Stabilizing a mathematical model of plant species interaction

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Abstract

In this paper, we will consider how to stabilize a mathematical model of plant species interaction which is modelled by using Lotka-Volterra system. We first identify the unstable steady states of the system, then we use the feedback control based on the solutions of the Riccati equation to stabilize the linearized system. We further stabilize the nonlinear system by using the feedback controller obtained in the stabilization of the linearized system. We introduce the backward Euler method to approximate the feedback control nonlinear system and obtain the error estimates. Four numerical examples are given which come from the application areas.

Key words:
Lotka-Volterra system, feedback controller, Riccati equation, backward Euler method
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1. Introduction

Stabilizing a nonlinear system is a very important topic in application. By using the feedback control based on the Riccati equation of the linearized system, Barbu and Coca and Yan [2] considers how to stabilize a semilinear parabolic equation. The method of [2] has been extended to stabilizing a semilinear parabolic system [16] and Navier-Stokes equation [17]. The purpose of this paper is to consider how to stabilize a nonlinear ordinary system by using the idea of [2] and [9]. The numerical approximation scheme of the feedback control nonlinear system is introduced and the error estimates are proved. To our knowledge, we didn't find any error estimates of the feedback control nonlinear system based on the solutions of the Riccati equation in literature.

Why are we interested to study this topic? In our recent research [3], [4], we know that ecological systems behave like other real world systems which are expected to run over a longer period of time to enable clearer qualitative characteristics to be observed. Therefore, an interesting problem in this context is that of stability of ecological systems. In particular, knowledge about the steady state solutions and stability may provide vital information for

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ecological studies in predicting the future states of the plant community [4]. Feedback control laws are also important in ecological studies as they can be used to control the outcome of competition between interacting populations described by a system of coupled nonlinear ordinary differential equations. In contrast, without control, one of the plant species is more likely to be driven to extinction. Within the ecological literature, ecosystem stability is an important feature of ecosystems [6], [11], [8].

What are the applications of our present study in ecological studies? The key contribution of our present work is to numerically estimate the depletion rates of two plant species by stabilizing unstable interacting ecological populations. Comparisons of these depletion rates can provide useful information to guide against severe depletion which other studies are yet to estimate as far as we know. These results would be of immense application in ecosystem monitoring and decision making against species extinction which would enhance the ideas and norms of ecological services in the sustainability of human life. Since our model equations of competition interaction are unstable, if they are to accurately model real ecosystems it is inevitable to find the mechanisms of stabilization [6].

There are some works related to this paper. For example, [12] has used an integral quadratic cost functional to obtain a quasi-optimum feedback control law for two competing species whose dynamics are described by the well established mathematical formulation of Volterra's competition equations. Next [13] also applied optimal control theory which has an integral linear cost functional to control a prey-predator system described by the Lotka-Voterra model equations. Similarly, [5] and [14] have studied optimal control of prey-predator systems which are described by the Lotka-Volterra equations. More recently, [10] have applied the methods from optimal control theory and from the theory of dynamical systems to the mathematical modelling of biological pest control.

The paper is organized as follows: In Section 2, we introduce the steady states of a nonlinear system and the stabilization theories by using the feedback control based on the solutions of the Riccati equation of the linearized system. In Section 3, we introduce the backward Euler method and prove an error estimate. In Section 4, we use our numerical method to consider four examples which come from the application areas. The conclusion is in Section 5.

2. Stabilization of steady states for a nonlinear system

Let us consider the steady states of the following general nonlinear system

$$\frac{dy}{dt} = y(t)(a_1 - b_1 y(t) - c_1 z(t)), \tag{1}$$

$$\frac{dz}{dt} = z(t)(a_2 - b_2 y(t) - c_2 z(t)), \tag{2}$$

with initial conditions $y(0) = y_0 > 0,\ z(0) = z_0 > 0$. Here $a_i, b_i, c_i,\ i = 1, 2$ are positive constants. The steady state (y_e, z_e) satisfies

$$y_e(a_1 - b_1 y_e - c_1 z_e) = 0, \tag{3}$$

$$z_e(a_2 - b_2 y_e - c_2 z_e) = 0, \tag{4}$$

which implies that there are four steady states

$$\begin{aligned} y_e &= 0, \quad z_e = 0, \\ y_e &= 0, \quad z_e = \frac{a_2}{c_2}, \\ y_e &= \frac{a_1}{b_1}, \quad z_e = 0, \\ y_e &= \frac{a_1c_2 - c_1a_2}{b_1c_2 - c_1b_2}, \quad z_e = \frac{b_1a_2 - a_1b_2}{b_1c_2 - c_1b_2}. \end{aligned}$$

To determine the stability of the steady state (y_e, z_e), we need to consider the linearized system of (1)- (2) about (y_e, z_e). Denote

$$\begin{aligned} F(y, z) &= y(a_1 - b_1y - c_1z), \\ G(y, z) &= z(a_2 - b_2y - c_2z). \end{aligned}$$

By using Taylor series, we have

$$\begin{aligned} F(y, z) &= F(y_e, z_e) + \frac{\partial F(y_e, z_e)}{\partial y}(y - y_e) + \frac{\partial F(y_e, z_e)}{\partial z}(z - z_e) + \text{ higher order terms}, \\ G(y, z) &= G(y_e, z_e) + \frac{\partial G(y_e, z_e)}{\partial y}(y - y_e) + \frac{\partial G(y_e, z_e)}{\partial z}(z - z_e) + \text{ higher order terms}. \end{aligned}$$

Hence we get the linearized system of (1)-(2)

$$\frac{d}{dt}\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = A\begin{bmatrix} y(t) - y_e \\ z(t) - z_e \end{bmatrix}, \quad \begin{bmatrix} y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \tag{5}$$

where

$$A = \begin{bmatrix} \frac{\partial F(y_e,z_e)}{\partial y} & \frac{\partial F(y_e,z_e)}{\partial z} \\ \frac{\partial G(y_e,z_e)}{\partial y} & \frac{\partial G(y_e,z_e)}{\partial z} \end{bmatrix}.$$

Lemma 2.1. *Assume that all the eigenvalues of A are negative, then the solution of* (5) *tends to the steady state* $\begin{bmatrix} y_e \\ z_e \end{bmatrix}$ *as $t \to \infty$ for some suitable initial value* $\begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$.

Proof 2.2. *Substituting $y - y_e$ and $z - z_e$ by Y and Z seperately and denoting* $u = \begin{bmatrix} Y \\ Z \end{bmatrix}$, *we can write the linearized system* (5) *into*

$$\frac{du}{dt} = Au, \quad u(0) = u_0, \tag{6}$$

where $u_0 = \begin{bmatrix} y_0 - y_e \\ z_0 - z_e \end{bmatrix}$.

It is sufficient to prove that the system (6) *tends to the steady state* $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *as* $t \to \infty$ *for some suitable initial value* $u_0 = \begin{bmatrix} y_0 - y_e \\ z_0 - z_e \end{bmatrix}$ *which we will prove now.*

Note that the solution of (6) *has the form*

$$u(t) = e^{tA} u_0. \tag{7}$$

Assume that A *has two eigenvalues* λ_1, λ_2*, and a corresponding basis of orthonormal eigenfunction* $\{e_i\}_{j=1}^2 \in \mathbf{R}^2$*. For any function* $g(\lambda)$*, we define the spectrum* $\sigma(A) = \{\lambda_j\}_{j=1}^2$ *of* A*, and set*

$$g(A)v = \sum_{j=1}^{2} g(\lambda_j)(v, e_j)e_j, \quad \textit{for } v \in \mathbf{R}^2,$$

where $(\cdot, \cdot)$ *denotes the inner product in* $\mathbb{R}^2$*. Denote* $\|\cdot\| = (\cdot, \cdot)^{1/2}$ *the Euclidean norm. We then have,*

$$\|g(A)\| = \max_j |g(\lambda_j)| = \max_{\lambda \in \sigma(A)} |g(\lambda)|. \tag{8}$$

In fact, by the definition of the operator norm,

$$\|g(A)\| = \sup_{v \neq 0} \frac{\|g(A)v\|}{\|v\|} \geq \frac{\|g(A)e_j\|}{\|e_j\|} = |g(\lambda_j)|, \ j = 1, 2,$$

which implies that

$$\|g(A)\| \geq \max_{\lambda \in \sigma(A)} |g(\lambda)|.$$

On the other hand, by Parseval's relation, we have

$$\begin{aligned} \|g(A)v\|^2 &= \sum_{j=1}^{2} \big((v, e_j)g(\lambda_j)\big)^2 \leq \big(\max_j |g(\lambda_j)|\big)^2 \sum_{j=1}^{2} (v, e_j)^2 \\ &= \big(\max_{\lambda \in \sigma(A)} |g(\lambda)|\big)^2 \|v\|^2, \end{aligned}$$

which implies that

$$\|g(A)\| = \sup_{v \neq 0} \frac{\|g(A)v\|}{\|v\|} \leq \max_{\lambda \in \sigma(A)} |g(\lambda)|.$$

Hence we have proved (8)*.*

By (7)*, we have*

$$\|u(t)\| = \|e^{tA} u_0\| \leq \|e^{tA}\| \|u_0\| \leq \max_{\lambda \in \sigma(A)} e^{t\lambda} \|u_0\|.$$

By assumption, A *has two negative eigenvalues, we therefore get*

$$\lim_{t \to \infty} \|u(t)\| = 0.$$

The proof is complete.

If A has at least one positive eigenvalue, then the steady state (y_e, z_e) is not stable, i.e., $(y(t), z(t))$ will not tend to (y_e, z_e) as $t \to \infty$. Then we will use the feedback control to stabilize the steady state. We can prove the following theorems following the approach in [9] and [2]. The first theorem is the stabilization of the linearized system (5) at an unstable steady state.

Theorem 2.3. *Assume that* $\begin{bmatrix} y_e \\ z_e \end{bmatrix}$ *is an unstable steady state of the system* (1) - (2)*, then there exists* $V : [0, \infty) \to \mathbb{R}^2$ *such that the following linearized feedback control system of* (5)

$$\frac{d}{dt}\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = A\begin{bmatrix} y(t) - y_e \\ z(t) - z_e \end{bmatrix} + BV(t), \quad \begin{bmatrix} y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \tag{9}$$

is exponentially stable at $\begin{bmatrix} y_e \\ z_e \end{bmatrix}$. *Here*

$$V(t) = -R^{-1}B^*\Pi \begin{bmatrix} y(t) - y_e \\ z(t) - z_e \end{bmatrix},$$

and Π *satisfies the Riccati equation*

$$A^*\Pi + \Pi A - \Pi BB^*\Pi + Q = 0. \tag{10}$$

Here $R = 1$ *and* Q *is any positive definite matrix and* $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *or* $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

More precisely, there exists $\rho > 0$ *such that for all* $\begin{bmatrix} y_0 \\ z_0 \end{bmatrix} : \left\| \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} - \begin{bmatrix} y_e \\ z_e \end{bmatrix} \right\| < \rho$, *there exists a unique solution* $\begin{bmatrix} y \\ z \end{bmatrix} \in C^1(0, \infty, \mathbb{R}^2)$, *such that, with some constant* C *and* $\gamma > 0$,

$$\left\| \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} - \begin{bmatrix} y_e \\ z_e \end{bmatrix} \right\| < Ce^{-\gamma t} \left\| \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \right\|.$$

The next theorem is the stabilization of the nonlinear system (1)-(2). We have

Theorem 2.4. *Assume that* $\begin{bmatrix} y_e \\ z_e \end{bmatrix}$ *is an unstable steady state of the system* (1) - (2). *Then*

$$V(t) = -R^{-1}B^*\Pi \begin{bmatrix} y(t) - y_e \\ z(t) - z_e \end{bmatrix},$$

will stabilize exponentially the nonlinear system at $\begin{bmatrix} y_e \\ z_e \end{bmatrix}$,

$$\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} F(y, z) \\ G(y, z) \end{bmatrix} + BV(t). \tag{11}$$

Here Π *is obtained by* (10).

More precisely, there exists $\rho > 0$ *such that for all* $\begin{bmatrix} y_0 \\ z_0 \end{bmatrix} : \left\| \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} - \begin{bmatrix} y_e \\ z_e \end{bmatrix} \right\| < \rho$*, there exists a unique solution* $\begin{bmatrix} y \\ z \end{bmatrix} \in C^1(0, \infty, \mathbb{R}^2)$*, such that, with some constant* C *and* $\gamma > 0$,

$$\left\| \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} - \begin{bmatrix} y_e \\ z_e \end{bmatrix} \right\| < Ce^{-\gamma t} \left\| \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \right\|.$$

3. error estimates

In this section, we will consider the numerical approximation of the nonlinear feedback control system (11) and prove an error estimate result.

Let $0 = t_0 < t_1 < t_2 < \dots$ be time points, and $k = t_j - t_{j-1}$ be time step. We use y^n to denote the approximation of $y(t_n)$. Using the backward Euler method, we define the following difference scheme of (11)

$$\begin{bmatrix} \frac{y^n - y^{n-1}}{k} \\ \frac{z^n - z^{n-1}}{k} \end{bmatrix} = \begin{bmatrix} F(y^n, z^n) \\ G(y^n, z^n) \end{bmatrix} - R^{-1}BB^*\Pi \begin{bmatrix} y^{n-1} - y_e \\ z^{n-1} - z_e \end{bmatrix}, \tag{12}$$

with initial value (y^0, z^0). Then we get the sequences $(y^n, z^n), n = 1, 2, \dots$. By Theorem 2.4, we have $(y^n, z^n) \to (y_e, z_e)$ as $n \to \infty$.

Let us consider the error estimate of (12). Substituting $y - y_e$ and $z - z_e$ by Y and Z, then (11) becomes

$$\frac{d}{dt} \begin{bmatrix} Y \\ Z \end{bmatrix} = \begin{bmatrix} F(Y + y_e, Z + z_e) \\ G(Y + y_e, Z + z_e) \end{bmatrix} + BV(t). \tag{13}$$

Here $V(t) = -RB^*\Pi \begin{bmatrix} Y \\ Z \end{bmatrix}$. Denote $u = \begin{bmatrix} Y \\ Z \end{bmatrix}$ and $F(u) = \begin{bmatrix} F(Y + y_e, Z + z_e) \\ G(Y + y_e, Z + z_e) \end{bmatrix}$. Then we can write (13) into

$$\frac{du}{dt} = F(u) - RBB^*\Pi u, \tag{14}$$

or

$$\frac{du}{dt} + A_1 u = F(u). \tag{15}$$

Here $A_1 = RBB^*\Pi$.

Denote U^n as the approximation of $u(t_n)$. We define the following backward Euler method for the abstract form (15)

$$\frac{U^n - U^{n-1}}{k} + A_1 U^n = F(U^{n-1}). \tag{16}$$

or, with $r(\lambda) = (1 + \lambda)^{-1}$,

$$U^n = r(kA_1)U^{n-1} + kr(kA_1)U^{n-1}. \tag{17}$$

In our paper, we assume that F satisfies the global Lipschitz condition and growth condition, i.e., there exist $C_1 > 0, C_2 > 0$ such that

$$\|F(u) - F(v)\| \leq C_1 \|u - v\|, \ \forall u, v \in \mathbb{R}^2. \tag{18}$$

and

$$\|F(u)\| \leq C_2 \|u\| \tag{19}$$

Note that A_1 is a positive definite matrix for our choice of $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, see [9] and [2], it is easy to prove the following lemma.

Lemma 3.1. *We have, with* $E(t) = e^{-tA_1}$,

$$\|A_1 E(t)\| \leq C t^{-1},$$

and

$$\|A_1^{-\gamma}(E(t) - I)\| \leq C t^{\gamma}, \quad 0 \leq \gamma \leq 1.$$

Here the operator $A_1^{-\gamma}$ *is defined by* $A_1^{-\gamma} v = \sum_{j=1}^{2} \mu_j^{-\gamma} (v, \varphi_j) \varphi_j$ *for any* $v \in \mathbf{R}^2$. *Here* $(\mu_j, \varphi_j), j = 1, 2$ *are the eigen pairs of the operator* $A_1 : \mathbf{R}^2 \to \mathbf{R}^2$.

Next we will prove a regularity result of the solution of (15).

Lemma 3.2. *Assume that* $u(t)$ *is the solution of* (15). *Then we have, for any* $0 \leq t_1 < t_2 \leq T$, *with* $l_k = |\ln(t_2 - t_1)|$,

$$\|u(t_2) - u(t_1)\| \leq C(t_2 - t_1) \Big(\|A_1 u_0\| + l_k \max_{0 \leq s \leq T} \|u(s)\| \Big).$$

Proof 3.3. *By Duhamel's principle, the solution of* (15) *may be written, with* $E(t) = e^{-tA_1}$,

$$u(t) = E(t) u_0 + \int_0^t E(t - s) F(u(s)) \, ds.$$

Thus, we have

$$\begin{aligned} u(t_2) - u(t_1) &= E(t_2) u_0 - E(t_1) u_0 + \int_0^{t_2} E(t_2 - s) F(u(s)) \, ds - \int_0^{t_1} E(t_1 - s) F(u(s)) \, ds \\ &= I + II. \end{aligned}$$

For I, *we have, by Lemma 3.1 and the stability of* $E(t)$, *that is* $\|E(t)\| \leq C$,

$$\begin{aligned} \|I\| = \|E(t_2) u_0 - E(t_1) u_0\| &= \|E(t_1) A_1^{-1} \big(E(t_2 - t_1) - I \big) A_1 u_0\| \\ &\leq C(t_2 - t_1) \|E(t_1) A_1 u_0\| \leq C(t_2 - t_1) \|A_1 u_0\|. \end{aligned}$$

Here A_1^{-1} *denote the inverse of* A_1.

For II, we have

$$\begin{aligned}
\|II\| &= \Big\| \int_0^{t_2} E(t_2-s)F(u(s)) - \int_0^{t_1} E(t_1-s)F(u(s))\,ds \Big\| \\
&\le \Big\| \int_0^{t_1} \big(E(t_2-s)-E(t_1-s)\big)F(u(s))\,ds \Big\| + \Big\| \int_{t_1}^{t_2} E(t_2-s)F(u(s))\,ds \Big\| \\
&= II_1 + II_2.
\end{aligned}$$

We first consider II_1. Let $k = t_2 - t_1$. Assume that $t_1 \le k$, we have

$$\begin{aligned}
II_1 &\le \int_0^{t_1} \|\big(E(t_2-s)-E(t_1-s)\big)F(u(s))\|\,ds \le \int_0^{t_2-t_1} C\|F(u(s))\|\,ds \qquad (20)\\
&\le C(t_2-t_1)\max_{0\le s\le T}\|F(u(s))\|.
\end{aligned}$$

Assume that $t_1 > k$, then we have, with $l_k = |\ln(k)|$, by Lemma 3.1,

$$\begin{aligned}
II_1 &\le \Big\| \int_0^{t_1-k} \big(E(t_2-s)-E(t_1-s)\big)F(u(s))\,ds \Big\| + \Big\| \int_{t_1-k}^{t_1} \big(E(t_2-s)-E(t_1-s)\big)F(u(s))\,ds \Big\| \\
&\le \Big\| \int_0^{t_1-k} A_1^{-1}\big(E(t_2-t_1)-I\big)A_1E(t_1-s)\big)F(u(s))\,ds \Big\| + C\int_{t_1-k}^{t_1} \|F(u(s))\|\,ds \\
&\le \Big\| \int_0^{t_1-k} A_1^{-1}\big(E(t_2-t_1)-I\big)A_1E(t_1-s)\big)F(u(s))\,ds \Big\| + C\int_{t_1-k}^{t_1} \|F(u(s))\|\,ds \\
&\le \int_0^{t_1-k} C(t_2-t_1)\|A_1E(t_1-s)\big)F(u(s))\|\,ds + C\int_{t_1-k}^{t_1} \|F(u(s))\|\,ds \\
&\le \int_0^{t_1-k} C(t_2-t_1)(t_1-s)^{-1}\|F(u(s))\|\,ds + C\int_{t_1-k}^{t_1} \|F(u(s))\|\,ds \\
&\le C(t_2-t_1)l_k\int_0^{t_1-k} \|F(u(s))\|\,ds + C\int_{t_1-k}^{t_1} \|F(u(s))\|\,ds \\
&\le C(t_2-t_1)l_k\max_{0\le s\le T}\|F(u(s))\|.
\end{aligned}$$

Together this with (20) *shows that*

$$II_1 \le C(t_2-t_1)l_k\max_{0\le s\le T}\|F(u(s))\|.$$

We now consider II_2. We have

$$II_2 \le C\int_{t_1}^{t_2} \|F(u(s)\|\,ds \le C(t_2-t_1)\max_{0\le s\le T}\|F(u(s))\|.$$

Together these estimates and (19) *completes the proof of the lemma.*

We have the following error estimates.

Theorem 3.4. *Let $T > 0$ and let $u(t_n), 0 \leq t_n \leq T$ and U^n be the solutions of* (15) *and* (17) *respectively. Assume that F satisfies the global Lipschitz condition* (18) *and growth condition* (19)*. Let k be the time step. Then there exists a constant $C(T)$ such that, with $l_k = |ln(k)|$,*

$$\|U^n - u(t_n)\| \leq C(T)k\Big(\|A_1 u_0\| + l_k \max_{0\leq s\leq T} \|u(s)\|\Big).$$

Proof 3.5. *By Duhamel's principle, the solution of* (15) *may be written, with $E(t) = e^{-tA_1}$,*

$$u(t) = E(t)u_0 + \int_0^t E(t-s)F(u(s))\,ds.$$

Further U^n can be written in the form

$$U^n = r(kA_1)^n U^0 + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} r(kA_1)^{n-j+1} F(U^j)\,ds.$$

Denoting $e^n = U^n - u(t_n)$ and $F_n = r(kA_1)^n - E(t_n)$, we have

$$\begin{aligned} e^n &= F_n u_0 + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} r(kA_1)^{n-j+1}\big(F(U^j) - F(u(t_j))\big)\,ds \\ &+ \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} r(kA_1)^{n-j+1}\big(F(u(t_j)) - F(u(s))\big)\,ds \\ &+ \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \big(r(kA_1)^{n-j+1} - E(t_n - t_{j-1})\big)F(u(s))\,ds \\ &+ \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \big(E(t_n - t_{j-1}) - E(t_n - s)\big)F(u(s))\,ds \\ &= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

Thus

$$\|e^n\| \leq \sum_{j=1}^{5} \|I_j\|.$$

Now we will estimate $\|I_j\|, j = 1, 2, 3, 4, 5$.

For I_1, we have, see [15],

$$\|I_1\| = \|F_n u_0\| = \|(r(kA_1) - e^{-kA_1})u_0\| \leq Ck\|u_0\|.$$

For I_2, noting that $\|r(kA_1)\| \leq 1$, we have, by (18),

$$\|I_2\| \leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|F(U^j) - F(u(t_j))\|\,ds \leq C \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|e_j\|\,ds.$$

For I_3, we have, by Lemma 3.2,

$$\|I_3\| \leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|u(t_j) - u(s)\| \, ds \leq C(T)k\Big(\|A_1 u_0\| + l_k \max_{0 \leq s \leq T} \|F(u(s))\|\Big).$$

For I_4, we have

$$\begin{aligned}
\|I_4\| &= \sum_{j=1}^{n} \Big\| \int_{t_{j-1}}^{t_j} r(kA)^{n-j+1} - E(t_n - t_{j-1}) F(u(s)) \, ds \Big\| \\
&= \sum_{j=1}^{n} \Big\| \int_{t_{j-1}}^{t_j} F_{n-j+1} F(u(s)) \, ds \Big\| \leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|F_{n-j+1} F(u(s))\| \, ds \\
&\leq \sum_{j=1}^{n} k \|F_j\| \max_{0 \leq s \leq T} \|F(u(s))\|.
\end{aligned}$$

Using the same idea as in [15], we have

$$k \sum_{j=1}^{n} \|F_j\| = k \sum_{j=1}^{n} \Big(\sup_{v \neq 0} \frac{\|F_j v\|}{\|v\|} \Big) = \sup_{v \neq 0} \frac{k \sum_{j=1}^{n} \|F_j v\|}{\|v\|} \leq Ck,$$

which implies that

$$\|I_4\| \leq Ck \max_{0 \leq s \leq T} \|F(u(s))\|.$$

For I_5, we have, by Lemma 3.1,

$$\begin{aligned}
\|I_5\| &= \Big\| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \big(E(t_n - t_{j-1}) - E(t_n - s)\big) F(u(s)) \, ds \Big\| \\
&\leq \Big\| \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \big(E(t_n - t_{j-1}) - E(t_n - s)\big) F(u(s)) \, ds \Big\| \\
&\quad + \Big\| \int_{t_{n-1}}^{t_n} \big(E(t_n - t_{n-1}) - E(t_n - s)\big) F(u(s)) \, ds \Big\| \\
&\leq \Big\| \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} A_1 E(t_n - s) A_1^{-1} \big(E(s - t_{j-1}) - I\big) F(u(s)) \, ds \Big\| + Ck \max_{0 \leq s \leq T} \|F(u(s))\| \\
&\leq \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} Ck(t_n - s)^{-1} \|F(u(s))\| \, ds + Ck \max_{0 \leq s \leq T} \|F(u(s))\| \\
&\leq Ck \Big(\int_{0}^{t_{n-1}} (t_n - s)^{-1} \, ds \Big) \max_{0 \leq s \leq T} \|F(u(s))\| + Ck \max_{0 \leq s \leq T} \|F(u(s))\| \\
&\leq Ckl_k \max_{0 \leq s \leq T} \|F(u(s))\|.
\end{aligned}$$

Combining these estimates and (19)*, we have, by using the Gronwall lemma*

$$\|e^n\| \leq C(T)k\Big(\|A_1u_0\| + l_k \max_{0\leq s\leq T} \|u(s)\|\Big).$$

The proof is complete.

4. Some Examples

4.1. Example 1

The first example is a system of nonlinear first order ordinary differential equations [3].

$$\frac{dN_1}{dt} = N_1(t)(0.168 - 0.0020339N_1(t) - 0.0005N_2(t)) \tag{21}$$

$$\frac{dN_2}{dt} = N_2(t)(0.002 - 0.00002N_1(t) - 0.000015N_2(t)) \tag{22}$$

with initial starting values $N_1 = 0.045$ grams per area of plant species cover, $N_2 = 0.045$ grams per area of plant species cover.

The detailed idea of this model formulation has been defined and discussed by [3]. The model parameters are estimated under the regime of a 70-day growing season. This system of equations is characterised by four steady states namely $(0, 0)$, $(82.59993, 0)$, $(0, 133.3333)$, and $(74, 34.5)$. The first three steady states are unstable and would need to be stabilized whereas the coexistence steady state is stable and may not require a further stabilization.

In Figure 1, we illustrate the stability of the steady state $(0, 0)$ in both uncontrolled and controlled cases. In Figure 2, we illustrate the stability of the steady state $(82.59993, 0)$ in both uncontrolled and controlled cases. In Figure 3, we illustrate the stability of the steady state $(0, 133.3333)$ in both uncontrolled and controlled cases.

4.2. Example 2

The second example below is a prey-predator system [1] whose uncontrolled dynamical equations are

$$\frac{dN_1}{dt} = N_1(t)\Big(r(1 - N_1(t)/K) - kN_2(t)((1 - e^{-\gamma N_1(t)})/N_1(t))\Big) = N_1(t)f_1(N_1(t), N_2(t)),$$
$$\frac{dN_2}{dt} = N_2(t)\Big(-b + \beta(1 - e^{-\mu N_1(t)}\Big)$$

where N_1 and N_2 denote the sizes of the prey, and the predator populations, respectively; $b, r, k, K, \beta, \gamma, \mu$ are positive constants. The unique steady state is $(N1e, N2e)$, where

$$N1e = (1/\mu)\log(\beta/(\beta - b)), \quad N2e = \frac{r}{k}\Big(1 - \frac{N1e}{K}\Big)\frac{N1e}{1 - e^{-\gamma N1e}}.$$

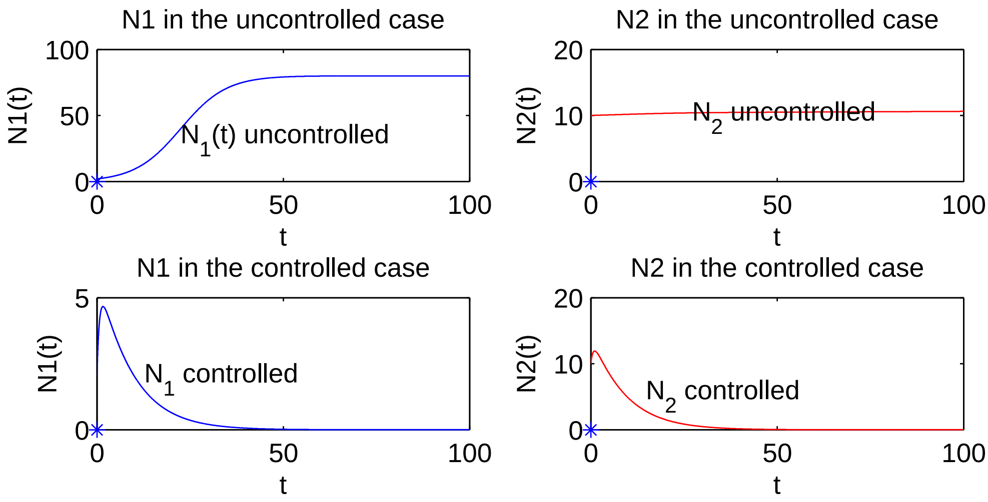

Figure 1: Uncontrolled and controlled cases at the steady state (0,0)

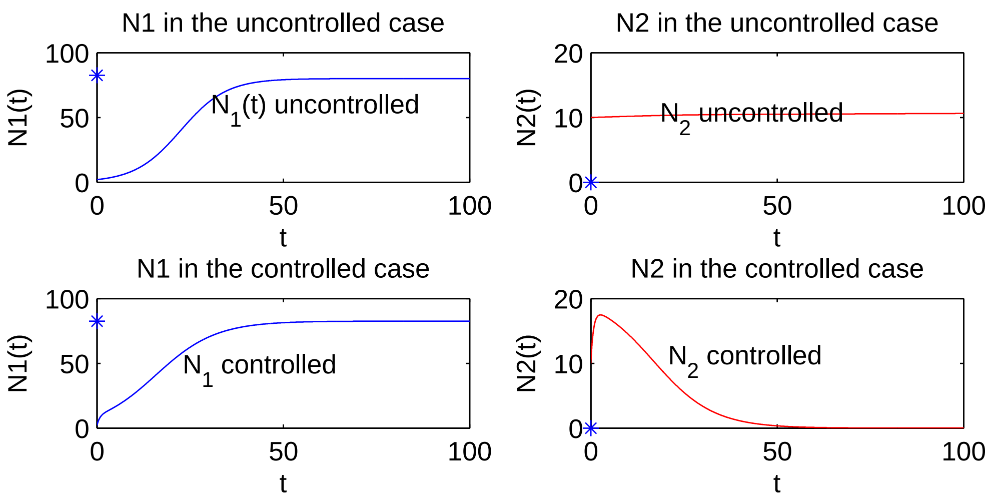

Figure 2: Uncontrolled and controlled cases at the steady state (82.59993,0)

This steady state is stable when

$$K < N1e\Big(1 + \Big(1 - \frac{\gamma N1e}{e^{\gamma N1e} - 1}\Big)^{-1}\Big),$$

and unstable, if this last inequality is reversed.

In our numerical simulation, we choose $b = 1,\ r = 1,\ k = 1,\ \beta = 2,\ \gamma = 1; \mu = 1$. Then we can calculate the steady state $(N1e, N2e)$. Denote

$$K_0 = N1e\Big(1 + \Big(1 - \frac{\gamma N1e}{e^{\gamma N1e} - 1}\Big)^{-1}\Big).$$

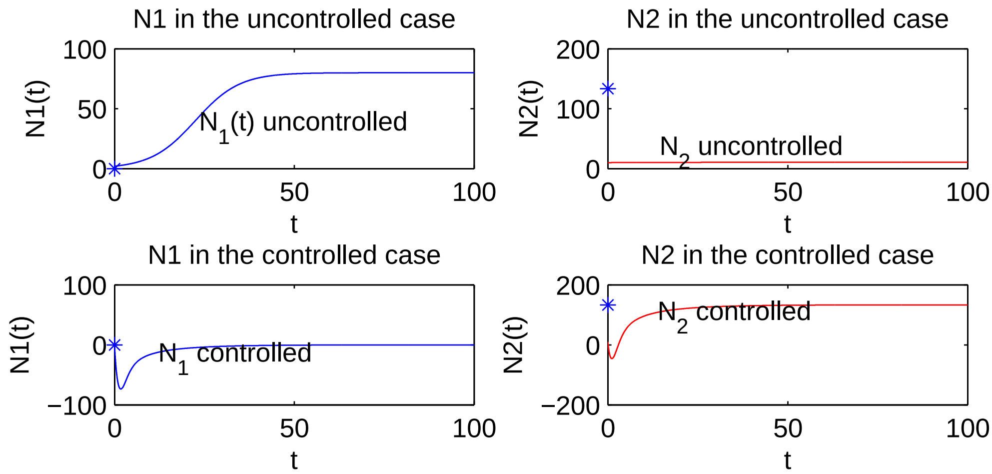

Figure 3: Uncontrolled and controlled cases at the steady state (0, 133.3333)

We choose $K = K_0 + 1 > K_0$. Then the nonlinear system is unstable at the steady state. By using the method in this paper, we can stabilize this steady state. Figure 4 shows the unstable and the stable solutions. Clearly we observe that the controlled system is stable at the steady state.

In the simulation, we choose the initial value $(2, 1)$. The final time is $T = 20$.

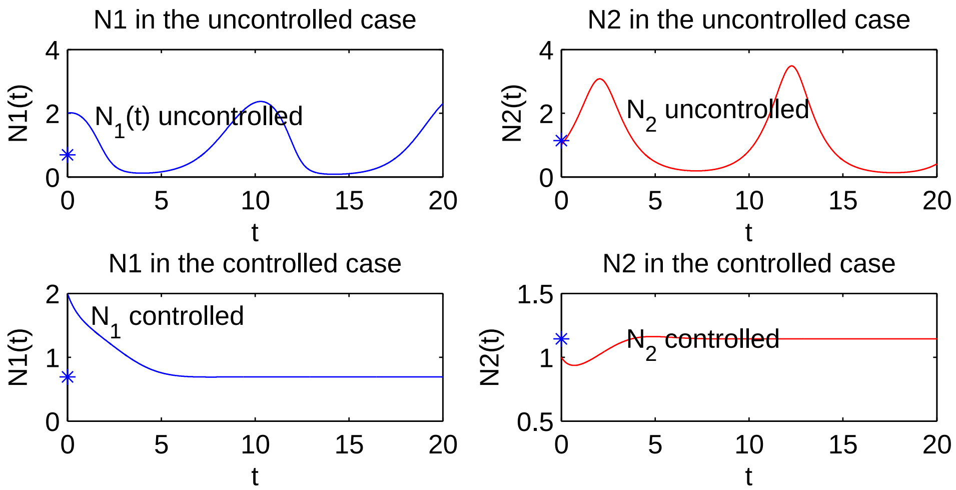

Figure 4: Uncontrolled and controlled cases at the steady state $(N1e, N2e)$

4.3. Example 3

The third example below is a competitive system governed by, [1]

$$\frac{dN_1}{dt} = \frac{r_1}{K_1}N_1(t)\Big(K_1 - N_1(t) - \alpha N_2(t)\Big),$$
$$\frac{dN_2}{dt} = \frac{r_2}{K_2}N_2(t)\Big(K_2 - N_2(t) - \beta N_1(t)\Big),$$

where $r_1, r_2, K_1, K_2, \alpha, \beta$ are positive numbers. The steady state

$$N1e = \frac{1}{1-\alpha\beta}(K_1 - \alpha K_2), \quad N2e = \frac{1}{1-\alpha\beta}(K_2 - \beta K_1),$$

is stable when $\alpha\beta < 1$ and unstable if $\alpha\beta > 1$.

In our numerical simulation, we choose $r_1 = 20,\ r_2 = 1,\ K_1 = 2,\ K_2 = 4$. We choose $\alpha = 1$ and $\beta = 3$ which implies that $\alpha\beta > 1$. Thus the nonlinear system is unstable at the steady state. By using the method in this paper, we can stabilize this steady state. Figure 5 shows the unstable and the stable solutions. Clearly we observe that the controlled system is stable at the steady state.

In the numerical simulation, we choose initial value $(1.5, 0.5)$. The final time is $T = 20$.

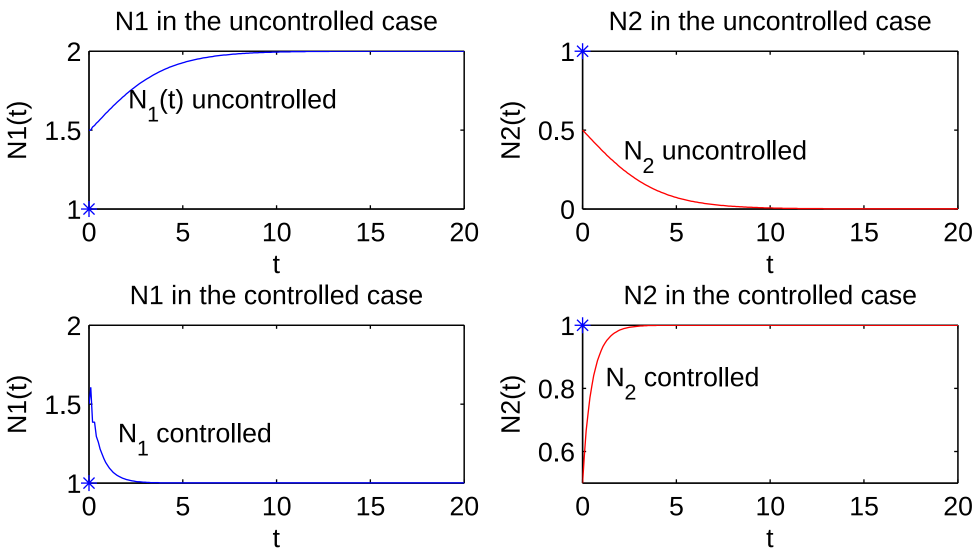

Figure 5: Uncontrolled and controlled cases at the steady state $(N1e, N2e)$

4.4. Example 4

The fourth example is a system of nonlinear first order ordinary differential equations [7]

$$\frac{dN_1}{dt} = r_1 N_1\Big(1 - \frac{N_1}{K_1} + \Big(\frac{b_1 N_2 - c_1 N_2^2}{1 + d_1 N_2^2}\Big)\frac{N_2}{K_1}\Big),$$
$$\frac{dN_2}{dt} = r_2 N_2\Big(1 - \frac{N_2}{K_2} + \Big(\frac{b_2 N_1 - c_2 N_1^2}{1 + d_2 N_1^2}\Big)\frac{N_1}{K_2}\Big),$$

where $r_1, r_2, K_1, K_2, b_1, b_2, c_1, c_2, d_1, d_2$ are positive numbers. The steady states are $(0, 0)$, $(0, K_2)$, and $(K_1, 0)$ which are unstable. By using the method in this paper, we can stabilize this steady states. Figure 6 - 8 show the unstable and the stable solutions for the different steady states. Clearly we observe that the controlled system are stable at the steady states.

In the numerical simulation, we choose $r_1 = 0.6, r_2 = 0.3, K_1 = 1, K_2 = 1, b_1 = 5, b_2 = 5, c_1 = 1, c_2 = 1, d_1 = 0.5, d_2 = 0.5$.

The initial value is $(0.5, 0.3)$. The final time $T = 100$.

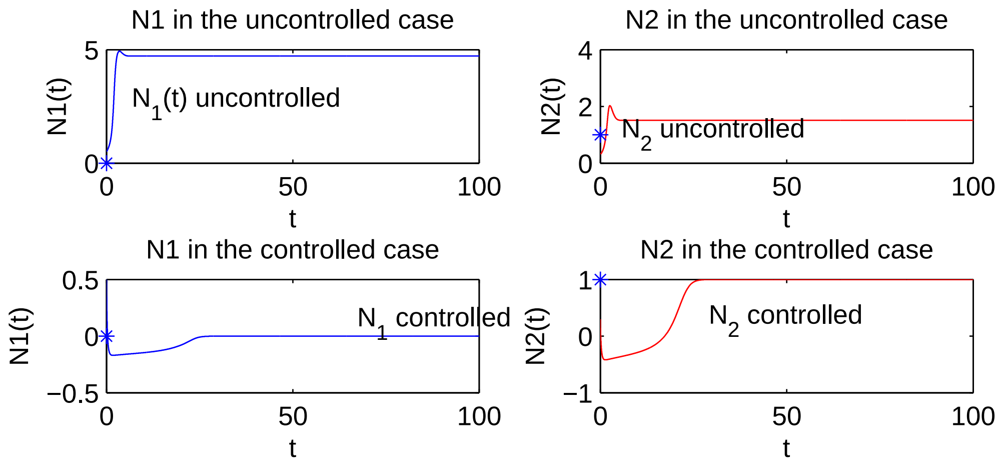

Figure 6: Uncontrolled and controlled cases at the steady state $(0, K_2)$

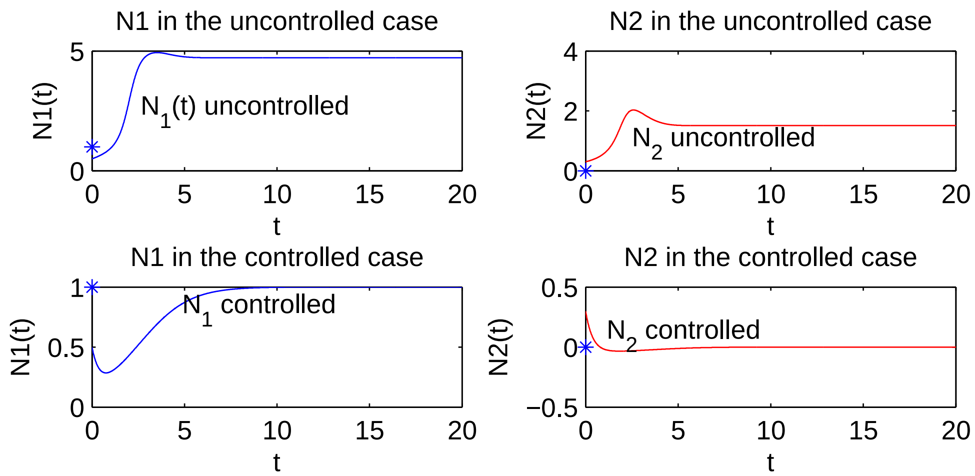

Figure 7: Uncontrolled and controlled cases at the steady state $(K_1, 0)$

From these numerical simulations, we observe that in the uncontrolled case we cannot guarantee where the arbitrary unstable steady state will converge to because it is unstable. However, in the controlled case we report that the arbitrary unstable steady state can

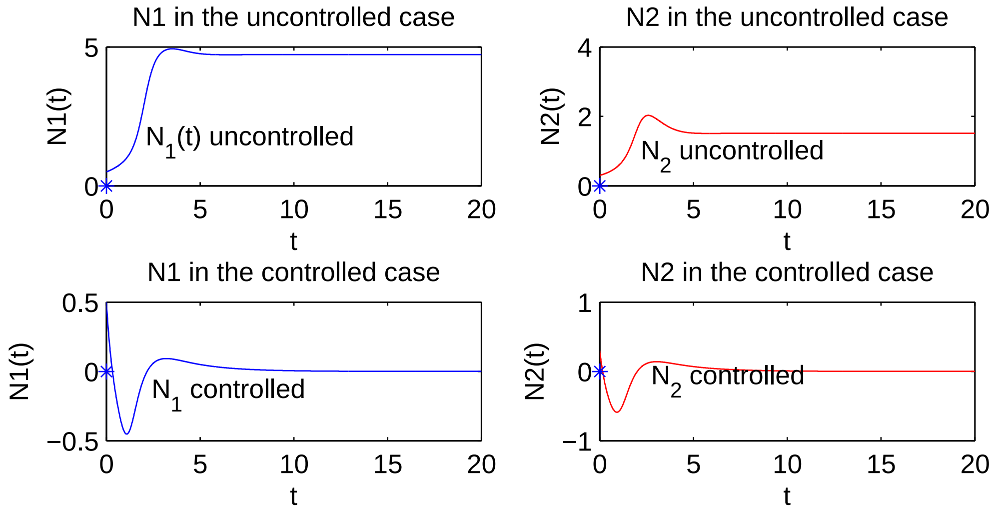

Figure 8: Uncontrolled and controlled cases at the steady state $(0, 0)$

converge to the given steady state. Hence, both the unstable trivial and nontrivial steady states can be said to be stabilized using our numerical method of constructing a controller.

5. Conclusion

In this paper, we have developed a powerful numerical tool with which to stabilize the unstable steady states which were constructed in our previous research [4]; [3]. In terms of ecological thinking, equilibrium models of coexistence are models where species would coexist indefinitely in spite of local and transient fluctuations in their population sizes. The application of our strong simultaneous stabilization as reported in this paper is a major contribution to this ecological thinking.

Our present analysis which we have not seen elsewhere can contribute to our further understanding of the role of numerical simulation and numerical stabilization of steady states into the study of computational and mathematical modelling of plant species interactions in harsh climates. Further extension of our analysis to capture the behaviour of plant species to spread as they grow would be our next investigation.

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