Numerical analysis of a two-parameter fractional telegraph equation

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Abstract

In this paper we consider the two-parameter fractional telegraph equation of the form

$$-{}^C D_{t_0^+}^{\alpha+1} u(t,x) + {}^C D_{x_0^+}^{\beta+1} u(t,x) - {}^C D_{t_0^+}^{\alpha} u(t,x) - u(t,x) = 0.$$

Here ${}^C D_{t_0^+}^{\alpha}$, ${}^C D_{t_0^+}^{\alpha+1}$, ${}^C D_{x_0^+}^{\beta+1}$ are operators of the Caputo-type fractional derivative, where $0 \leq \alpha < 1$ and $0 \leq \beta < 1$. The existence and uniqueness of the equations are proved by using the Banach fixed point theorem. A numerical method is introduced to solve this fractional telegraph equation and stability conditions for the numerical method are obtained. Numerical examples are given in the final section of the paper.

Keywords:
Fractional partial differential equation, fractional telegraph equation, finite difference method, stability, Mittag-Leffler function
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1. Introduction

The use of fractional partial differential equations in mathematical models has become increasingly popular in recent years. Some fractional partial differential equations such as the one-dimensional time-fractional diffusion-wave equation were successfully used for modeling relevant physical processes, see, for example, Caputo [1], Giona and Roman [9] , Hilfer [11], Mainardi [17], Mainardi and Tomirotti [18], Metzler et al. [19], Pipkin [23], Podlubny [24], etc.

The fractional telegraph equation has been considered recently by several authors. Cascaval et al. [2] considered the well-posedness and the asympototic behavior of the time-fractional telegraph equation by using the Riemann-Liouville approach. Orsingher and Beghin [22] discussed telegraph processes with Brownian time and showed that some processes are governed by time-fractional telegraph equations. Chen et al. [3] examined and derived a solution of the time-fractional telegraph equation with three kinds of nonhomogeneous boundary conditions by using the method of separation of variables. Recently, in [28] the second author dealt with a general operational approach to describe the fundamental solutions of the two-parameter fractional telegraph equation in the rectangular domains.

The Mittag-Leffler function was introduced by Mittag-Leffler, in connection with his method of summation of some divergent series. In his papers [20], [21], he investigated certain properties of this function. The Mittag-Leffler function arises naturally as a solution of fractional order integral or differential equations, and especially in the investigation of fractional generalizations of the kinetic equation, random walks, Lévy

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flights, super-diffusive transport and in the study of complex systems. This function also occurs in the solution of certain boundary value problems involving fractional integro-differential equations of Volterra type [27].

Numerical methods for fractional partial differential equations have been studied by many authors. The numerical methods include finite difference methods by Liu et al. [16], Langlands and Henry [13]. More recently, Lin and Xu [15] proposed a finite difference scheme in time and Legendre spectral method in space for time fractional partial differential equation. A space-time spectral method for space-time fractional partial differential equation has been studied in [14]. A Galerkin finite element approximation for variational solution to the steady state fractional advection dispersion equations has been studied in [7], [8] and [10]. In this paper, we will consider a finite difference method for the two-parameter fractional telegraph equation and a stability condition of the numerical method is obtained.

The paper is organized as follows. In the preliminaries, we recall some basic properties of the Mittag-Leffler function and some necessary elements of the fractional calculus. In Section 3 we will deduce the Green function associated with the fractional differential equation under consideration. In Section 4, we study the existence and uniqueness of solutions by using the Banach fixed point theorem. In Section 5, we introduce a numerical method to solve an equation of this type and in Section 6, we consider the stability of the numerical method. Finally, in Section 7, some numerical examples are given.

2. Preliminaries

2.1. Special Functions

The function

$$E_{\alpha,1}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma\left(\alpha k+1\right)}$$

was introduced by Mittag-Leffler in 1903 [20]. When $0<\alpha<2$ and μ is a real number such that

$$\frac{\pi\alpha}{2}<\mu<\min\left\{\pi,\pi\alpha\right\},$$

then for $N^*\in\mathbb{N}$, $N^*\neq 1$ the following asymptotic expansions hold:

$$E_{\alpha,1}(z) = \frac{1}{\alpha}z^{\frac{1-\alpha}{\alpha}}e^{z^{\frac{1}{\alpha}}}-\sum_{r=1}^{N^*}\frac{1}{\Gamma\left(1-\alpha r\right)\ z^r}+O\left(\frac{1}{z^{N^*+1}}\right),\quad |z|\to\infty,\ |\arg(z)|\leq\mu, \tag{1}$$

$$E_{\alpha,1}(z) = -\sum_{r=1}^{N^*}\frac{1}{\Gamma\left(1-\alpha r\right)\ z^r}+O\left(\frac{1}{z^{N^*+1}}\right),\quad |z|\to\infty,\ \mu\leq|\arg(z)|\leq\pi. \tag{2}$$

Here $\mathbb{N}$ denotes the set of natural numbers.

The so called three-parameter Mittag-Leffler function is

$$E^{\rho}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(\rho)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!},$$

where $Re(\alpha)>0, Re(\beta)>0$ and $Re(\rho)>0$ and $z\in\mathbb{C}$.

It is known [27] that the Mittag-Leffler function can be expressed as the inverse Laplace transform of a rational function, namely

$$\mathcal{L}^{-1}\left\{\frac{m!s^{\alpha-\gamma}}{(s^{\alpha}+a)^{m+1}}\right\}=t^{m\alpha+\gamma-1}E^{m}_{\alpha,\gamma}(-at^{\alpha}),$$

and

$$\mathcal{L}^{-1}\left\{\frac{s^{\rho-1}}{(s^\alpha+as^\beta+b)}\right\}=t^{\alpha-\rho}\sum_{r=0}^{\infty}(-a)^r t^{(\alpha-\beta)r}E^{r+1}_{\alpha,\alpha+1-\rho+(\alpha-\beta)r}(-bt^\alpha), \tag{3}$$

where $\left|\frac{as^\beta}{s^\alpha+b}\right|<1$ and $\alpha\geq\beta$.

2.2. Fractional Calculus

We recall some definitions of fractional derivatives and fractional integrals. Let $\Gamma(\cdot)$ denote the Gamma function. For any positive integer n and $n-1\leq\gamma<n$, the Caputo derivative and Riemann-Liouville derivative of order γ are defined, respectively, by

- Caputo derivative

$$^{C}D^{\gamma}_{a^+}v(x)=\frac{1}{\Gamma(n-\gamma)}\int_a^x\frac{v^{(n)}(t)}{(x-t)^{\gamma-n+1}}\,dt,\quad a\leq x\leq b,\quad n-1\leq\gamma<n,$$

  where $v^{(n)}(t)=\frac{d^n v(t)}{dt^n}$.

- Riemann-Liouville derivative

$$^{R}D^{\gamma}_{a^+}v(x)=\frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dx^n}\int_a^x\frac{v(t)}{(x-t)^{\gamma-n+1}}\,dt,\quad a\leq x\leq b,\quad n-1\leq\gamma<n.$$

We have the following Lemma

Lemma 2.1. *[27]*
Let $\gamma\geq 0$, $n-1\leq\gamma<n$, $n\in\mathbb{N}$ and $v\in C^n([a,b])$. Then $^{C}D^{\gamma}_{a^+}v(x)$ and $^{R}D^{\gamma}_{a^+}v(x)$ exist almost everywhere and

$$^{R}D^{\gamma}_{a^+}v(x)={}^{C}D^{\gamma}_{a^+}v(x)+\sum_{k=0}^{n-1}\frac{v^{(k)}(a)}{\Gamma(1+k-\gamma)}(x-a)^{k-\gamma},\quad a\leq x\leq b,\quad n-1\leq\gamma<n. \tag{4}$$

Denote $T_{n-1}[v;a](x)$ the $(n-1)$th Taylor expansion of the function $v(x)$ about a, i.e.,

$$T_{n-1}[v;a](x)=v(a)+\frac{v'(a)}{1!}(x-a)+\cdots+\frac{v^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1},\quad n\geq 1.$$

It is easy to calculate that

$$^{R}D^{\gamma}_{a^+}\big(T_{n-1}[v;a]\big)(x)=\sum_{k=0}^{n-1}\frac{v^{(k)}(a)}{\Gamma(1+k-\gamma)}(x-a)^{k-\gamma},\quad a\leq x\leq b,\quad n-1\leq\gamma<n,$$

which implies that, by Lemma 2.1,

$$^{C}D^{\gamma}_{a^+}v(x)={}^{R}D^{\gamma}_{a^+}\big(v-T_{n-1}[v;a]\big)(x),\quad a\leq x\leq b,\quad n-1\leq\gamma<n. \tag{5}$$

Lemma 2.2. *[12]*
Let $\gamma\geq 0$, $n-1\leq\gamma<n$, $n\in\mathbb{N}$ and $v\in C^n(\mathbb{R}^+)$. Assume that $v^{(n)}\in L^1(0,b)$ for any $b>0$ and $|v^{(n)}(x)|\leq Be^{bx}$ for some constant $B>0$. Further assume that the Laplace transform $\mathcal{L}v(s)$ and $\mathcal{L}v^{(n)}(s)$ exist, and $\lim_{x\to+\infty}v^{(k)}(x)=0$, $k=0,1,\dots,n-1$. Then we have

$$(\mathcal{L}\,{}^{C}D^{\alpha}_{0^+}v)(s)=s^\alpha\mathcal{L}v(s)-\sum_{k=0}^{n-1}s^{\alpha-k-1}\mathcal{L}v^{(k)}(0).$$

3. Green function for the fractional telegraph equation

The aim of this section is to obtain an expression for the Green function associated with the two-parameter fractional telegraph equation, with $0 \le \alpha < 1$, $0 \le \beta < 1$,

$$-\ {}^{C}D_{t_0^+}^{\alpha+1}u(t,x) + {}_{-\infty}^{C}D_{x}^{\beta+1}u(t,x) - {}^{C}D_{t_0^+}^{\alpha}u(t,x) - u(t,x) = 0, \quad t \ge t_0, \tag{6}$$

$$u(t_0,x) = u_{0x}(x),\ u_t(t_0,x) = u_{1x}(x), \tag{7}$$

where

$${}_{-\infty}^{C}D_{x}^{\beta+1}u(t,x) = \frac{1}{\Gamma(1-\beta)}\int_{-\infty}^{x}\frac{v^{(2)}(t)}{(x-t)^{\beta}}\,dt,$$

and where u_t denotes the partial derivative $\frac{\partial u}{\partial t}$.

Denote $U(s,x) = \displaystyle\int_{t_0}^{\infty} e^{-st}u(t,x)\ dt$ the Laplace transform (see [30]) of u with respect to t and denote $\bar{u}(t,w) = \frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}} e^{-iwx}u(t,x)\ dx$ the Fourier transform of u with respect to x. Applying the Laplace transform with respect to t and then the Fourier transform with respect to x in (6), we obtain, by Lemma 2.2,

$$-s^{\alpha+1}\bar{U}(s,w) + s^{\alpha}\overline{u_{0x}}(w) + s^{\alpha-1}\overline{u_{1x}}(w) + (iw)^{\beta+1}\bar{U}(s,w) - s^{\alpha}\bar{U}(s,w) + s^{\alpha-1}\overline{u_{0x}}(w) - \bar{U}(s,w) = 0, \tag{8}$$

or

$$\bar{U}(s,w) = \frac{s^{\alpha}\overline{u_{0x}}(w)}{s^{\alpha+1}+s^{\alpha}-(iw)^{\beta+1}+1} + \frac{s^{\alpha-1}\big(\overline{u_{1x}}(w)+\overline{u_{0x}}(w)\big)}{s^{\alpha+1}+s^{\alpha}-(iw)^{\beta+1}+1}. \tag{9}$$

Applying the inverse Laplace transforms in (9), we have, by (3),

$$\begin{aligned}
\bar{u}(t,w) &= \mathcal{L}^{-1}\left\{\frac{s^{\alpha}}{s^{\alpha+1}+s^{\alpha}-(iw)^{\beta+1}+1}\right\}\overline{u_{0x}}(w) \\
&\quad + \mathcal{L}^{-1}\left\{\frac{s^{\alpha-1}}{s^{\alpha+1}+s^{\alpha}-(iw)^{\beta+1}+1}\right\}(\overline{u_{1x}}(w)+\overline{u_{0x}}(w)) \\
&= \sum_{r=0}^{\infty}(-1)^r t^r\left\{E_{\alpha+1,1+r}^{r+1}\left(((iw)^{\beta+1}-1)t^{\alpha+1}\right) + tE_{\alpha+1,2+r}^{r+1}\left(((iw)^{\beta+1}-1)t^{\alpha+1}\right)\right\}\overline{u_{0x}}(w) \\
&\quad + \sum_{r=0}^{\infty}(-1)^r t^r\left\{tE_{\alpha+1,2+r}^{r+1}\left(((iw)^{\beta+1}-1)t^{\alpha+1}\right)\right\}\overline{u_{1x}}(w).
\end{aligned} \tag{10}$$

Applying the inverse Fourier transform in (10), we obtain the solution of (6),

$$\begin{aligned}
u(t,x) &= \frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}} e^{-iwx}\Big\{\sum_{r=0}^{\infty}(-1)^r t^r\Big(E_{\alpha+1,1+r}^{r+1}\big(((iw)^{\beta+1}-1)t^{\alpha+1}\big) \\
&\quad + tE_{\alpha+1,2+r}^{r+1}\big(((iw)^{\beta+1}-1)t^{\alpha+1}\big)\Big)\overline{u_{0x}}(w) + tE_{\alpha+1,2+r}^{r+1}\big(((iw)^{\beta+1}-1)t^{\alpha+1}\big)\Big\}\overline{u_{1x}}(w)\,dw.
\end{aligned} \tag{11}$$

Note that

$$u_{0x}(x) = \frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}} e^{-ixw}\overline{u_{0x}}(w)\ dw, \tag{12}$$

$$u_{1x}(x) = \frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}} e^{-ixw}\overline{u_{1x}}(w)\ dw, \tag{13}$$

are the inverse Fourier transforms of $\overline{u_{0x}}(w)$ and $\overline{u_{1x}}(w)$, respectively. By using the convolution theorem ([29], Theorem 40), we have, by (11),

$$u(t,x) = \int_{-\infty}^{+\infty} \Big(G_{\alpha,\beta}(t, x-y) u_{0x}(y) + F_{\alpha,\beta}(t, x-y) u_{1x}(y) \Big) \, dy, \tag{14}$$

where the corresponding Green functions $G_{\alpha,\beta}(t,\xi)$ and $F_{\alpha,\beta}(t,\xi)$ take the following forms

$$\begin{aligned} G_{\alpha,\beta}(t,\xi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iw\xi} \Big(\sum_{r=0}^{\infty} (-1)^r t^r \Big\{ & E^{r+1}_{\alpha+1,1+r}\big(((iw)^{\beta+1} - 1)t^{\alpha+1}\big) \\ & + t E^{r+1}_{\alpha+1,2+r}\big(((iw)^{\beta+1} - 1)t^{\alpha+1}\big) \Big\} \Big) \, dw, \end{aligned}$$

$$F_{\alpha,\beta}(t,\xi) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iw\xi} \Big(\sum_{r=0}^{\infty} (-1)^r t^r t E^{r+1}_{\alpha+1,2+r} \big(((iw)^{\beta+1} - 1)t^{\alpha+1}\big) \Big) \, dw, \tag{15}$$

respectively.

4. Existence and Uniqueness

In this section we will consider the existence and uniqueness of the solutions for the following two-parameter fractional telegraph equation, with $0 \le \alpha < 1,\ 0 \le \beta < 1,\ t_0 \le t \le T_0$, and $x_0 \le x \le X_0$,

$$-\left({}^C D^{\alpha+1}_{t_0^+} u\right)(t,x) + \left({}^C D^{\beta+1}_{x_0^+} u\right)(t,x) - \left({}^C D^{\alpha}_{t_0^+} u\right)(t,x) - u(t,x) = 0, \tag{16}$$

$$u(t_0, x) = u_{0x}(x),\ u_t(t_0, x) = u_{1x}(x), \tag{17}$$

$$u(t, x_0) = u_{0t}(t),\ u_x(t, x_0) = u_{1t}(t), \tag{18}$$

where u_t, u_x denote the partial derivatives $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}$, respectively.

Let m be a nonnegative integer. Denote $C^m = C^m([a,b])$ the space of m times continuously differentiable functions with the norm

$$\|v\|_{C^m} = \sum_{k=0}^{m} \|v^{(k)}\|_C = \sum_{k=0}^{m} \max_{x \in I} |v^{(k)}(x)|.$$

In particular, for $m = 0$, $C^0 = C([a,b])$ is the space of continuous functions v on $[a,b]$ with the norm $\|v\|_C = \max_{x\in I} |v(x)|$.

Lemma 4.1. *[12]*

Let $\gamma \ge 0,\ n-1 \le \gamma < n$. If $v \in C^n([a,b])$, then

$$\left| {}^C D^{\gamma}_{a+} v(x) \right| \le \frac{\|v^{(n)}\|_C}{\Gamma(n-\gamma)(n-\gamma+1)} (x-a)^{n-\gamma}. \tag{19}$$

Theorem 4.2. *Let*

$$\xi = \frac{2(T_0 - t_0)^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)} + \frac{(X_0 - x_0)^{1-\beta}}{\Gamma(1-\beta)(2-\beta)}.$$

Assume that $0 < \xi < 1$. Then the system (16) - (18) *has a unique solution.*

Proof: We denote by X the Banach space

$$X = \{u \mid u(\cdot, x) \in C^2([t_0, T_0]),\ u(t, \cdot) \in C^2([x_0, X_0])\}$$

and by Y the Banach space

$$Y = \{u \mid u(\cdot, x) \in C([t_0, T_0]),\ u(t, \cdot) \in C([t_0, T_0])\}.$$

Define the mapping $T : X \to Y$,

$$(Tu)(t, x) = -\ {}^C D_{t_{0+}}^{\alpha+1} u(t, x) +\ {}^C D_{x_{0+}}^{\beta+1} u(t, x) -^C D_{t_0^+}^{\alpha} u(t, x).$$

Then the system (16) - (18) can be written as

$$u(t, x) = (Tu)(t, x).$$

By (19), we have

$$\begin{aligned}
\|Tu_1 - Tu_2\|_Y &= \Big\| -\ {}^C D_{t_{0+}}^{\alpha+1} \big(u_1(t, x) - u_2(t, x)\big) +\ {}^C D_{x_{0+}}^{\beta+1} \big(u_1 - u_2\big) -\ {}^C D_{t_{0+}}^{\alpha} \big(u_1 - u_2\big) \Big\|_Y \\
&\leq \frac{(T_0 - t_0)^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)} \Big\| \frac{\partial^2}{\partial t^2} \big(u_1(t, x) - u_2(t, x)\big) \Big\|_Y + \frac{(X_0 - x_0)^{1-\beta}}{\Gamma(1-\beta)(2-\beta)} \Big\| \frac{\partial^2}{\partial x^2} \big(u_1(t, x) - u_2(t, x)\big) \Big\|_Y \\
&\quad + \frac{(T_0 - t_0)^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)} \Big\| \frac{\partial}{\partial t} \big(u_1(t, x) - u_2(t, x)\big) \Big\|_Y \\
&\leq \xi \|u_1 - u_2\|_X,
\end{aligned}$$

where

$$\xi \quad = \quad \frac{2(T_0 - t_0)^{1-\alpha}}{\Gamma(1-\alpha)(2-\alpha)} + \frac{(X_0 - x_0)^{1-\beta}}{\Gamma(1-\beta)(2-\beta)}.$$

By the Banach fixed point theorem, noting that $0 < \xi < 1$, we compete the proof of the theorem.

■

5. Numerical method

In this section, we will consider a numerical method for solving the following fractional telegraph equation, with $0 \leq \alpha < 1,\ 0 \leq \beta < 1,\ t_0 \leq t \leq T_0$ and $x_0 \leq x \leq X_0$,

$$-a\,{}^C D_{t_0^+}^{\alpha+1} u(t, x) +\ {}^C D_{x_0^+}^{\beta+1} u(t, x) - b\,{}^C D_{t_0^+}^{\alpha} u(t, x) - cu(t, x) = 0, \tag{20}$$

$$u(t_0, x) = u_{0x}(x),\ u_t(t_0, x) = u_{1x}(x), \tag{21}$$

$$u(t, x_0) = u_{0t}(t),\ u_x(t, x_0) = u_{1t}(t), \tag{22}$$

where a, b and c are some positive real constants.

By using (5), we see that the system (20) - (22) is equivalent to

$$-a\,{}^R D_{t_0^+}^{\alpha+1} \big(u - T_1[u; t_0]\big)(t, x) +\ {}^R D_{x_0^+}^{\beta+1} \big(u - T_1[u; x_0]\big)(t, x) - b\,{}^R D_{t_0^+}^{\alpha} \big(u - T_0[u; t_0]\big)(t, x) - cu(t, x) = 0. \tag{23}$$

Here the Taylor expansions are defined by $T_0[u; t_0](t, x) = u(t_0, x)$, $T_1[u; t_0](t, x) = u(t_0, x) + u_t(t_0, x)(t - t_0)$, and $T_1[u; x_0](t, x) = u(t, x_0) + u_x(t, x_0)(x - x_0)$.

Now let us consider how to approximate the Riemann-Liouville derivative ${}^{R}D_{a+}^{\gamma}x(t),\ 0 \le t \le 1$ at a particular time point, where $0 \le \gamma < 2$. We first consider the case for $0 \le \gamma < 1$. Recall that

$$ {}^{R}D_{0+}^{\gamma}x(t) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\int_0^t \frac{x(u)}{(t-u)^{\gamma}}\,du, \quad 0 \le \gamma < 1. \tag{24} $$

Based on the observation [4] we may interchange differentiation and integration in (24) to obtain

$$ {}^{R}D_{0+}^{\gamma}x(t) = \frac{1}{\Gamma(-\gamma)}\int_0^t \frac{x(u)}{(t-u)^{\gamma+1}}\,du, \quad 0 \le \gamma < 1, \tag{25} $$

where the integral must now be interpreted as a Hadamard finite-part integral.

For a given n, introducing an equispaced grid $t_j = j/n, j = 1, 2, \ldots, n$ on the interval $[0, 1]$ we have

$$ {}^{R}D_{0+}^{\gamma}x(t_j) = \frac{1}{\Gamma(-\gamma)}\int_0^{t_j} \frac{x(u)}{(t_j-u)^{\gamma+1}}\,du = \frac{t_j^{-\gamma}}{\Gamma(-\gamma)}\int_0^1 \frac{x(t_j - t_j w) - x(0)}{w^{\gamma+1}}\,dw, \quad 0 \le \gamma < 1. \tag{26} $$

Now, for every j, we replace the integral by a first-degree compound quadrature formula with equispaced nodes $0, 1/j, 2/j, \ldots, 1$,

$$ \int_0^1 g(u)u^{-\gamma-1}\,du = \sum_{k=0}^{j} \alpha_{kj} g(k/j) + R_{0j}[g], \quad 0 \le \gamma < 1, \tag{27} $$

with remainder term $R_{0j}[g]$ as proposed in [5]. The explicit expressions for the weights α_{kj} are given in the following Lemma

Lemma 5.1. *[5] Assume that $0 \le \gamma < 1$. For the weight α_{kj} of the quadrature formula (27) with $j \ge 1$, we have*

$$ \gamma(1-\gamma)j^{-\gamma}\alpha_{kj} = \begin{cases} -1, & \text{for } k = 0, \\ 2k^{1-\gamma} - (k-1)^{1-\gamma} - (k+1)^{1-\gamma}, & \text{for } k = 1, 2, \ldots, j-1, \\ (\gamma-1)k^{-\gamma} - (k-1)^{1-\gamma} + k^{1-\gamma}, & \text{for } k = j. \end{cases} $$

Thus we can obtain the approximation formula for (26) when $0 \le \gamma < 1$.

Now let us consider how to approximate the Riemann-Liouville derivative ${}^{R}D_{a+}^{\gamma}x(t),\ 0 \le t \le 1$ at a particular time point for $1 \le \gamma < 2$. In this case, we have

$$ {}^{R}D_{0+}^{\gamma}x(t) = \frac{1}{\Gamma(2-\gamma)}\frac{d^2}{dt^2}\int_0^t \frac{x(u)}{(t-u)^{\gamma-1}}\,du, \quad 1 \le \gamma < 2. \tag{28} $$

Based on the observation [4] we may interchange differentiation and integration in (24) to obtain

$$ {}^{R}D_{0+}^{\gamma}x(t) = \frac{1}{\Gamma(-\gamma)}\int_0^t \frac{x(u)}{(t-u)^{\gamma+1}}\,du, \quad 1 \le \gamma < 2. \tag{29} $$

where now the integral must be interpreted as a Hadamard finite-part integral. Thus

$$ {}^{R}D_{0+}^{\gamma}x(t_j) = \frac{1}{\Gamma(-\gamma)}\int_0^{t_j} \frac{x(u)}{(t_j-u)^{\gamma+1}}\,du = \frac{t_j^{-\gamma}}{\Gamma(-\gamma)}\int_0^1 \frac{x(t_j - t_j w) - x(0)}{w^{\gamma+1}}\,dw, \quad 1 \le \gamma < 2. $$

For every j, we replace the integral by a first-degree compound quadrature formula with equispaced nodes $0, 1/j, 2/j, \ldots, 1$ and obtain

$$ \int_0^1 g(u)u^{-\gamma-1}\,du = \sum_{k=0}^{j} \alpha_{kj} g(k/j) + R_{1j}[g], \quad 1 \le \gamma < 2, \tag{30} $$

with some remainder term $R_{1j}[g]$ as in the case when $0 \le \gamma < 1$. The weights α_{kj} are given in the following Lemma.

Lemma 5.2. *Assume that $1 \le \gamma < 2$. For the weight α_{kj} of the quadrature formula (30) with $j \ge 1$, we have*

$$\gamma(1-\gamma)j^{-\gamma}\alpha_{kj} = \begin{cases} -1, & \text{for } k=0, \\ \alpha, & \text{for } k=1,\ j=1, \\ -2^{1-\gamma}+2, & \text{for } k=1,\ j\ge 1, \\ 2k^{1-\gamma}-(k-1)^{1-\gamma}-(k+1)^{1-\gamma}, & \text{for } k=2,3,\dots,j-1, \\ (\gamma-1)k^{-\gamma}-(k-1)^{1-\gamma}+k^{1-\gamma}, & \text{for } k=j,\ j\ge 2. \end{cases}$$

Proof: For fixed j, let $0 < 1/j < 2/j < \cdots < k/j < \cdots < (j-1)/j < j/j = 1$ be a partition of $[0,1]$. We approximate $g(u)$ on $[0,1]$ by the piecewise linear interpolation $g_1(u)$. That is,

$$g_1(u) = \frac{u-\frac{k}{j}}{\frac{k-1}{j}-\frac{k}{j}} g\Big(\frac{k-1}{j}\Big) + \frac{u-\frac{k-1}{j}}{\frac{k}{j}-\frac{k-1}{j}} g\Big(\frac{k}{j}\Big), \quad \text{on} \quad \Big[\frac{k-1}{j}, \frac{k}{j}\Big].$$

Thus

$$\int_0^1 g(u)u^{-1-\gamma}\,du \approx \int_0^1 g_1(u)u^{-1-\gamma}\,du = Q_j(g), \quad j \ge 1.$$

Note that

$$Q_j(g) = \int_0^1 g_1(u)u^{-1-\gamma}\,du = \int_0^{\frac{1}{j}} g_1(u)u^{-1-\gamma}\,du + \sum_{k=2}^{j}\int_{\frac{k-1}{j}}^{\frac{k}{j}} g_1(u)u^{-1-\gamma}\,du.$$

By the definition of the Hadamard finite-part integral [4], we have

$$\begin{aligned} \int_0^{\frac{1}{j}} g_1(u)u^{-1-\gamma}\,du &= \sum_{l=0}^{1} \frac{g_1^{(l)}(0)\Big(\frac{1}{j}-0\Big)^{l+1-(1+\gamma)}}{(l+1-1-\alpha)l!} + \int_0^{\frac{1}{j}} u^{-1-\gamma}\Big(\frac{1}{1!}\int_0^u (u-y)g_1''(y)\,dy\Big)\,du \\ &= \frac{g(0)\cdot\Big(\frac{1}{j}\Big)^{-\gamma}}{-\gamma} + \frac{\Big((-j)g(0)+jg(\frac{1}{j})\Big)\cdot\Big(\frac{1}{j}\Big)^{1-\gamma}}{1-\gamma} \\ &= \frac{-1}{\gamma(1-\gamma)j^{-\gamma}}g(0) + \frac{1}{(1-\gamma)j^{-\gamma}}g\Big(\frac{1}{j}\Big). \end{aligned}$$

Further we have

$$\begin{aligned} \int_{\frac{k-1}{j}}^{\frac{k}{j}} g_1(u)u^{-1-\gamma}\,du &= g\Big(\frac{k-1}{j}\Big)\Big(\frac{k}{-\gamma}\Big(\frac{k}{j}\Big)^{-\gamma} - \frac{j}{-\gamma+1}\Big(\frac{k}{j}\Big)^{-\gamma+1} - \frac{k}{-\gamma}\Big(\frac{k-1}{j}\Big)^{-\gamma} + \frac{j}{-\alpha+1}\Big(\frac{k-1}{j}\Big)^{-\alpha+1}\Big) \\ &\quad + g\Big(\frac{k}{j}\Big)\Big(\frac{j}{-\gamma+1}\Big(\frac{k}{j}\Big)^{-\gamma+1} - \frac{k-1}{-\gamma}\Big(\frac{k}{j}\Big)^{-\gamma} - \frac{j}{-\gamma+1}\Big(\frac{k-1}{j}\Big)^{-\gamma+1} + \Big(\frac{k-1}{-\gamma}\Big)\Big(\frac{k-1}{j}\Big)^{-\gamma}\Big). \end{aligned}$$

Hence, we obtain after some simple calculations,

$$Q_j(g) = \int_0^1 g_1(u)u^{-1-\gamma}\,du = \int_0^{\frac{1}{j}} g_1(u)u^{-1-\gamma}\,du + \sum_{k=2}^{j}\int_{\frac{k-1}{j}}^{\frac{k}{j}} g_1(u)u^{-1-\gamma}\,du = \sum_{k=0}^{j}\alpha_{kj}\,g\Big(\frac{k}{j}\Big),$$

where α_{kj} satisfy the relations in the lemma.

■

Let $t_0 < t_1 < \cdots < t_m < \cdots < t_M = T_0$ be the time partition and Δt be the time step. Let $x_0 < x_1 < \cdots < x_n < \cdots < x_N = X_0$ be the space partition and Δx be the space step. Discretizing the equation (23) about point (t_m, x_n) by using Theorems 5.1, 5.2, we obtain

$$
\begin{aligned}
&- a\Big[\Delta t^{-\alpha-1}\sum_{k=0}^{m}\omega_{km}u(t_{m-k},x_n) - \frac{u_{0x}(x_n)t_m^{-\alpha-1}}{\Gamma(-\alpha)} - \frac{u_{1x}(x_n)t_m^{-\alpha}}{\Gamma(1-\alpha)}\Big]\\
&+ \Big[\Delta x^{-\beta-1}\sum_{j=0}^{n}\tilde{\omega}_{jn}u(t_m,x_{n-j}) - \frac{u_{0t}(t_m)x_n^{-\beta-1}}{\Gamma(-\beta)} - \frac{u_{1t}(t_m)x_n^{-\beta}}{\Gamma(1-\beta)}\Big]\\
&- b\Big[\Delta t^{-\alpha}\sum_{k=0}^{m}\tilde{\tilde{\omega}}_{km}u(t_{m-k},x_n) - \frac{u_{0x}(x_n)t_m^{-\alpha}}{\Gamma(1-\alpha)}\Big] - cu(t_m,x_n) + \text{higher order terms} = 0.
\end{aligned}
$$

Here the weights are

$$
\Gamma(1-\alpha)\omega_{km} = \begin{cases} 1, & \text{for } k=0,\\ -\alpha-1, & \text{for } k=1,\ m=1,\\ 2^{-\alpha}-2, & \text{for } k=1,\ m\geq 2,\\ -2k^{-\alpha}+(k-1)^{-\alpha}+(k+1)^{-\alpha}, & \text{for } k=2,3,\ldots,m-1,\\ (-\alpha)k^{-(\alpha+1)}+(k-1)^{-\alpha}-k^{-\alpha}, & \text{for } k=m,\ m\geq 2.\end{cases}
$$

$$
\Gamma(1-\beta)\tilde{\omega}_{jn} = \begin{cases} 1, & \text{for } j=0,\\ -\beta-1, & \text{for } j=1,\ n=1,\\ 2^{-\beta}-2, & \text{for } j=1,\ n\geq 2,\\ -2j^{-\beta}+(j-1)^{-\beta}+(j+1)^{-\beta}, & \text{for } j=2,3,\ldots,n-1,\\ (-\beta)j^{-(\beta+1)}+(j-1)^{-\beta}-j^{-\beta}, & \text{for } j=n,\ n\geq 2.\end{cases}
$$

$$
\Gamma(2-\alpha)\tilde{\tilde{\omega}}_{km} = \begin{cases} 1, & \text{for } k=0,\\ -2k^{1-\alpha}+(k-1)^{1-\alpha}+(k+1)^{1-\alpha}, & \text{for } k=1,2,\ldots,m-1,\\ -(\alpha-1)k^{-\alpha}+(k-1)^{1-\alpha}-k^{1-\alpha}, & \text{for } k=m.\end{cases}
$$

Denote $U_n^m \approx u(t_m,x_n)$ as the approximation of the exact solution $u(t_m,x_n)$. We can define the following finite difference scheme

$$
\begin{aligned}
&- a\Big[\Delta t^{-\alpha-1}\sum_{k=0}^{m}\omega_{km}U_n^{m-k} - \frac{u_{0x}(x_n)t_m^{-\alpha-1}}{\Gamma(-\alpha)} - \frac{u_{1x}(x_n)t_m^{-\alpha}}{\Gamma(1-\alpha)}\Big]\\
&+ \Big[\Delta x^{-\beta-1}\sum_{j=0}^{n}\tilde{\omega}_{jn}U_{n-j}^{m} - \frac{u_{0t}(t_m)x_n^{-\beta-1}}{\Gamma(-\beta)} - \frac{u_{1t}(t_m)x_n^{-\beta}}{\Gamma(1-\beta)}\Big]\\
&- b\Big[\Delta t^{-\alpha}\sum_{k=0}^{m}\tilde{\tilde{\omega}}_{km}U_n^{m-k} - \frac{u_{0x}(x_n)t_m^{-\alpha}}{\Gamma(1-\alpha)}\Big] - cU_n^m = 0,
\end{aligned}
\tag{31}
$$

or

$$
\begin{aligned}
U_n^m =&\Big[\Delta x^{-\beta-1}\tilde{\omega}_{0n} - a\Delta t^{-\alpha-1}\omega_{0m} - b\Delta t^{-\alpha}\tilde{\tilde{\omega}}_{0m} - c\Big]^{-1}\\
&\Big[a\Big(\Delta t^{-\alpha-1}\sum_{k=1}^{m}\omega_{km}u(t_{m-k},x_n) - \frac{u_{0x}(x_n)t_m^{-\alpha-1}}{\Gamma(-\alpha)} - \frac{u_{1x}(x_n)t_m^{-\alpha}}{\Gamma(1-\alpha)}\Big)\\
&- \Big(\Delta x^{-\beta-1}\sum_{j=1}^{n}\tilde{\omega}_{jn}u(t_m,x_{n-j}) - \frac{u_{0t}(t_m)x_n^{-\beta-1}}{\Gamma(-\beta)} - \frac{u_{1t}(t_m)x_n^{-\beta}}{\Gamma(1-\beta)}\Big)\\
&+ b\Big(\Delta t^{-\alpha}\sum_{k=1}^{m}\tilde{\tilde{\omega}}_{km}u(t_{m-k},x_n) - \frac{u_{0x}(x_n)t_m^{-\alpha}}{\Gamma(1-\alpha)}\Big)\Big]
\end{aligned}
\tag{32}
$$

Here U_n^m for $n, m = 1, 2, 3, \ldots$, can be obtained explicitly by using the initial values $U_0^k, k = 0, 1, \ldots$ and $U_j^0, j = 0, 1, \ldots$.

6. Stability

In this section we will consider the stability of the numerical method (31).

Theorem 6.1. *Let $t_0 < t_1 < \cdots < t_m < \cdots < t_M = T_0$ be the time partition and Δt be the time step. Let $x_0 < x_1 < \cdots < x_n < \cdots < x_N = X_0$ be the space partition and Δx be the space step. Then the numerical method* (31) *is stable under the condition*

$$\lambda < \frac{1}{\|B\|} \|aA_1 + b\Delta t A_2 + c\Delta t^{\alpha+1}\|, \tag{33}$$

where $\lambda = \Delta t^{\alpha+1}/\Delta x^{\beta+1}$, where $\|\cdot\|$ denotes the appropriate matrix norm in $\mathbf{R}^{(N+1)\times(N+1)}$ or $\mathbf{R}^{(M+1)\times(M+1)}$. Here the matrix B, A_1 and A_2 are introduced below.

Proof: Denote

$$U = \begin{bmatrix} U_0^0 & U_0^1 & \cdots & U_0^M \\ U_1^0 & U_1^1 & \cdots & U_1^M \\ \vdots & \vdots & \ddots & \vdots \\ U_N^0 & U_N^1 & \cdots & U_N^M \end{bmatrix}, \qquad A_1 = \begin{bmatrix} \omega_{00} & \omega_{11} & \cdots & \omega_{MM} \\ 0 & \omega_{01} & \cdots & \omega_{M-1\ M} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_{0M} \end{bmatrix},$$

$$B = \begin{bmatrix} \widetilde{\omega}_{00} & 0 & \cdots & 0 \\ \widetilde{\omega}_{11} & \widetilde{\omega}_{01} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \widetilde{\omega}_{NN} & \widetilde{\omega}_{N-1\ N} & \cdots & \widetilde{\omega}_{0N} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \tilde{\tilde{\omega}}_{00} & \tilde{\tilde{\omega}}_{11} & \cdots & \tilde{\tilde{\omega}}_{MM} \\ 0 & \tilde{\tilde{\omega}}_{01} & \cdots & \tilde{\tilde{\omega}}_{M-1\ M} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{\tilde{\omega}}_{0M} \end{bmatrix},$$

and

$$U_{0x} = [u_{0x}(x_0), \cdots, u_{0x}(x_N)], \quad U_{1x} = [u_{1x}(x_0), \cdots, u_{1x}(x_N)],$$
$$U_{0t} = [u_{0t}(t_0), \cdots, u_{0t}(t_M)], \quad U_{1t} = [u_{1t}(t_0), \cdots, u_{1t}(t_M)],$$

Then (31) can be written in the following matrix form

$$\begin{aligned} &-aUA_1 + \frac{\Delta t^{\alpha+1}}{\Delta x^{\beta+1}} BU - b\Delta t U A_2 - c\Delta t^{\alpha+1} U \\ &+ \Delta t^{\alpha+1}\Big[a\Big(\frac{1}{\Gamma(-\alpha)} U_{0x}^T T^{-\alpha-1} + \frac{1}{\Gamma(1-\alpha)} U_{1x}^T T^{-\alpha}\Big) \\ &- \Big(\frac{1}{\Gamma(-\beta)} X^{-\beta-1} U_{0t} + \frac{1}{\Gamma(1-\beta)} X^{-\beta} U_{1t}\Big) + b\Big(\frac{1}{\Gamma(1-\alpha)} U_{0x}^T T^{-\alpha}\Big)\Big] = 0 \end{aligned}$$

where

$$T^{-\alpha} = [t_0^{-\alpha}, \cdots, t_M^{-\alpha}], \quad T^{-\alpha-1} = [t_0^{-\alpha-1}, \cdots, t_M^{-\alpha-1}],$$

and

$$X^{-\beta} = [x_0^{-\beta}, \cdots, x_N^{-\beta}], \quad X^{-\beta-1} = [x_0^{-\beta-1}, \cdots, x_N^{-\beta-1}].$$

Denote $\lambda = \Delta t^{\alpha+1}/\Delta x^{\beta+1}$, we then have

$$U = (\lambda B) U \big(aA_1 + b\Delta t A_2 + c\Delta t^{\alpha+1}\big)^{-1} + F(X,T)\big(aA_1 + b\Delta t A_2 + c\Delta t^{\alpha+1}\big)^{-1},$$

where

$$F(X,T) = \Delta t^{\alpha+1}\Big[a\Big(\frac{1}{\Gamma(-\alpha)}U_{0x}^T T^{-\alpha-1} + \frac{1}{\Gamma(1-\alpha)}U_{1x}^T T^{-\alpha}\Big) - \Big(\frac{1}{\Gamma(-\beta)}X^{-\beta-1}U_{0t} + \frac{1}{\Gamma(1-\beta)}X^{-\beta}U_{1t}\Big) + b\Big(\frac{1}{\Gamma(1-\alpha)}U_{0x}^T T^{-\alpha}\Big)\Big]. \quad (34)$$

Further we denote

$$C = \lambda B, \quad D = (aA_1 + b\Delta t A_2 + c\Delta t^{\alpha+1})^{-1}, \quad E = F(X,T)(aA_1 + b\Delta t A_2 + c\Delta t^{\alpha+1})^{-1},$$

then (34) can be written into

$$U = CUD + E. \quad (35)$$

Now we use the Picard iteration to find the solution of (35). Let

$$\begin{aligned} U^{(0)} &= E, \\ U^{(1)} &= CU^{(0)}D + E, \\ U^{(2)} &= CU^{(1)}D + E, \\ &\vdots \end{aligned} \quad (36)$$

Then, we have, with some appropriate norm $\|\cdot\|$ in $\mathbf{R}^{(N+1)\times(M+1)}$,

$$\begin{aligned} \|U^{(n)} - U^{(n-1)}\| &= \big\|C\big(U^{(n-1)} - U^{(n-2)}\big)D\big\|, \\ &\leq \|C\|\|U^{(n-1)} - U^{(n-2)}\|\|D\|, \\ &\leq \cdots \quad (37) \\ &\leq (\|C\|\|D\|)^n\|U^{(1)} - U^{(0)}\|. \quad (38) \end{aligned}$$

Assume that

$$\|C\|\|D\| < 1. \quad (39)$$

Then we see that $U^{(n)}$ converges to the unique solution U.

Now let us investigate the condition (39). Note that

$$\|C\| = \lambda\|B\|,$$

and

$$\|D\| = \big\|\big(aA_1 + b\Delta t A_2 + c\Delta t^{\alpha+1}\big)^{-1}\big\|.$$

Thus the condition (39) is equivalent to

$$\lambda < \frac{1}{\|B\|}\big\|aA_1 + b\Delta t A_2 + c\Delta t^{\alpha+1}\big\|, \quad (40)$$

which is (33).

The proof of Theorem 6.1 is complete.

■

Corollary 6.2. *Let $t_0 < t_1 < \cdots < t_m < \cdots < t_M = T_0$ be the time partition and Δt be the time step. Let $x_0 < x_1 < \cdots < x_n < \cdots < x_N = X_0$ be the space partition and Δx be the space step.*

Assume that $b = 0$ and $c = 0$. Then the numerical method (31) *is stable under the condition*

$$\lambda < \frac{a\|A_1\|}{\|B\|}, \quad (41)$$

where $\lambda = \Delta t^{\alpha+1}/\Delta x^{\beta+1}$, where $\|\cdot\|$ denotes the appropriate matrix norm in $\mathbf{R}^{(N+1)\times(N+1)}$ or $\mathbf{R}^{(M+1)\times(M+1)}$. Here the matrix B and A_1 are defined as above.

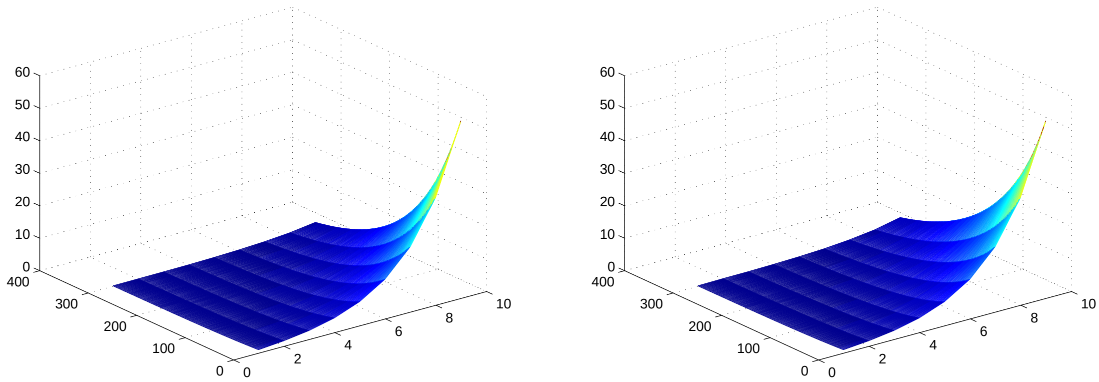

Figure 1: Analytical (left) and estimated (right) solutions in $T_0 = 3$ and $X_0 = 3$.

7. Numerical examples

In this section, we will introduce two examples of two-parameter fractional telegraph equations. We shall compare the numerical solutions with the exact solutions.

Example 7.1. *Consider the following two-parameter fractional telegraph equation*

$$\begin{aligned}
&-\,{}^C D_{0+}^{\alpha+1} u(t,x) + {}^C D_{0+}^{\beta+1} u(t,x) - {}^C D_{0+}^{\alpha} u(t,x) - u(t,x) = f(t,x), \quad 0 < t \le T_0,\\
&u(0,x) = e^x, \qquad u_t(0,x) = -e^x, \quad 0 \le x \le X_0,\\
&u(t,0) = e^{-t}, \qquad u_x(t,0) = e^{-t}, \quad 0 \le t \le T_0.
\end{aligned}$$

The analytical solution is given as

$$u(t,x) = e^{x-t}.$$

The right hand side of this equation can be obtained by using the fractional derivatives of $u(t,x)$.

We apply the numerical method discussed above. We choose $\Delta t = 0.001$, $\Delta x = 1/15$. The graphs of analytical and approximate solutions for $\alpha = 0.4$ and $\beta = 0.9$ are given in Fig.1.

Example 7.2. *Consider the following two-parameter fractional telegraph equation*

$$\begin{aligned}
&-\,{}^C D_{0+}^{\alpha+1} u(t,x) + {}^C D_{0+}^{\beta+1} u(t,x) - r\,{}^C D_{0+}^{\alpha} u(t,x) - s u(t,x) = f(t,x), \quad 0 < t \le T_0,\\
&u(0,x) = \sin x, \qquad u_t(0,x) = -\sin x, \quad 0 \le x \le X_0,\\
&u(t,0) = 0, \qquad u_x(t,0) = e^{-t}, \quad 0 \le t \le T_0,
\end{aligned}$$

where $r = 4$ and $s = 2$.

The analytical solution is given as

$$u(t,x) = e^{-t}\sin x.$$

The right hand side function $f(t,x)$ is approximated by using numerical quadrature formulae in the numerical approximation. We choose $\Delta t = 0.001$, $\Delta x = 1/15$. The graphs of analytical and approximation solutions for $\alpha = 0.4$ and $\beta = 0.9$ are given in Fig.2.

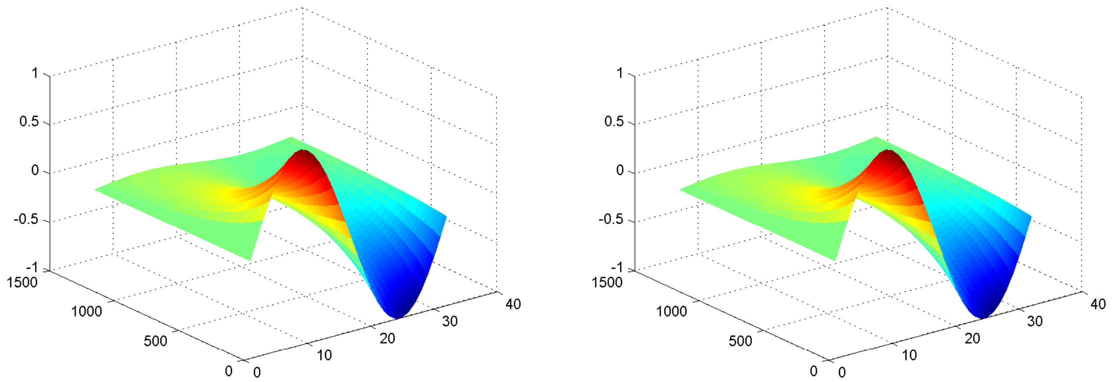

Figure 2: Analytical (left) and estimated (right) solutions in $T_0 = 3$ and $X_0 = 2\pi$

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