Annales Mathematicae et Informaticae
46 *(2016) pp. 115–120*
http://ami.ektf.hu

A note on the derived length of the group of units of group algebras of characteristic two*

Tibor Juhász

Institute of Mathematics and Informatics
Eszterházy Károly University
Eger, Hungary
juhasz.tibor@uni-eszterhazy.hu

Submitted November 25, 2016 — Accepted December 14, 2016

In memoriam Mihály Rados (1941–2016)

Abstract

Denote by FG the group algebra of a group G over a field F, by $U(FG)$ its group of units, and by $\mathrm{dl}(U(FG))$ the derived length of $U(FG)$. We know very little about $\mathrm{dl}(U(FG))$, especially when F has characteristic 2. In this short note, it is shown that, if F is of characteristic 2, G' is cyclic of order 2^n and the nilpotency class of G is less than $n+1$, then $\mathrm{dl}(U(FG))$ is equal to n or $n+1$. In addition, if $n > 1$ and $G' = \mathrm{Syl}_2(G)$, then $\mathrm{dl}(U(FG)) = n$.

Keywords: Group ring, group of units, derived length

MSC: 16S34, 16U60, 20C07, 20F14

1. Introduction

Let FG be the group algebra of a group G over a field F of prime characteristic p, and let $U(FG)$ be the group of units of FG. It is determined in [4] when $U(FG)$ is solvable, however, we know very little about the derived length of $U(FG)$.

*This research was supported by the European Union and the State of Hungary, co-financed by the European Social Fund in the framework of TÁMOP 4.2.4. A/2-11-1-2012-0001 'National Excellence Program'

Assume first that p is an odd prime. For this case, the group algebras FG with metabelian group of units are classified in [16], under restriction G is finite, and this result is extended to torsion G in [6]. In [7, 8] the finite groups G are described, such that $U(FG)$ has derived length 3. According to [1], if G is a finite p-group with cyclic commutator subgroup, then $\mathrm{dl}(U(FG)) = \lceil \log_2(|G'|+1) \rceil$, where $\lceil \cdot \rceil$ is the upper integer part function. The aim of [2] and [10] is to extend this result, and determine the value of $\mathrm{dl}(U(FG))$ for arbitrary groups G with G' is a cyclic p-group, where p is still an odd prime. As it turned out, if G is nilpotent and torsion, then the derived length of $U(FG)$ remains $\lceil \log_2(|G'|+1) \rceil$, but for non-nilpotent or non-torsion G it can be different. However, the description is not complete yet, for the open cases we refer the reader to [10].

For $p = 2$ and finite group G, necessary and sufficient conditions for $U(FG)$ to be metabelian is given in [9], and independently, in [14]. This result is extended in [6] as follows: if F is a field of characteristic 2, and G is a nilpotent torsion group, then $U(FG)$ is metabelian exactly when G' is a central elementary abelian group of order dividing 4. In [13], it is established that if G is a group of maximal class of order 2^n, then $\mathrm{dl}(U(FG))$ is less or equal to $n-1$. To the best of the author's knowledge, for $p = 2$ there is no other result concerning the derived length of $U(FG)$. The aim of this paper to draw the attention to this uncovered area by sharing the author's experience and an introductory result.

The group of units of a group algebra can be investigated via the Lie structure of the group algebra. For example, we can obtain an upper bound on the derived length of $U(FG)$, by the help of the strong Lie derived length of FG. Let $\delta^{(0)}(FG) = FG$, and for $i \geq 1$, denote by $\delta^{(i)}(FG)$ the associative ideal generated by all the Lie commutators $[x, y] = xy - yx$ with $x, y \in \delta^{(i-1)}(FG)$. FG is said to be strongly Lie solvable, if there exists i, for which $\delta^{(i)}(FG) = 0$, and the first such i is called the strong Lie derived length of FG, which will be denoted by $\mathrm{dl}^L(FG)$. For $x, y \in U(FG)$ we have that the group commutator $(x, y) = x^{-1}y^{-1}xy$ is equal to $1 + x^{-1}y^{-1}[x, y]$, which implies that $\delta_i(U(FG)) \subseteq 1 + \delta^{(i)}(FG)$ for all i, where $\delta_i(U(FG))$ denotes the ith term of the derived series of $U(FG)$. Therefore, if FG is strongly Lie solvable, then $\mathrm{dl}(U(FG)) \leq \mathrm{dl}^L(FG)$.

According to [15, Theorem 5.1], FG is strongly Lie solvable if and only if either G is abelian, or G' is a finite p-group and F is a field of characteristic p. By [11, Proposition 1], if FG is strongly Lie solvable such that G is nilpotent and $\gamma_3(G) \subseteq (G')^p$, then $\mathrm{dl}^L(FG) = \lceil \log_2(t(G')+1) \rceil$, where by $t(G')$ we mean the nilpotency index of the augmentation ideal of the subalgebra FG'.

Assume now that G is a group with cyclic commutator subgroup of order 2^n and F is a field of characteristic 2. Then G is nilpotent with nilpotency class $\mathrm{cl}(G) \leq n+1$, so we can apply the above formulas to get

$$\mathrm{dl}(U(FG)) \leq \mathrm{dl}^L(FG) = \lceil \log_2(2^n+1) \rceil = n+1.$$

Hence, if $n = 1$, then $\mathrm{dl}(U(FG)) = 2$. For the case when $n > 1$ and $\mathrm{cl}(G) \leq n$, we are able to give a lower bound on $\mathrm{dl}(U(FG))$ as well.

Theorem 1.1. *Let F be a field of characteristic 2, and let G be a group with cyclic commutator subgroup of order 2^n, where $n > 1$. Then $\mathrm{dl}(U(FG)) \geq n$, whenever G has nilpotency class at most n.*

According to [12, Theorem 1], under conditions of Theorem 1.1, $U(FG)$ is nilpotent and, by [5, Theorem 4.3], if $G' = \mathrm{Syl}_2(G)$, then $\mathrm{cl}(U(FG)) = 2^n - 1$. Using the well-known relation $\delta_i(U(FG)) \subseteq \gamma_{2^i}(U(FG))$ between terms of the derived series and the lower central series of groups, we have the following assertion.

Corollary 1.2. *Let F be a field of characteristic 2, and let G be a group with cyclic commutator subgroup of order 2^n, where $n > 1$. If $G' = \mathrm{Syl}_2(G)$ and $\mathrm{cl}(G) \leq n$, then $\mathrm{dl}(U(FG)) = n$.*

For instance, if

$$G = \langle a, b, c \mid c^{2^n} = 1, b^{-1}ab = ac, ac = ca, bc = cb \rangle,$$

with $n > 1$, and $\mathrm{char}(F) = 2$, then $\mathrm{dl}(U(FG)) = n$. This example also witnesses that for non-torsion G, $U(FG)$ can be metabelian, even if G' is cyclic of order 4.

The GAP system for computational discrete algebra [17] and its package, the LAGUNA [3] open the door to compute the derived length of $U(FG)$ for G of not too large size. Computing $\mathrm{dl}(U(FG))$ for some group G of order not greater than 512 and F of 2 elements, it seems that $\mathrm{dl}(U(FG))$ will always be at least n, even if $\mathrm{cl}(G) = n + 1$. However, it would also be interesting to know when $\mathrm{dl}(U(FG))$ is n or when it is $n + 1$.

2. Proof of Theorem 1.1

We will use the following notations. For a normal subgroup H of G we denote by $\mathfrak{I}(H)$ the ideal in FG generated by all elements of the form $h - 1$ with $h \in H$. For the subsets $X, Y \subseteq FG$ by $[X, Y]$ we mean the additive subgroup of FG generated by all Lie commutators $[x, y]$ with $x \in X$ and $y \in Y$.

Write $G' = \langle x \mid x^{2^n} = 1 \rangle$, and assume that $n > 1$. Then for any $m > 1$, $y \in \gamma_m(G)$ and $g \in G$ we have $g^{-1}yg = y^k$, where k is odd, thus $(y, g) = y^{k-1} \in \gamma_m(G)^2$. Hence, $\gamma_{m+1}(G) \subseteq \gamma_m(G)^2$ for all $m > 1$, so G is nilpotent of class at most $n + 1$. Evidently, if $\gamma_3(G) \subseteq (G')^4$, then $\mathrm{cl}(G)$ cannot exceed n. We show first the converse, that is, if $\mathrm{cl}(G) \leq n$, then

$$\gamma_3(G) \subseteq (G')^4. \tag{2.1}$$

This is clear, if $n = 2$. For $n \geq 3$, it is well known that the automorphism group of G' is the direct product of the cyclic group $\langle \alpha \rangle$ of order 2 and the cyclic group $\langle \beta \rangle$ of order 2^{n-2}, where the action of these automorphisms on G' is given by $\alpha(x) = x^{-1}$, $\beta(x) = x^5$. Consequently, for every $g \in G$ there exists $i \geq 0$, such that either $g^{-1}xg = x^{5^i}$ or $g^{-1}xg = x^{-5^i}$. Assume that there is a $g \in G$ such that $g^{-1}xg = x^{-5^i}$ for some i, and let $y \in \gamma_m(G)$ with $m > 1$. Then $(y, g) = y^{-1-5^i} \in$

$\gamma_{m+1}(G)$, and as $-1-5^i \equiv 2 \pmod 4$, we have that $\gamma_{m+1}(G) = (\gamma_m(G))^2$. This means that $\mathrm{cl}(G) = n+1$, which is a contradiction. Therefore, for any $g \in G$ there exists i such that $g^{-1}xg = x^{5^i}$ and $(x,g) = x^{-1+5^i} = x^{4k}$ for some integer k, which forces 2.1.

Let F be a field of characteristic 2. The next step is to show by induction that

$$[\omega(FG')^m, FG] \subseteq \mathfrak{I}(G')^{m+3} \tag{2.2}$$

for all $m \geq 1$. Let $y \in G'$ and $g \in G$. Then, using that $\gamma_3(G) \subseteq (G')^4$, we have

$$[y+1, g] = [y,g] = gy((y,g)+1) \in \mathfrak{I}(\gamma_3(G)) \subseteq \mathfrak{I}(G')^4.$$

Since the Lie commutators of the form $[y+1, g]$ span the subspace $[\omega(FG'), FG]$, (2.2) holds for $m = 1$. Assume now (2.2) for some $m \geq 1$. Then,

$$\begin{aligned}[\omega(FG')^{m+1}, FG] &\subseteq \omega(FG')^m[\omega(FG'), FG] + [\omega(FG')^m, FG]\omega(FG') \\ &\subseteq \mathfrak{I}(G')^{m+4},\end{aligned}$$

as desired. Furthermore, by using (2.2), for all $k, l \geq 1$ we have

$$\begin{aligned}&[\mathfrak{I}(G')^k, \mathfrak{I}(G')^l] \\ &\quad = [FG\omega(FG')^k, FG\omega(FG')^l] \\ &\quad \subseteq FG[\omega(FG')^k, FG\omega(FG')^l] + [FG, FG\omega(FG')^l]\omega(FG')^k \\ &\quad \subseteq FG[\omega(FG')^k, FG]\omega(FG')^l + FG[FG, \omega(FG')^l]\omega(FG')^k \\ &\qquad + [FG, FG]\omega(FG')^{k+l} \\ &\quad \subseteq \mathfrak{I}(G')^{k+l+1}.\end{aligned} \tag{2.3}$$

At this stage, it may be worth mentioning that without the assumption $\mathrm{cl}(G) \leq n$ we can only claim that $\gamma_3(G) \subseteq (G')^2$ and $[\omega(FG')^m, FG] \subseteq \omega(FG')^{m+1}$ instead of (2.1) and (2.2). Although those would be enough for (2.3), but not for what follows.

Denote by S the set of those $a \in G$, for which there exists $b \in G$, such that $\langle (a,b) \rangle = G'$. We are going to show that for all $k \geq 1$ and $a \in S$, there exists $w_k \in \mathfrak{I}(G')^{3 \cdot 2^{k-1}}$, such that

$$1 + a(x+1)^{3 \cdot 2^{k-1}-1} + w_k \in \delta_k(U(FG)). \tag{2.4}$$

This implies that $\delta_k(U(FG))$ contains non-identity element, while $3 \cdot 2^{k-1} - 1 < 2^n$, and then

$$\mathrm{dl}(U(FG)) \geq \left\lceil \log_2\left(\frac{2}{3}(2^n+1)\right)\right\rceil = n,$$

and the proof of Theorem 1.1 will be done.

Let $a \in S$. Then there exists $b \in G$ such that $(a,b) = x^i$, where i is odd. By (2.2), $[x+1, b] \in \mathfrak{I}(G')^4$, and

$$u := (1 + a(x+1), b) = 1 + (1 + a(x+1))^{-1}b^{-1}[a(x+1), b]$$

$$\equiv 1+(1+a(x+1))^{-1}b^{-1}[a,b](x+1)$$
$$\equiv 1+(1+a(x+1))^{-1}a(x^i+1)(x+1) \pmod{\Im(G')^3}.$$

Since $1+a(x+1)$ belongs to the normal subgroup $1+\Im(G')$, so does its inverse, and

$$u \equiv 1+a(x^i+1)(x+1) \pmod{\Im(G')^3}.$$

Using that $x^i+1 \equiv i(x+1) = x+1 \pmod{\omega(FG')^2}$, we obtain that

$$u \equiv 1+a(x+1)^2 \pmod{\Im(G')^3},$$

and (2.4) is confirmed for $k=1$. Assume, by induction, the truth of (2.4) for some $k \geq 1$, and let $a \in S$. Then there exists $b \in G$ such that $\langle (a,b)\rangle = G'$, and of course, b also belongs to S. Moreover, $b^{-1}a \in S$, because $(b^{-1}a,b)=(a,b)$. By the inductive hypothesis, there exist $w_k, w'_k \in \Im(G')^{3\cdot 2^{k-1}}$, such that

$$u := 1+b^{-1}a(x+1)^{3\cdot 2^{k-1}-1}+w_k \in \delta_k(U(FG))$$

and

$$v := 1+b(x+1)^{3\cdot 2^{k-1}-1}+w'_k \in \delta_k(U(FG)).$$

According to (2.3),

$$[u,v] \equiv [b^{-1}a(x+1)^{3\cdot 2^{k-1}-1}, b(x+1)^{3\cdot 2^{k-1}-1}] \pmod{\Im(G')^{3\cdot 2^k}}.$$

Applying (2.2), we have that $[(x+1)^{3\cdot 2^{k-1}-1}, b]$ and $[b^{-1}a,(x+1)^{3\cdot 2^{k-1}-1}]$ belong to $\Im(G')^{3\cdot 2^{k-1}+2}$, and

$$\begin{aligned}[u,v] &\equiv b^{-1}a[(x+1)^{3\cdot 2^{k-1}-1}, b](x+1)^{3\cdot 2^{k-1}-1}\\ &\quad + b[b^{-1}a,(x+1)^{3\cdot 2^{k-1}-1}](x+1)^{3\cdot 2^{k-1}-1} + [b^{-1}a,b](x+1)^{3\cdot 2^k-2}\\ &\equiv a(x^i+1)(x+1)^{3\cdot 2^k-2} \equiv a(x+1)^{3\cdot 2^k-1} \pmod{\Im(G')^{3\cdot 2^k}},\end{aligned}$$

where i is not divisible by 2. Since $u^{-1}, v^{-1} \in 1+\Im(G')$, so

$$(u,v) = 1+u^{-1}v^{-1}[u,v] \equiv 1+a(x+1)^{3\cdot 2^k-1} \pmod{\Im(G')^{3\cdot 2^k}}$$

and the induction is done.

Acknowledgements. The author would like to express his sincere thanks to Alexander Konovalov for the help in creating the GAP script, furthermore to my colleague, Tibor Tajti for providing a capable technical background to run it.

References

[1] Bagiński, C., A note on the derived length of the unit group of a modular group algebra, *Comm. Algebra* 30 (2002), 4905–4913.

[2] Balogh, Z., Li, Y., On the derived length of the group of units of a group algebra, *J. Algebra Appl.* 6 (2007), No. 6, 991–999.

[3] Bovdi, V., Konovalov, A., Rossmanith, R., Schneider, C., LAGUNA — Lie AlGebras and UNits of group Algebras, Version 3.7.0; 11 November 2014 (`http://www.cs.st-andrews.ac.uk/~alexk/laguna/`).

[4] Bovdi, A., Group algebras with a solvable group of units, *Comm. Algebra* 36 (2008), No. 2, 315–324.

[5] Bovdi, A., Kurdics, J., Lie properties of group algebra and the nilpotency class of the group of units, *J. Algebra* 212 (1999), No. 1, 28–64.

[6] Catino, F., Spinelli, E., On the derived length of the unit group of a group algebra, *J. Group Theory* 13 (2010), No. 4, 577–588.

[7] Chandra, H., Sahai, M., On group algebras with unit groups of derived length three in characteristic three, *Publ. Math.* 82 (2013), No. 3–4, 697–708.

[8] Chandra, H., Sahai, M., Group algebras with unit groups of derived length three, *J. Algebra Appl.* 9 (2010), No. 2, 305–314.

[9] Coleman, D.B., Sandling, R., Mod 2 group algebras with metabelian unit groups. *J. Pure Appl. Algebra* 131 (1998), 25–36.

[10] Juhász, T., The derived length of the unit group of a group algebra – The case $G' = \mathrm{Syl}_p(G)$, *J. Algebra Appl.*, In press, DOI: `http://dx.doi.org/10.1142/S0219498817501420`.

[11] Juhász, T., On the derived length of Lie solvable group algebras, *Publ. Math.* 68 (2006), No. 1–2, 243–256.

[12] Khripta, I.I., The nilpotency of the multiplicative group of a group ring, *Mat. Zametki* 11 (1972), 191–200.

[13] Konovalov, A., Wreath products in the unit group of modular group algebras of 2-groups of maximal class, PhD Thesis, Mathematical Institute, Ukrainian National Academy of Sciences, Kiev, 1995.

[14] Kurdics, J., On group algebras with metabelian unit groups, *Period. Math. Hung.* 32 (1996), No. 1–2, 57–64.

[15] Sehgal, S.K., Topics in group rings, Marcel Dekker, New York, 1978.

[16] Shalev, A., Meta-abelian unit groups of group algebras are usually abelian, *J. Pure Appl. Algebra* 72 (1991), 295–302.

[17] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.8.6; 2016 (`http://www.gap-system.org`).