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Generalized r-Whitney numbers of the first kind

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Abstract

In this paper, we consider a (p, q)-generalization of the r-Whitney number sequence of the first kind that reduces to it when $p = q = 1$. We obtain generalizations of some earlier results for the r-Whitney sequence, including recurrence and generating function formulas. We develop a combinatorial interpretation for our generalized numbers in terms of a pair of statistics on the set of r-permutations in which the elements within cycles of a permutation are assigned colors according to certain rules. This allows one to provide combinatorial proofs of various identities, including orthogonality relations. Finally, we consider the (p, q)-Whitney matrix of the first kind and find various factorizations for it.

Keywords: q-generalization, r-Whitney numbers, Stirling numbers, Whitney matrix

MSC: 05A19, 05A15, 60C05, 15A23

1. Introduction

We will make use of the following notation. If m and n are positive integers, then let $[m, n] = \{m, m + 1, \dots, n\}$ if $m \le n$, with $[m, n] = \varnothing$ if $m > n$. We will denote the special case $[1, n]$ by $[n]$. Given a positive integer k and an indeterminate q, let

$[k]_q = 1 + q + \cdots + q^{k-1}$, with $[0]_q = 0$. Throughout, empty sums will assume the value 0, and empty products the value 1.

Suppose $r \geq 0$ and $m \geq 1$ are given integers. Let $w(n, k) = w(n, k; r, m)$ denote the r-Whitney numbers of the first kind (see, e.g., [7, 15]), which are defined as connection constants in the polynomial identities

$$m^n x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} w(n,k)(mx+r)^k, \quad n \geq 0.$$

See also [1, 18] for further properties of the $w(n, k)$. Equivalently, the $w(n, k)$ array is determined by the recurrence

$$w(n,k) = w(n-1,k-1) - (r + m(n-1))w(n-1,k), \quad n,k \geq 1,$$

with initial values $w(n, 0) = (-1)^n \prod_{i=0}^{n-1}(r + mi)$ and $w(0, k) = \delta_{k,0}$ for $n, k \geq 0$. Note that $w(n, k; 0, 1) = s(n, k)$ and $w(n, k; 1, 1) = s(n + 1, k + 1)$, where $s(n, k)$ is the Stirling number of the first kind. In [10], a combinatorial interpretation for $w(n, k)$ when $r = 0$ is given in terms of the coefficients of the characteristic polynomial for the rank function on the Dowling lattice of rank n over a finite group of order m. An interpretation for $w(n, k)$ was given in [7] for arbitrary r.

Here, we consider a polynomial generalization of the r-Whitney numbers of the first kind, which we will denote by $w_{p,q}(n, k) = w_{p,q}(n, k; r, m)$. It is defined by the recurrence

$$w_{p,q}(n,k) = w_{p,q}(n-1,k-1) - ([r]_p + m[n-1]_q)w_{p,q}(n-1,k), \quad n,k \geq 1, \tag{1.1}$$

with initial values $w_{p,q}(n, 0) = (-1)^n \prod_{i=0}^{n-1}([r]_p + m[i]_q)$ and $w_{p,q}(0, k) = \delta_{k,0}$ for $n, k \geq 0$. Note that $w_{1,1}(n, k) = w(n, k)$ for all n and k. The numbers $w_{p,q}(n, k)$ when $p = 1$ differ slightly from the (q, r)-Whitney numbers $w_{m,r,q}(n, k)$ studied in [12] due to the absence of extra factors of q in the defining recurrence. In contrast to [12], where identities for $w_{m,r,q}(n, k)$ were shown by algebraic methods using q-boson operators, we provide a combinatorial interpretation for our $w_{p,q}(n, k)$ which allows one to explain identities bijectively. Moreover, on account of its simpler recurrence, it seems that the numbers $w_{p,q}(n, k)$ provide a more natural combinatorial generalization of the r-Whitney numbers than those studied in [12] which arose in a physical setting. Furthermore, the $w(n, k)$ form an orthogonal pair with the generalized version of the r-Whitney numbers of the second kind considered in [13]. This orthogonality relation generalizes one between the r-Whitney numbers of the first and second kind (see [7]).

In the next section, we give several algebraic properties satisfied by $w_{p,q}(n, k)$, including a connection between it and the elementary symmetric functions. We then develop in the third section a combinatorial interpretation for $w_{p,q}(n, k)$ in terms of statistics on a structure $\mathcal{A}$ enumerated by $|w(n, k)|$ and use this to provide bijective proofs of identities satisfied by $w_{p,q}(n, k)$, including orthogonality relations. The p- and q- variables will be seen here to play different combinatorial roles, with the former marking a statistic on $\mathcal{A}$ related only to those cycles containing the elements of $[r]$ no two of which are to belong to the same cycle, while the latter marks a statistic on $\mathcal{A}$ related to the positions of the elements of $[r+1, r+n]$ within all cycles. In the case $r = 0$ and $m = 1$, these statistics appear to be new on the set of permutations. In the final section, we consider the (p, q)-Whitney matrix of the first kind and find some factorizations of this matrix in analogy with the results of [16].

Let $W(n, k) = W(n, k; r, m)$ denote the r-Whitney number of the second kind (see [7]). We now recall a (p, q)-generalization of $W(n, k)$ from [13] defined by the recurrence

$$W_{p,q}(n, k) = W_{p,q}(n-1, k-1) + ([r]_p + m[k]_q)W_{p,q}(n-1, k), \quad n, k \geq 1,$$

with initial values $W_{p,q}(n, 0) = [r]_p^n$ and $W_{p,q}(0, k) = \delta_{k,0}$. Among the results, it was shown that the $W_{p,q}(n, k)$ are determined by the identities

$$(mx + [r]_p)^n = \sum_{k=0}^{n} W_{p,q}(n, k) m^k [x]_{\underline{q}}^{\underline{k}}, \quad n \geq 0, \tag{1.2}$$

where

$$[x]_{\underline{q}}^{\underline{n}} = \begin{cases} x(x - [1]_q) \cdots (x - [n-1]_q), & \text{if } n \geq 1; \\ 1, & \text{if } n = 0, \end{cases}$$

or equivalently by the generating function

$$\sum_{n \geq k} W_{p,q}(n, k) x^n = \frac{x^k}{(1 - ([r]_p + m[0]_q)x) \cdots (1 - ([r]_p + m[k]_q)x)}, \quad k \geq 0. \tag{1.3}$$

Various connections will be made between $w_{p,q}(n, k)$ and $W_{p,q}(n, k)$ in the next two sections. Note that $W_{p,q}(n, k)$ reduces to $W(n, k)$ when $p = q = 1$. Let us denote by $s_q(n, k)$ and $S_q(n, k)$ the $r = 0, m = p = 1$ case of $w_{p,q}(n, k)$ and $W_{p,q}(n, k)$, respectively. The $s_q(n, k)$ and $S_q(n, k)$ are q-Stirling polynomials of the first and second kind which were originally considered by Carlitz [5, 6] and have since been studied [20, 21] (see also [8, 9] for a related generalization).

2. Identities of the generalized r-Whitney numbers

In this section, we prove various algebraic properties of the array $w_{p,q}(n, k)$. We first show that the $w_{p,q}(n, k)$ serve as connection constants as follows.

Theorem 2.1. *If $n \geq 0$, then*

$$m^n [x]_{\underline{q}}^{\underline{n}} = \sum_{k=0}^{n} w_{p,q}(n, k)(mx + [r]_p)^k. \tag{2.1}$$

Proof. We proceed by induction on n. The equality clearly holds for $n = 0$. Now assume that the claim holds for n, and let us prove it for $n + 1$. From recurrence (1.1), we have

$$\sum_{k=0}^{n+1} w_{p,q}(n+1,k)(mx+[r]_p)^k = \sum_{k=0}^{n} w_{p,q}(n+1,k)(mx+[r]_p)^k + (mx+[r]_p)^{n+1}$$
$$= \sum_{k=0}^{n} w_{p,q}(n,k-1)(mx+[r]_p)^k - ([r]_p + m[n]_q)\sum_{k=0}^{n} w_{p,q}(n,k)(mx+[r]_p)^k$$
$$+ (mx+[r]_p)^{n+1}$$
$$= \sum_{k=0}^{n-1} w_{p,q}(n,k)(mx+[r]_p)^{k+1} - ([r]_p + m[n]_q)m^n[x]_q^{\underline{n}} + (mx+[r]_p)^{n+1}$$
$$= \sum_{k=0}^{n} w_{p,q}(n,k)(mx+[r]_p)^{k+1} - ([r]_p + m[n]_q)m^n[x]_q^{\underline{n}}$$
$$= (mx+[r]_p)m^n[x]_q^{\underline{n}} - ([r]_p + m[n]_q)m^n[x]_q^{\underline{n}}$$
$$= m^{n+1}[x]_q^{\underline{n+1}},$$

which completes the induction. □

We next give the generating function of the array $w_{p,q}(n,k)$ for fixed n.

Theorem 2.2. *If $n \geq 0$, then*

$$\sum_{k=0}^{n} w_{p,q}(n,n-k)x^k = \prod_{k=0}^{n-1}(1 - ([r]_p + m[k]_q)x). \tag{2.2}$$

Proof. We proceed by induction on n. The equality clearly holds for $n = 0$. Now assume that the claim holds for n, and let us prove it for $n + 1$. From recurrence (1.1), we have

$$\sum_{k=0}^{n+1} w_{p,q}(n+1,n+1-k)x^k$$
$$= \sum_{k=0}^{n} w_{p,q}(n,n-k)x^k - ([r]_p + m[n]_q)\sum_{k=0}^{n+1} w_{p,q}(n,n+1-k)x^k$$
$$= \prod_{k=0}^{n-1}(1 - ([r]_p + m[k]_q)x) - ([r]_p + m[n]_q)\sum_{k=0}^{n} w_{p,q}(n,n-k)x^{k+1}$$
$$= \prod_{k=0}^{n-1}(1 - ([r]_p + m[k]_q)x) - ([r]_p + m[n]_q)x\prod_{k=0}^{n-1}(1 - ([r]_p + m[k]_q)x)$$
$$= \prod_{k=0}^{n}(1 - ([r]_p + m[k]_q)x),$$

which completes the induction. □

Given a set of variables $x_1, x_2, \ldots, x_n$, the k-th elementary and complete symmetric functions are defined, respectively, by

$$e_k(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}, \quad 1 \le k \le n,$$
$$h_k(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}, \quad k \ge 1,$$

with initial conditions $e_0(x_1, x_2, \ldots, x_n) = h_0(x_1, x_2, \ldots, x_n) = 1$. Note that $e_k(x_1, x_2, \ldots, x_n) = 0$ if $k > n$. The generating functions for the e_k and h_k are given by

$$\sum_{k=0}^{n} e_k(x_1, x_2, \ldots, x_n) z^k = \prod_{i=1}^{n} (1 + x_i z),$$
$$\sum_{k \ge 0} h_k(x_1, x_2, \ldots, x_n) z^k = \prod_{i=1}^{n} \frac{1}{1 - x_i z}.$$

Using (2.2) and (1.3), it is not difficult to show that the (p, q)-Whitney numbers are the specializations of the elementary and complete symmetric functions given by

$$w_{p,q}(n+1, n+1-k)$$
$$= (-1)^k e_k([r]_p, m[1]_q + [r]_p, m[2]_q + [r]_p, \ldots, m[n]_q + [r]_p), \tag{2.3}$$
$$W_{p,q}(n+k, n)$$
$$= h_k([r]_p, m[1]_q + [r]_p, m[2]_q + [r]_p, \ldots, m[n]_q + [r]_p). \tag{2.4}$$

In particular, the q-Stirling numbers of the first and second kind satisfy

$$s_q(n+1, n+1-k) = (-1)^k e_k([1]_q, [2]_q, \ldots, [n]_q),$$
$$S_q(n+k, n) = h_k([1]_q, [2]_q, \ldots, [n]_q).$$

Lemma 2.3 (Merca [14]). *Let k and n be positive integers. Then*

$$f_k(t + x_1, t + x_2, \ldots, t + x_n) = \sum_{i=0}^{k} \binom{n - c_i}{k - i} f_i(x_1, x_2, \ldots, x_n) t^{k-i}$$

and

$$f_k(x_1, x_2, \ldots, x_n) = \sum_{i=0}^{k} \binom{n - c_i}{k - i} f_i(t + x_1, t + x_2, \ldots, t + x_n)(-t)^{k-i},$$

where $t, x_1, x_2, \ldots, x_n$ are variables, f_i is either the i-th elementary or complete symmetric function for all i, and

$$c_i = \begin{cases} i, & \text{if } f_i = e_i; \\ 1-k, & \text{if } f_i = h_i. \end{cases}$$

Using the prior lemma, one can obtain the following formulas for the (p,q)-Whitney numbers.

Proposition 2.4. *If $r \geq s \geq 0$, then*

$$w_{p,q}(n,k;r,m) = \sum_{i=k}^{n} \binom{i}{k} w_{p,q}(n,i;s,m)([s]_p - [r]_p)^{i-k} \tag{2.5}$$
$$= \sum_{i=k}^{n} (-1)^{i-k} p^{s(i-k)} \binom{i}{k} w_{p,q}(n,i;s,m)[r-s]_p^{i-k}$$

and

$$W_{p,q}(n,k;r,m) = \sum_{i=k}^{n} \binom{n}{i} W_{p,q}(i,k;s,m)([r]_p - [s]_p)^{n-i} \tag{2.6}$$
$$= \sum_{i=k}^{n} p^{s(n-i)} \binom{n}{i} W_{p,q}(i,k;s,m)[r-s]_p^{n-i}.$$

Proof. By (2.3), (2.4), and Lemma 2.3, we have

$$w_{p,q}(n,n-k;r,m) = (-1)^k e_k([r]_p, m[1]_q + [r]_p, \ldots, m[n-1]_q + [r]_p)$$
$$= (-1)^k \sum_{i=0}^{k} \binom{n-i}{k-i} e_i([s]_p, m[1]_q + [s]_p, \ldots, m[n-1]_q + [s]_p)([r]_p - [s]_p)^{k-i}$$
$$= \sum_{i=0}^{k} \binom{n-i}{k-i} (-1)^i e_i([s]_p, m[1]_q + [s]_p, \ldots, m[n-1]_q + [s]_p)([s]_p - [r]_p)^{k-i}$$
$$= \sum_{i=0}^{k} \binom{n-i}{k-i} w_{p,q}(n,n-i;s,m)([s]_p - [r]_p)^{k-i}$$

and

$$W_{p,q}(n+k,n;r,m) = h_k([r]_p, m[1]_q + [r]_p, \ldots, m[n]_q + [r]_p)$$
$$= \sum_{i=0}^{k} \binom{n+k}{k-i} h_i([s]_p, m[1]_q + [s]_p, \ldots, m[n]_q + [s]_p)([r]_p - [s]_p)^{k-i}$$
$$= \sum_{i=0}^{k} \binom{n+k}{k-i} W_{p,q}(n+i,n;s,m)([r]_p - [s]_p)^{k-i},$$

from which the identities may be obtained. $\square$

Taking $p = q = 1$ in (2.6) gives the following identity which was shown previously by a different method using Riordan matrix groups.

Corollary 2.5 (Cheon and Jung [7]). *If $r \geq s \geq 0$, then*

$$W(n,k;r,m) = \sum_{i=k}^{n} \binom{n}{i} W(i,k;s,m)(r-s)^{n-i}.$$

The following proposition shows how to express the (p,q)-Whitney numbers of both kinds in terms of the q-Stirling numbers and vice-versa.

Proposition 2.6. *If $n, k \geq 0$, then*

$$w_{p,q}(n,k) = \sum_{i=k}^{n} m^{n-i} \binom{i}{k} (-[r]_p)^{i-k} s_q(n,i), \tag{2.7}$$

$$s_q(n,k) = \frac{1}{m^{n-k}} \sum_{i=k}^{n} \binom{i}{k} [r]_p^{i-k} w_{p,q}(n,i), \tag{2.8}$$

$$W_{p,q}(n,k) = \sum_{i=k}^{n} m^{i-k} \binom{n}{i} [r]_p^{n-i} S_q(i,k), \tag{2.9}$$

$$S_q(n,k) = \frac{1}{m^{n-k}} \sum_{i=k}^{n} \binom{n}{i} (-[r]_p)^{n-i} W_{p,q}(i,k). \tag{2.10}$$

Proof. To show (2.9), note that

$$\begin{aligned}
W_{p,q}(n+k,n) &= h_k([r]_p, m[1]_q + [r]_p, \ldots, m[n]_q + [r]_p) \\
&= \sum_{i=0}^{k} \binom{n+k}{k-i} h_i(m[1]_q, m[2]_q, \ldots, m[n]_q)[r]_p^{k-i} \\
&= \sum_{i=0}^{k} \binom{n+k}{k-i} m^i h_i([1]_q, [2]_q, \ldots, [n]_q)[r]_p^{k-i} \\
&= \sum_{i=0}^{k} \binom{n+k}{k-i} m^i S_q(n+i,n)[r]_p^{k-i},
\end{aligned}$$

and then replace n by $n-k$, k by $n-k$, and i by $i-k$ in that order.

For (2.10), observe that

$$\begin{aligned}
S_q(n+k,n) &= h_k([1]_q, [2]_q, \ldots, [n]_q) \\
&= \frac{1}{m^k} h_k(m[1]_q, m[2]_q, \ldots, m[n]_q) \\
&= \sum_{i=0}^{k} \frac{1}{m^k} \binom{n+k}{k-i} h_i([r]_p, m[1]_q + [r]_p, \ldots, m[n]_q + [r]_p)(-[r]_p)^{k-i}
\end{aligned}$$

$$= \sum_{i=0}^{k} \frac{1}{m^k} \binom{n+k}{k-i} W_{p,q}(n+i,n)(-[r]_p)^{k-i}.$$

The proofs of (2.7) and (2.8) are similar. □

Let $s(n,k)$ and $S(n,k)$ denote the Stirling numbers of the first and second kind, respectively. Taking $p=q=1$ in (2.7) and (2.9) gives the following formulas.

Corollary 2.7 (Cheon and Jung [7])**.** *If $n,k \geq 0$, then*

$$w(n,k) = \sum_{i=k}^{n} m^{n-i} \binom{i}{k} (-r)^{i-k} s(n,i)$$

and

$$W(n,k) = \sum_{i=k}^{n} m^{i-k} \binom{n}{i} r^{n-i} S(i,k).$$

The $p=q=r=1$ case of (2.8) is also previously known.

Corollary 2.8 (Benoumhani [2])**.** *If $n,k \geq 0$, then*

$$s(n,k) = \frac{1}{m^{n-k}} \sum_{i=k}^{n} \binom{i}{k} w(n,i;1,m).$$

3. Combinatorial interpretation and properties

In this section, we develop a combinatorial interpretation for the array $w_{p,q}(n,k)$ and use it to explain various relations that it satisfies, including its orthogonality with $W_{p,q}(n,k)$. We first recall the concept of an r-permutation, see [3].

Definition 3.1. Given $0 \leq r \leq m$, by an r-permutation of $[m]$, it is meant a member of $\mathcal{S}_m$ in which the elements of $[r]$ belong to distinct cycles. If $n,k,r \geq 0$, then let $\Omega_r(n,k)$ denote the set of all r-permutations of $[n+r]$ having exactly $k+r$ cycles and let $\Omega_r(n) = \cup_{k=0}^{n} \Omega_r(n,k)$.

When $r=0$, a member of $\Omega_r(n)$ is the same as an ordinary permutation of $[n]$. Note that the cardinality of $\Omega_r(n,k)$ is given by the (signless) r-Stirling number of the first kind (see, e.g., [3]), while the cardinality of $\Omega_r(n)$ is seen to be $(r+1)^{\overline{n}}$.

Within a member of $\Omega_r(n,k)$, we will refer to the cycles containing an element of $[r]$ as *special* and to the remaining cycles comprised exclusively of elements of $I = [r+1, r+n]$ as *non-special*. (The members of $[r]$ themselves will also at times be described as *special*.) In addition, we will refer to an element within a member of $\Omega_r(n,k)$ that is the smallest within its cycle as *minimal*, and to all other elements as *non-minimal*. Throughout, we will assume that members of $\Omega_r(n,k)$ are expressed in *standard cycle form*, i.e., minimal elements first within each cycle, with cycles arranged left-to-right in ascending order of minimal elements.

We now consider a certain subset of the elements within a permutation expressed in standard form.

Definition 3.2. Suppose $\sigma \in \Omega_r(n)$ is in standard cycle form and $i \in I$, with i not the first element of a cycle of σ. Consider the word w obtained by writing all elements of the cycle C containing i, except for the first, left-to-right as they appear within C. Then we will say that i is a left-to-right cycle minimum (l-r cycle min) if i is a left-to-right minimum within w in the usual sense.

For example, let

$$\sigma = (1, 7, 13, 12, 4, 15)(2, 6, 10, 8, 5)(3, 9)(11, 14) \in \Omega_3(12, 1).$$

Then the first three cycles are special, the final cycle is non-special, and the l-r cycle min are $7, 4, 6, 5, 9, 14$. Note that the second element and the second smallest element within a cycle are always l-r cycle min, by definition. We now allow for certain elements within an r-permutation to be colored.

Definition 3.3. Given a positive integer m, let $\Omega_{r,m}(n, k)$ denote the set of r-permutations of $[n + r]$ having $k + r$ cycles in which elements within the following two classes are each assigned one of m colors: (i) non-minimal elements within non-special cycles, and (ii) non-minimal elements within special cycles that do not correspond to left-to-right cycle minima. Define $\Omega_{r,m}(n) = \cup_{k=0}^{n} \Omega_{r,m}(n, k)$.

Within the permutation σ above, the elements that would be assigned colors are (i) 14 and (ii) $13, 12, 15, 10, 8$.

Let $v(n, k) = v(n, k; r, m) = |\Omega_{r,m}(n, k)|$; note that $v(n, k) = (-1)^{n-k} w(n, k)$ upon comparing recurrences and initial values. See also Mihoubi and Rahmani [17] for an interpretation of $v(n, k; r, m)$ in terms of their partial r-Bell polynomials. In the formulation above, one may also regard m as an indeterminate marking the statistic on $\Omega_r(n)$ that records the sum of the number of non-minimal elements in non-special cycles and the number of non-minimal elements in special cycles not corresponding to l-r cycle min. For a combinatorial interpretation of $w(n, k)$ in terms of Dowling lattices, the reader is referred to [7, Section 2].

Definition 3.4. Suppose $\sigma \in \Omega_{r,m}(n)$ and that $i \in I$ belongs to cycle C of σ, with i not the first element of C. Then the predecessor of i is the first element of I to the left of i in C and smaller than i, provided such an element exists, which we will denote by *pred*(i). Define S_σ to be the set of all $i \in I$ that have a predecessor (possibly empty).

Observe that all non-minimal elements in non-special cycles of $\sigma \in \Omega_{r,m}(n)$ have predecessors, whereas only non-minimal elements not corresponding to l-r cycle min in special cycles have them. For example, if

$$\sigma = (1, 6, 4, 5)(2, 8, 7, 9)(3, 11, 13, 14)(10, 12)(15) \in \Omega_{3,1}(12, 2),$$

then we have $S_\sigma = \{5, 9, 12, 13, 14\}$. Given $\sigma \in \Omega_{r,m}(n)$ and $1 \leq i \leq r$, let ℓ_i denote the number of l-r cycle min within the i-th special cycle of σ. In the last example, we have $r = 3$, with $\ell_1 = \ell_2 = 2$ and $\ell_3 = 1$.

We now introduce a pair of statistics on the set $\Omega_{r,m}(n)$.

Definition 3.5. Define the statistics v_1 and v_2 on $\Omega_{r,m}(n)$ by letting

$$v_1(\sigma) = \sum_{i=1}^{r}(i-1)\ell_i$$

and

$$v_2(\sigma) = \sum_{i\in S_\sigma}(pred(i) - r - 1).$$

Note that the statistic v_1 appears to be new even in the case $r = 0$ and $m = 1$, though in this case it has the same distribution on $\mathcal{S}_n$ as a certain type of inversion statistic originally considered by Carlitz [6] and later studied [20]. We found no special cases in the literature of the statistic v_2. We now consider a (p,q)-generalization of the r-Whitney number of the first kind in terms of these statistics.

Definition 3.6. Define $v_{p,q}(n,k) = v_{p,q}(n,k;r,m)$ as the joint distribution polynomial for the v_1 and v_2 statistics on the set $\Omega_{r,m}(n,k)$, that is,

$$v_{p,q}(n,k) = \sum_{\sigma\in\Omega_{r,m}(n,k)} p^{v_1(\sigma)}q^{v_2(\sigma)}, \quad n,k \geq 0.$$

The $v_{p,q}(n,k)$ are determined recursively as follows.

Proposition 3.7. *The array $v_{p,q}(n,k)$ satisfies the recurrence*

$$v_{p,q}(n,k) = v_{p,q}(n-1,k-1) + ([r]_p + m[n-1]_q)v_{p,q}(n-1,k), \quad n,k \geq 1, \quad (3.1)$$

and has initial values $v_{p,q}(n,0) = \prod_{i=0}^{n-1}([r]_p + m[i]_q)$ and $v_{p,q}(0,k) = \delta_{k,0}$ for $n,k \geq 0$.

Proof. The initial condition $v_{p,q}(0,k) = \delta_{k,0}$ is clear from the definitions. To show $v_{p,q}(n,0) = \prod_{i=0}^{n-1}([r]_p + m[i]_q)$, we add the elements of I sequentially to the special cycles starting with $r+1$. The element $r+i$ contributes a factor of $[r]_p + m[i-1]_q$, upon considering whether it is inserted directly following a member of $[r]$ or a member of $[r+1, r+i-1]$; note that there are $1 + p + \cdots + p^{r-1} = [r]_p$ possibilities in the former case where $r+i$ would correspond to a l-r cycle min and $m(1 + q + \cdots + q^{i-2}) = m[i-1]_q$ possibilities in the latter. To show (3.1), first observe that the weight of all members of $\Omega = \Omega_{r,m}(n,k)$ in which $n+r$ belongs to its own cycle is $v_{p,q}(n-1,k-1)$, since neither the v_1 nor the v_2 statistic values are changed by its addition in this case. On the other hand, the weight of all members of Ω in which $n+r$ is a l-r cycle min within a special cycle is given by $[r]_p v_{p,q}(n-1,k)$. Finally, members of Ω in which $n+r$ directly follows some member of $[r+1, r+n-1]$ within a cycle are seen to have weight $m[n-1]_q v_{p,q}(n-1,k)$. Observe that the addition of $n+r$ to a cycle does not affect the predecessors of smaller elements already occupying the cycle. Combining the three previous cases gives (3.1). □

Note that $w_{p,q}(n,k) = (-1)^{n-k} v_{p,q}(n,k)$, upon comparing recurrences. One has the following further recurrence satisfied by $w_{p,q}(n,k)$.

Proposition 3.8. *If $n, k \geq 1$, then*

$$w_{p,q}(n,k) = \sum_{j=k}^{n} (-1)^{n-j} w_{p,q}(j-1,k-1) \prod_{i=0}^{n-j-1} ([r]_p + m[j+i]_q).$$

Proof. We show, equivalently, the relation

$$v_{p,q}(n,k) = \sum_{j=k}^{n} v_{p,q}(j-1,k-1) \prod_{i=0}^{n-j-1} ([r]_p + m[j+i]_q). \tag{3.2}$$

To do so, consider the smallest element, $r+j$, within the k-th non-special cycle of a member of $\Omega_{r,m}(n,k)$; note that $k \leq j \leq n$. Then the elements of $[r+j-1]$ may be positioned according to any member of $\Omega_{r,m}(j-1,k-1)$, and thus there are $v_{p,q}(j-1,k-1)$ possibilities concerning their arrangement. After placing the element $r+j$ in its own cycle, we insert the members of $[r+j+1, r+n]$ one-by-one starting with $r+j+1$. For $1 \leq i \leq n-j$, there are $[r]_p + m[j+i-1]_q$ possibilities concerning the placement of the element $r+j+i$, upon considering whether it directly follows a member of $[r]$ or a member of $[r+1, r+j+i-1]$. Thus, there are $\prod_{i=1}^{n-j}([r]_p + m[j+i-1]_q)$ possibilities concerning the placement of elements of $[r+j+1, r+n]$. Summing over j gives (3.2) and completes the proof. □

Using our combinatorial interpretation for $w_{p,q}(n,k)$, it is possible to prove bijectively the formulas for $w_{p,q}(n,k)$ and $s_q(n,k)$ given above in Proposition 2.6.

Combinatorial proofs of (2.7) *and* (2.8) *in Proposition 2.6.* We first prove formula (2.7), rewritten in the form

$$v_{p,q}(n,k) = \sum_{j=k}^{n} m^{n-j} \binom{j}{k} [r]_p^{j-k} c_q(n,j), \quad n \geq k \geq 0, \tag{3.3}$$

where $c_q(n,j) = (-1)^{n-j} s_q(n,j)$. To show (3.3), we count members of $\Omega_{r,m}(n,k)$ according to the number, $j-k$, of l-r cycle min in special cycles, where $k \leq j \leq n$. To form a member of $\Omega_{r,m}(n,k)$ for which the number of l-r cycle min in special cycles is $j-k$, we first consider $\rho \in \Omega_{0,1}(n,j)$ in standard cycle form, i.e., ρ is a permutation of $[n]$ having j cycles, and count all such ρ according to the value of the v_2 statistic. Note that there are $c_q(n,j)$ possibilities for ρ, each of whose $n-j$ non-minimal elements is assigned one of m colors. Next, we add r to all of the letters of ρ. We then select $j-k$ of the j cycles of ρ, remove the enclosing parentheses, and let $w_1, w_2, \ldots, w_{j-k}$ denote the resulting words, where $\min(w_1) < \min(w_2) < \cdots < \min(w_{j-k})$.

We insert the words w_i into r urns labeled $1, 2, \ldots, r$. Assign the weight of p^{i-1} to each word added to the i-th urn for $1 \leq i \leq r$, which we multiply to obtain the total weight. Thus, there are $[r]_p^{j-k}$ possibilities concerning the placement of the words w_i. Within urns, words are ordered from left-to-right in descending order of first (= smallest) elements and then concatenated, with the number labeling the urn written at the beginning. That is, if $w_{i_1}, \ldots, w_{i_s}$, with $i_1 < \cdots < i_s$, are the words in urn j, we form the long word $jw_{i_s} \cdots w_{i_1}$. The contents of urn j then becomes that of the j-th special cycle. Note that the first letter of each w_i becomes a l-r cycle min, by the ordering of words within urns. Taken together with the k cycles of ρ that were not selected, we obtain a member $\pi \in \Pi_{r,m}(n,k)$ in which the l-r cycle min in special cycles number $j-k$. Upon considering all possible ρ, the weight of such members of $\Pi_{r,m}(n,k)$ is seen to be $m^{n-j}\binom{j}{k}[r]_p^{j-k}c_q(n,j)$. Summing over all j then gives (3.3).

We illustrate the above procedure for transforming ρ into π, where $n = 20$, $k = 2$, $r = 4$ and $m = 1$. Let $j = 8$ and

$$\rho = (1,7,3)(2)(4,18,9,5)(6)(8,10,20,11)(12,16,13,19)(14)(15,17) \in \Omega_{0,1}(20,8).$$

Increase each element of ρ by $r = 4$ and then, in the resulting permutation of $[5,24]$, select $j-k = 6$ of the cycles, shown below:

$$(5,11,7), \quad (8,22,13,9), \quad (10), \quad (12,14,24,15), \quad (16,20,17,23), \quad (18),$$

which will be denoted by w_i, $1 \le i \le 6$, from left to right. Insert these words randomly into four urns U_i as shown:

$\mathbf{U_1}$	$\mathbf{U_2}$	$\mathbf{U_3}$	$\mathbf{U_4}$
w_6, w_2		w_4, w_3, w_1	w_5.

From this arrangement, we form the cycles $(1, w_6, w_2) = (1,18,8,22,13,9)$, (2), $(3, w_4, w_3, w_1) = (3,12,14,24,15,10,5,11,7)$ and $(4, w_5) = (4,16,20,17,23)$. Considering these cycles together with the two that were not selected, one obtains $\pi \in \Pi_{4,1}(20,2)$ given by

$$\pi = (1,18,8,22,13,9)(2)(3,12,14,24,15,10,5,11,7)(4,16,20,17,23)(6)(19,21).$$

We now prove formula (2.8), rewritten in the form

$$m^{n-k}c_q(n,k) = \sum_{j=k}^{n}(-1)^{j-k}\binom{j}{k}[r]_p^{j-k}v_{p,q}(n,j), \quad n \ge k \ge 0. \tag{3.4}$$

To show (3.4), let $\Omega'_{r,m}(n,k)$ denote the set obtained from members of $\Omega_{r,m}(n,k)$ by marking some subset of the l-r cycle min belonging to special cycles. Define the sign $\lambda \in \Omega'_{r,m}(n,k)$ to be $(-1)^{j-k}$, where $j-k$ denotes the number of marked l-r cycle min of λ, and define the weight of λ as we did before for members of $\Omega_{r,m}(n,k)$. We first show that the right-hand side of (3.4) gives the total (signed) weight of all the members of $\Omega'_{r,m}(n,k)$. To do so, it is enough to show that the weight of the members of $\Omega'_{r,m}(n,k)$ in which there are $j-k$ marked cycle min is $\binom{j}{k}[r]_p^{j-k}v_{p,q}(n,j)$ for $k \le j \le n$.

To form such members of $\Omega'_{r,m}(n,k)$, we first choose $\tau \in \Omega_{r,m}(n,j)$ for some $j \geq k$ and select exactly $j-k$ of the j non-special cycles of τ. We insert the contents of these $j-k$ cycles into the special cycles of τ as follows. Let $b = b_1 b_2 \cdots b_s$ denote the contents of a selected cycle in the order that the letters appear within the cycle. We will insert b into one of the r special cycles of τ so that b_1 will become a l-r cycle min. Let $C = j w_1 w_2 \cdots w_\ell$ denote the contents of the cycle in which we are to insert b, where $j \in [r]$ and w_i denotes all of the letters from the i-th largest cycle min of C up to but not including the $(i+1)$-st largest cycle min. That is, we have $\min(w_1) > \min(w_2) > \cdots > \min(w_\ell)$, with $\min(w_i)$ also the first letter of the subword w_i for each i.

If $b_1 > \min(w_1)$ or if C contains only j, then we write the letters in b directly after the letter j in C. Otherwise, let i_0 be the index $i \in [\ell]$ such that $\min(w_i) > b_1 > \min(w_{i+1})$, where $\min(w_{\ell+1}) = 0$. We then write the letters of b between the subwords w_{i_0} and w_{i_0+1} in C if $i_0 < \ell$ or after w_ℓ if $i_0 = \ell$. Next, we mark the letter b_1; note that b_1 is a cycle min, as are still the first letters of each of the w_i. Repeat the above procedure for each of the $j-k$ selected cycles, where cycles are inserted one after another, sequentially, and we consider also the subwords arising from previously inserted cycles when deciding on the position of the current cycle. Since the first letter of each selected cycle becomes a l-r cycle min, there are $[r]_p^{j-k}$ possibilities concerning the insertion of these cycles. Furthermore, since the predecessors of the non-minimal elements within the selected non-special cycles of τ remain the same once their contents have been added to the special cycles as described, the contribution of these non-minimal elements towards the q-weight (and also the m-weight) remains the same.

We illustrate the procedure described above for creating members of $\Omega'_{r,m}(n,k)$, where $n = 21$, $k = 2$, $r = 3$ and $m = 1$. Let $j = 6$ and $\tau \in \Omega_{3,1}(21,6)$ given by

$$\tau = (1,7,5,19)(2)(3,18,12,4)(6,9)(8,13,10)(11)(14,22,16)(15,24,21,17)(20,23).$$

Suppose now that we select the four non-special cycles $(6,9)$, (11), $(15,24,21,17)$ and $(20,23)$, and stipulate that $(6,9)$ and $(20,23)$ go in the first and second special cycle of τ, respectively, while the other two go in the third. This yields the permutation $\lambda \in \Omega'_{3,1}(21,2)$ given by

$$\lambda = (1,7,\underline{6},9,5,19)(2,\underline{20},23)(3,18,\underline{15},24,21,17,12,\underline{11},4)(8,13,10)(14,22,16),$$

where the marked cycle min are underlined. Upon considering the marked letters of λ, the above process is seen to be reversible. Allowing τ to vary thus yields all members of $\Omega'_{r,m}(n,k)$ having exactly $j-k$ marked cycle min, which are seen to have weight $\binom{j}{k}[r]_p^{j-k} v_{p,q}(n,j)$, as desired.

Now consider the smallest l-r cycle min belonging to a special cycle within a member of $\Omega'_{r,m}(n,k)$. Either mark it if it is unmarked, or remove the marking from it. For example, this would entail underlining the element 4 in the permutation λ above. This operation is a sign-changing, weight-preserving involution of $\Omega'_{r,m}(n,k)$, which is not defined whenever all of the special cycles are singletons. The sign of each such member of $\Omega'_{r,m}(n,k)$ is positive, and the weight of all such members is seen to be $m^{n-k}c_q(n,k)$, which implies (3.4) and completes the proof. □

We now provide a combinatorial proof of the orthogonality relations between $w_{p,q}(n,k)$ and $W_{p,q}(n,k)$. Before doing so, we first recall a combinatorial interpretation for the array $W_{p,q}(n,k)$ from [13]. Given $0 \le r \le m$, by an *r-partition* of $[m]$, we will mean a partition of the set $[m]$ in which the elements of $[r]$ belong to distinct blocks. If $n,k,r \ge 0$, then let $\Pi_r(n,k)$ denote the set of all r-partitions of $[n+r]$ having $k+r$ blocks and let $\Pi_r(n) = \cup_{k=0}^{n}\Pi_r(n,k)$. Note that when $r=0$, an r-partition of $[m]$ is the same as an ordinary partition. We will apply the terms *special* and *minimal* with regard to the members of $\Pi_r(n,k)$ in a manner completely analogous to how those terms were applied above towards members of $\Omega_r(n,k)$ (with "cycles" replaced by "blocks" at the appropriate points in the definitions).

Elements of r-partitions will be assigned colors in the following manner.

Definition 3.9 (Mansour et al. [13])**.** Given an integer $m \ge 1$, let $\Pi_{r,m}(n,k)$ denote the set of r-partitions of $[n+r]$ having $k+r$ blocks wherein within each non-special block, every non-minimal element is assigned one of m colors, and let $\Pi_{r,m}(n) = \cup_{k=0}^{n}\Pi_{r,m}(n,k)$.

Upon making a comparison of the recurrences and initial values, we see that $|\Pi_{r,m}(n,k)| = W(n,k;r,m)$ for all r and m. We now recall a couple of statistics on $\Pi_{r,m}(n,k)$.

Definition 3.10 (Mansour et al. [13])**.** Suppose $\pi \in \Pi_{r,m}(n,k)$ is represented as

$$\pi = A_1/A_2/\cdots/A_r/B_1/B_2/\cdots/B_k,$$

where A_i denotes the special block containing the element i for $i \in [r]$ and non-special blocks are denoted by B_j, with $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$. Define the statistics w_1 and w_2 on $\Pi_{r,m}(n,k)$ by letting

$$w_1(\pi) = \sum_{i=1}^{r}(i-1)(|A_i|-1)$$

and

$$w_2(\pi) = \sum_{i=1}^{k}(i-1)(|B_i|-1).$$

In [13], it was shown that

$$W_{p,q}(n,k) = \sum_{\pi\in\Pi_{r,m}(n,k)} p^{w_1(\pi)}q^{w_2(\pi)}, \quad n,k \ge 0.$$

Note that $W_{p,q}(n,k)$ reduces to $W(n,k)$ when $p=q=1$. Using (1.2) and (2.1), one can obtain orthogonality relations between the arrays $w_{p,q}(n,k)$ and $W_{p,q}(n,k)$. Here, we give bijective proofs by making use of our combinatorial interpretations for these arrays.

Theorem 3.11. *If $n \geq k \geq 0$, then*

$$\sum_{j=k}^{n} W_{p,q}(n,j)w_{p,q}(j,k) = \sum_{j=k}^{n} w_{p,q}(n,j)W_{p,q}(j,k) = \delta_{n,k}. \tag{3.5}$$

Proof. To show the first relation in (3.5), we consider sets $\mathcal{A}_j$ where $k \leq j \leq n$ of ordered pairs (α, β) in which $\alpha \in \Pi_{r,m}(n,j)$ and β is an arrangement of the blocks of α according to some member of $\Omega_{r,m}(j,k)$. Within β, blocks of α are ordered according to the size of their smallest elements, with the special blocks of α (i.e., those containing a member of $[r]$) regarded as special elements of β. Thus, the special cycles of β are those that contain a special block of α. Define the sign of $(\alpha,\beta) \in \mathcal{A}_j$ by $(-1)^{j-k}$ and the weight by $p^{w_1(\alpha)+v_1(\beta)}q^{w_2(\alpha)+v_2(\beta)}$. Let $\mathcal{A} = \cup_{j=k}^{n}\mathcal{A}_j$. For example, if $n = 10$, $k = 1$, $r = 2$, $m = 1$ and $j = 4$, then $(\alpha,\beta) \in \mathcal{A}_4$, where

$$\alpha = \{1,3,5\},\{2,4,8\},\{6\},\{7,11\},\{9\},\{10,12\}$$

and

$$\beta = (\{1,3,5\})(\{2,4,8\},\{9\},\{6\})(\{7,11\},\{10,12\}),$$

has sign $(-1)^{4-1} = -1$ and weight $p^{2+2}q^{4+1} = p^4q^5$. The first sum in (3.5) then gives the total (signed) weight of all the members of $\mathcal{A}$. To complete the proof, we define a sign-changing, weight-preserving involution of $\mathcal{A}$.

In order to do so, given $(\alpha,\beta) \in \mathcal{A}$, let x be the largest $i \in I$ such that it is *not* the case that a cycle of the form $(\{i\})$ containing only the block $\{i\}$ occurs in β. Let B be the block of α containing the element x. Note that B cannot have smallest element x, lest B be a singleton. If $|B| \geq 2$, then break off x and form the singleton $\{x\}$ to directly follow $B - \{x\}$ within its cycle of β. Observe that if $\{x\}$ occurs as a block of α, then it cannot be first within its cycle of β, by the ordering of blocks of α within β and the assumption on x (note that all $i > x$ must occur within β as 1-cycles of the form $(\{i\})$). Thus, if $\{x\}$ occurs, one may move it to the block within its cycle of β that directly precedes it. Combining the two previous mappings defines an involution of $\mathcal{A}$ if $n > k$ since at least one cycle of β in this case always contains at least two elements of $[n+r]$ altogether, with at least one member of I belonging to a block within such a cycle. If $n = k$, then $\mathcal{A}$ contains only a single element having weight 1.

Clearly, the involution defined in the previous paragraph changes the sign since the number of (non-special) blocks of α changes by one. We now show that it always preserves the weight. First suppose that B belongs to a non-special cycle of β. Then moving x as described in the first mapping preserves the sum $w_2(\alpha)+v_2(\beta)$ since if x belonged to the i-th non-special block of α to start with, then breaking off $\{x\}$ reduces $w_2(\alpha)$ by $i-1$ but increases $v_2(\beta)$ by the same amount since $\{x\}$ has predecessor $B - \{x\}$, which is now the i-th smallest non-special element of β. Note that $\{x\}$ then becomes the largest element within its cycle of β, and hence $\{x\}$ following $B - \{x\}$ does not affect a possible contribution to $v_2(\beta)$ from a block succeeding B in this cycle. Moreover, since all $i > x$ occur as singletons in

α, reordering the blocks of α after forming $\{x\}$ does not further affect the $w_2(\alpha)$ value. Note that the value of $w_1(\alpha)+v_1(\beta)$ is unaffected since neither statistic is. Finally, the color that the element x would have been assigned being a non-minimal element of a non-special block of α is transferred to the color assigned the block $\{x\}$ for having a predecessor. Thus, the weight of (α,β) is preserved by the involution in this case.

Now suppose that the block B belongs to a special cycle of β (i.e., one that has a block of α containing a member of $[r]$). If B is a non-special block of α that does not correspond to a l-r cycle min of β, then one may use the reasoning of the prior paragraph to show that the weight is preserved. The same also applies if B is indeed a l-r cycle min of β. Finally, suppose B is a special block of α. Then breaking off $\{x\}$ reduces the $w_1(\alpha)$ value by $\ell-1$ for some $\ell\in[r]$, while it increases $v_1(\beta)$ by the same amount since $\{x\}$ in this case becomes a l-r cycle min within the ℓ-th special cycle of β. Thus, the sum $w_1(\alpha)+v_1(\beta)$ is preserved. There is also no change in $w_2(\alpha)+v_2(\beta)$ since neither statistic is affected in this case, with neither the element x in α nor the block $\{x\}$ in β being assigned a color. Thus, the weight of (α,β) is once again preserved, which implies the first relation in (3.5).

A similar proof applies to the second relation in (3.5). We describe the main differences. Here, one would consider ordered pairs (γ,δ) in which $\gamma\in\Omega_{r,m}(n,j)$ and δ is an arrangement of the cycles of γ according to some member of $\Pi_{r,m}(j,k)$. The sign of (γ,δ) would be $(-1)^{n-j}$ and the weight $p^{v_1(\gamma)+w_1(\delta)}q^{v_2(\gamma)+w_2(\delta)}$. Note that a special block of δ is one that has an element of $[r]$ contained within one of its cycles.

To define the involution in this case, suppose that the blocks of δ are arranged from left-to-right in ascending order of smallest elements contained therein (the special blocks then being first). Consider the leftmost block of δ that contains at least two elements of $[n+r]$ altogether within its cycles. Let C denote this block and u and v be the smallest and second smallest elements of $[n+r]$ in C, respectively. If u and v belong to the same cycle of γ within C, then we split this cycle at v and form a new cycle starting with v, which we write directly following the cycle containing u in C. If u and v belong to different cycles of γ, whence v starts a cycle of γ, then we merge them into one large cycle. Upon considering whether or not C is a special block of δ, one may verify that this mapping is always a sign-changing, weight-preserving involution, which completes the proof. □

4. The (p,q)-Whitney matrix of the first kind

In [13], the (p,q)-Whitney matrix of the second kind was introduced and several properties of this matrix are proven. In this section, we introduce the (p,q)-Whitney matrix of the first kind and find some factorizations of it in analogy with the results of Mező and Ramírez [16].

Definition 4.1. The (p,q)-Whitney matrix of the first kind is the $n\times n$ matrix defined by

$$\mathcal{L}_{p,q}(n) := \mathcal{L}_{p,q}^{(m,r)}(n) = [w_{p,q}(i,j;r,m)]_{0\le i,j\le n-1}\,.$$

For example, $\mathcal{L}_{p,q}(4)$ is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -[r]_p & 1 & 0 & 0 \\ [r]_p^2 + m[r]_p & -m-2[r]_p & 1 & 0 \\ -[r]_p^3-(2+q)m[r]_p^2-(1+q)m^2[r]_p & (1+q)m^2+2(2+q)m[r]_p+3[r]_p^2 & -(2+q)m-3[r]_p & 1 \end{bmatrix}.$$

In particular, if $p = q = 1$, we obtain the r-Whitney matrix of the first kind [16]. If $m = p = 1$ and $r = 0$, we obtain the q-Stirling matrix of the first kind $S_{q,n}^{(1)} := [s_q(i,j)]_{0\le i,j\le n-1}$; see, e.g., [11, 19].

Recall the generalized $n\times n$ Pascal matrix $P_n\,[x]$ (see [4]) defined as

$$P_n\,[x] := \left[\binom{i}{j}x^{i-j}\right]_{0\le i,j\le n-1}.$$

If $x = 1$, we obtain the Pascal matrix P_n of order n. Moreover,

$$P_n^{-1}\,[x] = P_n\,[-x] = \left[(-1)^{i-j}\binom{i}{j}x^{i-j}\right]_{0\le i,j\le n-1}.$$

From identity (2.7), we have the following factorization:

$$\mathcal{L}_{p,q}(n) = S_{q,n}^{(1)}[m]P_n[-[r]_p], \quad n\ge 1, \tag{4.1}$$

where $S_{q,n}^{(1)}[x] := [s_q(i,j)x^{i-j}]_{0\le i,j\le n-1}$.

For example,

$$\mathcal{L}_{p,q}(4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -m & 1 & 0 \\ 0 & m^2(1+q) & -(2+q)m & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ -[r]_p & 1 & 0 & 0 \\ [r]_p^2 & -2[r]_p & 1 & 0 \\ -[r]_p^3 & 3[r]_p^2 & -3[r]_p & 1 \end{bmatrix}$$
$$= S_{q,4}^{(1)}\,[m]\,P_4\,[-[r]_p]\,.$$

Moreover, from identity (2.5), we obtain

$$\mathcal{L}_{p,q}^{(m,r)}(n) = \mathcal{L}_{p,q}^{(m,s)}(n)P_n[-p^s[r-s]_p], \quad 0\le s\le r.$$

Given $n\ge 1$, let $S_n[x]$ be the $n\times n$ matrix defined by $S_n[x] := \left[x^{i-j}\right]_{0\le j\le i\le n-1}$. The following factorization of the generalized Pascal matrix is due to Zhang [22, Theorem 1]:

$$P_n[x] = G_n[x]G_{n-1}[x]\cdots G_1[x], \quad n\ge 1, \tag{4.2}$$

where $G_n[x] = S_n[x]$ and $G_k[x] = I_{n-k}\oplus S_k[x]$ for $1\le k\le n-1$ with $\oplus$ denoting the matrix direct sum.

From the preceding, we obtain the following factorization of the (p,q)-Whitney matrix of the first kind.

Proposition 4.2. *If $n \geq 2$, then*

$$\mathcal{L}_{p,q}(n) = \overline{P}_1[-mq^{n-2}] \cdots \overline{P}_{n-2}[-mq]\overline{P}_{n-1}[-m]G_n[-[r]_p]G_{n-1}[-[r]_p] \cdots G_1[-[r]_p], \quad (4.3)$$

where

$$\overline{P}_k[x] = I_{n-k} \oplus P_k[x].$$

Proof. By (4.1), we have

$$\mathcal{L}_{p,q}(n) = S_{q,n}^{(1)}[m]P_n[-[r]_p].$$

The matrix $P_n[-[r]_p]$ can be factorized by means of (4.2), while the matrix $S_{q,n}^{(1)}[m]$ can be factorized by a result of Ernst [11] as

$$S_{q,n}^{(1)}[m] = \overline{P}_1[-mq^{n-2}] \cdots \overline{P}_{n-2}[-mq]\overline{P}_{n-1}[-m],$$

which implies (4.3). □

5. Conclusion

In this paper, we have introduced a (p, q)-generalization $w_{p,q}(n, k)$ of the r-Whitney numbers of the first kind that reduces to these numbers when $p = q = 1$. In addition to forming an orthogonal pair with a prior generalization of the r-Whitney numbers of the second kind, the $w_{p,q}(n, k)$ arise as the joint distribution for two statistics on a set of generalized permutations. When $r = 0$ and $m = 1$, these statistics are seen to be new on the usual set of permutations and the statistic marked by the q-variable has the same distribution on $\mathcal{S}_n$ as an earlier statistic considered by Carlitz. Since our $w_{p,q}(n, k)$ when $p = 1$ are closely related to the $w_{m,r,q}(n, k)$ studied in [12], which arose in a physical context, modifying slightly our combinatorial interpretation for $w_{p,q}(n, k)$ furnishes a comparable interpretation for the $w_{m,r,q}(n, k)$. Thus, one may obtain, via combinatorial arguments, generalizations of identities from [12].

Furthermore, using Theorem 2.2 and a generalized version of the central limit theorem, it is possible to show that the v_1 and v_2 statistics follow an asymptotically normal distribution as n increases without bound for all $r \geq 2$ and $m \geq 1$. In addition, from Theorem 2.2, it is seen that the array $w_{p,q}(n, k)$ when p and q are real is log-concave by Newton's criterion since the polynomial in identity (2.2) is real-rooted in that case. On the other hand, we seek a general asymptotic formula for $w_{p,q}(n, k)$ when p and q are positive. Non-trivial combinatorial (or algebraic) generalizations of the sequence satisfying recurrence (3.1) are also sought, as such generalizations would likely yield new statistics on the set of permutations. Finally, it would be interesting to find connections between the array $w_{p,q}(n, k)$ and other combinatorial structures.

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