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Coordination-Free Equilibria in Cheap Talk Games*

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Abstract

This paper characterizes generic equilibrium play in a multi-sender version of Crawford and Sobel's (1982) cheap talk model, when robustness to a broad class of beliefs about noise in the senders' observation of the state is required. Just like in the one-sender model, information transmission is partial, equilibria have an interval form, and they can be computed through a generalized version of Crawford and Sobel's forward solution procedure. Fixing the senders' biases, full revelation is not achievable even as the state space becomes large. Intuitive welfare predictions, such as the desirability of consulting senders with small and opposite biases, follow.

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1 Introduction

The transmission of information is an integral part of many economic models, whether implicitly or explicitly. In certain settings, such transmission is strategic: the side sending the information may choose the message in order to maximize its payoff. At the same time, the party receiving the information may be unable to offer incentives that significantly improve the informativeness of the message.

The seminal work of Crawford and Sobel (1982, henceforth CS) examines such a setting. A sender observes the state of the world $\theta \in [0, 1]$, sends a message to the receiver, who then takes an action. Both the sender and the receiver desire a higher action when θ is higher, but the optimal action for the sender differs from the optimal action for the receiver. Talk is cheap in the sense that neither player's utility depends on the sender's message. CS show that equilibria in this setting feature the sender revealing an interval of the state space. Moreover, there is a finite upper bound on the number of intervals that can be distinguished in equilibrium, and this bound increases as the sender's bias relative to the receiver becomes small.

This paper examines a model very similar to CS's, but with multiple senders simultaneously sending their messages. For example, a policymaker may seek the opinion of multiple experts. In multi-sender cheap talk games, because the actions that a given sender can induce depend on what messages other senders use, there exists a large set of equilibria, and there has so far been little progress in characterizing or refining it. Most existing work, reviewed later in the introduction, focuses on fully revealing equilibria, whose reasonability is questioned.

The main results of this paper show that, for an open and dense set of preferences and prior ("generically"), "robust" equilibria in this model have an interval structure, in that each message vector reveals an interval of states, just like equilibria in the one-sender CS model. Moreover, at each boundary between two intervals, only one sender's message changes,[1] so that senders do not coordinate locally about whether they are in the interval to the left or the one to the right of the boundary. The sender whose message changes must be indifferent at the boundary between inducing the action corresponding to the left interval and the action corresponding to the right interval, just like in CS equilibria. The latter property implies that the set of these *coordination-free* equilibria is finite and tractable: each such equilibrium can be computed through a generalized version of the CS forward solution procedure. This paper is the first that selects and characterizes a set of equilibria in simultaneous multi-sender cheap talk.

[1] *I.e.* there is no state θ such that two or more senders use a different message on each side of θ.

The proposed robustness concept requires equilibria to survive the possibility of small noise - where senders' observations are very close to θ (with high probability) - in the senders' observations of θ: an equilibrium is *(strongly) robust* if every player's strategy remains nearly optimal. Optimality is in an interim sense: each sender's message must be nearly optimal given her observed signal, and the receiver's action must be nearly optimal given the senders' messages. This paper's results hold whether senders are required to have common prior about the noise, or merely to have common knowledge that noise is small, which allows for the possibility of heterogeneous priors about the exact form of the noise.[2]

Many papers have studied the use of perturbations to the information structure to select equilibria. Most papers in the literature impose few restrictions on these perturbations. As a result, if heterogeneous prior is allowed, robust equilibria often fail to exist.[3] However, when considering cheap talk specifically, it is natural to restrict, for example, the set of payoff types: since senders care only about the state and the receiver's action, a message's payoff implication is entirely dependent on the receiver's strategy. This paper chooses to perturb information only about the parameter that already fails to be commonly known: the state θ.[4]

Theorems 1 and 2 are the main results of this paper. Theorem 2 shows that coordination-free equilibria (in which no sender ever finds a deviation leading to an out-of-equilibrium message vector to be nearly optimal) are robust, while, generically, pure-strategy equilibria that do not lead to the same play as a coordination-free equilibrium at almost all states fail robustness. The key to this result is that in coordination-free profiles, for any sender i and at any state θ, the message prescribed for i is nearly optimal whenever other senders send messages prescribed at a state near θ. Sender i therefore has little incentive to modify her course of action as a result of small noise. By contrast, in other equilibria, a sender i's message at a state can be substantially suboptimal in response to other senders' messages at nearby states, and with noise, i may believe that the latter messages are likely to be

[2]Online Appendix C shows that if the robustness concept were relaxed to require only that some "nearby" strategy profile be nearly optimal, then the results would still hold if heterogeneous priors about noise are allowed.

[3]For example, Oyama and Tercieux (2010) show, in finite complete information games, that generically, an equilibrium is robust only if it is the unique rationalizable action profile. Weinstein and Yildiz (2007) show a similar result when, instead, *interim* beliefs are concentrated around the complete information payoffs. (In fact, Weinstein and Yildiz show that, with the interim approach, imposing common prior would not change their result.)

[4]Even only perturbing information about θ, it can still be common 0-belief that payoffs are near the payoffs of the complete information game, just like in global games (see, for example, Carlsson and van Damme (1993)).

This paper also differs from most of the literature by requiring only approximate optimality. Haimanko and Kajii (2015) do so as well in order to guarantee existence of robust equilibria in the Kajii and Morris (1997) framework.

sent. The class of small noise examined in Theorem 2 has senders observing a signal near the true state with certainty. Theorem 1 shows that strong robustness, which allows for a small probability of the signal being far from the state, generically selects coordination-free equilibria where every combination of on-path messages form an on-path message vector.[5]

Several studies have examined simultaneous multi-sender cheap talk. Krishna and Morgan (2001a) note that if the senders' biases are sufficiently small relative to the state space, full revelation is achievable: for example, the receiver may threaten an action unappealing to all parties if the senders' messages diverge.[6] Battaglini (2002) notes that fully revealing equilibria in one-dimensional state spaces rely on implausible out-of-equilibrium beliefs: in the aforementioned example, were the receiver to face slightly divergent messages due to noise in the senders' observations, it would not be a best response to pick a "crazy" action if the senders are telling the truth or nearly doing so.[7] Ambrus and Lu (2014) exhibit equilibria that approach full revelation as the state space becomes large, do not rely on out-of-equilibrium beliefs, and, as a result, remain optimal for all players with certain classes of noise in the senders' observations. Rubanov (2015) obtains similar results with a fixed state space, as the number of senders becomes large. This paper requires, like Ambrus and Lu (2014) and Rubanov (2015), (approximate) optimality in the perturbed games, but considers a broader class of perturbations.[8] As a result, more equilibria, including the ones proposed in those two papers, are ruled out. In particular, as long as the biases of the senders are bounded away from zero, communication robust to the perturbations considered in this paper is bounded away from full revelation, even as the size of the state space and/or the number of senders become large.

In terms of the best coordination-free equilibrium for the receiver, in the popular uniform-

[5] A previous version of this paper showed that coordination-free equilibria also have a desirable property in a setting without noise. Consider the induced normal-form game played by senders, given a state and a receiver strategy. In a coordination-free equilibrium (of the cheap talk game), at almost all states, the Nash equilibrium (of the induced game) is unique among on-path message vectors. Therefore, coordination-free equilibria are robust to collusion among senders (in the sense that senders cannot jointly deviate to a different Nash equilibrium of the induced game) if off-path message vectors are ruled out - for example, the receiver might be able to harshly punish senders if an off-path message vector is observed. At the same time, equilibria that are not coordination-free generically do not have this property. For example, if all senders have positive bias, then full revelation is not robust in this sense: the senders could all switch to reporting $\theta + \varepsilon$.

[6] Thus, when the senders' biases are below a certain threshold determined by the size of the state space, there is no nontrivial bound on the amount of transmission transmitted, and the direction and magnitude of sender biases do not impact the receiver's maximum welfare.

[7] Ambrus and Takahashi (2008) propose a mild refinement that also rules out full revelation in multi-dimensional state spaces, subject to a technical condition on the shape of the state space.

[8] If noise were limited to "replacement noise" (where senders observe the *exact* state with high probability), then even with heterogeneous beliefs, the equilibria proposed in Section 3 of Ambrus and Lu (2014), presented in this paper as Example 2, and in Rubanov (2015) would survive.

quadratic specification:

- having multiple senders instead of one fails to improve the receiver's welfare if the additional senders' biases are no smaller and in the same direction as the first sender's; and

- if there are senders with a rightward bias and senders with a leftward bias, then as the state space becomes large, the pair of senders maximizing the receiver's welfare consists of the sender with the smallest rightward bias and the one with the smallest leftward bias.

Similar conclusions hold as long as the prior is near uniform and the players' utility functions are not too asymmetric about their peak. Krishna and Morgan (2001b) study sequential cheap talk with two senders, and also find that having senders with opposing biases is preferable for the receiver. This paper additionally shows, by allowing for more than two senders, that in coordination-free equilibria, consulting senders beyond the optimal pair typically yields only modest welfare gains for the receiver. Moreover, the characterization of coordination-free equilibria provides a way to find the optimal one for the receiver.

Many other issues related to CS have been investigated: multidimensional state space,[9] cheap talk that is sequential or occurs through an intermediary,[10] refinements in the one-sender case,[11] *etc.* This paper may contribute to the study of delegation: is it better for the receiver to retain the decision right and play a cheap talk game, or to delegate the action to the sender(s)? Melumad and Shibano (1991), Dessein (2002) and Alonso and Matoushek (2008) study this question in a single-sender setting. Since fully revealing equilibria exist when there are multiple senders (and when the state space is large relative to the biases), there has been little scope for studying this question in a multi-sender setting.[12] However, if one expects a coordination-free equilibrium to arise from multi-sender cheap talk, then the delegation issue becomes nontrivial since coordination-free equilibria cannot approach full revelation.

[9] Battaglini (2002, 2004) constructs a fully revealing equilibrium that satisfies a weak robustness condition when the state space is unbounded. Ambrus and Takahashi (2008) and Meyer, Moreno de Barreda and Nafziger (2016) study full revelation with bounded state spaces. See Lai, Lim and Wang (2015) and Vespa and Wilson (2016) for experimental evidence.

Multidimensional states are mostly outside the scope of this paper because, without strong functional form assumptions, it is difficult to construct equilibria (other than babbling and fully revealing) with a bounded state space due to boundary conditions. Section 7.2 explains that requiring only approximate optimality in the perturbed game is likely to wield limited power when the state space is multi-dimensional.

[10] Miura (2014) extends Krishna and Morgan's (2001b) two-sender analysis of sequential cheap talk to a two-dimensional state space. Ivanov (2010) and Ambrus, Azevedo and Kamada (2013) consider cheap talk with intermediaries.

[11] See, for example, Matthews, Okuno-Funiwara and Postlewaite (1991) and Chen, Kartik and Sobel (2008). The latter work selects the most informative equilibrium in the CS model, but Miura and Yamashita (2014) suggest that this may be problematic if there's a possibility that the model is slightly misspecified.

[12] Another obstacle in studying delegation with multiple experts is that there are many ways to delegate authority in such a setting. For example, there are many possible sets of rules by which a committee of experts may come to a decision.

2 Model

There are $n+1$ players: a set N of senders $1,...,n$ and a receiver R. The senders observe a common state of the world $\theta \in \Theta = [0,1]$, which is drawn from a probability distribution $F(.)$ with a continuous density $f(.)$ for which there exist $d > 0$ and $D < \infty$ such that $f(\theta) \in [d, D]$ for all θ. Each sender's pure strategy $m_i : \Theta \to M_i$ assigns a message from a set M_i to each state. M_i is assumed to be large relative to Θ, so that full revelation is possible. Upon observing message vector $m = (m_1, ..., m_n)$, the receiver takes action $a(m)$. Her pure strategies thus take the form $a : \times_{i=1}^{n} M_i \to \Theta$.[13] When referring to a specific strategy profile Γ, denote the receiver's action given m by $a^{\Gamma}(m)$.

All players' utilities depend on the state θ and the action $a \in \Theta$ taken by the receiver, but not (directly) on the message vector $m \in \times_{i=1}^{n} M_i$. Let $u_i(a,\theta)$ denote player i's utility when the action is a and state is θ, for $i = 1, ..., n, R$. The following standard assumptions are maintained throughout the paper:

1. all utility functions are Lipschitz continuous;
2. given θ, $u_R(.,\theta)$ is strictly concave with a maximum at $a = \theta$;
3. given θ, $u_i(.,\theta)$ is single-peaked, *i.e.* is strictly increasing to the left, and strictly decreasing to the right of its unique maximum, denoted $\theta + b_i(\theta)$;
4. $\exists \eta > 0$ such that, for all $i \in N$ and $\theta \in \Theta$, either $|b_i(\theta)| > \eta$ or $\theta + b_i(\theta) \in \{0,1\}$; and
5. for all $i \in N$, if $a < a'$, $\theta < \theta'$ and $u_i(a',\theta) \geq u_i(a,\theta)$, then $u_i(a',\theta') > u_i(a,\theta')$.

Assumption 2 implies that the receiver's best response is always unique. Assumption 4 and continuity imply that each i is either right-biased (for every θ, $b_i(\theta) > \eta$ or $\theta + b_i(\theta) = 1$) or left-biased (for every θ, $b_i(\theta) < -\eta$ or $\theta + b_i(\theta) = 0$). Assumption 5 is the commonly encountered single-crossing condition.

Messages m_i and m_i' are said to be *equivalent in strategy profile Γ* if $a^{\Gamma}(m_i, m_{-i}) = a^{\Gamma}(m_i', m_{-i})$ whenever the vector m_{-i} is composed of messages that are each sent with positive probability at some θ in Γ.[14] Throughout this paper, when a given strategy profile Γ is

[13] Identifying the action space with Θ allows for an easy statement of the assumption on the receiver's preferences (Assumption 2 below).

[14] This condition must hold even if m_{-i} itself is never sent in Γ. For example, suppose $a^{\Gamma}(1,1,1) = a^{\Gamma}(1,1,2) = a^{\Gamma}(1,2,1) = a^{\Gamma}(2,1,1) = a$, while $a^{\Gamma}(2,2,2) = a^{\Gamma}(2,2,1) = a^{\Gamma}(2,1,2) = a^{\Gamma}(1,2,2) = a' \neq a$. Also assume that of these eight message vectors, only $(1,1,1)$ and $(2,2,2)$ are sent in equilibrium. Then even though, on path, no sender's message affects the action, 1 and 2 are *not* equivalent for any sender: for example, if sender 1 sends 1 and sender 2 sends 2, then the actions induced by sender 3 through sending 1 and sending 2 are not the same.

discussed, $m_i = (\neq) m_i'$ means that m_i and m_i' are (not) equivalent in Γ, and $m = (\neq) m'$ means that each (some) component of m is (not) equivalent in Γ to the corresponding component of m'. That is, equivalent messages are treated as if they were the same message. As is standard for simultaneous multi-sender cheap talk in a continuous type space, this paper focuses on pure-strategy equilibria in the sense that if a sender mixes between m_i and m_i', then m_i and m_i' must be equivalent.[15]

Even when m_i and m_i' are not equivalent, we may have $a^\Gamma(m_i, m_{-i}) = a^\Gamma(m_i', m_{-i})$ for *some* m_{-i} sent with positive probability in Γ. The following assumption rules this out when (m_i, m_{-i}) and (m_i', m_{-i}) are both sent on path:

Assumption A: If two message vectors m and m' both occur on path and induce the same action, then m and m' cannot differ in exactly one component.

Assumption A implies that in a set of states where m_{-i} and the induced action are constant, m_i must also be constant, which allows for a simpler description of equilibria. If the set of states where message vector m is sent in a pure-strategy equilibrium Γ, henceforth denoted $\theta^\Gamma(m)$, is an interval (possibly degenerate) for all on-path m, then Γ must satisfy Assumption A: the receiver's optimality implies $a^\Gamma(m) \in \theta^\Gamma(m)$ for all on-path m, so that no two on-path m can induce the same action. Thus, all CS equilibria, all pure-strategy fully revealing equilibria and all equilibria proposed in the simultaneous unidimensional cheap talk literature cited in the introduction satisfy Assumption A. Online Appendix B analyzes the model without Assumption A.[16]

When referring to a specific pure-strategy profile Γ, it is convenient to denote sender i's message at θ by $m_i^\Gamma(\theta)$, and to let $M_i^\Gamma = \{m_i : m_i^\Gamma(\theta) = m_i \text{ for some } \theta \in \Theta\}$. Also let $m^\Gamma(\theta) = (m_1^\Gamma(\theta), m_2^\Gamma(\theta), ..., m_n^\Gamma(\theta))$ be the message vector sent at state θ.

The equilibrium concept is weak perfect Bayesian equilibrium (henceforth equilibrium). Pure strategy profile $\Gamma = (m_1, ..., m_n, a)$ and belief rule μ form an equilibrium if:

- for all $i \in N$ and all $\theta \in \Theta$, $m_i(\theta) \in \arg\max_{m_i' \in M_i} u_i(a(m_i', m_{-i}(\theta)), \theta)$,
- for all $m \in \times_{i=1}^n M_i$, $a(m) \in \arg\max_{a' \in \Theta} \int_{\theta \in \Theta} u_R(a', \theta) d\mu(m)$, and

[15] In the one-sender case, this is without loss of generality (except at the points where the sender's message changes) because any two messages leading to the same action are equivalent.

[16] The pure-strategy assumption and Assumption A can be summarized as follows: restrict attention to strategy profiles Γ consistent with each sender i having the following lexicographic preferences. Among messages yielding the highest expected utility, i picks her preferred one under some strict preference ranking $\succ_{i,\Gamma}$. For example, given the choice between two messages leading to the same outcome, the sender may pick the simpler one. ($\succ_{i,\Gamma}$ is allowed to - but does not have to - depend on Γ in the cheap talk spirit of messages acquiring their meaning endogenously.)

- $\mu(m)$ is obtained from $f(.), m_1(.), ..., m_n(.)$ through Bayes' rule whenever $m = m^\Gamma(\theta)$ for some $\theta \in \Theta$.[17]

It will sometimes be convenient to abuse notation by using Γ to denote the equilibrium containing strategy profile Γ.

For simplicity and without loss of generality for equilibrium play, assume throughout this paper that, in any strategy profile Γ, every message in M_i is equivalent to a message in M_i^Γ.

3 Examples and Basic Definitions

This section presents examples of equilibria in multi-sender cheap talk to illustrate that: i) they can be unintuitive, and ii) the set of equilibria is extremely difficult to characterize. In each of these examples, there are two senders, and the parameters follow the popular *uniform-quadratic* specification: $\theta \sim U[0,1]$ and $u_i(a,\theta) = -(a - (\theta + b_i))^2$ (with $b_R = 0$), so for all θ, sender i's utility is maximized at $\theta + b_i$. In both examples, every message vector in $\times_{i=1}^n M_i^\Gamma$ occurs in equilibrium, so any criterion that only places restrictions on out-of-equilibrium beliefs, such as Battaglini's (2002), would not rule out any of these equilibria.

These examples feature pairs of messages (m_1, m_2) that are sent in some nontrivial interval, but not at all (or almost all) states where they form mutual best responses. This paper's robustness concept selects equilibria where such an issue does not arise.

Example 1: No Interval Structure

Suppose $b_1 = b_2 = 0.04$. Each sender has two equilibrium messages, x and y. The prescribed message vector $m^\Gamma(\theta)$ is:

- (x,x) if $\theta \in [0, 0.01] \backslash \mathbb{Q}$;
- (y,y) if $\theta \in ([0, 0.01] \cap \mathbb{Q}) \cup (0.01, 0.18]$;
- (x,y) if $\theta \in (0.18, 0.51] \cup ((0.51, 1] \cap \mathbb{Q})$;
- (y,x) if $\theta \in (0.51, 1] \backslash \mathbb{Q}$.

It is easy to check that the receiver's optimal actions are $a^\Gamma(x,x) = 0.005$, $a^\Gamma(y,y) = 0.095$, $a^\Gamma(x,y) = 0.345$ and $a^\Gamma(y,x) = 0.755$, and that this profile is indeed part of an equilibrium. As a result, within $[0, 0.01]$, the action following a rational state is 0.095, while the action following an irrational state is 0.005; a similar situation arises within $(0.51, 1]$. Thus, outside of $[0.01, 0.51]$, there is no nontrivial interval of states following which the same messages are sent.

[17] Senders' strategies and μ must be such that the receiver's expected utility is well-defined. In particular, senders' strategies must be measurable, which implies that $\theta^\Gamma(m)$ is also measurable, for all m.

Clearly, (x,x) and (y,y) are both mutual best responses on $(0,0.01)$, and to avoid sending (x,y) or (y,x), the senders must coordinate in a very precise manner that would be difficult if they observe the true state with noise that has a continuous distribution. Senders face a similar problem on $(0.51,1)$.

Example 1 does not have "interval structure" in the following sense. For convenience, let $\lambda(.)$ denote the Lebesgue measure.

Definition: Given a pure-strategy profile Γ, a *cell in Γ* is a maximal interval[18] of states throughout which m^Γ remains constant. A *proper cell* is a cell with positive measure.

Definition: A pure-strategy profile Γ has *interval structure* if $\lambda(\cup_{I \text{ is a proper cell in } \Gamma} I) = \lambda(\Theta)$.

Example 2: Convoluted Finite Interval Structure

This example is taken from Ambrus and Lu (2014), and shows that even an equilibrium revealing a finite partition of the state space may seem implausible.

Suppose b_1 and b_2 are small, and divide Θ into q equally sized *blocks*, each of which is divided into q equally sized *cells*. Both sender's messages are labeled $1, 2, ..., q$ and are used as follows (see Figure 1):

- sender 1 sends message k in the k^{th} cell of each block;

- sender 2 sends message $k+l-1(\text{mod}\, q)$ in the k^{th} cell of the l^{th} block (when the formula gives 0, message q is sent).

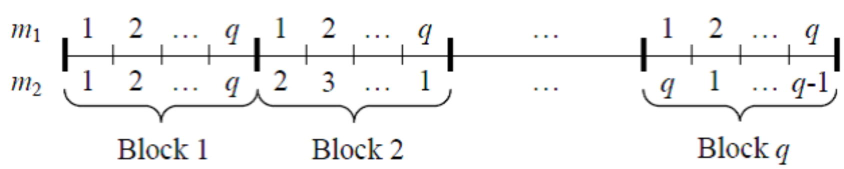

Figure 1: Example 2

This profile guarantees that every message vector in $\{1,...,q\} \times \{1,...,q\}$ occurs in exactly one cell, and that any deviation leads to an action at least almost a block away when q is large. Thus, if the biases b_1 and b_2 are small with respect to Θ, then the number of blocks can be large. Therefore, fixing the biases, this construction can yield an almost fully revealing equilibrium as Θ becomes large.

However, this construction appears to be asking too much of the senders, whose messages must change very frequently at boundaries between cells. These boundaries are arbitrarily

[18] That is, m^Γ does not remain constant in any connected strict superset of a cell. Cells can be degenerate intervals (*i.e.* they can consist of a single state).

set: for example, with large enough blocks, $(1,1)$ and $(2,2)$ are pairs of mutual best responses anywhere in block 1 sufficiently far from block 2, so the boundary θ between these cells can be moved. With noise in the senders' observations, near θ, both senders can be very uncertain about the message used by the other sender, which can lead to a coordination failure.[19]

4 Coordination-Free Equilibria

This section proposes a class of "coordination-free" equilibria and derives some of their properties. The main results of this paper show that coordination-freeness is essentially required for satisfying the robustness concepts proposed in Section 5.

4.1 Definition

Coordination-free equilibria are defined as follows.

Definition: An equilibrium Γ is *coordination-free* if it satisfies points 1 and 2 below, and *strictly coordination-free* if it satisfies points 1-3.

1. Every cell in Γ (other than, if present, $\{0\}$ or $\{1\}$) is proper, and if $\theta \neq 1$ (resp. $\theta \neq 0$) is the supremum (resp. infimum) of a cell, then it is also the infimum (resp. supremum) of another cell.
2. The message vectors sent in any two adjacent cells in Γ differ in exactly one component.
3. If an out-of-equilibrium message vector m' differs from the message vector m sent in cell C in only its ith component, then $u_i(a^\Gamma(m),\theta) \neq u_i(a^\Gamma(m'),\theta)$ for all $\theta \in \overline{C}$, where $\overline{C}$ denotes the closure of C.

Point 2 implies, by Assumption A, that the induced action differs in any two adjacent cells, and ensures that Γ does not rely on coordination in the following sense. Consider a boundary θ_b between two cells where sender i's equilibrium messages m_i and m_i' differ while senders $j \neq i$ play m_j. Point 2 implies that m_j is approximately optimal for j regardless of whether sender i sends m_i or m_i' and of whether the state is to the left or to the right of

[19] However, if the noise is in a class such that both senders observe the true state with probability near 1, then senders are able to successfully coordinate. This observation illustrates the difference between this paper and Ambrus and Lu (2014).

the boundary. That is, even if sender i makes a "mistake" and sends the wrong message, no other sender would have a strong incentive to coordinate with i by changing their message.

Point 3 ensures that Γ does not rely on coordination in another sense: since i is always strictly worse off switching to a message that leads to an out-of-equilibrium message vector m' (*i.e.* $m' \in \times_{i=1}^{n} M_i^{\Gamma}$ such that $m' \neq m^{\Gamma}(\theta')$ for any $\theta' \in \Theta$), the other senders may find it less plausible that i would make such a switch. Because, as Proposition 1 will show, there are finitely many cells in any coordination-free equilibrium, point 3 rules out only a finite number of values for $a^{\Gamma}(m')$.

In the one-sender case, every equilibrium is coordination-free, and strictly so since every message is (equivalent to) an on-path message.

Example 3: A Multi-Sender Strictly Coordination-Free Equilibrium

As in Section 3, consider a uniform-quadratic setting with two senders. Suppose $b_1 = 0.01$, $b_2 = 0.1$. *Each sender has two equilibrium messages,* x *and* y. *The prescribed message vector* $m^{\Gamma}(\theta)$ *is:*

- (x, x) *if* $\theta \in [0, 0.01]$;
- (y, x) *if* $\theta \in (0.01, 0.06]$;
- (y, y) *if* $\theta \in (0.06, 0.51]$;
- (x, y) *if* $\theta \in (0.51, 1]$.

The receiver maps every message to x *or* y, *derives beliefs from Bayes' rule, and optimally responds:* $a^{\Gamma}(x, x) = 0.005$, $a^{\Gamma}(y, x) = 0.035$, $a^{\Gamma}(y, y) = 0.285$ *and* $a^{\Gamma}(x, y) = 0.755$.

It is straightforward to check that the above describes a strictly coordination-free equilibrium. (Point 3 of the definition is vacuous here since every message vector in $\times_{i=1}^{n} M_i^{\Gamma}$ *occurs on path.)*

4.2 Basic Properties

This subsection presents some properties of coordination-free equilibria.

Proposition 1: The number of cells in any coordination-free equilibrium Γ is bounded above by $\frac{\lambda(\Theta)}{\eta}$, and every other cell must have size greater than η.

Proof: Let θ_b be the boundary between two adjacent cells in Γ. Since the message vectors sent in those cells, denoted m for the right cell and m' for the left cell, differ in exactly one component, there must exist a sender i such that $u_i(a^{\Gamma}(m), \theta_b) = u_i(a^{\Gamma}(m'), \theta_b)$. By single-crossing, $u_i(a^{\Gamma}(m), \theta) \gtreqless u_i(a^{\Gamma}(m'), \theta)$ for all $\theta \gtreqless \theta_b$. As a result, m cannot occur to the left of θ_b, and m' cannot occur to the right of θ_b, which implies $a^\Gamma(m) \geq \theta_b \geq a^\Gamma(m')$. Since $a^\Gamma(m) \neq a^\Gamma(m')$, they must occur on different sides of $\theta_b + b_i(\theta_b)$, so we also have $a^\Gamma(m) > \theta_b + b_i(\theta_b) > a^\Gamma(m')$ (and thus $\theta_b + b_i(\theta_b) \notin \{0, 1\}$, which implies $|b_i(\theta_b)| \geq \eta > 0$). Combining these observations yields:

i) $a^\Gamma(m) - a^\Gamma(m') > \eta$, implying the bound on the number of cells, and

ii) either $a^\Gamma(m) > \theta_b + \eta$ or $\theta_b - \eta > a^\Gamma(m')$, implying that at least one of the two cells has size greater than η. ■

Proposition 1 makes clear that for any fixed η, coordination-free equilibria are bounded away from full revelation: it is not possible to increase information transmission beyond a certain bound, be it by expanding the state space (as in Ambrus and Lu (2014)) or by adding senders (as in Rubanov (2015)). Of course, as $\eta \to 0$, the upper bound on information transmission approaches full revelation, just like in Crawford and Sobel (1982).

Another property of coordination-free equilibrium play (*i.e.* the collection of on-path induced actions and cell endpoints) is that it can be computed using the forward solution procedure of Crawford and Sobel (1982), given an ordered list stating the identity of the sender whose message changes at each boundary between two cells.[20] Given the receiver's leftmost action, denoted θ_1, the location of the leftmost cell's right endpoint, denoted θ_2, is determined by the prior and u_R. The indifferent sender at θ_2 is given by the list, and her indifference condition pins down the receiver's second leftmost action, and so on. If the last cell's right endpoint coincides with the right endpoint of Θ, play computed through this procedure is called *candidate play*. Clearly, play in a coordination-free equilibrium must be candidate play. Moreover, if all cell endpoints calculated through forward solution are monotonic in θ_2,[21] then the number of candidate play corresponding to each given ordered list of indifferent senders is either zero (if the list is too long) or one.

However, unlike in the one-sender case, not all candidate play corresponds to play in an equilibrium - see Online Appendix A for an example. The reason is that there may be an out-of-equilibrium vector m such that any value of $a^\Gamma(m)$ induces a profitable deviation by a sender. Proposition 7 in Online Appendix A identifies a sufficient condition under which this problem does not arise. This condition is typically satisfied by sufficiently informative candidate play if there is at least one sender with a small bias in each direction, and senders' preferences are not too asymmetric.

[20] In the one-sender case, this sender is trivially always the only sender, and the length of the list determines the number of intervals in equilibrium.

[21] This is Crawford and Sobel's (1982) condition (M), and is satisfied in the uniform-quadratic case.

A related observation is that if multiple senders have identical preferences, then deleting all but one of them does not change the set of possible candidate play. Doing so may, however, shrink the set of equilibria, because it removes ways of assigning messages. Nevertheless, Proposition 7 implies that under the assumptions of the previous paragraph, deleting duplicate senders does not prevent the most informative candidate play from being part of a coordination-free equilibrium, and therefore does not affect the receiver's maximum welfare achievable in a coordination-free equilibrium. Section 6 presents further results about receiver welfare and sender selection.

5 Robustness

This section introduces the possibility of small noise in the senders' observations of the state, and studies equilibria Γ that, for any $\delta > 0$, remain δ-equilibria with any sufficiently small noise. Such equilibria are called *strongly robust* when small noise means that senders' signals are near the true state with high probability, and *robust* when small noise means that senders' signals are near the true state for sure. These concepts are formally defined in Sections 5.1 to 5.3. The main results of the paper, Theorems 1 and 2, are stated below and fully presented in Sections 5.2 to 5.4.

The following definitions are used in Theorems 1 and 2.

Definition: An equilibrium Γ is *complete* if every $(m_1, ..., m_n) \in \times_{i=1}^{n} M_i^{\Gamma}$ occurs in equilibrium.

That is, Γ is complete if no combination of equilibrium messages is an out-of-equilibrium message vector. Examples 1, 2 and 3 all feature complete equilibria. Note that any complete and coordination-free equilibrium is also strictly coordination-free.

Definition: Two equilibria are *equivalent* if at all but a finite number of states, the same messages are sent, and the receiver's strategy is the same.

Definition: A statement holds *generically* if it holds for an open and dense set of vectors of primitives $(u_1, ..., u_n, u_R, f)$ (within the set satisfying the assumptions from Section 2) under the metric $d((u_1, ..., u_n, u_R, f), (u_1', ..., u_n', u_R', f')) = \sqrt{\sum_{i=1,...,n,R} \|u_i - u_i'\|^2 + \|f - f'\|^2}$, where $\|\cdot\|$ denotes the sup norm.

For Theorems 1 and 2, let Γ be an equilibrium of the noiseless cheap talk game.

Theorem 1:

(a) Generically, if Γ is strongly robust, then it is complete and coordination-free.

(b) If Γ is complete and coordination-free, and no cell in Γ is $\{0\}$ or $\{1\}$, then it is strongly robust.

Theorem 2:

(a) Generically, if Γ is robust, then it is equivalent to a strictly coordination-free equilibrium.

(b) If Γ is strictly coordination-free, then it is robust.

5.1 Noise and Perturbations

Conditional on state θ, each sender i observes a signal $s_i \in \Theta$, whose density is measurable on $\Theta \times \Theta$.[22] In the noiseless game, we simply have $s_i = \theta$. The meaning of "small noise" is formalized below:

Definition: *Noise has size less than* ε if:

1. for all $i \in N$ and $\theta \in \Theta$, $\Pr(|s_i - \theta| < \varepsilon|\theta) \geq 1 - \varepsilon$; and

2. for all $i \in N$ and $s_i \in \Theta$, $\Pr(|s_i - \theta| < \varepsilon|s_i) \geq 1 - \varepsilon$.

Definition: *Noise has local size less than* ε if for all $i \in N$ and $\theta \in \Theta$, $|s_i - \theta| < \varepsilon$ surely.

The first definition is inspired by the Ky Fan metric. It says that noise is small when at any state, each sender's signal is close to the state with high probability, and when after any signal, each sender puts a high probability on the state being close to the signal. It does not rule out the presence of atoms, so for example, the class of "replacement noise" sequences considered by Battaglini (2002), Section 3 of Ambrus and Lu (2014) and Rubanov (2015), where each sender observes the true state with probability approaching 1 and observes the realization of a continuous random variable with full support otherwise, has size converging to 0. A sequence of noise with size converging to 0 converges in probability to the trivial signal structure $s_i = \theta$.

The second definition is more stringent, as it requires that signals be always close to the state. Such small support is consistent with perturbations in the global games literature. Replacement noise does not have local size converging to 0 even as the probability of observing the true state approaches 1. However, a sequence of noise along which senders observe

[22] This paper's results do not depend on whether the signals are restricted to be independent, conditional on θ.

the closest element of a finer and finer finite grid has (local) size converging to 0. A sequence of noise with local size converging to 0 converges surely to the trivial signal structure $s_i = \theta$.

Denoting the probability that agent i puts on event X as $\Pr_i(X)$, given these definitions, *common knowledge that noise has size less than* ε means common knowledge that, for any $\theta, s_j \in \Theta$, agent i and sender j, $\Pr_i(|s_j - \theta| < \varepsilon|\theta) \geq 1 - \varepsilon$ and $\Pr_i(|s_j - \theta| < \varepsilon|s_j) \geq 1 - \varepsilon$. Similarly, *common knowledge that noise has local size less than* ε means common knowledge that $|s_j - \theta| < \varepsilon$ for all senders $j \in N$.

Finally, message vectors $m \in \times_{i=1}^{n} M_i^{\Gamma}$ that are off-path in the noiseless game can remain off-path with some small local noise Ξ. The receiver could then have any beliefs after such m, and thus any optimal action $a^{\Xi}(m)$, even when there is common knowledge that noise has local size less than ε, for any $\varepsilon > 0$. It is then impossible for $a^{\Xi}(m)$ to be near $a^{\Gamma}(m)$ for all beliefs consistent with Ξ; if robustness were to require this, it would rule out off-path message vectors and thus imply completeness, which would nullify the effect of the restriction to local noise. To address this issue, the robustness concept for Theorem 2 applies a different (and weaker) requirement to off-path message vectors, which assumes that perturbations on the receiver's off-path beliefs are small in the following sense:

Definition: A *perturbation on the receiver's off-path beliefs in* Γ *has size less than* γ if, for any message vector m that is off-path in Γ, the receiver's belief after m is such that the optimal action $a^*(m)$ satisfies $|a^*(m) - a^{\Gamma}(m)| < \gamma$.

5.2 Strong Robustness

This subsection presents the definition of strong robustness and Theorem 1.

Definition: Player i's strategy r_i is a δ*-best response* to opponent strategies r_{-i} if after any history h_i (signal s_i if i is a sender and message vector if i is the receiver), $E[u_i(r_i, r_{-i})|h_i] \geq E[u_i(r_i', r_{-i})|h_i] - \delta$ for any strategy r_i'.

Definition: An equilibrium Γ in the noiseless game is *strongly robust* if, for every $\delta > 0$, there exists $\varepsilon > 0$ such that whenever noise has size less than ε, each player's strategy r_i^{Γ} (m_i^{Γ} for senders and a^{Γ} for the receiver) is a δ-best response to r_{-i}^{Γ} evaluated under sender i's belief about the noise.

The above definition can be used either requiring common prior about the noise, or allowing a larger class of perturbations where it is common knowledge that noise has size less than ε (but players' priors may differ). The former leads to a weaker notion of robustness than the latter. Theorem 1 holds with either interpretation.

Given that small noise can change a sender's optimal message around cell boundaries even in the one-sender case, insisting on Γ remaining an exact best response in the noisy game appears too strong. One interpretation for instead requiring approximate best responses might be that players incur a small cost when deviating from their noiseless plan of action. For example, if a player is a committee, it may not be worth convening a meeting to determine a new strategy when a perturbation makes the original messages slightly suboptimal (for senders) or only slightly changes the optimal actions (for the receiver). Alternatively, the noiseless equilibrium may represent an established convention, and it may be costly for a player to compute an alternative, slightly preferable, course of action. Thus, an equilibrium is strongly robust when for any positive tolerance of suboptimality, if noise is small enough, everyone can be expected to stick with her noiseless equilibrium strategy.[23]

Recall the statement of Theorem 1.

Theorem 1: Consider an equilibrium Γ of the noiseless cheap talk game.

(a) Generically, if Γ is strongly robust, then it is complete and coordination-free.

(b) If Γ is complete and coordination-free, and no cell in Γ is $\{0\}$ or $\{1\}$, then it is strongly robust.

The appendix presents all omitted proofs. Theorem 1 holds with either common-prior small noise or only common knowledge that noise is small. The proof makes the weaker assumption for each part of the result (the former for Theorem 1a, and the latter for Theorem 1b).

Theorem 1a follows from Lemmata 1 and 2, discussed below, and Proposition 4a, stated below and discussed in Section 5.4.[24] Because Γ is trivially coordination-free when $|\{i \in N : |M_i^\Gamma| \geq 2\}| = 1$, the proof assumes $|\{i \in N : |M_i^\Gamma| \geq 2\}| \geq 2$.

Lemma 1: If Γ is strongly robust, then it has interval structure, and $\lambda(\theta^\Gamma(m)) > 0$ for every $m \in \times_{i=1}^n M_i^\Gamma$ (which implies completeness).

[23] See Jackson *et al.* (2012) for a discussion of ε-equilibria. Their main result, that selection criteria based on ε-equilibria of perturbed games have no power, does not apply to the setting considered here: interim optimality with a continuum of types and discontinuous strategies. It does highlight that conditions similar to the robustness concept used here are likely to be weak when applied to settings with finitely many types.

[24] The numbering of Proposition 4, as well as the position of its proof in the appendix, corresponds to its position in Section 5.4.

The proof of Lemma 1 has four steps. First, it is shown from the definition of robustness that $a^{\Xi}(m)$, the receiver's best response to sender strategies in Γ under noise Ξ, must be close to $a^{\Gamma}(m)$ whenever Ξ is small. That is, if noise is added and players keep playing Γ, then the receiver's action must be almost optimal.

Second, Γ must be complete. If not, then given an off-path message vector $m \in \times_{i=1}^{n} M_i^{\Gamma}$, one can easily specify small noise Ξ such that m is most likely to be observed far from $a^{\Gamma}(m)$. For example, consider a fully revealing equilibrium, and suppose that the noise is such that erroneous signals are much more likely when $\theta \approx 0.5$. Then a sender i with bias 0.1 might want to deviate when $s_i \approx 0.4$ in order to induce the receiver to take an action near 0.5. For simplicity, the noise used in the proof has an atomic distribution, but the argument can be made with more standard distributions as well. The key point is that when m is off-path, the optimal action after m is very sensitive to the form of the noise.

Third, steps 3 and 4 establish that Γ must have interval structure. Step 3 shows that every message vector in $\times_{i=1}^{n} M_i^{\Gamma}$ must be sent on a set of positive measure. Suppose instead that some m is sent on a set of measure zero. Then once again, the receiver's optimal response to m would be very sensitive to small noise because the set of states where the receiver expects m can change considerably. For example, if a message vector m is sent only at 0.1 and 0.9, then with noise, the receiver may believe the m is much more likely to occur when the state is near 0.9 than 0.1, which can make the best response to m very far from $a^{\Gamma}(m)$. Once again, this idea does not require extreme noise distributions like the one used in the proof for ease of exposition. Step 4 uses a similar argument to show that, even when every $m \in \times_{i=1}^{n} M_i^{\Gamma}$ were sent on a set of positive measure, if Γ fails to have interval structure, then the best response to m is very sensitive to noise that makes it difficult to observe signals inside $\theta^{\Gamma}(m)$ but outside a proper cell.

Lemma 2 refers to a property of cells defined as follows.

Definitions: For a given strategy profile Γ with interval structure,

- θ_b is a *left-natural boundary for sender* i if it is the right endpoint of a proper cell[25] in Γ, and, denoting the message vector sent in that cell by m, $\exists m_i'' \in M_i^{\Gamma}$ such that:

$$(m_i'', m_{-i}) = m^{\Gamma}(\theta) \text{ for some } \theta \in \Theta,$$
$$u_i(a^{\Gamma}(m_i, m_{-i}), \theta_b) = u_i(a^{\Gamma}(m_i'', m_{-i}), \theta_b), \text{ and } a^{\Gamma}(m_i, m_{-i}) < a^{\Gamma}(m_i'', m_{-i});$$

- θ_b is a *right-natural boundary for sender* i if it is the left endpoint of a proper cell in

[25]Formally, if S denotes the cell, $\theta_b = \sup S$.

Γ, and, denoting the message vector sent in that cell by m', $\exists m''_i \in M_i^\Gamma$ such that:

$$\begin{aligned} & (m''_i, m'_{-i}) = m^\Gamma(\theta) \text{ for some } \theta \in \Theta, \\ u_i(a^\Gamma(m''_i, m'_{-i}), \theta_b) &= u_i(a^\Gamma(m'_i, m'_{-i}), \theta_b), \text{ and } a^\Gamma(m''_i, m'_{-i}) < a^\Gamma(m'_i, m'_{-i}); \end{aligned}$$

- a proper cell is *natural* if its right endpoint is either left-natural or 1, and its left endpoint is either right-natural or 0.

Thus, a cell endpoint θ_b is a left(right)-natural boundary if, coming from the left(right), some sender would change her message at θ_b even if others do not, and the message vector resulting from this change occurs on path. That is, the cell stops at θ_b "naturally" in the sense that: (i) given the receiver's strategy, if the cell extends any further, a sender would have a profitable deviation; and (ii) this deviation would lead to an action of the receiver that is determined by on-path play (as opposed to off-path beliefs). In Example 2, the cell endpoints are not natural boundaries: the senders locally coordinate with each other to avoid designating a cell in another block, so a sender's best response at a boundary changes only because the other sender's message changes.

Definition: A strategy profile is *natural* if it has interval structure and all of its proper cells are natural.

Lemma 2: If Γ is strongly robust, then Γ is natural.

Lemma 2 shows that any right endpoint $\theta_b \neq 1$ of a proper cell C in a strongly robust Γ must be left-natural. Denote the message vector sent in C by m. The proof uses single-crossing and Assumption A to show that, if θ_b were not left-natural, then for any $i \in N$, sender i's unique approximate best response at θ_b to m_{-i} is m_i. Therefore, given s_i near θ_b and noise Ξ such that sender i believes that, with high probability, all other senders' signals are in C, sender i's only nearly optimal message is m_i. Since θ_b is the right endpoint of C, there must be at least at least one sender j whose message differs from m_j at or just to the right of θ_b. Therefore, there exists noise Ξ arbitrarily small such that j fails to play an approximate best response at or just to the right of θ_b.

Proposition 4a: Generically, if Γ is natural and every message vector in $\times_{i=1}^n M_i^\Gamma$ is sent on a set of positive measure, then Γ is coordination-free.

Proposition 4a is discussed in Section 5.4, which also shows that all coordination-free equilibria are natural. The two concepts differ when, in a natural equilibrium, a cell boundary fails to be left-natural and right-natural for the same sender.

The proof of Theorem 1b shows that, without noise, at all states in a coordination-free equilibrium Γ, any sender i's message is either optimal or very close to being optimal for all m_{-i} occurring in a small enough neighborhood - there is no strong reason to coordinate with other senders. (This cannot hold in any equilibrium that is not natural.) Thus, every sender follows a δ-best response by playing the original profile as long as she believes that noise is small. The same holds for the receiver since, as long as noise is less than ε, the message vector following every state at least ε away from boundaries must be the same as in the noiseless game with high probability.

5.3 Robustness

The completeness result in Theorem 1a is quite restrictive. For example, it implies that the number of cells in Γ is equal to $\Pi_{i \in N}|M_i^\Gamma|$, and thus rules out equilibria where: (i) two or more senders are informative, and (ii) the number of cells is a prime. As the intuition for step 2 of the proof of Lemma 1 suggests, completeness comes from the fact that when noise has full support, any pair of messages can be sent in any state. Therefore, a logical step in the analysis is to examine a more restrictive class of noise distributions: those with small *local* size. However, as explained in Section 5.1, restricting only noise on the senders' observation of the state does not get rid of completeness: perturbations on the receiver's off-path beliefs must also be restricted. The proposed robustness concept will therefore be similar to strong robustness on path, but differ from it off path.

Definition: Player i's strategy r_i is an *on-path δ-best response* to opponent strategies r_{-i} if after any history h_i that can be reached given i's belief about noise and r_{-i}, $E[u_i(r_i, r_{-i})|h_i] \geq E[u_i(r_i', r_{-i})|h_i] - \delta$ for any strategy r_i'.

Definition: An equilibrium Γ in the noiseless game is *robust* if:

1. for every $\delta > 0$, there exists $\varepsilon > 0$ such that whenever noise has local size less than ε, each player's strategy r_i^Γ is an on-path δ-best response to r_{-i}^Γ evaluated under sender i's belief about the noise, and

2. in the noiseless game, there exists $\gamma > 0$ such that whenever the perturbation on the receiver's off-path beliefs has size less than γ, every sender's strategy m_i^Γ is a best response to m_{-i}^Γ and $a^{\Gamma *}$, where $a^{\Gamma *}$ denotes the receiver's best-response to m^Γ and her perturbed off-path beliefs.

Point 1 of this definition is exactly the same as the definition of strong robustness, except for the qualifiers "on-path" (for δ-best response) and "local" (for noise size). In fact, without the word "local," robustness would exactly coincide with strong robustness: because the latter implies completeness, both the "on-path" qualifier and point 2 of the definition of robustness would have no effect. Therefore, strong robustness corresponds to robustness, but with a larger class of noise.

As argued previously, applying point 1 to message vectors m that are off-path even given noise is too strong: the receiver's beliefs after such m can be any distribution over Θ. At the same time, not perturbing beliefs after such m at all is unappealing: it seems unlikely that off-path beliefs are *less* subject to perturbations than on-path beliefs. Therefore, point 2 of the above definition requires Γ to survive small perturbations on off-path beliefs. Unlike in point 1, exact best response is required: since senders' strategies always remain nearly optimal for small changes in the receiver's actions, requiring δ-best response would render point 2 vacuous.

One interpretation of point 2 is that senders know approximately, but not exactly, what the receiver would do off path. Therefore, robustness requires that their strategies remain optimal even when their beliefs about the receiver's off-path actions differ slightly from the receiver's actual strategy.

Theorem 2: Consider an equilibrium Γ of the noiseless cheap talk game.

(a) Generically, if Γ is robust, then it is equivalent to a strictly coordination-free equilibrium.

(b) If Γ is strictly coordination-free, then it is robust.

Like Theorem 1, Theorem 2 holds whether robustness is understood to hold only with common-prior small local noise, or more broadly when there is common knowledge that noise is local and small. Once again, the proof makes the weaker assumption for each part of the theorem.

Theorem 2a follows from the following results.

Lemma 3: If Γ is robust, then Γ is natural and satisfies point 3 of the definition of strictly coordination-free equilibrium.

Proposition 4b: Generically, if Γ is natural, then Γ is equivalent to a coordination-free equilibrium.

Proposition 4b is discussed with Proposition 4a in Section 5.4.

Lemma 3 is the counterpart of Lemmata 1 and 2. The main difference between the proofs is the replacement of steps 2 and 3 in the proof of Lemma 1. Instead, step 2 in the proof of

Lemma 3 shows that $m \in \times_{i=1}^{n} M_i^{\Gamma}$ must be sent on a set of positive measure if m is sent at two or more states. The intuition is similar to the one for step 3 in the proof of Lemma 1, but the result does not apply to all m due to the lack of completeness.

Step 3 in the proof of Lemma 3 argues that the set of fully revealed states in Γ has measure zero. If not, then for any $\varepsilon > 0$, there would exist an interval I of size ε that includes a continuum of fully revealed states. Let θ be one such state, $m = m^{\Gamma}(\theta)$, and i be a sender with a continuum of messages in I. Then for any m_i' in this continuum of messages, point 2 of the definition of robustness implies that $a^{\Gamma}(m_i', m_{-i})$ must be far from θ: otherwise, slightly changing it would induce a sender deviation. The rest of step 3 uses this fact and step 2 to show that, in fact, there can only be countably many such m_i'.

The remainder of the proof of Lemma 3 follows the proof of Lemmata 1 and 2, with point 2 of the definition of robustness (which implies point 3 of the definition of strictly coordination-free equilibrium) ensuring that messages m_i' such that (m_i', m_{-i}) is out-of-equilibrium are not approximately optimal.

Theorem 2b is proved in the same way as Theorem 1b, but additional cases must be checked due to the potential presence of out-of-equilibrium message vectors and cells $\{0\}$ and $\{1\}$.

5.4 Relation between Coordination-Free and Natural Equilibria

Proposition 2 establishes basic properties of natural equilibria that are useful for the comparison with coordination-free equilibria.

Proposition 2:
(a) In any natural equilibrium Γ, $\theta^{\Gamma}(m)$ is connected whenever $\lambda(\theta^{\Gamma}(m)) > 0$.
(b) In each game, there is a finite upper bound for the number of cells in natural equilibria.

The reasoning for Proposition 2a is simple: by the definition of natural boundary, m ceases to be optimal at the endpoints of any proper cell where m is sent - and there must be such a cell because Γ has interval structure and m is sent on a positive-measure set. Single crossing then implies that $\theta^{\Gamma}(m)$ is connected. The proof of Proposition 2b notes that for every action a induced in a proper cell whose left endpoint θ_L is not 0, because θ_L must be right-natural, there must be another induced action a' at least η to the left of a such that the message vectors inducing a and a' differ in one component. An inductive argument, starting with the interval $[0, \eta]$, is then used to obtain the result.

The remainder of Section 5 argues that generically, a natural equilibrium features the same play as some coordination-free equilibrium at almost all states. Proposition 3 shows that one direction of this relation is simple.

Proposition 3: Every coordination-free equilibrium is natural.

Proof: If an equilibrium Γ is coordination-free, then at every cell endpoint θ,

$$u_i(a^\Gamma(m_i, m_{-i}), \theta) = u_i(a^\Gamma(m_i', m_{-i}), \theta) \text{ for some sender } i,$$

where (m_i, m_{-i}) is sent in the left cell and (m_i', m_{-i}) is sent in the right cell. Moreover, by definition, $a^\Gamma(m_i, m_{-i}) \neq a^\Gamma(m_i', m_{-i})$, so θ is a natural boundary. Thus Γ is natural. ■

The partial converse of Proposition 3 is more complicated.

Proposition 4: Generically,

(a) If Γ is natural and every message vector in $\times_{i=1}^n M_i^\Gamma$ is sent on a set of positive measure, then Γ is coordination-free.

(b) If Γ is natural, then Γ is equivalent to a coordination-free equilibrium.

The intuition for Proposition 4 is as follows. Suppose profile Γ is natural. Label the proper cells $0, ..., K$ from left to right, and let m^k denote the message vector sent in cell k. Let:

- $\theta_0 = 0$ and $\theta_{2K+2} = 1$;
- θ_{2k} denote the boundary between cell $k-1$ and cell k;
- θ_{2k+1} denote the action induced in cell k, *i.e.* $\theta_{2k+1} = a^\Gamma(m^k)$;
- i_{kR} (resp. i_{kL}) denote a sender for whom the boundary between cell $k-1$ and k is right-natural (resp. left-natural);
- $R_k < 2k+1$ be such that $u_{i_{kR}}(\theta_{2k+1}, \theta_{2k}) = u_{i_{kR}}(\theta_{R_k}, \theta_{2k})$ ($R_k \in \mathbb{N}$ exists because θ_{2k} is right-natural); and
- $L_k > 2k-1$ be such that $u_{i_{kL}}(\theta_{2k-1}, \theta_{2k}) = u_{i_{kL}}(\theta_{L_k}, \theta_{2k})$ ($L_k \in \mathbb{N}$ exists because θ_{2k} is left-natural).

Definition: The *structure* of Γ consists of $\{i_{kR}\}_{k=1}^K, \{i_{kL}\}_{k=1}^K, \{R_k\}_{k=1}^K$ and $\{L_k\}_{k=1}^K$.[26] $\{i_{kR}\}_{k=1}^K$ and $\{R_k\}_{k=1}^K$ form the *right-structure*, while $\{i_{kL}\}_{k=1}^K$ and $\{L_k\}_{k=1}^K$ form the *left-structure.*

[26] A given profile may be described by more than one structure, as there may happen to be multiple i_{kR}'s or i_{kL}'s at a boundary k.

By Proposition 2b, K is bounded above, so the number of possible structures is finite.

A natural equilibrium's right-structure and θ_1 (or θ_2) fully determine the action induced at every state that is not a boundary: given θ_1, the receiver's utility function and the prior density uniquely determine θ_2 (and *vice versa*). Then given θ_2, i_{1R}'s utility function and R_1 uniquely determine θ_3, and so on. θ_1 needs to be such that cell K's right endpoint is at 1. When $R_k = 2k - 1$ for all k, this procedure is the same as the forward solution used to compute single-sender and coordination-free equilibria.

Suppose Γ is coordination-free. Here, for all $k = 1, ..., K$, it must be that $R_k = 2k - 1$, $L_k = 2k + 1$ and $i_{kR} = i_{kL}$. Thus, the left-structure is redundant with the right-structure, and does not impose any additional condition for Γ to be an equilibrium.

By contrast, if play in a natural equilibrium Γ does not correspond to play in a coordination-free equilibrium in all cell interiors, then at some boundary k, multiple senders change their message. Proposition 2a implies that in a natural equilibrium, if a message vector is sent in a proper cell, then no other on-path message vector can induce the same action. This implies that $R_k \neq 2k - 1$ and $L_k \neq 2k + 1$. Thus in this case, the left-structure imposes a supplementary indifference condition. Since the right-structure has already fixed all boundaries and actions, extra conditions imposed by the left-structure are generically not satisfied.

Example 4 shows that one can build a natural equilibrium that is not coordination-free given non-generic primitives.

Example 4: Natural Equilibrium that Is Not Coordination-Free

Consider a two-player game that follows the uniform quadratic specification, except that player 1's bias $b_1(\theta)$ *is not constant. Instead,* $b_1(\theta) = 0.04$ *for* $\theta \leq 0.1$*, and is continuous and increasing for* $\theta > 0.1$*. Player 2's bias is constant at* $b_2 = -0.02$*.*

We look for a six-cell natural equilibrium with the following structure (following the above notation, from left to right, the cells are labeled 0, 1, 2, 3, 4 *and* 5*, the receiver's actions are* θ_1, θ_3, θ_5, θ_7, θ_9 *and* θ_{11}*, and the boundaries between cells are* θ_2, θ_4, θ_6, θ_8 *and* θ_{10}*):*

- right-structure: $i_{1R} = 1$, $i_{2R} = i_{3R} = i_{4R} = i_{5R} = 2$, $R_1 = 1$, $R_2 = 3$, $R_3 = 5$, $R_4 = 1$, $R_5 = 9$*;*

- left-structure: $i_{1L} = i_{4L} = 1$, $i_{2L} = i_{3L} = i_{5L} = 2$, $L_1 = 3$, $L_2 = 5$, $L_3 = 7$, $L_4 = 11$, $L_5 = 11$*.*

Note that at θ_2, θ_4, θ_6 *and* θ_{10}*, the structure identifies a single player that is indifferent between the adjacent actions. However, at* θ_8*, we must have* $u_2(\theta_1, \theta_8) = u_2(\theta_9, \theta_8)$ *and* $u_1(\theta_7, \theta_8) = u_1(\theta_{11}, \theta_8)$*. Since neither sender is indifferent between* θ_7 *and* θ_9 *at* θ_8*, such an equilibrium cannot be coordination-free.*

It is straightforward to show that in order for an equilibrium to have the specified right-structure, we must have $\theta_1 = \frac{1}{900}$, $\theta_2 = \frac{2}{900}$, $\theta_3 = \frac{75}{900}$, $\theta_4 = \frac{148}{900}$, $\theta_5 = \frac{185}{900}$, $\theta_6 = \frac{222}{900}$, $\theta_7 = \frac{223}{900}$, $\theta_8 = \frac{224}{900}$, $\theta_9 = \frac{411}{900}$, $\theta_{10} = \frac{598}{900}$ *and* $\theta_{11} = \frac{749}{900}$. *Therefore, the specified left-structure can be achieved only if*

$$u_1(\frac{223}{900}, \frac{224}{900}) = u_1(\frac{749}{900}, \frac{224}{900}).$$

If this non-generic condition is satisfied, then the following sender strategies are part of a natural equilibrium with the above structure:

- *if* $\theta \in [0, \frac{2}{900})$, $m(\theta) = (x, x)$;
- *if* $\theta \in [\frac{2}{900}, \frac{148}{900})$, $m(\theta) = (y, x)$;
- *if* $\theta \in [\frac{148}{900}, \frac{222}{900})$, $m(\theta) = (y, y)$;
- *if* $\theta \in [\frac{222}{900}, \frac{224}{900})$, $m(\theta) = (y, z)$;
- *if* $\theta \in [\frac{224}{900}, \frac{598}{900})$, $m(\theta) = (x, y)$; *and*
- *if* $\theta \in [\frac{598}{900}, 1]$, $m(\theta) = (x, z)$.

6 Best Coordination-Free Equilibrium for the Receiver

Let the receiver's maximum expected utility from a coordination-free equilibrium be $\overline{u_R}$. This section studies how $\overline{u_R}$ depends on the characteristics of senders available, and all of the analysis here also applies to strictly coordination-free equilibria.

It is not always the case that replacing a sender by a less biased one will increase $\overline{u_R}$. Example 5 shows that doing so may *decrease* $\overline{u_R}$, even in the uniform-quadratic specification.

Example 5: *Consider a uniform-quadratic game with two senders, where* $b_1 = 0.075$ *and* $b_2 = -0.0525$. *It is easy to check that the following strategy profile* Γ *is part of a coordination-free equilibrium:*

- *If* $\theta \in [0, 0.008)$, $m_1 = 1$ *and* $m_2 = 1$. $a^\Gamma(1,1) = 0.004$.
- *If* $\theta \in [0.008, 0.316)$, $m_1 = 2$ *and* $m_2 = 1$. $a^\Gamma(2,1) = 0.162$.
- *If* $\theta \in [0.316, 0.414)$, $m_1 = 2$ *and* $m_2 = 2$. $a^\Gamma(2,2) = 0.365$.
- *If* $\theta \in [0.414, 0.812)$, $m_1 = 3$ *and* $m_2 = 2$. $a^\Gamma(3,2) = 0.613$.
- *If* $\theta \in [0.812, 1]$, $m_1 = 3$ *and* $m_2 = 3$. $a^\Gamma(3,3) = 0.906$.
- $a^\Gamma(1,2) = a^\Gamma(1,3) = 0.2$, $a^\Gamma(2,3) = 0.5$ *and* $a^\Gamma(3,1) = 0$, *so that no deviation to an out-of-equilibrium vector is induced.*

The receiver's expected utility from Γ *is approximately* -0.008321.

Now suppose sender 2 becomes less biased, so that $b_2 = -0.05$. *It is easy to check that the following strategy profile* Γ' *is part of a coordination-free equilibrium:*

- If $\theta \in [0, 0.3)$*,* $m_1 = 1$ *and* $m_2 = 1$*.* $a^{\Gamma}(1,1) = 0.15$*.*
- If $\theta \in [0.3, 0.4)$*,* $m_1 = 1$ *and* $m_2 = 2$*.* $a^{\Gamma}(1,2) = 0.35$*.*
- If $\theta \in [0.4, 0.8)$*,* $m_1 = 2$ *and* $m_2 = 2$*.* $a^{\Gamma}(2,2) = 0.6$*.*
- If $\theta \in [0.8, 1]$*,* $m_1 = 2$ *and* $m_2 = 1$*.* $a^{\Gamma}(2,1) = 0.9$*.*

Because sender 2's bias is now smaller, the cell size decreases less from left to right at boundaries where sender 2 switches message. As a result, it can be shown that there can now be at most 4 nontrivial cells, and that of these, Γ' *is the best one for the receiver, with an expected utility of* $-\frac{1}{120} \approx -0.008333$*.*[27] *Therefore,* $\overline{u_R}$ *has decreased even though sender 2 has become less biased.*

Note that because the set of coordination-free equilibria is straightforward to compute, it was possible in Example 5 to find the receiver's best coordination-free equilibrium.

If all senders are biased in the same direction, the anomaly illustrated by Example 5 does not arise. In fact, Proposition 5 shows that in this case, even when multiple senders are available, only the least biased sender is informative in the receiver's optimal coordination-free equilibrium. The intuition is that if the sender whose message changes at a boundary is not the least biased, then replacing her by the least biased sender (while keeping the number of cells intact) must shrink the largest cells and expand the smallest ones, which increases the receiver's expected welfare.

Proposition 5: Consider the uniform-quadratic case where $\min_i b_i = b > 0$. In the receiver's optimal coordination-free equilibrium, any sender whose bias exceeds b babbles.

Under the uniform-quadratic specification, if there are senders biased in both directions, and the receiver must choose two senders, picking the least biased sender in each direction should be near optimal in the sense of maximizing $\overline{u_R}$. The intuition is as follows: the receiver would like to keep the size of cells to a minimum. If both chosen senders are biased in the same direction, then the cell sizes can only increase in that direction, eventually resulting in very large intervals. Therefore, the receiver should ensure that senders have opposite biases. Furthermore, the smaller a sender's bias, the slower cells grow in the direction of the bias, and the earlier that sender can be used to reduce cell size when going in the opposite direction.

[27] Γ' has cell sizes $(0.1, 0.2, 0.3, 0.4)$. The other coordination-free equilibria with four nontrivial intervals have cell sizes $(0.025, 0.225, 0.325, 0.425)$, $(0.05, 0.15, 0.35, 0.45)$ and $(0.075, 0.175, 0.275, 0.475)$, which all yield lower expected utility. With three intervals, even if there were an equilibrium where the intervals were equally sized, the receiver would still be worse off, with an expected utility of $-\frac{1}{108}$.

As Example 5 suggests, it is difficult to precisely characterize the optimal choice of senders in general. However, in the limit as Θ becomes large relative to the biases,[28] the above intuition is exactly confirmed.

Proposition 6: Take a sequence $\{\Theta_l\}$ where each Θ_l is a closed interval and $\lambda(\Theta_l) \overset{l\to\infty}{\to} \infty$. In the uniform-quadratic case with two senders that have biases $-b_2$ and b_1, where $b_1, b_2 > 0$ and $\frac{b_1}{b_2} \notin \mathbb{Q}$, then:

(a) in any sequence of coordination-free equilibria $\{\Gamma_l\}$ where each Γ_l has the maximum number of cells in a coordination-free equilibrium, the fraction of cells with size in any interval $I \subseteq [0, 4(b_1 + b_2)]$ converges to $\frac{\lambda(I)}{4(b_1+b_2)}$ as $l \to \infty$;

(b) letting $\overline{u_R}^l$ be the receiver's maximum expected utility for $\lambda(\Theta_l)$, we have $\overline{u_R}^l \overset{l\to\infty}{\to} -\frac{2}{3}(b_1 + b_2)^2$.

Proposition 6a states that if sender biases are $-b_2 < 0 < b_1$ and satisfy $\frac{b_1}{b_2} \notin \mathbb{Q}$, the distribution of cell sizes in coordination-free equilibria maximizing the number of cells converges to $U[0, 4(b_1 + b_2)]$ as $\lambda(\Theta) \to \infty$. The proposition is proved by showing that the number of cells is not maximized if any cell's size exceeds $4(b_1 + b_2)$. Given this restriction, the size of the leftmost cell uniquely determines the sequence of cell sizes because at each boundary, cell size must either increase by $4b_1$ or decrease by $4b_2$. As this sequence become longer, the distribution of its elements converges to $U[0, 4(b_1 + b_2)]$ whenever $\frac{b_1}{b_2} \notin \mathbb{Q}$.

Proposition 6b states that the receiver's limit maximum expected utility corresponds to the distribution of cell sizes from Proposition 6a. It implies that if the receiver is choosing two senders from a finite pool, then generically[29], in the limit $\lambda(\Theta) \to \infty$, the best choice is to pick the least biased sender in each direction. Moreover, the advantage from consulting senders with opposite biases is striking: if all senders' biases were in the same direction, then as $\lambda(\Theta) \to \infty$, $\overline{u_R} \to -\infty$.

Coordination-free equilibria can also be used to study games with more than two senders. For example, does limiting the number of senders to two (with biases $-b_2 < 0 < b_1$) dramatically increase cell sizes relative to consulting more senders with biases within $(-\infty, -b_2] \cup [b_1, \infty)$? Recall that in the uniform-quadratic case, from left to right, cell size either increases by $4b_1$ or decreases by $4b_2$ at each boundary. With two senders, the lower bound on

[28] Unlike in the rest of this paper, Proposition 6 allows Θ to vary. One could state an equivalent result holding $\Theta = [0, 1]$ fixed by scaling the players' preferences. Varying Θ and fixing preferences simplifies the exposition.

[29] That is, if for any two senders i and j, $\frac{b_i}{b_j} \notin \mathbb{Q}$.

the supremum of cell sizes as $\lambda(\Theta) \to \infty$ generically approaches $4(b_1 + b_2)$,[30] which is less than twice the corresponding bound of $\max\{4b_1, 4b_2\}$ from consulting more senders. Therefore, the receiver's loss from ignoring all senders but the least biased in each direction is not too large.

The intuition presented above extends to situations where preferences are not too asymmetric about their peak and the prior is not too far from uniform: cell sizes can be kept smaller when biases are small and opposite. Moreover, the minimum size of the largest cell is on the order of $\max\{b_1, b_2\}$ whether there are two senders with biases near $-b_2$ and b_1, or more senders with biases in $(-\infty, -b_2] \cup [b_1, \infty)$. Therefore, once again, the receiver can come reasonably close to achieving the minimum expected loss by consulting only two senders biased in opposite directions and with small biases relative to the available pool of senders.

7 Discussion

7.1 Alternative Robustness Concept

Online Appendix C studies a weakening of (strong) robustness of Γ where a strategy profile *close to* Γ, not necessarily Γ itself, is to be interim δ-optimal under noise. It is shown that Theorems 1 and 2 would still hold provided that, in the noisy setting, only common knowledge of small noise were assumed. (With, instead, only common-prior noise, it is not known whether Theorems 1a and 2a would remain true under this alternative robustness concept.)

The formal definitions of close strategy profiles and of the alternative robustness concepts are left to Online Appendix C. Lemma 1 is established in a similar way, but the suboptimality argument establishing Lemma 2, presented in Section 5, cannot be generalized to nearby profiles in a straightforward way. The proof of Lemma 2 in Online Appendix C relies on a noise structure where every sender i believes that $s_i = \theta$ for sure, but that $s_j = \max\{\theta - \varepsilon, 0\}$; thus, players do not share a common prior about the noise. With this noise structure, every sender i believes that other senders observe a signal (and thus send the message corresponding to a signal) slightly to the left of s_i. It is shown inductively, proceeding from left to right, that, if a cell C were not left-natural, then all senders would send the same messages as they

[30] If instead $\frac{b_1}{b_2} \in \mathbb{Q}$, the bound is reduced to $4(b_1 + b_2 - \gcd(b_1, b_2))$, where $\gcd(b_1, b_2)$ is the largest number k such that $\frac{b_1}{k}, \frac{b_2}{k} \in \mathbb{Z}$. For example, if $b_1 = 0.02$ and $b_2 = 0.03$, then for cells to be kept smaller than $4(b_1 + b_2) = 0.2$, their sizes, from left to right, would be $\varepsilon, 0.08 + \varepsilon, 0.16 + \varepsilon, 0.04 + \varepsilon, 0.12 + \varepsilon, \varepsilon, \ldots$ The lower bound on the size of the largest cell is therefore $0.16 = 4(0.02 + 0.03 - 0.01)$.

do in C well past the right endpoint of C, resulting in a strategy profile that is not close to the original one.

The proofs of Theorems 1b and 2b do not change since the robustness concept has been relaxed, while the proof of Lemma 3, and thus of Theorem 2a, is modified in a similar way as the proofs of Lemmata 1 and 2.

7.2 Multidimensional State Space

Extending (strong) robustness to a cheap talk game where Θ is multidimensional would not yield implications as strong as with a single-dimensional Θ. Consider a pure-strategy equilibrium Γ where, for any $\delta > 0$, there exists $\varepsilon > 0$ such that at any state θ, $m_i^\Gamma(\theta)$ is a δ-best response (given a^Γ) to any vector m_{-i} where every component is sent by the corresponding sender in Γ at some state in the ε-ball around θ. Then, with small enough noise, for $s_i = \theta$, sender i believes that $m_i^\Gamma(\theta)$ is a 2δ-best response.

For example, suppose $\Theta \subset \mathbb{R}^2$ and is bounded, $n = 2$, and curve $\mathcal{C}_i$, for $i = 1, 2$, is the boundary between sets where m_i and m_i' are sent. Then if $\mathcal{C}_1$ and $\mathcal{C}_2$ cross (as opposed to being tangent) at some θ^*, (m_1, m_2), (m_1', m_2), (m_1', m_2') and (m_1, m_2') are all sent in Γ arbitrarily close to θ^*. If, moreover, no message other than m_i and m_i' is sent near θ^*, then the property from the previous paragraph is satisfied around θ^*. As long as $\mathcal{C}_1$ and $\mathcal{C}_2$ cross whenever they meet, there are no further restrictions on their shape: they could cross multiple times, one (or both) of them could be a loop, etc. Therefore, with multidimensional Θ, generalizations of this paper's robustness concept would likely fail to yield an easy-to-characterize set of equilibria, unlike with $\Theta \subset \mathbb{R}$.

8 Conclusion

This paper has shown that strictly coordination-free equilibria remain interim nearly optimal for all players for sufficiently small local noise in senders' observation of the state, and any equilibrium satisfying this property is generically equivalent to a strictly coordination-free equilibrium. When the small noise is not required to be local, coordination-free equilibria where all combinations of on-path messages form on-path message vectors are selected. Both these results hold whether there is common prior about noise, or it is merely commonly known that noise is small.

Coordination-free equilibria have a similar structure to one-sender equilibria, in that the size of the first interval and the identity of the indifferent sender at each boundary determine play. This property implies that fixing senders' biases and increasing the size of the state space cannot make the size of the revealed intervals vanishingly small. The amount of information loss therefore remains nontrivial, unlike in the fully revealing and almost fully revealing equilibria examined by the existing literature. This paper's results may therefore enable nontrivial comparisons between cheap talk and other ways in which a decision maker can interact with multiple biased parties that hold relevant information.

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Appendix: Proofs

Proof of Lemma 1: Suppose Γ is strongly robust. Let a^{Ξ} denote the receiver's best response to $\{m_j^{\Gamma}\}_{j=1}^n$ given noise Ξ.

Step 1: For any $\delta > 0$, $\exists \varepsilon > 0$ such that for all noise Ξ with size less than ε, $|a^{\Gamma}(m) - a^{\Xi}(m)| < \delta$ for all $m \in \times_{i=1}^n M_i^{\Gamma}$.

By the definition of robustness, we know that for any $\delta > 0$, $\exists \varepsilon > 0$ such that for all noise Ξ with size less than ε, a^{Γ} is a δ-best response to $\{m_j^{\Gamma}\}_{j=1}^n$ under Ξ.

Because u_R is continuous and strictly concave in a, and Θ is compact, $\exists \gamma(\delta)$ such that, for all $m \in \times_{i=1}^n M_i^{\Gamma}$ and Ξ with size less than ε, $|a^{\Gamma}(m) - a^{\Xi}(m)| < \gamma(\delta)$, with $\lim_{\delta \to 0} \gamma(\delta) = 0$. Rewriting δ in lieu of $\gamma(\delta)$ yields the result. $\lozenge$

Step 2: Γ is complete.

Suppose instead that $m = (m_1, ..., m_n) \in \times_{i=1}^n M_i^{\Gamma}$ does not occur in Γ. Then consider noise Ξ where:

(i) at some $\theta \neq a^{\Gamma}(m)$, each sender i independently observes $s_i = \theta$ with probability $1-\varepsilon$, and $s_i = \theta_i$ with probability ε for some θ_i where sender i's message in Γ is m_i;

(ii) at all other states, each sender observes the true state.

Clearly, Ξ has size at most $\varepsilon \frac{D}{d}$. For any ε, θ is the only state at which m can occur in then noisy game, and m can indeed occur at θ. Thus $a^{\Xi}(m) = \theta$. By step 1, taking $\delta < |a^{\Gamma}(m) - \theta|$ implies that Γ cannot be robust. $\lozenge$

Step 3: For all $m \in \times_{i=1}^n M_i^{\Gamma}$, $\lambda(\theta^{\Gamma}(m)) > 0$.

Suppose instead that $\lambda(\theta^{\Gamma}(m)) = 0$. Then for any $\theta \in \theta^{\Gamma}(m)$ and any $\varepsilon > 0$, $\exists \theta_0(\theta) \in [\theta - \varepsilon, \theta + \varepsilon]$ such that for some $i \in N$, $m_i^{\Gamma}(\theta_0(\theta)) \neq m_i$. Let $i_{\varepsilon}^{\Gamma}(\theta)$ be some such i.

At least two senders have at least two equilibrium messages in Γ; assume without loss of generality that sender 1 is one of them. By step 2, every sender must send each of her equilibrium messages at a minimum of two states where the message vectors sent by other senders differ. Let θ' be such that $m_1^{\Gamma}(\theta') = m_1$ and $m^{\Gamma}(\theta') \neq m$, and let $\theta'' \neq a^{\Gamma}(m)$ be such that $m^{\Gamma}(\theta'') = (m_1'', m_2, ..., m_n)$ for some $m_1'' \in M_1^{\Gamma}$. Consider noise Ξ where:

(i) at state θ'', sender 1 observes $s_i = \theta''$ with probability $1-\varepsilon$, and $s_i = \theta'$ with probability ε;

(ii) at all states $\theta \neq \theta''$ where $m^{\Gamma}(\theta) = m$, consider a random variable $X \sim U[0,1]$; if the realization of X is θ, sender $i_{\varepsilon}^{\Gamma}(\theta)$ observes $s_i = \theta$, while if not, sender $i_{\varepsilon}^{\Gamma}(\theta)$ observes $s_i = \theta_0(\theta)$;

(iii) if neither (i) or (ii) applies, the true state is observed.

Clearly, Ξ has size at most $\varepsilon\frac{D}{d}$. At all states $\theta \neq \theta''$ where $m^\Gamma(\theta) = m$, the receiver observes m with probability 0; at θ'', she observes m with probability ε (if $m_1'' \neq m_1$) or 1 (if $m_1'' = m_1$); and at all other states, she cannot observe m. Because $\lambda(\theta^\Gamma(m)) = 0$, upon observing m, the receiver puts probability 1 on $\theta = \theta''$.[31] Picking $\delta < |a^\Gamma(m) - \theta''|$ completes the argument. $\Diamond$

Step 4: Γ has interval structure.

Let $S = \cup_{I \text{ is a proper cell in } \Gamma} I$. We proceed by contradiction: suppose $\lambda(S) < \lambda(\Theta)$. Define $C(m) = (\Theta \backslash S) \cap \{\theta \in \theta^\Gamma(m) : \theta < a^\Gamma(m)\}$ and $D(m) = (\Theta \backslash S) \cap \{\theta \in \theta^\Gamma(m) : \theta > a^\Gamma(m)\}$.

First, I argue that there exists $m \in \times_{i=1}^n M_i^\Gamma$ such that at least one of $C(m)$ and $D(m)$ has positive measure. Suppose not. Then $\lambda(C(m) \cup D(m)) = 0$ for all m. By step 3, only a countable number of m can be sent, implying both of the following:

$$\begin{aligned} \lambda(\{a^\Gamma(m) \text{ s.t. } m &\in \times_{i=1}^n M_i^\Gamma\}) = 0, \text{ and} \\ \sum_{m \in \times_{i=1}^n M_i^\Gamma} \lambda(C(m) \cup D(m)) &= 0. \end{aligned}$$

Combining the two relations above yields $\lambda(\Theta \backslash S) = 0$, which contradicts the hypothesis.

Assume that $\lambda(C(m)) > 0$ for some $m = (m_1, ..., m_n)$ (the argument for the case $\lambda(D(m)) > 0$ is symmetric). By definition, for any $\theta \in C(m)$, we have $\theta \notin S$, so for any $\varepsilon > 0$, $\exists \theta' \in [\theta - \varepsilon, \theta + \varepsilon]$ such that for some $i \in N$, $m_i^\Gamma(\theta') \neq m_i$. Let $i^\Gamma(\theta)$ be some such i, and consider the following noise Ξ:

(i) at states $\theta \in C(m)$, consider a random variable $X \sim U[0,1]$; if the realization of X is θ, sender $i^\Gamma(\theta)$ observes $s_i = \theta$, while if not, sender $i^\Gamma(\theta)$ observes $s_i = \theta'$ for some $\theta' \in [\theta - \varepsilon, \theta + \varepsilon]$ where $m_i^\Gamma(\theta') \neq m_i$;

(ii) otherwise, the true state is observed.

Clearly, Ξ has size at most ε, and because conditional on observing m under Ξ and $\{m_j^\Gamma\}_{j=1}^n$, the probability of $\theta \in C(m)$ is reduced (to zero), we have $a^\Xi(m) > a^\Gamma(m)$. Thus taking $\delta < a^\Xi(m) - a^\Gamma(m)$ completes the proof. ■

Proof of Lemma 2: We proceed by contradiction. Suppose instead, without loss of generality, that the right endpoint $\theta_b \neq 1$ of a proper cell C in Γ where $m = (m_1, ..., m_n)$ is sent is not left-natural. By the completeness of Γ and the definition of "left-natural," it follows that for any $i \in N$ and $m_i' \in M_i^\Gamma \backslash \{m_i\}$, either:

i. $u_i(a^\Gamma(m_i, m_{-i}), \theta_b) > u_i(a^\Gamma(m_i', m_{-i}), \theta_b)$; or

[31] Heuristically, the density that m is observed and that the state is in $\theta^\Gamma(m)$ is at most $D\lambda(\theta^\Gamma(m)) = 0$, while the density that m is observed and that the state is θ'' is at least $d\varepsilon > 0$.

ii. $u_i(a^\Gamma(m_i, m_{-i}), \theta_b) = u_i(a^\Gamma(m_i', m_{-i}), \theta_b)$ and $a^\Gamma(m_i, m_{-i}) \geq a^\Gamma(m_i', m_{-i})$.

Case ii can be ruled out: if $u_i(a^\Gamma(m_i, m_{-i}), \theta_b) = u_i(a^\Gamma(m_i', m_{-i}), \theta_b)$, then Assumption A implies $a^\Gamma(m_i, m_{-i}) \neq a^\Gamma(m_i', m_{-i})$. Furthermore, $a^\Gamma(m_i, m_{-i}) > a^\Gamma(m_i', m_{-i})$ would, by single-crossing, contradict m_i being a best response to m_{-i} immediately to the left of θ_b. Thus we have $u_i(a^\Gamma(m_i, m_{-i}), \theta_b) > u_i(a^\Gamma(m_i', m_{-i}), \theta_b)$.

There can be only finitely many actions $a^\Gamma(m_i'', m_{-i})$ for $m_i'' \in M_i^\Gamma$. If not, then at least two such actions, say $a < a'$, are within η of each other, which implies that sender i either strictly prefers a' whenever $\theta \geq a$ (if $b_i(.) > 0$), or strictly prefers a whenever $\theta \leq a'$ (if $b_i(.) < 0$). Given completeness, this contradicts the receiver always playing a best response.

Combining the observations from the two previous paragraphs, it follows that there exists $\delta > 0$ such that, at $\theta = \theta_b$, sender i's unique 2δ-best response to m_{-i} is m_i. Therefore, given $s_i \in [\theta_b, \theta_b + \varepsilon)$ for small enough ε and any noise Ξ such that sender i believes that, with sufficiently high probability, all other senders' signals are in cell C, sender i's unique δ-best response to m_{-i} is m_i. Therefore, if Γ is strongly robust, then every sender must send m_i both at and immediately to the right of θ_b. This contradicts θ_b being the right endpoint of C. ■

Proof of Theorem 1b: Fix $\delta > 0$.

Senders play a δ-best response

After signal s_i, sender i places probability at least $1 - n\varepsilon$ on all senders' signals being in $[s_i - 2\varepsilon, s_i + 2\varepsilon]$. Because utilities are continuous and bounded (the latter from Lipschitz continuity and Θ being bounded), this means that, for ε small enough, all senders are playing δ-best responses more than 2ε away from cell endpoints.

By the same token, close to a boundary between cells where (m_i, m_{-i}) and (m_i', m_{-i}) are sent, both m_i and m_i' are δ-best responses because player i places probability near 1 on others sending m_{-i}.

Close to a boundary between cells where (m_i, m_j, m_{-ij}) and (m_i, m_j', m_{-ij}) are sent, since m_i is a δ-best response to both (m_j, m_{-ij}) and (m_j', m_{-ij}), the same argument applies.

The receiver plays a δ-best response

Let $S_\varepsilon = \{\theta \in \Theta : m^\Gamma(\theta') = m^\Gamma(\theta) \text{ for all } \theta' \in [\theta - \varepsilon, \theta + \varepsilon]\}$ be the set of states more than ε away from a cell endpoint. Proposition 1 implies that for any $\gamma > 0$, $\exists \varepsilon(\gamma) > 0$ such that $\lambda(S_{\varepsilon(\gamma)}) > 1 - \gamma$. So for any $m \in \times_{i=1}^n M_i^\Gamma$, under any noise less than $\varepsilon(\gamma)$:

- when $\theta \in S_{\varepsilon(\gamma)} \cap \theta^\Gamma(m)$, the receiver sees m with probability at least $1 - n\varepsilon(\gamma)$;

- when $\theta \in S_{\varepsilon(\gamma)} \backslash \theta^\Gamma(m)$, the receiver sees m with probability at most $\varepsilon(\gamma)$.

Let $\lambda(\theta^\Gamma(m)) \equiv \lambda_m > 0$. Upon seeing m, the receiver puts probability at least

$$\frac{(1-n\varepsilon(\gamma))(\lambda_m-\gamma)d}{(1-n\varepsilon(\gamma))(\lambda_m-\gamma)d+\gamma D+\varepsilon(\gamma)D}$$

on the state being in $\theta^\Gamma(m)$. As $\gamma \to 0$, the above quantity still converges to 1. It follows that as $\varepsilon(\gamma) \to 0$, the receiver's optimal action converges to $a^\Gamma(m)$.

Because there are finitely many cells, $\min_{m:\lambda_m>0} \lambda_m$ exists (and is positive). It follows that for any $\delta > 0$, it is possible to pick $\gamma > 0$ such that under any noise less than $\varepsilon(\gamma)$, playing $a^\Gamma(m)$ is a δ-best response for the receiver to all m sent in equilibrium in the noiseless game. ■

Proof of Lemma 3: Like in the proof of Theorem 1a, we study the nontrivial case $|\{i \in N : |M_i^\Gamma| \geq 2\}| \geq 2$.

Step 1: Identical to step 1 in the proof of Lemma 1 for on-path m, and by the definition of small perturbations for off-path m.

Step 2: For any $m \in \times_{i=1}^n M_i^\Gamma$, if $\theta^\Gamma(m)$ contains two or more elements, then $\lambda(\theta^\Gamma(m)) >$ 0.

Since $\theta^\Gamma(m)$ contains at least two elements, there exists $\theta^* \in \theta^\Gamma(m)\backslash\{a^\Gamma(m)\}$.

Suppose instead that $\lambda(\theta^\Gamma(m)) = 0$. Then for any $\varepsilon > 0$ and any θ where $m^\Gamma(\theta) = m$, $\exists\theta_0(\theta) \in [\theta-\varepsilon, \theta+\varepsilon]$ such that for some $i \in N$, $m_i^\Gamma(\theta_0(\theta)) \neq m_i$. Let $i_\varepsilon^\Gamma(\theta)$ be any such i, and consider the following noise Ξ:

(i) at states $\theta \in \theta^\Gamma(m)\backslash\{\theta^*\}$, consider a random variable $X \sim U[0,1]$; if the realization of X is θ, sender $i_\varepsilon^\Gamma(\theta)$ observes $s_i = \theta$, while if not, sender $i_\varepsilon^\Gamma(\theta)$ observes $s_i = \theta'$ for some $\theta' \in [\theta-\varepsilon, \theta+\varepsilon]$ where $m_i^\Gamma(\theta') \neq m_i$;[32]

(ii) for all other senders, and for $i_\varepsilon^\Gamma(\theta)$ at all other states, the true state is observed.

Clearly, Ξ has size at most ε, and $a^\Xi(m) = \theta^* \neq a^\Gamma(m)$.[33] By step 1, taking $\delta < |\theta^* - a^\Gamma(m)|$ completes the proof. ◊

Step 3: The set of fully revealed states in Γ has measure zero.

Consider $\gamma > 0$ from point 2 of the definition of robustness. Pick $\delta < \gamma$, and let $\varepsilon > 0$ be the corresponding bound on noise from point 1 of the definition of robustness.

[32] It is not necessary for this proof to allow the possibility that sender $i_\varepsilon^\Gamma(\theta)$ observes $s_i = \theta$. However, doing so ensures that the proof remains consistent with an alternative definition of small local noise additionally requiring the sender to believe, after any signal, that the state is nearby.

[33] Heuristically, the density that m is observed and that the state is in $\theta^\Gamma(m)$ is at most $D\lambda(\theta^\Gamma(m)) = 0$, while the density that m is observed and that the state is θ^* is at least $d > 0$.

Let S denote the set of fully revealed states, and suppose instead that it has positive measure. Then there exists an interval I of size ε such that $\lambda(S \cap I) > 0$. Within $S \cap I$, a continuum of equilibrium message vectors is sent, so we may assume without loss of generality that sender 1 has a continuum of equilibrium messages within $S \cap I$; let T denote this set of messages.

Suppose $\theta \in S \cap I$, and let $m^{\Gamma}(\theta) = (m_1, m_2, ..., m_n)$. Consider $m' = (m'_1, m_2, ..., m_n)$ where $m'_1 \in T$. I now argue that m' must be sent at some state in Γ. Suppose instead that $\theta^{\Gamma}(m') = \varnothing$. Then consider the following noise Ξ:

(i) when the state is θ, sender 1 observes with equal probability θ and $\theta' \in S \cap I$ where $m_1^{\Gamma}(\theta') = m'_1$, while all other senders observe θ;

(ii) at all other states, everyone observes the true state.

Clearly, Ξ has size at most ε. By the hypothesis that $\theta^{\Gamma}(m') = \varnothing$, we have $a^{\Xi}(m') = \theta$: under Ξ and $\{m_j^{\Gamma}\}_{j=1}^{n}$, m' can only arise if the state is θ.

By step 1, $|a^{\Gamma}(m') - \theta| < \delta$; since $\delta < \gamma$, this implies $\theta \in (a^{\Gamma}(m') - \gamma, a^{\Gamma}(m') + \gamma)$. It follows that point 2 in the definition of robustness is violated: there exists $a \in [a^{\Gamma}(m') - \gamma, a^{\Gamma}(m') + \gamma]$ such that if the receiver's actions following m' were a, then sender 1 would have a profitable deviation at θ (specifically, take a slightly larger (smaller) than θ if sender 1 is biased to the right (left)).

Thus m' must be sent at some state in Γ. Because T contains a continuum of messages, there is a continuum of such m'. By step 2, each must be sent either at only one state or on a set of positive measure. Because the number of actions such m' can induce is finite (as they must be separated by at least η), only finitely many can be sent at only one state. Moreover, only countably many can be sent on a set of positive measure, which contradicts the continuum of m'. $\Diamond$

Step 4: Γ has interval structure.

Define S, $C(m)$ and $D(m)$ as in the proof of Lemma 1. We proceed by contradiction: suppose $\lambda(S) < \lambda(\Theta)$.

Once again, there exists $m \in \times_{i=1}^{n} M_i^{\Gamma}$ such that at least one of $C(m)$ and $D(m)$ has positive measure. Suppose not. Then $\lambda(C(m) \cup D(m)) = 0$ for all m. By step 2, only a countable number of m can be sent at two or more states, implying both of the following:

$$\begin{aligned} \lambda(\{a^{\Gamma}(m) \text{ s.t. } m \text{ is sent at two or more states}\}) &= 0, \text{ and} \\ \sum_{m \text{ is sent at two or more states}} \lambda(C(m) \cup D(m)) &= 0. \end{aligned}$$

Combining the two relations above with step 3 yields $\lambda(\Theta \backslash S) = 0$, which contradicts the hypothesis.

The remainder of this step is identical to the remainder of step 4 in the proof of Lemma 1. $\Diamond$

The remainder of the proof shows that Γ is natural by following the proof of Lemma 2. The same argument shows that among messages $m_i' \in M_i^\Gamma$ such that (m_i', m_{-i}) occurs on path in Γ, m_i is the unique approximate best response to m_{-i} at $\theta = \theta_b$. Furthermore, messages $m_i' \in M_i^\Gamma$ such that (m_i', m_{-i}) is off-path in Γ cannot be nearly optimal by point 2 of the definition of robustness (which also implies point 3 of the definition of strictly coordination-free equilibrium). Thus, the last paragraph of the proof of Lemma 2 applies here. ■

Proof of Theorem 2b: By Proposition 1, the number of distinct messages in a coordination-free equilibrium is finite. Combined with point 3 in the definition of strictly coordination-free equilibria, this ensures that point 2 in the definition of robustness is satisfied.

The rest of this proof is similar to the proof of Theorem 1b. The argument showing that senders play a δ-best response carries through. For the receiver's best response, there are three cases to consider:

a) To message vectors m sent in a cell that is not $\{0\}$ or $\{1\}$

The argument from the proof of Theorem 1b is valid (but can be simplified since the probabilities are 1 and 0 instead of $1 - n\varepsilon(\gamma)$ and $\varepsilon(\gamma)$).

b) To message vectors m not sent at any state in Γ in the noiseless game

If senders play according to a coordination-free Γ, combining message vectors from adjacent cells will not yield an out-of-equilibrium message vector because only one sender's message changes at each boundary. Thus, m must combine messages from non-adjacent cells, which are separated by more than ε for ε small enough. If noise is less than ε, then m must be unexpected to the receiver, who is therefore not required to play a δ-best response after m.

c) To a message vector m sent in cell $\{0\}$ (analogous argument for $\{1\}$)

As long as ε is less than the size of any proper cell in Γ, the receiver must believe that $\theta \in [0, \varepsilon]$ upon observing m if m is on-path according to the receiver's beliefs about the noise (if not, then as in case b, the receiver is not required to play a δ-best response). Thus, for ε sufficiently small, $a(m) = 0$ is a δ-best response. ■

Proof of Proposition 2a: Because $\lambda(\cup_{I \text{ is a proper cell in } \Gamma} I) = \lambda(\Theta)$ and m is sent on a set with positive measure, m must be sent in at least one proper cell. Suppose that $\theta^\Gamma(m)$ is not connected, and assume without loss of generality that $m^\Gamma(\theta) = m^\Gamma(\theta') = m$ for some θ in a proper cell and $\theta' > \theta$ outside that cell. Let θ_b be the right endpoint of the cell where θ lies. Because Γ is natural, for some sender i, we have $u_i(a^\Gamma(m_i, m_{-i}), \theta_b) = u_i(a^\Gamma(m_i', m_{-i}), \theta_b)$ and $a^\Gamma(m_i, m_{-i}) < a^\Gamma(m_i', m_{-i})$, with $m_i' \in M_i^\Gamma$. But by single-crossing, this implies that at $\theta' > \theta_b$, $u_i(a^\Gamma(m_i, m_{-i}), \theta') < u_i(a^\Gamma(m_i', m_{-i}), \theta')$, so sender i has a profitable deviation at θ'. ■

Proof of Proposition 2b: Let $\theta \in \Theta$, and consider a proper cell C with left endpoint $\theta_L \in (\theta, \theta + \eta]$, where message vector m is sent. By Proposition 2a, m is sent only in cell C. Because θ_L must be right-natural for some sender i, we must have $a^\Gamma(m) > \theta_L + b_i(\theta_L) > a^\Gamma(m')$ for some m' that is on-path and differs from m in component i only. (Thus $\theta_L + b_i(\theta_L) \notin \{0, 1\}$, which implies $|b_i(\theta_L)| \geq \eta > 0$.) Cell C can be of two types:

i. If $b_i(.) > 0$, we must have $a^\Gamma(m) > \theta_L + b_i(\theta_L) \geq \theta_L + \eta$, which implies that cell C has size greater than η. There is at most one such cell for a given θ.

ii. If $b_i(.) < 0$, we must have $a^\Gamma(m') < \theta_L + b_i(\theta_L) \leq \theta_L - \eta < \theta$. Therefore, if there are $k < \infty$ on-path actions in $[0, \theta]$, then at most k actions can be $a^\Gamma(m')$. By the definition of "right-natural," each of them corresponds to at most n values of θ_L. Thus, there are at most nk such cells for a given θ.

If there are k cells with a left endpoint in $[0, \theta]$, there are at most k on-path actions in $[0, \theta]$. By the above argument, there are then at most nk proper cells with a left endpoint in $(\theta, \theta + \eta]$. Because $\lambda(\cup_{I \text{ is a proper cell in } \Gamma} I) = \lambda(\Theta)$, every non-proper cell in $(\theta, \theta + \eta]$ must be the endpoint of a proper cell, so there are at most $2nk$ cells total whose left endpoint is in $(\theta, \theta + \eta]$. By induction, if a finite upper bound for the number of on-path actions in $[0, \theta]$ exists for some $\theta > 0$, then a finite upper bound for the overall number of cells exists as well.

To complete the proof, note that, by cases i and ii above, any proper cell whose left endpoint is not 0 must induce an action greater than η (for case ii, $a^\Gamma(m) > \theta_L > a^\Gamma(m') + \eta$). There are therefore at most three inducible actions in $[0, \eta]$: the first proper cell's action and endpoints. ■

Proof of Proposition 4b: The proof of Proposition 4a follows this proof because it uses Proposition 4b.

Define "structure" as in the main text. Because the number of structures is finite, and the intersection of finitely many open and dense sets is itself open and dense, it is appropriate to consider each structure separately. That is, it suffices to show that, fixing a natural equilibrium Γ that is not equivalent to a coordination-free equilibrium, the set P^Γ of primitives such that no natural equilibrium has the structure of Γ contains an open and dense set. To avoid clutter, the superscript Γ is suppressed in the remainder of the proof.

Label the sender indifference conditions (*i.e.* equalities of sender utility) imposed by the right-structure of Γ as $m = 1, ..., K$, and label the additional ones imposed by the left-structure as $m = K+1, ..., M$. Each of the latter conditions is "additional" in the sense that the right-structure condition corresponding to the same boundary must refer to different actions. As argued in the main text, if Γ is not equivalent to a coordination-free equilibrium, then $M \geq K+1$.

Let $d_m^{U,f}(\theta_2, \theta_4, ..., \theta_{2K})$ be the magnitude of the difference between the two sides of condition m when the preference profile is (U, f), the even θ's (cell boundaries) are $\theta_2, \theta_4, ..., \theta_{2K}$, and the odd θ's satisfy $\theta_{2k+1} = \arg\max_a \int_{\theta_{2k}}^{\theta_{2k+2}} u_R(a, \theta) f(\theta) d\theta$ for $k = 0, ..., K$. Therefore, if the receiver plays a best response, then condition m is satisfied with cell boundaries $\theta_2, \theta_4, ..., \theta_{2K}$ if and only if $d_m^{U,f}(\theta_2, \theta_4, ..., \theta_{2K}) = 0$. Consider the function $d(U, f) = \min_{0 \leq \theta_2 \leq \theta_4 \leq ... \leq \theta_{2K} \leq 1} \max_m d_m^{U,f}(\theta_2, \theta_4, ..., \theta_{2K})$, which is well-defined since $d_m^{U,f}$ is continuous in $(\theta_2, \theta_4, ..., \theta_{2K})$ and $\{(\theta_2, \theta_4, ..., \theta_{2K}) \in \mathbb{R}^K : 0 \leq \theta_2 \leq ... \leq \theta_{2K} \leq 1\}$ is compact. Let $P' = \{(U, f) : d(U, f) > 0\}$. By definition, $P' \subseteq P$: if $d(U, f) > 0$, then there is no cell boundary sequence $\theta_2, \theta_4, ..., \theta_{2K}$ such that all M conditions are satisfied. Therefore, it suffices to show that P' is open and dense.

P' is open: Since $d_m^{U,f}$ is continuous in (U, f), d is also continuous in (U, f). Therefore, P' must be open.

P' is dense: Let $t_{U,f}(\theta_2)$ denote the right endpoint of the rightmost cell implied by the right-structure given θ_2 when the primitives are (U, f); it is well-defined whenever ≤ 1 by the argument in the main text. Let $T(U, f) = \{\theta_2 \in [0, 1] : t_{U,f}(\theta_2) = 1\}$, U^α denote the preference profile $(u_1^\alpha, ..., u_n^\alpha, u_R^\alpha)$ where $u_i^\alpha(a, \theta) = u_i(\alpha a, \alpha\theta)$ for $\alpha \in (0, 1]$ and $a, \theta \in [0, \frac{1}{\alpha}]$, and $f^\alpha(\theta) = \alpha f(\alpha\theta)$ for $\alpha \in (0, 1]$ and $\theta \in [0, \frac{1}{\alpha}]$.

Note that for any θ_2 such that $t_{U,f}(\theta_2) \leq 1$, we have $t_{U^\alpha, f^\alpha}(\theta_2) = \frac{1}{\alpha} t_{U,f}(\theta_2)$. Therefore, $T(U^\alpha, f^\alpha) = \{\theta_2 \in [0, 1] : t_{U,f}(\theta_2) = \alpha\}$. By the bounds on f and the Lipschitz continuity of U, $t_{U,f}(.)$ is Lipschitz continuous where $t_{U,f}(\theta_2) \leq 1$. It follows that the graph of $t_{U,f}(.)$ has finite length, which implies that $T(U^\alpha, f^\alpha)$ cannot be infinite on a positive measure of α. Therefore, there exist α arbitrarily close to 1 such that $T(U^\alpha, f^\alpha)$ is finite.

By construction, $(U, f) \in P'$ when there exists no $\theta_2 \in T(U, f)$ such that the $M - K$ equalities corresponding to conditions $m \geq K+1$ are all satisfied. Focus on condition $K+1$, let θ_b be the corresponding boundary, and let j be the sender whose indifference is required by this condition. Given α such that $T(U^\alpha, f^\alpha)$ is finite, in order for condition $K+1$ to be satisfied, one of a finite number (one for each $\theta_2 \in T(U^\alpha, f^\alpha)$) of equalities of form $u_j(a, \theta_b) = u_j(a', \theta_b)$ must hold. Therefore, there exist arbitrarily small perturbations of u_j around a finite number of pairs (a, θ_b) such that these equalities do not hold. As long as, for all such pairs (a, θ_b), $u_j(a, \theta_b)$ does not appear in the finitely many equalities corresponding to conditions $1, ..., K$ for $\theta_2 \in T(U^\alpha, f^\alpha)$, these perturbations do not impact the right-structure for any $\theta_2 \in T(U^\alpha, f^\alpha)$. This is generically the case: for a given θ_2, no other condition can involve $u_j(a, \theta_b)$ (or else it would be redundant with condition $K+1$), so perturbing $u_j(a, \theta_b)$ can only impact the right-structure for a different $\theta_2' \in T(U^\alpha, f^\alpha)$ if a boundary in the right-structure corresponding to θ_2' happens to fall on θ_b. Therefore, for any (U, f), there exists an arbitrarily close $(U', f^\alpha) \in P'$, as desired. ■

Proof of Proposition 4a: Since every message vector in $\times_{i=1}^n M_i^\Gamma$ is sent on a set of positive measure, by Proposition 2a, $\theta^\Gamma(m)$ is a non-trivial interval for all $m \in \times_{i=1}^n M_i^\Gamma$. Then, if Γ were not coordination-free, it would also not be equivalent to a coordination-free equilibrium. By Proposition 4b, Γ would not natural, which violates the hypothesis. ■

Proof of Proposition 5: Consider a coordination-free equilibrium Γ with $K+1$ cells, labeled $0, 1, ..., K$, and let $i(k)$ be the sender whose message changes between cell $k-1$ and cell k, for $k = 1, ..., K$. Letting c_k be the size of cell k, we have $c_k = c_{k-1} + 4b_{i(k)}$. To see this, suppose that the boundary between cells $k-1$ and k is θ_b. Then the actions corresponding to these cells are $\theta_b - \frac{c_{k-1}}{2}$ and $\theta_b + \frac{c_k}{2}$, respectively. Thus $u_{i(k)}(\theta_b - \frac{c_{k-1}}{2}, \theta_b) = u_{i(k)}(\theta_b + \frac{c_k}{2}, \theta_b)$, which is equivalent to $b_{i(k)} + \frac{c_{k-1}}{2} = \frac{c_k}{2} - b_{i(k)}$, or $c_k = c_{k-1} + 4b_{i(k)}$, as desired. It follows that $c_k = c_0 + 4\sum_{l=1}^k b_{i(l)}$.

Since $\sum_{k=0}^K c_k = 1$, we have $(K+1)c_0 + 4\sum_{k=1}^K \sum_{l=1}^k b_{i(l)} = 1$, or $c_0 = \frac{1 - 4\sum_{k=1}^K \sum_{l=1}^k b_{i(l)}}{K+1}$.

Suppose $b_{i(k^*)} > b$ for some $k^* \in \{1, ..., K\}$, and let $b_j = b$. Consider alternative candidate play Γ' with $K+1$ cells labeled in the same way, where the sender whose message changes between cell $k-1$ and cell k is $i'(k) = \begin{cases} i(k) \text{ if } k \neq k^* \\ j \text{ if } k = k^* \end{cases}$. Let c_k' be the size of cell k. Note that Γ' exists since $c_0' = \frac{1 - 4\sum_{k=1}^K \sum_{l=1}^k b_{i'(l)}}{K+1} > \frac{1 - 4\sum_{k=1}^K \sum_{l=1}^k b_{i(l)}}{K+1} = c_0 \geq 0$. Furthermore, it must be that $c_k' > c_k$ for all $k = 0, ..., k^* - 1$, and $c_k' < c_k$ for all $k = k^*, ..., K$. Thus, the smallest cells become bigger, while the largest cells become smaller. Because the receiver's expected loss is $\sum_{k=0}^K \int_0^{c_k} (x - \frac{c_k}{2})^2 dx = \frac{1}{12}\sum_{k=0}^K c_k^3$, which is strictly convex, the expected loss is strictly smaller under Γ' than under Γ.

Iterating this argument implies that there exists a candidate play Γ^* with $K+1$ cells and where sender j is the only non-babbling sender, and that the receiver is strictly better off under Γ^* than under Γ. Furthermore, because only one sender has more than one equilibrium message in Γ^*, we need not worry about off-path play and know that Γ^* is part of an equilibrium. ■

Proof of Proposition 6a: First, note that by Proposition 7 (see Online Appendix A), as $\lambda(\Theta_l)$ becomes large and for a fixed maximum block size, every candidate play is part of a coordination-free equilibrium. This proof considers equilibria where no cell has size above $4(b_1 + b_2)$, which implies a bound on block size. Therefore, off-path play need not be considered.

By the first paragraph of the proof of Proposition 5, going from left to right, the size of cells either increases by $4b_1$ or decreases by $4b_2$ at each boundary.

Consider a coordination-free equilibrium Γ that has no cell with size above $4(b_1 + b_2)$. If a cell in Γ has size in $(0, 4b_2]$, then the indifferent sender at its right endpoint must be sender 1: the next cell cannot be smaller by $4b_2$. If a cell in Γ has size greater than $4b_2$, then the indifferent sender at the right endpoint must be sender 2: by assumption, the next cell cannot be bigger by $4b_1$. Thus, in any such Γ, the size of the leftmost cell uniquely determines the sizes of all cells: if a cell size is above $4b_2$, the next cell is smaller by $4b_2$; otherwise, it is bigger by $4b_1$.

It follows that, labeling cells sequentially with integers, and letting the size of cell 0 be $4c(b_1+b_2)$, the size of cell k is $4(b_1+b_2)\langle k\frac{b_1}{b_1+b_2}+c\rangle$, where $\langle x\rangle$ denotes the fractional part of x if $x \notin \mathbb{Z}$, and 1 if $x \in \mathbb{Z}$. Since $\frac{b_1}{b_2} \notin \mathbb{Q}$, we have $\frac{b_1}{b_1+b_2} \notin \mathbb{Q}$, and it follows that the numbers in the sequence $\langle k\frac{b_1}{b_1+b_2} + c\rangle_{k=0}^{K} \equiv \{x_k\}_{k=0}^{K}$ are uniformly distributed over $(0, 1]$ as $K \to \infty$, *i.e.* for each interval $I \subseteq (0, 1]$, $\lim_{K\to\infty} \frac{1}{K+1}|\{k : x_k \in I, 0 \leq k \leq K\}| = \lambda(I)$ (Hardy and Wright, 1960, Theorem 445).

It remains to be shown that if Γ maximizes the number of cells, then none of its cells has size exceeding $4(b_1 + b_2)$. Suppose a coordination-free equilibrium Γ has $K + 1$ cells, whose sizes are denoted $s_0, ..., s_K$, with $\max\{s_k\}_{k=0}^{K} > 4(b_1 + b_2)$. Then another coordination-free equilibrium Γ' with at least $K + 2$ cells, whose sizes are denoted $s'_0, ...$, can be built using the following steps:

i. Let $s'_0 = 4(b_1 + b_2)\langle\frac{s_0}{4(b_1+b_2)}\rangle$, and extrapolate additional cells so that none has size exceeding $4(b_1 + b_2)$ in the unique way described in paragraph 3 of this proof. This implies that $s'_k = 4(b_1 + b_2)\langle\frac{s_k}{4(b_1+b_2)}\rangle$ for all $k = 0, ..., K$. By assumption, the right endpoint of cell K in Γ' occurs at least $4(b_1 + b_2)$ to the left of the right endpoint of Θ. Therefore, at least one more cell will fit in the remaining space.

ii. Increase the sizes of the cells in Γ' so that they fill Θ. ■

Proof of Proposition 6b: By the first paragraph of the proof of Proposition 5, fixing the indifferent sender at each boundary in the uniform-quadratic case, the size of every cell is strictly monotonic in the size of the leftmost cell. Thus, all boundaries are strictly monotonic in the leftmost inducible action. It follows that each structure (as defined in Section 5.4) admits a unique candidate play. Because the number of structures is finite, the number of coordination-free equilibria is finite, so $\overline{u_R}^l$ is well-defined.

Consider the construction of Γ' from Γ in the final part of the proof of Proposition 6a. Step (i) enhances the receiver's average welfare (for the area covered by cells), as cells with sizes above $4(b_1 + b_2)$ are replaced by cells with sizes at or below $4(b_1 + b_2)$, while cells with sizes at or below $4(b_1+b_2)$ retain their sizes. Step (ii) decreases the receiver's average welfare. However, because the total amount by which cells in Γ' must be expanded is bounded above by $4(b_1+b_2)$, the effect on the size of each cell, and thus on receiver's average welfare, vanishes as $l \to \infty$. Therefore, letting $\overline{v_R}^l$ denote the receiver's maximum expected utility, when the state space is Θ_l, from a coordination-free equilibrium where no cell size exceeds $4(b_1 + b_2)$, we have:

$$\begin{aligned}\lim_{l\to\infty} \overline{u_R}^l &= \lim_{l\to\infty} \overline{v_R}^l = \frac{\int_0^{4(b_1+b_2)} -\frac{x^3}{12}dx}{\int_0^{4(b_1+b_2)} xdx} \\ &= -\frac{1}{24}\left[x^2\right]_0^{4(b_1+b_2)} = -\frac{2}{3}(b_1+b_2)^2,\end{aligned}$$

where the integrand $-\frac{x^3}{12}$ in the numerator is the contribution of a cell of size x to the expected loss when $\lambda(\Theta) = 1$, the denominator normalizes for the size of Θ, and the limit distribution of cell sizes is $U[0, 4(b_1 + b_2)]$ by Proposition 6a. ■

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Online Appendix for Coordination-Free Equilibria in Cheap Talk Games*

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Online Appendix A: Relation between Candidate Play and Equilibrium Play in Coordination-Free Equilibrium

First, consider the following example illustrating two ways in which candidate play can fail to be equilibrium play.

Example 6: *Consider candidate play with two senders whose biases satisfy* $b_1(.) < 0$ *and* $b_2(.) > 0$*, and three cells where the equilibrium message vectors are, from left to right,* (L, L)*,* (H, L) *and* (H, H) *(so that* (L, H) *is out-of-equilibrium). It is easy to think of two cases where any location of* $a^\Gamma(L, H)$ *induces a deviation:*

a) Sender 1 has a small leftward bias while sender 2 has a large rightward bias, such that the middle interval (H, L) *is very small, and the rightmost interval* (H, H) *is very big (see Figure A.1). Then choosing* $a^\Gamma(L, H) < a^\Gamma(H, H)$ *induces sender 1 to deviate from* H *to* L *near the left end of the* (H, H) *interval, choosing* $a^\Gamma(L, H) > a^\Gamma(H, H)$ *induces sender 1 to deviate near the right end of the* (H, H) *interval, while choosing* $a^\Gamma(L, H) = a^\Gamma(H, H)$ *induces sender 2 to deviate from* L *to* H *near the right end of the* (L, L) *interval.*

Figure A.1: Case a

b) Sender 2 dislikes $a^\Gamma(L, L)$ *so much that, at the boundary* θ_1 *between* (L, L) *and* (H, L)*,* $u_2(a, \theta_1) > u_2(a^\Gamma(L, L), \theta_1)$ *for all* $a > \theta_1$*. Similarly, sender 1 dislikes* $a^\Gamma(H, H)$ *so much that* $u_1(a, \theta_2) > u_1(a^\Gamma(H, H), \theta_2)$ *for all* $a < \theta_2$*. Since* $a^\Gamma(H, H) > a^\Gamma(L, L)$*, it is impossible for* $a^\Gamma(L, H)$ *to be simultaneously less than* $a^\Gamma(L, L)$ *and greater than* $a^\Gamma(H, H)$*, so once again a deviation is always desired.*

The following derives a condition under which candidate play is guaranteed to constitute equilibrium play.

Let $U_i(\theta) = \{u : \exists a_1 \neq a_2 \in \Theta \text{ s.t. } u_i(a_1, \theta) = u_i(a_2, \theta) = u\}$ be the set of utilities achieved for sender i at state θ by two distinct actions. By the single-peakedness of u_i, these actions must be on opposite sides of sender i's ideal action $\theta + b_i(\theta)$. Let $a_i^-(u, \theta) < a_i^+(u, \theta)$ be these actions. Then let $A_i = \max_{\theta \in \Theta} \max_{u \in U_i(\theta)} \max\{\frac{\theta + b_i(\theta) - a_i^-(u,\theta)}{a_i^+(u,\theta) - (\theta + b_i(\theta))}, \frac{a_i^+(u,\theta) - (\theta + b_i(\theta))}{\theta + b_i(\theta) - a_i^-(u,\theta)}\}$ be a measure of how asymmetric sender i's utility function can get around its peak $\theta + b_i(\theta)$: if u_i is perfectly symmetric, as in the quadratic case, then $A_i = 1$, and the more asymmetric it is, the higher A_i.

Given candidate play, call an *i-block* a maximal interval of states where each sender other than i sends a single message. Clearly, every block is a union of cells, and an i-block with more than one cell is formed when sender i's message changes at a boundary. For example, with two senders and three cells numbered 1, 2 and 3 from left to right, if the message pairs are (m_1, m_2), (m_1, m_2') and (m_1', m_2') in cells 1, 2 and 3 respectively, then there are two 1-blocks (cell 1; cells 2 and 3) and two 2-blocks (cells 1 and 2; cell 3). Note that a given i-block and a given j-block can overlap for at most one cell because each boundary can be crossed by only one block.

Proposition 7: Given candidate play Γ, let k_i^Γ denote the size of the largest i-block, and let $x_i^\Gamma = (1+A_i)(k_i^\Gamma + \max_{\theta\in\Theta} |b_i(\theta)|)$ for each i whose message changes at some boundary. If the sum of the two largest x_i^Γ is less than $\lambda(\Theta) = 1$, then there exists a strictly coordination-free equilibrium where:

- play is described by Γ; and
- each player i's messages can be ordered so that $m_i(\theta)$ is non-decreasing.

Proof of Proposition 7: Given candidate play Γ, assign messages as follows: in the leftmost cell, all senders send 1, and at every boundary where a sender's message changes, that sender's message increases by 1. This message assignment rules out the following scenario: in a cell where the assigned message vector is $m = (m_1, ..., m_n)$, a sender (without loss of generality, sender 1) wants to deviate to m_1', and $m' = (m_1', m_2, ..., m_n)$ occurs on the equilibrium path. To see this, assume without loss of generality that $a^\Gamma(m') > a^\Gamma(m)$. Then it must be that in the cell immediately to the right of the one where m is sent, the message vector is $m'' = (m_1'', m_2, ..., m_n)$ for some m_1'' possibly equal to m_1', so $a^\Gamma(m') \geq a^\Gamma(m'')$. Since within the cell where m is sent, sender 1 prefers $a^\Gamma(m)$ to $a^\Gamma(m'')$, by single-crossing, she also prefers $a^\Gamma(m)$ to $a^\Gamma(m')$ and cannot desire a deviation.

Therefore, the only concern is to place the receiver's actions after off-path message vectors without inducing a deviation. For any off-path message vector m, there are at most two senders whose deviation can induce m. To see this, normalize messages by subtracting a constant to each sender's messages such that $m = (0, ..., 0)$. If a sender i can induce m by deviating from a negative message when all others send 0, then when i sends 0, all other senders' messages must be nonnegative. Thus only one other sender can deviate to m, and must do so from a positive message. The symmetric argument holds as well, so at most one sender can deviate to m from a positive message, and at most one sender can deviate to m from a negative message.

Now suppose sender i can deviate to induce m. The set of states from which she can do this must constitute an i-block, which has size at most k_i^Γ. Denote the left and right endpoints of the i-block by θ_L and θ_R, the leftmost and rightmost inducible actions within the i-block by a_L and a_R, and assume without loss of generality that $b_i(.) > 0$. Then a deviation by i will not be induced if either:

- $a(m) > \theta_R + \max_{\theta\in\Theta} |b_i(\theta)| + A_i(\theta_R + \max_{\theta\in\Theta} |b_i(\theta)| - a_R)$; or
- $a(m) < \min\{a_L, \theta_L + \min_{\theta\in\Theta} |b_i(\theta)| - A_i(a_L - (\theta_L + \min_{\theta\in\Theta} |b_i(\theta)|))\}$.

Therefore, letting $D_i = \max_{\theta\in\Theta} |b_i(\theta)| - \min_{\theta\in\Theta} |b_i(\theta)|$, the maximum range where a deviation can be induced is:

$$\begin{aligned} & \max\{(A_i+1)\max_{\theta\in\Theta} |b_i(\theta)| + A_i(\theta_R - a_R) + (\theta_R - a_L), (A_i+1)(k_i^\Gamma + D_i) - A_i(a_R - a_L)\} \\ < \; & (A_i+1)(k_i^\Gamma + \max_{\theta\in\Theta} |b_i(\theta)|) = x_i^\Gamma, \end{aligned}$$

where the inequality follows from $\theta_R - a_R, \theta_R - a_L < k_i^\Gamma$, $D_i < \max_{\theta\in\Theta} |b_i(\theta)|$, and $a_R - a_L \geq 0$.

If the ranges for the two potential deviators do not cover Θ, then it is possible to place $a^\Gamma(m)$ without inducing a deviation. The result follows. ■

The proof of Proposition 7 shows that if messages are assigned as stated, then no deviation to an on-path message vector is ever desired, and at most two senders, each from one block, can deviate to an out-of-equilibrium message vector. An i-block of size k is associated with an interval of size at most $(1+A_i)(k+\max_{\theta\in\Theta} |b_i(\theta)|)$ where placing an out-of-equilibrium action would cause a deviation by sender i. Therefore, the total area where an out-of-equilibrium vector cannot be placed is at most the sum of the two largest x_i^Γ.

In the uniform-quadratic specification, as shown in the first paragraph of the proof of Proposition 5, cell size changes by $4b_i$ (from left to right) at a boundary where sender i's message changes. Thus, if $b_i > 0$, cells grow from left to right, and vice versa. It follows that:

- cells can be kept small if, in each direction, there is a sender with a small bias; and
- large i-blocks must contain large cells (relative to $|b_i|$) at one end.

Thus, the most informative candidate play must only have small i-blocks if, in each direction, there is a sender with a small bias. In this situation, any sufficiently informative candidate play Γ will have small k_i^Γ, and therefore small $x_i^\Gamma = 2(k_i^\Gamma + |b_i|)$, for all $i \in N$. Proposition 7 implies that such Γ corresponds to play in a strictly coordination-free equilibrium where messages are assigned so that each is used on a connected set of states. As a result, for each $i \in N$, there exists an order on M_i^Γ such that sender i's strategy is *monotonic.*[1]

A similar reasoning can be applied whenever the receiver's preferred action in each cell is not far from its center[2], and A_i is close to 1 for each sender. Therefore, in such settings, Proposition 7 (combined with Theorem 2 motivating strictly coordination-free equilibria) provides a justification for focusing on monotonic strategies when studying the most informative equilibria, if a sender with small bias is available in each direction.

Online Appendix B: Analysis Without Assumption A

This section dispenses with Assumption A, and allows for noise where players have heterogeneous prior, as long as there is common knowledge that noise is small. Then, if there is no state θ and pair of actions between which two senders are both indifferent at θ,[3] the implications of Theorems 1 and 2 about the function $a^\Gamma \circ m^\Gamma$ mapping state to action in a (strongly) robust equilibrium Γ still hold: it must generically correspond to candidate play computed by forward solution (and, for strong robustness, be complete).

Definition: Given a pure-strategy profile Γ, let a *supercell in* Γ be a maximal interval of states throughout which $a^\Gamma \circ m^\Gamma$ remains constant.

Definition: A proper supercell in Γ is *natural** if, denoting its endpoints as $\theta_1 < \theta_2$ and its induced action as a:

- (right-natural*) whenever $\theta_1 \neq 0$, $\exists \theta'$ such that $a^\Gamma(m^\Gamma(\theta')) = a$ and that, for some $i \in N$, $\exists m_i' \in M_i^\Gamma$ such that:

$$\begin{aligned} & (m_i', m_{-i}^\Gamma(\theta')) = m^\Gamma(\theta) \text{ for some } \theta \in \Theta, \text{ and} \\ u_i(a^\Gamma(m_i', m_{-i}^\Gamma(\theta')), \theta_1) \quad = \quad & u_i(a, \theta_1) \text{ and } a^\Gamma(m_i', m_{-i}^\Gamma(\theta')) < a; \text{ and} \end{aligned}$$

- (left-natural*) whenever $\theta_2 \neq 1$, $\exists \theta''$ such that $a^\Gamma(m^\Gamma(\theta'')) = a$ and that, for some

[1] Given a strictly coordination-free equilibrium Γ, it is not always possible to obtain a monotonic strictly coordination-free equilibrium through a reassignment of messages. Consider Example 3, and change m_1 in $(0.51, 1]$ to $z \neq x, y$ so that sender 1's strategy becomes monotonic. Message vector (x, y) is now out-of-equilibrium. If $a^\Gamma(x, y)$ is placed anywhere other than 0.285 and 0.755, then sender 1 would have a profitable deviation to x at some $\theta \in (0.06, 1]$. But placing $a^\Gamma(x, y)$ at 0.285 or 0.755 violates point 3 of the definition of strictly coordination-free equilibrium.

[2] This happens whenever F is not too far from being uniform and u_R is not too asymmetric.

[3] This assumption holds for generic biases (*i.e.* whenever no two biases are exactly equal) within the class of quadratic loss preferences from Section 3.

$j \in N$, $\exists m''_j \in M_i^\Gamma$ such that:

$$\begin{aligned} (m''_j, m^\Gamma_{-j}(\theta'')) &= m^\Gamma(\theta) \text{ for some } \theta \in \Theta \text{, and} \\ u_j(a^\Gamma(m''_j, m^\Gamma_{-j}(\theta'')), \theta_2) &= u_j(a, \theta_2) \text{ and } a^\Gamma(m''_j, m^\Gamma_{-j}(\theta'')) > a. \end{aligned}$$

Definition: An equilibrium is *natural** if its strategy profile has interval structure, and all of its proper supercells are natural*.

The definition of natural* proper supercell implies that, in Γ, $m^\Gamma(\theta')$ can be sent only at and to the right of θ_1. Since $a^\Gamma(m^\Gamma(\theta')) = a$, we have $\theta_1 \leq a$ and, by a similar argument, $a \leq \theta_2$. It follows that there is at most one proper supercell inducing a. As a result, a left-structure and a right-structure can be defined for a natural* equilibrium Γ in the same way as for natural equilibria, but using supercells rather than cells. The argument in the proof of Proposition 4(b) carries through: if $a^\Gamma \circ m^\Gamma$ does not correspond to candidate play, then the conditions imposed by the structures would be too numerous and thus, generically, would not be satisfied. In this context, that argument implies the following result:

Proposition 4*: Generically, if Γ is natural*, then the endpoints and induced action for all proper supercells in Γ can be computed by forward solution.

The results corresponding to Theorems 1 and 2 in the main text are as follows.

Theorem 1*: Suppose that whenever $u_i(a, \theta) = u_i(a', \theta)$, we have $u_j(a, \theta) \neq u_j(a', \theta)$ for all $j \neq i$ ("no simultaneous indifference," henceforth abbreviated NSI). Then:

(a) Generically, if Γ is strongly robust, then it is complete and corresponds to a forward solution (*i.e.* it has interval structure, and the endpoints and induced action for all proper supercells in Γ can be computed by forward solution.).

(b) If Γ is coordination-free and complete and has finitely many cells, and no cell in Γ is $\{0\}$ or $\{1\}$, then it is strongly robust.

Theorem 2*: Assume NSI. Then:

(a) Generically, if Γ is robust, then it corresponds to a forward solution.

(b) If Γ is strictly coordination-free and has finitely many cells, then it is robust.

The proofs of Theorems 1(b) and 2(b) remain valid for Theorems 1*(b) and 2*(b). These arguments rely on the number of cells being finite, which needs to be assumed here: while coordination-freeness still guarantees a finite number of supercells, within a given supercell, there could now be infinitely many cells. (For example, if $a^\Gamma(m_i, m_j, m_{-ij}) =$

$a^\Gamma(m_i', m_j, m_{-ij}) = a^\Gamma(m_i, m_j', m_{-ij})$, then there could be a supercell where the message vector sent switches infinitely many times between (m_i, m_j, m_{-ij}), (m_i', m_j, m_{-ij}) and (m_i, m_j', m_{-ij}). This can pose problems if $a^\Gamma(m_i, m_j, m_{-ij}) \neq a^\Gamma(m_i', m_j', m_{-ij})$.)

By Proposition 4*, to prove Theorem 1*(a), it suffices to show the following lemmata.

Lemma 1*: If Γ is strongly robust, then it is complete, has interval structure, and $\{m : m^\Gamma(\theta) = m \text{ for some } \theta \in \Theta\}$ is finite.

Lemma 2*: If Γ is strongly robust and NSI holds, then Γ is natural*.

Proof of Lemma 1*: The proof of Lemma 1, which shows $\lambda(\theta^\Gamma(m)) > 0$ for all $m \in \times_{i=1}^n M_i^\Gamma$ and interval structure, carries over.

Fix any $\delta \in (0, \frac{1}{4})$, and suppose instead that $\{m : m^\Gamma(\theta) = m \text{ for some } \theta \in \Theta\}$ is infinite. Then, for any $\varepsilon > 0$, $\exists m^0$ such that $\lambda(\theta^\Gamma(m^0)) \in (0, \varepsilon)$. Fix such m^0, and let $\Theta_0 = \theta^\Gamma(m^0)$ and $\Theta_i = \{\theta \in \Theta \backslash \Theta_0 : m_i^\Gamma(\theta) = m_i^0\}$. Let $\Theta' \subset \Theta \backslash \Theta_0$ be a nontrivial set of states such that the receiver's best response conditional on $\theta \in \Theta'$, denoted a', is outside $(a^\Gamma(m^0) - 2\delta, a^\Gamma(m^0) + 2\delta)$. ($\Theta'$ exists for ε sufficiently small.) Denote the *ex ante* probability that $\theta \in S$ by $F(S)$.

Since $\lambda(\Theta_0) < \varepsilon$, for any $\theta \in \Theta_0$, $\exists \theta_0(\theta) \in (\theta - \varepsilon, \theta + \varepsilon)$ such that for some $i \in N$, $m_i^\Gamma(\theta) \neq m_i^\Gamma(\theta_0(\theta))$. Let $i_\varepsilon^\Gamma(\theta)$ be some such i.

Consider noise Ξ where:

(i) at states $\theta \in \Theta'$, for each $i \in N$, with probability $\varepsilon \min\{\frac{1}{n}, \frac{F(\Theta_i)}{F(\Theta')}\}$, sender i observes $s_i \in \Theta_i$ according to density proportional to the prior, and with the remaining probability, $s_i = \theta$; observations are independent across senders;

(ii) at states $\theta \in \Theta_0$, consider a random variable X distributed according to a continuous density g, where $g(0) > 0$; if the realization of X is 0, sender $i_\varepsilon^\Gamma(\theta)$ observes $s_i = \theta$, while if not, sender $i_\varepsilon^\Gamma(\theta)$ observes $s_i = \theta_0(\theta)$;

(iii) if neither (i) or (ii) applies, the true state is observed.

It is straightforward to check that, by construction, Ξ has size at most ε. With *ex ante* probability $F(\Theta')\varepsilon^n \prod_{j=1}^n \min\{\frac{1}{n}, \frac{F(\Theta_j)}{F(\Theta')}\}$, the receiver observes m^0 and the state is in Θ'; with *ex ante* probability 0, the receiver observes m^0 and the state is in Θ_0; and with the remainder probability, the receiver does not observe m^0. Therefore, for ε sufficiently small, $a^\Xi(m^0) = a'$ is more than δ away from $a^\Gamma(m^0)$. By step 1 of the proof of Lemma 1, Γ is then not strongly robust. ■

Proof of Lemma 2*: Like for Lemma 2, we proceed by contradiction. Suppose instead, without loss of generality, that a proper supercell C in Γ with right endpoint θ_b is not left-natural*. Consider the following classes of noise:

Noise $(\varepsilon, N_1, ..., N_n, -)$

- Each sender i believes that $s_j = \max\{\theta - \varepsilon, 0\}$ for $j \in N_i \subseteq N \backslash \{i\}$, and that $s_j = \theta$ for $j \notin N_i$.
- The receiver believes that all senders observe the true state.
- These beliefs are common knowledge.

Noise $(\varepsilon, N_1, ..., N_n, +)$

- Each sender i believes that $s_j = \min\{\theta + \varepsilon, 1\}$ for $j \in N_i \subseteq N \backslash \{i\}$, and that $s_j = \theta$ for $j \notin N_i$.
- The receiver believes that all senders observe the true state.
- These beliefs are common knowledge.

Case A: $\theta_b \in C$

Fix an arbitrary $\varepsilon > 0$, and denote $m^\Gamma(\theta_b) = m$ and $m^\Gamma(\theta_b + \varepsilon) = m^\varepsilon$. Since $\theta_b \in C$, we have $a^\Gamma(m) = a$.

Since Γ is strongly robust, for any $\delta > 0$, there exists ε sufficiently small so that sender i's δ-optimality at $s_i = \theta_b$ under noise $(\varepsilon, N_i, N_{-i}, +)$ implies

$$u_i(a^\Gamma(m_i, m^*_{-i}), \theta_b) \geq u_i(a^\Gamma(m^\varepsilon_i, m^*_{-i}), \theta_b) - \delta,$$

where $m^*_j = m_j$ if $j \notin N_i$ and $m^*_j = m^\varepsilon_j$ if $j \in N_i$. Moreover, sender i's δ-optimality at $s_i = \theta_b + \varepsilon$ under noise $(\varepsilon, N \backslash (N_i \cup \{i\}), N_{-i}, -)$ implies

$$u_i(a^\Gamma(m^\varepsilon_i, m^*_{-i}), \theta_b + \varepsilon) \geq u_i(a^\Gamma(m_i, m^*_{-i}), \theta_b + \varepsilon) - \delta.$$

Therefore, we must have

$$u_i(a^\Gamma(m^\varepsilon_i, m^*_{-i}), \theta_b) \to u_i(a^\Gamma(m_i, m^*_{-i}), \theta_b) \text{ as } \varepsilon \to 0. \tag{1}$$

By Lemma 1*, $|M^\Gamma_i|$ is finite for all $i \in N$, so (1) implies that for any ε sufficiently small,

$$u_i(a^\Gamma(m^\varepsilon_i, m^*_{-i}), \theta_b) = u_i(a^\Gamma(m_i, m^*_{-i}), \theta_b). \tag{2}$$

Fix $\varepsilon > 0$ such that $a^\Gamma(m^\varepsilon) \neq a$ and (2) hold; such ε must exist, or else θ_b would not be an endpoint of C.

Observation 1:

(i) Suppose (2) holds, $m_i \neq m_i^\varepsilon$, $m_j \neq m_j^\varepsilon$, and $a^\Gamma(m_i, m_j, m_{-ij}^*) = a$ for some $m_{-ij}^* \in \times_{k \neq i,j}\{m_k, m_k^\varepsilon\}$. Then either $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*) = a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) = a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) \neq a$, or at least one of these three actions is equal to a.

(ii) If NSI additionally holds, then we have $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*) = a$, $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) = a$, or both.

Proof of Observation 1: (i) Suppose $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*) > a$. Then $u_i(a^\Gamma(m_i, m_j, m_{-ij}^*), \theta_b) = u_i(a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*), \theta_b)$ implies $\theta_b + b_i(\theta_b) \in (a^\Gamma(m_i, m_j, m_{-ij}^*), a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*))$. We also have $u_j(a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*), \theta_b) = u_j(a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*), \theta_b)$. If $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) \neq a$, then either:

a) $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) > a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$, so that $\theta_b + b_j(\theta_b) > a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$. In this case, since $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*) > a^\Gamma(m_i, m_j, m_{-ij}^*)$ and $u_j(a^\Gamma(m_i, m_j, m_{-ij}^*), \theta_b) = u_j(a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*), \theta_b)$, we have either $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) = a^\Gamma(m_i, m_j, m_{-ij}^*)$, or $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) > a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*)$. The latter is not possible since both $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*)$ and $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*)$ would be to the right of $\theta_b + b_i(\theta_b)$, and sender i must be indifferent between these actions at θ_b. Therefore, $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) = a^\Gamma(m_i, m_j, m_{-ij}^*) = a$.

b) $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) \in (a, a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*))$, so that $\theta_b + b_j(\theta_b) \in (a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*), a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*))$. It follows that either $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) = a$, or $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) > a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$. The latter is ruled out since sender i cannot be simultaneously indifferent between a and $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$, as well as between $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) > a$ and $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) > a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$.

c) $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) < a$, so that $\theta_b + b_j(\theta_b) \in (a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*), a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*))$. It follows that $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) < a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$. Then, $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) \neq a$ is not possible since sender i cannot be simultaneously indifferent between a and $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$, as well as between $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) < a$ and $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) < a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$.

d) $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) = a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$. Since sender i must be simultaneously indifferent between a and $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$, as well as $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) = a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$ and $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*)$, we have either $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) = a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*)$, or $a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) = a$.

A symmetric argument applies if $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*) < a$.

(ii) By NSI, i and j cannot be indifferent between $a^\Gamma(m_i, m_j, m_{-ij}^*) = a$ and the same other action at θ_b, so by (2), we cannot have $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*) = a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) \neq a$. Thus, if $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*), a^\Gamma(m_i, m_j^\varepsilon, m_{-ij}^*) \neq a$, by part (i), we have $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) = a$. But then, at θ_b, i is indifferent between $a^\Gamma(m_i, m_j, m_{-ij}^*) = a$ and $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$, while j is indifferent between $a^\Gamma(m_i^\varepsilon, m_j^\varepsilon, m_{-ij}^*) = a$ and $a^\Gamma(m_i^\varepsilon, m_j, m_{-ij}^*)$. This again cannot occur by NSI. □

Therefore, if $m_i \neq m_i^\varepsilon$, $m_j \neq m_j^\varepsilon$, and $a^\Gamma(m_i, m_j, m_{-ij}^*) = a$, then it is possible to change a component of (m_i, m_j, m_{-ij}^*) from its value in m to its value in m^ε without changing the induced action. Doing so and iterating the process yields $a^\Gamma(m_i, m^\varepsilon_{-i}) = a$ for some $i \in N$. Substituting this into (2) with $m^*_{-i} = m^\varepsilon_{-i}$ gives

$$u_i(a, \theta_b) = u_i(a^\Gamma(m^\varepsilon), \theta_b). \tag{3}$$

Moreover, by i's optimality at $\theta_b + \varepsilon$ in the noiseless game and single-crossing, we cannot have $a > a^\Gamma(m^\varepsilon)$. It follows that $a < a^\Gamma(m^\varepsilon)$, which, together with (3), implies that θ_b is left-natural* after all.

Case B: $\theta_b \notin C$

Now denote $m^\Gamma(\theta_b) = m'$ and $m^\Gamma(\theta_b - \varepsilon) = m^\varepsilon$. Since $\theta_b \notin C$, we have $a^\Gamma(m') \neq a$.

Since Γ is strongly robust, for any $\delta > 0$, there exists ε sufficiently small so that sender i's δ-optimality at $s_i = \theta_b$ under noise $(\varepsilon, N_i, N_{-i}, -)$ implies

$$u_i(a^\Gamma(m'_i, m^*_{-i}), \theta_b) \geq u_i(a^\Gamma(m^\varepsilon_i, m^*_{-i}), \theta_b) - \delta,$$

where $m^*_j = m'_j$ if $j \notin N_i$ and $m^*_j = m^\varepsilon_j$ if $j \in N_i$. Moreover, sender i's δ-optimality at $s_i = \theta_b - \varepsilon$ under noise $(\varepsilon, N \backslash (N_i \cup \{i\}), N_{-i}, +)$ implies

$$u_i(a^\Gamma(m^\varepsilon_i, m^*_{-i}), \theta_b - \varepsilon) \geq u_i(a^\Gamma(m'_i, m^*_{-i}), \theta_b - \varepsilon) - \delta.$$

By a similar reasoning as in Case A, we have that for any ε sufficiently small,

$$u_i(a^\Gamma(m^\varepsilon_i, m^*_{-i}), \theta_b) = u_i(a^\Gamma(m'_i, m^*_{-i}), \theta_b). \tag{4}$$

Fix $\varepsilon > 0$ such that (4) holds, and note that $a^\Gamma(m^\varepsilon) = a \neq a^\Gamma(m')$. The remainder of the proof is symmetric to the argument in Case A. ■

Similarly, by Proposition 4*, to prove Theorem 2*(a), it suffices to prove the following Lemma.

Lemma 3*: If Γ is robust and NSI holds, then Γ is natural*.

Proof of Lemma 3*: Steps 1 to 4 of the proof of Lemma 3 carry over to show interval structure. The following observation can be obtained by strengthening step 2 of the proof of Lemma 3:

Observation 2: For any $\delta > 0$, there exists $\varepsilon(\delta) > 0$ such that if $\theta^\Gamma(m) \neq \varnothing$ and $\lambda(\theta^\Gamma(m)) < \varepsilon(\delta)$, then $\sup \theta^\Gamma(m) - \inf \theta^\Gamma(m) < 3\delta$.

Proof of Observation 2: Suppose not, so that for any $\bar{\varepsilon} > 0$, $\exists \varepsilon \in (0, \bar{\varepsilon})$ such that $\theta^{\Gamma}(m) \neq \varnothing$, $\lambda(\theta^{\Gamma}(m)) < \varepsilon$, and $\sup \theta^{\Gamma}(m) - \inf \theta^{\Gamma}(m) \geq 3\delta$. For such m, there exists $\theta^* \in \theta^{\Gamma}(m)$ such that $|\theta^* - a^{\Gamma}(m)| > \delta$.

Since $\lambda(\theta^{\Gamma}(m)) < \varepsilon$, for any θ where $m^{\Gamma}(\theta) = m$, $\exists \theta_0(\theta) \in [\theta - \varepsilon, \theta + \varepsilon]$ such that for some $i \in N$, $m_i^{\Gamma}(\theta_0(\theta)) \neq m_i$. Let $i_{\varepsilon}^{\Gamma}(\theta)$ be any such i, and consider the following noise Ξ:

(i) at states $\theta \in \theta^{\Gamma}(m)\backslash\{\theta^*\}$, consider a random variable $X \sim U[0,1]$; if the realization of X is θ, sender $i_{\varepsilon}^{\Gamma}(\theta)$ observes $s_i = \theta$, while if not, sender $i_{\varepsilon}^{\Gamma}(\theta)$ observes $s_i = \theta'$ for some $\theta' \in [\theta - \varepsilon, \theta + \varepsilon]$ where $m_i^{\Gamma}(\theta') \neq m_i$;

(ii) for all other senders, and for $i_{\varepsilon}^{\Gamma}(\theta)$ at all other states, the true state is observed.

Clearly, Ξ has size at most ε, and $a^{\Xi}(m) = \theta^*$. By step 1 of the proof of Lemma 3, Γ is not robust. $\square$

In the remainder of this proof, adopt the notation from the proof of Lemma 2*.

Case A: $\theta_b \in C$, $a \neq \theta_b$

Note that (1) still holds. If (2) still holds for ε sufficiently small, then the argument in Lemma 2* carries through. For (2) not to hold for ε sufficiently small, it must be that for any $\bar{\varepsilon} > 0$, there are infinitely many distinct m^{ε} for $\varepsilon \in (0, \bar{\varepsilon})$. Observation 2 implies that this can only be the case if there exists a sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ converging to 0 as $k \to \infty$ such that $a^{\Gamma}(m^{\varepsilon_k}) \to \theta_b$ as $k \to \infty$.

An approximate version of Observation 1(i) that converges to Observation 1(i) as $\varepsilon \to 0$ can be obtained by making a similar argument and using (1). If at θ_b, i (resp. j) is indifferent between a and some action a_i (resp. a_j), then by NSI and the finiteness of N, $\min_{j \neq i} |a_i - a_j| > 0$. A similar argument as in the proof of Observation 1(ii) thus shows that if NSI holds, then as $\varepsilon \to 0$, we have $a^{\Gamma}(m_i^{\varepsilon}, m_j, m_{-ij}^*) \to a$, $a^{\Gamma}(m_i, m_j^{\varepsilon}, m_{-ij}^*) \to a$, or both.

Now consider $a^{\Gamma}(m_i^{\varepsilon}, m_{-i})$. By point 2 in the definition of robustness, if $(m_i^{\varepsilon}, m_{-i})$ were off-path, we could not have $u_i(a', \theta_b) = u_i(a, \theta_b)$ whenever $|a' - a^{\Gamma}(m_i^{\varepsilon}, m_{-i})| < \gamma$. This implies, by (1), that $(m_i^{\varepsilon}, m_{-i})$ must occur on path for all sufficiently small ε. Because any two distinct on-path actions induced by message vectors differing in only one component are separated by at least η (see the second-to-last paragraph of the proof of Theorem 1a), we have that for all i and sufficiently small ε, $u_i(a^{\Gamma}(m_i^{\varepsilon}, m_{-i}), \theta_b) = u_i(a, \theta_b)$. Taking $m_{-ij}^* = m_{-ij}$ in the previous paragraph implies that for at least $n-1$ senders i, we have $a^{\Gamma}(m_i^{\varepsilon}, m_{-i}) = a$ for sufficiently small ε.

(i) If $a^{\Gamma}(m_i^{\varepsilon}, m_{-i}) = a$ for sufficiently small ε for all i, then as $\varepsilon \to 0$, we must have $a^{\Gamma}(m_i^{\varepsilon}, m_j^{\varepsilon}, m_{-i}) \to a$ for all pairs (i, j). To see this, note that, at θ_b, i (resp. j) must be nearly indifferent between $a^{\Gamma}(m_i^{\varepsilon}, m_j^{\varepsilon}, m_{-i})$ and $a^{\Gamma}(m_i, m_j^{\varepsilon}, m_{-i}) = a$ (resp. $a^{\Gamma}(m_i^{\varepsilon}, m_j, m_{-i}) = a$), which, by NSI and the finiteness of N, can occur only if $a^{\Gamma}(m_i^{\varepsilon}, m_j^{\varepsilon}, m_{-i})$ is near a. Iterating this reasoning (which also applies when $a^\Gamma(m_i^\varepsilon, m_{-i}) \to a$ as $\varepsilon \to 0$) yields $a^\Gamma(m^\varepsilon) \to a \neq \theta_b$, which contradicts the existence of $\{\varepsilon_k\}_{k=1}^\infty$ noted earlier.

(ii) If instead there exist arbitrarily small ε such that $a^\Gamma(m_1^\varepsilon, m_{-1}) \neq a$, the above iterative reasoning can still be used with senders $2, ..., n$, so that $a^\Gamma(m_1, m_{-1}^\varepsilon) \to a$ as $\varepsilon \to 0$. Since $u_1(a^\Gamma(m_1, m_{-1}^\varepsilon), \theta_b) - u_1(a^\Gamma(m^\varepsilon), \theta_b) \to 0$ as $\varepsilon \to 0$, for small ε, $a^\Gamma(m^\varepsilon)$ must be near either a or $a' \neq a$, where $u_1(a', \theta_b) = u_1(a, \theta_b)$. The existence of $\{\varepsilon_k\}_{k=1}^\infty$, combined with $a \neq \theta_b$, implies that we must have $a' = \theta_b$. Since $u_1(a^\Gamma(m_1^\varepsilon, m_{-1}), \theta_b) = u_1(a, \theta_b)$ for sufficiently small ε, and there exist arbitrarily small ε such that $a^\Gamma(m_1^\varepsilon, m_{-1}) \neq a$, there also exist arbitrarily small ε such that $a^\Gamma(m_1^\varepsilon, m_{-1}) = a' = \theta_b$. Because, for sufficiently small ε, $(m_1^\varepsilon, m_{-1})$ occurs on path, if $a^\Gamma(m_1^\varepsilon, m_{-1}) = \theta_b \neq a$, $(m_1^\varepsilon, m_{-1})$ must occur both to the left and to the right of θ_b. This is not possible: because $u_1(a^\Gamma(m_1^\varepsilon, m_{-1}), \theta_b) = u_1(a, \theta_b)$, in one of the two cases, by single-crossing, sender 1 strictly prefers inducing $a^\Gamma(m_1, m_{-1}) = a$ to $a^\Gamma(m_1^\varepsilon, m_{-1}) \neq a$.

Case B: $\theta_b \notin C$

Adopt the notation of case B of the proof of Lemma 2*. Like in case A of this proof, if (4) holds for sufficiently small ε, then we are done. Once again, by Observation 2, if there is no ε sufficiently small such that (4) holds, there must exist a sequence $\{\varepsilon_k\}_{k=1}^\infty$ converging to 0 as $k \to \infty$ such that $a^\Gamma(m^{\varepsilon_k}) \to \theta_b$ as $k \to \infty$. Here, since m^ε are sent inside C for small ε, we have $a^\Gamma(m^\varepsilon) = a$ for small ε. Thus, $a = \theta_b$.

Proceeding like in case A (with $m^\Gamma(\theta_b)$ and $a^\Gamma(m^\Gamma(\theta_b))$ taking the place of m and a, respectively), in subcase (i), we have $a^\Gamma(m^\varepsilon) \to a^\Gamma(m^\Gamma(\theta_b))$. Here, this implies $a = a^\Gamma(m^\Gamma(\theta_b))$, which contradicts $\theta_b \notin C$.

In subcase (ii), we have that a must be equal to either $a^\Gamma(m^\Gamma(\theta_b))$ or $a' \neq a^\Gamma(m^\Gamma(\theta_b))$, where $u_1(a', \theta_b) = u_1(a^\Gamma(m^\Gamma(\theta_b)), \theta_b)$. Since $a \neq a^\Gamma(m^\Gamma(\theta_b))$ (because $\theta_b \notin C$), we have $a' = a = \theta_b$. The remainder of the argument is analogous to case A.

Case C: $\theta_b \in C$, $a = \theta_b$

Because C is proper with right endpoint θ_b, the receiver's optimality implies that there exists another proper supercell C' with left endpoint $\theta_b' > \theta_b$ where the induced action is also θ_b. Moreover, θ_b' cannot be right-natural: otherwise, some message vector inducing action θ_b would be sent only to the right of θ_b', which cannot be the case.

If $\theta_b' \in C'$, then the situation is symmetric to case A (now the endpoint θ_b' and induced action θ_b cannot be equal), so we are done. Moreover, we cannot have $\theta_b' \notin C'$: by the first paragraph of case B, this is possible only if θ_b' is the action induced in C', which is not the case here. ■

Online Appendix C: Near Robustness

This section introduces weaker robustness concepts, near robustness and strong near robustness, that require a "nearby" strategy profile, rather than the exact original strategy profile, to be approximately optimal under noise. As stated in the main paper, Theorems 1 and 2 remain true provided that heterogeneous priors about the noise are allowed.

The closeness of strategy profiles is defined as follows.

Definition: Given a profile Γ, messages m_i and m_i' are (Γ, δ)-*close* if for any $m_{-i} \in \times_{j \neq i} M_j^{\Gamma}$, $|a^{\Gamma}(m_i, m_{-i}) - a^{\Gamma}(m_i', m_{-i})| < \delta$.

Definition: Given a profile Γ, profile Γ' is δ-*close to* Γ if:

1. $M_i^{\Gamma'} = M_i^{\Gamma}$;
2. for any proper cell C in Γ or Γ', $m_i^{\Gamma'}(s_i)$ and $m_i^{\Gamma}(s_i)$ are (Γ, δ)-close for all $s_i \in [\inf C + \delta, \sup C - \delta]$;
3. letting a^{Ξ} and $a^{\Xi\prime}$ be the receiver's best responses given noise Ξ to the senders' strategies in Γ and Γ' respectively, $\exists \varepsilon > 0$ such that, for all $m \in \times_{i=1}^{n} M_i^{\Gamma}$, $|a^{\Xi}(m) - a^{\Xi\prime}(m)| < \delta$ whenever the size of Ξ is less than ε, and $a^{\Xi}(m)$ and $a^{\Xi\prime}(m)$ exist; and
4. $|a^{\Gamma}(m) - a^{\Gamma'}(m)| < \delta$ for all $m \in \times_{i=1}^{n} M_i^{\Gamma}$.

Points 1 and 4 in the definition of δ-closeness simply require that the senders use the same messages in Γ' as in Γ, and that the receiver takes a nearby action after every message vector. Point 2 restricts the senders' strategies by requiring the use of the similar messages in Γ and Γ' in proper cells at least δ away from boundaries.[4] However, this condition has no power when dealing with sender strategies that do not feature intervals: it is difficult to directly determine whether two sender profiles with complicated structures are "close."[5] Point 3 addresses this issue by using the receiver's best response to evaluate how close sender profiles are to each other. Noise is used because, in some cases, two sender profiles could generate the same receiver actions without noise while generating far apart actions with small noise; such profiles ought to be considered distant.

[4] The requirement can be weakened to allow a small probability of deviation and/or deviation on a small set of states.

[5] For example, suppose that within some interval, strategy m_i^{Γ} assigns m_i within the set of irrational numbers and m_i' elsewhere. Strategy $m_i^{\Gamma'}$ is identical to m_i^{Γ} everywhere except on the said interval, where it assigns m_i within the set of transcendental numbers and m_i' elsewhere. It is unclear by simple inspection how "close" m_i^{Γ} and $m_i^{\Gamma'}$ should be considered.

For instance, consider Example 2, and shift all cell boundaries and receiver actions by less than δ. The resulting profile is δ-close to the original one: points 1, 2 and 4 are clearly satisfied, and point 3 is as well because as $\varepsilon \to 0$, we must have $a^{\Xi}(m) \to a^{\Gamma}(m)$ and $a^{\Xi\prime}(m) \to a^{\Gamma'}(m)$.

The definitions for strong near robustness and near robustness parallel the ones for strong robustness and robustness.

Definition: An equilibrium Γ in the noiseless game is *strongly near-robust* if, for every $\delta > 0$, there exists $\varepsilon > 0$ such that whenever there is common knowledge that noise has size less than ε, there exists a δ-close strategy profile Γ' where each player's strategy $r_i^{\Gamma'}$ is a δ-best response to $r_{-i}^{\Gamma'}$ evaluated under sender i's belief about the noise.

Definition: An equilibrium Γ in the noiseless game is *near-robust* if:

1. for every $\delta > 0$, there exists $\varepsilon > 0$ such that whenever there is common knowledge that noise has local size less than ε, there exists a δ-close strategy profile Γ' where each player's strategy $r_i^{\Gamma'}$ is an on-path δ-best response to $r_{-i}^{\Gamma'}$ evaluated under sender i's belief about the noise, and

2. in the noiseless game, there exists $\gamma > 0$ such that whenever the perturbation on the receiver's off-path beliefs has size less than γ, every sender's strategy m_i^{Γ} is a best response to m_{-i}^{Γ} and $a^{\Gamma *}$, where $a^{\Gamma *}$ denotes the receiver's best-response to m^{Γ} and her perturbed off-path beliefs.[6]

A profile with the characteristics of Γ' will be called a *δ-supporting profile.* Γ' is interim δ-optimal, where each player's payoffs are evaluated under her own beliefs.

With these definitions, Theorems 1 and 2 hold with no change. The proofs of Theorems 1b and 2b still apply: they allow for heterogeneous priors, and Γ is δ-close to itself for all $\delta > 0$. The proofs of Lemmata 1 to 3, which imply Theorems 1a and 2a, are modified as follows.

Modified proof of Lemma 1: Suppose Γ is strongly near-robust. Given Γ and Γ', let a^{Ξ} and $a^{\Xi\prime}$ denote the receiver's best response to $\{m_j^{\Gamma}\}_{j=1}^n$ and $\{m_j^{\Gamma'}\}_{j=1}^n$, respectively, given noise Ξ.

[6] Point 2 is the same as in the definition of robustness.

Step 1: For any $\delta > 0$, $\exists\varepsilon > 0$ such that for all noise Ξ with size less than ε, $|a^{\Gamma}(m) - a^{\Xi}(m)| < \delta$ for all $m \in \times_{i=1}^{n} M_i^{\Gamma}$.

By the definitions of strong near robustness and δ-closeness, we know that for any $\delta > 0$, $\exists\varepsilon > 0$ such that for all noise Ξ with size less than ε, $\exists\Gamma'$ such that:

- $a^{\Gamma'}$ is a δ-best response to $\{m_j^{\Gamma'}\}_{j=1}^{n}$ under Ξ;
- $|a^{\Gamma}(m) - a^{\Gamma'}(m)| < \delta$ for all $m \in \times_{i=1}^{n} M_i^{\Gamma}$; and
- $|a^{\Xi}(m) - a^{\Xi'}(m)| < \delta$ for all $m \in \times_{i=1}^{n} M_i^{\Gamma}$.

Because u_R is continuous and strictly concave in a, and Θ is compact, the first point implies that $\exists\gamma(\delta)$ such that, for all $m \in \times_{i=1}^{n} M_i^{\Gamma}$, $|a^{\Gamma'}(m) - a^{\Xi'}(m)| < \gamma(\delta)$, with $\lim_{\delta\to 0}\gamma(\delta) = 0$. Therefore, $|a^{\Gamma}(m) - a^{\Xi}(m)| < 2\delta + \gamma(\delta)$ for all $m \in M_i^{\Gamma}$ and Ξ with size less than ε.

Rewriting δ in lieu of $2\delta + \gamma(\delta)$ yields the result. $\Diamond$

The remainder of the proof (steps 2 to 4) is unchanged. ■

Lemma 2 is now proved in two steps, numbered 5 and 6 (numbering continued from the proof of Lemma 1). Suppose a boundary θ_b in Γ is not left-natural, such as the boundary between the first two cells in Example 2, and consider the following beliefs about noise: each sender believes that she observes the true state while all other senders observe $s_i = \max\{\theta - \varepsilon, 0\}$, the receiver believes that all senders observe $s_i = \theta$, and these beliefs are common knowledge. Let m be the message vector sent to the left of the boundary - $(1, 1)$ in our example. The proof applies the definition of δ-closeness to show that in a δ-supporting profile Γ', for δ small enough, m must be sent in a neighborhood to the left of $\theta_b - \delta$. Then, for ε small enough, m must also be sent between $\theta_b - \delta$ and $\theta_b - \delta + \varepsilon$: upon observing a signal in that range, each sender believes opponents will send m_{-i}, and in turn must send m_i, which gives i expected payoff at least δ higher than any other message, for δ small enough. Because θ_b is not left-natural, this argument can be iterated past $\theta_b + \delta$, which means that no δ-supporting profile can exist. Therefore, Γ must be natural. This intuition bears parallels to the global games contagion argument (except for the heterogeneous prior).

Like for steps 2 to 4, the noise distribution used for steps 5 and 6 does not have to be atomic. For example, the argument carries through if each sender instead believes that other senders' signals are distributed according to $U[\max\{\theta - \varepsilon, 0\}, \theta]$.[7] Unlike for steps 2 to 4, the argument uses noise where the prior is heterogeneous.

[7] Point 2 of the definition of closeness can also be relaxed: if a message vector close to m must be sent in Γ' with probability near 1 in some interval I_m to the left of $\theta_b - \delta$, then the unraveling reasoning remains valid for sufficiently small ε (in particular, $\varepsilon < |I_m|$).

Modified proof of Lemma 2: We proceed by contradiction. Suppose instead, without loss of generality, that the right endpoint $\theta_b \neq 1$ of a proper cell C in Γ where $m = (m_1, ..., m_n)$ is sent is not left-natural. Denote the measure of this proper cell by λ.

Step 5: $\exists\overline{\delta} > 0$ such that for all $i \in N$ and $m_i' \in M_i^\Gamma \backslash \{m_i\}$, $u_i(a', \theta) + \overline{\delta} < u_i(a, \theta)$ for all $\theta \in [\theta_b - \overline{\delta}, \theta_b + \overline{\delta}]$, $a' \in [a^\Gamma(m_i', m_{-i}) - \overline{\delta}, a^\Gamma(m_i', m_{-i}) + \overline{\delta}]$, and $a \in [a^\Gamma(m_i, m_{-i}) - \overline{\delta}, a^\Gamma(m_i, m_{-i}) + \overline{\delta}]$.

By the definition of "left-natural," for any i and any $m_i' \in M_i^\Gamma \backslash \{m_i\}$, either:

(i) $u_i(a^\Gamma(m_i', m_{-i}), \theta_b) < u_i(a^\Gamma(m_i, m_{-i}), \theta_b)$,

(ii) (m_i', m_{-i}) is sent on path in Γ and $a^\Gamma(m_i, m_{-i}) = a^\Gamma(m_i', m_{-i})$, or

(iii) (m_i', m_{-i}) is not sent at any state in Γ and $u_i(a^\Gamma(m_i', m_{-i}), \theta_b) = u_i(a^\Gamma(m_i, m_{-i}), \theta_b)$.

Because u_i is continuous, if (i) holds, then $u_i(a', \theta) < u_i(a, \theta)$ for all θ in a non-degenerate interval around θ_b, and all a' and a sufficiently near $a^\Gamma(m_i', m_{-i})$ and $a^\Gamma(m_i, m_{-i})$ respectively. Therefore, it suffices to show that for any m_{-i}, there are finitely many $a^\Gamma(m_i', m_{-i})$ occurring on the equilibrium path. This must be true since any two such actions must be separated by at least η (see the second-to-last paragraph of the proof of Theorem 1a).

Case (ii) cannot arise by Assumption A.

Case (iii) cannot arise by step 2. $\Diamond$

Step 6: Let $\delta < \min\{\overline{\delta}, \frac{\lambda}{2}, \eta\}$. Then, for any $\varepsilon \in (0, \lambda - 2\delta)$, there is no δ-supporting profile for Γ under the following beliefs about the noise:

- Each sender believes that they observe the true state and that other senders observe $\max\{\theta - \varepsilon, 0\}$*.*

- The receiver believes that all senders observe the true state. (For the sake of completeness - this will not matter.)

- These beliefs are common knowledge.

By the definition of δ-closeness, in any δ-supporting profile Γ', it must be that for sufficiently small ε, for all i, and for all $s_i \in [\theta_b - \delta - \varepsilon, \theta_b - \delta)$, $m_i^{\Gamma'}(s_i)$ is (Γ, δ)-close to m_i. Now suppose $m_i^{\Gamma'}(s_i) \neq m_i$. Then, for some m_{-i}', $a^\Gamma(m_i^{\Gamma'}(s_i), m_{-i}') \neq a^\Gamma(m_i, m_{-i}')$. By completeness, both $(m_i^{\Gamma'}(s_i), m_{-i}')$ and (m_i, m_{-i}') occur on path in Γ, so by the same reasoning used at the end of case (i) of step 5, we must have $|a^\Gamma(m_i^{\Gamma'}(s_i), m_{-i}') - a^\Gamma(m_i, m_{-i}')| > \eta$. This contradicts $m_i^{\Gamma'}(s_i)$ being (Γ, δ)-close to m_i since $\delta < \eta$. Thus $m_i^{\Gamma'}(s_i) = m_i$ for all $s_i \in [\theta_b - \delta - \varepsilon, \theta_b - \delta)$.

Now suppose sender j observes $s_j \in [\theta_b - \delta, \theta_b - \delta + \varepsilon)$. She believes that all senders $i \neq j$ observed $s_i \in [\theta_b - \delta - \varepsilon, \theta_b - \delta)$, and therefore will send m_i. By step 5, her unique

δ-best response is m_j. Since this holds for all senders, we have that for all i and for all $s_i \in [\theta_b - \delta, \theta_b - \delta + \varepsilon)$, $m_i^{\Gamma'}(s_i) = m_i$.

Iterating the above argument, it follows by step 5 that for all i and for all $s_i \in [\theta_b - \delta, \theta_b + \overline{\delta}]$, $m_i^{\Gamma'}(s_i) = m_i$. By definition, Γ' can be δ-close to Γ only if, for all i, $m_i^{\Gamma}(s_i) = m_i$ for all $s_i \in [\theta_b, \theta_b + \overline{\delta} - \delta]$, which contradicts θ_b being the right endpoint of C. Therefore, Γ' is not a δ-supporting profile of Γ. $\Diamond$

Under the beliefs about noise in step 6, there is common knowledge that noise is less than ε. We therefore conclude that Γ is, in fact, not strongly near-robust. ■

Modified proof of Lemma 3: Modify step 1 as in the proof of Lemma 1. Steps 2 to 4 are unchanged. Step 5 and 6 follow the modified proof of Lemma 2, as adjusted below.

Step 5: Same statement as step 5 in the proof of Lemma 2, and same argument in cases (i) and (ii).

Case (iii) is ruled out by point 2 in the definition of near robustness and the continuity of u_i. $\Diamond$

Step 6: Same statement as step 6 in the proof of Lemma 2, except that δ is chosen to be also less than γ from the point 2 in the definition of near robustness. Then, to show that, for small enough ε, $m_i^{\Gamma'}(s_i) = m_i$ for all $s_i \in [\theta_b - \delta - \varepsilon, \theta_b - \delta)$, proceed again by contradiction. The argument is the same if, for some m'_{-i}, both $(m_i^{\Gamma'}(s_i), m'_{-i})$ and (m_i, m'_{-i}) occur on path but induce different actions in Γ. If not, then $(m_i^{\Gamma'}(s_i), m_{-i})$ is off path in Γ, and point 2 in the definition of near robustness implies $|a^{\Gamma}(m_i^{\Gamma'}(s_i), m_{-i}) - a^{\Gamma}(m_i, m_{-i})| \geq \gamma$. This again contradicts $m_i^{\Gamma'}(s_i)$ being (Γ, δ)-close to m_i since $\delta < \gamma$.

The remainder of the proof is identical to the analogous part of the modified proof of Lemma 2. ■