Stability of a numerical method for a fractional telegraph equation

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Abstract

In this paper, we introduce a numerical method for solving the time-space fractional telegraph equations. The numerical method is based on a quadrature formula approach and a stability condition for the numerical method is obtained. Two numerical examples are given and the stability regions are plotted.

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fractional telegraph equations, finite difference methods, stability regions
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1. Introduction

The telegraph equation is used in modeling reaction-diffusion and signal analysis for transmission and propagation of electrical signals. Let us consider an infinitesimal piece of telegraph wire as an electrical circuit. Let R denote the resistance of the conductors, expressed in ohms per unit length. Let L denote the inductance due to the magnetic field around the wire, expressed in Henries per unit length. Let C denote the capacitance between the two conductors, expressed by Farads per unit length. Let G denote the conductance of the dielectric material separating the two conductors, expressed in Siemens per unit length. The electric circuit consists of a resistor of resistance $R\,dx$, and a coil of inductance Ldx. Let $I(t,x)$ be the current through the wire at time t and let the position be denoted x. The voltage across the resistor is $IRdx$, and the voltage across the coil is $\frac{\partial I}{\partial t}Ldt$. Let $u(t,x)$ be the voltage at time t and position x. The change in voltage between the ends of the piece of wire is

$$du = -IRdx - \frac{\partial I}{\partial t}Ldx,$$

or

$$\frac{\partial u}{\partial x} = -IR - \frac{\partial I}{\partial t}L. \tag{1}$$

Suppose further that the current can escape from the wire to ground, either through a resistor of conductance $G\,dx$ or through a capacitor of capacitance $C\,dx$. The amount that escapes through the resistor is $uGdx$. Because the charge on the capacitor is $q = uCdx$, the amount that escapes from the capacitor is $\frac{\partial u}{\partial t}Cdx$. Thus we have

$$dI = -uGdx - \frac{\partial u}{\partial t}Cdx,$$

or

$$\frac{\partial I}{\partial x} = -uG - \frac{\partial u}{\partial t}C. \tag{2}$$

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Combining (1) and (2), we see that the voltage $u(t,x)$ satisfies the classical telegraph equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} + a\frac{\partial u(t,x)}{\partial t} + bu(t,x) = c^2\frac{\partial^2 u(t,x)}{\partial x^2},\ t>0,\ x>0, \tag{3}$$

where a,b,c are some positive constants. Similarly the current $I(t,x)$ also satisfies the classical telegraph equation (3), see Biazar and Ebrahimi [1], Debnath [7], Metaxas and Meredith [20].

It is natural to consider the fractional telegraph equation: fractional derivatives are more effective than their integer order counterparts for modelling the telegraph because fractional order derivatives can describe the memory and hereditary properties, see Podlubny [25]. A space or time or time-space fractional telegraph equation is obtained from the classical telegraph equation (3) by replacing the second order time derivative by a fractional time derivative of order $\alpha : 1<\alpha<2$, and the first order time derivative by a fractional time derivative of order $\gamma : 0<\gamma<1$, and the second space derivative by a fractional apace derivative of order $\beta : 1<\beta<2$. We therefore obtain the following initial-value problem for the time-space telegraph equation

$${}_0^C D_t^\alpha u(t,x) + a\,{}_0^C D_t^\gamma u(t,x) + bu(t,x) = c^2\,{}_0^C D_x^\beta u(t,x) + f(t,x), \quad t>0, \quad x>0, \tag{4}$$

$$u(0,x) = b_0(x), \qquad u_t(0,x) = b_1(x), \tag{5}$$

where ${}_0^C D_t^\alpha u(t,x)$, ${}_0^C D_t^\gamma u(t,x)$ and ${}_0^C D_x^\beta u(t,x)$ denote the time and space Caputo fractional derivatives which we will define in the next section, where a,b,c are some positive constants, where $u_t(0,x)$ denotes the time derivative of $u(t,x)$ at $t=0$.

There are different types of initial and boundary conditions in the literature to solve the time-space telegraph equation (4). We are concentrating on the initial conditions here and will cover boundary conditions in a sequel. We will review below some initial-value problems for the different types of telegraph equations in literature.

Let us first consider the initial-value wave equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} = \frac{\partial^2 u(t,x)}{\partial x^2}, \quad t>0,\ -\infty<x<+\infty, \tag{6}$$

$$u(0,x) = b_0(x), \quad u_t(0,x) = b_1(x). \tag{7}$$

The system (6)-(7) has the unique solution

$$u(t,x) = \frac{1}{2}\Big[b_0(x+t) + b_0(x-t) + \int_{x-t}^{x+t} b_1(\tau)\,d\tau\Big].$$

In Biazar et al. [2], the authors consider the following initial-value telegraph equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} + (\alpha+\beta)\frac{\partial u(t,x)}{\partial t} + \alpha\beta u(t,x) = c^2\frac{\partial^2 u(t,x)}{\partial x^2}, \quad t>0, \quad -\infty<x<+\infty, \tag{8}$$

$$u(0,x) = b_0(x), \quad u_t(0,x) = b_1(x), \tag{9}$$

The exact solution of (8)-(9) has the form

$$u(t,x) = b_0(x) + b_1(x)t + \int_0^t \int_0^\tau c^2\frac{\partial^2 u(s,x)}{\partial x^2} - (\alpha+\beta)\frac{\partial u(s,x)}{\partial s} - \alpha\beta u(s,x)\,ds\,d\tau.$$

By using the Adomian decomposition method, the authors give the solution $u(t,x)$ of (8)-(9) in a series form.

In Sevimlican [31], the author considers the following initial-value telegraph equation, with $0<\alpha<1$,

$${}_0^C D_t^{2\alpha} u(t,x) + \lambda\,{}_0^C D_t^\alpha u(t,x) = \nu\frac{\partial^2 u(t,x)}{\partial x^2}, \ t>0, \quad 0<x<1, \tag{10}$$

$$u(0,x) = b_0(x), \quad u_t(0,x) = b_1(x), \tag{11}$$

$$u(t,0) = s(t). \tag{12}$$

By using the variational iteration method, the authors obtain an iteration formula

$$u_{n+1}(t,x) = u_n(t,x) + \int_0^t (\tau - t)\Big[{}_0^C D_\tau^{2\alpha} u_n(\tau,x) + \lambda\, {}_0^C D_\tau^{\alpha} u_n(\tau,x) - \nu \frac{\partial^2 u_n(\tau,x)}{\partial x^2} \Big]\, d\tau,$$

for $n = 1, 2, \dots,$ with the initial guess $u_0(t,x) = b_0(x) + tb_1(x)$. Here $u_n(t,x)$ is the nth approximation of the exact solution $u(t,x)$, see also Dehghan et al. [6].

By using the Adomian method, Momani [23] obtains the exact solution of (10)-(12) in a series form

$$u(t,x) = \sum_{n=0}^{+\infty} u_n(t,x),$$

where, with $u_0(t,x) = b_0(x) + tb_1(x)$,

$$u_{n+1}(t,x) = J^{2\alpha}\Big[\nu \frac{\partial^2 u_n(t,x)}{\partial x^2} - \lambda\, {}_0^C D_t^{\alpha} u_n(t,x)\Big],\ n = 1, 2, \dots,$$

where

$$J^{2\alpha} u(t,x) = \frac{1}{\Gamma(2\alpha)} \int_0^x (x-\tau)^{2\alpha-1} u(t,\tau)\, d\tau.$$

In Garg et al. [12], the authors consider the following fractional time-space telegraph equation

$$ {}_0^C D_t^{\alpha} u(t,x) + a\, {}_0^C D_t^{\gamma} u(t,x) + bu(t,x) = c^2\, {}_0^C D_x^{\beta} u(t,x) + f(t,x), \quad t > 0, \quad 0 < x < 1, \tag{13}$$

$$u(t,0) = E_{1/2}(-t^{1/2}), \qquad u_x(t,0) = E_{1/2}(-t^{1/2}). \tag{14}$$

By using the generalized differential transform method, the authors obtain the following exact solution for $\alpha = 3/2,\ \beta = 3/2,\ \gamma = 1/2$,

$$u(t,x) = \Big(E_{3/2}(x^{3/2}) + xE_{3/2,2}(x^{3/2}))E_{1/2}(-t^{1/2}\Big),$$

where $E_{\alpha_1}(z), E_{\alpha_1,\beta_1}(z),\ \alpha_1, \beta_1 \in \mathbf{C}, \mathrm{Re}(\alpha_1) > 0, \mathrm{Re}(\beta_1) > 0$ are the Mittag-Leffler functions.

In Mainardi et al. [18], the authors consider the following signalling problem, with $1 < \alpha < 2$,

$${}_0^C D_t^{\alpha} u(t,x) = \frac{\partial^2 u(t,x)}{\partial x^2}, \quad t > 0,\ x > 0 \tag{15}$$

$$u(0,x) = 0, \quad u_t(0,x) = 0, \tag{16}$$

$$u(0,t) = h(t), \quad u(+\infty,t) = 0. \tag{17}$$

The exact solution of (15)-(17) is obtained by using Laplace transform.

A time-space fractional diffusion equation obtained from the standard diffusion equation by replacing the first order time derivative by a fractional derivative of order $\alpha, 0 < \alpha < 2$ and the second order space derivative by a fractional derivative of order $\beta > 0$, has also been treated by a number of authors- notably: Saichev and Zaslavsky [28], Gorenflo et al. [13], Scalas, Gorenflo and Mainardi [29], Metzler and Klafter [21], Mainardi et al. [18]. A number of researchers, such as Mainardi [16] [17], Ray [27], Chen et al. [4], Schot et al. [30], Erochenkova and Lima [10], McLean and Mustafa [19], Murio and Carlos [22], Kochubei [15], Sayed [9], have studied the space and/or time fractional diffusion equations.

The existence and uniqueness of the solutions of the time-space fractional telegraph equation (4)-(5) have been discussed by using the Banach fixed point theorem, see Yakubovich and Rodrigues [32]. In this paper, we will introduce a numerical method to solve (4)-(5) based on the quadrature formulae approach in [8]. A stability condition for the numerical method is obtained and the stability regions are plotted.

This paper is organised as follows. In preliminaries we recall some basic properties of fractional derivatives. Then, in Section 3, we show how the numerical method is constructed to deal with the time-space discretisation of the fractional telegraph equation. In Section 4, we prove the stability of the numerical method. Finally, in Section 5, we give some numerical examples.

2. Preliminaries

In this section, we will introduce the definitions of Riemann-Liouville and Caputo differential fractional operators and the relations between them. We refer to Podlubny [25] and Diethelm [8] for further details.

Definition 2.1. *Let $0 < q < 1$. The operator J^q, defined by*

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau)\, d\tau$$

for $0 \le t \le T$, is called the Riemann-Liouville fractional integral operator of order q.

Definition 2.2. *Let $n-1 \le q < n$, $n \ge 1$. The operator ${}_0^R D_t^q$, defined by*

$$ {}_0^R D_t^q f(t) = D^n J^{n-q} f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-q-1} f(\tau) d\tau, \tag{18}$$

for $0 \le t \le T$, is called the Riemann-Liouville differential fractional operator of order q, where $D^n = \frac{d^n}{dt^n}$ denotes the n-th order derivative.

Definition 2.3. *Let $n-1 \le q < n$, $n \ge 1$. The operator ${}_0^C D_t^q$, defined by*

$$ {}_0^C D_t^q f(t) = J^{n-q} D^n f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-\tau)^{n-q-1} \frac{d^n}{d\tau^n} f(\tau) d\tau, \tag{19}$$

for $0 \le t \le T$, is called the Caputo differential fractional operator of order q.

The following lemma gives the relation between the Riemann-Liouville and Caputo differential fractional operators.

Lemma 2.1. *Assume that $f \in C^n[0,T]$, $n \ge 1$, then we have*

$$ {}_0^C D_t^q f(t) = {}_0^R D_t^q \Big(f - T_{n-1}[f;0] \Big)(t) = {}_0^R D_t^q (f)(t) - {}_0^R D_t^q \Big(T_{n-1}[f;0] \Big)(t), \tag{20}$$

where $T_{n-1}[f;0])(t)$ is the Taylor expansion of function $f(t)$ about the origin up to $(n-1)$th order, that is,

$$T_{n-1}[f;0])(t) = f(0) + \frac{f'(0)}{1!} t + \frac{f''(0)}{2!} t^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!} t^{n-1} = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k.$$

We remark that the Riemann-Liouville fractional derivatives of $T_{n-1}[f;0])(t)$ are, see Podlubney [25]

$$ {}_0^R D_t^q \Big(T_{n-1}[f;0] \Big)(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0) t^{k-q}}{\Gamma(k+1-q)},\ n-1 \le q < n. \tag{21}$$

3. Discretisation of the time-space fractional telegraph equation

In this section, we will consider the discretisation of the time-space fractional telegraph equation (4)-(5). Let us first recall how to approximate a Riemann-Liouville fractional derivatives ${}_0^R D_t^q f(t)$ for $0 < q < 2$, $q \neq 1$ at some fixed point based on a quadrature formula approach, see Diethelm [8]. For a fixed positive integer n, let $0 = t_0 < t_1 < \cdots < t_j < \cdots < t_n$ be a partition of $[0,T]$ and let Δt denote the time step size. We then have the following lemma, Diethelm [8]

Lemma 3.1. *Let $0 < q < 2$, $q \neq 1$. For the fixed $t_j = \frac{j}{n}$, where n is fixed, we have*

$$\Big({}_0^R D_t^q f \Big)(t_j) = \Delta t^{-q} \sum_{k=0}^{j} w_{kj}^{(q)} f(t_j - t_k) + \frac{t_j^{-q}}{\Gamma(1-q)} R_j^{(q)}, \quad j = 1, 2, \dots, n,$$

where

$$\Gamma(2-q) w_{kj}^{(q)} = \begin{cases} 1, & k = 0, \\ -q, & k = 1, \ j = 1, \\ 2^{1-q} - 2, & k = 1, \ j > 1, \\ -2k^{1-q} + (k-1)^{1-q} + (k+1)^{1-q}, & k = 2, 3, \dots, j-1, \quad j \geq 2, \\ -(q-1)k^{-q} + (k-1)^{1-q} - k^{1-q}, & k = j, \quad j \geq 2, \end{cases}$$

where $R_j^{(q)}$ is the remainder and we have the similar estimates as in [8].

Now let us consider the discretisation of the time-space telegraph equation (4)-(5). In our numerical method, we also need the initial vaules $u(t,x)$ and $u_x(t,x)$ at $x = 0$. Therefore we will consider the following initial-value problem of the time-space telegraph equation, with $\alpha \in (1,2)$, $\gamma \in (0,1)$ and $\beta \in (1,2)$,

$$\begin{aligned} &{}_0^C D_t^\alpha u(t,x) + a\, {}_0^C D_t^\gamma u(t,x) + b u(t,x) = c^2\, {}_0^C D_x^\beta u(t,x) + f(t,x), \quad t > 0, \quad x > 0, && (22) \\ &u(0,x) = b_0(x), \qquad u_t(0,x) = b_1(x), && (23) \\ &u(t,0) = c_0(t), \qquad u_x(t,0) = c_1(t). && (24) \end{aligned}$$

Here $u_t(0,x)$ denotes the time derivative of $u(t,x)$ at $t = 0$ and $u_x(t,0)$ denotes the space derivative of $u(t,x)$ at $x = 0$.

We rewrite the equation (22)-(24) by using the Riemann-Liouville fractional derivatives and obtain, with $\alpha \in (1,2)$, $\gamma \in (0,1)$ and $\beta \in (1,2)$,

$$\begin{aligned} &{}_0^R D_t^\alpha \Big[u(t,x) - u(0,x) - \frac{u_t(0,x)}{1!} t \Big] + a\, {}_0^R D_t^\gamma \Big[u(t,x) - u(0,x) \Big] + b u(t,x) \\ &\quad = c^2\, {}_0^R D_x^\beta \Big[u(t,x) - u(t,0) - \frac{u_x(t,0)}{1!} x \Big] + f(t,x), \quad t > 0, \quad x > 0, \qquad (25) \end{aligned}$$

where a, b, c are some positive constants.

Let $0 < t_0 < t_1 < t_2 < \cdots < t_m < \cdots < t_M = T$ be the time partition and let Δt be the time step size. Let $0 < x_0 < x_1 < x_2 < \cdots < x_n < \cdots < x_N = X$ be the space partition and let Δx be the space step size. Here $T > 0$ and $X > 0$ are some positive real numbers. At the grid point (t_m, x_n), by Lemma 3.1, we have, with $1 < \alpha < 2$,

$$\begin{aligned} &{}_0^R D_t^\alpha \Big[u(t,x) - u(0,x) - \frac{u_t(0,x)}{1!} t \Big] \Big|_{t=t_m,\, x=x_n} \\ &= \Delta t^{-\alpha} \sum_{k=0}^{m} w_{km}^{(\alpha)} u(t_{m-k}, x_n) + \frac{t_m^{-\alpha}}{\Gamma(-\alpha)} R_m^{(\alpha)} - {}_0^R D_t^\alpha u(0, x_n) \Big|_{t=t_m} - {}_0^R D_t^\alpha \Big(u_t(0, x_n) t \Big) \Big|_{t=t_m}, \end{aligned}$$

and, with $1 < \beta < 2$,

$$\begin{aligned} &{}_0^R D_x^\beta \Big[u(t,x) - u(t,0) - \frac{u_x(t,0)}{1!} t \Big] \Big|_{t=t_m,\, x=x_n} \\ &= \Delta x^{-\beta} \sum_{j=0}^{n} w_{jn}^{(\beta)} u(t_m, x_{n-j}) + \frac{x_n^{-\beta}}{\Gamma(-\beta)} R_n^{(\beta)} - {}_0^R D_x^\beta u(t_m, 0) \Big|_{x=x_n} - {}_0^R D_x^\beta \Big(u_x(t_m, 0) x \Big) \Big|_{x=x_n}, \end{aligned}$$

and, with $0<\gamma<1$,

$$
\begin{aligned}
&{}_0^R D_t^{\gamma}\Big[u(t,x)-u(0,x)\Big]\Big|_{t=t_m,\,x=x_n}\\
&=\Delta t^{-\gamma}\sum_{k=0}^{m} w_{km}^{(\gamma)} u(t_{m-k},x_n)+\frac{t_m^{-\gamma}}{\Gamma(-\gamma)}R_m^{(\gamma)} -{}_0^R D_t^{\gamma} u(0,x_n)\Big|_{t=t_m},
\end{aligned}
$$

where $R_m^{(\alpha)}, R_n^{(\beta)}$ and $R_m^{(\gamma)}$ denote the remainders.

Further, by (21), we have, with $1<\alpha<2$,

$$
{}_0^R D_t^{\alpha} u(0,x_n)\Big|_{t=t_m}=u(0,x_n)\frac{t_m^{-\alpha}}{\Gamma(1-\alpha)},\quad {}_0^R D_t^{\alpha} u_t(0,x_n)t\Big|_{t=t_m}=u_t(0,x_n)\frac{t_m^{1-\alpha}}{\Gamma(2-\alpha)},
$$

and, with $1<\beta<2$,

$$
{}_0^R D_x^{\beta} u(t_m,0)\Big|_{x=x_n}=u(t_m,0)\frac{x_n^{-\beta}}{\Gamma(1-\beta)},\quad {}_0^R D_x^{\beta} u_x(t_m,0)x\Big|_{x=x_n}=u_x(t_m,0)\frac{x_n^{1-\beta}}{\Gamma(2-\beta)},
$$

and, with $0<\gamma<1$,

$$
{}_0^R D_t^{\gamma} u(0,x_n)\Big|_{t=t_m}=u(0,x_n)\frac{t_m^{-\gamma}}{\Gamma(1-\gamma)}.
$$

Let $U_n^m\approx u(t_m,x_n)$ denote the approximation of the exact solution $u(t_m,x_n)$. Removing the remainder terms, we obtain the following discretisation scheme for (25) :

$$
U_n^0=b_0(x_n),\quad n=0,1,2,\ldots,N,\quad U_0^m=c_0(t_m),\quad m=0,1,2,\ldots,M, \tag{26}
$$

and, for $n\geq 1$, $m\geq 1$,

$$
\begin{aligned}
&\Big(\Delta t^{-\alpha}\sum_{k=0}^{m}\omega_{km}^{(\alpha)}U_n^{m-k}-\frac{b_0(x_n)t_m^{-\alpha}}{\Gamma(1-\alpha)}-\frac{b_1(x_n)t_m^{1-\alpha}}{\Gamma(2-\alpha)}\Big)\\
-\quad & c^2\Big(\Delta x^{-\beta}\sum_{j=0}^{n}\omega_{jn}^{(\beta)}U_{n-j}^{m}-\frac{c_0(t_m)x_n^{-\beta}}{\Gamma(1-\beta)}-\frac{c_1(t_m)x_n^{1-\beta}}{\Gamma(2-\beta)}\Big)\\
+\quad & a\Big(\Delta t^{-\gamma}\sum_{k=0}^{m}\omega_{km}^{(\gamma)}U_n^{m-k}-\frac{b_0(x_n)t_m^{-\gamma}}{\Gamma(1-\gamma)}\Big)+bU_n^m=0,
\end{aligned}
\tag{27}
$$

that is, for $n\geq 1$, $m\geq 1$,

$$
\begin{aligned}
U_n^m \;=\;& \Big(\Delta t^{-\alpha}\omega_{0m}^{(\alpha)}-c^2\Delta x^{-\beta}\omega_{0n}^{(\beta)}+a\Delta t^{-\gamma}\omega_{0n}^{(\gamma)}+b\Big)^{-1}\\
&\times\Big(\Big(\Delta t^{-\alpha}\sum_{k=1}^{m}\omega_{km}^{(\alpha)}U_n^{m-k}-\frac{b_0(x_n)t_m^{-\alpha}}{\Gamma(1-\alpha)}-\frac{b_1(x_n)t_m^{1-\alpha}}{\Gamma(2-\alpha)}\Big)\\
-\;& c^2\Big(\Delta x^{-\beta}\sum_{j=1}^{n}\omega_{jn}^{(\beta)}U_{n-j}^{m}-\frac{c_0(t_m)x_n^{-\beta}}{\Gamma(1-\beta)}-\frac{c_1(t_m)x_n^{1-\beta}}{\Gamma(2-\beta)}\Big)\\
+\;& a\Big(\Delta t^{-\gamma}\sum_{k=1}^{m}\omega_{km}^{(\gamma)}U_n^{m-k}-\frac{b_0(x_n)t_m^{-\gamma}}{\Gamma(1-\gamma)}\Big)\Big).
\end{aligned}
\tag{28}
$$

From Lemma 3.1, let $s=\alpha,\beta,\gamma$, we have, with $0<s<2$ and $s\neq 1$,

$$
\Gamma(2-s)w_{km}^{(s)}=\begin{cases}
1, & k=0,\\
-s, & k=1,\ m=1,\\
2^{1-s}-2, & k=1,\ m>1,\\
-2k^{1-s}+(k-1)^{1-s}+(k+1)^{1-s}, & k=2,3,\ldots,m-1,\quad m\geq 2,\\
-(s-1)k^{-s}+(k-1)^{1-s}-k^{1-s}, & k=m,\quad m\geq 2.
\end{cases}
$$

We remark that the numerical method (28) is an explicit scheme. The stability analysis of the scheme (28) will be given in the next section.

4. Stability analysis

The key to our stability analysis is that we may write (27) in the following matrix form

$$\begin{aligned} & UB_1 - c^2\frac{\Delta t^\alpha}{\Delta x^\beta}AU + a\Delta t^{\alpha-\gamma}UB_2 + b\Delta t^\alpha U + \Delta t^\alpha\Big(- \frac{b_0(X)}{\Gamma(1-\alpha)}T^{-\alpha} - \frac{b_1(X)}{\Gamma(2-\alpha)}T^{1-\alpha} \\ + \;& c^2\frac{1}{\Gamma(1-\beta)}X^{-\beta}c_0(T) + c^2\frac{1}{\Gamma(2-\beta)}X^{1-\beta}c_1(T) - a\frac{b_0(X)}{\Gamma(1-\gamma)}T^{-\gamma}\Big) = 0, \end{aligned} \tag{29}$$

where

$$U = \begin{bmatrix} U_0^0 & U_0^1 & \cdots & U_0^M \\ U_1^0 & U_1^1 & \cdots & U_1^M \\ \vdots & \vdots & \ddots & \vdots \\ U_N^0 & U_N^1 & \cdots & U_N^M \end{bmatrix}, \quad B_1 = \begin{bmatrix} \omega_{00}^{(\alpha)} & \omega_{11}^{(\alpha)} & \cdots & \omega_{MM}^{(\alpha)} \\ 0 & \omega_{01}^{(\alpha)} & \cdots & \omega_{M-1\ M}^{(\alpha)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_{0M}^{(\alpha)} \end{bmatrix},$$

$$B_2 = \begin{bmatrix} \omega_{00}^{(\gamma)} & \omega_{11}^{(\gamma)} & \cdots & \omega_{MM}^{(\gamma)} \\ 0 & \omega_{01}^{(\gamma)} & \cdots & \omega_{M-1\ M}^{(\gamma)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_{0M}^{(\gamma)} \end{bmatrix},$$

$$A = \begin{bmatrix} \omega_{00}^{(\beta)} & 0 & \cdots & 0 \\ \omega_{11}^{(\beta)} & \omega_{01}^{(\beta)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{NN}^{(\beta)} & \omega_{N-1\ N}^{(\beta)} & \cdots & \omega_{0N}^{(\beta)} \end{bmatrix},$$

and

$$T^s = [t_0^s, \cdots, t_M^s], \quad \text{where } s = 1-\alpha, -\alpha, \text{ or } \gamma,$$
$$X^s = [x_0^s, \cdots, x_N^s]^T, \quad \text{where } s = 1-\beta \text{ or } -\beta,$$
$$b_0(X) = [b_0(x_0), \cdots, b_0(x_N)]^T, \quad b_1(X) = [b_1(x_0), \cdots, b_1(x_N)]^T,$$
$$c_0(T) = [c_0(t_0), \cdots, c_0(t_M)], \quad c_1(T) = [c_1(t_0), \cdots, c_1(t_M)].$$

Then we obtain, with $\lambda = \Delta t^\alpha / \Delta x^\beta$,

$$U = \lambda(c^2A)U\Big(B_1 + a\Delta t^{\alpha-\gamma}B_2 + b\Delta t^\alpha I\Big)^{-1} - F(X,T)\Big(B_1 + a\Delta t^{\alpha-\gamma}B_2 + b\Delta t^\alpha I\Big)^{-1}, \tag{30}$$

where

$$\begin{aligned} F(X,T) \;&=\; \Delta t^\alpha\Big(- \frac{b_0(X)}{\Gamma(1-\alpha)}T^{-\alpha} - \frac{b_1(X)}{\Gamma(2-\alpha)}T^{1-\alpha} \\ &+\; c^2\frac{1}{\Gamma(1-\beta)}X^{-\beta}c_0(T) + c^2\frac{1}{\Gamma(2-\beta)}X^{1-\beta}c_1(T) - a\frac{b_0(X)}{\Gamma(1-\gamma)}T^{-\gamma}\Big). \end{aligned} \tag{31}$$

We seek conditions under which (30) has a unique solution. Note that we can write (30) in the following form,

$$U = CUD + E, \tag{32}$$

where

$$C = \lambda(c^2 A), \quad D = (B_1 + a\Delta t^{\alpha-\gamma} B_2 + b\Delta t^{\alpha} I)^{-1}, \tag{33}$$

$$E = -F(X,T)\Big(B_1 + a\Delta t^{\alpha-\gamma} B_2 + b\Delta t^{\alpha} I\Big)^{-1}. \tag{34}$$

Applying the Picard iteration approach to (32), we get

$$\begin{array}{rcl} U^{(0)} & = & E, \\ U^{(1)} & = & CU^{(0)}D + E, \\ U^{(2)} & = & CU^{(1)}D + E, \\ & \cdots & \\ U^{(n+1)} & = & CU^{(n)}D + E, \end{array}$$

which implies that, with some suitable norm $\|\cdot\|$ in $\mathbf{R}^N$,

$$\begin{array}{rcl} \|U^{(n)} - U^{(n-1)}\| & = & \|C(U^{(n-1)} - U^{(n-2)})D\|, \\ & \leq & \|C\|\|U^{(n-1)} - U^{(n-2)}\|\|D\|, \\ & & \vdots \\ & \leq & (\|C\|\|D\|)^n \|U^{(1)} - U^{(0)}\|. \end{array} \tag{35}$$

Hence we see that (32) has a unique solution if

$$\|C\|\|D\| < 1.$$

Thus we obtain the following stability condition for the numerical method (30),

$$\lambda\|c^2 A\|\|(B_1 + a\Delta t^{\alpha-\gamma} B_2 + b\Delta t^{\alpha} I)^{-1}\| < 1,$$

that is

$$\lambda < \|c^2 A\|^{-1}\|B_1 + a\Delta t^{\alpha-\gamma} B_2 + b\Delta t^{\alpha+1} I\|.$$

Following the analysis above, we come to the following stability theorem for the numerical method (28).

Theorem 4.1. *Let $t_0 < t_1 < \cdots < t_m < \cdots < t_M = T$ be the time partition and Δt be the time step size. Let $x_0 < x_1 < \cdots < x_n < \cdots < x_N = X$ be the space partition and Δx be the space step size. Then (28) is stable if there exists an appropriate matrix norm such that*

$$0 < \lambda < \|B_1 + a\Delta t^{\alpha-\gamma} B_2 + b\Delta t^{\alpha} I\| / \|c^2 A\|, \tag{36}$$

hold, where $\lambda = \Delta t^{\alpha}/\Delta x^{\beta}$.

In particular, if b=0, the stability condition is reduced to

$$\lambda = \Delta t^{\alpha}/\Delta x^{\beta} < \|c^2 A\|^{-1}\|B_1 + a\Delta t^{\alpha-\gamma} B_2\|. \tag{37}$$

5. Numerical examples

In this section, we will introduce two examples of telegraph equations of fractional order. We compare the numerical solutions with the exact solutions and consider the convergence orders of the numerical method.

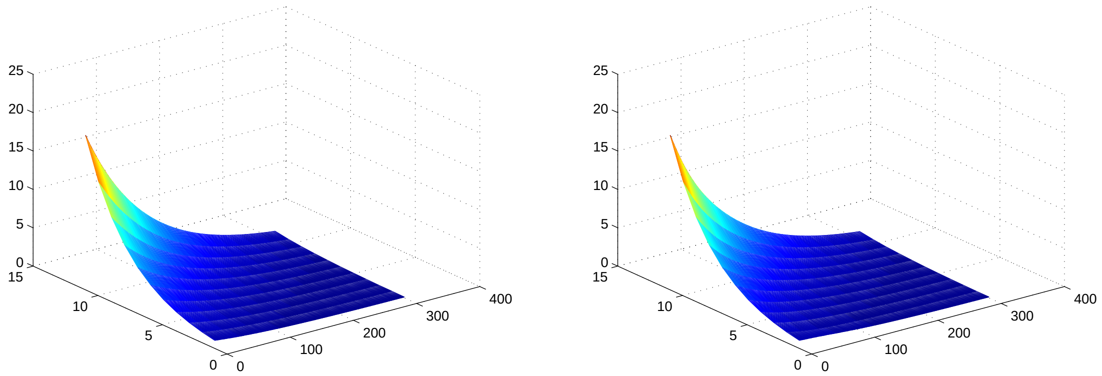

Figure 1: Example 5.1, analytical (left) and estimated (right) solutions at $T = 3$ and $X = 3$.

Example 5.1. *Consider the hyperbolic telegraph equation* (22)-(24) *with* $a = b = c = 1$. *Here, the initial conditions are*

$$\begin{aligned} u(0, x) &= e^{x}, & u_t(0, x) &= -e^{x}, \\ u(t, 0) &= e^{-t}, & u_x(t, 0) &= e^{-t}. \end{aligned}$$

The analytical solution is given as

$$u(t, x) = e^{x-t}.$$

The right hand side of this equation can be obtained by using the fractional derivatives of $u(t, x)$.

We apply the numerical methods discussed above. Choose $\Delta t = 0.01$, $\Delta x = 0.3$, and $T = 3$ and $X = 2$, the graphs of analytical and approximate solutions for $\alpha = 1.4$, $\beta = 1.9$ and $\gamma = 0.5$ are given in Figure 1.

Example 5.2. *Consider the hyperbolic telegraph equation* (22)-(24) *with* $a = c = 1$, $b = 10, \ 50$. *The initial conditions are*

$$\begin{aligned} u(0, x) &= \sin x, & u_t(0, x) &= -\sin x, \\ u(t, 0) &= 0, & u_x(t, 0) &= e^{-t}. \end{aligned}$$

The analytical solution is given as

$$u(t, x) = e^{-t} \sin x.$$

In this example, the right hand side function is approximated by using numerical quadrature formulae. Choose $\Delta t = 0.02$, $\Delta x = 2\pi/40$ and $T = 3$ and $X = 2\pi$. The graphs of analytical and approximate solutions for $\alpha = 1.4$, $\beta = 1.6$ and $\gamma = 0.5$ are given in Figure 2.

Below we will show the convergence orders of this method numerically. From the numerical experiments, as we would expect from the fractional ordinary differential equation case, we find that the convergence order is $2 - \beta$ with respect to the space step size Δx. The details of the parameters in these experiments have been given in Tables 1 - 3, where we choose $T = 3$ and $X = 2\pi$. We can see the different convergence orders at $T = 3$ in Figure 3. Similarly we can observe that the convergence order is $2 - \alpha$ with respect to the time step size Δt, again as expected.

Next we describe the stable region of the numerical method. Recall that the numerical method is stable if $1 - \|C\|_2 \|D\|_2 > 0$, unstable if $1 - \|C\|_2 \|D\|_2 < 0$, where C and D are defined in (33). Here $\| \cdot \|_2$ denotes the L_2 matrix norm. Note that the norms are equivalent in $\mathbf{R}^N$. For convenience, we choose L_2 matrix norm in our numerical simulations. Some stable and unstable values of Δx and Δt are given in Figures 4-5 and Tables 4-6.

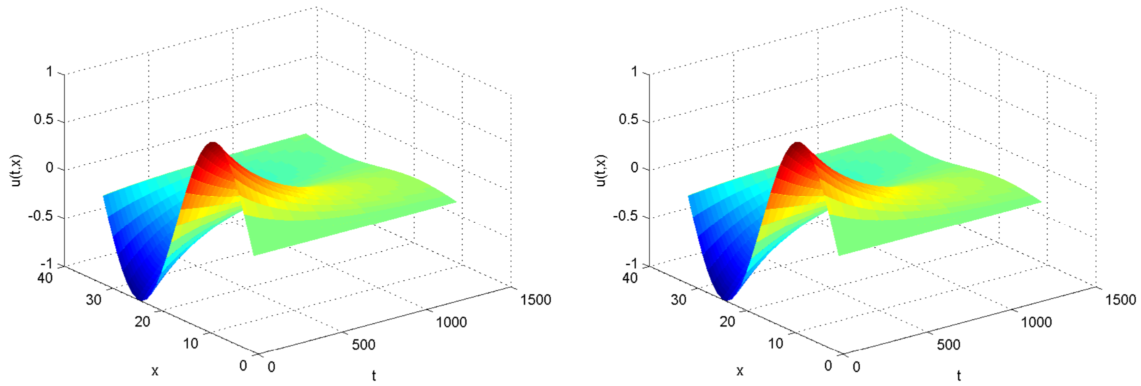

Figure 2: Example 5.2, analytical (left) and estimated (right) solutions at $T = 3$, $X = 2\pi$ and $c = 10$.

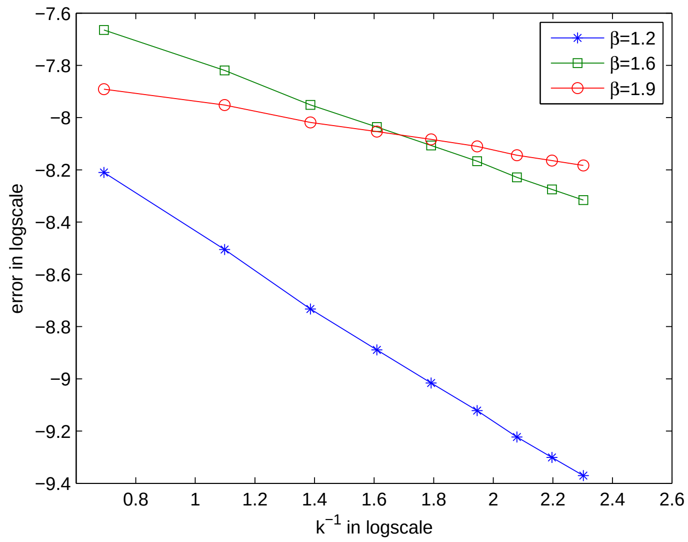

Figure 3: Example 5.2, the error as a function of Δt in logscale with different order β

Δx	L^2-norm of error	estimated convergence rate
1/2	2.7188×10^{-4}	
1/3	2.0244×10^{-4}	0.7274
1/4	1.6120×10^{-4}	0.7919
1/5	1.3789×10^{-4}	0.6999
1/6	1.2149×10^{-4}	0.6945
1/7	1.0926×10^{-4}	0.6882
1/8	9.8779×10^{-5}	0.7551
1/9	9.1338×10^{-5}	0.6650
1/10	8.5214×10^{-5}	0.6587

Table 1: Example 5.2, $\alpha = 1.4$, $\beta = 1.2$, $\gamma = 0.5$ and $\Delta t = 0.001$

Δx	L^2-norm of error	estimated convergence rate
1/2	4.6916×10^{-4}	
1/3	4.0192×10^{-4}	0.3815
1/4	3.5215×10^{-4}	0.4595
1/5	3.2356×10^{-4}	0.3794
1/6	3.0159×10^{-4}	0.3857
1/7	2.8399×10^{-4}	0.3902
1/8	2.6680×10^{-4}	0.4675
1/9	2.5492×10^{-4}	0.3869
1/10	2.4466×10^{-4}	0.3897

Table 2: Example 5.2, $\alpha = 1.4$, $\beta = 1.6$, $\gamma = 0.5$ and $\Delta t = 0.001$

Δx	L^2-norm of error	estimated convergence rate
1/2	3.7395×10^{-4}	
1/3	3.5198×10^{-4}	0.1558
1/4	3.2925×10^{-4}	0.1652
1/5	3.1800×10^{-4}	0.1726
1/6	3.0857×10^{-4}	0.2523
1/7	3.0047×10^{-4}	0.1740
1/8	2.9051×10^{-4}	0.1790
1/9	2.8462×10^{-4}	0.2588
1/10	2.7930×10^{-4}	0.1805

Table 3: Example 5.2, $\alpha = 1.4$, $\beta = 1.9$, $\gamma = 0.5$ and $\Delta t = 0.001$

In Figures 4 and 5, we show the different behaviours of the stable and unstable solutions of the equation (22) -(24) in Example 5.2. The solution of the equation in Figure 5 becomes nonsmooth and increases very fast, which means that the solution is unstable. The solution of the equation in Figure 4 is smooth and bounded even over a long time interval, which means the solution is stable.

We obtain some stability regions with respect to the different step sizes in Tables 4-6. Comparing the Tables 4 with 5, we observe, by similar parameters $\alpha = 1.4$, $\beta = 1.6$, $\gamma = 0.5$ and $\Delta t = 0.01$, that the stable region when $b = 50$ is wider than that with respect to $b = 10$ which is because the right hand side of (36) increases as b increases. On the other hand, in the Tables 5 and 6, for fixed $b = 50$, we obtain some clear information concerning the stable region of Δx with the different $\Delta t = 0.01$ or 0.001. We can observe that, in Table 5, $1 - \|C\|_2\|D\|_2 > 0$ when $\Delta x = 0.1$ which means the solution is stable, but in Table 6

$1 - \|C\|_2\|D\|_2 < 0$ when $\Delta x = 0.1$, which means the solution is unstable. In other words, the stable region of Δx with $\Delta t = 0.01$ is wider than that with $\Delta t = 0.001$, which can be observed in Figure 6. In this Figure, the numerical method is stable if Δx is larger than the x-coordinate of the cross point, where we plot the function of $1 - \|C\|_2\|D\|_2$ with respect to Δx. This is consistent with the theory since the value of the right hand side of the inequality (36) decreases when Δx decreases.

Δt	Δx	$1 - \|C\|_2\|D\|_2$	error
0.01	0.5	0.6678	2.7272×10^{-3}
0.01	0.4	0.5243	2.5071×10^{-3}
0.01	0.3	0.2448	2.2469×10^{-3}
0.01	0.2	-0.4469	0.1769
0.01	0.1	-3.3903	6.8542×10^{12}

Table 4: Example 5.2, $\alpha = 1.4$, $\beta = 1.6$, $\gamma = 0.5$, $b = 10$ and $\Delta t = 0.01$

Δt	Δx	$1 - \|C\|_2\|D\|_2$	error
0.01	0.5	0.9281	4.9520×10^{-4}
0.01	0.2	0.6867	3.4967×10^{-4}
0.01	0.1	4.9503	2.7087×10^{-4}
0.01	0.08	-0.3585	1.2477×10^{3}
0.01	0.07	-0.6822	1.0549×10^{10}
0.01	0.05	-1.1528	4.2644×10^{19}

Table 5: Example 5.2, $\alpha = 1.4$, $\beta = 1.6$, $\gamma = 0.5$, $b = 50$ and $\Delta t = 0.01$

Δt	Δx	$1 - \|C\|_2\|D\|_2$	error
0.001	0.2	0.6672	3.2356×10^{-4}
0.001	0.15	0.47343	2.8947×10^{-4}
0.001	0.1	-3.4934	2.5189×10^{10}
0.001	0.08	-5.4223	2.9147×10^{10}
0.001	0.07	-6.9513	9.1135×10^{20}

Table 6: Example 5.2, $\alpha = 1.4$, $\beta = 1.6$, $\gamma = 0.5$, $b = 10$ and $\Delta t = 0.001$

Then, we apply the stability condition discussed above in Theorem 4.1 to show some Figures about the stable regions in the $(\Delta t, \Delta x)$-plane with different parameter b.

In Figure 7, we plot the stable region in the case of $b = 0$. In this case, the stability condition is

$$\Delta x = \left(\Delta t^{\alpha} \frac{\|c^2 A\|}{\|B_1 + a\Delta t^{\alpha-\gamma} B_2\|}\right)^{\frac{1}{\beta}},$$

which is the boundary of the stable region. On the upper side of this curve, the numerical method is stable for pairs $(\Delta t, \Delta x)$. On the other side of curve, the numerical method is unstable for pairs $(\Delta t, \Delta x)$.

In Figures 8-9, we consider the stable region for $b \neq 0$. In this case, the boundary of the stable region is

$$\Delta x = \left(\Delta t^{\alpha} \frac{\|c^2 A\|}{\|B_1 + a\Delta t^{\alpha-\gamma} B_2 + b\Delta t^{\alpha} I\|}\right)^{\frac{1}{\beta}}. \tag{38}$$

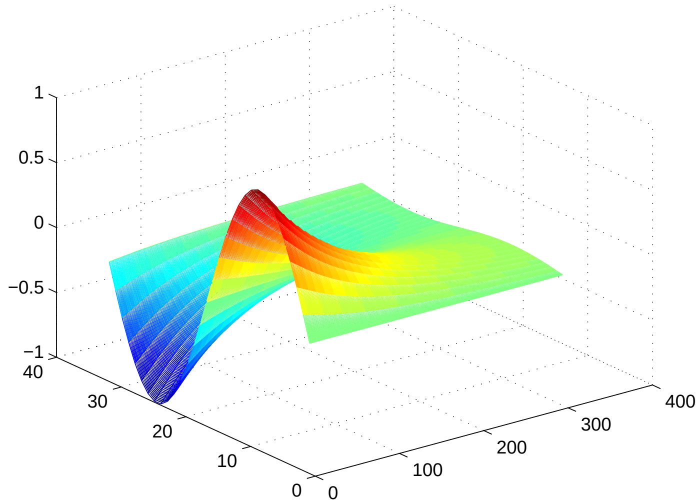

Figure 4: Example 5.2, $\alpha = 1.4$, $\beta = 1.6$, $\gamma = 0.5$, $\Delta t = 0.01$, $\Delta x = 0.2$ and $b = 50$

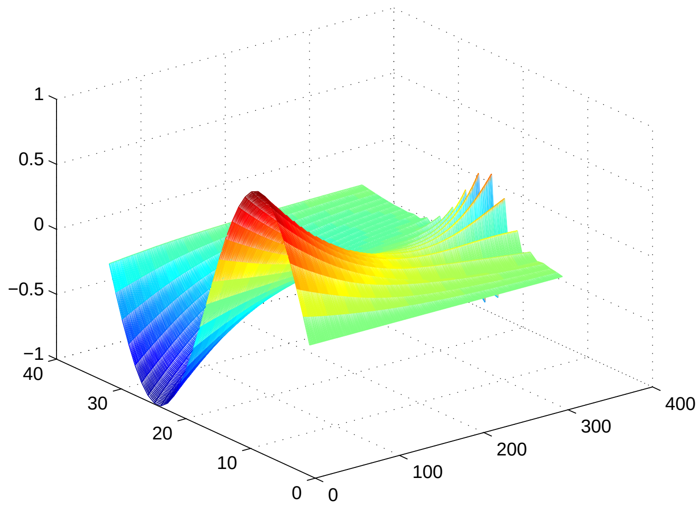

Figure 5: Example 5.2, $\alpha = 1.4$, $\beta = 1.6$, $\gamma = 0.5$, $\Delta t = 0.01$, $\Delta x = 0.2$ and $b = 10$

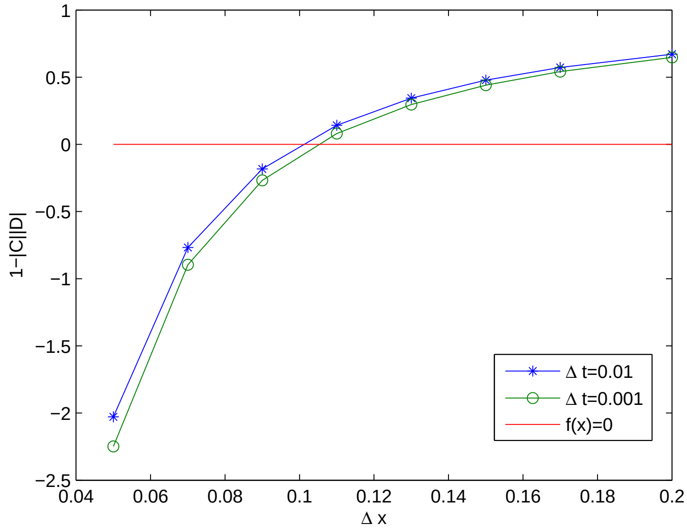

Figure 6: Example 5.2, $\alpha = 1.4$, $\beta = 1.6$, $\gamma = 0.5$, $\Delta t = 0.01$, $\Delta x = 0.2$ and $b = 10$

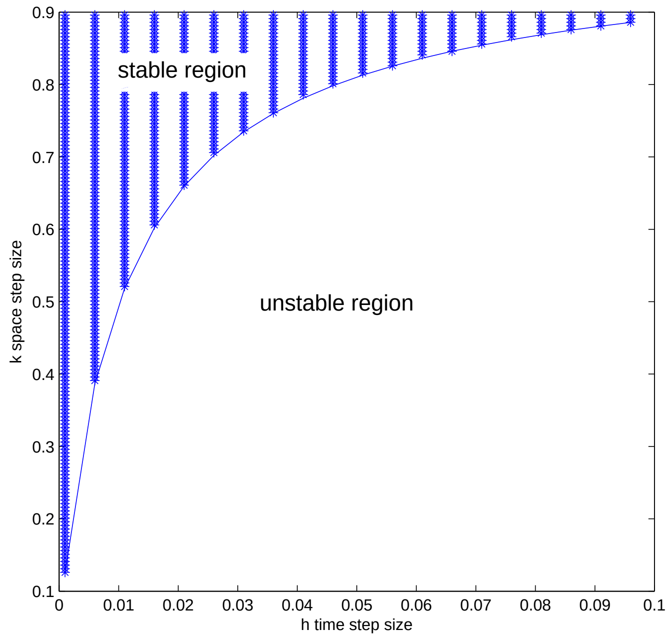

Figure 7: Stable region with $\alpha = 1.4$, $\beta = 1.6$, $\gamma = 0.5$, $b = 1$.

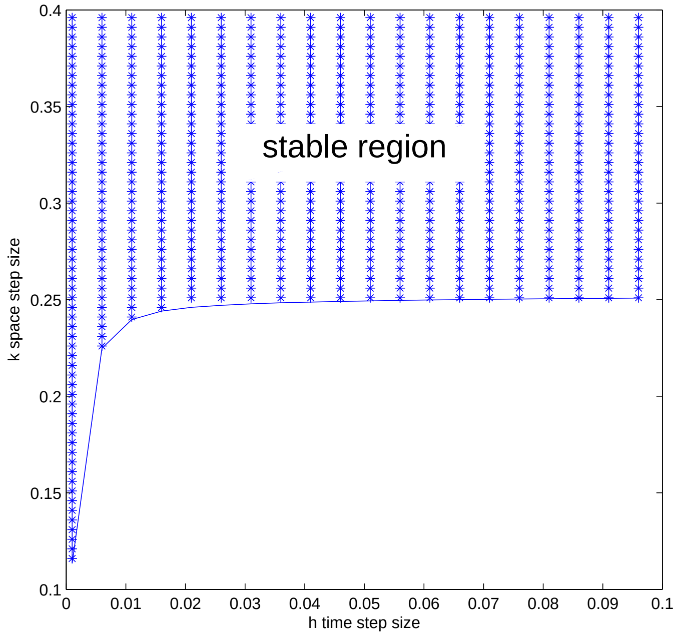

Figure 8: Stable region with $\alpha = 1.4$, $\beta = 1.6$, $\gamma = 0.5$, $b = 10$.

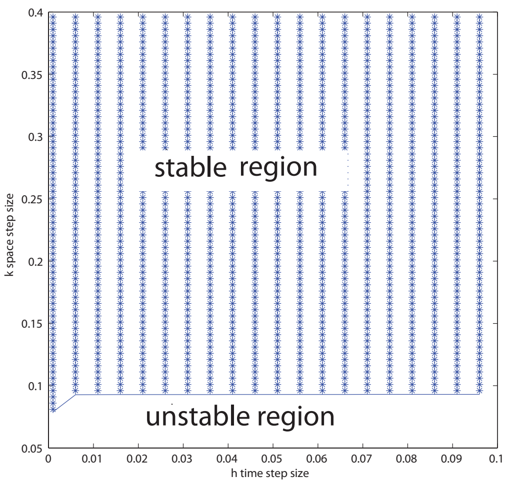

Figure 9: Stable region with $\alpha = 1.4$, $\beta = 1.6$, $\gamma = 0.5$, $b = 50$.

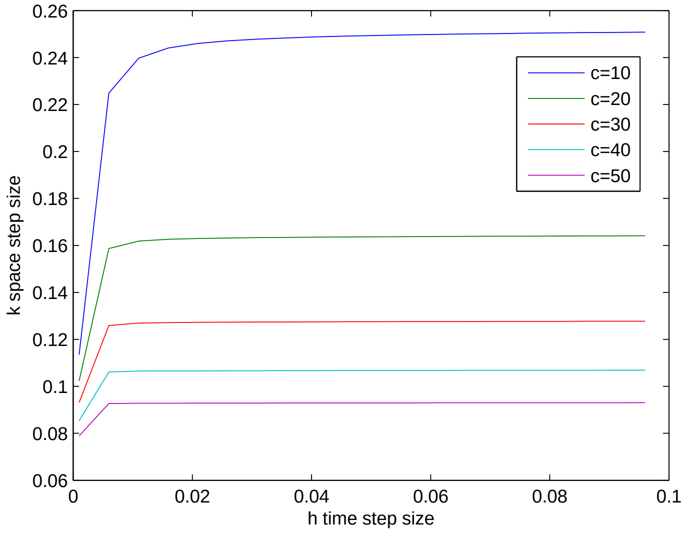

Figure 10: Stable boundaries for $\alpha = 1.4$, $\beta = 1.6$ and $\gamma = 0.5$ with varying b.

We observe that the stable region for $b = 10$ becomes bigger and wider than that for $b = 0$. In Figure 8 and Figure 9, the time step size Δt lies in $[0, 0.01]$ with $b = 10, 50$, respectively.

In Figure 10, we plot the different stable boundaries for the different parameters b. We observe that the stable region grows when the parameter b increases.

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