

$$\begin{aligned} \mathbf{C}(\lambda) &= \lambda\mathbf{A} + \mathbf{B} \\ &= \begin{bmatrix} 3\kappa_0 & 0 & -\lambda a \\ 0 & 3\lambda\kappa_1 & -a \\ -\lambda a & -a & 2(\lambda b - c) \end{bmatrix},\ \lambda \in \mathbb{R}. \end{aligned} \tag{3.2}$$

If we can find a degenerate element of this pencil, i.e. an element that is composed of a pair of straight lines, then we can reduce the original quartic problem to a quadratic one. The degenerate element of the pencil is provided by such a λ for which $\det(\mathbf{C}(\lambda)) = 0$, that yields the cubic polynomial

$$\kappa_1 a^2 \lambda^3 - 6b\kappa_0\kappa_1\lambda^2 + 6c\kappa_0\kappa_1\lambda + \kappa_0 a^2 = 0.$$

One real root of this cubic polynomial has to be determined, that always exists. After all, the quartic problem can only be reduced to a cubic one using this method, however this can exactly be solved in a simple way without a numerical procedure.

Parabolas (2.7) and (2.8) are of special position, their axes are perpendicular, and the pencil determined by them always contains an element which is a circle. Indeed, it is easy to check that absolute (or imaginary) circle points (points at which the line at infinity intersects any circle) $\begin{bmatrix} 1 & i & 0 \end{bmatrix}^T$ and $\begin{bmatrix} -1 & i & 0 \end{bmatrix}^T$

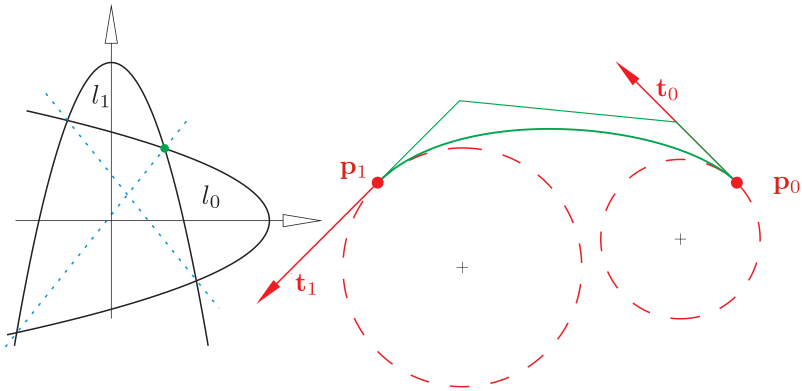

Figure 5: One positive solution (right) along with the corresponding pair of parabolas and a degenerated element (blue dotted) of the pencil determined by them (left). Settings are $\mathbf{p}_0 = [2\ 0]^T$, $\mathbf{p}_1 = [-1\ 0]^T$, $\mathbf{t}_0 = [-1\ 1]^T$, $\mathbf{t}_1 = [-1\ -1]^T$, $\kappa_0 = 1.5$, $\kappa_1 = 1$ (For interpretation of the references to color in this figure legend, the reader is referred to the pdf version of this article.)

are on the element of the pencil (3.2) that corresponds to

$$\lambda = \frac{\kappa_0}{\kappa_1},$$

provided $\kappa_1 \neq 0$, that we have already excluded. The matrix of this circle is

$$\begin{bmatrix} 3\kappa_0 & 0 & -\frac{\kappa_0}{\kappa_1}a \\ 0 & 3\kappa_0 & -a \\ -\frac{\kappa_0}{\kappa_1}a & -a & 2\left(\frac{\kappa_0}{\kappa_1}b - c\right) \end{bmatrix},$$

therefore the square of its radius is

$$r^2 = \frac{2}{3\kappa_0}\left(c - \frac{\kappa_0}{\kappa_1}b\right). \tag{3.3}$$

If (3.3) is positive (note that the circle can be imaginary as well) then parabolas (2.7) and (2.8) have real points on common, i.e. the system of quadratic equations has a solution. Note, that this is a criterion just for the solvability of the system and not for the positive solution.

In practice, positive solutions (both l_0 and l_1 are positive) are used in general, since only in this case will the direction of tangent vectors be kept. In the coordinate system (l_0, l_1) the axis of parabolas (2.7) and (2.8) coincide with the axes l_0 and l_1 of the coordinate system, respectively, thus the number of positive solutions can be $0, 1, 2$ or 3.

As it is specified in [2], in the generic case there can be a positive solution if products

$$\mathbf{t}_0 \wedge \mathbf{t}_1,\ (\mathbf{p}_1 - \mathbf{p}_0) \wedge \mathbf{t}_1,\ \mathbf{t}_0 \wedge (\mathbf{p}_1 - \mathbf{p}_0)$$

have the same sign that coincides with the sing of κ_0 and κ_1. In this case the existence of a positive solution can be guaranteed by adjusting the magnitude of curvatures, i.e. curvatures can be used as shape parameters. Fig. 5 illustrates a case of one positive solution.

References

[1] Klass, R. An offset spline approximation for plane cubic splines, *Computer–Aided Design*, Vol. 15 (1983), No. 5, 297–299.

[2] de Boor, C., Hölling, K., Sabin, M. High accuracy geometric Hermite interpolation, *Computer Aided Geometric Design*, Vol. 4 (1987), No. 4, 269–278.

[3] Ginnis, A.I., Kaklis, P.D. Planar C^2 cubic spline interpolation under geometric boundary conditions, *Computer Aided Geometric Design*, Vol. 19 (2002), No. 5, 345–363.

[4] Meek, D.S., Walton D.J. Planar G^2 Hermite interpolation with some fair, C-shaped curves, *Journal of Computational and Applied Mathematics*, Vol. 139 (2002), No. 1, 141–161.

[5] Habib, Z., Sakai, M. G^2 cubic transition between two circles with shape control, *Journal of Computational and Applied Mathematics*, Vol. 223 (2009), No. 1, 133–144.

[6] Sakai, M. Osculatory interpolation, *Computer Aided Geometric Design*, Vol. 18 (2001), No. 8, 739–750.

[7] Valasek, G., Vida, J. C^2 Geometric Spline Surfaces, *Proceedings of the Seventh Hungarian Conference on Computer Graphics and Geometry*, (2014), 7–12.

[8] Jaklič, G., Kozak, J., Krajnc, M., Vitrih, V., Žagar, E. On interpolation by planar cubic G^2 Pythagorean-hodograph spline curves, *Mathematics of Computation*, Vol. 79 (2010), No. 269, 305–326.