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Profinite properties of RAAGs and special groups

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Abstract

In this paper we prove that RAAGs are distinguished from each other by their pro-p completions for any choice of prime p, and that RACGs are distinguished from each other by their pro-2 completions. We also give a new proof that hyperbolic virtually special groups are good in the sense of Serre. Furthermore we give an example of a property of the underlying graph of a RAAG that translates to a property of the profinite completion.

Right-angled Artin groups (RAAGs) have been the subject of much recent interest, especially because of their rich subgroup structure; in particular every special group embeds in a RAAG [**7**]. Furthermore RAAGs are linear and have excellent residual properties. Here we will show that RAAGs, and the closely related right-angled Coxeter groups (RACGs), are in fact completely determined by their finite quotient groups. The proofs will rely principally on the cohomological rigidity result of Koberda [**9**].

First let us recall some definitions.

Definition 1. *Given a (finite simplicial) graph Γ, the* right-angled Artin group $A(\Gamma)$ *is the group with generating set $V(\Gamma)$ with the relation that vertices v, w commute imposed whenever v, w span an edge of Γ. The* right-angled Coxeter group $C(\Gamma)$ *is the quotient of $A(\Gamma)$ with the additional constraint that each generator has order 2.*

Definition 2. *Given a discrete group G, the* profinite completion *of G is the inverse limit of the system of groups*

$$\hat{G} = \varprojlim_{N \triangleleft_f G} G/N$$

where N ranges over the finite index normal subgroups of G. This is a compact Hausdorff topological group. Similarly one may define the pro-p completion $\hat{G}_{(p)}$ as the inverse limit of all finite quotients of G which are p-groups, for p a prime.

The isomorphism type of a right-angled Artin group $A(\Gamma)$ uniquely determines the graph Γ up to isomorphism; this fact was first established by Droms [**3**]. We use a stronger cohomological criterion proved by Koberda [**9**], who builds on earlier work of Subalka [**16**], Droms [**4**], Gubeladze [**6**] and Charney and Davis [**2**].

Theorem (Koberda [**9**]). *Let Γ, Γ' be finite graphs. Then $\Gamma \cong \Gamma'$ if and only if there is an isomorphism of cohomology groups*

$$H^*(A(\Gamma); \mathbb{Q}) \cong H^*(A(\Gamma'); \mathbb{Q})$$

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in dimensions one and two, which respects the cup product.

The proof relied solely on the following fact: each vertex $v \in \Gamma$ is dual to a cohomology class $f_v \in H^1(A(\Gamma);\mathbb{Q})$ for which the map

$$f_v \smile \bullet : H^1(A(\Gamma);\mathbb{Q}) \to H^2(A(\Gamma);\mathbb{Q})$$

has rank precisely the degree of the vertex v; moreover a vertex $w \in \Gamma$ is adjacent to v precisely if $f_v \smile f_w$ is non-zero.

Now the class $f_v \smile f_w$ is dual to an embedded 2-torus in the Salvetti complex of $A(\Gamma)$, hence gives a primitive element of $H^2(A(\Gamma);\mathbb{Z})$. It follows that changing the coefficient field $\mathbb{Q}$ to a finite field $\mathbb{Z}/p$ (for p a prime) changes neither the rank of the above map, nor the adjacency condition following it. Hence Koberda's cohomological rigidity result also holds with coefficient field $\mathbb{Z}/p$.

It remains to show that the pro-p completion of our right-angled Artin group detects the cohomology in dimensions one and two, and the cup product. As we will discuss later, it is frequently the case for groups arising in low-dimensional topology that the cohomology of a group is determined by its profinite completion. We always have substantial control over the cohomology in dimensions one and two. See [**18**] for definitions and basic properties of profinite cohomology; the definitions largely parallel those for discrete groups. In particular there is a natural notion of cup product and the natural map from G to its profinite completion induces a map on cohomology respecting the cup product.

PROPOSITION 3. *Let G be a discrete group and p a prime.*

- $H^1(\widehat{G}_{(p)};\mathbb{Z}/p) \to H^1(G;\mathbb{Z}/p)$ *is an isomorphism;*
- $H^2(\widehat{G}_{(p)};\mathbb{Z}/p) \to H^2(G;\mathbb{Z}/p)$ *is injective; and*
- *if H^{1+1} denotes that part of second cohomology generated by cup products of elements of H^1, then* $H^{1+1}(\widehat{G}_{(p)};\mathbb{Z}/p) \to H^{1+1}(G;\mathbb{Z}/p)$ *is an isomorphism;*

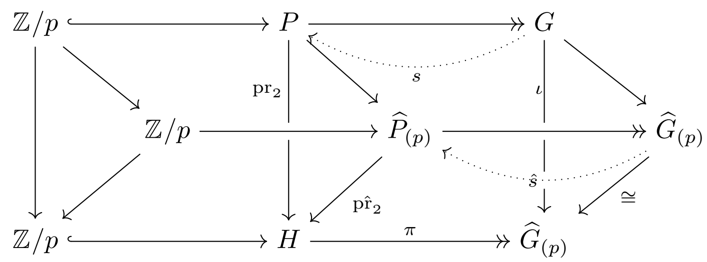
where all the maps are the natural ones induced by $G \to \widehat{G}_{(p)}$.

Proof. The first point is a trivial consequence of the fact that $H^1(-;\mathbb{Z}/p)$ is naturally isomorphic to $\mathrm{Hom}(-;\mathbb{Z}/p)$ in either the category of discrete groups or the category of pro-p groups. The third point follows from the first two and naturality of the cup product. The second is a special case of Exercise 2.6.1 of [**18**]; we give here an explicit proof in dimension two in terms of extensions.

Recall that $H^2(-;\mathbb{Z}/p)$ classifies central extensions of G by $\mathbb{Z}/p$ both for discrete and pro-p groups (see Section 6.8 of [**15**] for the profinite theory). Take a central extension H of $\widehat{G}_{(p)}$ by $\mathbb{Z}/p$ representing $\xi \in H^2(\widehat{G}_{(p)};\mathbb{Z}/p)$. Then the pull-back

$$P = \{(g,h) \in G \times H \text{ such that } \pi(h) = \iota(g)\}$$

(where $\pi : H \to \widehat{G}_{(p)}$ and $\iota : G \to \widehat{G}_{(p)}$ are the obvious maps) gives a central extension of G by $\mathbb{Z}/p$ representing $\iota^*(\xi)$. If $\iota^*(\xi) = 0$ in H^2 then the extension splits; so there is a group-theoretic section $s : G \to P$. Now s induces a map $\hat{s} : \widehat{G}_{(p)} \to \widehat{P}_{(p)}$. Furthermore H is a pro-p group so that the projection $\mathrm{pr}_2 : P \to H$ induces a map $\mathrm{pr}_2 : \widehat{P}_{(p)} \to H$; then $\hat{\mathrm{pr}}_2\hat{s}$ is a section of $H \to \widehat{G}_{(p)}$ and so ξ was a trivial extension also.



□

Remark. Note that a diagram chase applied to the lower parallelogram in the above diagram shows that in fact $H \cong \widehat{P}_{(p)}$. Thus the above analysis also illustrates why the map on H^2 may fail to be surjective; for any central extension of $\widehat{G}_{(p)}$ yielding a given extension P of G must be $\widehat{P}_{(p)}$; however there is no *a priori* reason that the map from $\mathbb{Z}/p$ to $\widehat{P}_{(p)}$ need be injective. See the example at the end of [**12**] for an example where the map on H^2 fails to be surjective.

Recall that for a RAAG, the dimension two cohomology is in fact generated by cup products; thus in dimensions one and two, the algebra $H^*(A(\Gamma);\mathbb{Z}/p)$ is determined by the pro-p completion $\widehat{A(\Gamma)}_{(p)}$; hence we have proved

Theorem 4. *Let Γ, Γ' be finite graphs and p a prime. Then $\widehat{A(\Gamma)}_{(p)} \cong \widehat{A(\Gamma')}_{(p)}$ if and only if $\Gamma \cong \Gamma'$.*

Note that *a fortiori* an isomorphism of profinite completions also forces the graphs to be isomorphic. In fact much more is true about the cohomology of the pro-p completion of $A(\Gamma)$; in particular:

Theorem 5 (Lorensen [**11**], [**12**]). *The map from a right-angled Artin group to its pro-p completion (or profinite completion) induces an isomorphism of mod-p cohomology for any prime p.*

We can extend Theorem 4 to right-angled Coxeter groups by noting that there are natural isomorphisms

$$H^1(C(\Gamma);\mathbb{Z}/2) \cong H^1(A(\Gamma);\mathbb{Z}/2)$$

and

$$H^2(C(\Gamma);\mathbb{Z}/2) \cong H^2(A(\Gamma);\mathbb{Z}/2) \oplus (\mathbb{Z}/2)^{|V(\Gamma)|}$$

where the second summand above derives from the relations $v^2 = 1$. The quotient map on H^2 which restricts to the isomorphism of the first summand with $H^2(A(\Gamma);\mathbb{Z}/2)$ is induced by the natural map $A(\Gamma) \to C(\Gamma)$. This quotient map is unique (i.e. does not depend on the presentation of $C(\Gamma)$ as a particular right-angled Coxeter group) in the following sense. Modulo 2, we have the relations $(a+b)^2 = a^2 + b^2$ so that the image of the squaring map $a \to a \smile a$ is a subgroup Σ of $H^2(C(\Gamma))$, the image of the diagonal subgroup of $(H^1(C(\Gamma)))^2$. The second summand $(\mathbb{Z}/2)^{|V(\Gamma)|}$ is precisely this subgroup Σ.

Thus the structure of the algebra $H^*(A(\Gamma))$ in dimensions one and two is determined by the behaviour of $H^*(C(\Gamma);\mathbb{Z}/2)$ in those dimensions, with the cup product map being given by the canonical map

$$(H^1(A(\Gamma)))^2 \stackrel{\cong}{\to} (H^1(C(\Gamma)))^2 \stackrel{\smile}{\to} H^2(C(\Gamma)) \to H^2(C(\Gamma))/\Sigma \cong H^2(A(\Gamma))$$

described above. Proposition 3 shows this algebra to be an invariant of the pro-2 completion, so that we have:

THEOREM 6. *Let Γ, Γ' be finite graphs. Then $\widehat{C(\Gamma)}_{(2)} \cong \widehat{C(\Gamma')}_{(2)}$ if and only if $\Gamma \cong \Gamma'$.*

Proposition 3 was sufficient to prove the Theorem; in fact right-angled Coxeter groups are 2-good, so that we have an isomorphism on cohomology in all dimensions. This follows from extension properties of 2-goodness applied to Proposition 9 of [**10**] or from the work on grpah products in [**17**].

We made heavy use of the cohomology of the profinite completions of RAAGs and RACGs, so let us digress and study the following property. A group G is *good* if the natural map on cohomology induced by $G \to \hat{G}$ is an isomorphism

$$H^n(\hat{G}; M) \stackrel{\cong}{\to} H^n(G; M)$$

for every finite G-module M and every $n \geq 1$.

Note that this map $G \to \hat{G}$ for any group G respects the cup product; for the cup product is defined for cohomology of profinite groups by precisely the same formulae as for abstract groups. Thus for a good group $G \to \hat{G}$ will not only induce an isomorphism of groups, but an isomorphism of graded algebras $H^*(\hat{G}; M) \cong H^*(G; M)$ under the cup product.

Goodness is preserved under taking extensions by good groups (under some mild conditions; see [**18**]), and passing to finite index subgroups or overgroups. Surface groups are good, as follows immemdiately from the exercises in [**18**]. Hence all virtually-fibred 3-manifold groups are good; due to work of Agol [**1**] and Wise [**19**] this includes all finite volume hyperbolic 3-manifolds.

Goodness is also preserved under taking amalgamated free products and HNN extensions, given suitable conditions. Recall that the *full profinite topology* on a group H is the topology induced by the map $H \to \hat{H}$. An amalgamated free product $G = A *_C B$ is said to be *efficient* if G is residually finite, if A, B, C are closed in the profinite topology on G, and if G induces the full profinite topology on A, B and C. In particular if G is LERF (and A, B, C are finitely generated) this condition will be satisfied. Similarly one may define efficient HNN extensions. Then:

THEOREM 7 (Proposition 3.6 of [**5**]). *An efficient amalgamated free product or HNN extension of good groups is good.*

We may now prove that all hyperbolic virtually special groups are good. This was proved by Schreve [**17**] and later by Minasyan and Zalesskii [**13**], both using virtual retraction properties; we give another proof using cube complex hierarchies. Hierarchies have also been used to prove goodness in other contexts, in particular in [**5**].

DEFINITION 8. *Let $\mathcal{QVH}$ be the smallest class of hyperbolic groups closed under the following operations.*

(i) *The trivial group is in $\mathcal{QVH}$.*

(ii) *If* $A, B, C \in \mathcal{QVH}$ *and* C *is quasiconvex in* $G = A *_C B$, *then* $G \in \mathcal{QVH}$.
(iii) *If* $A, B \in \mathcal{QVH}$ *and* B *is quasiconvex in* $G = A*_B$, *then* $G \in \mathcal{QVH}$.
(iv) *If* H *is commensurable to* G *and* $G \in \mathcal{QVH}$, *then* $H \in \mathcal{QVH}$.

We have the following useful characterisation of the groups in $\mathcal{QVH}$.

Theorem (Wise [**19**]). *A hyperbolic group is in* $\mathcal{QVH}$ *if and only if it is virtually special.*

Furthermore recall

Theorem (Haglund-Wise [**7**]). *Quasi-convex subgroups of hyperbolic virtually special groups are separable.*

With this in mind we can now prove.

Theorem 9. *Hyperbolic virtually special groups are good.*

Proof. As noted above, we are free to pass to an arbitrary finite index subgroup of G and prove goodness there. We define a measure of complexity for a special group H. Set $n(H)$ to be the minimal dimension of a CAT(0) cube complex X on which H acts with special quotient. After subdividing, a hyperplane in this complex is an embedded 2-sided cubical subcomplex, and H splits as an HNN extension or amalgamated free product over the stabiliser of this hyperplane. Iterating this process yields a rooted tree of groups in which each vertex has either two or three descendants (depending on whether the vertex splits as an HNN extension or amalgamated free product). Because H is in $\mathcal{QVH}$, this tree is finite and each branch terminates in the trivial group. Set $m(H)$ to be the length of a longest branch over all such trees with minimal diameter; that is, the length of a shortest hierarchy for H. Now define the *quasiconvex hierarchy complexity* $\mu(G)$ of a special group G to be the pair of integers $(n(G), m(G))$; order the pairs $(n, m) \in \mathbb{N} \times \mathbb{N}$ lexicographically.

Now assume that all hyperbolic special groups H with $\mu(H) < \mu(G)$ are good. We have a splitting of G either as $A *_C B$ or $A*_C$. Note that A, B, C are hyperbolic. Now C is the stabiliser of a hyperplane; this is a CAT(0) cube complex whose quotient by C is special and for which the intersections with other hyperplanes of X give a quasiconvex hierarchy; hence $n(C) < n(G)$, so $\mu(C) < \mu(G)$ and so C is good. Furthermore A and B have shorter hierarchies than G, so whether or not $n(A) = n(G)$, the complexities $\mu(A) < \mu(G)$ and $\mu(B) < \mu(G)$ do strictly decrease. Thus A, B, C are good. Furthermore quasiconvex subgroups of G are separable. All finite index subgroups of A, B, C are quasiconvex so the splitting is efficient and we may apply Theorem 7 to conclude that G is good. Note that the base case for the induction is simply the trivial group. □

Recalling that Haglund and Wise [**8**] proved that all Coxeter groups are virtually special, we have:

Corollary 10. *Hyperbolic Coxeter groups are good.*

For right-angled Artin groups, Theorem 5 guaranteed that in fact the mod-p cohomology is determined by the pro-p completion. This property, which is sometimes called *p-goodness*, is rather rarer than straightforward goodness; in particular proofs will often require strong separability constraints in which only p-group quotients are available. These constraints are difficult to obtain in general.

We move now to a result of a rather different flavour. Often, properties of the underlying graph of a right-angled Artin or Coxeter group are expressible as group theoretic properties. As an example of such a property carrying over to the profinite world, we prove the following Theorem.

THEOREM 11. *Let Γ be a graph. Then $\widehat{A(\Gamma)}$, respectively $\widehat{C(\Gamma)}$, splits as a non-trivial profinite free product $H_1 \amalg H_2$ if and only if Γ is disconnected.*

The proof will call upon the theory of actions on profinite trees developed by, among others, Ribes and Zalesskii. The theory is contained in the unpublished book [**14**]; the closely related pro-p version may be found in published form in [**15**].

Proof of Theorem 11. If Γ is disconnected the result follows directly from the abstract case. So suppose that Γ is connected and that $G = \widehat{A(\Gamma)}$ splits as a profinite free product $H_1 \amalg H_2$. The case when Γ is a point is easy, so suppose that Γ is not a point.

The splitting of G as a free profinite product induces an action of G on a profinite tree T, where vertex stabilisers are precisely the conjugates of the H_i (see Lemma 5.3.1 of [**14**]). All edge stabilisers are trivial, so that no element of G can fix more than one point of T. By Proposition 3.2.3 of [**14**], any abelian group acting on T either fixes a point or is a subgroup of $\hat{\mathbb{Z}}$. Each of the standard generators of $A(\Gamma)$ is contained in a copy of $\mathbb{Z}^2$ as there is some edge adjacent to the corresponding vertex; and these copies of $\mathbb{Z}^2$ are retracts of the whole RAAG, hence give an inclusion of $\hat{\mathbb{Z}}^2$ in the profinite completion. Hence every generator of G fixes some (unique) vertex of T, and so is contained in a (unique) conjugate of H_1 or H_2.

Note that for every edge $e = [v, w]$ of Γ, the subgroup $\overline{<v, w>}$ is a rank 2 free abelian group so that v, w fix the same vertex of T; by connectedness of Γ, it follows that all of $\widehat{A(\Gamma)}$ fixes a vertex of T and so $H_1 = 1$ or $H_2 = 1$.

The case of a right-angled Coxeter group is similar; indeed it is easier, as all the generators have finite order, and therefore fix a vertex of T by Theorem 3.1.7 of [**14**]. □

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