Synchronization of van der Pol Oscillators with Delayed Coupling

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ABSTRACT

The synchronization of self-sustained oscillators such as the van der Pol oscillator is a model for the adjustment of rhythms of oscillating objects due to their weak interaction and has wide applications in natural and technical processes. That these oscillators adjust their frequency or phase to an external forcing or mutually between several oscillators is a phenomenon which can be used in sound synthesis for various purposes. In this paper we focus on the influence of delays on the synchronization properties of these oscillators. As there is no general theory yet on this topic, we mainly present simulation results, together with some background on the non-delayed case. Finally, the theory is also applied in Neukom's studies 21.1-21.9.

1. INTRODUCTION

If several distinct natural or technical systems interact with each other, there is a tendency that these systems adjust to each other in some sense, i.e. that they synchronize their behavior. Put more precisely, by synchronization we mean the *adjustment of the rhythms of oscillating objects due to their mutual interaction*. Synchronization can occur in model systems such as a chain of coupled van der Pol oscillators but also in more complex physical, biological or social systems such as the coordination of clapping of an audience. Historically, synchronization was first described by Huygens (1629-1695) on pendulum clocks. In modern times, major advances were made by van der Pol and Appleton. Physically, we basically distinguish between synchronization by external excitation, mutual synchronization of two interacting systems and synchronization phenomena in chains or topologically more complex networks of oscillating objects. In this paper, we will focus on the case of either two interacting systems or a chain of a small number of oscillators.

Clearly, the synchronizability of such a system of coupled oscillators depends on the strength of the coupling between the two oscillators and the detuning, i.e. the frequency mismatch of the two systems. If the coupling between the two systems does not happen instantaneously, but with a delay, the question of the synchronizability becomes much more difficult to answer. The assumption of delayed feedback however is a vary natural one, since most natural and technical systems do not answer instantaneously to external inputs, but rather with a certain delay, due to physical, biological, or other kinds of limitations. The effect of using delays can be easily modeled in sound synthesis applications and therefore allows a fruitful exchange between theoretical and empirical results on the one hand and musical applications on the other hand.

In the absence of synchronization, other effects such as beats or amplitude death become important, and these effects depend (besides the coupling strength and the frequency detuning) on the delay of the coupling between the oscillators as well.

Self-sustained oscillators can be used in sound synthesis to produce interesting sounds and sound evolutions in different time scales. A single van der Pol oscillator, depending on only one parameter (μ, see (1)), produces a more or less rich spectrum, two coupled oscillators can synchronize after a while or produce beats depending on their frequency mismatch and strength of coupling [1,2]. In chains or networks of coupled oscillators in addition different regions can synchronize (so-called chimeras), which takes even more time. If the coupling is not immediate but after a delay it can take a long time for the whole system to come to a steady or periodic changing state. In addition all these effects can not only be used to produce sound but also to generate mutually dependent parameters of any sound synthesis technique.

This paper gives an introduction to the theory of synchronization [3,4] and shows how to get discrete systems, that is difference equations, from the differential equations and shows their usage in electroacoustic studies of one of the authors (Neukom, Studien 21.1 - 21.9.).

2. SYNCHRONIZATION OF COUPLED OSCILLATORS

2.1 Self-sustained oscillators

Self-sustained oscillators are a model of natural or technical oscillating objects which are active systems, i.e. which contain an inner energy source. The form of oscillation does not depend on external inputs; mathematically, this corresponds to the system being described by an autonomous (i.e. not explicitly time-dependent) dynamical system. Under perturbations, such an oscillator typically returns to the original amplitude, but a phase shift can remain even under weak external forces. Typical examples of self-sustained oscillators are the van der Pol oscillator

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\omega_0^2 x + \mu(1+\gamma-x^2)y \end{aligned} \quad (1)$$

or the Rössler or Lorenz oscillators. Note that in the van der Pol oscillator (1), the parameters μ and γ measure the strength of the nonlinearity; in particular, for $\mu = 0$ we obtain the standard harmonic oscillator. In the case of a single oscillators, we usually set $\gamma = 0$, in the case of several oscillators however, we can use distinct values of γ to describe the amplitude mismatch of the various oscillators. Assuming $\gamma = 0$, in the nonlinear case $\mu \neq 0$, the term $\mu(1-x^2)y$ means that for $|x| > 1$ and $|x| < 1$ there is negative or positive damping, respectively.

In the nonlinear case, these systems cannot be integrated analytically, and one has to use numerical algorithms (and also take into account the stiffness of e.g. the van der Pol system for large values of μ). One can also consider discrete systems of the type

$$\phi^{(k+1)} = F(\phi^{(k)}), \quad (2)$$

which often occurs in cases when one can measure a given systems only at given times $t_0, t_1, \ldots$.

We will discuss an implementation of the van der Pol model (1) in section 5.

2.2 Synchronization by external excitation

The question of synchronization arises when systems of the type (1) and (2) are externally forced or connected together. As a generalization of the van der Pol system (1), weakly nonlinear periodically forced systems are of the form

$$\dot{x} = F(x) + \epsilon p(t), \quad (3)$$

where the unforced system $\dot{x} = F(x)$ has a stable T_0-periodic limit cycle $x_0(t)$ and $p(t)$ is a T-periodic external force. The behavior of the system then primarily depends on the amplitude ϵ of the forcing and the frequency mismatch or detuning $\nu = \omega - \omega_0$, where ω_0 and ω are the frequencies of the oscillator (1) and the T-periodic external force $p(t)$, $\omega = \frac{2\pi}{T}$. One can show that in the simplest case of a sinusoidal forcing function the dynamics of the perturbed system (3) can be described by the *Adler equation*

$$\dot{\theta} = -\Delta + \epsilon \sin(\theta) \quad (4)$$

for the relative or slow phase $\theta = \phi - \omega t$. A stable steady state solution of (4) exists in the case

$$|\Delta| < |\epsilon| \quad (5)$$

and corresponds to a constant phase shift between the phases of the oscillator and the external forcing. The condition (5) describes the synchronization region in the Δ-ϵ-parameter space. Outside the synchronization region, one observes a beating regime with beat frequency

$$\Omega = 2\pi \left(\int_0^{2\pi} \frac{\mathrm{d}\psi}{\sqrt{\epsilon \sin(\psi) - \nu}} \right)^{-1}. \quad (6)$$

2.3 Mutual synchronization and chains of oscillators

Here we consider two coupled systems of the type (3), namely

$$\begin{aligned} \dot{x}_1 &= F_1(x_1, x_2) + \epsilon p_1(x_1, x_2), \\ \dot{x}_2 &= F_2(x_1, x_2) + \epsilon p_2(x_1, x_2) \end{aligned} \quad (7)$$

In the case of weak coupling, i.e. $\epsilon \ll 1$, (7) can be reduced to an equation for the phase difference $\psi = \phi_1 - \phi_2$ of the type (4), and the synchronization region is again of the type (5), where Δ in this case is the difference between the frequencies of the unperturbed oscillators x_1 and x_2. If the coupling becomes larger, the amplitudes have to be considered as well.

To be specific, we consider two coupled van der Pol oscillators, which we assume to connected by a purely dissipative coupling, which is measured by the parameter β:

$$\begin{aligned} \ddot{x}_1 + \omega_1^2 x_1 &= \mu(1 - x_1^2)\dot{x}_1 + \mu\beta(\dot{x}_2 - \dot{x}_1), \\ \ddot{x}_2 + \omega_2^2 x_2 &= \mu(1+\gamma - x_2^2)\dot{x}_2 + \mu\beta(\dot{x}_1 - \dot{x}_2). \end{aligned} \quad (8)$$

Here the two oscillators have the same nonlinearity parameter μ, and γ and $\Delta = \omega_2 - \omega_1$ describe the amplitude and frequency mismatches. In Figure 1, we show the results of a numerical computation of the synchronization region (which is usually called "Arnold tongue") of the system (8) in the case $\gamma = 0$ (no amplitude mismatch).

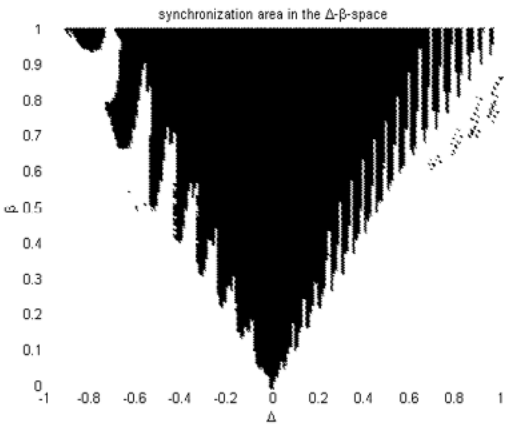

Figure 1. Synchronization area for two coupled van der Pol oscillators

If one considers an entire chain of oscillators (instead of $N = 2$ ones as in (8)), the model equations are for any $1 \le j \le N$

$$\ddot{x}_j + \omega_j^2 x_j = 2\epsilon(p - x_j^2)\dot{x}_j + 2\epsilon d(\dot{x}_{j-1} - 2\dot{x}_j + \dot{x}_{j+1}) \quad (9)$$

together with the (free end) boundary conditions $x_0(t) \equiv x_1(t)$, $x_{N+1}(t) \equiv x_N(t)$; sometimes we also use periodic boundary conditions, i.e. $x_0(t) \equiv x_N(t)$, $x_1(t) \equiv x_{N+1}(t)$. On the synchronization properties of chains as given by (9), in particular the dependence on the various coupling strengths (which can also vary instead of being constant as in (9)), there exists a vast literature, we only mention the study [5].

In this paper we restrict our attention to the model (8) of $N = 2$ oscillators; in our musical application however, we consider chains of the type (9) for $N = 8$. Our goal is to study how the Arnold tongue (Figure 1) is deformed when delays are introduced onto the model.

3. INFLUENCE OF DELAYS ON SYNCHRONIZATION

3.1 Arnold tongue of synchronization

If the coupling between the oscillators occurs with certain delays, we obtain instead of (8) the following model, again considering only dissipative coupling:

$$\begin{aligned}\ddot{x}_1(t) + \omega_1^2 x_1(t) &= \mu(1 - x_1(t)^2)\dot{x}_1(t) \\ &\quad +\mu\beta_{21}(\dot{x}_2(t-\tau_A) - \dot{x}_1(t-\tau_1)), \\ \ddot{x}_2(t) + \omega_2^2 x_2(t) &= \mu(1 + \gamma - x_2(t)^2)\dot{x}_2(t) \\ &\quad +\mu\beta_{12}(\dot{x}_1(t-\tau_B) - \dot{x}_2(t-\tau_1))\end{aligned} \tag{10}$$

Here we have 3 different delays, namely τ_1, τ_A and τ_B, which are the delays of self-connection, from oscillator x_2 to x_1 and from x_1 to x_2, respectively. Similarly, we have 2 different feedback factors, namely β_{21} and β_{12}, which describe the feedback strength from oscillator x_2 to x_1 and from x_1 to x_2, respectively.

To investigate the influence of the delays on the synchronization of the oscillators, we simulated the system (10) numerically again for $\gamma = 0$ (no amplitude mismatch) and for identical delays $\tau := \tau_1 = \tau_A = \tau_B$ ranging from $\tau = 0$ (no delay) to $\tau = 2$. In Figure 2 we show the Arnold tongue of the system for various values of τ in the described interval. Here we set $\beta := \beta_{12} = \beta_{21}$; note however that in section 4 we will also consider the case $\beta_{12} \neq \beta_{21}$.

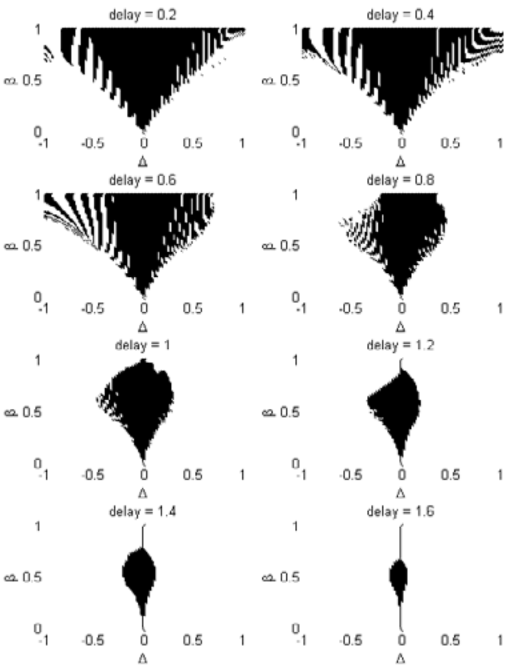

Figure 2. Synchronization area for two coupled van der Pol oscillators for various delay values

An analytical investigation of the synchronization area for growing values of τ is beyond the scope of this paper, but can be accomplished based on the analysis of the non-delayed case (see [3]) with additionally using methods for dealing with time-delay systems (see [6]). For a study of a single van der Pol oscillator with delayed self-feedback see [7].

3.2 Dependence of the beat frequency on the delay

As explained in section 2.2, outside the synchronization region, the dynamics of the system of coupled oscillators can be described by the beating frequency, namely the frequency of the relative phase of the two oscillators. In the case of an externally forced single oscillator, the beating frequency is given by the formula (6).

For large values of the coupling, besides the synchronization and beating regimes, one also observes the phenomenen of *oscillation death*. More precisely, oscillation death occures when the zero solution of the equations (10) becomes stable, which in the absence of delay it only is for large values of the coupling β. For some results on the dependence of the amplitude death region on the delay parameter τ in the case without detuning, we refer to [8]. We do not discuss this topic in detail, since it is of minor interest for our applications.

Of great relevance for our application however is the understanding of the beating regime, in particular the dependence of the beating frequency on the delay parameter τ. While an analytic discussion of the influence of the parameters τ (delay), Δ (detuning), β (coupling strength) and μ (nonlinearity) is beyond the scope of this paper, we present some results of numerical simulations. In this section we focus on investigating the combined influence of Δ and τ on the beat frequency Ω, i.e. the behavior of the beat frequency in the delay-detuning space, while in the following section, which is devoted to an implementation of the system of coupled van der Pol oscillators in Max, we focus on the behavior of Ω in the delay-coupling-space.

The following figures (Figures 3 and 4) show the beat frequency in the τ-Δ-space for $\mu = 1$ and $\beta = 0.5$; darker colors signify a higher beat frequency, i.e. the white region of the space belongs to the synchronization region.

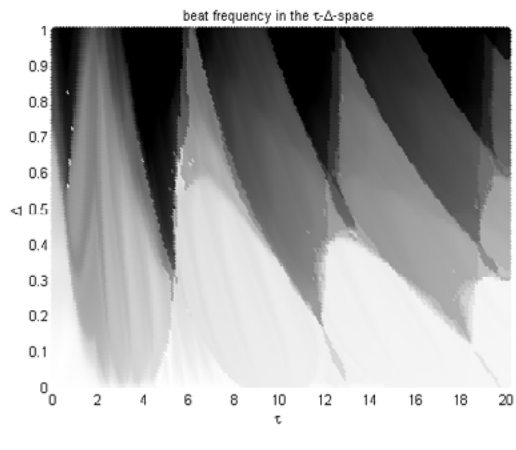

Figure 3. The beat frequency as a function of τ and Δ for $\mu = 1$ and $\beta = 0.5$

One can observe that for a given value of the delay τ, the beat frequency grows with the detuning, which is intuitively plausible, while for a given value of the detuning Δ, the beat frequency varies periodically with the delay τ, which is in good accordance with the results of the simulations in Max presented in section 4.

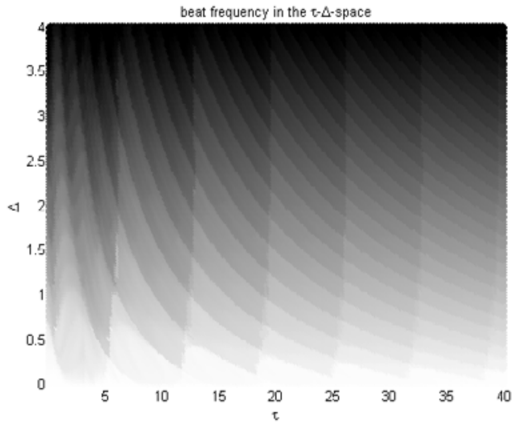

Figure 4. The beat frequency as a function of τ and Δ for $\mu = 1$ and $\beta = 0.5$

4. IMPLEMENTATION IN MAX

In order to experiment in real time we implemented the van der Pol oscillator in Max. While for the production of the figures in sections 2.3 and 3.1 we used the `ode45`- and `dde23`-method of `Matlab`, we will now show explicitly how to obtain discrete systems of the type (2), that is difference equations, from the differential equations: first by Euler's Method used in the studies 21 and then the classical Runge-Kutta method implemented in the Max-patch `icmc16_vdp.maxpat` which we used to produce the following figures. The examples are programmed as mxj~ externals. The following Java code samples are taken from the perform routine of these externals. The externals and Max patches can be downloaded from [9].

The implementation of Euler's Method is straightforward, the code is short and fast and with the sample period as time step quite precise [1]. First the acceleration is calculated according to the differential equation above (1). Then the velocity is incremented by the acceleration times *dt* and displacement x by velocity times *dt* ($dt = 1$).

```
a = (− c∗x + mu∗(1 − x∗x)∗v);
        // with c = (frequency∗2∗Pi/sr)^2
v += a;
x += (v + in[i]);
```

The classical Runge-Kutta method (often referred to as RK4) is a fourth-order method. The values x and v of the next sample are approximated in four steps. The following code sample from the mxj~ external `icmc_vdp` shows the calculation of the new values x and v using the function `f_` which calculates the acceleration.

```
double f_(double x, double v){return − c∗x + mu∗(1 −
    x∗x)∗v;}

k1 = f_(x, v);
l1 = v;
k2 = f_(x+l1/2, v+k1/2);
l2 = v+k1/2;
k3 = f_(x+l2/2, v+k2/2);
l3 = v+k2/2;
k4 = f_(x+l3, v+k3);
l4 = v+k3;
v += (k1 + 2∗k2 + 2∗k3 + k4)/6 + in[i];
x += (l1 + 2∗l2 + 2∗l3 + l4)/6;
```

The next code sample shows how a delayed mutual feedback of two oscillators is implemented. The velocities of the two oscillators (`v1` and `v2`) are stored in the circular buffers `bufv1` and `bufv2`. The differences of the delayed velocities are multiplied by the feedback factor `fbv21` and `fbv12` respectively and added to the new velocities.

```
v1 += ( (k11 + 2∗k21 + 2∗k31 + k41)/6 +
    fbv21∗(buf2v[pout2] − buf1v[pout1]) + in[i] );
x1 += (l11 + 2∗l21 + 2∗l31 + l41)/6;

v2 += ( (k12 + 2∗k22 + 2∗k32 + k42)/6 +
    fbv12∗(buf1v[pout1] − buf2v[pout2]) + in[i] );
x2 += (l12 + 2∗l22 + 2∗l32 + l42)/6;
```

In the Max-patch `icmc16_vdp.maxpat` the beats are measured and plotted in a lcd object (Figure 5) as a function of the delay (in samples).

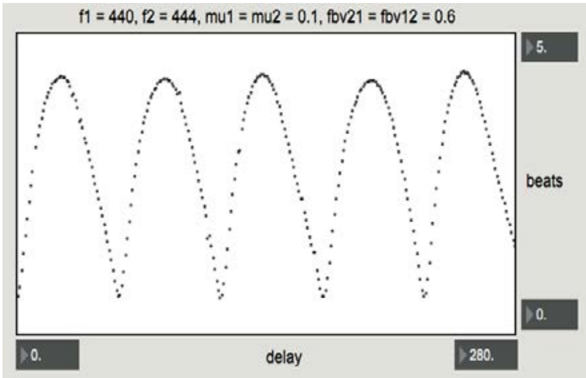

Figure 5. The beat frequency as a function of the delay

The following figures show the beat frequency as a function of the delay and the feedback in a 3D plot. More precisely, Figure 6 shows the results of the simulation of the Max-patch for the delay values 1,2, …280 samples and the feedback fbv21 = fbv12 = 0, 0.1, …, 0.7, and Figure 7 shows the analogous results for the delay values 1,2, …160 samples and the feedback fbv21 = 0.4, fbv12 = 0, 0.2, …, 3.0.

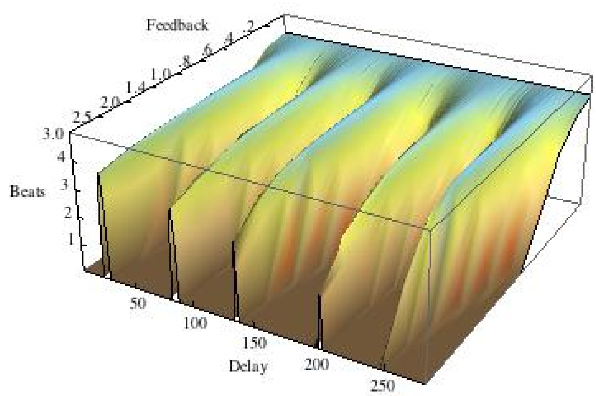

Figure 6. The beat frequency as a function of the delay and the feedback factor fbv21 = fbv12

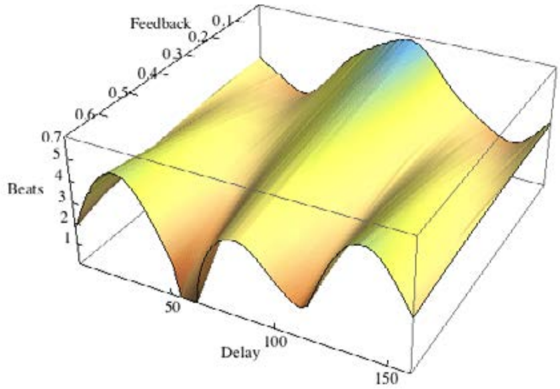

Figure 7. The beat frequency as a function of the delay and the feedback factor fbv21 (fbv12 constant)

Figure 8 shows the dependence of the beats on higher delay values.

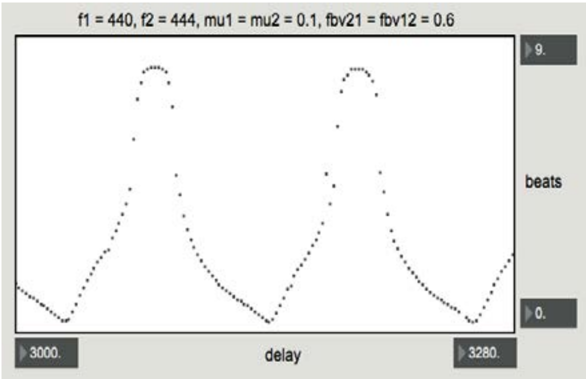

Figure 8. The beat frequency as a function of the delay for higher delay values

Especially in Figure 7, one can observe that an increase of the coupling generally leads to a decrease of the beating frequency, until it becomes zero, i.e. a transition to the synchronization region, with the exception of a periodic sequence of delay values with a higher beat frequency, which is however decreasing with increasing coupling as well. This transition towards synchronization can also be seen from the following sequence of plots (Figure 9):

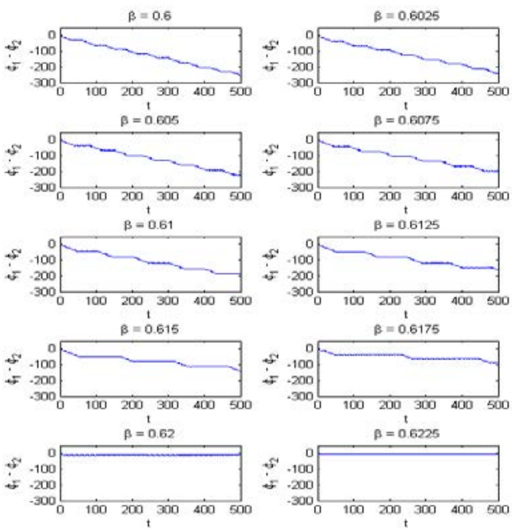

Figure 9. Transition from the beating to the synchronization regime

5. MUSICAL APPLICATIONS

In Neukoms 8-channel studies 21.1-21.9 eight van der Pol oscillators are arranged in a circle and produce the sound for the eight speakers, cf. equations (9) in the case of periodic boundary conditions. Each of these oscillators is coupled with its neighbors with variable delay times and gains in both directions. The main Max-patch contains eight joined sub-patches (Figure 10) which themselves contain the mxj∼ external `m_vdp_del` and the delay lines (Figure 11).

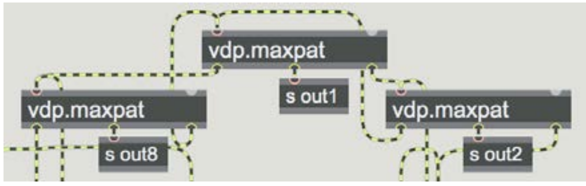

Figure 10. Three of the eight coupled sub-patches vdp.maxpat of the main Max-patch

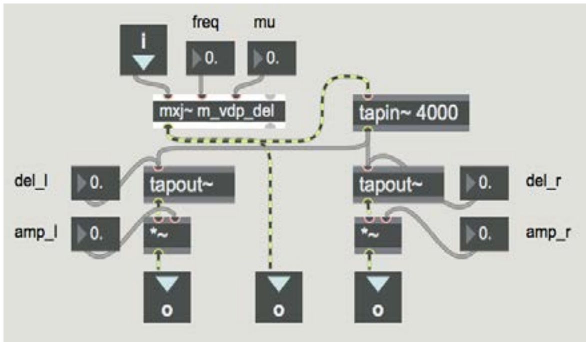

Figure 11. A simplified version of the vdp.maxpat showing the individual delays and gains to the left and to the right outlet and the direct output to the middle outlet

Two additional chains of eight van der Pol oscillators produce control functions which are used for amplitude and frequency modulation. If the frequencies of the oscillators are lower than about 20 Hz the modulations produce pulsations and vibratos. Depending on the coupling strength and the delay some or all pulsations and vibratos synchronize their frequencies. The relative phase which is not audible in audio range plays an important role in the sub-audio range: the pulsations of the single sound sources can have the same frequency while being asynchronous in a rhythmic sense. With growing coupling strength they can produce regular rhythmic patterns which are exactly in or out of phase.

The coupled van der Pol oscillators can be used as a system for purely algorithmic composition. Without changing any parameters the produced sound changes over a long time without exact repetitions. They also can be used as a stable system for improvisation with a wide range of sounds, rhythms and temporal behavior.

Some sound samples of a binaural version of Neukom's studies can be downloaded from [9].

6. REFERENCES

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