TECHNISCHE UNIVERSITÄT
CHEMNITZ

Faculty of Mathematics
Professorship Geometry

An Invitation to Generalized Minkowski Geometry

Dissertation submitted to the Faculty of Mathematics at Technische Universität Chemnitz in accordance with the requirements for the degree

doctor rerum naturalium (Dr. rer. nat.)

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submitted December 3, 2018
defended February 14, 2019

Jahn, Thomas
An Invitation to Generalized Minkowski Geometry
Dissertation, Faculty of Mathematics
Technische Universität Chemnitz, 2019

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Bibliographical description

Thomas Jahn
An Invitation to Generalized Minkowski Geometry
Dissertation, 170 pages, Technische Universität Chemnitz, Faculty of Mathematics, 2019

Report
The present thesis contributes to the theory of generalized Minkowski spaces as a continuation of Minkowski geometry, i.e., the geometry of finite-dimensional normed spaces over the field of real numbers. In a generalized Minkowski space, distance and length measurement is provided by a gauge, whose definition mimics the definition of a norm but lacks the symmetry requirement. This seemingly minor change in the definition is deliberately chosen. On the one hand, many techniques from Minkowski spaces can be adapted to generalized Minkowski spaces because several phenomena in Minkowski geometry simply do not depend on the symmetry of distance measurement. On the other hand, the possible asymmetry of the distance measurement set up by gauges is nonetheless meaningful and interesting for applications, e.g., in location science. In this spirit, the presentation of this thesis is led mainly by minimization problems from convex optimization and location science which are appealing to convex geometers, too. In addition, we study metrically defined objects, which may receive a new interpretation when we measure distances asymmetrically. To this end, we use a combination of methods from convex analysis and convex geometry to relate the properties of these objects to the shape of the unit ball of the generalized Minkowski space under consideration.

Keywords
Apollonius circle, ball convexity, Birkhoff orthogonality, bisector, Cassini curve, ellipse, gauge, generalized Minkowski space, hyperbola, isosceles orthogonality, successive radii

Acknowledgments

At first, I would like to express my sincere gratitude and deep appreciation to the people who shaped me as a mathematician during my research.
I am indebted to my supervisor Prof. Dr. Horst Martini for his unlimited advice and support, for his trust and confidence, and for giving me the opportunity to meet lots of inspiring colleagues at wonderful conferences. Thankfully, I became a coauthor of some of these colleagues. In particular, I owe thanks to Christian Richter and Bernardo González Merino who did not only impress me with their geometric intuition but also with their kindheartedness.
Furthermore, I am grateful to Prof. Dr. Gert Wanka and his former research group at Technische Universität Chemnitz (including Sebastian Banert, Radu Ioan Boț, Ernö Robert Csetnek, Sorin-Mihai Grad, Christopher Hendrich, and Oleg Wilfer) for the scientific exchange across research groups in a friendly and supportive atmosphere.
Moreover, I would like to thank Prof. Dr. Valeriu Soltan and Prof. Dr. Senlin Wu for the time and effort they invested in reviewing this thesis.

1 Introduction

Abstraction and formalization of aspects of everyday experience is a basic ingredient for mathematics. For instance, distance measurement is a way of quantifying the concept of the spatial gap between physical objects or the temporal gap between events. In this sense, distances have been implemented in a variety of mathematical contexts. Common features of those abstract distance notions are their nonnegativity and the validity of the triangle inequality, meaning that the distance between two points is less or equal than the accumulated distance taking a detour via a third point. The comparison of arithmetic combinations of its values is therefore a defining property of distance. Comparing distances immediately leads to optimization, i.e., asking for nearest or farthest points. This may be utilized for approximation tasks, i.e., for the process of replacing real-world data by a sufficiently simple model. In many applications, distances are symmetric in the sense that it does not matter if we measure the distance from a first to a second point or vice versa. Sometimes, however, an asymmetric formalization of distances is required by real-world applications, e.g., when we model transportation costs for uphill and downhill traveling.

Norms are a common tool for defining distances between points in vector spaces which are not only symmetric but also translation-invariant and homogeneous. The shape of balls is then independent of their centers and radii. Therefore, one can show that norms correspond to centrally symmetric convex bodies in a natural way. Aiming for applications, where symmetric distance measurement is undesirable or inappropriate, a reasonable generalization of norms should still yield translation-invariant distances but allow for asymmetry. Gauges meet these requirements, yet norms have been studied more in greater detail and are considerably better understood today. Minkowski introduced gauges in his monograph *Geometrie der Zahlen* [167], which appeared in its first edition in 1896. In contrast to that, the term *Minkowski space* usually stands for a finite-dimensional normed space in modern convex geometry. Mirroring this, we shall call a finite-dimensional real vector space equipped with a gauge a *generalized Minkowski space*.

In some situations, the translation invariance of distances allows for another viewpoint on generalized Minkowski spaces. Namely, the reference to the gauge appearing in certain definitions may be replaced by a reference to its unit ball or, more generally, by a reference to a general convex body with non-empty interior. This approach is chosen, e.g., in the study of self-circumferences of convex sets in the plane [84] or in the study of optimal containment of pairs of convex sets [35, 37, 142]. Explicit reference to a gauge appears, e.g., in the study of bisectors [144] or in certain convex optimization problems [195]. Topics such as the inscribability of simplices in smooth hypersurfaces [94] may also be seen in this context.

The present thesis is a contribution to the theory of generalized Minkowski spaces. In the sequel, we discuss generalizations of a variety of metrically defined objects from Minkowski spaces and look for replacements for the corresponding theorems. As it turns out, the seemingly minor change in the definition of a norm amounts to a much richer theory in some places in the sense that we will encounter phenomena which are exclusive to norms, and others which are exclusive to non-norms. Continuing the interaction between analysis and geometry, the shape of a distinguished convex body, namely the unit ball, will play a crucial role for analytic properties of the gauge such as differentiability. In this context, rotundity, which is one way of formalizing the roundness of convex bodies, will be a recurring topic.

1.1 Description of the content

The presentation of thesis at hand follows the one given in our articles. For the sake of consistency, some notational changes and rearrangements of results had to be performed. In the remainder of the present chapter, we provide basic definitions and results from convex geometry and convex analysis. This way, we fix the language of generalized Minkowski spaces analogously to the one of normed spaces.

Chapter 2 deals with Birkhoff orthogonality. This binary relation is a generalization of usual Euclidean orthogonality. It is closely connected to the approximate solutions of a variant of the best approximation problem. Based on the results of [121, Section 3], we give dual descriptions of ε-Birkhoff orthogonality, show how smoothness and rotundity of the gauge impacts 0-Birkhoff orthogonality, and study gauges whose Birkhoff orthogonality relations are identical or mutually inverses.

The second part of this thesis deals with intersections of translates of a fixed convex body in a threefold way. In Chapter 3, we recall the definitions of the circumradius and the circumcenters of a subset of a generalized Minkowski space. Thereafter, we give the main results of [91] on the dimension of the set of circumcenters and its relation to the boundary structure of the unit ball. Following [118] and [120, Section 3], we also discuss the definitions of the inradius, the diameter, and the minimum width of a convex subset of a generalized Minkowski space, and generalize interpolating quantities which are known as successive radii in Minkowski geometry. For these quantities, we prove basic identities and inequalities relating them to each other or showing their behavior under manipulations of the input sets.

In Chapter 4, we are concerned with an abstract convexity notion called ball convexity. Its definition involves intersections of equal-sized balls. Following [120, Section 4] and [124], we introduce the notion of ball hull as a replacement for the closed convex hull in classical convexity, and find analogs of results like Straszewicz's theorem in the ball convexity context.

The subject of Chapter 5 is another abstract orthogonality notion called isosceles orthogonality. Its geometry is intertwined with the notion of bisectors. Based on methods developed in [121, Section 4] and [125], we study gauges whose isosceles orthogonality relations are identical or mutually inverses, investigate the relationship of Birkhoff orthogonality and isosceles orthogonality, and describe the shape of bisectors. Following [123, Sections 4 and 6], hyperboloids and apollonoids are introduced as the loci of points whose difference and ratio to two fixed points is constant, respectively. Both families of sets contain bisectors as a special case and may be used to characterize Euclidean space.

In the last part of this thesis, we discuss two types of multifocal sets. In Chapter 6, we revisit the results of [122] and [123, Section 3], i.e., we study the pointwise sum of distances to finitely many given convex bodies. Dual descriptions of the minimizers of this function are derived and the behavior of the set of minimizers under manipulations of the input data is studied. Using metrically defined segments, we identify necessary and sufficient conditions for a gauge to be a norm. Finally, we investigate the geometry of level sets of the sum objective function.
Chapter 7 answers first questions regarding the geometry of the pointwise product of distances to finitely many given points, following the presentation of [123, Section 5]. For instance, we show that the sublevel sets of these functions split into finitely many star-shaped components for small and for large parameters. Else, the number of connected components may be infinite (countable or uncountable).

1.2 Vector spaces

In this section, we introduce the terminology and notation used in this thesis, outline the setting by giving basic definitions, and recall some preliminary results from convex geometry, convex analysis, and optimization theory. These topics are extensively covered, e.g., in the books [202, 207,243]. The readers' attention shall also be turned to the monograph [56] which is devoted to the functional analysis of so-called asymmetric normed spaces. (We tacitly assume some degree of acquaintance with finite-dimensional vector spaces and basic topology.)
Our investigations take place in a finite-dimensional vector space X over the field $\mathbb{R}$ of real numbers. Latin and Greek lower-case letters stand for vectors, numbers, and functions in the sequel, whereas Latin upper-case letters denote sets or collections of sets. We refer to the zero element of X as the *origin* 0. (The notation suppresses the vector space X, which will be evident from the context.) By $\operatorname{card}(I)$, we denote the cardinality of a set I. The *dimension* of X, i.e., the maximal cardinality of a linearly independent subset of X, shall be denoted by $\dim(X)$. Unless otherwise stated, we assume that $\dim(X) \geq 1$. For $1 \leq j \leq \dim(X)$, we use the abbreviation $\mathscr{L}_j^X$ for the family of j-dimensional linear subspaces of X and write $\mathscr{L}_0^X := \{\{0\}\}$. The *linear hull* $\operatorname{lin}(K)$ of a set $K \subset X$ is the smallest linear subspace of X which contains K. (As usual, *smallest* and *largest sets* are always meant to be with respect to the partial order given by set inclusion.) For $K, K' \subset X$, $x \in X$, and $\lambda \in \mathbb{R}$, we define *Minkowski addition* and *Minkowski subtraction* by

$$K \pm \lambda K' := \{x \pm \lambda y \mid x \in K, y \in K'\}$$

and

$$K \sim K' := \{x \in X \mid x + K' \subset K\},$$

respectively, where $x \pm K$ shall be used as an abbreviation for the *translate* $\{x\} \pm K$ of $\pm K$ by x. If $K' = x + \lambda K$ for some $x \in X$ and $\lambda > 0$, we say that $K \subset X$ and $K' \subset X$ are *homothetic* or that K' is a *homothetic image* of K. Accordingly, a mapping $f : X \to X$, $f(z) := x + \lambda z$ with $x \in X$ and $\lambda > 0$ is said to be a *homothety*. The *affine hull* $\operatorname{aff}(K)$ of a set $K \subset X$ is the smallest affine subspace of X which contains K, that is, a set of the form $x + L$, where $x \in X$ and L is a linear subspace of X. A set $K \subset X$ is said to be a *cone* if there exists a point $x \in X$ such that

$K-x=\lambda(K-x)$ for all $\lambda>0$. In this case, the point x is an *apex* of the cone. The *conical hull* of a set $K\subset X$ is $\mathrm{cone}(K):=\{\lambda x\mid x\in K,\lambda>0\}$. The *dimension* $\dim(K)$ of $K\subset X$ is the dimension of the linear subspace $\mathrm{aff}(K)-\mathrm{aff}(K)$ of X. Affine subspaces of X of dimension $\dim(X)-1$ are called *hyperplanes*. Sets of dimension 1 are, for instance, the *straight line*

$$\langle x,y\rangle:=\{\lambda x+(1-\lambda)y\mid\lambda\in\mathbb{R}\},$$

the *line segment*

$$[x,y]:=\{\lambda x+(1-\lambda)y\mid 0\le\lambda\le 1\},$$

and the *ray*

$$[x,y\rangle:=\{\lambda y+(1-\lambda)x\mid\lambda\ge 0\},$$

where x and y denote distinct points of the vector space X. In case $X=\mathbb{R}$, the line segment $[x,y]$ is more commonly called a *closed interval*. Corresponding to which of the endpoints $x,y\in X$ is contained, we use the analogous notation (x,y), $[x,y)$, or $(x,y]$ for *open* and *half-open line segments* or *intervals*. We shall also introduce the *extended real line* $\overline{\mathbb{R}}=\mathbb{R}\cup\{+\infty,-\infty\}$, which we use with the conventions $0(+\infty):=+\infty$, $0(-\infty):=0$, and $(+\infty)+(-\infty):=+\infty$. Finally, affine subspaces L_1, L_2 of X are said to be *parallel* if L_1 is a translate of L_2 or, equivalently, if $L_1-L_1=L_2-L_2$.

1.3 Gauges and topology

The concept of *convexity* has attracted the interest of researchers since antiquity. Early contributions are attributed to Euclid and Archimedes, see [96]. The formation of convex geometry as a discipline in the sense of a systematic study of convex sets was initiated by Minkowski [167]. However, the term *convex set* and the modern definition first appear in Steinitz's paper [220].

Definition 1.1. A set $K\subset X$ is said to *convex* if $\lambda K+(1-\lambda)K=K$ for all $\lambda\in[0,1]$. For arbitrary subsets K of X, we denote by $\mathrm{co}(K)$ the *convex hull* of K, that is, the smallest convex subset of X which contains K.

In his famous monograph *Geometrie der Zahlen* [167], Minkowski investigated a class of functions which he called *einhellige Strahldistanzen*. These functions are nowadays known as *gauges*, *convex distance functions*, or *Minkowski functionals*.

Definition 1.2. A *gauge* on X is a function $\gamma:X\to\mathbb{R}$ satisfying the conditions

(a) $\gamma(x)\ge 0$ for all $x\in X$ *(nonnegativity)*,
(b) $\gamma(x)=0\Rightarrow x=0$ *(definiteness)*,
(c) $\gamma(\lambda x)=\lambda\gamma(x)$ for all $x\in X$ and $\lambda\ge 0$ *(positive homogeneity)*,
(d) $\gamma(x+y)\le\gamma(x)+\gamma(y)$ for all $x,y\in X$ *(subadditivity, triangle inequality)*.

If $\gamma(x)=1$, then x is said to be a *unit vector*.

With the establishment of functional analysis at the beginning of the 20th century, Minkowski's *wechselseitige Strahldistanzen*, i.e., gauges satisfying the symmetry condition $\gamma(x) = \gamma(-x)$ for all $x \in X$, were called *norms* and became a popular research topic among analysts. In honor of Minkowski, a finite-dimensional normed space $(X, \|\cdot\|)$ over the field of real numbers is also called a *Minkowski space* when we focus on its geometric properties. Accordingly, the geometric theory of Minkowski spaces is called *Minkowski geometry*.

Definition 1.3. A pair (X,γ) consisting of a finite-dimensional real vector space X and a gauge $\gamma : X \to \mathbb{R}$ is called a *generalized Minkowski space*. If X is two-dimensional, we refer to (X,γ) as a *generalized Minkowski plane*. If γ is a norm, we drop the word *generalized*.

Busemann [44, p. 202] and Zaustinsky [245, p. 5] investigate functions $\rho : X \times X \to \mathbb{R}$ which have non-negative values, i.e., $\rho(x, y) \geq 0$ for all $x, y \in X$, satisfy the triangle inequality $\rho(x,z) \leq \rho(x, y) + \rho(y,z)$ for all $x, y,z \in X$, and for which $\rho(x, y) = 0$ implies $x = y$. An additional requirement concerns the topology of these *general metric spaces* (X, ρ): For all $x \in X$ and sequences $(x_i)_{i\in\mathbb{N}}$ of elements $x_i \in X$ indexed by the set of natural numbers $\mathbb{N}$, we have $\rho(x, x_i) \to 0$ if and only if $\rho(x_i, x) \to 0$. Busemann and Zaustinsky, *loc. cit.*, state that this way, the topology coincides with the one induced by the metric $X \times X \ni (x, y) \mapsto \max\{\rho(x, y), \rho(y, x)\}$. Because of this, the distinction between general metric spaces in the above sense and usual metric spaces is not topological by nature. For a gauge $\gamma : X \to \mathbb{R}$, we may interpret $\rho(x, y) = \gamma(y - x)$ as the *distance from* x *to* y. Then the requirements of Busemann's general metric spaces are met, see [245, Section 5]. In particular, the topology induced by γ via the notion of convergence $x_i \to x$ in (X,γ), defined by $\gamma(x_i - x) \to 0$ in $\mathbb{R}$ with the standard topology, coincides with the topology induced in the same way by the *opposite gauge* $\gamma^\vee : X \to \mathbb{R}$, $\gamma^\vee(x) := \gamma(-x)$, see [56, p. 7]. (We write $x = \lim_{i\to+\infty} x_i$ if $x_i \to x$, and call x the *limit* of the sequence $(x_i)_{i\in\mathbb{N}}$.) Therefore, it also coincides with the topology on X induced by the norm $X \ni x \mapsto \max\{\gamma(x), \gamma^\vee(x)\}$. Since all norms defined on a finite-dimensional vector space X induce the same topology, all gauges defined on X do the same. An alternative approach to the topology of a generalized Minkowski space (X,γ) is provided by the collection of open sets. For this, we start with a reasonable implementation of the concept of balls and spheres.

Definition 1.4. Let (X,γ) be a generalized Minkowski space. By

$$U_\gamma(x,\lambda) := \{y \in X \mid \gamma(y - x) < \lambda\},$$
$$B_\gamma(x,\lambda) := \{y \in X \mid \gamma(y - x) \leq \lambda\},$$

and

$$S_\gamma(x,\lambda) := \{y \in X \mid \gamma(y - x) = \lambda\},$$

we denote the *open ball*, the *closed ball* and the *sphere with radius* $\lambda > 0$ *and center* $x \in X$, respectively. If the gauge γ is clear from the context, we omit it from the notation. The sets $U_\gamma(0, 1)$, $B_\gamma(0, 1)$, and $S_\gamma(0, 1)$ are *open unit ball*, *(closed) unit ball*, and the *unit sphere* of (X,γ), respectively.

Analogously to the theory of metric spaces, a subset V of X is said to be a *neighborhood* of a point $x \in X$ if there exists a number $\lambda > 0$ such that $B(x,\lambda) \subset V$. An *open set* is a neighborhood of each of its points. (One readily checks that open balls are open sets.) The family of open sets then forms a *topology* on X, that is, $\emptyset$ and X are open sets, the intersection of two open sets is an open set, and the union of an arbitrary number of open sets is again an open set. It turns out that under the assumption that for each two points $x, y \in X$, there exist neighborhoods V_x of x and V_y of y such that $x \notin V_y$ and $y \notin V_x$ (also known as the T_1 separation property), the topology of the space coincides with the one of a Euclidean space of the same dimension, see [82, Theorem 9]. The T_1 property is guaranteed by the definiteness of gauges [82, Lemma 12]. By choosing a basis and identifying X with $\mathbb{R}^{\dim(X)}$, the proof of [82, Theorem 9] resembles the standard proof of the fact that each two norms on $\mathbb{R}^d$ are equivalent. This technique therefore helps to push topological obstructions aside. In this case, one may think of gauges as functions on $\mathbb{R}^{\dim(X)}$ in which the topology is set up by the Euclidean norm. Nonetheless, topological and metrical notions like the *closure* $\operatorname{cl}(K)$ of a set $K \subset X$, its *interior* $\operatorname{int}(K)$, its *boundary* $\operatorname{bd}(K)$, its relative interior $\operatorname{ri}(K)$, i.e., its interior with respect to $\operatorname{aff}(K)$, its boundedness, its compactness, its connectedness, the convergence of sequences, and continuity of functions may equivalently be defined with respect to γ if necessary. In particular, for any generalized Minkowski spaces (X_1, γ_1), (X_2, γ_2),

- closed balls are indeed closed sets, which need not be the case if one neglects the T_1 property, see [56, Proposition 1.1.8],
- "compact" is synonymous for "closed and bounded" (known as the Heine–Borel theorem, for which $\dim(X) < +\infty$ is important),
- linear operators $X_1 \to X_2$ are continuous,
- $S(x, \lambda)$ is the boundary of both $U(x, \lambda)$ and $B(x, \lambda)$,
- $B(x, \lambda)$ is the closure of $U(x, \lambda)$,
- $U(x, \lambda)$ is the interior of $B(x, \lambda)$, see [56, Proposition 2.2.7].

A classical result says that the closed unit ball $B(0, 1)$ of a Minkowski space $(X, \|\cdot\|)$ is non-empty, *centrally symmetric* (i.e., $B(0, 1) = -B(0, 1)$), convex, closed, and bounded subsets of X for which 0 is an interior point, and conversely, for every such set $B \subset X$, the *Minkowski functional*

$$\gamma_B : X \to \mathbb{R}, \quad \gamma_B(x) := \inf\{\lambda > 0 \mid x \in \lambda B\} \tag{1.1}$$

is a norm, cf. [228, Propositions 1.1.6 and 1.1.8]. In this statement, the metrical notion of boundedness appears. This has to be understood in the Euclidean sense via above mentioned the isomorphism $X \cong \mathbb{R}^{\dim(X)}$ of vector spaces, since it is vacuously true if we call a subset K of a generalized Minkowski space (X, γ) a *bounded set* if there exists $\lambda > 0$ such that $K \subset B(0, \lambda)$. Alternatively, boundedness of a convex set $K \subset X$ can be expressed algebraically as the absence of rays contained in X. Closedness of closed unit balls can be seen in the context of the fact that $B_\gamma(0, 1)$ is the largest set whose Minkowski functional is γ, see [243, p. 4] and [56, Section 2.2.2]. As a generalization of [228, Propositions 1.1.6 and 1.1.8], gauges on a finite-dimensional vector space X are precisely the Minkowski functionals of non-empty, convex, closed, and bounded subsets of X which have the origin in their interiors. This correspondence enables the combination of analytic and geometric tools for the study of generalized Minkowski spaces. In the sequel, we will use a coordinate-free approach as often as possible in order to view the gauge as the constituent of the topology and geometry of the vector space X, this way emulating the role played by the norm in a normed space. (Nonetheless, explicit examples will take place in $X = \mathbb{R}^d$.) In this spirit, we use the notation $\mathscr{C}^X$ for the family of non-empty closed convex sets in a generalized Minkowski space (X, γ). Bounded sets that belong to $\mathscr{C}^X$ form the class $\mathscr{K}^X$ of *convex bodies*. We also write $\mathscr{C}_0^X$ and $\mathscr{K}_0^X$ for the classes of sets having non-empty interior and belonging to $\mathscr{C}^X$ and $\mathscr{K}^X$, respectively.

1.4 Convexity and polarity

Encouraged by the development of optimization theory, the analytic treatment of convexity received a great impetus in the early 1960s. Rockafellar's monograph [202] is the eponymous work of the young discipline but also Fenchel and Moreau are considered founding fathers of *convex analysis*, see [111, pp. 245–249]. The term *convex function*, which is central to convex analysis, was coined by Jensen [129].

Definition 1.5. A function $f : X \to \overline{\mathbb{R}}$ is called *convex* if $f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)$ for all $x, y \in X$ and all $\lambda \in [0,1]$.

As gauges are positively homogeneous and subadditive functions, they are convex functions, see [202, Theorem 4.7]. Furthermore, Minkowski functionals of convex sets are convex functions. One readily checks that *sublevel sets*

$$f_{\leq\alpha} := \{x \in X \mid f(x) \leq \alpha\},$$
$$f_{<\alpha} := \{x \in X \mid f(x) < \alpha\}$$

of convex functions $f : X \to \overline{\mathbb{R}}$ are convex sets, independently of the level $\alpha \in \mathbb{R}$. (Applied to a gauge γ, we obtain the convexity of the balls $B(0,\alpha)$ and $U(0,\alpha)$.) *Level sets*

$$f_{=\alpha} := \{x \in X \mid f(x) = \alpha\}$$

will also appear throughout this thesis, e.g., $\gamma_{=\alpha} = S(0,\alpha)$. If $f : X \to \mathbb{R}$ and $-f : X \to \mathbb{R}$, $(-f)(x) := -f(x)$ are convex functions, then f is said to be an *affine function*. Whenever $\phi : X \to \mathbb{R}$ is a *linear functional*, i.e., $\phi(\lambda x + y) = \lambda\phi(x) + \phi(y)$ for all $x, y \in X$ and $\lambda \in \mathbb{R}$, it is a continuous function. Therefore the dual space, which is an important concept in classical functional analysis, can be introduced the following way.

Definition 1.6. The *dual space* of the vector space X is the vector space X^* of linear functionals $\phi : X \to \mathbb{R}$. For $\phi \in X^*$ and $x \in X$, we shall write $\langle \phi \mid x \rangle$ for $\phi(x)$.

For $d \in \mathbb{N}$, we will always identify $\mathbb{R}^d$ with $(\mathbb{R}^d)^*$ via $\mathbb{R}^d \ni (\alpha_1, \dots, \alpha_d) \mapsto (\mathbb{R}^d \ni (\beta_1, \dots, \beta_d) \mapsto \sum_{i=1}^d \beta_i \alpha_i)$. This way, the bilinear mapping $\langle \cdot \mid \cdot \rangle : X^* \times X \to \mathbb{R}$ can be understood as an inner product after identifying both X^* and X with $\mathbb{R}^d$ where $d = \dim(X) = \dim(X^*)$. In this sense, the *annihilator* $K^\perp := \{\phi \in X^* \mid \langle \phi \mid x \rangle = 0 \ \forall\, x \in K\}$ of a set $K \subset X$ is a replacement for its orthogonal complement. The concept of the dual norm is replaced by the polar function in the sense that the latter is indeed a gauge defined on the dual space, see [202, Section 15].

Definition 1.7. The *polar function* of a gauge $\gamma : X \to \mathbb{R}$ is given by

$$\gamma^\circ : X^* \to \mathbb{R}, \quad \gamma^\circ(\phi) := \inf\{\lambda > 0 \mid \langle \phi \mid x \rangle \leq \lambda\gamma(x) \ \forall\, x \in X\}.$$

From the definition it follows that gauges satisfy the Cauchy–Schwarz-like inequalities

$$-\gamma^\circ(-\phi)\gamma(x) \leq \langle \phi \,|\, x \rangle \leq \gamma^\circ(\phi)\gamma(x) \tag{1.2}$$

for all $\phi \in X^*$ and $x \in X$. Other representations of the polar function are

$$\begin{aligned}\gamma^\circ(\phi) &= \sup\left\{ \frac{\langle \phi \,|\, y \rangle}{\gamma(y)} \,\middle|\, y \in X, y \neq 0 \right\} \\ &= \sup\{\langle \phi \,|\, y \rangle \,|\, y \in X, \gamma(y) = 1\} \\ &= \sup\{\langle \phi \,|\, y \rangle \,|\, y \in X, \gamma(y) \leq 1\},\end{aligned}$$

see again [56, Proposition 2.1.7] and [202, Section 15]. Thus the polar gauge γ° can also be viewed as the support function of the unit ball of γ.

Definition 1.8. The *support function* of a set $K \subset X$ is given by

$$h(\cdot, K) : X^* \to \overline{\mathbb{R}}, \quad h(\phi, K) := \sup\{\langle \phi \,|\, x \rangle \,|\, x \in K\}.$$

The *polar set* of K is $K^\circ := \{\phi \in X^* \,|\, h(\phi, K) \leq 1\}$.

The close relationship between a gauge $\gamma : X \to \mathbb{R}$ and its opposite gauge $\gamma^\vee$ yields the following formulas when combined with polarity, see [202, Theorem 15.1].

Lemma 1.9. *Let (X, γ) be a generalized Minkowski space. Then*

(a) $B_{\gamma^\vee}(0,1) = -B_\gamma(0,1)$,
(b) $B_{\gamma^\circ}(0,1) = B_\gamma(0,1)^\circ$,
(c) $(\gamma^\vee)^\circ = (\gamma^\circ)^\vee$,
(d) $(-B_\gamma(0,1))^\circ = -B_\gamma(0,1)^\circ$.

The second item of the Lemma 1.9 reveals the role of the polar set of the unit ball as the unit ball of the polar function of the gauge. In the sense of (1.1), the polar function γ° is the Minkowski functional of $B_\gamma(0,1)^\circ$. The *Hahn–Banach theorem* is a fundamental link between functional analysis and convex geometry. Its numerous appearances include norm-preserving extension of linear functionals and separation of convex sets by hyperplanes, particularly the existence of a supporting hyperplane passing through a given boundary point of a convex closed set. We give the appropriate version for generalized Minkowski spaces, taken from [56, Theorem 2.2.2].

Theorem 1.10. *Let (X, γ) be a generalized Minkowski space.*

(a) If L is a linear subspace of X and $\phi_0 : L \to \mathbb{R}$ is a linear functional on the generalized Minkowski space $(L, \gamma|_L)$, then there exists a linear functional $\phi : X \to \mathbb{R}$ such that $\phi|_L = \phi_0$ and $\gamma^\circ(\phi) = \sup\{\langle \phi_0 \,|\, x \rangle \,|\, x \in L, \gamma(x) \leq 1\}$.
(b) If $x \in X \setminus \{0\}$, then there exists a linear functional $\phi : X \to \mathbb{R}$ such that $\gamma^\circ(\phi) = 1$ and $\langle \phi \,|\, x \rangle = \gamma(x)$.

Since X is finite-dimensional, the *bidual space* $(X^*)^*$ can and will always be identified with X. In this sense, the gauge $(\gamma^\circ)^\circ$ can be defined on X and, as shown next, coincides with γ. This fact is stated for the special case of norms in [40, Corollary 1.4], see also [56, Corollary 2.2.4] for the corresponding result in asymmetric seminormed spaces. For this reason, generalized Minkowski spaces naturally come in pairs (X, γ) and (X^*, γ°).

Lemma 1.11. *In any generalized Minkowski space* (X,γ)*, we have*

$$\gamma(x) = \max\{\langle \phi \,|\, x\rangle \mid \phi \in X^*, \gamma^\circ(\phi) \leq 1\}$$

for all $x \in X$.

Proof. Fix $x \in X$. Taking the supremum over $\phi \in X^*$ with $\gamma^\circ(\phi) \leq 1$ in (1.2), we obtain

$$\gamma(x) = \sup\{\gamma^\circ(\phi)\gamma(x) \mid \phi \in X^*, \gamma^\circ(\phi) \leq 1\} \geq \sup\{\langle \phi \,|\, x\rangle \mid \phi \in X^*, \gamma^\circ(\phi) \leq 1\}.$$

By Theorem 1.10(b), there exists a functional $\phi_0 \in X^*$ such that $\gamma^\circ(\phi_0) = 1$ and $\langle \phi_0 \,|\, x\rangle = \gamma(x)$, i.e., $\gamma(x) = \langle \phi_0 \,|\, x\rangle \leq \sup\{\langle \phi \,|\, x\rangle \mid \phi \in X^*, \gamma^\circ(\phi) \leq 1\}$. □

Lemma 1.11 motivates the following terminology which is chosen as a compromise between a resemblance of the classical notion from Banach space theory and a reference to the actual gauge.

Definition 1.12. A functional $\phi \in X^*$ is called a *γ-norming functional* for $x \in X$ if $\gamma^\circ(\phi) = 1$ and $\langle \phi \,|\, x\rangle = \gamma(x)$.

Elements of the dual space and the concept of polarity serve as tools for describing the boundary structure of convex sets. For instance, the support function $h(\cdot, K)$ of a convex set $K \in \mathscr{C}^X$ encodes signed (Euclidean) distances from the origin to so-called supporting hyperplanes of K. On the other hand, sets of the form $\phi_{=\alpha}$ with $\phi \in X^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$ are precisely the hyperplanes in X. (A *half-space* is a set of the form $\phi_{\leq\alpha}$.) In this spirit, the hyperplane $\phi_{=\alpha}$ *supports* K *at* $x \in K$ if $\alpha = h(\phi, K) = \langle \phi \,|\, x\rangle$ or, equivalently, if $K \subset \phi_{\leq\alpha}$ and $x \in \phi_{=\alpha}$. In this case, the half-space $\phi_{\leq\alpha}$ is said to be a *supporting half-space* of K, and ϕ is a *outer normal* of K at x. If $\dim(X) = 2$, we refer to half-spaces and supporting hyperplanes as *half-planes* and *supporting lines*, respectively. The intersection F of K and one of its supporting hyperplanes is called an *exposed j-face* of K if $\dim(F) = j$. If $\{x\}$ is an exposed 0-face of K, we call x an *exposed point* of K. A *chord* of K, that is, a line segment joining two boundary points of K is called an *affine diameter* if its endpoints belong to distinct parallel supporting hyperplanes of K. An *extreme j-face*, $0 \leq j \leq \dim(X)$, of a closed convex set $K \subset X$ is a subset $F \subset K$ of dimension $\dim(F) = j$ such that, whenever the relative interior of a line segment $[x, y] \subset K$ meets F, then $[x, y] \subset F$ [207, Sections 1.4 and 2.1]. The point $x \in K$ is said to be an *extreme point* of K if $\{x\}$ is an extreme 0-face. According to [207, Theorem 2.1.2], every point $x \in K$ belongs to the relative interior $\mathrm{ri}(F_x)$ of a unique extreme face F_x of K. The point x is called *j-extreme* if $\dim(F_x) \leq j$. Clearly, a point x is an extreme point if and only if it is 0-extreme. The *j-skeleton* of K is the set

$$\mathrm{ext}_j(K) := \{x \in K \mid x \text{ is } j\text{-extreme}\},$$

see [207, Section 2.1]. Exposed j-faces are always extreme j-faces. The converse statement is true for $j = \dim(X) - 1$.

Lemma 1.13. *Every extreme* $(\dim(X)-1)$*-face of a set* $K \in \mathscr{C}^X$ *is exposed. In particular,*

$$\mathrm{ext}_{\dim(X)-2}(K) = \mathrm{bd}(K) \setminus \left(\bigcup_F \mathrm{ri}(F)\right),$$

where the union runs over all exposed $(\dim(X)-1)$*-faces* F *of* K.

Proof. Let F be an extreme $(\dim(X)-1)$-face of K. Assume that F is not an exposed face. Then the hyperplane $\mathrm{aff}(F)$ does not support K, and $F = \mathrm{aff}(F) \cap K$ contains interior points of K. Thus F is not an extreme face, a contradiction. By [207, Theorem 2.1.2], we know that $\mathrm{bd}(K)$ is the disjoint union of the sets $\mathrm{ri}(F)$, where F is an extreme j-face of K with $j \le \dim(X)-1$. Therefore, the set $\mathrm{ext}_{\dim(X)-2}(K)$ which is the union of the sets $\mathrm{ri}(F)$ for which F is an extreme j-face of K with $j \le \dim(X)-2$ coincides with the set of boundary points of K which do not belong the relative interior of any extreme $(\dim(X)-1)$-face of K. But extreme $(\dim(X)-1)$-faces of K are also exposed $(\dim(X)-1)$-faces of K, as we showed at the beginning of this proof. □

In order to handle the infinite variety of shapes of convex sets, it is necessary to introduce classes of convex bodies whose defining criteria are easy to describe both geometrically and analytically, and which are nonetheless meaningful for applications. This is an issue inherited from Banach space theory, whose infinite-dimensional version offers even more possibilities for fine-tuned classifications of their norms and unit balls. Two notions central to Banach space geometry regardless of the dimension are known as *smoothness* and *rotundity* which put, in some sense, the “roundness” of a convex set in concrete terms. Unsurprisingly, these notions do not depend on central symmetry. A set $K \in \mathscr{C}^X$ is called *smooth* if distinct supporting hyperplanes support K at distinct sets of points. A set $K \in \mathscr{C}^X$ is called *rotund* if every supporting hyperplane of K supports K at only one point. For an illustration, see Figure 1.1. The Hahn–Banach theorem implies that for each boundary point x of a set $K \in \mathscr{C}^X$, there is a hyperplane which supports K at x. Therefore, the set K is rotund if and only if no boundary points of K do not “share” a supporting hyperplane of K.

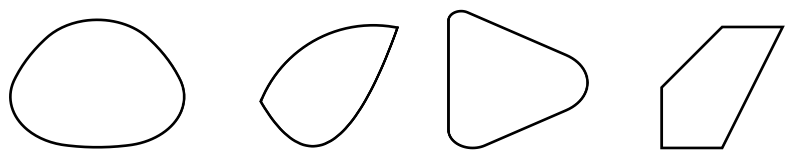


Figure 1.1. Rotundity and smoothness: A convex set may have both properties (left), neither of them (right), or exactly one of them (middle).

1.5 Optimization theory and set-valued analysis

Apart from Minkowski functionals and support functions, there are alternative ways to embed convex sets in the class of convex functions. The following examples are important in convex analysis.

Definition 1.14. The *indicator function* of a set $K \subset X$ is defined as

$$\delta(\cdot, K) : X \to \overline{\mathbb{R}}, \quad \delta(x, K) := \begin{cases} 0 & \text{if } x \in K, \\ +\infty & \text{else.} \end{cases}$$

The *distance function* of K with respect to a gauge $\gamma : X \to \mathbb{R}$ is defined as

$$\underline{\operatorname{dist}}_\gamma(\cdot, K) : X \to \overline{\mathbb{R}}, \quad \underline{\operatorname{dist}}_\gamma(x, K) := \inf\{\gamma(x-y) \mid y \in K\}.$$

One readily checks that both the indicator function and the distance function of a convex set $K \subset X$ are indeed convex functions.

Example 1.15. Let (X, γ) be a generalized Minkowski space, $x \in X$, $\phi \in X^* \setminus \{0\}$, and $\alpha \in \mathbb{R}$. Then

$$\underline{\operatorname{dist}}_{\gamma^\vee}(x, \phi_{=\alpha}) = \begin{cases} \frac{\alpha - \langle \phi \mid x \rangle}{\gamma^\circ(\phi)} & \text{if } \langle \phi \mid x \rangle \leq \alpha, \\ \frac{\langle \phi \mid x \rangle - \alpha}{\gamma^\circ(-\phi)} & \text{else,} \end{cases}$$

see [195, Theorem 1.1].

Operations which map convex functions to convex functions are of peculiar interest. Notable examples are the pointwise sum $\sum_{i \in I} f_i : X \to \overline{\mathbb{R}}$ and the pointwise supremum $\sup_{i \in I} f_i : X \to \overline{\mathbb{R}}$ of functions $f_i : X \to \overline{\mathbb{R}}$, $i \in I$, defined by $\left(\sum_{i \in I} f_i\right)(x) := \sum_{i \in I} f_i(x)$ and $(\sup_{i \in I} f_i)(x) := \sup\{f_i(x) \mid i \in I\}$, respectively, see [202, Theorems 5.2 and 5.5]. In particular, if $\operatorname{card}(I) < +\infty$, we write $\max_{i \in I} f_i$ for $\sup_{i \in I} f_i$. The *infimal convolution* $f_1 \,\square\, f_2 : X \to \overline{\mathbb{R}}$, given by $(f_1 \,\square\, f_2)(x) := \inf\{f_1(y) + f_2(x-y) \mid y \in X\}$, is another binary operation which maps convex functions to convex functions, see [202, Theorem 5.4]. A special case of the infimal convolution is given by the distance function:

$$\underline{\operatorname{dist}}_\gamma(\cdot, K) = \delta(\cdot, K) \,\square\, \gamma. \tag{1.3}$$

Pointwise multiplication of a convex function $f : X \to \overline{\mathbb{R}}$ with a constant $\lambda > 0$ yields the convex function $\lambda f : X \to \overline{\mathbb{R}}$, $(\lambda f)(x) := \lambda f(x)$.

Convexity-preserving operations on convex functions are used to set up mathematical models for *single facility location problems*. These problems ask for the optimal location of a new facility in order to minimize the costs for the transportation of goods from or to the existing facilities. Most commonly, facilities are modeled as points of the plane $\mathbb{R}^2$, and transportation costs are assumed to be a function of distances between facilities. For this, the usage of ℓ_p norms, defined by

$$\|x\|_p := \left(\sum_{i=1}^d |\xi_i|^p\right)^{1/p} \quad \text{and} \quad \|x\|_\infty := \max\{|\xi_1|, \ldots, |\xi_d|\}$$

for $x = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$ and $p \geq 1$, and so-called block norms, i.e., norms whose unit ball is a polygon, are popular. Basic single facility location problems identify transportation costs and distance with respect to a norm, and implement the combination of the transportation costs between the new facility and an existing one by taking their sum or their maximum. In mathematical terms, it is required to solve scalar optimization problems of the form

$$\inf_{x \in \mathbb{R}^2} \sum_{i=1}^n \|p_i - x\| \tag{1.4}$$

and

$$\inf_{x \in \mathbb{R}^2} \max_{i \in \{1, \ldots, n\}} \|p_i - x\| \tag{1.5}$$

where $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ is a norm, and $p_i \in \mathbb{R}^2$ is the location of the ith existing facility. The variable $x \in \mathbb{R}^2$ models the location new facility, and minimizers of the objective function are optimal locations. Problem (1.4) is known as *minsum location problem*, *median problem*, *Weber problem*, or *Fermat–Torricelli problem*, whereas problem (1.5) is usually called *minimax location problem*, *center problem*, or *Sylvester problem*. A great number of variants and generalizations of these problems emerged since their proposals. In Chapters 3 and 6, we address the problems

$$\inf_{x\in X} \left\{ \delta(x, K_0) + \sum_{i=1}^{n} \underline{\operatorname{dist}}_{\gamma_i}(x, K_i) \right\} \tag{1.6}$$

and

$$\inf_{x\in X} \sup_{y\in K} \gamma(y - x) \tag{1.7}$$

where $K, K_0, \ldots, K_n \in \mathscr{C}^X$ are closed convex sets and $\gamma, \gamma_1, \ldots, \gamma_n : X \to \mathbb{R}$ are gauges. However, we leave the construction of algorithms for numerically solving scalar optimization problems with convex objective functions to researchers working in convex optimization. Instead, we will study the geometry of the sublevel sets of the objective functions of (1.6) and (1.7). If the values of a convex function $f : X \to \overline{\mathbb{R}}$ are bounded below, then for $\varepsilon \geq 0$, its set of *ε-minimizers* $\{x \in X \mid f(x) \leq f(y) + \varepsilon \;\forall\, y \in X\}$ is a non-empty sublevel set of f and it can be characterized using appropriate ε-versions of the (Fenchel) subdifferential and the (Gâteaux) directional derivative. We recall the definitions from [243, p. 82 and Theorem 2.1.14].

Definition 1.16. Let $f : X \to \overline{\mathbb{R}}$, $x \in X$, and $\varepsilon \geq 0$.

(a) If $f(x) \in \mathbb{R}$, the *ε-subdifferential* of f at x is the set

$$\partial_\varepsilon f(x) := \{\phi \in X^* \mid \langle \phi \mid y - x\rangle \leq f(y) - f(x) + \varepsilon \;\forall\, y \in X\}.$$

Else we define $\partial_\varepsilon f(x) := \emptyset$.

(b) The *ε-directional derivative* of f at x is given by

$$f'_\varepsilon(x; \cdot) : X \to \overline{\mathbb{R}}, \quad f'_\varepsilon(x; y) := \inf_{\lambda>0} \frac{f(x + \lambda y) - f(x) + \varepsilon}{\lambda}.$$

If $\varepsilon = 0$, we omit it from the notation. If $f'(x; \cdot)$ is a linear functional, i.e., there exists $\nabla f(x) \in X^*$ such that $f'(x; y) = \langle \nabla f(x) \mid y\rangle$ for all $y \in X$, we say that f is *Gâteaux differentiable* at x and call $\nabla f(x)$ the *Gâteaux derivative* of f at x.

As an example, we compute the ε-subdifferential of gauges and indicator functions, cf. [243, Theorem 2.4.14].

Example 1.17. Let (X, γ) be a generalized Minkowski space, $x \in X$, and $\varepsilon \geq 0$. Then

$$\partial_\varepsilon \gamma(x) = \begin{cases} \{\phi \in X^* \mid \gamma^\circ(\phi) \leq 1\} = B(0,1)^\circ & \text{if } x = 0, \\ \{\phi \in X^* \mid \langle \phi \mid x\rangle \geq \gamma(x) - \varepsilon, \gamma^\circ(\phi) \leq 1\} & \text{else.} \end{cases} \tag{1.8}$$

Given a set $K \subset X$ and a point $x \in K$, we shall use the notation

$$\operatorname{nor}_\varepsilon(x, K) := \partial_\varepsilon \delta(x, K) = \{\phi \in X^* \mid \langle \phi \mid y - x\rangle \leq \varepsilon \;\forall\, y \in K\}.$$

For $\varepsilon = 0$, the set $\operatorname{nor}(x, K) := \operatorname{nor}_0(x, K)$ is a cone (with apex 0) called the *normal cone* of K at x.

A mild assumption on functions $f : X \to \overline{\mathbb{R}}$ is *properness*, i.e., the property that $f(x) > -\infty$ for all $x \in X$ together with the nonemptiness of the *domain* $\mathrm{dom}(f) := \{x \in X \mid f(x) < +\infty\}$. It is known that proper and convex functions $f : X \to \overline{\mathbb{R}}$ are continuous on $\mathrm{ri}(\mathrm{dom}(f))$ and their ε-subdifferentials $\partial_\varepsilon f(x)$ are non-empty, closed, and convex at each point $x \in X$, see [243, Theorems 2.1.5(v), 2.4.2(i), and 2.4.9]. For these functions, Gâteaux differentiability can also be expressed in terms of subdifferentials, see [243, Corollary 2.4.10].

Lemma 1.18. *Let $f : X \to \overline{\mathbb{R}}$ be a proper and convex function which is continuous at $x \in \mathrm{dom}(f)$. Then f is Gâteaux differentiable at x if and only if $\partial f(x)$ is a singleton. In this situation, the unique element of $\partial f(x)$ is the Gâteaux derivative of f at x.*

Furthermore, there is a characterization of ε-minimizers as announced before, cf. [20, Theorem 16.3 and Proposition 17.14], [110, Theorem XI.1.1.5], [202, p. 264], or [243, Theorem 2.4.4(i)].

Lemma 1.19. *Let $f : X \to \mathbb{R}$ be a proper and convex function, $x \in \mathrm{dom}(f)$, and $\varepsilon \geq 0$. The following statements are equivalent:*

(a) $f(x) \leq f(y) + \varepsilon$ *for all* $y \in X$,
(b) $0 \in \partial_\varepsilon f(x)$,
(c) $f'_\varepsilon(x; y) \geq 0$ *for all* $y \in X$.

For $\varepsilon = 0$, Lemma 1.19 gives a characterization of 0-minimizers of a convex function $f : X \to \overline{\mathbb{R}}$ in terms of the subdifferential. While for $\varepsilon > 0$, ε-minimizers of a bounded below function f always exist, the existence of a 0-minimizer of a convex function is not automatic. However, it is guaranteed under the additional assumptions of *coercitivity*, i.e., the boundedness of the sublevel set $f_{\leq \alpha}$ independently of $\alpha \in \mathbb{R}$, and *lower semicontinuity*, i.e., the closedness of the sublevel set $f_{\leq \alpha}$ independently of $\alpha \in \mathbb{R}$, cf. [20, Theorem 11.10] as well as [243, Exercise 1.15 and Theorem 2.5.1]. For proper, convex, and lower semicontinuous functions $f : X \to \mathbb{R}$, the ε-directional derivative and the ε-subdifferential are linked via

$$\partial_\varepsilon f(x) = \{\phi \in X^* \mid \langle \phi \mid y \rangle \leq f'(x; y) \; \forall \, y \in X\}$$

and

$$f'_\varepsilon(x; y) = \sup\{\langle \phi \mid y \rangle \mid \phi \in \partial_\varepsilon f(x)\}, \tag{1.9}$$

see [243, Theorems 2.4.4(i) and 2.4.9] or [110, Theorem X.1.1.4]. Substituting $-y$ for y in (1.9) and using the closedness and convexity of $\partial_\varepsilon f(x)$, one concludes

$$\{\langle \phi \mid y \rangle \mid \phi \in \partial_\varepsilon f(x)\} = [-f'(x; -y), f'(x; y)]. \tag{1.10}$$

For a proper, convex, and lower semicontinuous function $f : X \to \mathbb{R}$, there is also a link between its ε-subdifferential and its conjugate function

$$f^* : X^* \to \overline{\mathbb{R}}, \quad f^*(\phi) := \sup\{\langle \phi \mid x \rangle - f(x) \mid x \in X\},$$

and a result known as *Fenchel–Moreau theorem* on the *biconjugate function* $f^{**} : X \to \overline{\mathbb{R}}$ which is given by $f^{**}(x) := \sup\{\langle \phi \mid x \rangle - f^*(\phi) \mid \phi \in X^*\}$, see [202, Theorems 12.2 and 23.5] and [243, Theorems 2.3.3 and 2.4.2(ii)]. (Note that the definition of the conjugate function is sensible for all functions $f : X \to \overline{\mathbb{R}}$.)

Lemma 1.20. *Let $f : X \to \overline{\mathbb{R}}$ be a proper, convex, and lower semicontinuous function and let $\varepsilon \geq 0$. Then $f(x) + f^*(\phi) \leq \langle \phi \,|\, x \rangle + \varepsilon$ if and only if $\phi \in \partial_\varepsilon f(x)$. Furthermore, we have $f^{**} = f$.*

Basic examples for conjugate functions are as follows.

Example 1.21. (a) Let $K \subset X$. Then $\delta(\cdot, K)^* = h(\cdot, K)$.
(b) Let (X, γ) be a generalized Minkowski space. Let $g : \mathbb{R} \to \mathbb{R}$ be an even function, i.e., $g(\alpha) = g(-\alpha)$ for all $\alpha \in \mathbb{R}$. Then $(g \circ \gamma)^* = g^* \circ \gamma^\circ$, cf. [20, Example 13.8]. Indeed,

$$\begin{aligned}
(g \circ \gamma)^*(\phi) &= \sup_{x \in X} \{\langle \phi \,|\, x \rangle - g(\gamma(x))\} \\
&= \sup_{\lambda \geq 0} \sup_{x \in S(0,1)} \{\langle \phi \,|\, \lambda x \rangle - g(\gamma(\lambda x))\} \\
&= \sup_{\lambda \geq 0} \sup_{x \in S(0,1)} \{\lambda \langle \phi \,|\, x \rangle - g(\lambda \gamma(x))\} \\
&= \sup_{\lambda \geq 0} \{\lambda \gamma^\circ(\phi) - g(\lambda)\} \\
&= \sup_{\lambda \in \mathbb{R}} \{\lambda \gamma^\circ(\phi) - g(\lambda)\} \\
&= g^*(\gamma^\circ(\phi)).
\end{aligned}$$

(c) In particular, we have $\gamma^* = \delta(\cdot, B_{\gamma^\circ}(0,1))$ and $(\frac{1}{2}\gamma^2)^* = \frac{1}{2}(\gamma^\circ)^2$.

For applying Lemma 1.19 to composed functions, it is promising to have tractable formulas for the conjugate and the ε-subdifferential, preferably such formulas in which the conjugates and ε-subdifferentials of the single functions are evaluated separately. This may sometimes require certain regularity conditions. For multiplication of a function with a positive constant, translations of arguments, and pointwise addition of a linear functional, the following rules can be found in [243, Theorems 2.3.1 and 2.4.2(vi)].

Lemma 1.22. *Let $f, g : X \to \overline{\mathbb{R}}$, $x, x_0 \in X$, $\phi, \phi_0 \in X^*$, and $\lambda > 0$. Then*

(a) $(\lambda f)^*(\phi) = \lambda f^*(\frac{1}{\lambda}\phi)$, $\partial_\varepsilon(\lambda f)(x) = \lambda \partial_{\frac{\varepsilon}{\lambda}} f(x)$,
(b) $f(\cdot + x_0)^*(\phi) = f^*(\phi) - \langle \phi \,|\, x_0 \rangle$, $\partial_\varepsilon f(\cdot + x_0)(x) = \partial_\varepsilon f(x + x_0)$,
(c) $(f + \phi_0)^*(\phi) = f^*(\phi - \phi_0)$, $\partial_\varepsilon(f + \phi_0)(x) = \phi_0 + \partial_\varepsilon f(x)$.

Under conjugation, the sum of convex functions turns into the infimal convolution of the single conjugates. For $\varepsilon = 0$, the subdifferential of a sum of convex functions is the sum of the subdifferentials of the summands, see [243, Theorem 2.8.7]. This will be important in Chapter 6, namely for the investigation of functions of the form $\sum_{i=1}^n \underline{\mathrm{dist}}_{\gamma_i}(\cdot, K_i) : X \to \mathbb{R}$, where $\gamma_1, \ldots, \gamma_n : X \to \mathbb{R}$ are gauges and $K_1, \ldots, K_n \in \mathscr{C}^X$.

Theorem 1.23. *Let $f, g : X \to \overline{\mathbb{R}}$ be convex and proper functions. If there exists $x_0 \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ such that g is continuous at x_0, then*

$$(f + g)^*(\phi) = (f^* \,\square\, g^*)(\phi)$$

and

$$\partial_\varepsilon(f+g)(x) = \bigcup_{\varepsilon_1\in[0,\varepsilon]} (\partial_{\varepsilon_1} f(x) + \partial_{\varepsilon-\varepsilon_1} g(x))$$

for all $\phi \in X^*$, $x \in \operatorname{dom}(f+g)$, *and* $\varepsilon \geq 0$.

Although scalar optimization problems with maximum objective will play a major role in Chapter 3, its investigation by means of conjugate duality and variational analysis will appear only implicitly in Lemma 3.9. Explicit accounts in these directions for variants of the optimization problem (1.7) are [179] and [240, Chapter 4]. Due to Lemma 1.19, ε-subdifferentials yield dual characterizations of sublevel sets of convex functions. In the case of functions of the form $\sup_{y\in K} \gamma(y-\cdot) : X \to \mathbb{R}$, where (X,γ) is a generalized Minkowski space and $K \in \mathscr{K}^X$, sublevel sets are intersections of balls of equal radii. These sets will be studied in detail in Chapter 4 without any subdifferential calculus. For the sake of completeness, we refer to [98] for a subdifferential formula for the supremum of infinitely many convex functions, and we cite the case $\varepsilon = 0$ of [243, Corollary 2.8.11] to show what subdifferential formulas for the supremum of finitely many convex functions look like.

Theorem 1.24. *Let* $f_1,\dots,f_n : X \to \overline{\mathbb{R}}$ *be convex and proper functions. For every point* $x \in \operatorname{dom}(\max\{f_1,\dots,f_n\})$, *denote by* $I(x)$ *the set* $\{i \in \{1,\dots n\} \mid f_i(x) = \max\{f_1,\dots,f_n\}(x)\}$. *Then, if* $\bigcap_{i=1}^n \operatorname{dom}(f_i) \neq \emptyset$, *we have*

$$\max\{f_1,\dots,f_n\}^*(\phi) = \min\left\{ \left(\sum_{i=1}^n \lambda_i f_i\right)^*(\phi) \,\middle|\, \lambda_1,\dots,\lambda_n \geq 0, \sum_{i=1}^n \lambda_i = 1 \right\}$$

and

$$\partial \max\{f_1,\dots,f_n\}(x) = \bigcup_{\substack{\lambda_1,\dots,\lambda_n\geq 0:\\ \sum_{i=1}^n \lambda_i = 1,\\ \lambda_i = 0\ \forall\, i\notin I(x)}} \partial\left(\sum_{i=1}^n \lambda_i f_i\right)(x)$$

for all $\phi \in X^*$ *and* $x \in \operatorname{dom}(\max\{f_1,\dots,f_n\})$.

Conjugates and subdifferentials of infimal convolutions of convex functions will also play a role in Chapters 2 and 6, as there will appear distance functions of convex sets, cf. (1.3). The following result is adapted from [243, Theorem 2.3.1(ix) and Corollary 2.4.7].

Theorem 1.25. *Let* $f_1, f_2 : X \to \overline{\mathbb{R}}$ *convex and proper functions. Then* $(f_1 \square f_2)^* = f_1^* + f_2^*$. *Furthermore, if there exist* $x_1 \in \operatorname{dom}(f_1)$ *and* $x_2 \in \operatorname{dom}(f_2)$ *such that* $(f_1 \square f_2)(x_1+x_2) = f_1(x_1) + f_2(x_2)$, *we have*

$$\partial_\varepsilon(f_1 \square f_2)(x_1+x_2) = \bigcup_{\varepsilon_1\in[0,\varepsilon]} \left(\partial_{\varepsilon_1} f_1(x_1) \cap \partial_{\varepsilon-\varepsilon_1} f_2(x_2)\right).$$

Convex functions have the convenient property that local minimizers are always global ones. In contrast that, maximizers of convex functions $f : X \to \overline{\mathbb{R}}$ are only considered relative to a subset K of $\operatorname{dom}(f)$, that is, it is required to find the supremum $\sup\{f(x) \mid x \in K\}$. One readily checks that $\sup\{f(x) \mid x \in K\} = \sup\{f(x) \mid x \in \operatorname{co}(K)\}$, i.e., there is no loss of generality in assuming that K is convex. The following result is taken from [202, Corollary 32.3.2].

Proposition 1.26. *If $f : X \to \overline{\mathbb{R}}$ is a convex function and $K \in \mathscr{K}^X$ a subset of* $\mathrm{dom}(f)$*, then the supremum* $\sup\{f(x) \mid x \in K\}$ *is finite, and it is attained at some extreme point of K.*

In particular, the set $\arg\max_{x\in K} f(x) := \{z \in K \mid f(z) \geq f(x)\ \forall\, x \in K\}$ is a union of extremal faces of K when $f : X \to \overline{\mathbb{R}}$ is a convex function. For $f = \phi \in X^*$ a linear functional, this means that hyperplanes may support non-empty convex compact sets K only at boundary points. Furthermore, if there exists a point $z \in \mathrm{ri}(K) \cap \arg\max_{x\in K} f(x)$, then $f(x)$ is constant as x traverses K.
Finally, the subdifferential ∂f of a function $f : X \to \mathbb{R}$ is an example for a *set-valued operator* or a *multifunction*, see [243, p. 12] for basic terminology. In this thesis, all set-valued operators will be of the form $A : X \to 2^{X^*}$, abbreviated as $A : X \rightrightarrows X^*$. This means that A assigns to each element $x \in X$ an element of the power set 2^{X^*} of X^*, i.e., a subset of X^*. The *graph* and the *set of zeros* of a set-valued operator $A : X \rightrightarrows X^*$ are the sets

$$\begin{aligned} \mathrm{Graph}(A) &:= \{(x,\phi) \in X \times X^* \mid \phi \in A(x)\}, \\ \mathrm{Zer}(A) &:= \{x \in X \mid 0 \in A(x)\}, \end{aligned}$$

respectively. For instance, a point $x \in X$ is an ε-minimizer of a proper and convex function $f : X \to \overline{\mathbb{R}}$ if and only if $x \in \mathrm{Zer}(\partial_\varepsilon f)$. The *inverse* $A^{-1} : X^* \rightrightarrows X$ of A is given by its graph $\mathrm{Graph}(A^{-1}) = \{(\phi, x) \in X^* \times X \mid \phi \in A(x)\}$.
A set-valued operator $A : X \rightrightarrows X^*$ is said to be *maximally monotone* if $(x_1, \phi_1) \in \mathrm{Graph}(A)$ if and only if, for all $(x_2, \phi_2) \in \mathrm{Graph}(A)$, we have $\langle \phi_1 - \phi_2 \mid x_1 - x_2 \rangle \geq 0$. The following result is taken from [202, Theorem 24.9] and [243, Theorem 2.4.4(iv)].

Lemma 1.27. *Let $f : X \to \overline{\mathbb{R}}$ be a proper, convex, and lower semicontinuous function, and let $\varepsilon \geq 0$. Then $\partial f : X \rightrightarrows X^*$ is a maximally monotone operator and $(\partial_\varepsilon f)^{-1} = \partial_\varepsilon f^*$.*

The formula for the inverse of the subdifferential will appear implicitly throughout Chapter 2, and the maximal monotonicity of subdifferentials is the main ingredient of the proof of Theorem 2.29.

2

Birkhoff orthogonality and best approximation

Motivated by the versatility of orthogonality in the theory of inner-product spaces, mathematicians introduced various generalizations of the notion of orthogonality for non-Hilbert spaces, see [5]. The most popular one is named after Birkhoff [29], although there are earlier papers on this orthogonality type by Radon [199] and Blaschke [31]. Its usage in the setting of normed spaces reaches from angular measures [49], approximation theory [108], curve theory [156, 216], orthocentric systems [186], matrix theory [22, 28, 147, 204], and orthogonal decompositions of Banach spaces [3, 30, 130] to random processes [205, Section 2.9].
Following the presentation of the results in [121, Section 3], we introduce an ε-version of Birkhoff orthogonality in generalized Minkowski spaces.

Definition 2.1. Let (X,γ) be a generalized Minkowski space. We say that the point $x \in X$ is *ε-Birkhoff orthogonal to* $y \in X$ and write $x \perp_B^{\varepsilon} y$ if $\gamma(x) \leq \gamma(x+\lambda y)+\varepsilon$ for all $\lambda \in \mathbb{R}$. If $\varepsilon = 0$, we shall omit ε from the notation and simply refer to it as *Birkhoff orthogonality*.

Here is an example.

Example 2.2. For instance, take $X = \mathbb{R}^2$ and consider the gauge $\gamma : X \to \mathbb{R}$,

$$\gamma(\xi_1,\xi_2) := \begin{cases} |\xi_1|+|\xi_2| & \text{if } \xi_1,\xi_2 > 0, \\ \sqrt{\xi_1^2+\xi_2^2} & \text{else.} \end{cases}$$

Then, for $x = (2.5, 0)$, $y_1 = (-1.5, 1)$, $y_2 = (-1.5, -1)$, and $\varepsilon = 1$, we have $x \perp_B^{\varepsilon} y_1$ but not $x \perp_B^{\varepsilon} y_2$, see Figure 2.1 for an illustration.

Our approach to approximate Birkhoff orthogonality extends the one in [97, 100]. Another approach used in the literature is methodically closer to so-called *semi-inner products* which serve as a substitute notion for the inner product, see [51, 52, 68, 147], and [134, 177] for applications. When γ is a norm, then Birkhoff orthogonality is known to be intertwined with the *best approximation problem*. This problem asks for the evaluation of the distance function, which is in our case $\underline{\mathrm{dist}}_{\gamma^\vee}(\cdot,K) : X \to \mathbb{R}$,

$$\underline{\mathrm{dist}}_{\gamma^\vee}(x,K) = \inf\{\gamma(y-x) \mid y \in K\}, \tag{2.1}$$

where $x \in X$ is a given point and $K \subset X$ is a given set. Using an analogous version of best approximation problems, Zaustinsky [245, Section 8] introduces an orthogonality relation of straight lines in the setting of general metric spaces (X,ρ). Because of the possible asymmetry of general metrics, a distinction between the distance from a set $K \subset X$ to a point $x \in X$ and the

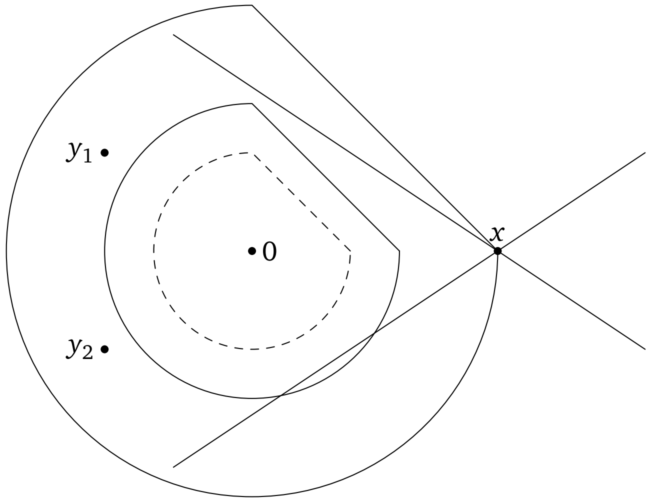


Figure 2.1. Illustration of Example 2.2: For the unique gauge γ on $X = \mathbb{R}^2$ whose unit ball is depicted in dashed lines, we have $x \perp_B^\varepsilon y_1$ but not $x \perp_B^\varepsilon y_2$ for the given choice of ε.

distance from a point $x \in X$ to set $K \subset X$ are made. Via $\rho(x, y) = \gamma(x - y)$, this corresponds to the evaluation of $\underline{\mathrm{dist}}_\gamma(x, K)$ and $\underline{\mathrm{dist}}_{\gamma^\vee}(x, K)$ in our setting, respectively. A best approximation of $x \in X$ in $K \subset X$ is then called a *foot on K toward x* or *from x* in [245], depending on which distance is used.

Techniques from convex analysis and convex optimization such as ε-subdifferential calculus have been applied to the convex optimization problem (2.1) successfully when $\gamma : X \to \mathbb{R}$ is a norm and $K \subset X$ is a convex set. For instance, closedness and convexity of K are in that case sufficient for the existence of solutions of (2.1), i.e., for the existence of points $y \in K$ such that $\gamma(y - x) = \underline{\mathrm{dist}}_{\gamma^\vee}(x, K)$, see [243, Theorem 3.8.1]. In this chapter, we extend such convex-analytic aspects of best approximation problems and Birkhoff orthogonality to generalized Minkowski spaces. These tools will help us examine “how much” ε-Birkhoff orthogonality resembles usual Euclidean orthogonality. In the theory of abstract orthogonality notions in Minkowski spaces, this is done by checking the presence of the following properties for a given binary relation $\perp \subset X \times X$:

(a) Nondegeneracy: For all $x \in X$ and $\lambda, \mu \in \mathbb{R}$, we have $\lambda x \perp \mu x$ if and only if $\lambda\mu x = 0$.
(b) Symmetry: For all $x, y \in X$, the relation $x \perp y$ implies $y \perp x$.
(c) Right additivity: For all $x, y, z \in X$, the relations $x \perp y$ and $x \perp z$ taken together imply $x \perp (y + z)$.
(d) Left additivity: For all $x, y, z \in X$, the relations $x \perp z$ and $y \perp z$ taken together imply $(x + y) \perp z$.
(e) Right homogeneity: For all $x, y \in X$ and $\lambda > 0$, the relation $x \perp y$ implies $x \perp \lambda y$.
(f) Left homogeneity: For all $x, y \in X$ and $\lambda > 0$, the relation $x \perp y$ implies $\lambda x \perp y$.
(g) Right existence: For all $x, y \in X$, there exists a number $\alpha \in \mathbb{R}$ such that $x \perp (\alpha x + y)$.
(h) Left existence: For all $x, y \in X$, there exists a number $\alpha \in \mathbb{R}$ such that $(\alpha x + y) \perp x$.

The nondegeneracy property fails for ε-Birkhoff orthogonality when $\varepsilon > 0$: For $x \in X$ and $\lambda, \mu \in \mathbb{R}$, we have that $\lambda x \perp_B^\varepsilon \mu x$ if and only if $\gamma(\lambda x) \le \varepsilon$ or $\mu = 0$. But ε-Birkhoff orthogonality possesses the following homogeneity property: For $x, y \in X$, $\lambda > 0$, and $\mu \in \mathbb{R}$, we have that $x \perp_B^{\varepsilon} y$ implies $\lambda x \perp_B^{\lambda\varepsilon} \mu y$. The remainder of this section is subdivided into three parts which address existence properties of ε-Birkhoff orthogonality, additivity of Birkhoff orthogonality, and symmetry of ε-Birkhoff orthogonality, respectively.

2.1 Dual characterizations

Since various numerical methods for solving convex optimization problems can be derived from inclusion problems like $0 \in \partial f(x)$, rephrasing optimality (in the sense of minimizing a certain convex function $f : X \to \overline{\mathbb{R}}$) via Lemma 1.19 is central to convex optimization. As subdifferentials are subsets of the dual space X^*, whose elements are continuous linear functionals $\phi : X \to \mathbb{R}$, Fermat's rule provides a *dual characterization* of solutions of convex optimization problems. Another family of set-valued operators which are of interest for giving analytic descriptions of the geometry of normed spaces is given by the so-called *duality mappings* $J_f : X \rightrightarrows X^*$, see, e.g., [41], [54, Chapters I and II], or [243, Section 3.7]. In the following, we will show how ε-Birkhoff orthogonality relates to convex optimization problems and certain proximality notions and give dual descriptions thereof. Some of our results require $\varepsilon = 0$. In such a case, their novelty compared to the existing literature lies in the usage of gauges instead of norms. In this spirit, we start by introducing duality mappings in generalized Minkowski spaces (X, γ), see also [56, Section 2.4.7] for a related discussion in the context of asymmetric moduli of rotundity and smoothness.

Definition 2.3. Let (X, γ) be a generalized Minkowski space, and let $f : [0, +\infty) \to [0, +\infty)$ be a *weight*, i.e., a continuous, non-decreasing, and non-negative function. The *duality mapping with weight* f is the set-valued operator $J_f\gamma : X \rightrightarrows X^*$,

$$J_f\gamma(x) := \{\phi \in X^* \mid \langle \phi \mid x \rangle \geq f(\gamma(x))\gamma(x), \gamma^\circ(\phi) \leq f(\gamma(x))\}.$$

Remark 2.4. (a) Our notation alters the classical one, cf. [41], in order to emphasize the dependency on γ.

(b) If $\gamma = \|\cdot\|$ is a norm, then $\gamma^\circ = \|\cdot\|_*$ is the dual norm. Therefore, Definition 2.3 is an extension of the classical notion.

(c) In the literature, the weight $f : [0, +\infty) \to [0, +\infty)$ is assumed to be strictly increasing and to meet the additional requirements $f(0) = 0$ and $\lim_{\alpha \to +\infty} f(\alpha) = +\infty$. In this case, we would have $J_f\gamma(0) = \{0\} \neq \partial\gamma(0)$ independently of f and γ. With our definition, we have $J_f\gamma = \partial\gamma$ when $f(\alpha) = 1$ for all $\alpha \geq 0$.

When $f : [0, +\infty) \to [0, +\infty)$ is the identity, Ciorănescu [54, Chapters I and II] refers to $J_f(x)$ as the *normalized duality mapping* at x. Peypouquet [192, Section 1.1.2] uses this term for $\partial \|\cdot\|(x)$ at non-zero points $x \in X$. In any case, normalization is not important as duality mappings turn out to be rescalings of each other at non-zero points $x \in X$. The following result extends [54, Theorem I.4.4] to generalized Minkowski spaces.

Theorem 2.5. *Let $f : [0, +\infty) \to [0, +\infty)$ be a weight, and let $g : [0, +\infty) \to [0, +\infty)$, $g(\alpha) := \int_0^\alpha f(\xi)\mathrm{d}\xi$. Then g is a non-decreasing convex function, $g \circ \gamma : X \to \mathbb{R}$ is a convex function, and $J_f\gamma(x) = \partial(g \circ \gamma)(x)$ for all $x \in X$.*

Proof. The convexity of g is a consequence of the fundamental theorem of calculus, see [243, Theorem 2.1.7]. Furthermore, the function g is Gâteaux differentiable and its right derivative is the non-negative function $g'(\cdot;1)=f$ which means that g is non-decreasing. By [243, Theorem 2.1.3(vi)], we know that $g\circ\gamma$ is a convex function. The chain rule for subdifferentials [243, Theorem 2.8.10] yields

$$\begin{aligned}\partial(g\circ\gamma)(x) &= \{\alpha\phi \mid \alpha\in\partial g(\gamma(x)), \phi\in\partial\gamma(x)\}\\ &= g'(\gamma(x);1)\partial\gamma(x) = f(\gamma(x))\partial\gamma(x) = J_f\gamma(x),\end{aligned}$$

which completes the proof. $\square$

A consequence of Theorem 2.5 is that $f_2(\gamma(x))J_{f_1}(x) = f_1(\gamma(x))J_{f_2}(x)$ for all $x\in X$ and for all weights $f_1, f_2 : [0,+\infty)\to[0,+\infty)$, cf. [54, Proposition I.4.7(f)].

By definition, Birkhoff orthogonality is linked to the best approximation problem (2.1) for K a straight line. More precisely, we have $x \perp_B (\alpha x + y)$ if and only if x is a minimizer of the function $f := \gamma + \delta(\cdot, x + \mathrm{lin}(\{\alpha x + y\})) : X \to \overline{\mathbb{R}}$. Using Lemma 1.19 and Theorem 1.23, we obtain $0 \in \partial f(x) = \partial\gamma(x) + \mathrm{nor}(x, x+\mathrm{lin}(\{\alpha x+y\})) = \partial\gamma(x) + \mathrm{lin}(\{\alpha x + y\})^\perp$. Now, if $x\neq 0$, this is equivalent to the existence of a linear functional $\phi\in X^*$ such that $\gamma^\circ(\phi)=1$, $\gamma(x) = \langle\phi\,|\,x\rangle$, and $\langle\phi\,|\,\alpha x + y\rangle = 0$, see [70, Remark 15] and [128, Corollary 2.2] for the corresponding version in normed spaces. Alternatively, we have $x\perp_B(\alpha x+y)$ if and only if 0 is a minimizer of the function $f:\mathbb{R}\to\mathbb{R}$, $f(\lambda) := \gamma(x+\lambda(\alpha x+y))$. Necessary and sufficient conditions for its ε-minimizers can be derived analogously, as we will show in our next result. Note that given $x\in X$ and $\varepsilon\geq\gamma(x)$, we have $x\perp_B^\varepsilon y$ for all $y\in X$. Furthermore, (1.8) gives $\partial_\varepsilon\gamma(0) = B(0,1)^\circ$ independently of ε. Therefore, the restriction to $0\leq\varepsilon<\gamma(x)$ is justified when asking for statements which link ε-subdifferentials to ε-Birkhoff orthogonality.

Theorem 2.6. *Let (X,γ) be a generalized Minkowski space. Furthermore, let $x,y\in X$, $\alpha\in\mathbb{R}$, $0\leq\varepsilon<\gamma(x)$, and define $f:\mathbb{R}\to\mathbb{R}$, $f(\lambda):=\gamma(x+\lambda(\alpha x+y))$. The following statements are equivalent:*

(a) $x\perp_B^\varepsilon(\alpha x+y)$,
(b) $f(0)\leq f(\lambda)+\varepsilon$ *for all* $\lambda\in\mathbb{R}$,
(c) *there exists* $\phi\in X^*$ *such that* $\gamma^\circ(\phi)=1$, $\langle\phi\,|\,x\rangle\geq\gamma(x)-\varepsilon$, *and* $\alpha = -\frac{\langle\phi\,|\,y\rangle}{\langle\phi\,|\,x\rangle}$,
(d) $\gamma'_\varepsilon(x;\pm(\alpha x+y))\geq 0$.

Proof. The equivalence (a)$\Longleftrightarrow$(b) is a direct consequence of the definition. The equivalences (b)$\Longleftrightarrow$(c)$\Longleftrightarrow$(d) follow from Lemma 1.19. To this end, we show that (c) and (d) are reformulations of the conditions $0\in\partial_\varepsilon f(0)$ and $f'_\varepsilon(0;\eta)\geq 0$ for all $\eta\in\mathbb{R}$, respectively. First, the chain rule for subdifferentials [243, Theorem 2.8.10] yields

$$\begin{aligned}\partial_\varepsilon f(\lambda) &= \bigcup_{\substack{\varepsilon_1\in[0,\varepsilon],\\ \phi\in\partial_{\varepsilon-\varepsilon_1}\gamma(x+\lambda(\alpha x+y))}} \partial_{\varepsilon_1}\langle\phi\,|\,x+\cdot(\alpha x+y)\rangle(\lambda)\\ &= \bigcup_{\substack{\varepsilon_1\in[0,\varepsilon],\\ \phi\in\partial_{\varepsilon-\varepsilon_1}\gamma(x+\lambda(\alpha x+y))}} \{\langle\phi\,|\,\alpha x+y\rangle\}\end{aligned}$$

$$= \{\langle \phi \,|\, \alpha x + y\rangle \mid \phi \in \partial_\varepsilon \gamma(x + \lambda(\alpha x + y))\},$$

where we use that $g : \mathbb{R} \to \mathbb{R}$, $g(\lambda) := \langle \phi \,|\, x + \lambda(\alpha x + y)\rangle$ is an affine function. Taking (1.9) into account, we obtain

$$\begin{aligned} f'_\varepsilon(\lambda;\eta) &= \sup\{\mu\eta \mid \mu \in \partial_\varepsilon f(\lambda)\} \\ &= \sup\{\langle \phi \,|\, \alpha x + y\rangle\, \eta \mid \phi \in \partial_\varepsilon \gamma(x + \lambda(\alpha x + y))\} \\ &= \sup\{\langle \phi \,|\, \eta(\alpha x + y)\rangle \mid \phi \in \partial_\varepsilon \gamma(x + \lambda(\alpha x + y))\} \\ &= \gamma'_\varepsilon(x + \lambda(\alpha x + y), \eta(\alpha x + y)). \end{aligned}$$

In particular,

$$\begin{aligned} f'_\varepsilon(0;\eta) &= \gamma'_\varepsilon(x;\eta(\alpha x + y)), \\ \partial_\varepsilon f(0) &= \{\langle \phi \,|\, \alpha x + y\rangle \mid \phi \in \partial_\varepsilon \gamma(x)\}. \end{aligned}$$

Taking the positive homogeneity of the ε-directional derivative in the second variable into account, it is sufficient to consider $\eta = \pm 1$ for checking whether $f'_\varepsilon(0;\eta) \geq 0$ for all $\eta \in \mathbb{R}$. Moreover, we have

$$\begin{aligned} & 0 \in \partial_\varepsilon f(0) \\ \Longleftrightarrow\ & \text{there exists } \phi \in \partial_\varepsilon \gamma(x) \text{ such that } \langle \phi \,|\, \alpha x + y\rangle = 0 \\ \Longleftrightarrow\ & \text{there exists } \phi \in X^* \text{ such that } \gamma^\circ(\phi) \leq 1, \langle \phi \,|\, x\rangle \geq \gamma(x) - \varepsilon, \alpha = -\frac{\langle \phi \,|\, y\rangle}{\langle \phi \,|\, x\rangle} \\ \Longleftrightarrow\ & \text{there exists } \phi \in X^* \text{ such that } \gamma^\circ(\phi) = 1, \langle \phi \,|\, x\rangle \geq \gamma(x) - \varepsilon, \alpha = -\frac{\langle \phi \,|\, y\rangle}{\langle \phi \,|\, x\rangle}. \end{aligned}$$

This completes the proof. □

For $\varepsilon = 0$, the characterization of Birkhoff orthogonality in terms of directional derivatives given in Theorem 2.6(a)⟺(d) can be rewritten in a form resembling [128, Theorem 3.2].

Corollary 2.7. *Let (X,γ) be a generalized Minkowski space, $x, y \in X$, and $\alpha \in \mathbb{R}$. Then $x \perp_B (\alpha x + y)$ if and only if*

$$-\gamma'(x;-y) \leq -\alpha\gamma(x) \leq \gamma'(x;y). \tag{2.2}$$

Proof. Due to Theorem 2.6, we have $x \perp_B (\alpha x + y)$ if and only if $\gamma'(x;\pm(\alpha x + y)) \geq 0$. Using (1.8), we have $\langle \phi \,|\, x\rangle = \gamma(x)$ for all $\phi \in \partial\gamma(x)$ and thus

$$\begin{aligned} \gamma'(x;\alpha x + \mu y) &= \sup\{\alpha\langle \phi \,|\, x\rangle + \mu\langle \phi \,|\, y\rangle \mid \phi \in \partial\gamma(x)\} \\ &= \sup\{\alpha\gamma(x) + \mu\langle \phi \,|\, y\rangle \mid \phi \in \partial\gamma(x)\} \\ &= \alpha\gamma(x) + \mu \sup\{\langle \phi \,|\, y\rangle \mid \phi \in \partial\gamma(x)\} \\ &= \alpha\gamma(x) + \mu\gamma'(x;y) \end{aligned}$$

for all $x, y \in X$, $\alpha \in \mathbb{R}$, and $\mu \geq 0$. □

Mazur [163, p. 76] gives the following quite similar result, which includes the special case of (1.10) for f a gauge, see also [164, Proposition 5.4.16]. We give a proof in the language used here.

Proposition 2.8. *Let (X,γ) be a generalized Minkowski space and let $x \in S(0,1)$. Then we have $-\gamma'(x;-y) \leq \langle \phi \mid y\rangle \leq \gamma'(x;y)$ for all $y \in X$ whenever the hyperplane $\phi_{=1}$ supports $B(0,1)$ at $x \in S(0,1)$. Moreover, if $y \in X$ and $\alpha \in \mathbb{R}$ satisfy $-\gamma'(x;-y) \leq \alpha \leq \gamma'(x;y)$, then there exists a linear functional $\phi \in X^*$ such that $\langle \phi \mid y\rangle = \alpha$ and $\phi_{=1}$ supports $B(0,1)$ at x.*

Proof. Let $x \in X$ and $\phi \in X^*$ and assume that the hyperplane $\phi_{=1}$ supports $B(0,1)$ at x. Then $\gamma^\circ(\phi) = h(\phi, B(0,1)) = \langle \phi \mid x\rangle = \gamma(x) = 1$. From (1.8), we obtain $\phi \in \partial\gamma(x)$. Now (1.10) yields $-\gamma'(x;-y) \leq \langle \phi \mid y\rangle \leq \gamma'(x;y)$ for all $y \in X$. Conversely, let $y \in X$ and $\alpha \in \mathbb{R}$ be chosen such that $-\gamma'(x;-y) \leq \alpha \leq \gamma'(x;y)$. Then, by (1.10), there exists $\phi \in \partial\gamma(x)$ such that $\langle \phi \mid y\rangle = \alpha$. Using (1.8), we have $\gamma^\circ(\phi) = 1 = \gamma(x) = \langle \phi \mid x\rangle$, i.e., the hyperplane $\phi_{=1}$ supports $B(0,1)$ at x. □

We obtain the following corollary of Theorems 2.5 and 2.6. For a special case of its part (b), see [54, Proposition I.4.10] and [197, Equation (18)].

Corollary 2.9. *Let (X,γ) be a generalized Minkowski space and let $f : [0,+\infty) \to [0,+\infty)$ be a weight. For $x, y \in X$ with $0 \leq \varepsilon < \gamma(x)$ and $f(\gamma(x)) \neq 0$, we have*

(a) $x \perp_B^\varepsilon y$ if and only if there exists $\phi \in \partial_\varepsilon\gamma(x)$ such that $\langle \phi \mid y\rangle = 0$,
(b) $x \perp_B y$ if and only if there exists $\phi \in J_f\gamma(x)$ such that $\langle \phi \mid y\rangle = 0$.

In the situation of Corollary 2.9(b),

$$\begin{aligned}\{\alpha \in \mathbb{R} \mid x \perp_B (\alpha x + y)\} &= \left\{-\frac{\langle \phi \mid y\rangle}{\langle \phi \mid x\rangle} \,\middle|\, \phi \in J_f(x)\right\} \\ &= \left\{-\frac{\langle \phi \mid y\rangle}{\gamma(x) f(\gamma(x))} \,\middle|\, \phi \in J_f(x)\right\} \qquad (2.3)\end{aligned}$$

is a non-empty compact interval, see [70, Corollary 11 and Remark 16] as well as [197, Corollary 7 and Remark 8]. Rephrasing (2.3), duality maps can be written in terms of Birkhoff orthogonality analogously to [197, Theorem 5] on which, in turn, the following theorem is patterned.

Theorem 2.10. *Let (X,γ) be a generalized Minkowski space and let $f : [0,+\infty) \to [0,+\infty)$ be a weight. For $x \neq 0$ with $f(\gamma(x)) \neq 0$, we have*

$$J_f\gamma(x) = \left\{\phi \in X^* \,\middle|\, x \perp_B \left(y - \frac{\langle \phi \mid y\rangle}{\gamma(x) f(\gamma(x))} x\right) \forall\, y \in X\right\}.$$

Proof. Let $\phi \in J_f\gamma(x)$. Then $\left\langle \phi \,\middle|\, y - \frac{\langle \phi \mid y\rangle}{\gamma(x) f(\gamma(x))} x\right\rangle = 0$. Taking Corollary 2.9 into account, we have $x \perp_B \left(y - \frac{\langle \phi \mid y\rangle}{\gamma(x) f(\gamma(x))} x\right)$. Conversely, let $\phi \in X^*$ such that $x \perp_B \left(y - \frac{\langle \phi \mid y\rangle}{\gamma(x) f(\gamma(x))} x\right)$ for all $y \in X$. By Corollary 2.9, there exists a linear functional $\phi_y \in J_f\gamma(x)$ for each $y \in X$, i.e., $\langle \phi_y \mid x\rangle \geq f(\gamma(x))\gamma(x)$, $\gamma^\circ(\phi_y) \leq f(\gamma(x))$, such that

$$\left\langle \phi_y \,\middle|\, y - \frac{\langle \phi \mid y\rangle}{\gamma(x) f(\gamma(x))} x\right\rangle = 0.$$

Thus $\langle \phi_y \,|\, y\rangle = \langle \phi \,|\, y\rangle$ for all $y \in X$. For $y = x$, it follows that $\langle \phi \,|\, x\rangle = \langle \phi_x \,|\, x\rangle \geq f(\gamma(x))\gamma(x)$, i.e., $\gamma^\circ(\phi) \geq f(\gamma(x))$. Furthermore, we have $\langle \phi \,|\, y\rangle = \langle \phi_y \,|\, y\rangle \leq \gamma^\circ(\phi_y)\gamma(y) \leq f(\gamma(x))\gamma(y)$ for all $y \in X$, so $\gamma^\circ(\phi) \leq f(\gamma(x))$. Summarizing, we obtain $\gamma^\circ(\phi) = f(\gamma(x))$ and the proof is complete. □

Extending [128, Corollary 2.2 and Lemma 3.1] to generalized Minkowski spaces, there is an analogous statement about the numbers $\alpha \in \mathbb{R}$ for which $x \perp_B^\varepsilon (\alpha x + y)$.

Proposition 2.11. *Let (X, γ) be a generalized Minkowski space, $x, y \in X$, and $0 \leq \varepsilon < \gamma(x)$. The set of numbers $\alpha \in \mathbb{R}$ for which $x \perp_B^\varepsilon (\alpha x + y)$ is a non-empty compact interval. In particular, if $x \perp_B^\varepsilon (\alpha x + y)$, then $|\alpha| \leq \max\left\{\frac{\gamma(y)}{\gamma(x)-\varepsilon}, \frac{\gamma(-y)}{\gamma(x)-\varepsilon}\right\}$.*

Proof. Choose a linear functional $\phi \in \partial_\varepsilon \gamma(x) \neq \emptyset$, i.e., $\gamma^\circ(\phi) \leq 1$ and $\langle \phi \,|\, x\rangle \geq \gamma(x) - \varepsilon > 0$. If $\langle \phi \,|\, z\rangle = 0$, then $x \perp_B^\varepsilon z$, see Corollary 2.9. Assume that for all $\alpha \in \mathbb{R}$, the point x is not ε-Birkhoff orthogonal to $\alpha x + y$. In particular, the line $\{\alpha x + y \mid \alpha \in \mathbb{R}\}$ does not intersect the hyperplane $\phi_{=0}$. Consequently, we have $\langle \phi \,|\, x\rangle = 0$, a contradiction. By Definition 2.1, we have $x \perp_B^\varepsilon (\alpha x + y)$ if and only if $\gamma(x) - \varepsilon \leq \gamma(x + \lambda(\alpha x + y))$ for all $\lambda \in \mathbb{R}$. In particular, for $\alpha \neq 0$ and $\lambda = -\frac{1}{\alpha}$, we obtain $\gamma(x) - \varepsilon \leq \gamma(-\frac{1}{\alpha} y)$. If $\alpha > 0$, then $\gamma(x) - \varepsilon \leq \frac{1}{\alpha}\gamma(-y)$. In case $\alpha < 0$, we have $\gamma(x) - \varepsilon \leq -\frac{1}{\alpha}\gamma(y)$. This yields $|\alpha| \leq \max\left\{\frac{\gamma(y)}{\gamma(x)-\varepsilon}, \frac{\gamma(-y)}{\gamma(x)-\varepsilon}\right\}$ for $\alpha \neq 0$, which holds trivially for $\alpha = 0$. Using Corollary 2.9, it follows that

$$\begin{aligned}
\left\{\alpha \in \mathbb{R} \,\middle|\, x \perp_B^\varepsilon (\alpha x + y)\right\} &= \left\{-\frac{\langle \phi \,|\, y\rangle}{\langle \phi \,|\, x\rangle} \,\middle|\, \phi \in \partial_\varepsilon \gamma(x)\right\} \\
&= \left\{-\frac{\langle \phi \,|\, y\rangle}{\langle \phi \,|\, x\rangle} \,\middle|\, \phi \in \partial_\varepsilon \gamma(x), \langle \phi \,|\, x\rangle = \gamma(x) - \varepsilon\right\} \\
&= \left\{-\frac{\langle \phi \,|\, y\rangle}{\gamma(x) - \varepsilon} \,\middle|\, \phi \in \partial_\varepsilon \gamma(x), \langle \phi \,|\, x\rangle = \gamma(x) - \varepsilon\right\}.
\end{aligned}$$

Since $X^* \ni \phi \mapsto -\frac{\langle \phi \,|\, y\rangle}{\gamma(x)-\varepsilon}$ is a linear functional and $\{\phi \in \partial_\varepsilon \gamma(x) \mid \langle \phi \,|\, x\rangle = \gamma(x) - \varepsilon\}$ is a compact convex set, $\{\alpha \in \mathbb{R} \mid x \perp_B^\varepsilon (\alpha x + y)\}$ is a compact convex set, too. □

A feature of gauges is their possible asymmetry. More precisely, we may think of $\gamma(x - y)$ as the distance from y to x which need not coincide with $\gamma(y - x)$. In this sense, the definition of Birkhoff orthogonality states that $x \perp_B^\varepsilon y$ if and only if x is "approximately closest" to 0 among the points of the form $x + \lambda y$ with $\lambda \in \mathbb{R}$. This kind of proximality is comprised in the notion of ε-best approximation, which is naturally accompanied by the notion of ε-best co-approximation. In normed spaces, the former has been introduced by Buck [43], the latter by Hasani et al. [100], despite the fact that best co-approximations (for $\varepsilon = 0$) have already been investigated by Franchetti and Furi [79].

Definition 2.12. Let (X, γ) be a generalized Minkowski space and $K \subset X$. A point $x \in K$ is called an *ε-best approximation of $y \in X$ in K* if $\gamma(x - y) \leq \gamma(z - y) + \varepsilon$ for all $z \in K$. A point $x \in K$ is called an *ε-best co-approximation of $y \in X$ in K* if $\gamma(x - z) \leq \gamma(y - z) + \varepsilon$ for all $z \in K$. The sets of ε-best approximations and ε-best co-approximations of y in K shall be denoted by $P_K^\varepsilon(y)$ and $Q_K^\varepsilon(y)$, respectively.

For a convex set $K \subset X$, the set $Q_K^\varepsilon(y)$ can be readily checked for closedness and convexity as

$$Q_K^\varepsilon(y) = K \cap \left(\bigcap_{z \in K} B(z, \gamma(y-z)+\varepsilon) \right)$$

is the intersection of closed convex sets. Similarly, the set $P_K^\varepsilon(y)$ is convex when $K \subset X$ is. To this end, let $x_1, x_2 \in P_K^\varepsilon(y)$. For all $z \in K$ and $\lambda \in [0,1]$, we then have $\gamma((\lambda x_1 + (1-\lambda)x_2 - y) \leq \lambda\gamma(x_1-y) + (1-\lambda)\gamma(x_2-y) \leq \lambda(\gamma(z-y)+\varepsilon) + (1-\lambda)(\gamma(z-y)+\varepsilon) = \gamma(z-y)+\varepsilon$. Thus $\lambda x_1 + (1-\lambda)x_2 \in P_K^\varepsilon(y)$. Alternatively, the convexity of $P_K^\varepsilon(y)$ can be seen from the representation $P_K^\varepsilon(y) = B(x, \underline{\mathrm{dist}}_{\gamma^\vee}(y,K)) \cap K$ which also yields the closedness of $P_K(y)$, provided K is closed. For non-empty closed convex sets $K \subset X$, one can also show the nonemptiness of $P_K^\varepsilon(y)$ analogously to [243, Theorem 3.8.1]. However, for certain subsets $K \subset X$ which are important for applications, explicit descriptions of $P_K(x)$ may be cumbersome, even if the projection is performed with respect to the Euclidean norm, cf. [240, Section 5.2]. On the other hand, there are also trivial examples.

Example 2.13. For any point y of a generalized Minkowski space (X, γ), the set of ε-best approximations of y in X and the set of ε-best co-approximations of y in X coincide with $B(y, \varepsilon)$.

For ε-best approximations, we have the following stability property, cf. [243, Remark 3.8.1] and [245, Theorem 8.5].

Proposition 2.14. *Let (X, γ) be a generalized Minkowski space, $K \subset X$ a non-empty set, $x \in X$, and $y \in P_K^\varepsilon(x)$. Then $y \in P_K^\varepsilon(\lambda y + (1-\lambda)x)$ for all $\lambda \in [0,1]$.*

Proof. Let $\lambda \in [0,1]$. For $x_\lambda := \lambda y + (1-\lambda)x$, we have to show that $\gamma(y - x_\lambda) \leq \gamma(z - x_\lambda) + \varepsilon$ for all $z \in K$. Now assume that there exists a point $z_0 \in K$ such that $\gamma(z_0 - x_\lambda) < \gamma(y - x_\lambda) - \varepsilon$. Then $z_0 \neq y$ and

$$\gamma(z_0 - x) \leq \gamma(z_0 - x_\lambda) + \gamma(x_\lambda - x) < \gamma(y - x_\lambda) - \varepsilon + \gamma(x_\lambda - x) = \gamma(y - x) - \varepsilon,$$

contradicting the assumption $\gamma(y-x) \leq \gamma(z-x) + \varepsilon$ for all $z \in K$. □

The next result links ε-Birkhoff orthogonality and ε-best approximations in linear subspaces to ε-subdifferentials. The proof follows the lines of [217, Theorem 6.12] which is the corresponding result for normed spaces, see also [56, Theorem 2.5.1], [97, Lemma 1.1], and [100, Theorem 2.3].

Proposition 2.15. *Let L be a non-trivial linear subspace of a generalized Minkowski space (X, γ), $x, y \in X$, $y \notin L$, $x \in L$, and $\varepsilon \geq 0$. Then $P_L^\varepsilon(y)$ is non-empty, compact, and convex. Moreover, the following statements are equivalent.*

(a) The point x is an ε-best approximation of y in L.
(b) We have $(x-y) \perp_B^\varepsilon z$ for all $z \in L$.
(c) There exists a linear functional $\phi \in X^*$ such that $\gamma^\circ(\phi) = 1$, $\langle \phi \,|\, z \rangle = 0$ for all $z \in L$, and $\phi \in \partial_\varepsilon \gamma(x-y)$.

We prepare the proof of Proposition 2.15 by giving a geometric description of the ε-subdifferential of a gauge. A non-relaxed analog for normed spaces is presented in [128, Theorem 2.1].

Lemma 2.16. *Let (X,γ) be a generalized Minkowski space, $x \in X$, $\phi \in X^*$, $\gamma^\circ(\phi) = 1$, and $0 \leq \varepsilon < \gamma(x)$. The following statements are equivalent:*

(a) $\phi \in \partial_\varepsilon \gamma(x)$,
(b) $\langle \phi \,|\, x \rangle \geq \gamma(x) - \varepsilon$,
(c) $\langle \phi \,|\, x \rangle > 0$ *and* $x \perp_B^\varepsilon z$ *for all* $z \in \phi_{=0}$,
(d) $\sup \{ \langle \phi \,|\, z \rangle \,|\, z \in X, \gamma(z) \leq \gamma(x) - \varepsilon \} \leq \langle \phi \,|\, x \rangle$.

Proof. The equivalence (a)$\Leftrightarrow$(b) is a consequence of (1.8). For the implication (a)$\Rightarrow$(c), note that we have

$$\gamma(y) + \varepsilon \geq \gamma(x) + \langle \phi \,|\, y - x \rangle \tag{2.4}$$

for all $y \in X$. Substituting $y = x + \lambda z$ in (2.4) for arbitrary choices of $z \in \phi_{=0}$ and $\lambda \in \mathbb{R}$, we obtain

$$\gamma(x + \lambda z) + \varepsilon \geq \gamma(x) + \langle \phi \,|\, \lambda z \rangle = \gamma(x),$$

that is, $x \perp_B^\varepsilon z$. From (b), we also obtain $\langle \phi \,|\, x \rangle \geq \gamma(x) - \varepsilon > 0$. For showing the implication (c)$\Rightarrow$(b), set $\mu := \frac{\langle \phi \,|\, x \rangle}{\gamma(x) - \varepsilon} > 0$. For every point $y \in X$, there are a number $\lambda \in \mathbb{R}$ and a point $z \in \phi_{=0}$ such that $y = \lambda x + z$. If $\lambda < 0$, then $\langle \phi \,|\, y \rangle = \lambda \langle \phi \,|\, x \rangle < 0 < \mu\gamma(y)$. If $\lambda \geq 0$, we obtain

$$\langle \phi \,|\, y \rangle = \lambda \langle \phi \,|\, x \rangle = \lambda\mu(\gamma(x) - \varepsilon) \leq \mu\gamma(\lambda x + z) = \mu\gamma(y).$$

Thus $1 = \gamma^\circ(\phi) \leq \mu$, which is equivalent to $\langle \phi \,|\, x \rangle \geq \gamma(x) - \varepsilon$. Finally, for (d)$\Leftrightarrow$(b), note that the identity $\gamma^\circ = h(\cdot, B(0,1))$ yields

$$\sup\{\langle \phi \,|\, z \rangle \,|\, z \in B(0, \gamma(x) - \varepsilon)\} = \gamma^\circ(\phi)(\gamma(x) - \varepsilon) = \gamma(x) - \varepsilon,$$

which completes the proof. □

In case $\varepsilon = 0$, the previous result can be slightly improved to

Lemma 2.17. *Let (X,γ) be a generalized Minkowski space, $x \in X$, $\phi \in X^*$, and $\phi \neq 0$. The following statements are equivalent:*

(a) $\frac{\phi}{\gamma^\circ(\phi)} \in \partial\gamma(x)$,
(b) $\langle \phi \,|\, x \rangle = \gamma^\circ(\phi)\gamma(x)$,
(c) $\langle \phi \,|\, x \rangle \geq 0$ *and* $x \perp_B z$ *for all* $z \in \phi_{=0}$,
(d) ϕ *attains its supremum on* $B(0, \gamma(x))$ *at* x.

The nontriviality of Lemma 2.16 is a consequence of the nonemptiness of the ε-subdifferential of γ, see [243, Theorem 2.4.9]. A related result in normed spaces can be found in [128, Theorem 2.2].

Lemma 2.18. *Let (X,γ) be a generalized Minkowski space, $x \in X$, and $\varepsilon \geq 0$. Then there exists a linear functional $\phi \in X^* \setminus \{0\}$ such that $x \perp_B^\varepsilon z$ for all $z \in \phi_{=0}$.*

Proof of Proposition 2.15. The convex, hence continuous, and coercive function $\gamma(\cdot - y) : L \to \mathbb{R}$ possesses a 0-minimizer, i.e., $P_L^0(y) \neq \emptyset$. Therefore, the set $P_L^\varepsilon(y)$ is also non-empty and, being a sublevel set of $\gamma(\cdot - y) : L \to \mathbb{R}$, compact and convex. The equivalence (a)$\Leftrightarrow$(b) follows from $x + \lambda L = L$ for all $\lambda \in \mathbb{R}$. For (a)$\Rightarrow$(c), consider $\mu := \inf_{z \in L} \gamma(z - y) > 0$. Using the Hahn–Banach theorem, there exists a hyperplane H which separates L and $B(y, \mu)$. As $L \cap B(y, \mu) = P_L^0(y)$ is non-empty, the hyperplane H contains L and is therefore a linear subspace itself. Thus there exists a linear functional $\phi \in X^*$ such that $H = \phi_{=0}$. Moreover, the hyperplane H is a supporting hyperplane of $B(y, \mu)$. Hence we may choose ϕ such that $B(y, \mu) \subset \phi_{\leq 0}$. We conclude that $B(0, \mu) \subset \phi_{\leq 0} - y$. Choosing $x_0 \in P_L^0(y)$ and applying Lemma 2.17, the linear functional ϕ restricted to $B(0, \mu)$ attains its supremum $\langle \phi \,|\, -y \rangle$ at $x_0 - y$, so $\frac{\phi}{\gamma^\circ(\phi)} \in \partial\gamma(x_0 - y)$. This means that we may choose $\phi \in X^*$ such that $\gamma^\circ(\phi) = 1$ and $\langle \phi \,|\, x_0 - y \rangle = \gamma(x_0 - y)$. Since $x \in P_L^\varepsilon(y)$, we have

$$\gamma(x - y) \leq \mu + \varepsilon = \gamma(x_0 - y) + \varepsilon = \langle \phi \,|\, x_0 - y \rangle + \varepsilon = \langle \phi \,|\, x - y \rangle + \varepsilon.$$

Finally, we show (c)$\Rightarrow$(a). By virtue of (1.2) and (1.8), we have

$$\gamma(x - y) \leq \langle \phi \,|\, x - y \rangle + \varepsilon = \langle \phi \,|\, v - y \rangle + \varepsilon \leq \gamma(v - y) + \varepsilon$$

for all $v \in L$. □

In contrast to that, there are sufficient conditions for ε-best co-approximations of points in linear subspaces in terms of ε-Birkhoff orthogonality and ε-subdifferentials which need not be necessary ones. Closely related results in finite-dimensional normed spaces are, for instance, [79, p. 1046, (1)], [100, Theorems 2.3, 2.6, and 2.10], and [189, Proposition 2.1].

Proposition 2.19. *Let L be a non-trivial linear subspace of a generalized Minkowski space (X, γ), $x, y \in X$, $y \notin L$, $x \in L$, and $\varepsilon \geq 0$. Then each of the following three equivalent statements*

(a) *We have $z \perp_B^\varepsilon (y - x)$ for all $z \in L$.*
(b) *For all $z \in L$, the point x is an ε-best approximation of z in $\langle x, y \rangle$.*
(c) *For all $z \in L$, there exists a linear functional $\phi \in X^*$ such that $\gamma^\circ(\phi) = 1$, $\langle \phi \,|\, y - x \rangle = 0$, and $\phi \in \partial_\varepsilon \gamma(z)$.*

implies that

(d) *The point x is an ε-best co-approximation of y in L.*

Proof. As $\tilde{z}$ traverses L, the point $z = x - \tilde{z}$ does the same, and vice versa. So $\gamma(\tilde{z}) \leq \gamma(\tilde{z} + \lambda(y - x)) + \varepsilon$ for all $\tilde{z} \in L$ and $\lambda \in \mathbb{R}$ if and only if $\gamma(x - z) \leq \gamma(x + \lambda(y - x) - z) + \varepsilon$ for all $z \in L$ and $\lambda \in \mathbb{R}$. (Note that $x + \lambda(y - x)$ is an arbitrary point of $\langle x, y \rangle$.) This shows (a)$\Leftrightarrow$(b). For (b)$\Leftrightarrow$(c), we know from Proposition 2.15(a)$\Leftrightarrow$(c) that a point $x_0 \in L'$ is an ε-best approximation of $y_0 \in X$ in an affine subspace L' if and only if there exists a linear functional $\phi \in X$ such that $\gamma^\circ(\phi) = 1$, $\langle \phi \,|\, v - x_0 \rangle = 0$ for all $v \in L'$, and $\phi \in \partial_\varepsilon \gamma(x_0 - y_0)$. Statement (b) can be equivalently written as

(b′) For all $z \in L$, the point x is an ε-best approximation of $z + x$ in $\langle x, y \rangle$.

because $z+x \in L$ for all $z \in L$. Now choose $L' := \langle x, y\rangle$, $y_0 := z+x$, and $x_0 := x$. Thus, we have (b′) if and only if, for all $z \in L$, there exists a linear functional $\phi \in X^*$ such that $\gamma^\circ(\phi) = 1$, $\langle \phi \,|\, v - x\rangle = 0$ for all $v \in \langle x, y\rangle$, and $\phi \in \partial_\varepsilon \gamma(z)$. Finally, we show the implication (b)⇒(d). In

$$\gamma(x-z) \leq \gamma(w-z) + \varepsilon \text{ for all } z \in L \text{ and } w \in \langle x, y\rangle\,,$$

we choose $w = y$ to obtain

$$\gamma(x-z) \leq \gamma(y-z) + \varepsilon \text{ for all } z \in L.$$

This yields the assertion. □

If γ is a norm and $\varepsilon = 0$, the implication (d)⇒(a) is true as well. However, we cannot expect this implication to be valid for gauges in general, even for $\varepsilon = 0$. For instance, take $X = \mathbb{R}^2$, $\gamma : X \to \mathbb{R}$, $\gamma(\xi_1, \xi_2) := \max\{-\xi_2, \xi_2 - \xi_1, \xi_2 + \xi_1\}$, and $L := \{(\xi_1, 0) \,|\, \xi_1 \in \mathbb{R}\}$, and $y := (0,1)$. Then the unit ball $B(0,1)$ is the triangle with vertices $(0,1)$, $(-2,-1)$, $(2,-1)$, and the set of 0-best co-approximations of y in L is

$$Q_L^0(y) = L \cap \left(\bigcap_{z \in L} B(z, \gamma(y-z)) \right) = [(-1,0),(1,0)]\,.$$

Now take $x := (0,0)$, $z := (1,0)$, and $w := (0,-0.5)$. Then $\gamma(w-z) = 0.5 < 1 = \gamma(x-z)$, so x is not a 0-best approximation of z in $\langle x, y\rangle$. Furthermore, for $\lambda := -0.5$, we have $\gamma(z) = 1 > 0.5 = \gamma(z + \lambda(y-x))$, so z is not Birkhoff orthogonal to $y - x$.
On the other hand, item (d) from Proposition 2.19 implies that $\gamma(z) \leq \gamma(z + \lambda(y-x)) + \varepsilon$ for all $z \in L$ and $\lambda \in [0,1]$, see again [100, Theorem 2.3] for the analogous statement in normed spaces. As a corollary of Proposition 2.15 for one-dimensional subspaces, we obtain the following result, see also [128, Theorem 2.3].

Corollary 2.20. *Let (X, γ) be a generalized Minkowski space of dimension* $\dim(X) \geq 2$. *Furthermore, let $x, y \in X$, $\varepsilon \geq 0$, and $\alpha \in \mathbb{R}$. The following statements are equivalent.*

(a) The point $\alpha x + y$ is an ε-best approximation of 0 *in* $y + \operatorname{lin}(\{x\})$.
(b) The point αx is an ε-best approximation of $-y$ *in* $\operatorname{lin}(\{x\})$.
(c) We have $(\alpha x + y) \perp_B^\varepsilon x$.
(d) There exists a linear functional $\phi \in X^*$ such that $\gamma^\circ(\phi) = 1$, $\langle \phi \,|\, x\rangle = 0$ and $\langle \phi \,|\, y\rangle \geq \gamma(\alpha x + y) - \varepsilon$.

Moreover, the set of numbers $\alpha \in \mathbb{R}$ such that $(\alpha x + y) \perp_B^\varepsilon x$ is a compact interval provided $x \neq 0$.

Note that in Corollary 2.20, the implication (c)⇒(d) fails for $\dim(X) = 1$ and $x \in X \setminus \{0\}$. Emulating key properties of usual inner products, *semi-inner products* enable Hilbert-like arguments in arbitrary Banach spaces. Prominent examples include the *superior* and *inferior semi-inner product* associated with a given norm $\|\cdot\|$, which are in fact directional derivatives of the convex function $\frac{1}{2}\|\cdot\|^2$. New approaches to classical concepts have been developed, not only to optimization problems like the Fermat–Torricelli problem [57] and the best approximation problem [69] but also to geometric concepts like orthogonality [70, Chapters 8–11]. In particular, several results connecting semi-inner products to Birkhoff orthogonality have been derived. We close this subsection by demonstrating how Birkhoff orthogonality in generalized Minkowski spaces can be characterized in terms of natural analogs of the superior and inferior semi-inner products. To this end, consider the functions $g:\mathbb{R}\to\mathbb{R}$, $g(\alpha):=\frac{1}{2}\alpha^2$, and $[\cdot\,|\,\cdot]_s, [\cdot\,|\,\cdot]_i : X\times X\to\mathbb{R}$ defined by

$$\begin{aligned}
[y\,|\,x]_s &:= \lim_{\lambda\downarrow 0}\frac{\gamma(x+\lambda y)^2-\gamma(x)^2}{2\lambda} = (g\circ\gamma)'(x;y),\\
[y\,|\,x]_i &:= \lim_{\lambda\uparrow 0}\frac{\gamma(x+\lambda y)^2-\gamma(x)^2}{2\lambda} = -(g\circ\gamma)'(x;-y) = -[-y\,|\,x]_s\,.
\end{aligned}$$

Note that $[\cdot\,|\,\cdot]_s$, $[\cdot\,|\,\cdot]_i$ need not be semi-inner products in the sense of [70, Definition 6] since $[x\,|\,y]_p^2 \leq [x\,|\,x]_p\,[y\,|\,y]_p$ for $p\in\{s,i\}$ may be invalidated by the asymmetry of γ. In normed spaces, this estimate is checked in [70, Proposition 6]. The proof uses the reverse triangle inequality which is not valid for gauges. However, we can show an upper bound for $[\cdot\,|\,\cdot]_s$:

$$\begin{aligned}
[y\,|\,x]_s &= \lim_{\lambda\downarrow 0}\frac{\gamma(x+\lambda y)^2-\gamma(x)^2}{2\lambda}\\
&= \lim_{\lambda\downarrow 0}\left(\frac{\gamma(x+\lambda y)+\gamma(x)}{2}\cdot\frac{\gamma(x+\lambda y)-\gamma(x)}{\lambda}\right)\\
&= \lim_{\lambda\downarrow 0}\frac{\gamma(x+\lambda y)+\gamma(x)}{2}\lim_{\lambda\downarrow 0}\frac{\gamma(x+\lambda y)-\gamma(x)}{\lambda}\\
&= \gamma(x)\lim_{\lambda\downarrow 0}\frac{\gamma(x+\lambda y)-\gamma(x)}{\lambda} \qquad (2.5)\\
&\leq \gamma(x)\lim_{\lambda\downarrow 0}\frac{\gamma(\lambda y)}{\lambda}\\
&\leq \gamma(x)\gamma(y),
\end{aligned}$$

where (2.5) can be written as $[y\,|\,x]_s=\gamma(x)\gamma'(x;y)$, which is an application of the chain rule for directional derivatives [214, Proposition 3.6] to the function $g\circ\gamma$. Similarly, a lower bound for the function $[\cdot\,|\,\cdot]_i$ is given by $[y\,|\,x]_i=-\gamma(x)\gamma'(x;-y)\geq-\gamma(x)\gamma(-y)$. Following the lines of [70, Proposition 5], we may also check that

$$\begin{aligned}
[x\,|\,x]_s &= \lim_{\lambda\downarrow 0}\frac{\gamma(x+\lambda x)^2-\gamma(x)^2}{2\lambda} = \lim_{\lambda\downarrow 0}\frac{(1+\lambda)^2\gamma(x)^2-\gamma(x)^2}{2\lambda}\\
&= \gamma(x)^2\lim_{\lambda\downarrow 0}\frac{(1+\lambda)^2-1}{2\lambda} = \gamma(x)^2
\end{aligned}$$

and

$$\begin{aligned}
[x\,|\,x]_i &= \lim_{\lambda\uparrow 0}\frac{\gamma(x+\lambda x)^2-\gamma(x)^2}{2\lambda} = \lim_{\lambda\uparrow 0,\lambda>-1}\frac{(1+\lambda)^2\gamma(x)^2-\gamma(x)^2}{2\lambda}\\
&= \gamma(x)^2\lim_{\lambda\uparrow 0,\lambda>-1}\frac{(1+\lambda)^2-1}{2\lambda} = \gamma(x)^2
\end{aligned}$$

for $x\in X$. For $x,y\in X$, $\alpha\in\mathbb{R}$, and $\mu\geq 0$, a generalization of [70, Theorem 16] can be established by using the computation in the proof of Corollary 2.7 and the chain rule for directional derivatives:

$$\begin{aligned}(g\circ\gamma)'(x;\alpha x+\mu y) &= \gamma(x)\gamma'(x;\alpha x+\mu y)\\ &= \gamma(x)(\alpha\gamma(x)+\mu\gamma'(x;y))\\ &= \alpha\gamma(x)^2+\mu\gamma(x)\gamma'(x;y).\end{aligned}$$

In the classical theory in normed spaces, computations like these provide the basis for proving characterizations of Birkhoff orthogonality in terms of superior and inferior semi-inner products. In our context, we may define a weight $f : [0,+\infty) \to [0,+\infty)$ via $f(\alpha) := \alpha$, yielding $J_f\gamma(x) = \partial(g\circ\gamma)(x) = \gamma(x)\partial\gamma(x)$. In combination with $[y\,|\,x]_s = (g\circ\gamma)'(x;y) = \gamma(x)\gamma'(x;y)$, results on Birkhoff orthogonality in (X,γ) in terms of $[\cdot\,|\,\cdot]_s$ and $[\cdot\,|\,\cdot]_i$ can therefore be derived using the above theory by suitably multiplying by $\gamma(x)$. For instance, the analog of [70, Corollary 12] in generalized Minkowski spaces is the equivalence of the statements

(a) $x \perp_B (\alpha x + y)$,
(b) $[y\,|\,x]_i \leq -\alpha\gamma(x)^2 \leq [y\,|\,x]_s$.

Proof. In (2.2), multiply by $\gamma(x)$. □

For $\alpha = 0$, this yields the equivalence of $x \perp_B y$ and $[y\,|\,x]_i \leq 0 \leq [y\,|\,x]_s$, see [70, Theorem 50], [108, p. 54], or [197, Equation (17)] for the corresponding result in normed spaces. Finally, setting $f : [0,+\infty) \to [0,+\infty)$, $f(\alpha) := \alpha$, in (2.3) yields $x \perp_B \left(y - \frac{[y\,|\,x]_s}{\gamma(x)^2}x\right)$ for $x \neq 0$, cf. [197, Equation (19)].

2.2 Smoothness and rotundity

As Birkhoff orthogonality is intertwined with the concept of supporting hyperplanes of balls, it can be used to characterize their smoothness and rotundity. In the theory of normed spaces, these characterizations are usually supplemented by further reformulations involving linear functionals or the triangle inequality. This can be also done in generalized Minkowski spaces, which we demonstrate for smoothness by extending [128, Theorems 4.2 and 5.1], see also [164, Corollary 5.4.3, Theorem 5.4.17, and Corollary 5.4.18]. An illustration of the following theorem is given in Figure 2.2.

Theorem 2.21. *Let (X,γ) be a generalized Minkowski space. The following statements are equivalent.*

(a) The unit ball $B(0,1)$ is smooth.
(b) The gauge γ is Gâteaux differentiable on $X\setminus\{0\}$.
(c) If $x,y,z\in X$, $x\perp_B y$, and $x\perp_B z$, then $x\perp_B (y+z)$.
(d) For every $x,y\in X$, $x\neq 0$, there exists a unique number $\alpha\in\mathbb{R}$ such that $x\perp_B(\alpha x+y)$.
(e) For all $x\in X$ with $\gamma(x)=1$, there exists a unique linear functional $\phi\in X^*$, $\gamma^\circ(\phi)=1$ such that $\langle\phi\,|\,x\rangle = 1$.

In this case, the linear functional ϕ is the Gâteaux derivative of γ at x (items (e) and (b)), $\gamma'(x;y) = -\alpha\gamma(x)$ (item (d)), and the unique supporting hyperplane of $B(0,1)$ at one of its boundary points x consists of all points y such that $x\perp_B(y-x)$ (item (a)).

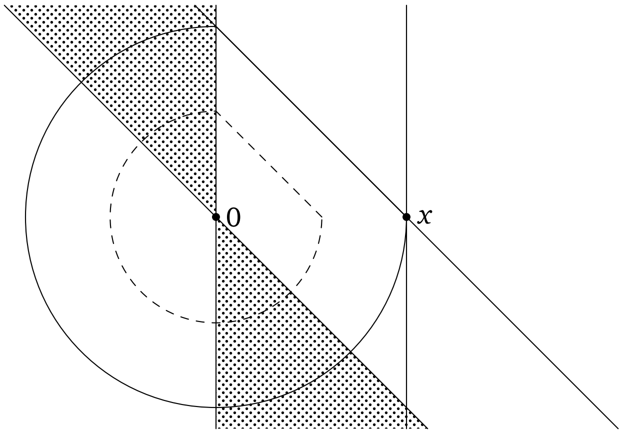


Figure 2.2. Illustration of Theorem 2.21: For the non-smooth gauge $\gamma : X \to \mathbb{R}$ and the point $x \in X$ specified in Example 2.2, the sum of two members of the set $\{y \in X \mid x \perp_B y\}$ (shaded region) may not belong to this set.

Proof. For (e)⇔(a), see Lemma 2.16. In order to show the implication (a)⇒(c), note that unique supporting hyperplane of $B(0,\gamma(x))$ passing through x has the form $H := \phi_{=\langle \phi \mid x \rangle}$, where the linear functional $\phi \in X^* \setminus \{0\}$ is uniquely determined up to a constant factor. Using Lemma 2.16(c)⇔(d), we conclude that $x \perp_B z$ if and only if $z \in H - x$. That is, if $x \perp_B y$ and $x \perp_B z$, we also have $y+z \in H-x$ because $H-x$ is a linear subspace of X, and thus $x \perp_B (y+z)$. The implication (c)⇒(d) can be proved as follows. Assume that there exist numbers $\alpha, \beta \in \mathbb{R}$ such that $x \perp_B (\alpha x + y)$ and $x \perp_B (\beta x + y)$. Due to the homogeneity of Birkhoff orthogonality and (c), we obtain $x \perp_B (\alpha x - \beta x)$, which implies $\alpha = \beta$. Next, for the implication (d)⇒(a), let $\phi_1, \phi_2 \in X^*$ be linear functionals such that the hyperplanes $H_i := (\phi_i)_{=\langle \phi_i \mid x \rangle}$, $i \in \{1,2\}$, are distinct supporting hyperplanes of $B(0,\gamma(x))$ at x. For $y \in X$, set $\alpha_i := -\frac{\langle \phi_i \mid y \rangle}{\langle \phi_i \mid x \rangle}$. Since $H_1 \neq H_2$, there exists a point $y \in X$ such that the intersection points of the straight line $\{\alpha x + y \mid \alpha \in \mathbb{R}\}$ with H_1 and H_2 do not coincide, yielding $\alpha_1 \neq \alpha_2$. Finally, we show the equivalence (b)⇔(a). By Lemma 1.18, we have $\partial\gamma(x) = \{\nabla\gamma(x)\}$ for all $x \in S(0,1)$. In particular, $\gamma^\circ(\nabla\gamma(x)) = 1$, and Lemma 2.16 yields the uniqueness of supporting hyperplanes of $B(0,1)$ at x. Conversely, if there is a unique supporting hyperplane of $B(0,\gamma(x))$ at $x \in S(0,\gamma(x))$, then Lemma 2.16 implies that $\partial\gamma(x)$ is a singleton. By Lemma 1.18, the gauge γ is Gâteaux differentiable at x. □

Following Klee's terminology of a smooth norm(ed space) [137], we say that the gauge γ and the generalized Minkowski space (X,γ) are *smooth* if one of the equivalent conditions in Theorem 2.21 is satisfied. Krein [2, p. 177] introduces the term *normal (нормальный)* for a point x of a normed space (X,γ) for which the equation $\langle \phi \mid x \rangle = \gamma^\circ(\phi)\gamma(x)$ fixes $\phi \in X^*$ up to multiplication with non-negative scalars. In view of Theorem 2.21(e), the gauge γ is smooth if and only all points of $S(0,1)$ are normal in this sense.

Generalizing [128, Theorems 4.3 and 5.2], we present characterizations of gauges with rotund unit balls, see also [164, Definition 5.1.1, Propositions 5.1.2, 5.1.10, and 5.1.11, Theorem 5.1.15, and Corollary 5.1.16]. The proof is prepared by a refinement of the triangle inequality in the sense that its equality cases determine the line segments on the unit sphere. For Minkowski spaces, the latter can be found in [154, Proposition 1].

Lemma 2.22. *For all points $x, y \in X \setminus \{0\}$ of a generalized Minkowski space (X,γ), we have $\gamma(x+y)=\gamma(x)+\gamma(y)$ if and only if $\left[\frac{x}{\gamma(x)},\frac{y}{\gamma(y)}\right]\subset S(0,1)$.*

Proof. We may assume that $x+y\neq 0$, because otherwise $y=-x$, and the claim is trivial. If $\gamma(x+y)=\gamma(x)+\gamma(y)$, then

$$\frac{x+y}{\gamma(x+y)}=\frac{\gamma(x)}{\gamma(x+y)}\frac{x}{\gamma(x)}+\frac{\gamma(y)}{\gamma(x+y)}\frac{y}{\gamma(y)},$$

i.e., the unit vector $\frac{x+y}{\gamma(x+y)}$ is a convex combination of the unit vectors $\frac{x}{\gamma(x)}$ and $\frac{y}{\gamma(y)}$, and hence, $\left[\frac{x}{\gamma(x)},\frac{y}{\gamma(y)}\right]\subset S(0,1)$. Conversely, if $\left[\frac{x}{\gamma(x)},\frac{y}{\gamma(y)}\right]\subset S(0,1)$, we have

$$\frac{x+y}{\gamma(x)+\gamma(y)}=\frac{\gamma(x)}{\gamma(x)+\gamma(y)}\frac{x}{\gamma(x)}+\frac{\gamma(y)}{\gamma(x)+\gamma(y)}\frac{y}{\gamma(y)},$$

i.e., $\frac{x+y}{\gamma(x)+\gamma(y)}$ is a point of the line segment $\left[\frac{x}{\gamma(x)},\frac{y}{\gamma(y)}\right]$. Hence $\frac{x+y}{\gamma(x)+\gamma(y)}$ is a unit vector or, equivalently, we have $\gamma(x+y)=\gamma(x)+\gamma(y)$. □

Rotundity of the unit ball can now be expressed in terms of the gauge. In particular, there is a characterization via Birkhoff orthogonality.

Theorem 2.23. *Let (X,γ) be a generalized Minkowski space. The following statements are equivalent.*

(a) The unit ball $B(0,1)$ is rotund.
(b) If $[x,y]\subset S(0,1)$ for some $x,y\in X$, then $x=y$.
(c) Each point of $S(0,1)$ is an extreme point of $B(0,1)$.
(d) We have $\gamma(\lambda x+(1-\lambda)y)<1$ whenever $x\neq y$, $\gamma(x)=\gamma(y)=1$, and $0<\lambda<1$.
(e) We have $\gamma(\frac{1}{2}(x+y))<1$ whenever $x\neq y$ and $\gamma(x)=\gamma(y)=1$.
(f) For each $x\in S(0,1)$, $\gamma(x\pm y)=1$ implies $y=0$.
(g) Whenever $x,y\in X$ and $\gamma(x+y)=\gamma(x)+\gamma(y)$, then there exists a number $\lambda>0$ such that $x=\lambda y$.
(h) For every $x,y\in X$, $x\neq 0$, there exists a unique number $\alpha\in\mathbb{R}$ such that $(\alpha x+y)\perp_B x$.
(i) Each linear functional $\phi\in X^*$ possesses at most one maximizer on $B(0,1)$.
(j) For each $\phi\in B(0,1)^\circ\setminus\{0\}$, there is at most one point $x\in S(0,1)$ such that $\langle\phi\,|\,x\rangle=1$.

Note that the equivalence (a)⟺(g) is already stated in [167, pp. 35–38].

Proof. The implication (d)⇒(e) and the equivalences (d)⟺(b)⟺(c) are trivial. The equivalence (b)⟺(g) is a consequence of Lemma 2.22. For (a)⟺(b), assume first that (b) fails. Then the unit sphere $S(0,1)$ contains a non-singleton line segment F. The disjoint convex sets $U(0,1)$ and F can be separated by a hyperplane. This hyperplane is a supporting hyperplane of $B(0,1)$ and contains F, contradicting (a). Conversely, if (a) fails, then there is a supporting hyperplane of

$B(0,1)$ which contains at least two distinct points $x, y \in B(0,1)$. By convexity, the line segment $[x, y]$ is contained in the $S(0,1)$, contradicting (b). Next, we show (e)$\Rightarrow$(d). Choose distinct points $x, y \in S(0,1)$. Since $\gamma(\frac{1}{2}(x+y)) < 1$, we have $\frac{1}{2}(x+y) \in U(0,1) = \operatorname{int}(B(0,1))$. Thus $[\frac{1}{2}(x+y), x) \subset U(0,1)$ and $[\frac{1}{2}(x+y), y) \subset U(0,1)$, see [207, Lemma 1.1.9]. For (f)$\Leftrightarrow$(e), assume first that there are points $x, y \in X$ such that $y \neq 0$ and $\gamma(x) = \gamma(x \pm y) = 1$. Let $x_1 := x+y$ and $x_2 := x-y$. Then $x = \frac{1}{2}(x_1+x_2)$, $x_1 \neq x_2$, and $\gamma(x_1) = \gamma(x_2) = \gamma(\frac{1}{2}(x_1+x_2)) = 1$. Conversely, choose $x_1, x_2 \in S(0,1)$ such that $x_1 \neq x_2$, $\gamma(x_1) = \gamma(x_2) = 1$, and $\gamma(\frac{1}{2}(x_1+x_2)) = 1$. Let $x := \frac{1}{2}(x_1+x_2)$ and $y := \frac{1}{2}(x_1-x_2) \neq 0$. Then $\gamma(x \pm y) = 1$. For (b)$\Leftrightarrow$(h), fix $x, y \in X$, $x \neq 0$. Let $L := \{\alpha x + y \mid \alpha \in \mathbb{R}\}$ and $\mu := \underline{\operatorname{dist}}_{\gamma^\vee}(0, L)$. Then $B(0,\mu) \cap L$ is the set of points $\alpha x + y$ for which $(\alpha x + y) \perp_B x$. It is non-empty, closed, and convex. By choice of μ, we have $B(0,\mu) \cap L \subset S(0,\mu)$. Thus, if $B(0,\mu) \cap L$ is not a singleton, there is a ball which contains a non-singleton line segment in its boundary. Conversely, if $S(0,1)$ contains a non-singleton line segment $[y, z]$, set $x := z - y$. Then the straight line $\{\alpha x + y \mid \alpha \in \mathbb{R}\}$ does not meet the interior of $B(0,1)$ and, hence, $(\alpha x + y) \perp_B x$ for $\alpha \in \{0,1\}$. In order to show (b)$\Leftrightarrow$(i), note that the family of supporting hyperplanes of $B(0,1)$ coincides with the family of hyperplanes

$$H := \{y \in X \mid \langle \phi \mid y \rangle = h(\phi, B(0,1))\},$$

where $\phi \in X^*$. The set of maximizers of ϕ on $B(0,1)$ is then $H \cap B(0,1)$, which is a subset of the boundary of $B(0,1)$. Finally, the equivalence (i)$\Leftrightarrow$(j) follows from Lemma 2.17. $\square$

Following Day's terminology of a rotund norm(ed space) [61, § VII.2], we say that the gauge γ and the generalized Minkowski space (X, γ) are *rotund* if one of the equivalent conditions in Theorem 2.23 is satisfied. Note that the concept of rotundity in normed spaces has been studied by Fréchet [80] as early as 1926. Notable contributions are due to Clarkson [55] and Krein [2, p. 178] where normed spaces whose norm has property (g) from Theorem 2.23 are called *strictly convex* and *strictly normed (строго нормированный)*, respectively. For normed spaces (X, γ), Krein [2, p. 179] also proves the equivalence (a)$\Leftrightarrow$(g) and shows that (g) implies that for each $\phi \in X^*$, the equation $\langle \phi \mid x \rangle = \gamma(x)\gamma^\circ(\phi)$ uniquely determines $x \in X$ up to multiplication with non-negative scalars, see Theorem 2.23(j).

Remark 2.24. Rotundity of a gauge γ should not be confused with another notion of strict convexity useful in optimization theory, referring to functions $f : X \to \overline{\mathbb{R}}$ with $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y)$ for all $x, y \in X$, $x \neq y$, and $0 < \lambda < 1$. (An early contribution to the investigation of this notion is [53].) However, a gauge γ is rotund if and only if $\gamma^2 : X \to \overline{\mathbb{R}}$, $\gamma^2(x) := \gamma(x)^2$ is strictly convex is the sense just described.

Proof. We have

$$\begin{aligned}\gamma(\lambda x + (1-\lambda)y)^2 &\leq (\lambda\gamma(x) + (1-\lambda)\gamma(y))^2 \\ &= \lambda\gamma(x)^2 + (1-\lambda)\gamma(y)^2 - \lambda(1-\lambda)(\gamma(x) - \gamma(y))^2 \\ &< \lambda\gamma(x)^2 + (1-\lambda)\gamma(y)^2.\end{aligned}$$

for all $x, y \in X$ with $\gamma(x) \neq \gamma(y)$ and $0 < \lambda < 1$. Using Theorem 2.23(d), the gauge γ is rotund if and only if $\gamma(\lambda x + (1-\lambda)y) < \gamma(x)$ for all $x, y \in X$ with $\gamma(x) = \gamma(y)$ if and only if $\gamma(\lambda x + (1-\lambda)y)^2 < \lambda\gamma(x)^2 + (1-\lambda)\gamma(y)^2$ for all $x, y \in X$ with $\gamma(x) = \gamma(y)$ if and only if γ^2 is strictly convex. $\square$

An open problem in approximation theory requires finding a characterization of those Hilbert spaces X in which every Chebyshev set, i.e., a set $K \subset X$ such that $P_K(x)$ is a singleton for every point $x \in X$, is convex, see [65, Chapter 12]. Rotundity plays a many-faceted role for this problem. Relatedly, it turns out that rotundity is equivalent to uniqueness of best approximations in convex sets, cf. [179, Lemma 3.10], [243, Theorem 3.8.1], and [245, Theorem 8.7].

Proposition 2.25. *Let (X,γ) be a generalized Minkowski space, and let $K \in \mathscr{C}^X$. Then $P_K(x)$ is a singleton for all $x \in X$ if and only if γ is rotund.*

Proof. By $K \in \mathscr{C}^X$, we have $P_K(x) \neq \emptyset$ for all $x \in X$, see again the discussion after Definition 2.12. Now assume that there exist points $x, y_1, y_2 \in X$ such that $y_1, y_2 \in P_K(x)$ and $y_1 \neq y_2$. Set $x_1 := \frac{1}{\mathrm{dist}_{\gamma^\vee}(x,K)}(y_1-x)$ and $x_2 := \frac{1}{\mathrm{dist}_{\gamma^\vee}(x,K)}(y_2-x)$. Then $x_1 \neq x_2$ and $\gamma(x_1)=\gamma(x_2)=1$. By convexity of $P_K(x)$, we have $\frac{1}{2}(y_1+y_2) \in P_K(x)$, which means that $\gamma(\frac{1}{2}(x_1+x_2))=1$. By Theorem 2.23, the gauge γ is not rotund. Conversely, if γ is not rotund, there exists $x_1, x_2 \in S(0,1)$ such that $x_1 \neq x_2$ and $[x_1,x_2] \subset S(0,1)$. For $K := \langle x_1, x_2 \rangle \in \mathscr{C}^X$, we then have $[x_1,x_2] \subset P_K(0)$, so $P_K(0)$ is not a singleton. □

Theorem 2.21 shows that right additivity characterizes smoothness in all dimensions. However, left additivity does not play the same role for rotundity if the dimension of the generalized Minkowski space is at least two. (In one-dimensional generalized Minkowski spaces, all gauges are rotund and have left-additive Birkhoff orthogonality relations.)

Theorem 2.26. *Let (X,γ) be a generalized Minkowski space of dimension $\dim(X) \geq 2$. If Birkhoff orthogonality is left-additive, then γ is a norm.*

Proof. Let $\dim(X)=2$. If γ is not a norm and Birkhoff orthogonality is left-additive, then there is an affine diameter of $B(0,1)$ not passing through the origin, see [218, 4.1]. In other words, there are points $x,y,z \in X$ such that x and y are linearly independent, $\gamma(x)=\gamma(y)=1$, $z \neq 0$, $x \perp_B z$, $y \perp_B z$, and $x-y \notin \mathrm{lin}(\{z\})$. Thus, there are numbers $\lambda, \mu \in \mathbb{R}$ with $0<\lambda<1$ and $\lambda x + (1-\lambda)y = \mu z$. But left additivity and homogeneity imply $(\alpha x + \beta y) \perp_B z$ for all numbers $\alpha, \beta > 0$. Therefore $\mu z \perp_B z$, which implies $z=0$. If $\dim(X) \geq 3$ and Birkhoff orthogonality is left-additive, then it is left-additive on each two-dimensional linear subspace of X. This implies, using the first part of the proof, that the restriction of γ to any two-dimensional subspace of X is a norm on that subspace. Hence, the gauge γ is a norm. □

Therefore, left additivity for gauges reduces to the case of norms which can be found in [127, pp. 561–562].

Corollary 2.27. *Let (X,γ) be a generalized Minkowski space of dimension $\dim(X) \geq 2$. Assume that Birkhoff orthogonality is left-additive. If $\dim(X)=2$, then γ is rotund. Else γ is a norm induced by an inner product.*

Theorem 2.23(j) and Theorem 2.21(e) indicate that rotundity and smoothness are dual notions. The proof presented in [164, Propositions 5.4.4 and 5.4.5] for normed spaces can be translated verbatim to generalized Minkowski spaces.

Proposition 2.28. *A generalized Minkowski space (X,γ) of dimension $\dim(X) \geq 2$ is smooth if its dual space (X^*,γ°) is rotund. Conversely, (X,γ) is rotund if its dual space (X^*,γ°) is smooth.*

Proof. Suppose that γ is not smooth. Then for some $x \in S(0,1)$, there are distinct linear functionals $\phi_1, \phi_2 \in X^*$ for which the hyperplanes $(\phi_1)_{=1}$ and $(\phi_2)_{=1}$ support $B(0,1)$ at x. In particular, we have $\langle \phi_1 \,|\, x\rangle = \langle \phi_2 \,|\, x\rangle = 1$ and thus $\langle \frac{1}{2}(\phi_1+\phi_2) \,|\, x\rangle = 1$, i.e., $1 \leq \gamma^\circ(\frac{1}{2}(\phi_1+\phi_2)$. By convexity of γ°, we have $\gamma^\circ(\frac{1}{2}(\phi_1+\phi_2)) \leq 1$. Thus $\gamma^\circ(\frac{1}{2}(\phi_1+\phi_2) = 1$ and γ° is not rotund by Theorem 2.23. Conversely, suppose that γ is not rotund. Then for some $\phi \in S(0,1)^\circ$, there are distinct points $x_1, x_2 \in S(0,1)$ at which the hyperplane $\phi_{=1}$ supports $B(0,1)$. Thus $\langle \phi \,|\, x_1\rangle = \langle \phi \,|\, x_2\rangle = 1$. Now $\chi_1 := \langle \cdot \,|\, x_1\rangle$ and $\chi_2 := \langle \cdot \,|\, x_2\rangle$ are members of the bidual space $(X^*)^* = X$, and the hyperplanes $(\chi_1)_{=1}$ and $(\chi_2)_{=1}$ in X^* support $B(0,1)^\circ$ at ϕ. Hence γ° is not smooth. □

Note that Megginson [164, Definition 5.4.23] uses the term *spherical image* for $\partial\gamma(x)$ at points $x \in S(0,1)$. In view of Lemma 2.17, the spherical image is a singleton for all $x \in S(0,1)$ if and only if γ is smooth, and γ is rotund if and only if $\partial\gamma(x) \cap \partial\gamma(y) = \emptyset$ whenever $x, y \in S(0,1)$, $x \neq y$, see [164, Propositions 5.4.24 and 5.4.25] for the corresponding result in normed spaces. The latter property is called *injectivity* of the set-valued map $\partial\gamma$ in [14, Definition 5.4.9]. Proposition 2.28 can also be understood in terms of the duality mapping $J_f\gamma : X \rightrightarrows X^*$ where the weight f is given by $f : [0,+\infty) \to [0,+\infty)$, $f(\alpha) := \alpha$. We write $g : [0,+\infty) \to [0,+\infty)$, $g(\alpha) := \frac{1}{2}\alpha^2$ for the antiderivative of the weight, and obtain $J_f\gamma = \partial(g \circ \gamma)$. With $(X^*)^* = X$, we also see that $J_f(\gamma^\circ) : X^* \rightrightarrows X$ coincides with $\partial(g \circ \gamma^\circ)$, which is the inverse of $J_f\gamma$, see Lemma 1.20 and Example 1.21. In particular,

$$\Pi(X) := \{(x,\phi) \in X \times X^* \mid \gamma(x) = \gamma^\circ(\phi) = \langle \phi \,|\, x\rangle = 1\}$$

is a subset of Graph$(J_f\gamma)$. The set $\Pi(X)$ encodes information about the geometry of the unit ball the generalized Minkowski space (X,γ), that is, about a special convex body. It serves as a tool also in other contexts. For instance, in the study of parallel sets of convex bodies, Schneider [207, Section 4.1] refers to the set $\Pi(X)$ as the *normal bundle* of $B_\gamma(0,1)$. Furthermore, if X denotes a Banach space, Capel et al. [46] investigate the numerical radius of bounded linear operators $T : X \to X$, the former of which is defined using the set $\Pi(X)$. In our setting, the geometric information hidden in the set $\Pi(X)$ can be outlined as follows. For a given point $x \in X$ with $\gamma(x) = 1$, the set $\{\phi \in X^* \mid (x,\phi) \in \Pi(X)\} = \partial\gamma(x)$ is the set of unit outer normals ϕ of $B_\gamma(0,1)$ at x. If it is a singleton for each x, then γ is smooth. Similarly, for given $\phi \in X^*$, $\gamma^\circ(x) = 1$, the set $\{x \in X \mid (x,\phi) \in \Pi(X)\} = \partial\gamma^\circ(x)$ is the set of boundary points x of $B_\gamma(0,1)$ at which the hyperplane $\phi_{=1}$ supports $B_\gamma(0,1)$. If it is a singleton for each $\phi \in X^*$ with $\gamma^\circ(x) = 1$, the gauge γ is rotund.

2.3 Orthogonality reversion and symmetry

Norms whose Birkhoff orthogonality relations coincide are studied in [197, Theorem 10] and [208], and the two-dimensional special case is implicitly stated, e.g., in [73, pp. 165–166] and [228, p. 90]. The analogous investigation for gauges on $\mathbb{R}^2$ is done in [206, 4A]. As the proof of [197, Theorem 10] is not based on the symmetry property of norms but on general facts like the maximal monotonicity of subdifferentials of convex functions (cf. Lemma 1.27), we may translate the result to our setting and omit the proof.

Theorem 2.29. *Let $\gamma_1, \gamma_2 : X \to \mathbb{R}$ two gauges whose Birkhoff orthogonality relations shall be denoted $\perp_{B,1}$ and $\perp_{B,2}$, respectively. Furthermore, let $f : [0, +\infty) \to [0, +\infty)$, $f(\alpha) := \alpha$. The following statements are equivalent.*

(a) *There exists a number $\lambda > 0$ such that $\gamma_1(x) = \lambda\gamma_2(x)$ for all $x \in X$.*
(b) *For all $x, y \in X$, we have $x \perp_{B,1} y$ if and only if $x \perp_{B,2} y$.*
(c) *For all $x \in X \setminus \{0\}$, we have $\frac{1}{\gamma_1(x)^2} J_f \gamma_1(x) = \frac{1}{\gamma_2(x)^2} J_f \gamma_2(x)$.*

The identification of pairs of norms whose Birkhoff orthogonality relations are inverses of each other yields the notion of *antinorm* in two-dimensional spaces, see [45, p. 867], [108, Proposition 3.1], and [153]. For normed spaces of dimension at least three, this class reduces to pairs of norms whose unit balls are homothetic ellipsoids, see [113, Theorem 3.2]. Closely related are norms whose Birkhoff orthogonality relation is symmetric. In the two-dimensional case, these norms are named after Radon [199]. From [113, Theorem 3.2] it follows that in higher dimensions, symmetry of Birkhoff orthogonality characterizes Euclidean spaces. However, the three-dimensional case of this result goes back to Blaschke [31]. In the present section, we prove that there are no asymmetric analogs of the antinorm and of Radon norms.

Theorem 2.30. *Let $\dim(X) \geq 2$ and $\varepsilon \geq 0$. Furthermore, let $\gamma_1, \gamma_2 : X \to \mathbb{R}$ be gauges whose ε-Birkhoff orthogonality relations shall be denoted $\perp_{B,1}^{\varepsilon}$ and $\perp_{B,2}^{\varepsilon}$, respectively. Assume that for all $x, y \in X$, we have $x \perp_{B,1}^{\varepsilon} y$ if and only if $y \perp_{B,2}^{\varepsilon} x$. Then γ_1 is a norm and $\varepsilon = 0$.*

Proof. Let $x, y \in X \setminus \{0\}$. Due to homogeneity and the assumption, we have

$$x \perp_{B,1}^{\varepsilon} y \iff y \perp_{B,2}^{\varepsilon} x \iff y \perp_{B,2}^{\varepsilon} \frac{-x}{\gamma_1(-x)} \iff \frac{-x}{\gamma_1(-x)} \perp_{B,1}^{\varepsilon} y. \tag{2.6}$$

Case 1: $\varepsilon = 0$. If $\gamma_1(x) = 1$, then x and $\frac{-x}{\gamma_1(-x)}$ are the endpoints of a chord of the convex body $S_{\gamma_1}(0,1)$ which passes through the origin 0. From (2.6) and the separation theorems for convex sets it follows every chord passing through 0 is an affine diameter of $B_{\gamma_1}(0,1)$. Since 0 is an interior point of $B_{\gamma_1}(0,1)$, the claim follows by taking [218, 4.1] into account.
Case 2: $\varepsilon > 0$. Fix $x \in X$ such that $\gamma_1(x) > \varepsilon$. Then there exists $y \in X \setminus \{0\}$ such that $x \perp_{B,1}^{\varepsilon} y$ and, without loss of generality, $\gamma_1(y) < \varepsilon$. But then $\gamma_1(y) < \varepsilon \leq \gamma_1(y + \lambda z) + \varepsilon$, so $y \perp_{B,1}^{\varepsilon} z$ for all $z \in X$. By assumption, we have $z \perp_{B,2}^{\varepsilon} y$ for all $z \in X$, that is,

$$\gamma_2(z) \leq \gamma_2(z + \lambda y) + \varepsilon \tag{2.7}$$

for all $z \in X$ and $\lambda \in \mathbb{R}$. In particular, if we choose $n \in \mathbb{N}$ large enough such that $n\gamma_2(y) > \varepsilon$ and set $z := ny$ and $\lambda := -n$, then (2.7) becomes $n\gamma_2(y) \leq \varepsilon$, a contradiction. □

Corollary 2.31. *Let (X, γ) be a generalized Minkowski space of dimension $\dim(X) \geq 2$ and let $\varepsilon \geq 0$. If ε-Birkhoff orthogonality is a symmetric relation, then $\varepsilon = 0$ and γ is a norm.*

3

Centers and radii

The best approximation problem (2.1), which is the main ingredient of Chapter 2, is an instance of *constrained convex optimization* as we require to find the infimum of the convex function $\gamma(y-\cdot): X \to \mathbb{R}$ over a convex set $K \subset X$. It can be embedded in the following larger class of constrained optimization problems. Let $K', K \subset X$ be arbitrary sets. Consider the quantity

$$\inf_{x\in K'} \sup_{y\in K} \gamma(y-x). \tag{3.1}$$

Indeed, if $K=\{y\}$ is a singleton and K' is a convex set, then (3.1) reduces to the evaluation of a distance function, i.e., a best approximation problem in the sense of (2.1):

$$\underline{\operatorname{dist}}_\gamma(y,K') = \inf\{\gamma(y-x) \mid x\in K'\}.$$

Generally, the convex function $X \ni x \mapsto \sup_{y\in K}\gamma(y-x)$, yields the least possible radius of a ball centered at x and containing K. Thus (3.1) is the least possible value of this quantity as x traverses K'. This setting has been investigated by Amir and Ziegler [11] for the case where γ is a norm. If $K'=X$, then (3.1) is the least possible radius of any ball containing K, without any constraints on the location of its center. The optimal radius

$$R(K, B_\gamma(0,1)) = \inf_{x\in X} \sup_{y\in K} \gamma(y-x) \tag{3.2}$$

in this case shall be called the *circumradius* of K with respect to $B_\gamma(0,1)$. The centers x appearing in (3.2) are then the *circumcenters* of K with respect to $B_\gamma(0,1)$, and their collection shall be denoted by $\operatorname{cc}(K, B_\gamma(0,1))$. The notation using two set variables is reminiscent of the fact that we can and, in view of the theory presented in this chapter, have to extend the definition of circumradius and circumcenters in the following way, which has already been investigated in [37–39]. The contents of this chapter merge [91], [118], and [120, Section 3].

Definition 3.1. Let $K\subset X$ and $C \in \mathscr{C}_0^X$. The *circumradius of K with respect to C* is

$$R(K,C) := \inf_{x\in X} \inf\{\lambda > 0 \mid K \subset x + \lambda C\}.$$

If $K \subset x + R(K,C)C$, then x is a *circumcenter* of K with respect to C and $x+R(K,C)C$ is a *circumball* of K. The *set of circumcenters of K with respect to C* is the set

$$\operatorname{cc}(K,C) := \{x \in X \mid K \subset x + R(K,C)\,C\}.$$

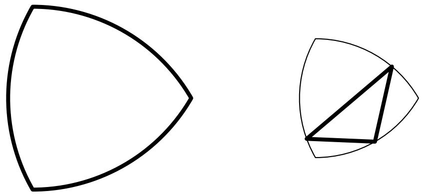


Figure 3.1. Circumradius: The set C is a Reuleaux triangle (bold line, left), the set K is a triangle (bold line, right). The circumradius $R(K, C)$ is determined by the smallest homothetic image of C that contains K (thin line).

An illustration of this definition is given by Figure 3.1. The nonemptiness assumption on the interior of C guarantees that $R(K, C) < +\infty$ if K is bounded, cf. [38, Lemma 2.2]. If, in addition, the set C is bounded and $0 \in \operatorname{int}(C)$, then $C = B_\gamma(0, 1)$ for the gauge $\gamma : X \to \mathbb{R}$, $\gamma(x) := \inf\{\lambda > 0 \mid x \in \lambda C\}$, and the circumradius takes the form (3.2).
The interest in finding the balls of least possible radius containing a given set (its *circumballs*) has been vocalized for the first time by Sylvester [221]. Since then, the problem received attention from several mathematical communities, resulting in various names under which the problem is known (center problem, minimal enclosing ball problem, minimax location problem, Sylvester problem, optimal containment problem). The attention has not been restricted to the solution of the original problem which Sylvester posed for the Euclidean plane, but has also yielded extensions of the problem obtained by transferring conceptual details of the problem to other contexts in order to apply specific methods. In his paper [246], Zindler proves not only the uniqueness of circumcenters in two-dimensional and three-dimensional Euclidean space. Zindler also shows that the points where the given set touches the boundary of its circumball are well spread in the sense that they do not lie in a hemisphere. Finally, Zindler proposes in [246] the study of an analogous problem in multiple dimensions where Euclidean balls have been replaced by homothetic images of an arbitrary but fixed convex body C. This is the setting of this chapter, as defined in Definition 3.1. As a first step to the full generality of this extension, the problem has been addressed in normed spaces (i.e., $C = -C$) as early as 1962 [83]. Variants of Sylvester's problem and Zindler's extension also appear in approximation theory [11], location science [74], computational geometry [213, Theorem 14], and convex analysis [179]. An elementary account of the location of circumcenters of triangles in normed planes is [8]. Termed optimal containment problems, the generalized problem has been addressed in [39, 72], but also in [37] where one can find a corresponding result on touching points being well spread. In the present chapter, the study of geometric properties of the circumradius and the set of circumcenters is complemented by the investigation of inradius, diameter, minimum width, and several series of successive radii which interpolate between the former quantities.

3.1 Circumradius

While circumcenters of a given set $K \subset X$ serve as approximations of the latter in the sense that they are simultaneously close to all of its points, its circumradius measures in a sense the "size" of K compared to $C \in \mathscr{C}_0^X$. In this section, we will present some basic identities and inequalities of the circumradius, and we will discuss the geometry of the set of circumcenters. We start with the behavior of the circumradius under manipulations of the input sets.

Proposition 3.2. *Let $K, K' \subset X$, $C, C' \in \mathscr{C}^X$, $x, y \in X$, and $\alpha, \beta > 0$. Then*

(a) $R(K', C') \leq R(K, C)$ *if* $K' \subset K$ *and* $C \subset C'$,
(b) $R(K, C) = R(\mathrm{cl}(K), C) = R(\mathrm{co}(K), C)$,
(c) $R(K + K', C) \leq R(K, C) + R(K', C)$,
(d) $R(x + K, y + C) = R(K, C)$,
(e) $R(\alpha K, \beta C) = \frac{\alpha}{\beta} R(K, C)$,
(f) $R(K, C') \leq R(K, C) R(C, C')$.

Proof. For $x \in X$ and $\lambda \geq 0$, we have $\mathrm{cl}(K) \subset x + \lambda C$ if and only if $K \subset x + \lambda C$ if and only if $\mathrm{co}(K) \subset x + \lambda C$. This proves (b). For (c), note that if $K \subset z + \lambda C$ and $K' \subset z' + \lambda' C$ for some $z, z' \in X$, $\lambda, \lambda' > 0$, then $K + K' \subset (z + z') + (\lambda + \lambda')C$. Finally, for (f), there exist numbers $\lambda, \lambda' > 0$ and points $z, z' \in X$ such that $K \subset z + \lambda C$ and $C \subset z' + \lambda' C'$. Substituting the latter inclusion into the former one, we obtain $K \subset z + \lambda z' + \lambda\lambda' C'$. □

Remark 3.3. (a) For $K, K' \in \mathscr{C}^X$ and $C \in \mathscr{K}_0^X$, we do not always have $R(K + K', C + K') = R(K, C)$. For example, take $X = \mathbb{R}^d$ and

$$C := [-e_1, e_1] + \ldots + [-e_d, e_d],$$
$$K := \mathrm{co}(\{0, e_1, \ldots, e_d\}),$$
$$K' := [0, e_d],$$

where $e_i \in X$ denotes the vector whose entries are 0 except for the ith one, which is 1. Then $R(K, C) = \frac{1}{2}$ and $R(K + K', C + K') = \frac{2}{3}$. However, the following implication is true: If $K, C' \in \mathscr{C}^X$, $K \in \mathscr{K}^X$, and $R(K, C) = 1$, then $R(K + K', C + K') = R(K, C)$. Indeed, for all $\lambda > 1$, there exists a point $z \in X$ such that $K \subset z + \lambda C$. It follows that $K + K' \subset z + \lambda C + K' \subset z + \lambda(C + K')$. In other words, $R(K + K', C + K') \leq 1$. Conversely, assume that $R(K + K', C + K') < 1$. Then there are a number $\lambda < 1$ and a point $z \in X$ such that $K + K' \subset z + \lambda(C + K') \subset z + \lambda C + K'$. By virtue of the cancellation rule [207, Remark 1.7.6], we have $K \subset z + \lambda(C + K') \subset z + \lambda C$, which is a contradiction to $R(K, C) = 1$.
(b) Proposition 3.2(c) trivially holds with equality if $K' = \alpha C$ for some $\alpha > 0$.

Proposition 3.2 says that the circumradius of K with respect to C is invariant under translations of both K and C. Thus we may assume $0 \in \mathrm{int}(C)$ and write the circumradius in the form (3.2) whenever it is useful. In Section 3.3 and Chapter 4, this will be our standing assumption. In addition, we will sometimes change the name of the "container" from C to B as an abbreviation of $B(0, 1)$ when we want to emphasize its role as the unit ball of a given generalized Minkowski space. Next, we show that 0 is a circumcenter of K with respect to C if both K and C are centrally symmetric, cf. [92, (1.1)].

Proposition 3.4. *Let $K \subset X$, $C \in \mathscr{K}_0^X$, and assume that $C = -C$ and $K = -K$. Then*

$$R(K, C) = \sup\{\gamma_C(x) \mid x \in K\}$$

with $\gamma_C : X \to \mathbb{R}$ as defined in (1.1).

Proof. If $K \subset z + \lambda C$ for suitable $z \in X$ and $\lambda > 0$, then $K \subset -z + \lambda C$ due to the central symmetry of K and C. It follows that

$$K \subset \frac{1}{2}K + \frac{1}{2}K \subset \frac{1}{2}(z + \lambda C) + \frac{1}{2}(-z + \lambda C) = \lambda C.$$

In other words, the circumradius is already determined by the sets λC with $\lambda > 0$:

$$R(K, C) = \inf\{\lambda > 0 \mid K \subset \lambda C\} = \sup\{\gamma_C(x) \mid x \in K\}.$$

This completes the proof. □

For applications in location science, it is of peculiar interest to find circumballs of finite sets $P = \{p_1, \dots, p_n\} \subset X$ with respect the unit ball $B(0,1)$ of a generalized Minkowski space (X, γ). For this setting, we give a description of circumballs of P in terms of a certain subdivision of X into closed convex sets. This subdivision arises from so-called *elementary cones*. These are the sets $\mathrm{cone}(F) \cup \{0\}$ where F is an exposed face of $B(0,1)$. Taking Lemma 2.22 into account, the elementary cones are just the maximal convex cones on which the gauge γ coincides with a given linear functional. (Dually, for a given convex body $K \subset X$, Ewald considers in [76, Chapter 5] the family of normal cones $\mathrm{nor}(x, K)$ at points $x \in \mathrm{bd}(K)$ and shows that the restriction of $h(\cdot, K)$ to each of these cones is linear.) Suitable translates of the elementary cones prove useful in the study of geometric objects which are defined by combinations of distances from a fixed configuration of points. In particular, these cones will appear again in Sections 5.2 and 5.4, and in Chapter 6. Our definition is inspired by [71, Definition 3.1].

Definition 3.5. Let (X, γ) be a generalized Minkowski space. Given a linear functional $\phi \in B(0,1)^\circ$ and a point $x \in X$, set

$$C_\gamma(x, \phi) := x + \{y \in X \mid \langle \phi \mid y \rangle = \gamma(y)\} = \{y \in X \mid \langle \phi \mid y - x \rangle = \gamma(y - x)\}.$$

Note that if $\gamma^\circ(\phi) < 1$, then $C_\gamma(x, \phi) = \{x\}$. Else, if $\gamma^\circ(\phi) = 1$, we have $C_\gamma(x, \phi) = x + \mathrm{cone}(B_\gamma(0,1) \cap \phi_{=1})$ where $B_\gamma(0,1) \cap \phi_{=1}$ is an exposed face of $B_\gamma(0,1)$. By definition, we have $\gamma(z - x) = \langle \phi \mid z - x \rangle$ for $z \in C_\gamma(x, \phi)$, i.e., the restriction of $\gamma(\cdot - x)$ to $C_\gamma(x, \phi)$ is an affine function. Given a generalized Minkowski space (X, γ), there exists a (possibly infinite) collection of linear functionals ϕ_i, $i \in I$, such that $\gamma^\circ(\phi_i) = 1$ for all $i \in I$ and $B(0,1) = \bigcap_{i \in I} (\phi_i)_{\leq 1}$. (A convex body is the intersection of all of its supporting half-spaces.) For a given finite set $P = \{p_1, \dots, p_n\}$ and $i_1, \dots, i_n \in I$, define

$$G_{i_1, \dots, i_n} := \bigcap_{j=1}^{n} C_{\gamma^\vee}(p_j, \phi_{i_j}),$$

Then $G_{i_1, \dots, i_n}$ is the (possibly empty) intersection of closed and convex sets, thus closed and convex itself. Therefore

$$R(P, B(0,1)) = \inf_{i_1, \dots, i_n \in I} \inf_{x \in G_{i_1, \dots, i_n}} \sup_{p \in P} \gamma(p - x)$$

which means that we may compute the balls of least possible radius of a ball which contains P and whose center is a point of $G_{i_1, \dots, i_n}$, iterate over all choices of $i_1, \dots, i_n \in I$, and a ball with least possible radius among them will be a circumball of P with respect to $B(0,1)$. This makes less sense the more exposed faces the unit ball $B(0,1)$ has. If $B(0,1)$ is rotund, then every set $G_{i_1,\dots,i_n}$ is a singleton, and the decomposition yields no improvement. On the other hand, if $B(0,1)$ is a polytope, i.e., a bounded intersection of finitely many closed half-spaces, then I can be chosen to be finite, and we can decompose the problem of finding circumballs of P into finitely many problems of the same kind, but with polyhedral feasible sets $G_{i_1,\dots,i_n}$. This may yield a simplification because the restriction of $\sup_{p\in P}\gamma(p-\cdot):X\to\mathbb{R}$ to $G_{i_1,\dots,i_n}$ is then the pointwise supremum of n affine functions, whereas the function $\sup_{p\in P}\gamma(p-\cdot):X\to\mathbb{R}$ itself is the pointwise supremum of nk affine functions, where k is the number of exposed faces of $B(0,1)$. The remainder of this section is devoted to the investigation of dimension of the set of circumcenters of a bounded set $K\subset X$ with respect to a convex body $C\in\mathscr{K}_0^X$. By Proposition 3.2(b), we may assume that $K\in\mathscr{K}^X$.

Theorem 3.6. *Let $K\in\mathscr{K}^X$, $C\in\mathscr{K}_0^X$. Then* $\mathrm{cc}(K,C)$ *is a non-empty, bounded, closed, and convex set of dimension* $\dim(\mathrm{cc}(K,C))\le\dim(X)-1$.

Proof. Convexity, boundedness, and closedness follow from the representation of the set of circumcenters as an intersection of convex, bounded, and closed sets:

$$\mathrm{cc}(K,C)=\bigcap_{x\in K}(x-R(K,C)\,C).$$

By Proposition 3.2(d), we may assume $0\in\mathrm{int}(C)$. Nonemptiness of $\mathrm{cc}(K,C)$ is then a consequence of the coercitivity of the function $\sup_{y\in K}\gamma_C(y-\cdot):X\to\mathbb{R}$. By convexity of $\mathrm{cc}(K,C)$, it is sufficient to prove the emptiness of its interior for showing that the dimension does not exceed $\dim(X)-1$. Assume there is a point $x\in\mathrm{int}(\mathrm{cc}(K,C))$. Then there exist a number $\lambda>0$ and a point $y\in K$ with $x-\lambda C\subset\mathrm{cc}(K,C)$ and $\gamma_C(y-x)=R(K,C)$. On the one hand, we have

$$z:=y-\left(1+\frac{\lambda}{2\gamma_C(y-x)}\right)(y-x)=x-\frac{\lambda}{2\gamma_C(y-x)}(y-x)\in x-\lambda C\subset\mathrm{cc}(K,C),$$

but, on the other hand, we have $\gamma_C(y-z)=\left(1+\frac{\lambda}{2\gamma_C(y-x)}\right)\gamma_C(y-x)>\gamma_C(y-x)=R(K,C)$. This implies $y\notin z+R(K,C)\,C$, a contradiction. □

Analogously to [92, (1.11)], we may use Helly's theorem and the finite-intersection property for suitable families of sets to show that the circumradius of a subset $K\in\mathscr{K}^X$ of a vector space X of dimension $d=\dim(X)$ with respect to another set $C\in\mathscr{K}_0^X$ is determined by the circumradii of subsets of K of cardinality $d+1$. Namely, we have

$$R(K,C)=\sup\{R(\{x_1,\dots,x_{d+1}\},C)\mid x_1,\dots,x_{d+1}\in K\}.$$

This supremum is in fact a maximum because the set $K^n\subset X^n$ is non-empty, convex, and compact, and for fixed $n\in\mathbb{N}$, the function $f:X^n\to\mathbb{R}$ given by $f(x_1,\dots,x_n):=R(\{x_1,\dots,x_n\},C)$ is convex due to Proposition 3.2, hence continuous. In particular, the function f attains its supremum on K^n at an extreme point of K^n, i.e., there exists $x_1,\dots,x_{d+1}\in\mathrm{ext}(K)$ such that $R(K,C)=R(\{x_1,\dots,x_{d+1}\},C)$. The determination of the circumradius (and the circumcenters) by finite sets also plays a role in the sequel, where we investigate convex bodies $C\in\mathscr{K}_0^X$ such that $\mathrm{cc}(K,C)$ is unique for all $K\in\mathscr{K}^X$. The following theorem is a special case of [11, Lemma 1.2], see also [83, Теорема VI] for the equivalence of its first and its third item.

Theorem 3.7. *Let $(X, \|\cdot\|)$ be a normed space with unit ball B. Then the following statements are equivalent.*

(a) There exists a convex body $K \in \mathcal{K}^X$ not having a unique circumcenter.
(b) There exists a two-element set $\{x_1, x_2\} \subset X$ not having a unique circumcenter.
(c) The boundary of B contains a non-singleton line segment.

Here (b) says that it is sufficient to have uniqueness of circumcenters for two-element sets of to get the uniqueness for all convex bodies. Condition (c) is a simple geometric property of the unit ball.
Note that the formulation of Theorem 3.7 is adapted to the notation used here. The result [11, Lemma 1.2] of Amir and Ziegler actually takes place in Banach spaces of arbitrary dimension. In infinite-dimensional Banach spaces, compact sets need not have circumcenters. This is why the phrase “a unique circumcenter” has to be replaced by “at most one circumcenter” in that setting. Furthermore, [11, Lemma 1.2] actually refers to constrained centers in the sense of (3.1) with K' being linear subspace of X. For this, item (c) of Theorem 3.7 has to be specified to line segments which are contained in a translate of K'. In [83, Теорема VI], Garkavi also considers normed spaces of arbitrary dimensions and the notion of *uniform convexity in every direction (равномерно выпуклый по каждому направлению)*, which coincides with rotundity in the finite-dimensional case. Next, we generalize Theorem 3.7 in two ways. On the one hand, we drop the symmetry condition of the unit ball B. That is, we replace B by an arbitrary full-dimensional convex body C and ask for covers of bounded sets by smallest possible homothetic images of C, as originally proposed in [246]. On the other hand, we will use the above mentioned result of Brandenberg and König [37, Theorem 2.3] to characterize the situation where the dimension of the set of circumcenters of every convex body K is at most $j \in \{0, \ldots, \dim(X) - 2\}$ in terms of the boundary structure of C.

Theorem 3.8. *Let (X, γ) be a generalized Minkowski space of dimension $d = \dim(X) \geq 2$ with unit ball C, and let $j \in \{0, \ldots, d-2\}$. Then the following are equivalent:*

(a) There exists a convex body $K \in \mathcal{K}^X$ such that $\dim(\mathrm{cc}(K, C)) > j$.
(b) There exist points $x_1, \ldots, x_{d-j} \in X$ such that $\dim(\mathrm{cc}(\{x_1, \ldots, x_{d-j}\}, C)) > j$.
(c) There exist points $x_1, \ldots, x_{d-j} \in \mathrm{bd}(C)$ and a convex body $A_j \in \mathcal{K}^X$ with $\dim(A_j) > j$ such that $x_i + A_j \subset \mathrm{bd}(C)$ for $i \in \{1, \ldots, d-j\}$ and there exist linear functionals $\phi_i \in \mathrm{nor}(x_i, C) \setminus \{0\}$, $i \in \{1, \ldots, d-j\}$, with $0 \in \mathrm{co}(\{\phi_1, \ldots, \phi_{d-j}\})$.

Note that the sufficiency of rotundity for the uniqueness of circumcenters in the situation of Theorem 3.8 is also stated in [179, Corollary 3.7]. For the proof of Theorem 3.8, we shall use the following tool taken from [37, Theorem 2.3]. It formalizes the wellspreadness of the touching points $K \cap \mathrm{bd}(C)$ when $K \in \mathcal{K}^X$ is optimally contained in $C \in \mathcal{K}_0^X$, that is $R(K, C) = 1$. For Euclidean spaces of dimension at most three, this has already been stated in [246].

Lemma 3.9. *Let $C \in \mathcal{K}_0^X$ be the unit ball of a generalized Minkowski space (X, γ) of dimension $d = \dim(X)$ and let $K \in \mathcal{K}^X$ be such that $K \subset C$. The following statements are equivalent.*

(a) We have $R(K, C) = 1$.

(b) *There exist points* $x_1, \ldots, x_k \in K \cap \operatorname{bd}(C)$ *for some* $k \in \{2, \ldots, d+1\}$ *and linear functionals* $\phi_i \in \operatorname{nor}(x_i, C) \setminus \{0\}$*,* $i \in \{1, \ldots, k\}$*, such that* $0 \in \operatorname{co}(\{\phi_1, \ldots, \phi_k\})$*.*

Alternatively, Lemma 3.9 can be seen in the light of convex optimization. Via (3.2), we may write the circumradius as the infimal value of the function $\sup_{y \in K} \gamma_C(y - \cdot) : X \to \mathbb{R}$ after suitably translating C. The circumcenters $x \in \operatorname{cc}(K, C)$ are then the minimizers of this function, and their optimality can be expressed in terms of linear functionals via Fermat's rule, see Lemma 1.19, which eventually reduces to item (b) of Lemma 3.9. A Lagrange duality approach is used in [240, Section 4.2.4] to reiterate the geometric interpretation of wellspreadness for Euclidean space. There, the linear functionals ϕ can be replaced by the points x_i, and the condition $0 \in \operatorname{co}(\{\phi_1, \ldots, \phi_k\})$ appearing in item (b) of Lemma 3.9 says that the Euclidean circumcenter of K can be written as a convex combination of the touching points x_i. Relatedly, the authors of [148] investigate circumcenters of convex bodies in Minkowski spaces that can be written as convex combinations of some touching points. Note that in the case when K is a finite set, the special role played by the touching points, i.e., the points of K which lie in the boundary of a fixed circumball, can already be conjectured from the subdifferential formula stated in Theorem 1.24.

Proof of Theorem 3.8. The implication (b)⇒(a) is evident. To see (c)⇒(b), we use the points $x_1, \ldots, x_{d-j}$ from (c). By Lemma 3.9, the second part of (c) implies

$$R(\{x_1, \ldots, x_{d-j}\}, C) = R(\operatorname{co}(\{x_1, \ldots, x_{d-j}\}), C) = 1.$$

The first part of (c) gives $\{x_1, \ldots, x_{d-j}\} + A_j \subset C$, i.e., $\{x_1, \ldots, x_{d-j}\} \subset C - v$ for all $v \in A_j$. Therefore, we have $-A_j \subset \operatorname{cc}(\{x_1, \ldots, x_{d-j}\}, C)$ and $\dim(\operatorname{cc}(\{x_1, \ldots, x_{d-j}\}, C)) > j$. For proving (a)⇒(c), we may suppose that $R(K, C) = 1$ and $K \subset C$. By Lemma 3.9, there are points $x_i \in K \cap \operatorname{bd}(C)$ and linear functionals $\phi_i \in \operatorname{nor}(x_i, C) \setminus \{0\}$, $i \in \{1, \ldots, d+1\}$, such that $0 \in \operatorname{co}(\{\phi_1, \ldots, \phi_{d+1}\})$. So

$$0 = \sum_{i=1}^{d+1} \lambda_i \phi_i \tag{3.3}$$

for suitable numbers $\lambda_i \geq 0$. Moreover, we may suppose that

$$\lambda_i > 0 \quad \text{for all} \quad i \in \{1, \ldots, d+1\}. \tag{3.4}$$

(Assume that $\lambda_i = 0$ for $i \in I \subsetneq \{1, \ldots, d+1\}$ and $\lambda_i > 0$ for $i \in \{1, \ldots, d+1\} \setminus I$. Choose $j \in \{1, \ldots, d+1\} \setminus I$, and replace each of the points x_i, $i \in I$, by x_j, replace each of the functionals ϕ_i, $i \in I$, by ϕ_j and each of the numbers λ_j, λ_i, $i \in I$, by $\frac{\lambda_j}{\operatorname{card}(I)+1}$.) Put $L := \operatorname{lin}(\{\phi_1, \ldots, \phi_{d+1}\})$. Let $v \in \operatorname{cc}(K, C)$, i.e., $K \subset C + v$. For every $i \in \{1, \ldots, d+1\}$, we have $x_i - v \in K - v \subset C$. This yields $\langle \phi_i \,|\, x_i - v \rangle \leq h(\phi_i, C) = \langle \phi_i \,|\, x_i \rangle$ because $\phi_i \in \operatorname{nor}(x_i, C)$. So $\langle \phi_i \,|\, v \rangle \geq 0$ for $i \in \{1, \ldots, d+1\}$. Taking (3.3) and (3.4) into account, we obtain

$$0 = \langle 0 \,|\, v \rangle = \sum_{i=1}^{d+1} \lambda_i \langle \phi_i \,|\, v \rangle$$

and, in turn,

$$\langle \phi_i \,|\, v \rangle = 0 \quad \text{for all} \quad i \in \{1, \ldots, d+1\}. \tag{3.5}$$

Consequently, we have $L \subset \mathrm{cc}(K,C)^{\perp}$ and $\dim(L) \le d - \dim(\mathrm{cc}(K,C)) < d-j$. As a consequence of Carathéodory's theorem, the condition $0 \in \mathrm{co}(\{\phi_1,\ldots,\phi_{d+1}\})$ can be strengthened to $0 \in \mathrm{co}(\{\phi_1,\ldots,\phi_{d-j}\})$ after a possible reordering of $x_1,\ldots,x_{d+1}$, and the second claim of (c) is verified. Finally, to verify that the first claim of (c) is satisfied with $A_j = -\mathrm{cc}(K,C)$, we shall show that $x_i - v \in \mathrm{bd}(C)$ for all $i \in \{1,\ldots,d-j\}$ and $v \in \mathrm{cc}(K,C)$. We have $x_i - v \in K - v \subset (C+v) - v = C$. By $\phi_i \in \mathrm{nor}(x_i, C)$, we obtain $h(\phi_i, C) = \langle \phi_i \,|\, x_i \rangle$. Using (3.5), we conclude $\langle \phi_i \,|\, x_i - v \rangle = \langle \phi_i \,|\, x_i \rangle = h(\phi_i, C)$. The observations $x_i - v \in C$ and $\langle \phi_i \,|\, x_i - v \rangle = h(\phi_i, C)$ show that $x_i - v \in \mathrm{bd}(C)$. This completes the proof. □

Remark 3.10. (a) A weaker version of condition (b) in Theorem 3.8 with $\{x_1,\ldots,x_{d-j}\}$ replaced by $\{x_1,\ldots,x_{d+1}\}$ is known to be equivalent to item (a) of Theorem 3.8, due to Lemma 3.9.

(b) Note that the number of points in item (b) in Theorem 3.8 is best possible. Example 3.11 gives, for all $d \ge 2$ and $j \in \{0,\ldots,d-2\}$, a d-dimensional generalized Minkowski space such that every set of cardinality at most $d-j-1$ has a unique circumcenter, whereas there exists a set of $d-j$ points whose set of circumcenters has dimension larger than j.

Example 3.11. Let $X = \mathbb{R}^d$. Suppose that $d \ge 2$ and $j \in \{0,\ldots,d-2\}$. Consider the $(j+1)$-dimensional cube $\square_{j+1} = [-e_{d-j}, e_{d-j}] \times \ldots \times [-e_d, e_d]$, and let $\triangle_{d-j-1} \subset \mathrm{lin}(\{e_1,\ldots,e_{d-j-1}\})$ be the vertices of a $(d-j-1)$-dimensional simplex which is regular in the Euclidean sense and satisfies $0 = \sum_{i=1}^{d-j} \frac{1}{d-j} x_i$. Choose a convex body $C \in \mathscr{K}_0^X$ with

$$\triangle_{d-j-1} + \square_{j+1} \subset C \subset (d-j-1)(-\triangle_{d-j-1}) + 2\square_{j+1} \tag{3.6}$$

such that

(i) C is smooth,
(ii) all points of $\mathrm{bd}(C) \setminus \left(\bigcup_{i=1}^{d-j} (x_i + \square_{j+1}) \right)$ are exposed points of C,

see [86] for a justification of the existence of C. Note that properties (3.6), (i), and (ii) imply

(i′) $\mathrm{nor}(x_i + v, C) = \{\lambda x_i \,|\, \lambda \ge 0\}$ for all $i \in \{1,\ldots,d-j\}$ and $v \in \square_{j+1}$,
(ii′) $x_i + \square_{j+1}$, $i \in \{1,\ldots,d-j\}$, are the only non-singleton exposed faces of C.

We see that C satisfies (c) from Theorem 3.8 with $A_j = \square_{j+1}$, because $0 = \sum_{i=1}^{d-j} \frac{1}{d-j} x_i$. So there are sets K of cardinality $d-j$, such as $K = \{x_1,\ldots,x_{d-j}\}$, satisfying $\dim(\mathrm{cc}(K,C)) > j$. Now we show that every set of cardinality at most $d-j-1$ has a unique circumcenter. Assume that this is not the case. Then there exists a set $K' = \{y_1,\ldots,y_{d-j-1}\}$ and a point $z \in X \setminus \{0\}$ such that

$$R(K',C) = 1 \qquad \text{and} \qquad K' + \lambda z \subset C \text{ for all } \lambda \in [-1,1]. \tag{3.7}$$

Applying Lemma 3.9 to the situation $\lambda = 0$, we obtain points $y_i \in K' \cap \mathrm{bd}(C)$, $i \in \{1,\ldots,k\}$, and linear functionals $\phi_i \in \mathrm{nor}(y_i, C) \setminus \{0\}$ such that $0 \in \mathrm{co}(\{\phi_1,\ldots,\phi_k\})$. The inclusion $y_i \in K' \cap \mathrm{bd}(C)$ and property (3.7) show that $[y_i - z, y_i + z] \subset \mathrm{bd}(C)$ for $i \in \{1,\ldots,k\}$. By (ii′), the points $y_1,\ldots,y_k$ belong to the $d-j$ cubes mentioned there. Now (i′) says that the outer normals $\phi_1,\ldots,\phi_k$ are positive multiples of not more than $k \le d-j-1$ of the points $x_1,\ldots,x_{d-j}$. This contradicts $0 \in \mathrm{co}(\{\phi_1,\ldots,\phi_k\})$.

We obtain an analog of Theorem 3.8 for normed spaces.

Theorem 3.12. *Let* $(X, \|\cdot\|)$ *be a Minkowski space of dimension* $\dim(X) \geq 2$ *with unit ball* B*. Furthermore, let* $j \in \{0, \ldots, \dim(X)-2\}$*. Then the following statements are equivalent.*

(a) *There exists* $K \in \mathscr{K}^X$ *such that* $\dim(\mathrm{cc}(K, B)) > j$*.*
(b) *There exists a set* $\{x_1, x_2\} \subset X$ *such that* $\dim(\mathrm{cc}(\{x_1, x_2\}, B)) > j$*.*
(c) *The ball* B *possesses an exposed face of dimension larger than* j*.*

Proof. Theorem 3.8 implies (a)⇒(c) and the implication (b)⇒(a) is obvious. It is sufficient to verify (c)⇒(b). For this, let $F \subset \mathrm{bd}(B)$ be an exposed face set of B with $\dim(F) > j$ and let $x_F \in \mathrm{ri}(F)$. Clearly, we have $R(\{x_F, -x_F\}, B) = 1$ because $\|x_F - (-x_F)\| = 2\|x_F\| = 2$ and $\{x_F, -x_F\} \subset B$. Since $x_F \in \mathrm{ri}(F)$, the set $A := (F - x_F) \cap (-F + x_F)$ contains 0 in its relative interior and satisfies $\dim(A) = \dim(F) > j$. For every point $v \in A$, we have

$$x_F - v \in x_F - A \subset x_F - (-F + x_F) = F \subset B$$

and

$$-x_F - v \in -x_F - A \subset -x_F - (F - x_F) = -F \subset B.$$

Thus $\{x_F, -x_F\} \subset B + v$ for all $v \in A$. This yields $A \subset \mathrm{cc}(\{x_F, -x_F\}, B)$ and, in turn, we have $\dim(\mathrm{cc}(\{x_F, -x_F\}, B)) \geq \dim(A) > j$. □

The equivalence of items (a) and (c) of Theorem 3.12 is also a consequence of [234, Corollary 2.8 and Proposition 2.1]. A localized version of that equivalence is [234, Theorem 2.7].

Remark 3.13. The dependence of conditions (a), (b), (c) from Theorem 3.7 in the case of generalized Minkowski spaces (X, γ) with unit ball B is as follows:

- Trivially, (b) implies (a). If $\dim(X) = 2$, then (a) also implies (b), because (b) coincides with item (b) from Theorem 3.8 for $j = 0$. If $\dim(X) \geq 3$ then (a) does not imply (b), see Example 3.11.
- (a) implies (c): For $j = 0$, item (c) from Theorem 3.8 implies item (c) from Theorem 3.7. So, if some $K \in \mathscr{K}^X$ has at least two circumcenters, then the unit ball is not rotund.
- (c) does not imply (a): A d-dimensional simplex $B \subset X = \mathbb{R}^d$ is a striking example of a unit ball of a generalized Minkowski space (X, γ) which is not rotund, but every set $K \in \mathscr{K}^X$ has a unique circumcenter with respect to B. The unique circumball of K is the intersection of the $d+1$ half-spaces having the same outer normals as the $(d-1)$-facets of the simplex and whose bounding hyperplanes support K.

Finally, let us point out that for $j = 0$, condition (c) from Theorem 3.8 gives rise to the following criterion in generalized Minkowski planes, cf. [233, Lemma 1].

Corollary 3.14. *In a generalized Minkowski plane* (X, γ)*, every convex body* $K \in \mathscr{K}^X$ *has a unique circumcenter with respect to the unit ball* $B(0,1)$ *if and only if the unit ball* $B(0,1)$ *does not have a pair of parallel supporting lines* H_1*,* H_2*, for which the exposed faces* $B(0,1) \cap H_1$*,* $B(0,1) \cap H_2$ *are non-singletons.*

3.2 Inradius, diameter, and minimum width

Instead of asking for the smallest homothetic image of a convex body which contains a second convex body of fixed size and position, we may interchange the roles of the convex bodies and ask for the largest homothetic image of a convex body which is contained in a second convex body of fixed size and position. Of course, both approaches result in the situation of *optimal containment* as described in Lemma 3.9: Simultaneously, there is no smaller homothetic image of the “outer” convex body and no larger homothetic image of the “inner” convex body such that the inclusion is preserved. Historically, the notions of the *inradius* and the *incenter* of a convex body have been investigated in Euclidean spaces first. Although that covers the interchangeability of the involved convex bodies, Zindler proved forerunners of Lemma 3.9 only in the Euclidean plane [246, Satz 2 and Satz 4] and in three-dimensional Euclidean space [246, Satz 12 and Satz 13].

Definition 3.15. Let $K \subset X$ and $C \in \mathscr{C}^X$. The *inradius of K with respect to C* is

$$r(K,C) := \sup_{x\in X} \sup \{\lambda \geq 0 \mid x + \lambda C \subset K\}.$$

If $x + r(K,C)\,C \subset K$, then x is a *incenter* of K with respect to C. The *set of incenters of K with respect to C* is the set $\operatorname{ic}(K,C) := \{x \in X \mid x + r(K,C)\,C \subset K\}$.

Note that both incenters and circumcenters are sometimes called *Chebyshev centers*, see [34, Section 4] and [148]. The definition of both notions are strikingly similar, see Figure 3.2.

Figure 3.2. Inradius: The set C is a triangle (bold line, left), the set K is a Reuleaux triangle (bold line, right). The inradius $r(K,C)$ is determined by the largest homothetic image of C that is contained in K (thin line).

This similarity is continued in the behavior of the inradius under basic manipulations of the input sets.

Proposition 3.16. *Let $K, K' \subset X$, $C, C' \in \mathscr{C}^X$, $x, y \in X$, and $\alpha, \beta > 0$. Then*

(a) $r(K',C') \geq r(K,C)$ *if* $K' \subset K$ *and* $C \subset C'$,
(b) $r(K,C) = r(\operatorname{cl}(K),C)$ *if K is convex,*
(c) $r(K+K',C) \geq r(K,C) + r(K',C)$,
(d) $r(x+K, y+C) = r(K,C)$,
(e) $r(\alpha K, \beta C) = \frac{\alpha}{\beta} r(K,C)$,
(f) $r(K,C') \geq r(K,C)\, r(C,C')$.

For $K, C \in \mathscr{K}_0^X$, the equation $R(C,K) = r(K,C)^{-1}$ is well-known [207, p. 388, Remark 4], and it implies

$$\mathrm{ic}(K,C) = -r(K,C)\,\mathrm{cc}(C,K)\,. \tag{3.8}$$

Thus, for understanding circumcenter sets and incenter sets, it is sufficient to describe the shape of one of them, where K and C are arbitrary sets from $\mathscr{K}_0^X$. A direct consequence of the definition and formula (3.8) is the following statement, see also Proposition 3.4.

Proposition 3.17. *If $K, C \in \mathscr{K}_0^X$ are centrally symmetric, then so are $\mathrm{cc}(K,C)$ and $\mathrm{ic}(K,C)$.*

Early results concerning the dimension of the set of incenters are [246, Satz 5, Satz 14, and Satz 15] which concern the two-dimensional and three-dimensional Euclidean space. In the original version of Bonnesen and Fenchel's monograph from 1934 (see [33, p. 59] in the 1987 edition), one finds the following statement without any proof: The set of incenters of an arbitrary convex body in $\mathbb{R}^d$ dimensions is an arbitrary convex body of dimension at most $d-1$.

Proposition 3.18. *For any $K, C \in \mathscr{K}_0^X$, the set $\mathrm{ic}(K,C)$ is non-empty, bounded, closed, and convex of dimension $\dim(\mathrm{ic}(K,C)) \leq \dim(X) - 1$. Moreover, if K is a polytope, then so is $\mathrm{ic}(K,C)$.*

Proof. Nonemptiness, boundedness, closedness, and convexity of $\mathrm{ic}(K,C)$ can be deduced from Theorem 3.6 and (3.8). Alternatively, we use that $\mathrm{ic}(K,C)$ can be written as the Minkowski difference $K \sim r(K,C)C$ and thus can be written as the intersection of closed half-spaces, see [207, p. 147, (3.19)]. This yields also the addendum. □

By virtue of (3.8) and Proposition 3.18, the set $\mathrm{cc}(K,C)$ is a polytope whenever $K, C \in \mathscr{K}_0^X$ and C is a polytope.
Another quantity which is used to measure the size of a given set $K \subset X$ is its *diameter*. In Euclidean geometry, the diameter of a given set is usually defined as the maximum distance of two points of this set. Averkov [15, Theorem 2] shows that there are several other representations of the diameter which coincide in Minkowski spaces. In the sequel, we investigate analogously defined prototypes for the diameter in generalized Minkowski spaces and how they relate to each other. We start with a classical approach, namely the diameter as the maximum distance between points of a set. For a convex set, this coincides with the maximum length of line segments contained in the chosen set. As line segments are independent of the order of their generating endpoints, it might be desirable to assign a length to a line segment which follows the same principle. This can be done using the circumradius.

Lemma 3.19. *Given $x, y, z \in X$, $\alpha > 0$, and $C \in \mathscr{K}_0^X$ with $0 \in \mathrm{int}(C)$, we define $g : X \times X \to \mathbb{R}$, $g(x,y) = 2R(\{x,y\},C)$. Then*

(a) $g(x,y) \geq 0$ *with equality if and only if* $x = y$,
(b) $g(x,y) = g(y,x)$,
(c) $g(x+z, y+z) = g(x,y)$,
(d) $g(\alpha x, \alpha y) = \alpha g(x,y)$,
(e) $g(x,y) \leq g(x,z) + g(z,y)$.

Proof. The nonnegativity follows from the definition. The characterization of the equality case is a consequence of the more general result that $R(K,C)=0$ if and only if K is contained in a translate of the cone $\{y \in X \mid y + C \subset C\}$ when the set of extreme points of C is bounded, see [38, Lemma 2.2]. The symmetry $g(x,y)=g(y,x)$ is clear. The invariance under translations and the compatibility with scaling follow from Proposition 3.2(d), (e). Finally, since g is translation-invariant and symmetric, we only have to check $g(0,x+y) \leq g(0,x)+g(0,y)$ for the triangle inequality. But we have

$$\begin{aligned} g(0,x+y) &= R(\{0,x+y\},C) \leq R(\{0,x,y,x+y\},C) \\ &\leq R(\{0,x\},C)+R(\{0,y\},C) = g(0,x)+g(0,y) \end{aligned}$$

by Proposition 3.2(c). □

Since the triangle inequality for g turns out to be true, the mapping $X \ni x \mapsto 2R(\{0,x\},C)$, defines a norm on X. The unit ball of this norm is $\frac{1}{2}(C-C)$. This fact can be proved as follows. First, we show that if $x \in \frac{1}{2}(C-C)$, then $R(\{0,x\},\frac{1}{2}C) \leq 1$. Namely, there exist $x_1, x_2 \in \frac{1}{2}C$ such that $x = x_1 - x_2$. Thus $R(\{0,x\},\frac{1}{2}C) = R(\{x_1,x_2\},\frac{1}{2}C) \leq 1$. The reverse implication can be done analogously. If $R(\{0,x\},\frac{1}{2}C) > 1$, then there is no point $z \in X$ such that $\{0,x\} \in z + \frac{1}{2}C$ or, equivalently, such that $\{-z, x-z\} \in \frac{1}{2}C$. Thus there is no representation $x = (x-z)-(-z) \in \frac{1}{2}C - \frac{1}{2}C$. Next, we show that for centrally symmetric sets K, the maximal circumradius of two-element subsets is attained at antipodal points of K, cf. Proposition 3.4.

Proposition 3.20. *Let $K \subset X$ be a bounded set and let $C \in \mathscr{K}_0^X$. If $K = -K$, then*

$$\sup\{R(\{-x,x\},C) \mid x \in K\} = \sup\{R(\{x,y\},C) \mid x,y \in K\}.$$

Proof. Using Proposition 3.2, we have

$$\begin{aligned} &\sup\{R(\{x,y\},C) \mid x,y \in K\} \\ \leq\, &\sup\{R(\{0,x,y,x+y\},C) \mid x \in K\} \\ \leq\, &\sup\{R(\{0,x\},C) \mid x \in K\} + \sup\{R(\{0,y\},C) \mid y \in K\} \\ =\, &2\sup\{R(\{0,x\},C) \mid x \in K\} \\ =\, &\sup\{R(\{0,2x\},C) \mid x \in K\} \\ =\, &\sup\{R(\{-x,x\},C) \mid x \in K\} \\ \leq\, &\sup\{R(\{x,y\},C) \mid x,y \in K\}. \end{aligned}$$

This yields the assertion. □

Another representation of the diameter in Minkowski spaces involves the maximal chord-length function and the radius function of a convex body which are the maximal Euclidean length of chords with given direction and the Euclidean distance from the origin 0 to the boundary, respectively. More precisely, the *maximal chord-length function* of $K \subset X$ is defined by $l_K : X \to \overline{\mathbb{R}}$, $l_K(x) := \sup\{\alpha > 0 \mid \alpha x \in K - K\}$. The *radius function* $\varrho_K : X \to \overline{\mathbb{R}}$, defined by $\varrho_K(x) := \sup\{\alpha > 0 \mid \alpha x \in K\}$, is the pointwise inverse of the Minkowski functional γ_K. Under reasonable assumptions, this formulation of the diameter coincides with the interpretation as the maximum distance of two points of the given set when distances are measured by the gauge γ_C.

Theorem 3.21. *For $K \subset X$ and $C \in \mathscr{K}_0^X$ with $0 \in \operatorname{int}(C)$, the following numbers are equal:*

(a) $\sup\{\gamma_C(x-y) \mid x, y \in K\}$,

(b) $\sup\{\langle \phi \mid x\rangle \mid \phi \in C^\circ, x \in K-K\}$.

If $K \in \mathscr{C}^X$, then the following number also belongs to this set of equal quantities:

(c) $\sup\left\{\frac{l_K(x)}{\varrho_C(x)} \,\middle|\, x \in X \setminus \{0\}\right\}$.

Proof. Using Lemma 1.11, have

$$\begin{aligned}
\sup_{x,y\in K} \gamma_C(x-y) &= \sup_{x,y\in K} \sup_{u\in C^\circ} \langle \phi \mid x-y\rangle \\
&= \sup_{\phi\in C^\circ} \sup_{x,y\in K} \langle \phi \mid x-y\rangle \\
&= \sup_{\phi\in C^\circ} \sup_{x,y\in K} (\langle \phi \mid x\rangle + \langle -\phi \mid y\rangle) \\
&= \sup_{\phi\in C^\circ} (h(\phi,K) + h(-\phi,K)) \\
&= \sup_{\phi\in C^\circ} h(\phi, K-K) \\
&= \sup\{\langle \phi \mid x\rangle \mid \phi \in C^\circ, x \in K-K\}.
\end{aligned}$$

If $K \in \mathscr{C}^X$, then

$$\begin{aligned}
\sup_{x,y\in K} \gamma_C(x-y) &= \sup_{x\in X\setminus\{0\}} \sup_{\alpha>0:\, \alpha x\in K-K} \gamma_C(\alpha x) \\
&= \sup_{x\in X\setminus\{0\}} \sup_{\alpha>0:\, \alpha x\in K-K} \alpha\gamma_C(x) \\
&= \sup_{x\in X\setminus\{0\}} l_K(x)\gamma_C(x) \\
&= \sup_{x\in X\setminus\{0\}} \frac{l_K(x)}{\varrho_C(x)}.
\end{aligned}$$

Now the proof is complete. □

Combining the quantities from Lemma 3.19 and Theorem 3.21, we obtain a chain of inequalities.

Theorem 3.22. *If $K \subset X$ and $C \in \mathscr{K}_0^X$ with $0 \in \operatorname{int}(C)$, then*

$$\begin{aligned}
2\sup\left\{\frac{h(\phi,K-K)}{h(\phi,C-C)} \,\middle|\, \phi \in X^* \setminus \{0\}\right\} &= 2\sup\{R(\{x,y\},C) \mid x,y \in K\} \\
&= R\left(K-K, \frac{1}{2}(C-C)\right) \qquad (3.9) \\
&\le R(K-K, C) \\
&\le \sup\{\gamma_C(x-y) \mid x,y \in K\},
\end{aligned}$$

with equality if $C = -C$. If $K \in \mathscr{C}^X$, then we also have

$$\sup\{R(\{x,y\},C) \mid x,y \in K\} = \sup\left\{\frac{l_K(x)}{l_C(x)} \,\middle|\, x \in X \setminus \{0\}\right\}. \qquad (3.10)$$

Proof. If $C=-C$, then we have

$$\begin{aligned}
2\sup\left\{\frac{h(\phi,K-K)}{h(\phi,C-C)}\,\middle|\,\phi\in X^*\setminus\{0\}\right\} &= \sup\left\{\frac{h(\phi,K-K)}{h(\phi,C)}\,\middle|\,\phi\in X^*\setminus\{0\}\right\}\\
&= \sup\left\{\frac{h(\phi,K-K)}{h(\phi,C)}\,\middle|\,\phi\in C^\circ\setminus\{0\}\right\}\\
&= \sup\left\{h\left(\frac{\phi}{h(\phi,C)},K-K\right)\,\middle|\,\phi\in C^\circ\setminus\{0\}\right\}\\
&= \sup\{h(\phi,K-K)\mid\phi\in C^\circ\setminus\{0\}\}\\
&= \sup\{h(\phi,K-K)\mid\phi\in C^\circ\}\\
&= \sup\{\gamma_C(x)\mid x\in K-K\}\\
&= R(K-K,C)\\
&= R\left(K-K,\frac{1}{2}(C-C)\right)\\
&= \sup\left\{\gamma_{\frac{1}{2}(C-C)}(x)\,\middle|\,x\in K-K\right\}\\
&= 2\sup\{R(\{0,x\},C)\mid x\in K-K\}\\
&= 2\sup\{R(\{x,y\},C)\mid x,y\in K\}
\end{aligned}$$

by taking Propositions 3.2 and 3.4, Lemma 3.19, and Theorem 3.21 into account. Note that Lemma 3.19 is independent of central symmetry of C and, therefore, can be similarly used in the general case, which comes next. So we drop the assumption $C=-C$ now, apply the calculations performed for the symmetric case, and obtain

$$\begin{aligned}
2\sup\left\{\frac{h(\phi,K-K)}{h(\phi,C-C)}\,\middle|\,\phi\in X^*\setminus\{0\}\right\} &= 2\sup\left\{\frac{h(\phi,(K-K)-(K-K))}{h(\phi,(C-C)-(C-C))}\,\middle|\,\phi\in X^*\setminus\{0\}\right\}\\
&= R\left((K-K)-(K-K),\frac{1}{2}((C-C)-(C-C))\right)\\
&= R\left(K-K,\frac{1}{2}(C-C)\right)\\
&= 2\sup\{R(\{x,y\},C)\mid x,y\in K\}\\
&= 2\sup\{R(\{0,x\},C)\mid x\in K-K\}\\
&= \sup\{R(\{-x,x\},C)\mid x\in K-K\}\\
&\le R(K-K,C)\\
&\le \inf\{\lambda>0\mid K-K\subset\lambda C\}\\
&= \sup\{\gamma_C(x)\mid x\in K-K\}\\
&= \sup\{\gamma_C(x-y)\mid x,y\in K\}
\end{aligned}$$

For $K\in\mathscr{C}^X$, we have

$$2\sup\{R(\{x,y\},C)\mid x,y\in K\} = 2\sup_{x\in X\setminus\{0\}}\sup\left\{\frac{\alpha}{l_C(x)}\,\middle|\,\alpha>0,\alpha x\in K-K\right\}$$

$$= 2 \sup_{x \in X \setminus \{0\}} \frac{\sup \{\alpha \mid \alpha > 0, \alpha x \in K - K\}}{l_C(x)}$$
$$= 2 \sup \left\{ \frac{l_K(x)}{l_C(x)} \,\middle|\, x \in X \setminus \{0\} \right\}.$$

This yields the addendum (3.10). □

The following examples show that the inequalities in (3.9) need not be strict if K and C are not centrally symmetric but, on the other hand, may be strict even if K is centrally symmetric. An illustration of these examples is provided by Figure 3.3.

Example 3.23. (a) Let $X = \mathbb{R}^2$, denote by $B := \{x \in X \mid \|x\|_2 \leq 1\}$ the unit ball of the Euclidean norm, and let

$$C := -K := ((2,0) + 2\sqrt{3}B) \cap ((-1, \sqrt{3}) + 2\sqrt{3}B) \cap ((-1, -\sqrt{3}) + 2\sqrt{3}B).$$

Then $K - K = C - C = 2\sqrt{3}B$, i.e.,

$$2 \sup \left\{ \frac{h(\phi, K-K)}{h(\phi, C-C)} \,\middle|\, \phi \in X^* \setminus \{0\} \right\} = 2,$$

but $R(K-K, C) = \sup \{\gamma_C(x-y) \mid x, y \in K\} = \frac{1}{2}(3 + \sqrt{3}) \approx 2.366025$.

(b) Let $X = \mathbb{R}^2$, $C := \operatorname{co}\bigl(\{(2,0), (-1, \sqrt{3}), (-1, -\sqrt{3})\}\bigr)$, and

$$K := \operatorname{co}\bigl(\{(-\sqrt{3}, -\sqrt{3}), (-\sqrt{3}, \sqrt{3}), (\sqrt{3}, -\sqrt{3}), (\sqrt{3}, \sqrt{3})\}\bigr).$$

Then

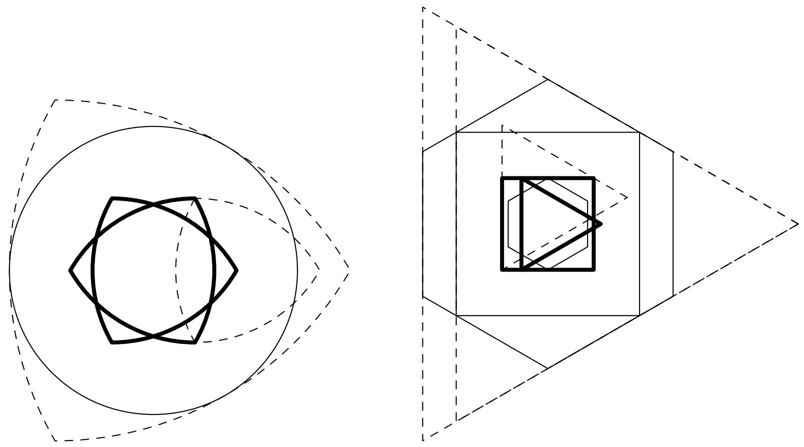
$$\sup \{\gamma_C(x-y) \mid x, y \in K\} = 3 + \sqrt{3} \qquad \approx 4.732,$$
$$R(K-K, C) = 2 + \frac{4}{\sqrt{3}} \qquad \approx 4.3094,$$
$$2 \sup \{R(\{x,y\}, C) \mid x, y \in K\} = \frac{2}{3}(3 + \sqrt{3}) \approx 3.1547,$$
$$2 \sup \left\{ \frac{h(\phi, K-K)}{h(\phi, C-C)} \,\middle|\, \phi \in X^* \setminus \{0\} \right\} = \frac{2}{3}(3 + \sqrt{3}) \approx 3.1547.$$

Although each of the prototypes of the diameter from Theorem 3.22 may have its own benefits, we follow [38, Definition 2.5] and define the notion of diameter via circumradii of two-element subsets which is, by Lemma 3.19, the usual diameter with respect to the norm with unit ball $\frac{1}{2}(C - C)$.

Definition 3.24. The *diameter* of K with respect to C is

$$D(K, C) := 2 \sup \{R(\{x,y\}, C) \mid x, y \in K\}.$$

By Theorem 3.22, the diameter is invariant under symmetrization of its arguments: $D(K, C) = D(K - K, C - C)$. This has already been stated in [38, Lemma 2.8]. The diameter also behaves conveniently under hull operations and Minkowski sums in the first arguments, as well as under independent translation and scaling of both arguments.



(a) C and K are Reuleaux triangles.

(b) C is an equilateral triangle, K is a square.

Figure 3.3. Illustration of Example 3.23: The sets C and K are depicted in bold lines, while homothetic images of $K-K$ and $C-C$ are depicted in thin solid lines, and homothetic images of C are depicted in dashed lines.

Proposition 3.25. *Let $K, K' \subset X$, $C, C' \in \mathscr{C}^X$, $x, y \in X$, and $\alpha, \beta > 0$. Then we have*

(a) $D(K', C') \le D(K, C)$ *if* $K' \subset K$ *and* $C \subset C'$,
(b) $D(K, C) = D(\operatorname{cl}(K), C) = D(\operatorname{co}(K), C)$,
(c) $D(K + K', C) \le D(K, C) + D(K', C)$,
(d) $D(x + K, y + C) = D(K, C)$,
(e) $D(\alpha K, \beta C) = \frac{\alpha}{\beta} D(K, C)$,
(f) $D(K, C') \le D(K, C) D(C, C')$.

Proof. Statement (a) is a consequence of Proposition 3.2(a), and it yields $D(K, C) \le D(\operatorname{cl}(K), C)$ and $D(K, C) \le D(\operatorname{co}(K), C)$. Furthermore, we obtain

$$\begin{aligned}
D(\operatorname{co}(K), C) &= \sup \{R(\{0, x\}, C) \mid x \in \operatorname{co}(K-K)\} \\
&= \sup \left\{ R\left(\left\{ 0, \sum_{i=1}^{n} \lambda_i x_i \right\}, C \right) \,\middle|\, \begin{array}{c} n \in \mathbb{N}, x_i \in K-K, \lambda_i \ge 0, \\ i \in \{1, \ldots, n\}, \sum_{i=1}^{n} \lambda_i = 1 \end{array} \right\} \\
&\le \sup \left\{ R\left(\sum_{i=1}^{n} \lambda_i \{0, x_i\}, C \right) \,\middle|\, \begin{array}{c} n \in \mathbb{N}, x_i \in K-K, \lambda_i \ge 0, \\ i \in \{1, \ldots, n\}, \sum_{i=1}^{n} \lambda_i = 1 \end{array} \right\} \\
&\le \sup \left\{ \sum_{i=1}^{n} \lambda_i R(\{0, x_i\}, C) \,\middle|\, \begin{array}{c} n \in \mathbb{N}, x_i \in K-K, \lambda_i \ge 0, \\ i \in \{1, \ldots, n\}, \sum_{i=1}^{n} \lambda_i = 1 \end{array} \right\} \\
&\le \sup \left\{ \sum_{i=1}^{n} \lambda_i D(K, C) \,\middle|\, \begin{array}{c} n \in \mathbb{N}, \lambda_i \ge 0, \\ i \in \{1, \ldots, n\}, \sum_{i=1}^{n} \lambda_i = 1 \end{array} \right\}
\end{aligned}$$

$$
\begin{aligned}
&= D(K,C) \sup \left\{ \sum_{i=1}^{n} \lambda_i \;\middle|\; \begin{array}{c} n \in \mathbb{N}, \lambda_i \geq 0, \\ i \in \{1,\ldots,n\}, \sum_{i=1}^{n} \lambda_i = 1 \end{array} \right\} \\
&= D(K,C)
\end{aligned}
$$

and

$$
\begin{aligned}
D(\mathrm{cl}(K),C) &= \sup \{R(\{x_i,y_i\},C) \mid x_i,y_i \in K, i \in \mathbb{N}, x_i \to x, y_i \to y\} \\
&\leq \sup \left\{ \sup \{R(\{w,z\},C) \mid w,z \in K\} \;\middle|\; \begin{array}{c} x_i,y_i \in K, i \in \mathbb{N}, \\ x_i \to x, y_i \to y \end{array} \right\} \\
&= \sup \{R(\{w,z\},C) \mid w,z \in K\}.
\end{aligned}
$$

This yields claim (b). In order to prove item (c), we observe that

$$
\begin{aligned}
D(K+K',C) &= \sup\{R(\{w+w',z+z'\},C) \mid w,z \in K, w',z' \in K'\} \\
&\leq \sup\{R(\{w+w',z+z',w+z,w'+z'\},C) \mid w,z \in K, w',z' \in K'\} \\
&\leq \sup\{R(\{w,z\},C)) + R(\{w',z'\},C) \mid w,z \in K, w',z' \in K'\} \\
&= \sup \{R(\{w,z\},C) \mid w,z \in K\} + \sup\{R(\{w',z'\},C) \mid w',z' \in K'\} \\
&= D(K,C) + D\big(K',C\big).
\end{aligned}
$$

In order to prove (f), we use Proposition 3.2(f) to show

$$
\begin{aligned}
D(K,C') &= \sup\{R(\{x,y\},C') \mid x,y \in K\} \\
&\leq \sup\{R(\{x,y\},C)R(C,C') \mid x,y \in K\} \\
&= D(K,C)R(C,C').
\end{aligned}
$$

Recall that for $C = -C \in \mathscr{K}_0^X$, we have $R(K,C) \leq D(K,C)$ by Bohnenblust's inequality, see [38, Theorem 4.1]. Taking the invariance of the diameter under symmetrization of its arguments, i.e., $D(K,C) = D(K-K,C-C)$, into account, we obtain

$$
\begin{aligned}
D(K,C') &= D(K-K,C'-C') \\
&\leq D(K-K,C-C)R(C-C,C'-C') \\
&\leq D(K-K,C-C)\,D(C-C,C'-C') \\
&= D(K,C)\,D(C,C').
\end{aligned}
$$

This completes the proof. □

In the proof of Proposition 3.25, we used (a generalization of) Bohnenblust's inequality as presented in [38, Theorem 4.1]. This inequality gives a lower bound of the diameter in terms of the circumradius. The classical upper bound of the diameter in terms of the circumradius is still valid in generalized Minkowski spaces. Namely $D(K,C) \leq 2R(K,C)$ for all sets $K \subset X$ and $C \in \mathscr{K}_0^X$ with $0 \in \mathrm{int}(C)$, with equality if, e.g., $C = -C$ and $K = -K$. This is follows immediately from Proposition 3.2(a) and Theorem 3.22.

We close this section with a discussion of the notion of *minimum width* of a given set $K \subset X$. In Euclidean space, it is intimately related to the notion of diameter. While the latter is the supremum of the distances of parallel supporting hyperplanes (see Theorem 3.21), the former is classically defined as the corresponding infimum. In Minkowski spaces, the minimum width can be introduced by several coinciding expressions, see [15, Theorem 3]. In this spirit, we regard those expressions as prototypes for defining minimum width respect to a convex body $C \in \mathscr{K}_0^X$, collect relations between them, and fix a definition afterwards. As a first instance, we may refer to $C \in \mathscr{K}_0^X$ as the reference body for measuring the minimum width of a set $K \subset X$ by taking the ratio of the Euclidean width functions, i.e., the support functions of the difference sets.

Lemma 3.26. *Let $K \subset X$, $C \in \mathscr{K}_0^X$. If $C = -C$, then we have*

$$2\inf\left\{\frac{h(\phi,K-K)}{h(\phi,C-C)} \,\middle|\, \phi \in X^* \setminus \{0\}\right\} = \inf\left\{\frac{h(\phi,K-K)}{\gamma_{C^\circ}(\phi)} \,\middle|\, \phi \in X^* \setminus \{0\}\right\}.$$

In other words, the minimal ratio of the (Euclidean) width functions is equal to the minimal distance of parallel supporting hyperplanes of K, measured by the norm γ_C. If $h(\phi,K-K) > 0$ for all $\phi \in X^* \setminus \{0\}$, also the reverse implication is true.

Proof. The first statement is clear by the relations $\gamma_{C^\circ} = h(\cdot,C)$ and $C-C = 2C$. For the reverse statement, assume that $h(\phi,K-K) > 0$ for all $\phi \in X^* \setminus \{0\}$ and

$$2\inf\left\{\frac{h(\phi,K-K)}{h(\phi,C-C)} \,\middle|\, \phi \in X^* \setminus \{0\}\right\} < \inf\left\{\frac{h(\phi,K-K)}{h(\phi,C)} \,\middle|\, \phi \in X^* \setminus \{0\}\right\}. \tag{3.11}$$

Then

$$2\frac{h(\phi,K-K)}{h(\phi,C-C)} < \frac{h(\phi,K-K)}{h(\phi,C)}$$

for all $\phi \in X^* \setminus \{0\}$, which is equivalent to $\frac{1}{2}h(\cdot,C-C) > h(\cdot,C)$ and, in turn, to $h(\cdot,C) > h(\cdot,-C)$ which is impossible. We arrive at the same conclusion if we assume the reverse inequality in (3.11). □

Secondly, the minimum width in Minkowski spaces coincides with the inradius of the difference body in Minkowski spaces.

Lemma 3.27. *If $K, C \in \mathscr{K}_0^X$, then*

$$r(K-K,C) = R(C,K-K)^{-1} = (\sup\{\langle \phi \,|\, x\rangle \mid \phi \in (K-K)^\circ, x \in C\})^{-1}.$$

Proof. Note that $K-K$ is centrally symmetric and apply Theorem 3.22. □

Note that the claim of the previous result fails if K is not convex. For example, if K is a finite set, then $r(K-K,C) = 0$. But

$$\begin{aligned}(\mathrm{co}(K)-\mathrm{co}(K))^\circ &= (\mathrm{co}(K-K))^\circ = \{\phi \in X^* \mid h(\phi,\mathrm{co}(K-K)) \leq 1\} \\ &= \{\phi \in X^* \mid h(\phi,K-K) \leq 1\} = (K-K)^\circ.\end{aligned}$$

It follows that

$$\sup\{\langle\phi \mid x\rangle \mid \phi \in (K-K)^\circ, x \in C\} = \sup\{\langle\phi \mid x\rangle \mid \phi \in (\operatorname{co}(K)-\operatorname{co}(K))^\circ, x \in C\},$$

which is apparently not equal to zero in general. If one drops the requirement of central symmetry of C, the quantities considered in Lemmas 3.26 and 3.27 need not coincide.

Lemma 3.28. *Let $K \in \mathscr{K}^X$, $C \in \mathscr{K}_0^X$. Then*

$$r(K-K,C) \leq 2\inf\left\{\frac{h(\phi,K-K)}{h(\phi,C-C)} \,\middle|\, \phi \in X^* \setminus \{0\}\right\} \tag{3.12}$$

with equality if $C=-C$.

Proof. For all $\alpha < r(K-K,C)$, there exists $z \in X$ such that $z+\alpha C \subset K-K$. Thus $\alpha(C-C) = (z+\alpha C)-(z+\alpha C) \subset (K-K)-(K-K) = 2(K-K)$. It follows that $\alpha h(\phi,C-C) \leq 2h(\phi,K-K)$ for all $\phi \in X^* \setminus \{0\}$, i.e., $\alpha \leq 2\frac{h(\phi,K-K)}{h(\phi,C-C)}$ for all $\phi \in X^* \setminus \{0\}$. Passing α to $r(K-K,C)$, we obtain (3.12). Now let $C=-C$ and assume that (3.12) is a strict inequality, i.e., there exists a number $\alpha \in \mathbb{R}$ such that

$$r(K-K,C) < \alpha < 2\inf\left\{\frac{h(\phi,K-K)}{h(\phi,C-C)} \,\middle|\, \phi \in X^* \setminus \{0\}\right\}.$$

Like above, we obtain $h(\cdot,\alpha C-\alpha C) < h(\cdot,(K-K)-(K-K))$. Dividing by 2, we have $h(\cdot,\alpha C) < h(\cdot,K-K)$. It follows that $r(K-K,C)C \subsetneq \alpha C \subsetneq K-K$. This is a contradiction to the definition of the inradius $r(K-K,C)$. □

Example 3.29. If $C \neq -C$, we may have a strict inequality in (3.12). In the situation of Example 3.23(a), we obtain

$$2\inf\left\{\frac{h(\phi,K-K)}{h(\phi,C-C)} \,\middle|\, \phi \in X^* \setminus \{0\}\right\} = 2,$$

but obviously $r(K-K,C) = \sqrt{3} \neq 2$, see Figure 3.4.

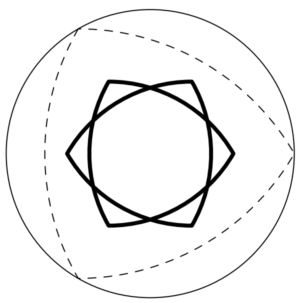


Figure 3.4. Illustration of Example 3.29: K and C are Reuleaux triangles (bold lines), $K-K$ is a Euclidean ball (thin solid line), $r(K-K,C)$ is determined by a homothetic image of C (thin dashed line).

Summarizing, we obtain the following theorem on the notion of minimum width in normed spaces.

Theorem 3.30. *For $K \in \mathscr{C}^X$ and $C \in \mathscr{K}_0^X$ with $C = -C$, the following numbers are equal:*

(a) $r(K-K,C)$,
(b) $2\inf\left\{\frac{h(\phi,K-K)}{h(\phi,C-C)} \,\middle|\, \phi \in X^* \setminus \{0\}\right\}$,
(c) $\inf\left\{\frac{h(\phi,K-K)}{\gamma_{C^\circ}(\phi)} \,\middle|\, \phi \in X^* \setminus \{0\}\right\}$,
(d) $(\sup\{\langle \phi \mid x\rangle \mid \phi \in (K-K)^\circ, x \in C\})^{-1}$.

Proof. This is a combination of the previous results. If $h(\cdot,K-K) \equiv +\infty$, then $K = X$ and all the numbers equal $+\infty$. Similarly, if there is $\phi \in X^* \setminus \{0\}$ such that $h(\phi,K-K) = 0$, then all the numbers equal 0. (Only for the last item, use the conventions $1/0 = +\infty$ and $1/(+\infty) = 0$.) □

We follow [38, Definition 2.6] in defining the notion of minimum width via an infimal ratio of support functions.

Definition 3.31. The *minimum width* of $K \subset X$ with respect to $C \in \mathscr{C}^X$ is

$$\Delta(K,C) := 2\inf\left\{\frac{h(\phi,K-K)}{h(\phi,C-C)} \,\middle|\, \phi \in X^* \setminus \{0\}\right\} = 2\inf\left\{R(K,C+L) \,\middle|\, L \in \mathscr{L}^X_{\dim(X)-1}\right\}.$$

By definition, the minimum with is invariant under symmetrization of its arguments: $\Delta(K,C) = \Delta(K-K,C-C)$. This has already been stated in [38, Lemma 2.8]. Analogously to Propositions 3.2, 3.16, and 3.25, we check basic properties of the minimum width with respect to a convex body C.

Proposition 3.32. *Let $K,K' \subset X$, $C,C' \in \mathscr{C}^X$, $x,y \in X$, and $\alpha,\beta > 0$. Then we have*

(a) $\Delta(K',C') \geq \Delta(K,C)$ *if $K' \subset K$ and $C \subset C'$,*
(b) $\Delta(K,C) = \Delta(\mathrm{cl}(K),C)$ *if K is convex,*
(c) $\Delta(K+K',C) \geq \Delta(K,C) + \Delta(K',C)$,
(d) $\Delta(x+K,y+C) = \Delta(K,C)$,
(e) $\Delta(\alpha K,\beta C) = \frac{\alpha}{\beta}\Delta(K,C)$,
(f) $\Delta(K,C') \geq \Delta(K,C)\,\Delta(C,C')$.

Proof. For $x \in X$, $\lambda \geq 0$, and $L \in \mathscr{L}^X_{\dim(X)-1}$, we have $\mathrm{cl}(K) \subset x + L + \lambda C$ if and only if $K \subset x + L + \lambda C$. This proves (b). In order to prove item (c), we observe

$$\begin{aligned}
&\Delta(K+K',C)\\
&= \inf\left\{\frac{h(\phi,K+K'-K-K')}{h(\phi,C-C)} \,\middle|\, \phi \in X^* \setminus \{0\}\right\}\\
&= \inf\left\{\frac{h(\phi,K-K)}{h(\phi,C-C)} + \frac{h(\phi,K'-K')}{h(\phi,C-C)} \,\middle|\, \phi \in X^* \setminus \{0\}\right\}\\
&\geq \inf\left\{\frac{h(\phi,K-K)}{h(\phi,C-C)} \,\middle|\, \phi \in X^* \setminus \{0\}\right\} + \inf\left\{\frac{h(\phi,K'-K')}{h(\phi,C-C)} \,\middle|\, \phi \in X^* \setminus \{0\}\right\}\\
&= \Delta(K,C) + \Delta(K',C).
\end{aligned}$$

Finally, we have

$$\begin{aligned}
&\Delta(K, C') \\
&= \inf\left\{ \frac{h(\phi, K-K)}{h(\phi, C-C)} \frac{h(\phi, C-C)}{h(\phi, C'-C')} \,\middle|\, \phi \in X^* \setminus \{0\} \right\} \\
&\geq \inf\left\{ \frac{h(\phi, K-K)}{h(\phi, C-C)} \,\middle|\, \phi \in X^* \setminus \{0\} \right\} \inf\left\{ \frac{h(\phi, C-C)}{h(\phi, C'-C')} \,\middle|\, \phi \in X^* \setminus \{0\} \right\} \\
&= \Delta(K, C')\Delta(C, C').
\end{aligned}$$

This completes the proof. □

3.3 Successive radii

For determining the circumradius of a set $K \subset X$ with respect to a convex body $C \in \mathscr{K}_0^X$, it is required to find the least possible scaling factor λ such that a translate of λC contains K. The convex sets among which we search for the ones that optimally contain K are homothetic images of a fixed convex body C. A variation of this concept can be achieved by covering K by a union of equal-sized homothetic images of C, where the size is to be minimized. Such a union will always be a Minkowski sum $A + \lambda C$ for some $A \subset X$. Thus we would consider the quantity

$$\begin{aligned}
f(K, A, C) &= \inf\{\lambda > 0 \mid K \subset A + \lambda C\} \\
&= \inf\{\lambda > 0 \mid y \in A + \lambda C \ \forall\, y \in K\} \\
&= \inf\{\lambda > 0 \mid (y - \lambda C) \cap A \neq \emptyset \ \forall\, y \in K\}. \qquad (3.13)
\end{aligned}$$

If C is the unit ball of a gauge $\gamma : X \to \mathbb{R}$ and $A = \{z\}$ is a singleton, we may interpret $f(K, A, C)$ as the least possible radius of a ball centered at z and containing K. Conversely, if $K = \{x\}$ is a singleton, then $f(K, A, C)$ is nothing but $\underline{\mathrm{dist}}_{\gamma^\vee}(x, A)$. Generally, the number $f(K, A, C)$ is not only the least possible common radius of balls centered at points of A whose union covers K, but also A intersects all of the convex bodies $y - \lambda C$ with $y \in K$ for $\lambda > f(K, A, C)$. Mostly when A is an affine subspace, then A is called a *transversal* of the convex bodies $y - \lambda C$ or it is said to *stab* those convex bodies if $(y - \lambda C) \cap A \neq \emptyset$ for all $y \in K$, see [9, Section 2.7], [60, Section 4], and [239] for surveys on geometric transversal theory. Another variation of the definition of the circumradius of a set $K \subset X$ can be performed by computing the extremal circumradii of K with respect to the members of a certain family of convex bodies. For instance, we may consider *cylinders*, i.e., Minkowski sums of $C \in \mathscr{K}_0^X$ and linear subspaces L of X of fixed dimension $j \in \{0, \ldots, \dim(X) - 1\}$, and compute the extremal values of $R(K, C + L)$ as L traverses $\mathscr{L}_j^X$. (Note that we did this already in Definition 3.31.) Transversals and circumradii with respect to cylinders are connected via $\inf_{x \in X} f(K, x + L, C) = R(K, C + L)$, with f as in (3.13) and L being a linear subspace of X. In Euclidean space, this connection is exploited in the contexts of location science and computational geometry, see [1, 151]. The aspect of computing circumradii of $K \subset X$ with respect to cylinders has been introduced by Gritzmann and Klee [92, 93] in Minkowski spaces, as part of an extension of the theory of successive radii. This theory goes back to the late 1970s [198] and it deals with Euclidean circumradii and inradii of sections of K by and projections of K onto affine subspaces of specified dimension. This research direction has been continued in [23, 24, 103, 191], and successive radii are linked to intrinsic volumes [104] and successive minima of convex bodies [105] but also to certain aspects of operator theory [90]. In Euclidean space, projecting K onto a j-dimensional linear subspace and computing the circumradius with respect to the Euclidean ball is equivalent to computing the circumradius of K with respect to the Minkowski sum of the Euclidean ball and the orthogonal complement of the linear subspace. Therefore, the latter procedure can be used as a definition for generalizations.

In the following, we replace the centrally symmetric unit ball of a Minkowski space by an arbitrary convex body with non-empty interior and give analogous definitions of these quantities in this generalized setting. We start with the most basic notions which are called *inner* and *outer j-radii* in [92, 93], *external* and *internal radii* in [191], and *Kolmogorov* and *Bernstein diameters* in [198]. In this section, our default assumptions are $K \subset X$, $C \in \mathcal{K}_0^X$, and $0 \in \operatorname{int}(C)$.

Definition 3.33. Let $j \in \{1, \ldots, \dim(X)\}$. Define

(a)
$$r_\sigma^j(K,C) := \sup_{L \in \mathcal{L}_j^X} \inf_{x \in X} \sup_{y \in X} \sup \{\lambda \geq 0 \mid y + \lambda(C \cap (x+L)) \subset K\}$$
$$= \sup_{L \in \mathcal{L}_j^X} \inf_{x \in X} r(K, C \cap (x+L)).$$

(b)
$$R_j^\pi(K,C) := \inf\{\lambda \geq 0 \mid \exists\, L \in \mathcal{L}_{\dim(X)-j}^X \; \exists\, x \in X : K \subset x + L + \lambda C\}$$
$$= \inf_{L \in \mathcal{L}_{\dim(X)-j}^X} \inf_{x \in X} \inf\{\lambda \geq 0 \mid K \subset x + L + \lambda C\}$$
$$= \inf_{L \in \mathcal{L}_{\dim(X)-j}^X} \inf_{x \in X} \sup_{y \in K} \underline{\operatorname{dist}}_{\gamma_C}(y, x+L)$$
$$= \inf_{L \in \mathcal{L}_{\dim(X)-j}^X} R(K, C+L),$$

Note that both terminology and notation are not unified in the literature. Here we adapt the notation of [23] which consists of a letter R or r, referring to whether circumradii or inradii are used in the corresponding Euclidean definition, and a superscript and a subscript index. One of the latter two is a number $j \in \{1, \ldots, \dim(X)\}$ referring to the dimension of the involved linear subspaces, the other one is a letter π or σ referring to projections and sections used in the classical (Euclidean) definitions. The usage of cylinders in the definitions of circumradii instead of projections onto and sections by linear subspaces helps us avoid the question for the "correct" reference body with respect to which we would have to compute circumradii in an affine subspace. In Euclidean space, there is no ambiguity because projections of Euclidean balls onto linear subspaces and sections of Euclidean balls by linear subspaces "all look alike".

Example 3.34. Take $X = \mathbb{R}^d$ and denote by $\operatorname{proj}(K,L)$ the orthogonal projection of $K \subset X$ onto a linear subspace $L \subset X$, that is, $\operatorname{proj}(K,L) = \bigcup_{x \in K} P_K(x)$ where the projection $P_K(x)$ is performed with respect to the Euclidean norm. If C is the Euclidean ball, we have

$$R_j^\pi(K,C) = \inf_{L \in \mathcal{L}_j^X} R(\operatorname{proj}(K,L), C) = \inf_{L \in \mathcal{L}_j^X} R(\operatorname{proj}(K,L), \operatorname{proj}(C,L)),$$

see [37, Definition 3.1 and Remark 3.2]. Else, circumradii of projections onto linear subspaces are not the same as circumradii of projections with respect to projections of C. For instance, take $X = \mathbb{R}^3$, $L := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = 0\}$, and

$$C := [(1,0,1), (-1,0,-1)] + \{(x_1, x_2, 0) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq 1\}.$$

If $K := \operatorname{proj}(C, L)$, then $R(K, C) = 2$ but $R(K, \operatorname{proj}(C, L)) = 1$, see Figure 3.5.

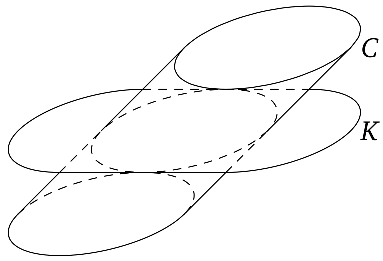


Figure 3.5. Illustration of Example 3.34.

In Definition 3.35, we extend more series of successive radii appearing in their Euclidean version in [23, 88] to generalized Minkowski spaces by substituting the reference to the Euclidean ball with a reference to a convex body $C \in \mathcal{K}_0^X$ with $0 \in \operatorname{int}(C)$. The Euclidean versions of R_π^j, r_j^σ, and r_π^j appear also in [90, 104], while R_π^j and R_σ^j have already been investigated in their generalized form in [37].

Definition 3.35. Let $j \in \{1, \dots, \dim(X)\}$. Define

(a) $R_\sigma^j(K, C) := \sup_{L \in \mathcal{L}_j^X} \sup_{x \in X} R(K \cap (x + L), C)$,
(b) $R_j^\sigma(K, C) := \inf_{L \in \mathcal{L}_j^X} \sup_{x \in X} R(K \cap (x + L), C)$,
(c) $r_\pi^j(K, C) := \sup_{L \in \mathcal{L}_{\dim(X)-j}^X} r(K + L, C)$,
(d) $r_j^\pi(K, C) := \inf_{L \in \mathcal{L}_{\dim(X)-j}^X} r(K + L, C)$,
(e)
$$\begin{aligned} R_\pi^j(K, C) &:= \sup_{L \in \mathcal{L}_{\dim(X)-j}^X} R(K, C + L) \\ &= \sup_{L \in \mathcal{L}_{\dim(X)-j}^X} \inf\{\lambda \geq 0 \mid \exists\, x \in X : K \subset x + L + \lambda C\} \\ &= \sup_{L \in \mathcal{L}_{\dim(X)-j}^X} \inf_{x \in X} \inf\{\lambda \geq 0 \mid K \subset x + L + \lambda C\} \\ &= \sup_{L \in \mathcal{L}_{\dim(X)-j}^X} \inf_{x \in X} \sup_{y \in K} \underline{\operatorname{dist}}_{\gamma_C}(y, x + L), \end{aligned}$$
(f)
$$\begin{aligned} r_j^\sigma(K, C) &:= \inf_{L \in \mathcal{L}_j^X} \inf_{x \in X} \sup_{y \in X} \sup\{\lambda \geq 0 \mid y + \lambda(C \cap (x + L)) \subset K\} \\ &= \inf_{L \in \mathcal{L}_j^X} \inf_{x \in X} r(K, C \cap (x + L)). \end{aligned}$$

In [37, Theorem 3.3], it is shown that $R_\pi^j(K,C) = R_\sigma^j(K,C)$ for all $j \in \{1,\dots,\dim(X)\}$ and all $K \in \mathscr{C}^X$. In fact, this equality is valid for arbitrary sets $K \subset X$ due to the invariance of $R_\pi^j(K,C)$ and $R_\sigma^j(K,C)$ under taking the convex hull in the first arguments. Next we establish the monotonicity of R_π^j and R_j^π with respect to the dimension index j. In the Euclidean setting, these results are stated in [88, 104], the first one also in [89, 90, 92, 105, 191, 229], and the second one also in [23, 37].

Theorem 3.36. *We have* $R_\pi^1(K,C) \le \dots \le R_\pi^{\dim(X)}(K,C)$ *and* $R_1^\pi(K,C) \le \dots \le R_{\dim(X)}^\pi(K,C)$.

Proof. Fix $j \in \{1,\dots,\dim(X)-1\}$. Let $L \in \mathscr{L}_{\dim(X)-j}^X$ and $L' \in \mathscr{L}_{\dim(X)-j-1}^X$ be such that $L' \subsetneq L$. For all $x \in X$, we have

$$\inf\{\lambda \ge 0 \,|\, K \subset x + L + \lambda C\} \le \inf\{\lambda \ge 0 \,|\, K \subset x + L' + \lambda C\}.$$

Since $x \in X$ is arbitrary, we obtain

$$\inf_{x\in X} \inf\{\lambda \ge 0 \,|\, K \subset x + L + \lambda C\} \le \inf_{x\in X} \inf\{\lambda \ge 0 \,|\, K \subset x + L' + \lambda C\}.$$

It follows that

$$\inf_{x\in X} \inf\{\lambda \ge 0 \,|\, K \subset x + L + \lambda C\} \le \sup_{\substack{L' \in \mathscr{L}_{\dim(X)-j-1}^X: \\ L' \subsetneq L}} \inf_{x\in X} \inf\{\lambda \ge 0 \,|\, K \subset x + L' + \lambda C\}.$$

Consequently,

$$\begin{aligned} R_\pi^j(K,C) &= \sup_{L \in \mathscr{L}_{\dim(X)-j}^X} \inf_{x\in X} \inf\{\lambda \ge 0 \,|\, K \subset x + L + \lambda C\} \\ &\le \sup_{L \in \mathscr{L}_{\dim(X)-j}^X} \sup_{\substack{L' \in \mathscr{L}_{\dim(X)-j-1}^X: \\ L' \subsetneq L}} \inf_{x\in X} \inf\{\lambda \ge 0 \,|\, K \subset x + L' + \lambda C\} \\ &= R_\pi^{j+1}(K,C) \end{aligned}$$

and

$$\begin{aligned} R_j^\pi(K,C) &= \inf_{L \in \mathscr{L}_{\dim(X)-j}^X} \inf_{x\in X} \inf\{\lambda \ge 0 \,|\, K \subset x + L + \lambda C\} \\ &\le \inf_{L \in \mathscr{L}_{\dim(X)-j}^X} \inf_{\substack{L' \in \mathscr{L}_{\dim(X)-j-1}^X: \\ L' \subsetneq L}} \inf_{x\in X} \inf\{\lambda \ge 0 \,|\, K \subset x + L' + \lambda C\} \\ &= R_{j+1}^\pi(K,C). \end{aligned}$$

We obtain $R_\pi^j(K,C) \le R_\pi^{j+1}(K,C)$ and $R_j^\pi(K,C) \le R_{j+1}^\pi(K,C)$. $\square$

The monotonicity of R_j^σ and R_σ^j with respect to j is established in [23, 88] for the Euclidean setting.

Theorem 3.37. *We have* $R_\sigma^1(K,C) \le \dots \le R_\sigma^{\dim(X)}(K,C)$ *and* $R_1^\sigma(K,C) \le \dots \le R_{\dim(X)}^\sigma(K,C)$.

Proof. Fix $j \in \{1,\dots,\dim(X)-1\}$. Let $L \in \mathcal{L}_j^X$ and $L' \in \mathcal{L}_{j+1}^X$ be such that $L \subsetneq L'$. For all $x \in X$, we have $K \cap (x+L) \subset K \cap (x+L')$, and hence

$$\inf_{y\in X} \inf\{\lambda \geq 0 \,|\, K \cap (x+L) \subset y + \lambda C\} \leq \inf_{y\in X} \inf\{\lambda \geq 0 \,|\, K \cap (x+L') \subset y + \lambda C\}.$$

It follows that

$$\inf_{y\in X} \inf\{\lambda \geq 0 \,|\, K \cap (x+L) \subset y + \lambda C\} \leq \sup_{\substack{L'\in\mathcal{L}_{j+1}^X:\\ L\subsetneq L'}} \inf_{y\in X} \inf\{\lambda \geq 0 \,|\, K \cap (x+L') \subset y + \lambda C\}.$$

Consequently,

$$\begin{aligned} R_\sigma^j(K,C) &= \sup_{L\in\mathcal{L}_j^X} \sup_{x\in X} \inf_{y\in X} \inf\{\lambda \geq 0 \,|\, K \cap (x+L) \subset y + \lambda C\} \\ &\leq \sup_{L\in\mathcal{L}_j^X} \sup_{\substack{L'\in\mathcal{L}_{j+1}^X:\\ L\subsetneq L'}} \sup_{x\in X} \inf_{y\in X} \inf\{\lambda \geq 0 \,|\, K \cap (x+L') \subset y + \lambda C\} \\ &= R_\sigma^{j+1}(K,C) \end{aligned}$$

and

$$\begin{aligned} R_j^\sigma(K,C) &= \inf_{L\in\mathcal{L}_j^X} \sup_{x\in X} \inf_{y\in X} \inf\{\lambda \geq 0 \,|\, K \cap (x+L) \subset y + \lambda C\} \\ &\leq \inf_{L\in\mathcal{L}_j^X} \inf_{\substack{L'\in\mathcal{L}_{j+1}^X:\\ L\subsetneq L'}} \sup_{x\in X} \inf_{y\in X} \inf\{\lambda \geq 0 \,|\, K \cap (x+L') \subset y + \lambda C\} \\ &= R_{j+1}^\sigma(K,C). \end{aligned}$$

This completes the proof. □

The monotonicity of the inradii counterparts can be shown by similar arguments. For the corresponding results in Euclidean space, see again [23, 88–90, 92, 104, 105, 191].

Theorem 3.38. *We have*

$$\begin{aligned} &r_\pi^1(K,C) \geq \ldots \geq r_\pi^{\dim(X)}(K,C), && r_1^\sigma(K,C) \geq \ldots \geq r_{\dim(X)}^\sigma(K,C), \\ &r_1^\pi(K,C) \geq \ldots \geq r_{\dim(X)}^\pi(K,C), && r_\sigma^1(K,C) \geq \ldots \geq r_\sigma^{\dim(X)}(K,C). \end{aligned}$$

The next identities for the special cases $j \in \{1, \dim(X)\}$ connect the eight series of successive radii with the circumradius, the inradius, the diameter, and the minimum width. Their Euclidean counterparts are stated in every of the aforementioned papers on successive radii, in which the corresponding quantities appear.

Lemma 3.39. *We have*

$$\begin{aligned} R_{\dim(X)}^\pi(K,C) = R_\pi^{\dim(X)}(K,C) = R_\sigma^{\dim(X)}(K,C) = R_{\dim(X)}^\sigma(K,C) &= R(K,C), \\ r_\sigma^{\dim(X)}(K,C) = r_{\dim(X)}^\sigma(K,C) = r_{\dim(X)}^\pi(K,C) = r_\pi^{\dim(X)}(K,C) &= r(K,C), \\ R_1^\pi(K,C) = R_1^\sigma(K,C) = r_1^\pi(K,C) = r_1^\sigma(K,C) &= \tfrac{1}{2}\Delta(K,C), \\ R_\pi^1(K,C) = R_\sigma^1(K,C) = r_\pi^1(K,C) = r_\sigma^1(K,C) &= \tfrac{1}{2}D(K,C). \end{aligned}$$

Proof. The first two lines and $R_1^\pi(K,C) = \frac{1}{2}\Delta(K,C)$ are direct from the definitions. Theorem 3.22 and [37, Theorem 3.3] yield

$$\begin{aligned} D(K,C) &= 2\sup\left\{ \frac{h(\phi,K-K)}{h(\phi,C-C)} \,\middle|\, \phi \in X^* \setminus \{0\} \right\} \\ &= 2\sup\{R(K,C+L) \mid L \in \mathscr{L}^X_{\dim(X)-1}\} = 2R^1_\pi(K,C) = 2R^1_\sigma(K,C). \end{aligned}$$

For $r^1_\pi(K,C) = \frac{1}{2}D(K,C)$ and $r_1^\pi(K,C) = \frac{1}{2}\Delta(K,C)$, note that $r(K+L,C) = r(K+L,C+L) = R(K+L,C+L) = R(K,C+L) = \frac{h(\phi,K-K)}{h(\phi,C-C)}$ for $\phi \in X^* \setminus \{0\}$ and $L := \phi_{=0} \in \mathscr{L}^X_{\dim(X)-1}$. For the remaining equalities we use that $r([x,y],[w,z]) = R([x,y],[w,z])$ for $x,y,z,w \in X$ with $x \neq y$, $z \neq w$, and $\mathrm{lin}(\{y-x\}) = \mathrm{lin}(\{z-w\})$. Consider

$$\begin{aligned} r^1_\sigma(K,C) &= \sup_{L \in \mathscr{L}^X_1} \inf_{x \in X} r(K, C \cap (x+L)) \\ &= \sup_{L \in \mathscr{L}^X_1} \inf_{x \in X} r\left(K, \frac{1}{2}(C-C) \cap (x+L)\right) \\ &= \sup_{L \in \mathscr{L}^X_1} \inf_{x \in X} r\left(\frac{1}{2}(K-K), \frac{1}{2}(C-C) \cap (x+L)\right) \\ &= \sup_{L \in \mathscr{L}^X_1} r\left(\frac{1}{2}(K-K), \frac{1}{2}(C-C) \cap L\right) \\ &= \sup_{L \in \mathscr{L}^X_1} r\left(\frac{1}{2}(K-K) \cap L, \frac{1}{2}(C-C) \cap L\right) \\ &= \sup_{L \in \mathscr{L}^X_1} R\left(\frac{1}{2}(K-K) \cap L, \frac{1}{2}(C-C) \cap L\right) \\ &= \sup_{L \in \mathscr{L}^X_1} R\left(\frac{1}{2}(K-K) \cap L, \frac{1}{2}(C-C)\right) \\ &= \frac{1}{2}D\left(\frac{1}{2}(K-K), \frac{1}{2}(C-C)\right) \\ &= \frac{1}{2}D(K,C), \\ r_1^\sigma(K,C) &= \inf_{L \in \mathscr{L}_1} \inf_{x \in X} r(K, C \cap (x+L)) \\ &= \inf_{L \in \mathscr{L}^X_1} \inf_{x \in X} r\left(K, \frac{1}{2}(C-C) \cap (x+L)\right) \\ &= \inf_{L \in \mathscr{L}^X_1} \inf_{x \in X} r\left(\frac{1}{2}(K-K), \frac{1}{2}(C-C) \cap (x+L)\right) \\ &= \inf_{L \in \mathscr{L}^X_1} r\left(\frac{1}{2}(K-K), \frac{1}{2}(C-C) \cap L\right) \\ &= \inf_{L \in \mathscr{L}^X_1} r\left(\frac{1}{2}(K-K) \cap L, \frac{1}{2}(C-C) \cap L\right) \end{aligned}$$

$$
\begin{aligned}
&= r\left(\frac{1}{2}(K-K), \frac{1}{2}(C-C)\right)\\
&= \frac{1}{2}\Delta\left(\frac{1}{2}(K-K), \frac{1}{2}(C-C)\right)\\
&= \frac{1}{2}\Delta(K,C)\,,
\end{aligned}
$$

and in a similar fashion,

$$
\begin{aligned}
R_1^\sigma(K,C) &= \inf_{L\in\mathscr{L}_1^X} \sup_{x\in X} R(K\cap(x+L), C)\\
&= \inf_{L\in\mathscr{L}_1^X} \sup_{x\in X} R\left(\frac{1}{2}(K-K)\cap(x+L), C\right)\\
&= \inf_{L\in\mathscr{L}_1^X} \sup_{x\in X} R\left(\frac{1}{2}(K-K)\cap(x+L), \frac{1}{2}(C-C)\right)\\
&= \inf_{L\in\mathscr{L}_1^X} R\left(\frac{1}{2}(K-K)\cap L, \frac{1}{2}(C-C)\right)\\
&= \inf_{L\in\mathscr{L}_1^X} R\left(\frac{1}{2}(K-K)\cap L, \frac{1}{2}(C-C)\cap L\right)\\
&= \inf_{L\in\mathscr{L}_1^X} r\left(\frac{1}{2}(K-K)\cap L, \frac{1}{2}(C-C)\cap L\right)\\
&= \frac{1}{2}\Delta(K,C)\,.
\end{aligned}
$$

This completes the proof. □

4 Intersections of translates of a convex body

In the proof of Theorem 3.6, we used that the circumradius and the set of circumcenters can be written in terms of intersections of translates of a convex body, namely

$$R(K,C) = \inf\left\{\lambda > 0 \;\middle|\; \bigcap_{y\in K}(y-\lambda C) \neq \emptyset\right\},$$
$$\operatorname{cc}(K,C) = \bigcap_{y\in K}(y - R(K,C)\,C).$$

Following and extending the presentation of [120, Section 4] and [124], we use this as a template for our next definition.

Definition 4.1. Let $K,B \in \mathscr{K}_0^X$ with $0 \in \operatorname{int}(B)$. The *ball intersection* of K with respect to B and parameter λ is defined as

$$\operatorname{bi}(K,B,\lambda) := \bigcap_{x\in K}(x-\lambda B).$$

Similarly to the theory developed in Chapter 3, the assumption $0 \in \operatorname{int}(B)$ in Definition 4.1 is not essential but it will lead to easier formulations of some results as it enables the utilization of the gauge $\gamma_B : X \to \mathbb{R}$. For instance, ball intersections of K with respect to B are the sublevel sets of the convex function $\sup_{y\in K}\gamma_B(y-\cdot) : X \to \mathbb{R}$, whose infimal value is the circumradius, see again (3.2):

$$\left\{x \in X \;\middle|\; \sup_{y\in K}\gamma_B(y-x) \leq \lambda\right\} = \operatorname{bi}(K,B,\lambda).$$

Note that we call $\operatorname{bi}(K,B,\lambda)$ the ball intersection with respect to B but it is actually an intersection of translates of $-\lambda B$, i.e., an intersection of balls of $\gamma_{-B} = \gamma_B^\vee$. Note that in the definition of ball intersections of [149], signs are changed: in their language, the set $\operatorname{bi}(K,-B,\lambda)$ is called the λB-*ball intersection* of K. In Minkowski spaces, ball intersections frequently appear in the context of diametrically complete bodies and convex bodies of constant width, see, for instance, [219, Section 2]. There it is naturally complemented by the notion of the ball hull, which is also an intersection of equal-sized balls.

Definition 4.2. Let $K,B \in \mathscr{K}_0^X$ with $0 \in \operatorname{int}(B)$. The *ball hull* with respect to B and parameter λ is defined as

$$\operatorname{bh}(K,B,\lambda) := \bigcap_{x\in X:\,K\subset x+\lambda B}(x+\lambda B).$$

If there is no point $x \in X$ such that $K \subset x + \lambda B$, i.e., if $R(K,B) > 1$, then $\operatorname{bh}(K,B,\lambda) := X$ shall be understood as the intersection of an empty family of sets.

Another notion called ball hull which is used in Banach space theory [173,174,227] and approximation theory [42] refers to the intersection of all closed balls (of arbitrary radii) containing a given set. Figure 4.1 shows ball hulls according to our definition for polygonal sets K and B.

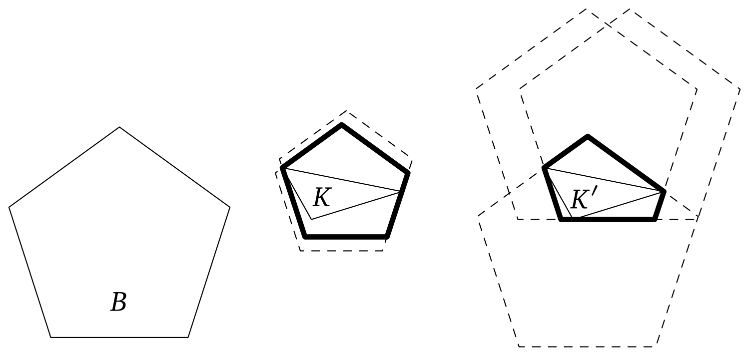


Figure 4.1. The ball hulls $\mathrm{bh}(K,B,0.6)$ and $\mathrm{bh}(K',B,1)$ are depicted in bold lines, where K and K' are triangles, and B is a pentagon.

The combination of the notions of ball hull and ball intersection yields almost the same formulas as in normed spaces, i.e., in the case $B=-B$, see [219, Propositions 2.1.1 and 2.1.2].

Theorem 4.3. *Let $K,K',B\in\mathcal{K}_0^X$ such that $0\in\mathrm{int}(B)$ and $K\subset K'$. Furthermore, let $\lambda,\lambda'\in\mathbb{R}$ with $\lambda\leq\lambda'$.*

(a) We have $\mathrm{bi}(K,B,\lambda)\supset\mathrm{bi}(K',B,\lambda)$ and $\mathrm{bh}(K,B,\lambda)\subset\mathrm{bh}(K',B,\lambda)$.

(b) We have $\mathrm{bi}(K,B,\lambda)\subset\mathrm{bi}(K,B,\lambda')$ and $\mathrm{bh}(K,B,\lambda)\supset\mathrm{bh}(K,B,\lambda')$.

(c) If $\mathrm{bh}(K,B,\lambda)$ and $\mathrm{bi}(K,B,\lambda)$ are non-empty, we have $\mathrm{bh}(K,B,\lambda)=\mathrm{bi}(\mathrm{bi}(K,B,\lambda),-B,\lambda)$ and $\mathrm{bi}(K,B,\lambda)=\mathrm{bi}(\mathrm{bh}(K,B,\lambda),B,\lambda)$.

(d) If $\sup\{\gamma_B(x-y)\mid x,y\in K\}=\lambda$, we have $K\subset\mathrm{bh}(K,B,\lambda)\subset\mathrm{bi}(K,-B,\lambda)$ and $K\subset\mathrm{bh}(K,-B,\lambda)\subset\mathrm{bi}(K,B,\lambda)$.

(e) If $\sup\{\gamma_B(x-y)\mid x,y\in K\}=\sup\{\gamma_B(x-y)\mid x,y\in\mathrm{bi}(K,B,\lambda)\}=\lambda$, then we have $\mathrm{bh}(K,B,\lambda)=\mathrm{bi}(K,B,\lambda)=\mathrm{bh}(K,-B,\lambda)=\mathrm{bi}(K,-B,\lambda)$.

Proof. Statement (a) follows directly from Definitions 4.1 and 4.2. In order to prove the first part of (b), observe that $x-\lambda B\subset x-\lambda' B$ for all $x\in K$. This implies

$$\mathrm{bi}(K,B,\lambda)=\bigcap_{x\in K}(x-\lambda B)\subset\bigcap_{x\in K}(x-\lambda' B)=\mathrm{bi}(K,B,\lambda').$$

For the second part of (b), note that $K\subset x+\lambda' B$ if $K\subset x+\lambda B$. Therefore $\{x\in X\mid x+\lambda B\supset K\}\subset\{x\in X\mid x+\lambda' B\supset K\}$, which yields $\mathrm{bh}(K,B,\lambda)\supset\mathrm{bh}(K,B,\lambda')$. The first part of (c) follows from

$$y\in\mathrm{bh}(K,B,\lambda)\Longleftrightarrow y\in\bigcap_{x\in X:\,K\subset x+\lambda B}(x+\lambda B)$$

$$\begin{aligned}&\Longleftrightarrow y \in \bigcap_{x\in \mathrm{bi}(K,B,\lambda)} (x+\lambda B)\\ &\Longleftrightarrow y \in \mathrm{bi}(\mathrm{bi}(K,B,\lambda),-B,\lambda),\end{aligned}$$

because, for $x \in X$, we have $K \subset x+\lambda B$ if and only if $z \in x+\lambda B$ for all $z \in K$ if and only if $x \in z-\lambda B$ for all $z \in K$ if and only if $x \in \mathrm{bi}(K,B,\lambda)$. The second part of (c) is a consequence of $\mathrm{bi}(K,B,\lambda) = \{x \in X \mid K \subset x+\lambda B\}$ and the equivalence of $K \subset x+\lambda B$ and $\mathrm{bh}(K,B,\lambda) \subset x+\lambda B$. Indeed, if K is contained in a translate of λB, then $\mathrm{bh}(K,B,\lambda)$ is a subset of this translate by Definition 4.2. Conversely, since $\mathrm{bh}(K,B,\lambda)$ is the intersection of all translates of λB that contain K, it follows that K itself is a subset of $\mathrm{bh}(K,B,\lambda)$, and therefore it is contained in each translate of λB that contains $\mathrm{bh}(K,B,\lambda)$. Now assume that $\sup\{\gamma_B(x-y) \mid x,y \in K\} = \lambda$. Then, for all $x,y \in K$, we have $y-x \in \lambda B$. In other words, $x+\lambda B$ is a translate of λB which contains K for all $x \in K$. Hence $\mathrm{bh}(K,B,\lambda) \subset \mathrm{bi}(K,-B,\lambda)$. A similar argument can be employed for the proof the second part of (d). Combining (c) and (d), we finally obtain

$$\begin{aligned}\mathrm{bh}(K,-B,\lambda) \subset \mathrm{bi}(K,B,\lambda) &\subset \mathrm{bi}(\mathrm{bi}(K,B,\lambda),-B,\lambda) = \mathrm{bh}(K,B,\lambda)\\ &\subset \mathrm{bi}(K,-B,\lambda) \subset \mathrm{bi}(\mathrm{bi}(K,-B,\lambda),B,\lambda) = \mathrm{bh}(K,-B,\lambda),\end{aligned}$$

and the proof is complete. □

Remark 4.4. Since we have $K \subset x+\lambda B$ if and only if $\mathrm{bh}(K,B,\lambda) \subset x+\lambda B$, it follows that the relation $\mathrm{bh}(K,B,\lambda) = \mathrm{bh}(\mathrm{bh}(K,B,\lambda),B,\lambda)$ is valid. Let us also prove the relation

$$\mathrm{bi}(K,B,\lambda) = \mathrm{bh}(\mathrm{bi}(K,B,\lambda),-B,\lambda).$$

The inclusion $\mathrm{bi}(K,B,\lambda) \subset \mathrm{bh}(\mathrm{bi}(K,B,\lambda),-B,\lambda)$ is trivial because $\mathrm{bh}(A,-B,\lambda)$ is, by definition, the intersection of supersets of $A := \mathrm{bi}(K,B,\lambda)$. Thus it remains to show the reverse inclusion. By definition, we have $K \subset \{y \in X \mid y-\lambda B \supset \mathrm{bi}(K,B,\lambda)\}$ and obtain

$$\mathrm{bi}(K,B,\lambda) = \bigcap_{y\in K}(y-\lambda B) \supset \bigcap_{y\in X:\, y-\lambda B\supset \mathrm{bi}(K,B,\lambda)} (y-\lambda B) = \mathrm{bh}(\mathrm{bi}(K,B,\lambda),-B,\lambda).$$

The family of sets that are intersections of balls (of the same radius) has the intriguing property that the intersection of any two sets of the family belongs it, too. This property is shared by the family of sets that are intersections of closed half-spaces, i.e., closed convex sets. In this sense, intersections of balls yield an abstraction of the eponymous concept of convex geometry: convexity. One is then also interested in finding analogs of related convexity concepts like support, separation, the Krein–Milman theorem as well as Helly's, Radon's, and Carathéodory's theorems, see [17, 18, 136, 140]. A selection of these topics is covered in the present chapter, referring to the notion of *ball convexity* with respect to a convex body studied by Lángi et al. [140]. Note that the interest in intersections of balls goes beyond the aspect of abstract convexity. Intersections of finitely many equal-sized balls in two-dimensional and three-dimensional Euclidean space have already been studied in [106, 107] in the contexts of Borsuk's problem and diameters of finite sets. More recent papers contribute to the study of the combinatorial structure of intersections of finitely many congruent balls [26, 27, 139, 188] and their perimeter as well as their approximation properties in the two-dimensional case [25, 78]. Starting with the following definition, all ball hulls in the remainder of this chapter are meant to be with respect to the unit ball B of a generalized Minkowski space (X,γ) and with parameter 1.

Definition 4.5. Let K be a subset of a generalized Minkowski space (X,γ) with unit ball B. The set K is said to be *b-convex* if $K = \mathrm{bh}(K,B,1)$ or, equivalently, if K is an intersection of closed balls of radius 1.

According to Definition 4.5, both the empty set $\emptyset$ and the vector space X are b-convex. (The latter can be understood as the intersection of an empty family of balls.)
In classical convexity, the boundary of a convex body K consists of exposed faces which carry information about the closed half-spaces which are essential to form K by taking their intersection. Using Straszewicz's theorem [207, Theorem 1.4.7], a convex body can be retrieved from the information of its exposed points, namely by taking their closed convex hull. Moreover, suitably chosen closed half-spaces separate convex sets in the sense that they contain one but not the other. Under mild conditions, these concepts can be adapted to ball convexity. We start with some definitions.

Definition 4.6. Let (X,γ) be a generalized Minkowski space with unit ball B. A *b-convex body* K is a non-empty bounded b-convex set. A *supporting sphere* $S(x,1)$ of K is characterized by $K \subset B(x,1)$ and $K \cap S(x,1) \neq \emptyset$. The corresponding *b-exposed face* is $K \cap S(x,1)$. If a b-exposed face is a singleton $\{x_0\}$, then x_0 is called a *b-exposed point* of K, and $\text{b-exp}(K,B)$ denotes the set of all b-exposed points. A set K is called *b-bounded* if $R(K,B) < 1$.

Note that $\emptyset$ and X are the only b-convex sets that are not b-convex bodies, and the non-empty facets of ball-polyhedra in [139, Definition 5.3] are a special case of the b-exposed faces defined here. We close this section by summarizing several basic facts about ball hulls and circumradii, and we give a lemma on intersections of compact sets with the boundaries of their circumballs.

Lemma 4.7. *Let (X,γ) be a generalized Minkowski space with unit ball B. The following statements are true for all sets $K, K' \subset X$ and points $x \in X$.*

(a) *We have $K \subset \mathrm{cl}(\mathrm{co}(K)) \subset \mathrm{bh}(K,B,1)$ and $\mathrm{bh}(K,B,1) = \mathrm{bh}(\mathrm{cl}(K),B,1) = \mathrm{bh}(\mathrm{co}(K),B,1) = \mathrm{bh}(\mathrm{bh}(K,B,1),B,1)$.*
(b) *The ball $B(x,\lambda)$ is a b-convex body for every $\lambda \in [0,1]$.*
(c) *If $R(K,B) \leq 1$, then $R(\mathrm{bh}(K,B,1),B) = R(K,B)$. In particular, the set $\mathrm{bh}(K,B,1)$ is b-bounded if K is b-bounded.*
(d) *If K is closed and $K \subset U(x,\lambda)$ for some $\lambda > 0$, then $K \subset B(x,\lambda')$ for some $\lambda' \in (0,\lambda)$ and $R(K,B) < \lambda$. In particular, a closed subset of X is b-bounded if and only if it is contained in an open ball of radius 1.*

Proof. Statement (a) is direct from Remark 4.4 and Theorem 4.3(a). For (b), the triangle inequality gives the following representation of $B(x,\lambda)$ as an intersection of balls of radius 1:

$$B(x,\lambda) = \bigcap_{y \in x+(\lambda-1)B(0,1)} B(y,1).$$

To see (c), first note that $R(K,B) \leq R(\mathrm{bh}(K,B,1),B)$ by (a). If $B(x,R(K,B))$ is a circumball of K, then $\mathrm{bh}(K,B,1) \subset \mathrm{bh}(B(x,R(K,B)),B,1) = B(x,R(K,B))$ by (b) and Theorem 4.3(a). Hence $R(\mathrm{bh}(K,B,1),B) \leq R(B(x,R(K,B)),B) = R(K,B)$. For (d), suppose that K contains at least two points. Consider the continuous function $f := \underline{\mathrm{dist}}_{\gamma^\vee}(\cdot, S(x,\lambda)) : X \to \mathbb{R}$. Since K is a compact set, the function f attains its infimum on K at a point y_0, i.e., $f(y) \geq f(y_0) \in (0,\lambda)$ for all $y \in K$. This shows that $K \subset B(x,\lambda')$ where $\lambda' := \lambda - f(y_0) \in (0,\lambda)$. $\square$

For a circumball of a set K in a generalized Minkowski space, only the points of K which lie in the boundary of the circumball are essential, see again the discussion after Theorem 3.8.

Lemma 4.8. *Let $B(x_0,R(K,B))$ be a circumball of a non-empty compact subset K with respect to the unit ball B of a generalized Minkowski space (X,γ). Then $R(K\cap S(x_0,R(K,B)),B)=R(K,B)$.*

Proof. Without loss of generality, we set $B(x_0,R(K,B))=B(0,1)$. Assume that, contrary to our claim, we have $R(K\cap S(0,1),B)<1$. Then there exists a point $x_1\in X$ such that

$$K\cap S(0,1)\subset U(x_1,1). \tag{4.1}$$

Since $R(K,B)=1$, we have $K\not\subset U(\frac{1}{i}x_1,1)$ for all $i\in\mathbb{N}$, cf. Lemma 4.7(d). For $i\in\mathbb{N}$, choose

$$y_i\in K\setminus U\left(\frac{1}{i}x_1,1\right). \tag{4.2}$$

We obtain $y_i\in K\setminus U(x_1,1)$ for all $i\in\mathbb{N}$ because $y_i\in K\setminus U(\frac{1}{i}x_1,1)\subset B(0,1)\setminus U(\frac{1}{i}x_1,1)$ implies $\gamma(y_i)\leq 1$, $\gamma(y_i-\frac{1}{i}x_1)\geq 1$, and in turn

$$\gamma(y_i-x_1)=\gamma\left(i\left(y_i-\frac{1}{i}x_1\right)-(i-1)y_i\right)\geq i\gamma\left(y_i-\frac{1}{i}x_1\right)-(i-1)\gamma(y_i)\geq i-(i-1)=1.$$

Since $K\setminus U(x_1,1)$ is a compact set, the sequence $(y_i)_{i\in\mathbb{N}}$ has a convergent subsequence $(y_{i_j})_{j\in\mathbb{N}}$ whose limit is a point $y_0\in K\setminus U(x_1,1)$. We know that $\gamma(y_0)\leq 1$ from $K\subset B(0,1)$, whereas (4.2) implies $\gamma(y_{i_j}-\frac{1}{i_j}x_1)\geq 1$. Taking the limit $j\to+\infty$, we obtain $\gamma(y_0)\geq 1$. This way we see that $y_0\in K\cap S(0,1)\setminus U(x_1,1)$, which contradicts (4.1) and completes the proof. □

4.1 Support and separation of b-convex sets

The following results on the separation of b-convex bodies and points by spheres are analogs of theorems on the separation of convex bodies by hyperplanes in classical convexity.

Proposition 4.9. *Let K be a b-convex body in a generalized Minkowski space (X,γ) with unit ball B.*

(a) For every $x_0\in\operatorname{bd}(K)$, there exists a supporting sphere $S(y_0,1)$ of K such that $x_0\in S(y_0,1)$.
(b) For every $x_0\in X\setminus K$, there exists a supporting sphere $S(y_0,1)$ of K such that $x_0\notin B(y_0,1)$.
(c) If K is b-bounded then, for every $x_0\in X\setminus K$, there exists a point $y_0\in X$ such that $K\subset U(y_0,1)$ and $x_0\notin B(y_0,1)$. In particular, we have $K\subset B(y_0,\lambda)$ for some $\lambda\in(0,1)$.

Proof. For proving (a), note that the assumption

$$x_0\in\operatorname{bd}(K)=\operatorname{bd}(\operatorname{bh}(K,B,1))=\operatorname{bd}\left(\bigcap_{y\in X:\,K\subset B(y,1)}B(y,1)\right)$$

yields the existence of points $y_i\in X$ such that $K\subset B(y_i,1)$ for all $i\in\mathbb{N}$ and $1-\gamma(x_0-y_i)\to 0$. From $K\subset B(y_i,1)$, we get that the sequence $(y_i)_{i\in\mathbb{N}}$ is contained in the compact set $K-B$, and thus has a convergent subsequence. Without loss of generality, we may assume that $(y_i)_{i\in\mathbb{N}}$ is a convergent sequence, and we denote its limit by y_0. Then the above observations imply $K \subset B(y_0,1)$ and $\gamma(x_0-y_0)=1$, i.e., $x_0 \in S(y_0,1)$. For (b), we have $x_0 \notin K = \bigcap_{y\in X:\,K\subset B(y,1)} B(y,1)$. Hence there is a point $y_1 \in X$ such that $K \subset B(y_1,1)$ and $x_0 \notin B(y_1,1)$. We consider the translated balls $B_\alpha := B(y_1+\alpha(y_1-x_0),1)$ for $\alpha \geq 0$. We know that $K \subset B_0$. Let $\alpha_0 \geq 0$ be maximal such that $K \subset B_{\alpha_0}$. (By convexity of balls and K, we also have $K \subset B_\alpha$ for $0 \leq \alpha \leq \alpha_0$.) By maximality of α_0, we know that $\mathrm{bd}(B_{\alpha_0}) = S(y_1+\alpha_0(y_1-x_0),1) =: S(y_0,1)$ is a supporting sphere of K. Moreover, we have $x_0 \notin B(y_0,1)$ because $x_0 \notin B(y_1,1)$ and

$$\gamma(x_0-y_0) = \gamma(x_0-(y_1+\alpha_0(y_1-x_0))) = (1+\alpha_0)\gamma(x_0-y_1) > 1+\alpha_0 \geq 1.$$

This proves (b). For the proof of (c), the b-boundedness of K yields the existence of a point $y_1 \in X$ such that $K \subset U(y_1,1)$. By (b), there is a point $y_2 \in X$ with $K \subset B(y_2,1)$ and $x_0 \notin B(y_2,1)$. We find $\varepsilon \in (0,1)$ small enough such that $x_0 \notin B(y_0,1)$, where $y_0 := y_2+\varepsilon(y_1-y_2)$. Then we obtain

$$K \subset U(y_0,1) \tag{4.3}$$

because for arbitrary $x \in K$, the inclusions $K \subset U(y_1,1)$ and $K \subset B(y_2,1)$ imply $\gamma(x-y_1) < 1$, $\gamma(x-y_2) \leq 1$, and in turn

$$\begin{aligned}\gamma(x-y_0) &= \gamma(x-(y_2+\varepsilon(y_1-y_2)))\\ &= \gamma(\varepsilon(x-y_1)+(1-\varepsilon)(x-y_2))\\ &\leq \varepsilon\gamma(x-y_1)+(1-\varepsilon)\gamma(x-y_2)\\ &< \varepsilon+(1-\varepsilon)\\ &= 1.\end{aligned}$$

Finally, (4.3) and Lemma 4.7(d) yield $K \subset B(y_0,\lambda)$ for some $\lambda \in (0,1)$. □

The following result resembles the fact that each boundary point of a convex body belongs to one of its exposed faces.

Corollary 4.10. *Every b-convex body K in a generalized Minkowski space (X,γ) satisfies*

$$\mathrm{bd}(K) = \bigcup_{F \text{ is a b-exposed face of } K} F.$$

Proof. Definition 4.6 and Proposition 4.9(a) yield one inclusion each. □

Proposition 4.9 gives rise to alternative representations of ball hulls.

Corollary 4.11. *Every b-bounded subset K of a generalized Minkowski space (X,γ) with unit ball B satisfies*

$$\mathrm{bh}(K,B,1) = \bigcap_{x\in X:\,K\subset U(x,1)} B(x,1) = \bigcap_{x\in X,\lambda<1:\,K\subset B(x,\lambda)} B(x,1) = \bigcap_{x\in X,\lambda<1:\,K\subset B(x,\lambda)} B(x,\lambda).$$

Proof. We assume that K is non-empty and put

$$A_1 := \mathrm{bh}(K,B,1) = \bigcap_{x\in X:\,K\subset B(x,1)} B(x,1), \qquad A_2 := \bigcap_{x\in X:\,K\subset U(x,1)} B(x,1),$$

$$A_3 := \bigcap_{x\in X,\lambda<1:\,K\subset B(x,\lambda)} B(x,1), \qquad A_4 := \bigcap_{x\in X,\lambda<1:\,K\subset B(x,\lambda)} B(x,\lambda).$$

The inclusions $A_1 \subset A_2 \subset A_3$ and $A_4 \subset A_3$ are trivial. It remains to show that $A_3 \subset A_1$ and $A_1 \subset A_4$. For proving $A_3 \subset A_1$, we consider an arbitrary point $x_0 \in X \setminus A_1$ and have to show that $x_0 \notin A_3$. Application of Proposition 4.9(c) to x_0 and A_1, which is b-bounded by Lemma 4.7(c), yields the existence of a point $y_0 \in X$ and a number $\lambda \in (0,1)$ such that $K \subset A_1 \subset B(y_0,\lambda)$ and $x_0 \notin B(y_0,1)$. This implies $x_0 \notin A_3$. For $A_1 \subset A_4$, note that $K \subset B(x,\lambda)$ if and only if $A_1 \subset B(x,\lambda)$. Indeed, if $K \subset B(x,\lambda)$, then we have $A_1 = \mathrm{bh}(K,B,1) \subset \mathrm{bh}(B(x,\lambda),B,1) = B(x,\lambda)$ by Theorem 4.3(a) and Lemma 4.7(b). Conversely, if $A_1 \subset B(x,\lambda)$, then $K \subset \mathrm{bh}(K,B,1) = A_1 \subset B(x,\lambda)$ by Lemma 4.7(a). From this equivalence, we conclude

$$A_1 \subset \bigcap_{x\in X,\lambda<1:\,A_1\subset B(x,\lambda)} B(x,\lambda) = \bigcap_{x\in X,\lambda<1:\,K\subset B(x,\lambda)} B(x,\lambda) = A_4.$$

This completes the proof. □

Corollary 4.12. *Every b-bounded closed subset K of a generalized Minkowski space (X,γ) with unit ball B satisfies*

$$\mathrm{bh}(K,B,1) = \bigcap_{x\in X:\,K\subset U(x,1)} U(x,1).$$

Proof. By Corollary 4.11,

$$\mathrm{bh}(K,B,1) = \bigcap_{x\in X:\,K\subset U(x,1)} B(x,1) \supset \bigcap_{x\in X:\,K\subset U(x,1)} U(x,1).$$

For the converse inclusion, note that $K \subset U(x,1)$ implies $\mathrm{bh}(K,B,1) \subset U(x,1)$. Indeed, if $K \subset B(x,1)$, then $K \subset B(x,\lambda)$ for some $\lambda \in (0,1)$ by Lemma 4.7(d), and, by Theorem 4.3(a) and Lemma 4.7(b), we have $\mathrm{bh}(K,B,1) \subset \mathrm{bh}(B(x,\lambda),B,1) = B(x,\lambda) \subset U(x,1)$. From this implication, we conclude

$$\bigcap_{x\in X:\,K\subset U(x,1)} U(x,1) \supset \bigcap_{x\in X:\,\mathrm{bh}(K,B,1)\subset U(x,1)} U(x,1) \supset \mathrm{bh}(K,B,1).$$

This completes the proof. □

The assumption of b-boundedness is essential in Corollaries 4.11 and 4.12. For example, if K is a closed ball of radius 1, then $\mathrm{bh}(K,B,1) = K$, whereas the four other intersections represent X, since they are intersections over empty index sets. To see that the assumption of closedness in Corollary 4.12 cannot be dropped, consider the example $K := U(x_0,\lambda_0)$ with $x_0 \in X$ and $\lambda_0 \in (0,1)$. Then $\mathrm{bh}(U(x_0,\lambda_0),B,1) = \mathrm{bh}(B(x_0,\lambda_0),B,1) = B(x_0,\lambda_0)$ by Lemma 4.7(a) and (b). In contrast to that, the triangle inequality yields

$$\bigcap_{x\in X:\,U(x_0,\lambda_0)\subset U(x,1)} U(x,1) = \bigcap_{x\in x_0+(\lambda_0-1)B(0,1)} U(x,1) = U(x_0,\lambda_0).$$

In classical convexity two disjoint convex sets can be separated by a hyperplane. An analogous claim for ball convexity would say that, given two disjoint b-convex bodies $K_1, K_2 \subset X$, there exists a separating sphere $S(x_0, 1)$ for K_1 and K_2, i.e., $K_1 \subset B(x_0, 1)$ and $K_2 \cap B(x_0, 1) = \emptyset$. Such separating spheres may not exist, even in the case where γ is a norm.

Example 4.13. Let $X = \mathbb{R}^3$ and $\gamma : X \to \mathbb{R}$ be the gauge with unit ball

$$B := \operatorname{co}(\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}).$$

Furthermore, let $0 < \varepsilon < \frac{1}{2}$ and consider the line segments $K_1 := \left[\left(\frac{1}{4}, \frac{1}{4}, 0\right), \left(-\frac{1}{4}, -\frac{1}{4}, 0\right)\right]$ and $K_2 := \left[\left(\frac{1}{4}, -\frac{1}{4}, \varepsilon\right), \left(-\frac{1}{4}, \frac{1}{4}, \varepsilon\right)\right]$. Then K_1 and K_2 are disjoint b-bounded b-convex bodies in (X, γ), and there is no sphere $S(x_0, 1)$ such that $K_1 \subset B(x_0, 1)$ and $K_2 \cap U(x_0, 1) = \emptyset$.

Proof. The set K_1 is b-convex, because $K_1 = B\left(\left(-\frac{3}{4}, \frac{1}{4}, 0\right), 1\right) \cap B\left(\left(\frac{3}{4}, -\frac{1}{4}, 0\right), 1\right)$, and b-bounded, since $R(K_1, B) = \frac{1}{2}$. Similarly, K_2 is b-bounded and b-convex. If $K_1 \subset B(x_0, 1)$, then $B(x_0, 1)$ contains at least one of the points of the line segment $\left[\left(\frac{1}{4}, -\frac{1}{4}, \frac{1}{2}\right), \left(-\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)\right]$, and we obtain $K_2 \cap U(x_0, 1) \neq \emptyset$, since $0 < \varepsilon < \frac{1}{2}$. □

4.2 Ball convexity and rotundity

Rotundity plays an important role not only for the best approximation problem (2.1) and the optimal containment problem (3.2), but also impacts ball convexity. We start with necessary conditions of rotundity in terms of circumballs. Their nonsufficiency is clear from Theorem 3.8.

Lemma 4.14. *Let (X, γ) be a generalized Minkowski space with unit ball B. The following statements are equivalent.*

(a) Every bounded non-empty subset of X has a unique circumcenter.
(b) The circumradius of the intersection of any two distinct balls of the same radius $\lambda > 0$ is smaller than λ.

Proof. First, if (a) fails, then there is a bounded set K with circumradius $R(K, B) > 0$ which possesses two circumballs $B(x_1, R(K, B))$ and $B(x_2, R(K, B))$ with $x_1 \neq x_2$. From this, we obtain $B(x_1, R(K, B)) \cap B(x_2, R(K, B)) \supset K$ and $R(B(x_1, R(K, B)) \cap B(x_2, R(K, B)), B) \geq R(K, B)$, contradicting (b). Second, if (b) fails, then there are points $x_1, x_2 \in X$, $x_1 \neq x_2$, and a number $\lambda > 0$ such that $R(B(x_1, \lambda) \cap B(x_2, \lambda), B) \geq \lambda$. Since $B(x_1, \lambda) \cap B(x_2, \lambda) \subset B(x_1, \lambda)$ implies

$$R(B(x_1, \lambda) \cap B(x_2, \lambda), B) \leq \lambda,$$

we know that $B(x_1, \lambda)$ and $B(x_2, \lambda)$ are circumballs of $B(x_1, \lambda) \cap B(x_2, \lambda)$, contradicting (a). □

The conditions in the following result, which is a higher-dimensional version of Corollary 3.14, are necessary but not sufficient for those in Lemma 4.14 as can be see in Example 3.11.

Proposition 4.15. *Let (X, γ) be a generalized Minkowski space with unit ball B. The following statements are equivalent.*

(a) *Every two-element set* $\{x_1, x_2\} \subset X$ *has a unique circumcenter.*
(b) *The unit ball* B *does not have a pair of parallel supporting hyperplanes* H, H', *for which the exposed faces* $B \cap H$, $B \cap H'$ *are non-singletons.*

Proof. First, assume that (a) fails. Then there exist points $y_1, y_2 \in X$ such that $y_1 \neq y_2$ and $\{x_1, x_2\} \in y_i + R(\{x_1, x_2\}, B)B$ for $i \in \{1, 2\}$. Thus there exist points $z_1, z_2 \in B$ such that $z_1 \neq z_2$ and $[z_i, z_i + \lambda(x_1 - x_2)] \subset B$ for $i \in \{1, 2\}$ and some $\lambda > 0$. In particular, this inclusion implies $\lambda \leq \lambda_0 := R(\{x_1, x_2\}, B)^{-1}$. Furthermore, we have

$$[z_i, z_i + \lambda(x_1 - x_2)] \subset B_0 := B \cap (z_2 + \operatorname{lin}(\{x_1 - x_2, z_1 - z_2\}))$$

for $i \in \{1, 2\}$ and some $\lambda > 0$, i.e., the set $\{x_1, x_2\}$ has at least two circumcenters with respect to B_0. By Corollary 3.14, the line segments $[z_1, z_2]$ and $[z_1 + \lambda_0(x_1 - x_2), z_2 + \lambda_0(x_1 - x_2)]$ are contained in the boundary of B_0 relative to $\operatorname{aff}(B_0)$ and, in turn, in $\operatorname{bd}(B) = S(0, 1)$. By [218, 3.1], the line segments $[z_1, z_1 + \lambda_0(x_1 - x_2)]$ and $[z_2, z_2 + \lambda_0(x_1 - x_2)]$ are affine diameters of B. This means that there are parallel supporting hyperplanes H_1 and H_1' supporting B at z_1 and $z_1 + \lambda_0(x_1 - x_2)$, respectively, and there are parallel supporting hyperplanes H_2 and H_2' supporting B at z_2 and $z_2 + \lambda_0(x_1 - x_2)$, respectively. By the above arguments, we have $H_1 = H_2$ and $H_1' = H_2'$. Thus, there is a pair of supporting hyperplanes of B, each of which contains at least two boundary points of B. Conversely, if (b) fails, there are parallel supporting hyperplanes H, H' of B and points $u_1, u_2, v \in X$, $v \neq 0$, such that $[z_1, z_1 + v] \subset B \cap H$ and $[z_2, z_2 + v] \subset B \cap H'$. This means that $[z_1, z_2]$ and $[z_1 + v, z_2 + v]$ are affine diameters of B. By [218, 3.1], these line segments are longest chords of B in their (common) direction $z_1 - z_2$, and $\{z_1, z_2\}$ has more than one circumball. □

Now we come to further necessary conditions for rotundity all of which involve notions related to ball convexity and some of which are also sufficient for rotundity.

Proposition 4.16. *Let* (X, γ) *be a generalized Minkowski space with unit ball* B. *Consider the following statements.*

(a) *The gauge* γ *is rotund.*
(b) *Every b-convex body that is not b-bounded is a closed ball of radius* 1.
(c) *Every b-convex body that is not b-bounded has only one supporting sphere.*
(d) *For every boundary point* x *of a b-convex body* K *that is not b-bounded, there exists only one supporting sphere of* K *that contains* x.
(e) *For all* $x \in X$, $\lambda \in (0, 1)$, *and* $x_0 \in S(x, \lambda)$, *the ball* $B(x, \lambda)$ *has only one supporting sphere that contains* x_0.
(f) *There exist* $x \in X$ *and* $\lambda \in (0, 1)$ *such that for all* $x_0 \in S(x, \lambda)$, *the ball* $B(x, \lambda)$ *has only one supporting sphere that contains* x_0.
(g) *For all* $x \in X$ *and* $\lambda \in (0, 1)$, *each supporting sphere of* $B(x, \lambda)$ *meets* $B(x, \lambda)$ *in only one point.*
(h) *There exist* $x \in X$ *and* $\lambda \in (0, 1)$ *such that each supporting sphere of* $B(x, \lambda)$ *meets* $B(x, \lambda)$ *in only one point.*
(i) *For all* $x \in X$ *and* $\lambda \in (0, 1)$, *we have* $\operatorname{b-exp}(B(x, \lambda), B) = S(x, \lambda)$.
(j) *There exist* $x \in X$ *and* $\lambda \in (0, 1)$ *such that* $\operatorname{b-exp}(B(x, \lambda), B) = S(x, \lambda)$.

(k) Every b-convex body is rotund.
(l) For any two distinct points $x_1, x_2 \in X$, the set $\mathrm{bh}(\{x_1, x_2\}, B, 1)$ *is rotund.*
(m) Every b-convex body that contains at least two points has non-empty interior.
(n) For any two distinct points $x_1, x_2 \in X$, the set $\mathrm{bh}(\{x_1, x_2\}, B, 1)$ *has non-empty interior.*

Then the implications (a)⇒(b)⇒(c)⇒(d), (a)⇒(k)⇒(l)⇒(n), (k)⇒(m)⇒(n), and the equivalences (a)⇔(e)⇔(f)⇔(g)⇔(h)⇔(i)⇔(j) are true.

Proof. The implications (b)⇒(c)⇒(d), (e)⇒(f), (g)⇒(h), (i)⇒(j), (k)⇒(l), and (m)⇒(n) are evident. For showing (a)⇒(b), note that every b-convex body K is a non-empty intersection of a non-empty family of closed balls of radius 1. If the family consisted of more than one ball, then its intersection K would be b-bounded because (a) implies item (b) from Lemma 4.14. Hence the only b-convex bodies that are not b-bounded are closed balls of radius 1. Next, we show the implications (a)⇒(e) and (a)⇒(g). We use the fact that *if two balls $B(y, \lambda)$ and $B(y', \lambda')$ in a rotund generalized Minkowski space are on the same side of a common supporting hyperplane H with respective touching points y_0 and y'_0, then the homothety $f := \frac{\lambda'}{\lambda}(\cdot - y_0) + y'_0 : X \to X$ maps $B(y, \lambda)$ onto $B(y', \lambda')$ and satisfies $f(y) = y'$.* Coming back to the proof of (a)⇒(e) and (a)⇒(g), we consider an arbitrary supporting sphere $S(y, 1)$ of $B(x, \lambda)$ and suppose that x_0 belongs to the b-exposed face $B(x, \lambda) \cap S(y, 1)$. The supporting hyperplane of $B(y, 1)$ at x_0 supports $B(x, \lambda)$ as well. Now the above fact says that $B(y, 1)$ is the image of $B(x, \lambda)$ under the homothety $f_0 := \frac{1}{\lambda}(\cdot - x_0) + x_0 : X \to X$. This shows in particular that the supporting sphere $S(y, 1)$ is uniquely determined by the touching point x_0, which proves (e) because there exists at least one supporting sphere at x_0 according to Proposition 4.9(a). To show (g), we have to prove that every point $x_1 \in B(x, \lambda) \cap S(y, 1)$ coincides with x_0. By the same argument as above, the ball $B(y, 1)$ is the image of $B(x, \lambda)$ under the homothety $f_1 := \frac{1}{\lambda}(\cdot - x_1) + x_1 : X \to X$. We obtain

$$x_0 + \frac{1}{\lambda}(x - x_0) = f_0(x) = y = f_1(x) = x_1 + \frac{1}{\lambda}(x - x_1),$$

which gives $x_0 = x_1$ and completes the proof of (g). For (f)⇒(a), suppose that (a) fails. Then, for every $x \in X$ and every $\lambda \in (0, 1)$, the sphere $S(x, \lambda)$ contains a line segment $[x_0, x_1] \subset S(x, \lambda)$ with $x_0 \neq x_1$. Let $f_0 := \frac{1}{\lambda}(\cdot - x_0) + x_0 : X \to X$ and $f_1 := \frac{1}{\lambda}(\cdot - x_1) + x_1 : X \to X$. Then the images of $S(x, \lambda)$ under f_0 and f_1 are distinct supporting spheres of $B(x, \lambda)$ at $x_0 \in S(x, \lambda)$ because $x_0 = f_0(x_0) = f_1(\lambda x_0 + (1 - \lambda)x_1)$ and $x_0, \lambda x_0 + (1 - \lambda)x_1 \in S(x, \lambda)$. Hence (f) is disproved. The implications (g)⇒(i) and (h)⇒(j) follow from Proposition 4.9(a). For the proof of (j)⇒(a), assume that (a) fails. Then every sphere $S(x, \lambda)$ with $x \in X$ and $\lambda \in (0, 1)$ contains a non-singleton line segment $[x_1, x_2]$. We shall show that $\frac{x_1 + x_2}{2} \notin \text{b-exp}(B(x, r), B)$, this way disproving (j). Indeed, if $S(y, 1)$ is a supporting sphere of $B(x, \lambda)$ with touching point $\frac{x_1 + x_2}{2} \in S(y, 1)$, then $[x_1, x_2] \subset B(x, \lambda) \subset B(y, 1)$, and the point $\frac{x_1 + x_2}{2}$ (from the relative interior) of $[x_1, x_2]$ is in $S(y, 1) = \mathrm{bd}(B(y, 1))$. Hence $[x_1, x_2] \subset S(y, 1)$. This shows that the b-exposed face $B(x, r) \cap S(y, 1)$ that contains $\frac{x_1 + x_2}{2}$ necessarily contains the whole line segment $[x_1, x_2]$. Thus $\frac{x_1 + x_2}{2} \notin \text{b-exp}(B(x, r), B)$. Similarly, we show (a)⇒(k). Assume that (k) fails, i.e., there are a b-convex body K and two points $x_1 \neq x_2$ such that $[x_1, x_2] \subset \mathrm{bd}(K)$. By Proposition 4.9(a), there is a supporting sphere $S(y, 1)$ of K such that $\frac{x_1 + x_2}{2} \in S(y, 1)$. As above, we have $[x_1, x_2] \subset K \subset B(y, 1)$ and $\frac{x_1 + x_2}{2} \in S(y, 1) = \mathrm{bd}(B(y, 1))$, which yields $[x_1, x_2] \subset S(y, 1)$ and contradicts (a). Finally, for (k)⇒(m) and (l)⇒(n), note that if a convex body K contains distinct points x_1 and x_2 and has empty interior, then K is not rotund, because $[x_1, x_2] \subset K = \mathrm{bd}(K)$. □

Out of the remaining implications in Proposition 4.16, some can be readily invalidated by an example. For this, consider a generalized Minkowski space $(\mathbb{R}^d, \gamma)$ whose unit ball $B = B(0,1)$ is a d-dimensional simplex, cf. Remark 3.13. Then (b) and (m) are valid but (l) is not.

4.3 Representation of ball hulls from inside

In this section we deal with the b-convexity of unions of increasing sequences of b-convex bodies and with the representation of ball hulls of sets as unions of ball hulls of finite subsets. We start with a result on the convergence of hyperplane sections of convex bodies with respect to the Hausdorff distance. In Euclidean geometry, the latter defines a metric on the family of compact sets. Since all gauges defined on a finite-dimensional vector space X induce the same topology on X, their respective Hausdorff distances

$$(K, K') \mapsto \max\left\{ \sup_{x \in K} \inf_{y \in K'} \gamma(x-y), \sup_{x \in K'} \inf_{y \in K} \gamma(x-y) \right\}$$

induce the same topology on the family of compact sets.

Lemma 4.17. *Let (X,γ) be a generalized Minkowski space, $K \in \mathscr{K}^X$, and $H \subset X$ be a hyperplane. Furthermore, let $(y_i)_{i\in\mathbb{N}}$ be a convergent sequence of points $y_i \in X$ with limit $y_0 \in X$ and $H \cap (K + y_i) \neq \emptyset$ for all $i \in \mathbb{N}$. Then $H \cap (K + y_i) \to H \cap (K + y_0)$ in the Hausdorff distance.*

Proof. First note that $H \cap (K + y_0) \neq \emptyset$ and thus $H \cap (K + y_0) \in \mathscr{K}^X$. Indeed, for every $i \in \mathbb{N}$, we may choose a point $z_i \in H \cap (K + y_i)$, i.e., $z_i = x_i + y_i$ with $x_i \in K$. Due to the compactness of K, may assume that $(x_i)_{i\in\mathbb{N}}$ is a convergent sequence whose limit x_0 is a point of K. We obtain $z_0 := x_0 + y_0 \in H \cap (K + y_0)$ from $z_i \to z_0$ and the closedness of H. By [207, Theorem 1.8.8], our claim $H \cap (K + y_i) \to H \cap (K + y_0)$ is now equivalent to the following conditions taken together:

(a) for every $v_0 \in H \cap (K + y_0)$, there exist $v_i \in H \cap (K + y_i)$, $i \in \mathbb{N}$, such that $v_i \to v_0$,
(b) if $(v_{i_j})_{j\in\mathbb{N}}$ is a sequence with $i_1 < i_2 < \ldots$, $v_{i_j} \in H \cap (K + y_{i_j})$, and $v_{i_j} \to v_0 \in X$, then $v_0 \in H \cap (K + y_0)$.

Proof of (a). Suppose that $H = \phi_{=\alpha}$ for some $\phi \in X^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$. Fix $v_0 \in H \cap (K + y_0)$, i.e., $v_0 = x_0 + y_0$ with $x_0 \in K$ and

$$\langle \phi \mid v_0 \rangle = \alpha, \text{ i.e., } \langle \phi \mid x_0 \rangle = \alpha - \langle \phi \mid y_0 \rangle . \tag{4.4}$$

Choose x^*, $x^{**} \in K$ such that $\langle \phi \mid x^* \rangle = -h(\phi, -K)$ and $\langle \phi \mid x^{**} \rangle = h(\phi, K)$. For every $i \in \mathbb{N}$, the relation $H \cap (K + y_i) \neq \emptyset$ yields the existence of a point $\tilde{x}_i \in K$ such that

$$\langle \phi \mid \tilde{x}_i + y_i \rangle = \alpha. \tag{4.5}$$

Choose $v_i := x_i + y_i \in H \cap (K + y_i)$ as follows. From $\tilde{x}_i \in K$, we know that $\langle \phi \mid \tilde{x}_i \rangle \in [\langle \phi \mid x^* \rangle, \langle \phi \mid x^{**} \rangle]$.

Case 1: $\langle \phi \,|\, \tilde{x}_i \rangle = \langle \phi \,|\, x_0 \rangle$. In this case, put $x_i := x_0$. Then $v_i = x_0 + y_i \in K + y_i$ and $v_i \in H$ because $\langle \phi \,|\, v_i \rangle = \langle \phi \,|\, x_0 + y_i \rangle = \langle \phi \,|\, \tilde{x}_i + y_i \rangle = \alpha$ by (4.5).
Case 2: $\langle \phi \,|\, \tilde{x}_i \rangle \in [\langle \phi \,|\, x^* \rangle, \langle \phi \,|\, x_0 \rangle)$. Then

$$x_i := \frac{\langle \phi \,|\, \tilde{x}_i \rangle - \langle \phi \,|\, x^* \rangle}{\langle \phi \,|\, x_0 \rangle - \langle \phi \,|\, x^* \rangle} x_0 + \frac{\langle \phi \,|\, x_0 \rangle - \langle \phi \,|\, \tilde{x}_i \rangle}{\langle \phi \,|\, x_0 \rangle - \langle \phi \,|\, x^* \rangle} x^* \tag{4.6}$$

satisfies $x_i \in \langle x_0 \,|\, x^* \rangle \subset K$. Hence $v_i = x_i + y_i \in K + y_i$ and

$$\langle \phi \,|\, x_i \rangle = \frac{\langle \phi \,|\, \tilde{x}_i \rangle - \langle \phi \,|\, x^* \rangle}{\langle \phi \,|\, x_0 \rangle - \langle \phi \,|\, x^* \rangle} \langle \phi \,|\, x_0 \rangle + \frac{\langle \phi \,|\, x_0 \rangle - \langle \phi \,|\, \tilde{x}_i \rangle}{\langle \phi \,|\, x_0 \rangle - \langle \phi \,|\, x^* \rangle} \langle \phi \,|\, x^* \rangle = \langle \phi \,|\, \tilde{x}_i \rangle.$$

This implies $v_i \in H$ because $\langle \phi \,|\, v_i \rangle = \langle \phi \,|\, x_i + y_i \rangle = \langle \phi \,|\, \tilde{x}_i + y_i \rangle = \alpha$ by (4.5).
Case 3: $\langle \phi \,|\, \tilde{x}_i \rangle \in (\langle \phi \,|\, x_0 \rangle, \langle \phi \,|\, x^{**} \rangle]$. Then

$$x_i := \frac{\langle \phi \,|\, x^{**} \rangle - \langle \phi \,|\, \tilde{x}_i \rangle}{\langle \phi \,|\, x^{**} \rangle - \langle \phi \,|\, x_0 \rangle} x_0 + \frac{\langle \phi \,|\, \tilde{x}_i \rangle - \langle \phi \,|\, x_0 \rangle}{\langle \phi \,|\, x^{**} \rangle - \langle \phi \,|\, x_0 \rangle} x^{**} \tag{4.7}$$

satisfies $x_i \in [x_0, x^{**}] \subset K$. Hence $v_i = x_i + y_i \in K + y_i$ and

$$\langle \phi \,|\, x_i \rangle = \frac{\langle \phi \,|\, x^{**} \rangle - \langle \phi \,|\, \tilde{x}_i \rangle}{\langle \phi \,|\, x^{**} \rangle - \langle \phi \,|\, x_0 \rangle} \langle \phi \,|\, x_0 \rangle + \frac{\langle \phi \,|\, \tilde{x}_i \rangle - \langle \phi \,|\, x_0 \rangle}{\langle \phi \,|\, x^{**} \rangle - \langle \phi \,|\, x_0 \rangle} \langle \phi \,|\, x^{**} \rangle = \langle \phi \,|\, \tilde{x}_i \rangle.$$

This yields $v_i \in H$ because $\langle \phi \,|\, v_i \rangle = \langle \phi \,|\, x_i + y_i \rangle = \langle \phi \,|\, \tilde{x}_i + y_i \rangle = \alpha$ by (4.5).
Finally, for proving $v_i \to v_0$, we use the following arguments. We obtain $x_i \to x_0$ by partitioning the sequence $(x_i)_{i \in \mathbb{N}}$ into three subsequences corresponding to the above Cases 1–3, where each of the subsequences (if it is infinite) converges to x_0. In Case 1, this is trivial. In the other two cases, it follows from $\langle \phi \,|\, \tilde{x}_i \rangle = \alpha - \langle \phi \,|\, y_i \rangle \to \alpha - \langle \phi \,|\, y_0 \rangle = \langle \phi \,|\, x_0 \rangle$, for which we use (4.4) and (4.5), and from the definitions (4.6) and (4.7). This yields $v_i = x_i + y_i \to x_0 + y_0 = v_0$.
Proof of (b). The inclusion $v_{i_j} \in H \cap (K + y_{i_j})$ implies $x_{i_j} := v_{i_j} - y_{i_j} \in K$. Hence $x_{i_j} = v_{i_j} - y_{i_j} \to v_0 - y_0 \in K$ because K is closed, and we obtain $v_0 \in K + y_0$. Moreover, the inclusion $v_{i_j} \in H$ yields $v_{i_j} \to v_0 \in H$ because H is closed. Hence $v_0 \in H \cap (K + y_0)$. □

Remark 4.18. Note that the hyperplane H cannot be replaced by an affine subspace L of arbitrary dimension in Lemma 4.17. For instance, take $X = \mathbb{R}^3$ and

$$\begin{aligned} K &:= \left\{ (\xi_1, \xi_2, \xi_3) \in X \;\middle|\; \sqrt{\xi_1^2 + \xi_2^2} + |\xi_3| \le 1 \right\} \\ &= \operatorname{co}(\{(\cos(\alpha), \sin(\alpha), 0) \mid 0 \le \alpha < 2\pi\} \cup \{(0,0,\pm 1)\}). \end{aligned}$$

Consider the affine subspace $L := \operatorname{aff}(\{(1,0,0),(0,0,1)\})$ of X, let $y_i := (1 - \cos(\frac{1}{i}), \sin(\frac{1}{i}), 0)$, and $y_0 := \lim_{i \to +\infty} y_i = (0,0,0)$. Then $L \cap (K + y_i) = \{(1,0,0)\}$ for all $i \in \mathbb{N}$, and $L \cap (K + y_0) = L \cap K = [(1,0,0),(0,0,1)]$. Hence $L \cap (K + y_i)$ does not converge to $L \cap (K + y_0)$ in the described way.

In classical convexity, if the union of an increasing sequence of convex bodies $K_i \in \mathcal{K}^X$ is bounded, its closure is again a member of $\mathcal{K}^X$. Next, we give a sufficient condition for an analogous situation in the context of ball convexity.

Theorem 4.19. *Let $K_1 \subset K_2 \subset \dots$ be an increasing sequence of b-convex bodies in a generalized Minkowski space (X,γ) with unit ball B such that*

$$\dim\left(\mathrm{bh}\left(\bigcup_{i\in\mathbb{N}} K_i, B, 1\right)\right) \in \{0, 1, \dim(X)-1, \dim(X)\}. \tag{4.8}$$

Then

$$\mathrm{cl}\left(\bigcup_{i\in\mathbb{N}} K_i\right) = \mathrm{bh}\left(\bigcup_{i\in\mathbb{N}} K_i, B, 1\right). \tag{4.9}$$

In particular, $\mathrm{cl}\big(\bigcup_{i\in\mathbb{N}} K_i\big)$ *is a b-convex body.*

Proof. Since the inclusion "$\subset$" in (4.9) is obvious, we prove "$\supset$" now.
Case 1. For $\dim\big(\mathrm{bh}\big(\bigcup_{i\in\mathbb{N}} K_i, B, 1\big)\big) = 0$, the set $\bigcup_{i\in\mathbb{N}} K_i = \{x_0\}$ is a singleton. Thus $K_i = \{x_0\}$ for all $i \in \mathbb{N}$, and (4.9) is trivial.
Case 2. For $\dim\big(\mathrm{bh}\big(\bigcup_{i\in\mathbb{N}} K_i, B, 1\big)\big) = 1$, the set $F := \mathrm{bh}\big(\bigcup_{i\in\mathbb{N}} K_i, B, 1\big)$ is a non-singleton line segment. Every line segment contained in a translate of F is also b-convex, so each of the sets K_i is of this kind. Furthermore, the set $\bigcup_{i\in\mathbb{N}} K_i$ is a bounded, convex, and not necessarily closed subset of $\mathrm{aff}(F)$, and

$$\mathrm{bh}\left(\bigcup_{i\in\mathbb{N}} K_i, B, 1\right) = \mathrm{cl}\left(\bigcup_{i\in\mathbb{N}} K_i\right).$$

Case 3. $\dim\big(\mathrm{bh}\big(\bigcup_{i\in\mathbb{N}} K_i, B, 1\big)\big) \geq \dim(X) - 1 > 0$. Assume that we have "$\not\supset$" in (4.9). Then there exists a point $x_0 \in \mathrm{bh}\big(\bigcup_{i\in\mathbb{N}} K_i, B, 1\big) \setminus \mathrm{cl}\big(\bigcup_{i\in\mathbb{N}} K_i\big)$, and we find a number $\varepsilon_0 > 0$ such that $B(x_0, \varepsilon_0) \cap \big(\bigcup_{i\in\mathbb{N}} K_i\big) = \emptyset$. Consequently, there exists a point $x_1 \in \mathrm{ri}\big(\mathrm{bh}\big(\bigcup_{i\in\mathbb{N}} K_i, B, 1\big)\big) \setminus \big(\bigcup_{i\in\mathbb{N}} K_i\big)$, i.e.,

$$x_1 \in \mathrm{ri}\left(\mathrm{bh}\left(\bigcup_{i\in\mathbb{N}} K_i, B, 1\right)\right) \tag{4.10}$$

and

$$x_1 \notin K_i \quad \text{for} \quad i \in \mathbb{N}. \tag{4.11}$$

Property (4.11) and Proposition 4.9(b) yield the existence of points $y_i \in X$ such that $x_1 \notin B(y_i, 1)$ and $K_i \subset B(y_i, 1)$ for $i \in \mathbb{N}$. Since $K_1 \subset K_i \subset B(y_i, 1)$ for all $i \in \mathbb{N}$, we have $y_i \in K_1 - B$ for all $i \in \mathbb{N}$ with $K_1 - B$ being a compact set. Thus there exists a convergent subsequence $(y_{i_j})_{j\in\mathbb{N}}$ whose limit shall be denoted by $y_0 \in X$. By continuity of γ and the inclusions $K_1 \subset K_2 \subset \dots$, we have

$$x_1 \notin U(y_0, 1) \quad \text{and} \quad \bigcup_{i\in\mathbb{N}} K_i \subset B(y_0, 1). \tag{4.12}$$

Subcase 3.1: $\dim\big(\mathrm{bh}\big(\bigcup_{i\in\mathbb{N}} K_i, B, 1\big)\big) = \dim(X)$. With (4.12), we have $\mathrm{bh}\big(\bigcup_{i\in\mathbb{N}} K_i, B, 1\big) \subset B(y_0, 1)$ and

$$x_1 \notin \mathrm{int}\left(\mathrm{bh}\left(\bigcup_{i\in\mathbb{N}} K_i, B, 1\right)\right) = \mathrm{ri}\left(\mathrm{bh}\left(\bigcup_{i\in\mathbb{N}} K_i, B, 1\right)\right),$$

a contradiction to (4.10).
Subcase 3.2: $\dim\big(\mathrm{bh}\big(\bigcup_{i\in\mathbb{N}} K_i, B, 1\big)\big) = \dim(X)-1$. Let $H := \mathrm{aff}\big(\mathrm{bh}\big(\bigcup_{i\in\mathbb{N}} K_i, B, 1\big)\big)$. From $y_{i_j} \to y_0$, we get $B(y_{i_j}, 1) \to B(y_0, 1)$ in the Hausdorff distance, and Lemma 4.17 yields $B(y_{i_j}, 1) \cap$

$H \to B(y_0,1) \cap H$. Then, by $x_1 \notin B(y_{i_j},1)$, we obtain $x_1 \notin \mathrm{int}_H(B(y_0,1) \cap H)$ where int_H denotes the interior with respect to the subspace topology in H. On the other hand, we have $\mathrm{bh}\big(\bigcup_{i\in\mathbb{N}} K_i, B, 1\big) \subset B(y_0,1) \cap H$ by (4.12) and the choice of H. With this and the choice of H, we obtain $x_1 \notin \mathrm{int}_H\big(\mathrm{bh}\big(\bigcup_{i\in\mathbb{N}} K_i, B, 1\big)\big) = \mathrm{ri}\big(\mathrm{bh}\big(\bigcup_{i\in\mathbb{N}} K_i, B, 1\big)\big)$, a contradiction to (4.10). This completes the proof of "$\supset$" in (4.9).

To show that $\mathrm{cl}\big(\bigcup_{i\in\mathbb{N}} K_i\big) = \mathrm{bh}\big(\bigcup_{i\in\mathbb{N}} K_i, B, 1\big)$ is a b-convex body, it is sufficient to verify that $\mathrm{bh}\big(\bigcup_{i\in\mathbb{N}} K_i, B, 1\big) \neq X$, i.e., that $\bigcup_{i\in\mathbb{N}} K_i$ is contained in some ball of radius 1. This is obvious by the second part of (4.12), which can be shown analogously in Cases 1 and 2. $\square$

Remark 4.20. Note that the technical assumption (4.8) is satisfied in each of the following situations:

(a) $\dim(X) \leq 3$,
(b) (X,γ) is rotund,
(c) $\dim\big(\bigcup_{i\in\mathbb{N}} K_i\big) \geq \dim(X) - 1$ or, equivalently, $\dim(K_{i_0}) \geq \dim(X) - 1$ for some $i_0 \in \mathbb{N}$.

Proof. Situation (a) is trivial. In situation (b), Proposition 4.16(a)⇒(m) shows that

$$\dim\left(\mathrm{bh}\left(\bigcup_{i\in\mathbb{N}} K_i, B, 1\right)\right) = \dim(X)$$

as soon as $\bigcup_{i\in\mathbb{N}} K_i$ is not a singleton. Finally, situation (c) implies (4.8) because $\bigcup_{i\in\mathbb{N}} K_i \subset \mathrm{bh}\big(\bigcup_{i\in\mathbb{N}} K_i, B, 1\big)$. $\square$

In classical convexity, a convex body $K \in \mathscr{K}^X$ is the union of its polytopal subsets. One may ask if an analogous statement is true for ball convexity. The claim of the following result is shown in [140, Theorem 1] under the strong assumption that $\dim(\mathrm{bh}(K,B,1)) = \dim(X)$. The authors ask in [140, Problem 2.6] if this assumption can be dropped. It turns out that the answer is negative, see Example 4.23 below. However, the condition can be weakened.

Theorem 4.21. *Let K be a non-empty subset of a generalized Minkowski space (X,γ) with unit ball B such that*

$$\dim(\mathrm{bh}(K,B,1)) \in \{0, 1, \dim(X) - 1, \dim(X)\}. \tag{4.13}$$

Then

$$\mathrm{bh}(K,B,1) = \mathrm{cl}\left(\bigcup_{K' \subset K \text{ finite}} \mathrm{bh}(K',B,1)\right). \tag{4.14}$$

Proof. The inclusion "$\supset$" in (4.14) is evident. For "$\subset$", first note that there are is a countable collection of points $x_i \in K$, $i \in \mathbb{N}$, such that $\mathrm{cl}(K) = \mathrm{cl}(\{x_1, x_2, \ldots\})$. (To be more constructive, identify X with $\mathbb{R}^{\dim(X)}$, let $v_i \in X$, $i \in \mathbb{N}$, be the points with only rational coordinates and choose points $x_i \in K$ such that $\gamma(x_i - v_i) < \underline{\mathrm{dist}}_{\gamma^\vee}(v_i, K) + \frac{1}{i}$ for $i \in \mathbb{N}$. For $x_0 \in \mathrm{cl}(K)$, there exists a subsequence $(v_{i_j})_{j\in\mathbb{N}}$ with $v_{i_j} \to x_0$. But then we also have $\gamma(x_{i_j} - x_0) \leq \gamma(x_{i_j} - v_{i_j}) + \gamma(v_{i_j} - x_0) \leq \underline{\mathrm{dist}}_{\gamma^\vee}(v_i, K) + \frac{1}{i} + \gamma(v_{i_j} - x_0) = 0$, i.e., $x_{i_j} \to x_0$.) Putting $K_i := \mathrm{bh}(\{x_1, \ldots, x_i\}, B, 1)$ for $i \in \mathbb{N}$, we obtain $K_1 \subset K_2 \subset \ldots$. If K_{i_0} is not a b-convex body for some $i_0 \in \mathbb{N}$, then $\mathrm{bh}\big(\{x_1, \ldots, x_{i_0}\}, B, 1\big) =$

$K_{i_0} = X$, and (4.14) is obvious as both sides are X. Hence we may assume that all sets K_i are b-convex bodies. Moreover,

$$\begin{aligned}
\mathrm{bh}\left(\bigcup_{i\in\mathbb{N}} K_i, B, 1\right) &= \mathrm{bh}\left(\bigcup_{i\in\mathbb{N}} \mathrm{bh}(\{x_1,\dots,x_i\}, B, 1), B, 1\right) \\
&= \mathrm{bh}\left(\bigcup_{i\in\mathbb{N}} \{x_1,\dots,x_i\}, B, 1\right) \\
&= \mathrm{bh}(\{x_1, x_2, \dots\}, B, 1) \\
&= \mathrm{bh}(\mathrm{cl}(\{x_1, x_2, \dots\}), B, 1) \\
&= \mathrm{bh}(\mathrm{cl}(K), B, 1) \\
&= \mathrm{bh}(K, B, 1),
\end{aligned} \tag{4.15}$$

which gives, using (4.13),

$$\dim\left(\mathrm{bh}\left(\bigcup_{i\in\mathbb{N}} K_i, B, 1\right)\right) = \dim\left(\mathrm{bh}(K, B, 1)\right) \in \{0, 1, \dim(X) - 1, \dim(X)\}.$$

Now apply Theorem 4.19 and use (4.15) to obtain

$$\begin{aligned}
\mathrm{bh}(K, B, 1) &= \mathrm{bh}\left(\bigcup_{i\in\mathbb{N}} K_i, B, 1\right) \\
&= \mathrm{cl}\left(\bigcup_{i\in\mathbb{N}} K_i\right) \\
&= \mathrm{cl}\left(\bigcup_{i\in\mathbb{N}} \mathrm{bh}(\{x_1,\dots,x_i\}, B, 1)\right) \\
&\subset \mathrm{cl}\left(\bigcup_{K'\subset K \text{ finite}} \mathrm{bh}\big(K', B, 1\big)\right).
\end{aligned}$$

This completes the proof. □

Remark 4.22. As in Remark 4.20, we see that (4.13) holds in each of the following situations:

(a) $\dim(X) \leq 3$,
(b) (X, γ) is rotund,
(c) $\dim(K) \geq \dim(X) - 1$.

An example shows that the restrictions (4.8) in Theorem 4.19 and (4.13) in Theorem 4.21 are essential, even in normed spaces.

Example 4.23. Let $X = \mathbb{R}^4$. We denote the Euclidean norm of $\mathbb{R}^2$ by $\|\cdot\|_2$ and consider convex bodies

$$B_1 := \{(\alpha_1, \alpha_2, \alpha_3, 0) \mid \|(\alpha_1, \alpha_2)\|_2 \leq 1, \|(\alpha_2, \alpha_3)\|_2 \leq 1\},$$

$$B_2 := \left\{ (\beta_1, 0, \beta_3, \beta_4) \,\middle|\, \|(\beta_1, \beta_4)\|_2 \le 1, |\beta_3| \le 1 \right\}.$$

We define the unit ball $B = B(0,1)$ of a Minkowski space $(X, \|\cdot\|)$ by

$$\begin{aligned} B &:= \operatorname{co}(B_1 \cup B_2) \\ &= \left\{ (\alpha_1 + \beta_1, \xi_2, \alpha_3 + \beta_3, \xi_4) \,\middle|\, \begin{array}{c} \max\left\{ \|(\alpha_1, \xi_2)\|_2, \|(\xi_2, \alpha_3)\|_2 \right\} \\ + \max\left\{ \|(\beta_1, \xi_4)\|_2, |\beta_3| \right\} \le 1 \end{array} \right\}. \end{aligned} \tag{4.16}$$

For this Minkowski space, we shall see that

$$\operatorname{bh}(\{(\lambda,0,0,0),(-\lambda,0,0,0)\}, B, 1) = [(\lambda,0,0,0),(-\lambda,0,0,0)] \text{ for } \lambda \in (0,1) \tag{4.17}$$

and

$$\operatorname{bh}(\{(1,0,0,0),(-1,0,0,0)\}, B, 1) = \left\{ (\xi_1, 0, 0, \xi_4) \,\middle|\, \|(\xi_1, \xi_4)\|_2 \le 1 \right\}. \tag{4.18}$$

Consequently, the line segments $K_i := [(1-\frac{1}{i},0,0,0),(-1+\frac{1}{i},0,0,0)]$, $i \in \mathbb{N}$, form an increasing sequence of b-convex bodies, and we obtain

$$\begin{aligned} \operatorname{cl}\left(\bigcup_{i \in \mathbb{N}} K_i \right) &= [(1,0,0,0),(-1,0,0,0)], \\ \operatorname{bh}\left(\bigcup_{i \in \mathbb{N}} K_i, B, 1 \right) &= \left\{ (\xi_1,0,0,\xi_4) \,\middle|\, \left\|(\xi_1,\xi_4)\right\|_2 \le 1 \right\}. \end{aligned}$$

Hence (4.9) fails and $\operatorname{cl}\left(\bigcup_{i \in \mathbb{N}} K_i\right)$ is not b-convex. Similarly, for $K := ((1,0,0,0),(-1,0,0,0))$, we have

$$\begin{aligned} \operatorname{bh}(K, B, 1) &= \left\{ (\xi_1,0,0,\xi_4) \,\middle|\, \left\|(\xi_1,\xi_4)\right\|_2 \le 1 \right\}, \\ \operatorname{cl}\left(\bigcup_{K' \subset K \text{ finite}} \operatorname{bh}(K', B, 1) \right) &= [(1,0,0,0),(-1,0,0,0)], \end{aligned}$$

and (4.14) fails.

Proof of (4.17) *and* (4.18). *Step 1: Verification of* (4.17). Let $\lambda \in (0,1)$ be fixed. Consider the linear functional $\phi \in X^*$, $\langle \phi \,|\, (\xi_1,\xi_2,\xi_3,\xi_4) \rangle := \sqrt{1-\lambda^2}\xi_2 + \lambda\xi_3$, of Euclidean norm $\|\phi\|_2 = \|(\sqrt{1-\lambda^2}, \lambda)\|_2 = 1$. Partially based on the Cauchy–Schwarz inequality, we obtain $B_1 \subset \phi_{\ge -1} \cap \phi_{\le 1}$, $B_2 \subset \phi_{\ge -\lambda} \cap \phi_{\le \lambda}$, and $B_2 \cap \phi_{=1} = \emptyset$. These yield

$$B \subset \phi_{\ge -1} \cap \phi_{\le 1} \tag{4.19}$$

and

$$B \cap \phi_{=1} = B_1 \cap \phi_{=1} = \left[\left(\lambda, \sqrt{1-\lambda^2}, \lambda, 0\right), \left(-\lambda, \sqrt{1-\lambda^2}, \lambda, 0\right) \right]. \tag{4.20}$$

Next, note that $(\pm\lambda, \pm\sqrt{1-\lambda^2}, \pm\lambda, 0) \in B_1 \subset B$ for arbitrary choices of signs. This implies $(\pm\lambda,0,0,0) \in B((0,\pm\sqrt{1-\lambda^2},\pm\lambda,0),1)$. In particular, we have

$$\{(\lambda,0,0,0),(-\lambda,0,0,0)\} \subset B\left(\left(0,\sqrt{1-\lambda^2},\lambda,0\right),1\right) \cap B\left(\left(0,-\sqrt{1-\lambda^2},-\lambda,0\right),1\right),$$

and, by (4.19) and (4.20),

$$\begin{aligned}
&\mathrm{bh}(\{(\lambda,0,0,0),(-\lambda,0,0,0)\},B,1)\\
&\subset B\Big(\Big(0,\sqrt{1-\lambda^2},\lambda,0\Big),1\Big)\cap B\Big(\Big(0,-\sqrt{1-\lambda^2},-\lambda,0\Big),1\Big)\\
&=\Big((B\cap\phi_{\geq -1})+\Big(0,\sqrt{1-\lambda^2},\lambda,0\Big)\Big)\cap\Big((B\cap\phi_{\leq 1})+\Big(0,-\sqrt{1-\lambda^2},-\lambda,0\Big)\Big)\\
&=\Big(B+\Big(0,\sqrt{1-\lambda^2},\lambda,0\Big)\Big)\cap\phi_{\geq 0}\cap\Big(B+\Big(0,-\sqrt{1-\lambda^2},-\lambda,0\Big)\Big)\cap\phi_{\leq 0}\\
&\subset\Big(B+\Big(0,-\sqrt{1-\lambda^2},-\lambda,0\Big)\Big)\cap\phi_{=0}\\
&=(B\cap\phi_{=1})+\Big(0,-\sqrt{1-\lambda^2},-\lambda,0\Big)\\
&=[(\lambda,0,0,0),(-\lambda,0,0,0)].
\end{aligned}$$

This gives the inclusion "$\subset$" in (4.17). The reverse inclusion is evident, since the ball hull is closed and convex.
Step 2: Verification of the equivalence of

$$\{(1,0,0,0),(-1,0,0,0)\}\subset B(z,1)=B+z \tag{4.21}$$

and

$$z=(0,0,\tau_3,0)\quad\text{with}\quad\tau_3\in[-1,1]. \tag{4.22}$$

Suppose that the point $z=(\tau_1,\tau_2,\tau_3,\tau_4)$ satisfies (4.21). The inclusion $B=\mathrm{co}(B_1\cup B_2)\subset\mathrm{co}([-1,1]^4)=[-1,1]^4$ together with (4.21) yields

$$\tau_1=0\quad\text{and}\quad\tau_3\in[-1,1]. \tag{4.23}$$

Now the assumption $(-1,0,0,0)\in B+z$ implies $(-1,-\tau_2,-\tau_3,-\tau_4)\in B$ and, by symmetry, also $(1,\tau_2,\tau_3,\tau_4)\in B$. By (4.16), there are numbers $\alpha_1,\beta_1,\alpha_3,\beta_3\in\mathbb{R}$ such that $\alpha_1+\beta_1=1$, $\alpha_3+\beta_3=\tau_3$, and

$$\max\left\{\|(\alpha_1,\tau_2)\|_2,\|(\tau_2,\alpha_3)\|_2\right\}+\max\left\{\|(\beta_1,\tau_4)\|_2,|\beta_3|\right\}\leq 1. \tag{4.24}$$

From $\alpha_1+\beta_1=1$ and (4.24), we obtain $1\leq|\alpha_1|+|\beta_1|\leq\|(\alpha_1,\tau_2)\|_2+\|(\beta_1,\tau_4)\|_2\leq 1$ and, in turn, we have $\tau_2=\tau_4=0$. By (4.23), the implication (4.21)$\Rightarrow$(4.22) is proved. For the converse implication (4.22)$\Rightarrow$(4.21), we have to show that $(\pm 1,0,-\tau_3,0)\in B$ for all $\tau_3\in[-1,1]$. But this is evident from $(\pm 1,0,-\tau_3,0)\in B_2\subset B$. Note that the equivalence (4.21)$\Longleftrightarrow$(4.22) yields

$$\mathrm{bh}(\{(1,0,0,0),(-1,0,0,0)\},B,1)=\bigcap_{\tau_3\in[-1,1]}(B+(0,0,\tau_3,0)). \tag{4.25}$$

Step 3: Verification of "$\subset$" from (4.18). The linear functional $\tilde{\phi}\in X^*$, $\langle\tilde{\phi}\,|\,(\xi_1,\xi_2,\xi_3,\xi_4)\rangle:=\xi_3$, satisfies $B_1\cup B_2\subset\tilde{\phi}_{\geq -1}\cap\tilde{\phi}_{\leq 1}$. Hence

$$B\subset\tilde{\phi}_{\geq -1}\cap\tilde{\phi}_{\leq 1} \tag{4.26}$$

and

$$B\cap\tilde{\phi}_{=1}=\mathrm{co}((B_1\cap\tilde{\phi}_{=1})\cup(B_2\cap\tilde{\phi}_{=1}))$$

$$
\begin{aligned}
&= \operatorname{co}(\{(\xi_1,0,1,0) \mid |\xi_1| \le 1\} \cup \{(\xi_1,0,1,\xi_4) \mid \|(\xi_1,\xi_4)\|_2 \le 1\}) \\
&= \{(\xi_1,0,1,\xi_4) \mid \|(\xi_1,\xi_4)\|_2 \le 1\}.
\end{aligned} \tag{4.27}
$$

For the claim "⊂" from (4.18), use (4.25), (4.26), and (4.27) to obtain

$$
\begin{aligned}
&\operatorname{bh}(\{(1,0,0,0),(-1,0,0,0)\},B,1) \\
&\subset (B+(0,0,1,0)) \cap (B+(0,0,-1,0)) \\
&= ((B \cap \tilde{\phi}_{\ge -1}) + (0,0,1,0)) \cap ((B \cap \tilde{\phi}_{\le 1}) + (0,0,-1,0)) \\
&= (B+(0,0,1,0)) \cap \tilde{\phi}_{\ge 0} \cap (B+(0,0,-1,0)) \cap \tilde{\phi}_{\le 0} \\
&\subset (B+(0,0,-1,0)) \cap \tilde{\phi}_{=0} \\
&= (B \cap \tilde{\phi}_{=1}) + (0,0,-1,0) \\
&= \{(\xi_1,0,0,\xi_4) \mid \|(\xi_1,\xi_4)\|_2 \le 1\}.
\end{aligned}
$$

Step 4: Verification of "⊃" from (4.18). Let $\xi_1,\xi_4,\tau_3 \in \mathbb{R}$ be such that $\|(\xi_1,\xi_4)\|_2 \le 1$ and $|\tau_3| \le 1$. Then $(\xi_1,0,-\tau_3,\xi_4) \in B_2 \subset B$ and, in turn, $(\xi_1,0,0,\xi_4) \in B+(0,0,\tau_3,0)$. By (4.25), this implies "⊃" from (4.18). □

4.4 Minimal representation of ball convex bodies as ball hulls

In this section we will present, as announced, minimal representations of ball convex bodies in terms of their ball exposed faces. This is motivated by Straszewicz's theorem [207, Theorem 1.4.7] from classical convexity which says that a convex body $K \in \mathscr{K}^X$ coincides with the closed convex hull of its exposed points. Furthermore, the set of exposed points of $K \in \mathscr{K}^X$ is the smallest subset of K whose closed convex hull is K. In the context of ball convexity, we will describe subsets K' of a b-convex body $K \subset X$ for which $\operatorname{bh}(K',B,1) = K$. Under some extra assumptions, we may take the set $\operatorname{b-exp}(K,B)$ for K', and $\operatorname{cl}(\operatorname{b-exp}(K,B))$ is then the unique minimal closed set whose ball hull is K, see Theorem 4.27 below. We start with a characterization of in the above sense generating subsets of a b-convex body under the assumption of b-boundedness.

Theorem 4.24. *Let (X,γ) be a generalized Minkowski space with unit ball B, $K \subset X$ a b-bounded b-convex body, and $K' \subset K$. Then* $\operatorname{bh}(K',B,1) = K$ *if and only if every b-exposed face of K meets* $\operatorname{cl}(K')$.

Proof. For the proof of "⇒", suppose that there is a b-exposed face F of K such that $F \cap \operatorname{cl}(K') = \emptyset$. We have to show that $\operatorname{bh}(K',B,1) \ne K$. The b-exposed face F has a representation $F = K \cap S(y,1)$, where $S(y,1)$ is a supporting sphere of K. By $F \cap \operatorname{cl}(K') = \emptyset$, we obtain $\operatorname{cl}(K') \subset U(y,1)$ and, by Lemma 4.7(d), $\operatorname{cl}(K') \subset B(y,\lambda)$ for some $\lambda < 1$. We fix $x_0 \in F$. Then $\gamma(x_0 - y) = 1$, because $F \subset S(y,1)$, and $x_0 \notin B(y-(1-\lambda)(x_0-y),1)$, since $\gamma(x_0-(y-(1-\lambda)(x_0-y))) = (2-\lambda)\gamma(x_0-y) > 1$. But $K' \subset B(y,\lambda) \subset B(y-(1-\lambda)(x_0-y),1)$ due to the triangle inequality. Thus $x_0 \notin B(y-(1-r)(x_0-y),1) \supset \operatorname{bh}(K',B,1)$ and $x_0 \in F \subset K$, showing that $\operatorname{bh}(K',B,1) \ne K$. For the converse implication "⇐", we suppose that $\operatorname{bh}(K',B,1) \ne K$ and show that $\operatorname{cl}(K')$ misses at least one b-exposed face F_0 of K. Since $\operatorname{bh}(K',B,1) \ne K$ and $\operatorname{bh}(K',B,1) \subset K$ by Lemma 4.7, there exists a point $x_0 \in K \setminus \mathrm{bh}(K',B,1)$. Using Proposition 4.9(b), we separate x_0 from the b-convex body $\mathrm{bh}(K',B,1) \subset K$ by a sphere $S(y_0,1)$, i.e.,

$$\mathrm{cl}(K') \subset \mathrm{bh}(K',B,1) \subset B(y_0,1) \text{ and } x_0 \notin B(y_0,1). \tag{4.28}$$

The b-boundedness of K yields the existence of a point $y_1 \in X$ such that

$$\mathrm{cl}(K') \subset K \subset U(y_1,1). \tag{4.29}$$

We consider the balls $B_\alpha := B(y_0+\alpha(y_1-y_0),1)$ for $\alpha \in [0,1]$. Then $K \not\subset B_0$ by (4.28) and $K \subset B_1$ by (4.29). Consequently, there exists $\alpha_0 := \min\{\alpha \in [0,1] \,|\, K \subset B_\alpha\} \in (0,1]$. By the definition of α_0 and a compactness argument, the set $F_0 := K \cap \mathrm{bd}(B_{\alpha_0})$ is non-empty, so that $S_{\alpha_0} := \mathrm{bd}(B_{\alpha_0})$ is a supporting sphere of K and F_0 is a b-exposed face of K. Now it remains to show that $F_0 \cap \mathrm{cl}(K') = \emptyset$. Suppose that this is not the case, i.e., there exists $z_0 \in F_0 \cap \mathrm{cl}(K')$. The inclusions (4.28) and (4.29) yield $\gamma(z_0-y_0) \le 1$ and $\gamma(z_0-y_1) < 1$. Finally, the inclusion $z_0 \in F_0 \subset S_{\alpha_0} = S(y_0+\alpha_0(y_1-y_0),1)$ implies

$$\begin{aligned}1 &= \gamma(z_0-(y_0+\alpha_0(y_1-y_0)))\\ &= \gamma(\alpha_0(z_0-y_1)+(1-\alpha_0)(z_0-y_0))\\ &\le \alpha_0\gamma(z_0-y_1)+(1-\alpha_0)\gamma(z_0-y_0)\\ &< \alpha_0+(1-\alpha_0)\\ &= 1.\end{aligned}$$

This contradiction completes the proof. □

Note that the proof of "⇒" in Theorem 4.24 did not require b-boundedness of K. However, b-boundedness is essential for "⇐" in Theorem 4.24. To see this, consider a closed ball $K := B(y,1)$. (Proposition 4.16(a)⇒(b) says that these are the only b-convex bodies that are not b-bounded, provided that the gauge γ is rotund.) Then the only supporting sphere of K is $S(y,1)$, and the only b-exposed face is $K \cap S(y,1) = S(y,1)$. Then every singleton $K' = \{x_0\} \subset S(y,1)$ satisfies the condition from Theorem 4.24, but $\mathrm{bh}(K',B,1) = \{x_0\}$ is not K. We illustrate Theorem 4.24 with the help of an example.

Example 4.25. Let $X = \mathbb{R}^2$ and $B := [-1,1]^2$. Then all b-convex bodies are of the form $[\alpha_1,\beta_1] \times [\alpha_2,\beta_2]$ with $0 \le \beta_i - \alpha_i \le 2$, $i \in \{1,2\}$. We restrict our consideration to b-bounded b-convex bodies K with non-empty interior. These are rectangles $K = [\alpha_1,\beta_1] \times [\alpha_2,\beta_2]$ with $0 < \beta_i - \alpha_i < 2$, $i \in \{1,2\}$. The b-exposed faces of K are the line segments

$$\begin{aligned}F_1 &:= [\alpha_1,\beta_1] \times \{\beta_2\}, & F_2 &:= \{\alpha_1\} \times [\alpha_2,\beta_2],\\ F_3 &:= [\alpha_1,\beta_1] \times \{\alpha_2\}, & F_4 &:= \{\beta_1\} \times [\alpha_2,\beta_2],\end{aligned}$$

and the unions $F_1 \cup F_2$, $F_2 \cup F_3$, $F_3 \cup F_4$, $F_4 \cup F_1$. Theorem 4.24 states that a set $K' \subset K$ satisfies $\mathrm{bh}(K',B,1) = K$ if and only if $\mathrm{cl}(K') \cap F_i \ne \emptyset$ for $i \in \{1,2,3,4\}$. Consequently, a set K' with $\mathrm{bh}(K',B,1) = K$ must contain at least one point from each of F_1, F_2, F_3, F_4. Such a set K' may consist of two (if K' is composed of two vertices symmetric with respect to the center of K), three, or four points (if K' contains exactly one point from the relative interior of each set F_i). This example can be generalized to $X = \mathbb{R}^d$ and $B := [-1,1]^d$. Given a b-bounded b-convex body K with non-empty interior, we may conclude that K is a box $[\alpha_1, \beta_1] \times \ldots \times [\alpha_d, \beta_d]$. Furthermore, each set $K' \subset K$ with $\mathrm{bh}(K', B, 1) = K$ must contain a point from every $(d-1)$-face of K and may consist of $2, \ldots, 2d$ elements.

As a corollary, we obtain a necessary condition in terms of b-exposed points.

Corollary 4.26. *If a subset K' of a b-bounded b-convex body K in a generalized Minkowski space (X, γ) with unit ball B satisfies* $\mathrm{bh}(K', B, 1) = K$*, then* $\text{b-exp}(K, B) \subset \mathrm{cl}(K')$.

Proof. If $x \in \text{b-exp}(K, B)$, then $\{x\}$ is a b-exposed face of K. Now Theorem 4.24 yields $\{x\} \cap \mathrm{cl}(K') \neq \emptyset$, i.e., $x \in \mathrm{cl}(K')$. □

Under the additional assumption of rotundity, the necessary condition from Corollary 4.26 is also sufficient.

Theorem 4.27. *A subset K' of a b-bounded b-convex body K in a rotund generalized Minkowski space (X, γ) with unit ball B satisfies* $\mathrm{bh}(K', B, 1) = K$ *if and only if* $\text{b-exp}(K, B) \subset \mathrm{cl}(K')$*. In particular, we have* $K = \mathrm{bh}(\text{b-exp}(K, B), B, 1)$*, and* $\mathrm{cl}(\text{b-exp}(K, B))$ *is the unique minimal closed subset of X whose ball hull is K.*

Proof. The implication $\mathrm{bh}(K', B, 1) = K \Rightarrow \text{b-exp}(K, B) \subset \mathrm{cl}(K')$ is given by Corollary 4.26. To see the converse implication, it is enough to show that

$$K \subset \mathrm{bh}(\text{b-exp}(K, B), B, 1). \tag{4.30}$$

Indeed, if $\text{b-exp}(K, B) \subset \mathrm{cl}(K')$ and if (4.30) is verified, then Theorem 4.3(a) and Lemma 4.7(a) imply $K \subset \mathrm{bh}(\text{b-exp}(K, B), B, 1) \subset \mathrm{bh}(\mathrm{cl}(K'), B, 1) = \mathrm{bh}(K', B, 1) \subset \mathrm{bh}(K, B, 1) = K$. Now, for showing (4.30), let us assume that there exists a point $x_0 \in K \setminus \mathrm{bh}(\text{b-exp}(K, B), B, 1)$. The b-boundedness of K implies b-boundedness of the set $\tilde{K} := \mathrm{bh}(\text{b-exp}(K, B), B, 1) \subset \mathrm{bh}(K, B, 1) = K$. Separation of x_0 and $\tilde{K}$ by Proposition 4.9(c) and the b-boundedness of K yield the existence of points $y_0, y_1 \in X$ and a number $\lambda \in (0, 1)$ such that

$$\tilde{K} \subset B(y_0, \lambda), \ x_0 \notin B(y_0, 1), \tag{4.31}$$

and

$$\tilde{K} \subset K \subset U(y_1, \lambda). \tag{4.32}$$

Similarly as in the proof of Theorem 4.24, we define $B_\alpha^\lambda := B(y_0 + \alpha(y_1 - y_0), \lambda)$ for $\alpha \in [0, 1]$ and, taking $K \not\subset B_0^\lambda$ from (4.31) and $K \subset B_1^\lambda$ from (4.32) into account, find

$$\alpha_0 := \inf\{\alpha \in [0, 1] \mid K \subset B_\alpha^\lambda\} \in (0, 1].$$

Then there exists a point $x_1 \in K \cap S_{\alpha_0}^\lambda$ where $S_{\alpha_0}^\lambda := \mathrm{bd}(B_{\alpha_0}^\lambda)$. Next we show that

$$x_1 \notin \tilde{K}. \tag{4.33}$$

Indeed, if we assume $x_1 \in \tilde{K}$, we obtain $\gamma(x_1 - y_0) \leq \lambda$ and $\gamma(x_1 - y_1) < \lambda$. Then the inclusion $x_1 \in S_{\alpha_0}^{\lambda} = S(y_0 + \alpha_0(y_1 - y_0), \lambda)$ gives

$$\begin{aligned}\lambda &= \gamma(x_1 - (y_0 + \alpha_0(y_1 - y_0))) \\ &= \gamma(\alpha_0(x_1 - y_1) + (1 - \alpha_0)(x_1 - y_0)) \\ &\leq \alpha_0\gamma(x_1 - y_1) + (1 - \alpha_0)\gamma(x_1 - y_0) \\ &< \alpha_0\lambda + (1 - \alpha_0)\lambda \\ &= \lambda,\end{aligned}$$

a contradiction. Since $B_{\alpha_0}^{\lambda}$ is a b-convex body by Lemma 4.7(b) and since $x_1 \in S_{\alpha_0}^{\lambda} = \mathrm{bd}(B_{\alpha_0}^{\lambda})$, Proposition 4.9(a) yields the existence of a point $y_2 \in X$ such that $B_{\alpha_0}^{\lambda} \subset B(y_2, 1)$ and $x_1 \in B_{\alpha_0}^{\lambda} \cap S(y_2, 1)$. From Proposition 4.16(a)⇒(g) and the rotundity of γ, we obtain $B_{\alpha_0}^{\lambda} \cap S(y_2, 1) = \{x_1\}$. Using the inclusions $x_1 \in K$ and $K \subset B_{\alpha_0}^{\lambda}$, it follows that $K \subset S(y_2, 1)$ and $K \cap S(y_2, 1) = \{x_1\}$. Hence $x_1 \in \text{b-exp}(K, B)$. But (4.33) implies $x_1 \notin \mathrm{bh}(\text{b-exp}(K, B), B, 1)$. This final contradiction establishes (4.30) and completes the proof. □

By Theorem 4.27, every b-bounded b-convex body in a rotund generalized Minkowski space gives rise to a unique minimal closed subset whose ball hull is that b-bounded b-convex body. This need not be the case if the gauge fails to be rotund, see Example 4.25. Furthermore, Theorem 4.27 makes a statement about the closure of the set of b-exposed points of a b-bounded b-convex body. There, the closure operator is essential, as the following example shows.

Example 4.28. The set b-exp(K, B) of all b-exposed points of a b-bounded b-convex body K is not necessarily closed. An example for this in the Euclidean plane $(\mathbb{R}^2, \|\cdot\|_2)$ with unit ball B is given by the convex body K bounded by the circular arcs

$$\begin{aligned}A_1 &:= \left\{\left(-\frac{\sqrt{3}}{4} + \sin(\alpha), \cos(\alpha)\right) \,\middle|\, \frac{\pi}{3} \leq \alpha \leq \frac{2\pi}{3}\right\}, \\ A_2 &:= \left\{\left(\frac{1}{2}\sin(\alpha), -\frac{1}{4} + \frac{1}{2}\cos(\alpha)\right) \,\middle|\, \frac{2\pi}{3} < \alpha < \frac{4\pi}{3}\right\}, \\ A_3 &:= \left\{\left(\frac{\sqrt{3}}{4} + \sin(\alpha), \cos(\alpha)\right) \,\middle|\, \frac{4\pi}{3} \leq \alpha \leq \frac{5\pi}{3}\right\}, \\ A_4 &:= \left\{\left(\frac{1}{2}\sin(\alpha), \frac{1}{4} + \frac{1}{2}\cos(\alpha)\right) \,\middle|\, -\frac{\pi}{3} < \alpha < \frac{\pi}{3}\right\}.\end{aligned}$$

Thus K is a b-bounded b-convex body with b-exp$(K, B) = A_2 \cup A_4$. Non-singleton b-exposed faces are A_1 and A_3.

Next, we point out two consequences of Theorem 4.27.

Corollary 4.29. *If K is a b-bounded b-convex body in a rotund generalized Minkowski space (X, γ) with unit ball B, then every b-exposed face of K meets the closure of* b-exp(K, B).

Proof. This follows from Theorems 4.24 and 4.27. □

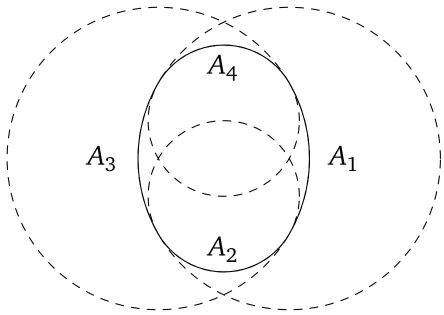


Figure 4.2. Illustration for Example 4.28: The set of b-exposed points of a b-bounded b-convex body need not be closed.

Corollary 4.30. *If K' is a non-empty b-bounded subset of a rotund generalized Minkowski space (X,γ) with unit ball B, then* $\text{b-exp}(\text{bh}(K',B,1),B) \subset \text{cl}(K')$.

Proof. Taking items (a) and (c) of Lemma 4.7 into account, we know that $K := \text{bh}(K',B,1)$ is a b-bounded b-convex body and that $K' \subset K$. Now Theorem 4.27 implies $\text{cl}(K') \supset \text{b-exp}(K,B) = \text{b-exp}(\text{bh}(K',B,1),B)$. □

In Example 4.25, we saw that b-bounded b-convex bodies need not have b-exposed points. This shows that the claims of Theorem 4.27 and Corollary 4.29 fail in general if the underlying gauge is not rotund. A similar reason justifies the assumption of b-boundedness in Theorem 4.27 and Corollary 4.29.

Proposition 4.31. *If a b-convex body K in a generalized Minkowski space (X,γ) with unit ball B is not b-bounded, then* $\text{b-exp}(K,B) = \emptyset$.

Proof. As K is not b-bounded, we have $R(K,B) = 1$. Hence every supporting sphere $S(x,1)$ of K is the boundary of a circumball $B(x,1)$. By Lemma 4.8, the set $K \cap S(x,1)$ contains at least two points. Therefore, b-exposed faces of K are not singletons. □

5

Isosceles orthogonality and bisectors

Resembling the facts that parallelograms whose diagonals are equal in length are rectangles and that isosceles triangles are mirror-symmetric, *isosceles orthogonality* is another approach of extending classical Euclidean orthogonality to other spaces. James introduced this notion in [126] to normed spaces by calling points x and y of a normed space $(X, \|\cdot\|)$ isosceles orthogonal if $\|y+x\| = \|y-x\|$, thus if y is at equal distance from $-x$ and x. The set of all points which is at equal distance from two fixed points is called the *bisector* of the those points. Bisectors therefore consist of intersections of congruent spheres or, equivalently, of intersections of the boundaries of translates of a convex body, and have already been investigated for non-symmetric convex bodies and non-symmetric distance functions, see [115–117]. In several contexts, bisectors serve as building blocks of Voronoi-type diagrams which have a great impact as tools in discrete and computational geometry. (For recent research in this direction see, e.g., [66].) Although the geometry of bisectors and the geometry of isosceles orthogonality are essentially the same, these topics are often investigated independently. This also results in distinct sets of problems in which researchers are interested, and it might be the cause why isosceles orthogonality lacks a generalization to generalized Minkowski spaces. In this chapter, we fill this gap by giving descriptions of the geometry and topology of bisectors in this setting and by getting acquainted with the properties of an appropriate version of isosceles orthogonality. The present chapter combines the contents of [121, Section 4], [123, Sections 4 and 6], and [125].

Definition 5.1. Let (X, γ) be a generalized Minkowski space. We say that the point $y \in X$ is *isosceles orthogonal to* $x \in X$ and write $y \perp_I x$ if $\gamma(y+x) = \gamma(y-x)$. The *bisector* of points $x, y \in X$ is the set $\mathrm{bsc}_\gamma(x,y) := \{z \in X \mid \gamma(z-x) = \gamma(z-y)\}$ As before, the index γ is omitted when there is no ambiguity.

The fact that the geometry of bisectors and isosceles orthogonality is essentially the same can be made precise by observing that $y \perp_I x$ if and only if $y \in \mathrm{bsc}(-x,x)$, and $\mathrm{bsc}(x,y) = \frac{x+y}{2} + \left\{z \in X \,\middle|\, z \perp_I \frac{x-y}{2}\right\}$. As announced, one may write the bisector $\mathrm{bsc}(x,y)$ as a union of intersections of congruent spheres centered at $x, y \in X$. Radii close to zero are not important for this intersection as the corresponding spheres are disjoint. A sharp lower bound for the involved radii can be given in terms of a circumradius.

Proposition 5.2. *Let (X, γ) be a generalized Minkowski space. The bisector of two points $x, y \in X$ is*

$$\begin{aligned}\mathrm{bsc}(x,y) &= \{z \in X \mid \gamma(z-x) = \gamma(z-y)\} \\ &= \bigcup_{\lambda \geq 0} \{z \in X \mid \gamma(z-x) = \gamma(z-y) = \lambda\}\end{aligned}$$

$$= \bigcup_{\lambda \geq 0} (S(x,\lambda) \cap S(x,\lambda))$$
$$= \bigcup_{\lambda \geq R(\{x,y\},-B(0,1))} (S(x,\lambda) \cap S(x,\lambda)).$$

In particular, we have $\mathrm{cc}(\{x,y\},-B(0,1)) \subset \mathrm{bsc}(x,y)$.

The proof of Proposition 5.2 is straightforward. For its parts referring to the circumradius and the set of circumcenters of $\{x,y\}$ with respect to the reflected ball $-B(0,1)$, Väisälä's account [233, Lemma 5] might give the reader a start. The latter can be understood as an elementary treatment of the fact that a line segment in a generalized Minkowski plane is a chord of any of its circumballs, which is also a consequence of Lemma 3.9. In fact, this carries over verbatim to higher dimensions. Therefore we reproduce [233, Lemma 5] and its proof for generalized Minkowski spaces of arbitrary dimension.

Lemma 5.3. *Let* (X,γ) *be a generalized Minkowski space,* $x,y,z \in X$*, and* $x \neq y$.

(a) *We have* $R(\{x,y\},B(0,1)) < \min\{\gamma(x-y),\gamma(y-x)\}$.
(b) *If* $\gamma(x-z) < \gamma(y-z)$*, then* $R(\{x,y\},B(0,1)) < \gamma(y-z)$.
(c) *If* $\gamma(x-z) \leq \gamma(y-z) = R(\{x,y\},B(0,1))$*, then* $R(\{x,y\},B(0,1)) = \gamma(x-z)$.
(d) *If* $z \in \mathrm{cc}(\{x,y\},B(0,1))$*, then* $\gamma(x-z) = \gamma(y-z) = R(\{x,y\},B(0,1))$.

Proof. Statement (a) follows from $\{x,y\} \subset B(x,\gamma(y-x))$ and $\{x,y\} \subset B(y,\gamma(x-y))$. For (b), choose $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < \min\{\gamma(z-y),\gamma(y-z)-\gamma(x-z)\}$ and let $w \in [z,y]$ be the point with $\gamma(z-w) = \varepsilon$, i.e., $w = z - \frac{\varepsilon}{\gamma(z-y)}(z-y)$. Now

$$\gamma(x-w) \leq \gamma(x-z) + \gamma(z-w) = \gamma(x-z) + \varepsilon < \gamma(y-z).$$

As $\gamma(y-w) < \gamma(y-z)$, we have $\{x,y\} \subset U(w,\gamma(y-z))$. Hence $R(\{x,y\},B(0,1)) < \gamma(y-z)$. Finally, (c) and (d) are direct consequences of (b). □

5.1 Symmetry and directional convexity of bisectors

Theorem 2.29 states that for coinciding Birkhoff orthogonality relations on X, the underlying gauges are positive multiples of each other. An analogous result is true for isosceles orthogonality.

Lemma 5.4. *Let* $\gamma_1,\gamma_2 : X \to \mathbb{R}$ *be two gauges whose isosceles orthogonality relations shall be denoted* $\perp_{I,1}$ *and* $\perp_{I,2}$*, respectively. Assume that for all* $x,y \in X$*, the relation* $x \perp_{I,1} y$ *implies* $x \perp_{I,2} y$ *or, equivalently,* $\mathrm{bsc}_{\gamma_1}(x,y) \subset \mathrm{bsc}_{\gamma_2}(x,y)$*. Then there exists a number* $\alpha > 0$ *such that* $\gamma_1(x) = \alpha\gamma_2(x)$ *for all* $x \in X$.

Proof. Take $z,w \in X$ with $\gamma_1(z) = \gamma_1(w)$. For $x := z+w$ and $y := z-w$, we have $x \perp_{I,1} y$ because $\gamma_1(x+y) = 2\gamma_1(z) = 2\gamma_1(w) = \gamma_1(x-y)$. By assumption, we obtain $x \perp_{I,2} y$, i.e., $2\gamma_2(z) = \gamma_2(x+y) = \gamma_2(x-y) = 2\gamma_2(w)$. □

The converse implication of Lemma 5.4 is trivially true. In this sense, the shape of bisectors is only determined by the shape of the unit ball of the gauge, not by its size. In Theorem 2.30, we showed that if ε-Birkhoff orthogonality relations of two generalized Minkowski spaces (X,γ_1) and (X,γ_2) are inverses of each other, then γ_1 and γ_2 are norms and $\varepsilon = 0$. Here is a result on isosceles orthogonality relations which are mutual inverses.

Theorem 5.5. *Let $\gamma_1, \gamma_2 : X \to \mathbb{R}$ be two gauges whose isosceles orthogonality relations shall be denoted $\perp_{I,1}$ and $\perp_{I,2}$, respectively. Assume that for all $x, y \in X$, the relation $x \perp_{I,1} y$ implies $y \perp_{I,2} x$. Then there exists a number $\alpha > 0$ such that $\gamma_1(x) = \alpha\gamma_2(x)$ for all $x \in X$, and γ_2 is a norm.*

Proof. For all $y \in X$, we have $y \perp_{I,1} 0$. By assumption, we have $0 \perp_{I,2} y$ for all $y \in X$, i.e., $\gamma_2(0+y) = \gamma_2(0-y)$ for all $y \in X$. Thus γ_2 is a norm. Since isosceles orthogonality is a symmetric relation in normed spaces, we have that $x \perp_{I,1} y$ implies $x \perp_{I,2} y$. Now apply Lemma 5.4. □

As a direct consequence of Theorem 5.5, we obtain the following statement.

Corollary 5.6. *Let (X,γ) be a generalized Minkowski space. If isosceles orthogonality is a symmetric relation, then γ is a norm.*

In other words, symmetry of isosceles orthogonality as a relation implies central symmetry of all bisectors. The validity of the converse statements of Theorem 5.5 and Corollary 5.6 is evident. Next, we will show the directional convexity of bisectors. For this, we use the following extension of [154, Proposition 7] to generalized Minkowski spaces. It establishes an inequality between the sum of the lengths of two non-adjacent sides of a plane non-degenerate quadrilateral and the sum of the lengths of its diagonals. (Here, a *plane non-degenerate quadrilateral* in X is the convex hull of four points none of which lies in the convex hull of the remaining three points, with the property that said convex hulls have dimension 2, see Figure 5.1 for an illustration.) We will use this result again in Section 5.2 for a two-dimensional reiteration of the directional convexity, and in Section 5.4 for proving the directional convexity of hyperboloids.

Lemma 5.7. *Suppose that $z, x, y, w \in X$ are in this successive order the vertices of a plane non-degenerate quadrilateral in a generalized Minkowski space (X,γ). Then*

$$\gamma(z-x) + \gamma(w-y) \leq \gamma(z-y) + \gamma(w-x),$$

with equality if and only if $\left[\frac{z-y}{\gamma(z-y)}, \frac{w-x}{\gamma(w-x)}\right] \subset S(0,1)$.

Proof. Let $\{s\} := [z,y] \cap [x,w]$, see Figure 5.1. Using the triangle inequality, we have

$$\begin{aligned}\gamma(z-x) + \gamma(w-y) &\leq \gamma(z-s) + \gamma(s-x) + \gamma(w-s) + \gamma(s-y)\\ &= \gamma(z-y) + \gamma(w-x).\end{aligned}$$

Note that $\frac{z-s}{\gamma(z-s)} = \frac{s-y}{\gamma(s-y)} = \frac{z-y}{\gamma(z-y)}$ and $\frac{w-s}{\gamma(w-s)} = \frac{s-x}{\gamma(s-x)} = \frac{w-x}{\gamma(w-x)}$. Thus, if the line segment $\left[\frac{z-y}{\gamma(z-y)}, \frac{w-x}{\gamma(w-x)}\right]$ is a subset of $S(0,1)$, then $\gamma(z-x) = \gamma(z-s) + \gamma(s-x)$ and $\gamma(w-y) = \gamma(w-s) + \gamma(s-y)$ by Lemma 2.22. It follows that

$$\gamma(z-x) + \gamma(w-y) = \gamma(z-s) + \gamma(s-x) + \gamma(w-s) + \gamma(s-y) = \gamma(z-y) + \gamma(w-x).$$

Conversely, if

$$\gamma(z-x)+\gamma(w-y)=\gamma(z-y)+\gamma(w-x)=\gamma(z-s)+\gamma(s-x)+\gamma(w-s)+\gamma(s-y),$$

then $\gamma(z-x)=\gamma(z-s)+\gamma(s-x)$ and $\gamma(w-y)=\gamma(w-s)+\gamma(s-y)$. (In both lines, the triangle inequality gives "$\leq$". Since the sum of these two inequalities is in fact an equality, we have equality in the single inequalities.) Using Lemma 2.22 again, we have $\left[\frac{z-y}{\gamma(z-y)},\frac{w-x}{\gamma(w-x)}\right]\subset S(0,1)$. $\square$

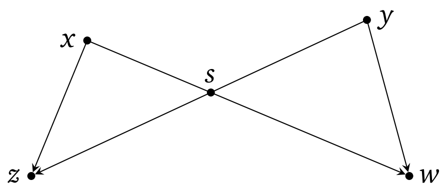


Figure 5.1. Illustration for Lemma 5.7.

The intersection of a bisector $\operatorname{bsc}(-x,x)$ and every translate of $\langle -x,x\rangle$ is non-empty and convex. (We shall refer to this property as *directional convexity*.) This result supersedes the finite-dimensional case of [126, Theorem 4.4], where the nonemptiness part is stated for normed spaces. Our proof is patterned after [126, Lemma 4.4].

Lemma 5.8. *Let (X,γ) be a generalized Minkowski space, $x,y\in X$, $x\neq 0$. Then, the set of numbers $\alpha\in\mathbb{R}$ with $(\alpha x+y)\perp_I x$ is a non-empty, closed, bounded, and convex, i.e., a singleton or a line segment.*

Proof. For fixed $y\in X$, consider the function $f:\mathbb{R}\to\mathbb{R}$ given by $f(\alpha):=\gamma(\alpha x+y+x)-\gamma(\alpha x+y-x)$. Note that $f_{=0}=\{\alpha\in\mathbb{R}\mid(\alpha x+y)\perp_I x\}$. Closedness of this set is due to continuity of f. For $\alpha>0$ and $\lambda\in\mathbb{R}$, we have

$$\gamma((\alpha+\lambda)x+y))-\gamma(\alpha x+y)=\gamma\left(\alpha x+\frac{\alpha}{\alpha+\lambda}y\right)-\gamma(\alpha x+y)+\lambda\gamma\left(x+\frac{1}{\alpha+\lambda}y\right)$$

provided $\alpha+\lambda>0$. Using the subadditivity of γ, we obtain

$$0\leq\left|\gamma\left(\alpha x+\frac{\alpha}{\alpha+\lambda}y\right)-\gamma(\alpha x+y)\right|\leq\max\left\{\gamma\left(\frac{\lambda}{\alpha+\lambda}y\right),\gamma\left(-\frac{\lambda}{\alpha+\lambda}y\right)\right\},$$

yielding

$$\lim_{\alpha\to+\infty}\left(\gamma\left(\alpha x+\frac{\alpha}{\alpha+\lambda}y\right)-\gamma(\alpha x+y)\right)=0.$$

It follows that

$$\lim_{\alpha\to+\infty}(\gamma((\alpha+\lambda)x+y))-\gamma(\alpha x+y))=\lim_{\alpha\to+\infty}\lambda\gamma\left(x+\frac{1}{\alpha+\lambda}y\right)=\lambda\gamma(x).$$

From this equation, we conclude

$$\begin{aligned}
&\lim_{\alpha\to+\infty}(\gamma((\alpha x+y)+x)-\gamma((\alpha x+y)-x))\\
=&\lim_{\alpha\to+\infty}(\gamma((\alpha+1)x+y)-\gamma((\alpha-1)x+y))\\
=&\lim_{\alpha\to+\infty}(\gamma((\alpha+2)x+y)-\gamma(\alpha x+y))\\
=&\,2\gamma(x)>0 \qquad (5.1)
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\alpha\to-\infty}(\gamma((\alpha x+y)+x)-\gamma((\alpha x+y)-x))\\
=&\lim_{\alpha\to+\infty}(\gamma((-\alpha+1)x+y)-\gamma((-\alpha-1)x+y))\\
=&\lim_{\alpha\to+\infty}(\gamma((\alpha-2)(-x)+y)-\gamma(\alpha(-x)+y))\\
=&-2\gamma(-x)<0. \qquad (5.2)
\end{aligned}$$

Using the intermediate value theorem, the continuity of f yields the existence of a zero of f. Moreover, (5.1) and (5.2) imply that the set of zeros of f is bounded. Now fix $y\in X$ and $\alpha>0$. We show that $\gamma(y+x)-\gamma(y-x)\leq\gamma(\alpha x+y+x)-\gamma(\alpha x+y-x)$. If y is a multiple of x, the claim is easily seen. Else the points y, $-x$, x, and $\alpha x+y$ are (in this cyclic order) the vertices of a convex quadrangle. Apply Lemma 5.7 and obtain

$$\gamma(y+x)+\gamma(\alpha x+y-x)\leq\gamma(y-x)+\gamma(\alpha x+y-x)$$

or, equivalently,

$$\gamma(y+x)-\gamma(y-x)\leq\gamma(\alpha x+y+x)-\gamma(\alpha x+y-x).$$

Hence f is increasing, and its sublevel sets are intervals. This yields the convexity part of the claim. □

In view of Corollary 5.6 and Lemma 5.8, the following question has to be answered separately: For given points $x,y\in X$, $x\neq 0$, in a generalized Minkowski space (X,γ), is there a number $\alpha\in\mathbb{R}$ such that $x\perp_I(\alpha x+y)$? Surprisingly, the answer turns out to be negative in general. For instance, take $X=\mathbb{R}^2$, $\gamma:X\to\mathbb{R}$, $\gamma(\xi_1,\xi_2):=\max\{-\xi_2,2\xi_2-\xi_1,2\xi_2+\xi_1\}$, $x:=(1,0)$, and $y:=(0,1)$.

5.2 Topological properties of bisectors

The directional convexity of bisectors stated in Lemma 5.8 can also be established differently, namely, without using the intermediate value theorem. For Minkowski spaces, this is done in, e.g., [154, Proposition 15] and [112, Lemmas 1 and 2]. These techniques can be carried over to generalized Minkowski spaces and they yield insight to the occurrence of non-singleton line segments in a bisector $\operatorname{bsc}(x,y)$ which are contained in translates of $\langle x,y\rangle$. This is closely related to the construction of bisectors given in [144, Section 2.1.1] for generalized Minkowski planes. The latter turns out to be sufficient for giving a construction of bisectors in generalized Minkowski spaces of arbitrary dimension which carries further information on topological properties of bisectors. First, we show that under certain circumstances, bisectors may have interior points. Here, the cones $C_\gamma(x,\phi)$ introduced in Section 3.1 come into play again.

Proposition 5.9. *Let (X,γ) be a generalized Minkowski space, $x \in X \setminus \{0\}$, $\phi \in X^*$, $\gamma^\circ(\phi) = 1$, and $\langle \phi \,|\, x\rangle = 0$. Then we have $C_\gamma(-x,\phi) \cap C_\gamma(x,\phi) \subset \operatorname{bsc}_\gamma(-x,x)$.*

Proof. Consider

$$\begin{aligned}
&z \in C_\gamma(-x,\phi) \cap C_\gamma(x,\phi)\\
&\iff \langle \phi \,|\, z+x\rangle = \gamma(z+x) \wedge \langle \phi \,|\, z-x\rangle = \gamma(z-x)\\
&\implies \gamma(z+x) = \langle \phi \,|\, z+x\rangle + \langle \phi \,|\, -2x\rangle = \langle \phi \,|\, z-x\rangle = \gamma(z-x)\\
&\implies z \in \operatorname{bsc}_\gamma(-x,x)\,.
\end{aligned}$$

This yields the assertion. □

In Proposition 5.9, the assumptions $\phi \in X^*$, $\gamma^\circ(\phi) = 1$, and $\langle \phi \,|\, x\rangle = 0$ imply that there is a common supporting hyperplane $\phi_{=1}$ of $B(-x,1)$ and $B(-x,1)$ such that $\langle -x,x\rangle$ is contained in $\phi_{=0}$ (which is a translate of $\phi_{=1}$). The respective exposed faces $F_x := B(x,1) \cap \phi_{=1}$ and $F_{-x} := B(-x,1) \cap \phi_{=1} = -2x + F_x$ are translates of each other. Consequently, if $\dim(F_x) = \dim(F_{-x}) = \dim(X) - 1$, then $C_\gamma(-x,\phi) \cap C_\gamma(x,\phi)$ has non-empty interior, and so does $\operatorname{bsc}(-x,x)$. In view of Lemma 5.8, the sets $C_\gamma(-x,\phi) \cap C_\gamma(x,\phi)$ are portions of the bisector $\operatorname{bsc}(-x,x)$ for which the intersection with many translates of $\langle -x,x\rangle$ are non-singleton line segments. The cases in which the straight line $y + \langle -x,x\rangle$ intersects the bisector $\operatorname{bsc}(-x,x)$ in precisely one point are (at least partially) specified in Theorem 5.11 below. For this, we may restrict our considerations to the smallest linear subspace of X which contains $y + \langle -x,x\rangle$. If $\{x,y\}$ is a linearly dependent subset of X, then $y + \langle -x,x\rangle = \langle -x,x\rangle$, and we are interested in the intersection of $\langle -x,x\rangle$ and $\operatorname{bsc}(-x,x)$. This is the singleton $\{m_{-x,x}\}$ consisting of the *metric midpoint* $m_{-x,x} := \frac{\gamma(-x)-\gamma(x)}{\gamma(-x)+\gamma(x)}x$ of $-x$ and x. Else, if $\{x,y\}$ is a linearly independent subset of X, we only have to consider the portion of the bisector which is contained in the then two-dimensional *half-flat* $\operatorname{hfl}(x,y) := \langle -x,x\rangle + [0,y\rangle$. This is because

$$\bigcup_{\substack{\{x,y\}\subset X \text{ linearly}\\ \text{independent}}} (\operatorname{bsc}(-x,x) \cap \operatorname{hfl}(x,y)) = \operatorname{bsc}(-x,x)\,.$$

As can be conjectured from Proposition 5.9, the shape of the bisector is influenced by the absence or presence of line segments in the unit sphere which are contained in translates of the straight line passing through the points defining the bisector. For the two-dimensional description of the bisector in Theorem 5.11, we define the quantity

$$M_y(x) := \sup\left\{ \gamma(w-z) \,\middle|\, \begin{array}{c} [w,z] \subset S(0,1) \cap \operatorname{hfl}(x,y), \\ \exists\, \lambda \geq 0 : w - z = \lambda x \end{array} \right\}$$

which is the maximal length of a line segment contained in a set of the form $S(0,1) \cap (w + \langle -x,x\rangle)$ with $w \in \operatorname{hfl}(x,y)$. The following statement is a special case of Proposition 1.26, see [154, Lemma 5] for the analogous result in Minkowski spaces.

Lemma 5.10. *Let (X,γ) a generalized Minkowski space, $x,y,z \in X$, $\lambda \in (0,1)$, and $w := \lambda y + (1-\lambda)z$. Then $\gamma(w-x) \leq \max\{\gamma(y-x),\gamma(z-x)\}$, with equality if and only if $\gamma(w-x) = \gamma(y-x) = \gamma(z-x)$. In the case of equality, we have $\gamma(w-x) = \min\{\gamma(v-x) \mid v \in \langle y,z\rangle\}$ and $\gamma(w-x) = \gamma(v-x)$ for all $v \in [y,z]$.*

Proof. We have

$$\begin{aligned}
\gamma(w-x) &= \gamma(\lambda y + (1-\lambda)z) - x) \\
&= \gamma(\lambda(y-x) + (1-\lambda)(z-x)) \\
&\leq \lambda\gamma(y-x) + (1-\lambda)\gamma(z-x) && (5.3) \\
&\leq \max\{\gamma(y-x),\gamma(z-x)\}\,. && (5.4)
\end{aligned}$$

If (5.4) holds with equality, then $\gamma(y-x) = \gamma(z-x)$, and if (5.3) holds with equality as well, then these numbers are equal to $\gamma(w-x)$. In other words, the points $y,w,z \in S(x,\gamma(w-x))$ are collinear. Hence $[y,z] \subset S(x,\gamma(w-x))$ or, equivalently, $\gamma(w-x) = \gamma(v-x)$ for all $v \in [y,z]$. Now let $v \in \langle y,z\rangle$ be such that $z = \alpha y + (1-\alpha)v$ for some $\alpha \in (0,1)$. Applying the chain of inequalities above to y, z, and v, we obtain

$$\gamma(z-x) \leq \max\{\gamma(y-x),\gamma(v-x)\}\,. \tag{5.5}$$

Suppose $\gamma(v-x) < \gamma(y-x)$. Then (5.5) holds with equality, i.e., $\gamma(v-x) = \gamma(y-x)$. This is a contradiction. Thus $\gamma(v-x) \geq \gamma(y-x)$, which shows that $\gamma(w-x) = \min\{\gamma(v-x) \mid v \in \langle y,z\rangle\}$. □

Now we are able to give an analog of [131, Theorem 2.6] for generalized Minkowski spaces.

Theorem 5.11. *Let (X,γ) be a generalized Minkowski space, $x,y \in X\setminus\{0\}$. If $M_y(x) \leq \frac{2\gamma(x)}{\gamma(y)}$, then there exists a unique real number α such that $(y+\alpha x) \perp_I x$, i.e., the set $(y+\langle -x,x\rangle)\cap \operatorname{bsc}(-x,x)$ is a singleton.*

Proof. The existence of a number $\alpha \in \mathbb{R}$ with $y + \alpha x \in \operatorname{bsc}(-x,x)$ follows from Lemma 5.8. Suppose that there are two numbers α_1, $\alpha_2 \in \mathbb{R}$ such that $\alpha_1 < \alpha_2$ and $\{y+\alpha_1 x, y+\alpha_2 x\} \subset \operatorname{bsc}(-x,x)$. Let $f : \mathbb{R} \to \mathbb{R}$, $f(\alpha) := \gamma(y+\alpha x)$. Since f is a convex function, the equations

$$\begin{aligned}
f(\alpha_1+1) &= \gamma(y+\alpha_1 x + x) = \gamma(y+\alpha_1 x - x) = f(\alpha_1 - 1), \\
f(\alpha_2+1) &= \gamma(y+\alpha_2 x + x) = \gamma(y+\alpha_2 x - x) = f(\alpha_2 - 1)
\end{aligned}$$

imply that f is constant on the interval $[\alpha_1 - 1, \alpha_2 + 1]$. By Lemma 5.10, this constant equals $\beta := \min\{f(\alpha) \mid \alpha \in \mathbb{R}\}$. Therefore, the line segment $[y+(\alpha_1-1)x, y+(\alpha_2+1)x]$ is contained in $S(0,\beta)$, and we have

$$M_y(x) \geq \frac{1}{\beta}(\alpha_2 - \alpha_1 + 2)\gamma(x) \geq \frac{1}{\gamma(y)}(\alpha_2-\alpha_1+2)\gamma(x) > 2\frac{\gamma(x)}{\gamma(y)}.$$

This completes the proof. □

The sufficiency of two-dimensional descriptions of bisectors is also used in the construction given in [144, Section 2.1.1]. We reproduce this construction here because it carries information about the topology of bisectors. Furthermore, we reiterate the proof of the directional convexity of bisectors.

Remark 5.12 (Construction of the bisector I)**.** Let (X,γ) be a generalized Minkowski plane. Since $B_{\alpha\gamma}(0,1)=\frac{1}{\alpha}B_\gamma(0,1)$ for all numbers $\alpha>0$ and gauges $\gamma : X\to\mathbb{R}$, we have $\mathrm{bsc}_\gamma(x,y)=\mathrm{bsc}_{\alpha\gamma}(x,y)$ for all $x,y\in X$ with $x\neq y$, and vice versa. (See again Lemma 5.4.) Thus, we may assume that $B(x,1)$ and $B(y,1)$ are disjoint. Choose $v_x\in S(x,1)$ and $v_y\in S(y,1)$ such that $\langle v_x,v_y\rangle$ is a parallel to $\langle x,y\rangle$ and such that $[x,v_x\rangle$ and $[y,v_y\rangle$ have exactly one intersection point z, see Figure 5.2. The sets $\{z,v_x,v_y\}$ and $\{z,x,y\}$ are homothetic images of each other, so

$$\frac{\gamma(z-x)}{\gamma(v_x-x)}=\frac{\gamma(z-y)}{\gamma(v_y-y)},$$

i.e., $z\in\mathrm{bsc}(x,y)$. Conversely, for $z\in\mathrm{bsc}(x,y)\setminus\langle x,y\rangle=\mathrm{bsc}(x,y)\setminus\{m_{x,y}\}$, we set $\{v_x\}:=[x,z\rangle\cap S(x,1)$ and $\{v_y\}:=[y,z\rangle\cap S(y,1)$. Again, the straight lines $\langle v_x,v_y\rangle$ and $\langle x,y\rangle$ parallel.

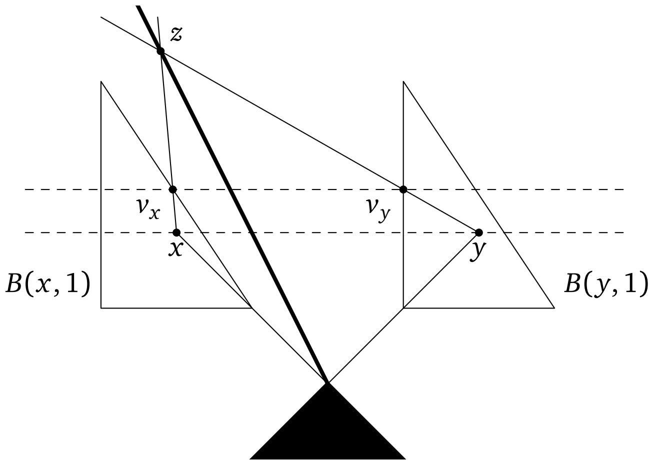


Figure 5.2. The construction of the bisector $\mathrm{bsc}(x,y)$ (bold line) as described in Remark 5.12.

Remark 5.13 (Construction of the bisector II)**.** In the situation of Remark 5.12, let H^+ be one of the closed half-planes bounded by $\langle x,y\rangle$. We will now examine the qualitative difference of the shape caused by the presence or absence of a non-singleton line segment in $S(x,1)\cap H^+$ which is contained in a translate of $\langle x,y\rangle$. By convexity, such a line segment is contained in the exposed face $F_x=S(x,1)\cap L$, where L is the unique translate of $\langle x,y\rangle$ which is contained in H^+ and which is a supporting line of $B(x,1)$. In this case, the straight line L is also a supporting line of $B(y,1)$ and $F_y=S(y,1)\cap L=y-x+F_x$. First, assume that $F_x=\{a_x\}$ and $F_y=\{a_y\}$ are singletons. Then the rays $[x,a_x\rangle$ and $[y,a_y\rangle$ are disjoint, see Figure 5.3.

In the situation of Remark 5.13, the set $\mathrm{bsc}(x,y)\cap H^+$ is homeomorphic to a half-open interval, see [144, Lemma 2.1.1.1].

Theorem 5.14. *In the situation of Remark 5.13, the set* $(\mathrm{bsc}(x,y)\cap H^+)\setminus\{m_{x,y}\}$ *is contained in the interior of* $\mathrm{co}([x,a_x\rangle\cup[y,a_y\rangle)$*. Moreover, there is a homeomorphism* $f:[0,1)\to\mathrm{bsc}(x,y)\cap H^+$.

The proof can be done analogously to the one of [144, Lemma 2.1.1.1]. It is based on the observation that the points $v_x\in S(x,1)$, $v_y\in S(y,1)$ appearing in the construction of $\mathrm{bsc}(x,y)\cap H^+$ given in Remark 5.12 have to lie in $\mathrm{co}([x,a_x\rangle\cup[y,a_y\rangle)\setminus([x,a_x\rangle\cup[y,a_y\rangle)$, see [144, Corollary 2.1.1.4].

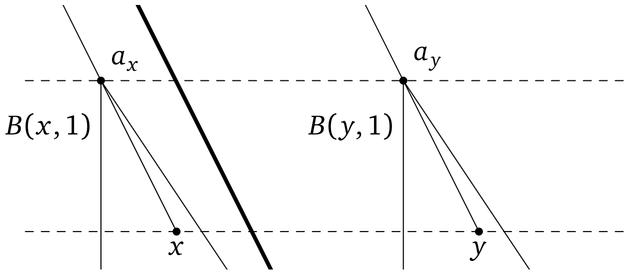


Figure 5.3. The bisector $\mathrm{bsc}(x,y)$ (bold line) as described in Remark 5.13 and Theorem 5.14.

Under the assumptions of Remark 5.13, that is, F_x and F_y are singletons, we can employ the elementary result Lemma 5.7 to show that $\mathrm{bsc}(x,y)\cap H^+$ is contained in an intersection of cones which yields an alternative proof of its directional convexity. The proof is patterned after [154, Proposition 17], which is the special case of Minkowski spaces.

Proposition 5.15. *In the situation of Remark 5.13, the set* $\mathrm{bsc}(x,y)\cap H^+$ *is contained in the cone* $\{z+\lambda(x-z)+\mu(y-z)\mid\lambda\mu\geq 0\}$ *for each choice of* $z\in(\mathrm{bsc}(x,y)\cap H^+)\setminus\{m_{x,y}\}$*. Moreover, the set* $\mathrm{bsc}(x,y)\cap(v+\langle x,y\rangle)$ *is a singleton for all* $v\in H^+$.

Proof. Assume the converse statement. Then there is a point $w:=z+\lambda(x-z)+\mu(y-z)\in\mathrm{bsc}(x,y)\cap H^+$ with $\lambda\mu<0$, say $\lambda<0$ and $\mu>0$ like in Figure 5.4. From Lemma 5.7, it follows that $\gamma(w-x)+\gamma(z-y)\geq\gamma(w-y)+\gamma(z-x)$, in which we actually have equality since $z,w\in\mathrm{bsc}(x,y)$. By Lemma 5.7, it follows that $\left[\frac{w-x}{\gamma(w-x)},\frac{z-y}{\gamma(z-y)}\right]\subset S(0,1)$, which is impossible. Indeed, if $F_x=\{a_x\}$ and $F_y=\{a_y\}$ are the exposed faces of $B(x,1)$ and $B(y,1)$ determined by the common supporting line of $B(x,1)$ and $B(y,1)$ in H^+, then $\frac{w-x}{\gamma(w-x)}$ and $\frac{z-y}{\gamma(z-y)}$ are points of $S(0,1)\cap H^+$ which lie on different sides of the straight line through 0 and $a_x-x=a_y-y$. The claimed directional convexity can be shown by the same argument, since $w\in z-x+\langle x,y\rangle$ implies that $w=z+\lambda(x-z)+\mu(y-z)\in\mathrm{bsc}(x,y)\cap H^+$ with $\lambda\mu<0$. $\square$

Actually, Proposition 5.15 can be sharpened. The set $(\mathrm{bsc}(x,y)\cap H^+)\setminus\{z\}$ is contained in the cone $\{z+\lambda(x-z)+\mu(y-z)\mid\lambda\mu>0\}$ for each choice of $z\in(\mathrm{bsc}(x,y)\cap H^+)\setminus\{m_{x,y}\}$.

Remark 5.16 (Construction of the bisector III)**.** Now assume that the exposed faces F_x and F_y are non-singletons in Remark 5.12. Without loss of generality, we write $F_x=:[a_x,b_x]$, $F_y=:[b_y,a_y]$ where $y-x=\lambda(b_x-a_x)$ for some $\lambda>0$, $b_y:=a_y+y-x$, and $a_y:=b_x+y-x$. Then the rays $[x,b_x\rangle$ and $[y,b_y\rangle$ have a common point s, and the rays $[x,a_x\rangle$ and $[y,a_y\rangle$ are disjoint, see Figure 5.5.

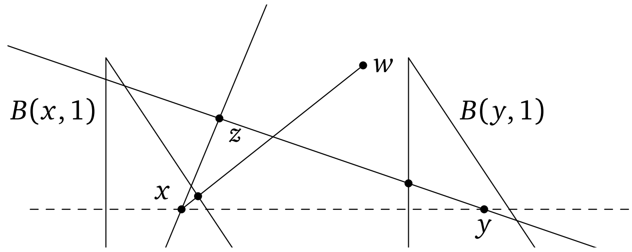


Figure 5.4. Illustration for Proposition 5.15.

In this situation, the set $\mathrm{bsc}(x,y) \cap H^+$ consists of two parts one of which is homeomorphic to the closed unit interval, while the other one is a cone. The proof of [144, Lemma 2.1.1.1] can be adapted accordingly.

Theorem 5.17. *In the situation of Remark 5.16, the set* $(\mathrm{bsc}(x,y) \cap H^+) \setminus \{m_{x,y}\}$ *is contained in the interior of* $\mathrm{co}([x,a_x\rangle \cup [y,a_y\rangle)$*. Moreover, we have* $\mathrm{bsc}(x,y) \cap H^+ = B_1 \cup B_2$ *where*

- *there is a homeomorphism* $f : [0,1] \to B_1$*,*
- $B_2 = \mathrm{co}([s,2s-x\rangle \cup [s,2s-y\rangle)$*,*
- $B_1 \cap B_2 = \{s\}$*.*

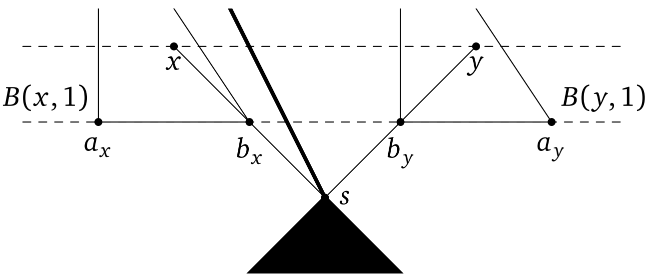


Figure 5.5. The bisector $\mathrm{bsc}(x,y)$ (bold line) as described in Remark 5.16 and Theorem 5.17.

The possible occurrence of conical subsets of bisectors as in Theorem 5.17 might not be desirable for applications. This is the reason why some authors impose an assumption of *general position* in the study of bisectors $\mathrm{bsc}(x,y)$, i.e., they assume that the unit sphere does not contain any line segments which are also contained in translates of $\langle x,y\rangle$. This situation is considered general as its opposite can be avoided by applying small perturbations to x and y or to the unit ball. (See [183, Section 3.7] for stability results on bisectors.) The two cases for "bisector halves" in generalized Minkowski planes (X,γ) described in Theorems 5.14 and 5.17 are independent in the sense that the may appear simultaneously for a fixed pair of points $x,y \in X$, see Figure 5.2. If γ is a norm, then both "halves" of the bisector $\mathrm{bsc}(x,y)$ look alike because of its central symmetry: $\mathrm{bsc}(x,y) - \frac{x+y}{2} = -\left(\mathrm{bsc}(x,y) - \frac{x+y}{2}\right)$, see also [232]. Local versions of the cone property and the elementary approach to the directional convexity can be proved similarly to Proposition 5.15.

Proposition 5.18. *In the situation of Theorem 5.17, the set* $\operatorname{bsc}(x,y)\cap H^+$ *is contained in the cone* $\{z+\lambda(x-z)+\mu(y-z) \mid \lambda,\mu\in\mathbb{R}, \lambda\mu\geq 0\}$ *for all choices* $z\in B_1\setminus\{m_{x,y}\}$*. In particular, we have* $B_1\subset \operatorname{co}(\{x,y,s\})$*. Moreover, the set* $\operatorname{bsc}(x,y)\cap(v+\langle x,y\rangle)$ *is a singleton for each* $v\in[s-x,0)$*.*

As a corollary, line segments in the bisector $\operatorname{bsc}(x,y)$ which are contained in a translate of $\langle x,y\rangle$ are at constant and coinciding distances from each point in $[x,y]$, see [157, Lemma 1] for the special case of Minkowski planes.

Corollary 5.19. *Let* (X,γ) *be a generalized Minkowski plane,* $x,y\in X$*, and* $x\neq y$*. Assume that the points* $w,z\in\operatorname{bsc}(x,y)$ *are contained in a translate of* $\langle x,y\rangle$*. For all* $u\in[w,z]$ *and all* $v\in[x,y]$*, we have* $\gamma(u-v)=\gamma(w-x)$*.*

Conversely, the occurrence of non-singleton line segments in $\operatorname{bsc}(x,y)$ which are at constant and coinciding distances from x and y means that $S(x,\lambda)\cap S(y,\lambda)$ contains a line segment for some $\lambda>0$. This line segment is therefore contained in a common supporting line H of $S(x,\lambda)$ and $S(y,\lambda)$. Depending on whether y is located in the same or in the opposite half-space bounded by H compared to x, we obtain the distinction stated in the following corollary.

Corollary 5.20. *Let* x *and* y *be distinct points of a generalized Minkowski plane* (X,γ)*. Let* $w,z\in\operatorname{bsc}(x,y)$ *be distinct points in the same open half-plane bounded by* $\langle x,y\rangle$*. Assume that* $\gamma(w-x)=\gamma(z-x)$*. Then there are the following mutually exclusive cases.*

(a) *We have* $\gamma(z-x)=R(\{x,y\},-B(0,1))$*.*
(b) *The straight lines* $\langle w,z\rangle$ *and* $\langle x,y\rangle$ *are parallel.*

Summarizing Theorems 5.14 and 5.17, bisectors in generalized Minkowski planes are unions of convex cones and homeomorphic images of intervals which can be glued together. This already yields the connectedness of bisectors, which we formulate immediately for generalized Minkowski spaces of arbitrary dimension employing the same argument. The connectedness of bisectors in Minkowski spaces has been established in [112, Lemma 1].

Theorem 5.21. *Let* (X,γ) *be a generalized Minkowski space and* $x\in X\setminus\{0\}$*. Then* $\operatorname{bsc}(-x,x)$ *is connected.*

Proof. Whenever $\{x,y\}$ is a linearly independent subset of X, the set $\operatorname{bsc}(-x,x)\cap\operatorname{hfl}(x,y)$ is connected. This is a consequence of Theorems 5.14 and 5.17 as well of the facts that homeomorphisms preserve connectedness and that the union of non-disjoint connected sets is connected, see [178, Theorem 23.3 and 23.5]. The point $m_{-x,x}$ is common to all sets $\operatorname{bsc}(-x,x)\cap\operatorname{hfl}(x,y)$, so their union is connected again by [178, Theorem 23.3]. □

Note that we can say even more about the homeomorphism f appearing in Theorem 5.14. By [178, Theorem 18.2(d)], we know that for all $\alpha\in[0,1)$, the restriction $f|_{[0,1)\setminus\{\alpha\}}:[0,1)\setminus\{\alpha\}\to(\operatorname{bsc}(x,y)\cap H^+)\setminus\{f(\alpha)\}$ is a homeomorphism. Since homeomorphisms preserve (dis)connectedness [178, Theorem 23.5], we know that $f(0)$ is the only point of $\operatorname{bsc}(x,y)$ which can be removed without losing connectedness. Therefore, we have $f(0)=m_{x,y}$. Similarly, in the situation of Theorem 5.14, we may choose the homeomorphism $f:[0,1]\to B_1$ such that $f(0)=s$ and $f(1)=m_{x,y}$.

5.3 Characterizations of norms

An intriguing and, surprisingly, characteristic property of bisectors in Euclidean spaces is their hyperplanarity.

Proposition 5.22. *Let (X,γ) be a generalized Minkowski space of dimension* $\dim(X) \geq 2$. *The following statements are equivalent.*

(a) The gauge γ is a norm induced by an inner product.
(b) The bisector $\operatorname{bsc}(x,y)$ *is a hyperplane for all distinct points $x,y \in X$.*
(c) The bisector $\operatorname{bsc}(x,y)$ *is convex for all distinct points $x,y \in X$.*

Proof. The implications (a)⇒(b)⇒(c) are clear. For the equivalence (a)⇔(c) see [241]. □

Closely related, isosceles orthogonality is homogeneous or additive exactly in Euclidean spaces.

Theorem 5.23. *Let (X,γ) be a generalized Minkowski space. The following statements are equivalent.*

(a) The gauge γ is a norm induced by an inner product.
(b) Isosceles orthogonality is right-homogeneous.
(c) Isosceles orthogonality is right-additive.
(d) Isosceles orthogonality is left-homogeneous.
(e) Isosceles orthogonality is left-additive.

Proof. It is sufficient to show that (b), (c), (d), and (e) imply that γ is a norm. The claim then follows from [126, Theorems 4.7 and 4.8]. First, assume that (b) holds. If $y \perp_I x$, then $y \perp_I \lambda x$ for all $\lambda > 0$ or, equivalently, $\lambda^{-1} y \in \operatorname{bsc}(-x,x)$ for all $\lambda > 0$. Taking the limit $\lambda \to +\infty$, we obtain $0 \in \operatorname{bsc}(-x,x)$. (Note that the bisector is a closed set due to the continuity of γ.) Since x was chosen arbitrarily, γ is a norm. Second, assume that (d) holds. Given a point $x \in X \setminus \{0\}$, there exists a unique number $\alpha \in \mathbb{R}$ for which $\alpha x \perp_I x$. If γ is not a norm, then x can be chosen such that $\alpha \neq 0$. By left homogeneity, we have $\lambda\alpha x \perp_I x$ for all $\lambda > 0$. But this is impossible for $|\lambda\alpha| > 1$. Finally, also (c) and (e) imply that γ is a norm. For this, we may employ the above arguments, but with $\lambda \in \mathbb{N}$ instead of $\lambda > 0$. □

After introducing several substitutes for orthogonality, it is natural to ask how these substitutes relate to each other. In Minkowski spaces, the plethora of orthogonality relations paves the way for a multitude of characterizations of Minkowski spaces in which one orthogonality type implies the other. Frequently, these are characterizations of inner-product spaces, which sometimes require some extra assumptions, see [6]. This also applies to Birkhoff orthogonality and isosceles orthogonality in Minkowski spaces: If one implies the other, then the space is an inner-product space, see also [185, Theorems 1 and 2] and [10, (10.2) and (10.9)]. The same is true for generalized Minkowski spaces.

Theorem 5.24. *Let (X,γ) be a generalized Minkowski space.*

(a) If Birkhoff orthogonality implies isosceles orthogonality, then γ is a norm.
(b) If isosceles orthogonality implies Birkhoff orthogonality, then γ is a norm.

Proof. For (a), note that we have $0 \perp_B y$ for all $y \in X$, thus $\gamma(y) = \gamma(-y)$ for all $y \in X$. For (b), assume that γ is not a norm. Then there exists a point $y \in X$ such that $\gamma(y) \neq \gamma(-y)$. Furthermore, there is a unique point $x \in \langle -y, y \rangle$ such that $x \perp_I y$, namely $x = m_{-y,y} = \frac{\gamma(-y)-\gamma(y)}{\gamma(-y)+\gamma(y)} y \neq 0$. Due to the hypothesis, we have $\frac{\gamma(-y)-\gamma(y)}{\gamma(-y)+\gamma(y)} y \perp_B y$, which is impossible. □

Note that Theorem 5.24 is essentially two-dimensional due to our knowledge of the shapes of the sets $\{x \in X \mid x \perp_B y\} \cap L$ and $\{x \in X \mid x \perp_I y\} \cap L = \operatorname{bsc}(-y, y) \cap L$ where $L \in \mathscr{L}_2^X$. The former is a union of straight lines passing through 0, the latter consists of at most two cones whose apices (which cannot be 0) are connected by a single curve (which need not pass through 0). If $\{x \in X \mid x \perp_B y\}$ contains $\operatorname{bsc}(-y, y)$, or vice versa, the sets $\{x \in X \mid x \perp_B y\} \cap L$ and $\operatorname{bsc}(-y, y) \cap L$ have the same inclusion property for arbitrary $L \in \mathscr{L}_2^X$. But then, $\{x \in X \mid x \perp_B y\} \cap L$ and $\operatorname{bsc}(-y, y) \cap L$ must coincide with a single straight line. If this is the case independently of x and y, Proposition 5.22 yields that the restriction of γ to L is a norm induced by an inner product. Since $L \in \mathscr{L}_2^X$ was chosen arbitrarily, the gauge γ is a norm induced by an inner product, see [10, (1.4′)].

5.4 Voronoi diagrams, hyperboloids, and apollonoids

Clearly, bisectors can be expressed as the locus of points whose difference of distances measured from two fixed points equals zero. Equivalently, it is the locus of points whose ratio of distances measured from two fixed points equals one. By changing the differences and ratios, we obtain families of sets which in Euclidean space are known as hyperbolas and Apollonian circles, respectively.

Definition 5.25. Let (X, γ) be a generalized Minkowski space, $x, y \in X$, $x \neq y$, and $\alpha \in \mathbb{R}$. We refer to the sets

$$\begin{aligned} H_{x,y,=\alpha} &:= \{z \in X \mid \gamma(z-x) = \alpha + \gamma(z-y)\}, \\ H_{x,y,\leq\alpha} &:= \{z \in X \mid \gamma(z-x) \leq \alpha + \gamma(z-y)\} \end{aligned}$$

as *hyperboloids*. Sets of the form

$$\begin{aligned} A_{x,y,=\alpha} &:= \{z \in X \mid \gamma(z-x) = \alpha\gamma(z-y)\}, \\ A_{x,y,\leq\alpha} &:= \{z \in X \mid \gamma(z-x) \leq \alpha\gamma(z-y)\} \end{aligned}$$

shall be called *apollonoids*. In both cases, the points x and y shall be called the *foci* of those sets. The term *hyperbola* is set aside exclusively for hyperboloids in generalized Minkowski planes.

The term *apollonoid* is supposed to resemble *Apollonius* and *hyperboloid*, in order to leave open the question if $A_{x,y,=\alpha}$ is a sphere for arbitrary gauges. (It turns out not to be the case, see Theorem 5.29 below.) When we refer to *apollonoids*, we avoid the ambiguity between $A_{x,y,=\alpha}$ and $A_{x,y,\leq\alpha}$ by clearly pointing out which set we study. (The same applies to hyperboloids.) In Definition 5.25, it is enough to consider hyperboloids for $\alpha < 0$ and apollonoids for $\alpha \in (0, 1]$ because

$$H_{x,y,=\alpha} = H_{y,x,=-\alpha} \quad \text{and} \quad A_{x,y,=\alpha} = A_{y,x,=\frac{1}{\alpha}}.$$

Moreover, it is sufficient to study hyperboloids and apollonoids whose foci are negatives of each other because

$$H_{x,y,=\alpha} = \frac{x+y}{2} + H_{\frac{x-y}{2},\frac{y-x}{2},=\alpha} \quad \text{and} \quad A_{x,y,=\alpha} = \frac{x+y}{2} + A_{\frac{x-y}{2},\frac{y-x}{2},=\alpha}.$$

As already mentioned above, we have $H_{x,y,=0} = A_{x,y,=1} = \operatorname{bsc}(x,y)$. In Euclidean plane geometry, both hyperbolas and Apollonian circles are closely linked to Euclid's contemporary Apollonius of Perga. Apollonius contributed to the study of conics in which he coined the terms ellipse, parabola, and hyperbola for plane curves, see [143, pp. 8 and 10]. Hyperbolas have a multitude of appearances in science, e.g., some lampshades cast hyperbolic shadows, and certain escape trajectories in astrodynamics are hyperbolic in shape. In 20th century mathematics, the concept of taking differences of distances to fixed objects has been taken up, for instance, by Zamfirescu who investigates in [244] sets which are defined by taking sums and differences of Euclidean distances to two fixed convex sets in $\mathbb{R}^d$, the distances themselves being defined as in Definition 1.14. The authors of [95, 165] propose the study of (positively or negatively) weighted distances to a finite number of points in Minkowski spaces. Purely geometric approaches to hyperbolas in Minkowski planes are presented in [77, 114]. Hyperboloids in the sense of Definition 5.25 also appear implicitly in Definition 2.12.

The fact that points which have a constant ratio of distances to two fixed points form either a straight line or a circle was already known to Aristotle, see [101, p. 340] and [238, p. 376]. Nonetheless, these circles are commonly named after Apollonius, see [58, Section 6.6] as well as [190, Sections 18.3]. Apollonian circles play a role in the reflection of light, resulting in circular shapes of halos and rainbows, see again [238, p. 376]. Furthermore, they occur in the study of vibrations of circular drumheads [180, 181]. The mathematical relevance of Apollonian circles lies in their property of forming a special non-intersecting family of circles, cf. [58, Section 6.5] and [190, Section 29]. Such families of Euclidean circles are important in Möbius geometry and in Poincaré's models of the hyperbolic plane, see [58, Section 16.7] and [190, Sections 56 and 57]. Zalgaller and Merkulova [242] study the loci of constant ratio of distances to two fixed points in hyperbolic and spherical geometry. Instead of changing the ambient space and therefore the measurement of distances, one might also obtain generalizations of Apollonian circles by altering the number or nature of the foci. In [223], Apollonian curves are introduced as the locus of points having a given ratio of products of distances to two finite sets of foci. In contrast to that, Makuchowski [146] investigates the loci of points in the Euclidean plane which have a constant ratio of distances to two Euclidean circles. Special cases of this construction include usual ellipses and hyperbolas. In addition, each parabola has a point and a straight line from which its points have equal distances, see [196]. In [228, Figures 4.11a and 4.11b], Thompson shows Apollonian "circles" for two non-Euclidean Minkowski planes and notices that the closed curves among them need not be the boundaries of convex sets and that, apparently, their orthogonal trajectories are not circles as well. In the context of location theory, Nickel and Puerto combine in [183, Section 3.2] additive and multiplicative weights to obtain extensions of bisectors in generalized Minkowski spaces. Given a finite but non-singleton set $P \subset X$, the *Voronoi cell*

$$V_{x,P} := \{z \in X \mid \gamma(z-x) \leq \gamma(z-y) \ \forall\, y \in P \setminus \{x\}\} = \bigcap_{y \in P\setminus\{x\}} H_{x,y,\leq 0} = \bigcap_{y \in P\setminus\{x\}} A_{x,y,\leq 1}$$

of $x \in P$ encodes the proximity to x compared to the other elements of P. The sets $H_{x,y,\leq 0} = A_{x,y,\leq 1}$ are called *Leibnizian half-spaces* in Minkowski geometry, see [112]. Voronoi cells play an important role in computational geometry, see [213, Section IV]. In particular, for any finite but non-singleton subset P of a generalized Minkowski space (X,γ) with unit ball B, there is a circumcenter of P with respect to $-B$ which lies on the boundary of at least two Voronoi cells, see again Lemma 4.8. In Euclidean space, Voronoi cells are polygons (as Leibnizian half-spaces are usual half-spaces), and have been rediscovered, e.g., in meteorology as Thiessen polygons and in solid-state physics as Wigner–Seitz cells. Two parametrizations of the Voronoi cell of $x \in P$ are given by

$$V_{x,P}^{\alpha,\star} := \{z \in X \mid \gamma(z-x) \leq \alpha \star \gamma(z-y)\ \forall\, y \in P \setminus \{x\}\} \tag{5.6}$$

In (5.6), the symbol $\star$ denotes either addition or multiplication, and α is a real number. If $\star$ denotes addition, we have $V_{x,P}^{\alpha,\star} = \bigcap_{y \in P\setminus\{x\}} H_{x,y,\leq\alpha}$. Else, we have $V_{x,P}^{\alpha,\star} = \bigcap_{y \in P\setminus\{x\}} A_{x,y,\leq\alpha}$. Generalizing a lemma from [50, p. 237], we show now that those parametrized Voronoi cells are star-shaped sets. Recall that a set $K \subset X$ is said to be *star-shaped with respect to $x \in X$* if $[x,y] \in K$ for all $y \in K$.

Proposition 5.26. *Let (X,γ) be a generalized Minkowski space, $P \subset X$, $2 \leq \operatorname{card}(P) < +\infty$, and $x \in P$. Let us denote by $\star$ either addition or multiplication. If $\star$ denotes addition, let $\alpha \in \mathbb{R}$. Otherwise let $\alpha \in [0,1]$. Then the set $V_{x,P}^{\alpha,\star}$ is star-shaped with respect to x.*

Proof. Let $z \in V_{x,P}^{\alpha,\star}$ and $\mu \in [0,1]$. We have

$$\begin{aligned}\gamma(z-x) &\leq \alpha \star \gamma(z-y)\\ &= \alpha \star \gamma(\mu z + (1-\mu)x - y + (1-\mu)(z-x))\\ &\leq \alpha \star \gamma(\mu z + (1-\mu)x - y) + (1-\mu)\gamma(z-x),\end{aligned}$$

and thus

$$\begin{aligned}\alpha \star \gamma(\mu z + (1-\mu)x - y) &\geq \gamma(z-x) - (1-\mu)\gamma(z-x)\\ &= \gamma(\mu(z-x))\\ &= \gamma(\mu z + (1-\mu)x - x)\end{aligned}$$

for all $y \in P \setminus \{x\}$. Therefore $\mu z + (1-\mu)x \in V_{x,P}^{\alpha,\star}$. □

As a corollary, we obtain the starshapedness of Voronoi cells, hyperboloids, and apollonoids.

Corollary 5.27. *Let (X,γ) be a generalized Minkowski space, $P \subset X$, and $2 \leq \operatorname{card}(P) < +\infty$. Then the Voronoi cell $V_{x,P}$ of $x \in P$ is star-shaped with respect to x. In addition, for any two points $x, y \in X$ and any two numbers $\alpha \in \mathbb{R}$, $\beta \in [0,1]$, the sets $H_{x,y,\leq\alpha}$ and $A_{x,y,\leq\beta}$ are star-shaped with respect to x.*

In contrast to starshapedness, the convexity of hyperboloids and apollonoids turns out to be a characteristic property of Euclidean space. We split these characterizations into two theorems which have to be seen in the context of Proposition 5.22. Their proofs are postponed to the end of this section.

Theorem 5.28. *Let (X,γ) be a generalized Minkowski space of dimension* $\dim(X) \geq 2$. *The following statements are equivalent.*

(a) The gauge γ is a norm induced by an inner product.
(b) For any two distinct points $x,y \in X$ and any $\alpha \in (-\gamma(x-y),0)$, the set $H_{x,y,\leq\alpha}$ is convex.
(c) For any two distinct points $x,y \in X$ and any $\alpha \in (-\infty,0]$, the set $H_{x,y,\leq\alpha}$ is convex.
(d) For any two distinct points $x,y \in X$, the set $H_{x,y,\leq 0}$ is convex.

Theorem 5.29. *Let (X,γ) be a generalized Minkowski space of dimension* $\dim(X) \geq 2$. *The following statements are equivalent.*

(a) The gauge γ is a norm induced by an inner product.
(b) For any two distinct points $x,y \in X$ and any $\alpha \in (0,1)$, the set $A_{x,y,=\alpha}$ is a sphere in (X,γ).
(c) For any two distinct points $x,y \in X$ and any $\alpha \in (0,1)$, the set $A_{x,y,\leq\alpha}$ is convex.

Note that special cases of Theorem 5.29 can be found in [59,223]. One-dimensional generalized Minkowski spaces have to be excluded from Theorems 5.28 and 5.29 because there, the sets $H_{x,y,\leq\alpha}$ and $A_{x,y,\leq\alpha}$ are convex for all gauges. Let us start our investigation of hyperboloids and apollonoids with the fact that in Definition 5.25, not all numbers $\alpha \in \mathbb{R}$ yield non-empty sets $H_{x,y,=\alpha}$.

Lemma 5.30. *Let (X,γ) be a generalized Minkowski space, $x,y \in X$, and $x \neq y$. Then the function $f := \gamma(\cdot-x)-\gamma(\cdot-y) : X \to \mathbb{R}$ is bounded from above by $\gamma(y-x)$ and from below by $-\gamma(x-y)$. These extremal values are attained, e.g., at y and x, respectively. In particular, $H_{x,y,=\alpha} = \emptyset$ if and only if $\alpha \notin [-\gamma(x-y),\gamma(y-x)]$.*

Proof. The estimates are obtained from the triangle inequalities $\gamma(z-x) \leq \gamma(z-y)+\gamma(y-x)$ and $\gamma(z-y) \leq \gamma(z-x)+\gamma(x-y)$. □

Hyperboloids $H_{x,y,=\alpha}$ for extremal parameters α are cones.

Proposition 5.31. *The set of maximizers of the function f from Lemma 5.30 is the cone*

$$\operatorname*{arg\,max}_{z\in X} f(z) = \{y\} \cup \left\{ z \in X \setminus \{y\} \;\middle|\; \left[\frac{z-y}{\gamma(z-y)}, \frac{y-x}{\gamma(y-x)}\right] \subset S(0,1) \right\}.$$

In particular, if γ is rotund, then $\operatorname{arg\,max}_{z\in X} f(z)$ is a ray.

Proof. Let $z \in \operatorname{arg\,max}_{z\in X} f(z) \setminus \{y\}$. Then $f(z) = \gamma(y-x)$, which is equivalent to $\gamma(z-x) = \gamma(z-y)+\gamma(y-x)$. Due to Lemma 2.22, this is equivalent to $\left[\frac{z-y}{\gamma(z-y)}, \frac{y-x}{\gamma(y-x)}\right] \subset S(0,1)$. □

In particular, hyperboloids are unbounded. In contrast to that, apollonoids are bounded except for bisectors.

Lemma 5.32. *Let (X,γ) be a generalized Minkowski space, $x,y \in X$, $x \neq y$, and $\alpha \in (0,1)$. Then $A_{x,y,\leq\alpha}$ is bounded. More precisely, we have*

$$A_{x,y,\leq\alpha} \subset B\left(x, \frac{\alpha}{1-\alpha}\gamma(x-y)\right).$$

Proof. Every point $z \in A_{x,y,\leq\alpha}$ satisfies $\gamma(z-x) \leq \alpha\gamma(z-y) \leq \alpha(\gamma(z-x)+\gamma(x-y))$. Thus, we have $\gamma(z-x) \leq \frac{\alpha}{1-\alpha}\gamma(x-y)$ and, in turn,

$$A_{x,y,\leq\alpha} \subset B\left(x, \frac{\alpha}{1-\alpha}\gamma(x-y)\right).$$

This completes the proof. □

The connectedness of hyperboloids can be shown similarly to the connectedness of bisectors, see Theorem 5.21.

Proposition 5.33. *Let (X,γ) be a generalized Minkowski space. For all points $x \in X \setminus \{0\}$ and numbers $\alpha \in [-2\gamma(-x), 2\gamma(x)]$, the set $H_{-x,x,=\alpha}$ is connected.*

In its proof, we use the following lemma, communicated to the author by Gerd Wachsmuth and Constantin Christof.

Lemma 5.34. *Let $\dim(X) = 2$ and let $g : X \to \mathbb{R}$ be a continuous function. Assume that there exists a point $v \in X \setminus \{0\}$ such that for all $y \in X$, the function $f : \mathbb{R} \to \mathbb{R}$, $f(\lambda) := g(y+\lambda v)$ is increasing and the set $\{\lambda \in \mathbb{R} \mid f(\lambda) = 0\}$ is non-empty and bounded. Then $\{x \in X \mid g(x) = 0\}$ is a connected set.*

Proof of Proposition 5.33. If $\alpha \in \{-2\gamma(-x), 2\gamma(x)\}$, the claim is a direct consequence of Proposition 5.31. Else, consider the functions $g : X \to \mathbb{R}$, $g(y) := \gamma(y+x) - \gamma(y-x) - \alpha$, and $f : \mathbb{R} \to \mathbb{R}$, $f(\lambda) := g(y+\lambda x)$. From the proof of Lemma 5.8, it follows that the function f is increasing and that $f_{=0}$ is non-empty and bounded because $-2\gamma(-x) < \alpha < 2\gamma(x)$, see again Lemma 5.30. For each linear subspace $L \in \mathscr{L}_2^X$ with $x \in L$, the set $L \cap H_{-x,x,=\alpha}$ is connected by Lemma 5.34. Using a gluing argument like in Theorem 5.21, we conclude that

$$H_{-x,x,=\alpha} = \bigcup_{L \in \mathscr{L}_2^X : x \in L} \left(H_{-x,x,=\alpha} \cap L\right)$$

is indeed connected. □

As a generalization of Proposition 5.9, we are able to identify subsets of hyperboloids which are intersections of cones.

Lemma 5.35. *Let (X,γ) be a generalized Minkowski space, $x \in X \setminus \{0\}$, $\phi \in X^*$, and $\gamma^\circ(\phi) = 1$. Then we have $C_\gamma(-x,\phi) \cap C_\gamma(x,\phi) \subset H_{-x,x,=2\langle\phi \,|\, x\rangle}$.*

Proof. Consider

$$\begin{aligned}
& z \in C_\gamma(-x,\phi) \cap C_\gamma(x,\phi) \\
& \iff \langle\phi \,|\, z+x\rangle = \gamma(z+x) \wedge \langle\phi \,|\, z-x\rangle = \gamma(z-x) \\
& \implies \gamma(z+x) = \langle\phi \,|\, z+x\rangle = \langle\phi \,|\, z-x\rangle + 2\,\langle\phi \,|\, x\rangle = \gamma(z-x) + 2\,\langle\phi \,|\, x\rangle \\
& \implies z \in H_{-x,x,=2\langle\phi \,|\, x\rangle}.
\end{aligned}$$

This yields the assertion. □

Similarly to Proposition 5.9, Lemma 5.35 is trivial if $C_\gamma(-x,\phi)\cap C_\gamma(x,\phi)\neq\emptyset$, but yields the existence of hyperbolas with non-empty interiors in other situations.
As a corollary of the proof of Lemma 5.8, the function $\gamma(\cdot-x)-\gamma(\cdot-y):X\to\mathbb{R}$ has constant sign on either side of the bisector.

Corollary 5.36. *Let (X,γ) be a generalized Minkowski space, $x,y\in X$, $x\neq y$, $z\in\operatorname{bsc}(x,y)$, and $\lambda>0$. Then, we have $\gamma((z+\lambda(x-y))-x)<\gamma((z+\lambda(x-y))-y)$.*

Rotundity has a strong impact on the shape of bisectors. In particular, in a rotund generalized Minkowski space (X,γ), bisectors $\operatorname{bsc}(x,y)$ are homeomorphic to $\mathbb{R}$ and each translate of $\langle x,y\rangle$ contains exactly one point of $\operatorname{bsc}(x,y)$, see again Lemma 5.8 and Theorem 5.14. Also, the intersection of the cones $C_\gamma(-x,\phi)$ and $C_\gamma(x,\phi)$ appearing in Lemma 5.35 is empty or a cone if $\dim(X)=2$. These facts can be used to give a characterization of rotundity in terms of hyperbolas and bisectors.

Theorem 5.37. *Let (X,γ) be a generalized Minkowski plane. The following statements are equivalent.*

(a) The gauge γ is not rotund.
(b) For all distinct points $x,y\in X$, there is a number $\alpha\in[-\gamma(x-y),\gamma(y-x)]$ and a straight line L parallel but not identical to $\langle x,y\rangle$ such that the set $L\cap H_{x,y,=\alpha}$ contains at least two points.
(c) For all distinct points $x,y\in X$, there is a number $\alpha\in[-\gamma(x-y),\gamma(y-x)]$ such that $H_{x,y,=\alpha}$ has non-empty interior.
(d) There is a convex cone $C\subset X$ with non-empty interior such that for all distinct points $x,y\in X$, there exists a number $\alpha\in[-\gamma(x-y),\gamma(y-x)]$ such that $H_{x,y,=\alpha}$ contains a translate of C.
(e) There are distinct points $x,y\in X$, a number $\alpha\in[-\gamma(x-y),\gamma(y-x)]$ and a straight line L parallel but not identical to $\langle x,y\rangle$ such that the set $L\cap H_{x,y,=\alpha}$ contains at least two points.
(f) There are distinct points $x,y\in X$ and a number $\alpha\in[-\gamma(x-y),\gamma(y-x)]$ such that $H_{x,y,=\alpha}$ has non-empty interior.
(g) There are distinct points $x,y\in X$ and a number $\alpha\in(-\gamma(x-y),\gamma(y-x))\setminus\{0\}$ such that $H_{x,y,=\alpha}$ has non-empty interior.
(h) There are distinct points $x,y\in X$ such that $H_{x,y,=0}=\operatorname{bsc}(x,y)$ has non-empty interior.

Proof. Lemma 5.35 shows the implications from (a) to (d), (g), and (h). Indeed, (a)⇒(d) is obvious. For (a)⇒(g), apply Lemma 5.35 to a non-singleton line segment $F:=S(0,1)\cap\phi_{=1}\subset S(0,1)$ with $\phi\in X^*$, $\gamma^\circ(\phi)=1$, and two points $x,y\in X$ such that F is not contained in a translate of $\langle x,y\rangle$ and $\frac{x-y}{\gamma(x-y)},\frac{y-x}{\gamma(y-x)}\notin F$. Thus, we have $\langle\phi\,|\,y-x\rangle\notin\{0,-\gamma(x-y),\gamma(y-x)\}$. Similarly, for (a)⇒(h), choose points $x,y\in X$ such that $F:=S(0,1)\cap\phi_{=1}$ is contained in a translate of $\langle x,y\rangle$. The implications (d)⇒(c)⇒(b)⇒(e), (g)⇒(f)⇒(e), and (h)⇒(e) are obvious. (Note that (a)⇔(h) is already stated in [144, Corollary 2.1.1.2].) It remains to show that (e)⇒(a). By (e), there exist distinct points $x,y,w,z\in X$ that are, in this cyclic order, the vertices of a convex quadrilateral with parallel edges $[x,y]$ and $[z,w]$ such that $z,w\in H_{x,y,=\alpha}$ for some real number α. The last inclusion yields $\gamma(z-x)+\gamma(w-y)=\gamma(z-y)+\gamma(w-x)$. Now Lemma 5.7 implies $\left[\frac{z-y}{\gamma(z-y)},\frac{w-x}{\gamma(w-x)}\right]\subset S(0,1)$. Thus γ is not rotund. □

Problem 5.38. *For generalized Minkowski planes (X,γ), nonrotundity of γ implies that there is a point $v \in X \setminus \{0\}$ such that for all $x, y \in X$, there exists a number $\alpha \in [-\gamma(x-y), \gamma(y-x)]$ for which $H_{x,y,=\alpha}$ contains a non-singleton line segment in direction v. This is a consequence of Theorem 5.37. Is the reverse implication also true?*

Apollonoids apart from bisectors are bounded and thus cannot contain any cones. However, they may contain non-singleton line segments.

Proposition 5.39. *Let (X,γ) be generalized Minkowski space, $x, y \in X$, and $x \neq y$. If the gauge γ is not rotund, then there is a number $\alpha > 0$ such that $A_{x,y,=\alpha}$ contains a non-singleton line segment.*

Proof. If γ is not rotund, there exits a linear functional $\phi \in X^*$ with $\gamma^\circ(\phi) = 1$ such that $S(0,1) \cap \phi_{=1}$ is not a singleton. The restriction of the functions $\gamma(\cdot - x), \gamma(\cdot - y) : X \to \mathbb{R}$ to sets of the form $C_\gamma(x,\phi) \cap C_\gamma(y,\phi) \cap \phi_{=\alpha}$, where $\alpha \in \mathbb{R}$, is constant. Thus, the same applies to the function $\frac{\gamma(\cdot - x)}{\gamma(\cdot - y)} : X \setminus \{y\} \to \mathbb{R}$. $\square$

We prepare the proofs of Theorems 5.28 and 5.29 by two similar lemmas on non-rotund generalized Minkowski planes.

Lemma 5.40. *Let (X,γ) be a non-rotund generalized Minkowski plane. Then there exist distinct points $x, y \in X$ and a number $\alpha \in (-\gamma(x-y), 0)$ such that $H_{x,y,\leq\alpha}$ is not convex.*

Proof. Choose $x, y \in X$ and $\phi \in X^*$ with $\gamma^\circ(\phi) = 1$ such that $[u,v] := S(0,1) \cap \phi_{=1}$ is not contained in a translate of $\langle x, y\rangle$, the condition $\left\{\frac{x-y}{\gamma(x-y)}, \frac{y-x}{\gamma(y-x)}\right\} \cap [u,v] = \emptyset$ is satisfied, and $\langle x, x+v\rangle$ and $\langle y, y+u\rangle$ are the supporting lines of the cone $C_\gamma(x,\phi) \cap C_\gamma(y,\phi)$, see Figure 5.6. Like in Lemma 5.35, the function $\gamma(\cdot - x) - \gamma(\cdot - y) : X \to \mathbb{R}$ is constant on $C_\gamma(x,\phi) \cap C_\gamma(y,\phi)$. Denote this constant by α. By construction, we have $\alpha \notin \{-\gamma(x-y), 0, \gamma(y-x)\}$. If $\alpha > 0$, change the roles of x and y, and those of u and v accordingly. Assume that, contrary to our claim, the set $H_{x,y,\leq\alpha}$ is convex. According to the proof of Lemma 5.8, the function $\gamma(\cdot - x) - \gamma(\cdot - y) : X \to \mathbb{R}$ is non-decreasing in direction $y - x$ on each translate of $\langle x, y\rangle$. In particular, if $z \in \langle x, x+v\rangle \cap (C_\gamma(y,\phi) \setminus \langle y, y+u\rangle)$, then $\gamma(w-x) - \gamma(w-y) > \gamma(z-x) - \gamma(z-y)$ for all $w \in [z, z+(y-x)\rangle \setminus \{z\}$. This holds since $\left[\frac{z-y}{\gamma(z-y)}, \frac{w-x}{\gamma(w-x)}\right] \not\subset S(0,1)$ because $\frac{z-y}{\gamma(z-y)} \in (u,v)$ but $\frac{w-x}{\gamma(w-x)} \notin [u,v]$, cf. Lemma 5.7. It follows that $\langle x, x+v\rangle$ is a supporting line of $C_\gamma(x,\phi) \cap C_\gamma(y,\phi)$ as well as of $H_{x,y,\leq\alpha}$. But for $\lambda := \frac{\gamma(y-x)-\alpha}{\gamma(y-x)+\gamma(x-y)}$, we obtain $\lambda x + (1-\lambda)y \in (x,y) \cap H_{x,y,\leq\alpha}$, a contradiction. $\square$

Lemma 5.41. *Let (X,γ) be a non-rotund generalized Minkowski plane. Then there exist distinct points $x, y \in X$ and a number $\alpha \in (0,1)$ such that $A_{x,y\leq\alpha}$ is not convex.*

Proof. As γ is non-rotund, there exist points $a, b \in X$ and a linear functional $\phi \in X^*$ with $\gamma^\circ(\phi) = 1$ such that $[a,b] := S(0,1) \cap \phi_{=1}$. Let $H := \{z \in X \mid \langle -\phi \mid z\rangle = \gamma^\circ(-\phi)\}$ be the supporting line of $B(0,1)$ which is a proper translate of $\phi_{=1}$. We investigate two cases which are illustrated in Figure 5.7.
Case 1: $B(0,1) \cap H$ is a singleton $\{c\}$. Let $f := \frac{1}{2}\left(\cdot - \frac{a+b}{2}\right) + c : X \to X$ and consider $x := f(0)$ and $y := 0$. Since $S(y,1)$ meets the straight line $L := \langle f(a), f(b)\rangle$ only in $c := \frac{f(a)+f(b)}{2}$,

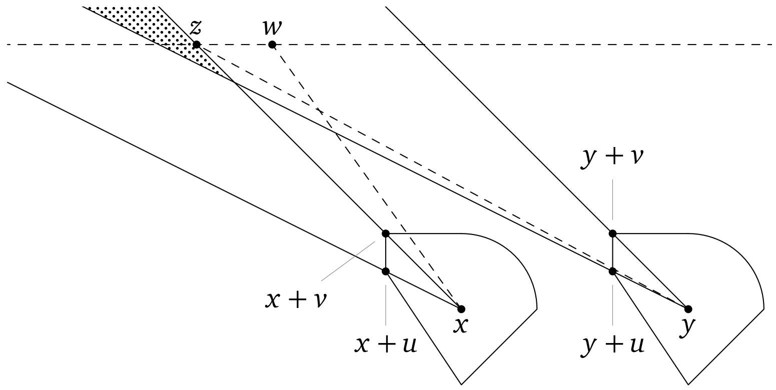


Figure 5.6. Proof of Lemma 5.40: Given the balls $B(x,1)$ and $B(y,1)$ and $\phi \in X^*$, we may construct $C_\gamma(x,\phi) \cap C_\gamma(x,\phi)$ (shaded region).

the sphere $S(y,1+\varepsilon)$ meets the line segment $[f(a),f(b)]$ in exactly two points u,v such that $c \in (u,v)$ if $\varepsilon > 0$ is chosen small enough. Now put $\alpha := \frac{1}{2(1+\varepsilon)}$. Then $u,v \in A_{x,y,\leq\alpha}$ because $u,v \in [f(a),f(b)] \cap S(y,1+\varepsilon) \subset S(x,\frac{1}{2}) \cap S(y,1+\varepsilon)$, which gives $\gamma(u-x) = \frac{1}{2} = \alpha(1+\varepsilon) = \alpha\gamma(u-y)$. In the same way, we obtain $\gamma(v-x) = \alpha\gamma(v-y)$ and, in turn, we have $u,v \in A_{x,y,\leq\alpha}$. However $c \notin A_{x,y,\leq\alpha}$, since $\gamma(c-x) = \frac{1}{2} = \alpha(1+\varepsilon) \nleq \alpha = \alpha\gamma(c-y)$. The inclusions $u,v \in A_{x,y,\leq\alpha}$, $c \notin A_{x,y,\leq\alpha}$, and $c \in (u,v)$ show that $A_{x,y,\leq\alpha}$ is not convex.
Case 2: $B(0,1) \cap H$ is a non-singleton line segment $[c_1,c_2]$. We may suppose that a,b,c_1,c_2 lie in this cyclic order on $S(0,1)$. Let $f : X \to X$ be a homothety with factor $\frac{1}{2}$ such that $c_1 \in (f(a),f(b))$ and $f(a) \in (c_1,c_2)$. We consider $x := f(0)$ and $y := 0$. If $\varepsilon > 0$ is chosen small enough, then $S(y,1+\varepsilon)$ meets the line segment $(c_1,f(b))$ in a point u. In particular, we have $c_1 \in (f(a),u)$, put $\alpha := \frac{1}{2}$, and note that $u \in \operatorname{int}(A_{x,y,\leq\alpha})$ because $\gamma(u-x) = \frac{1}{2} < \alpha(1+\varepsilon) = \alpha\gamma(u-y)$ and γ is continuous. On the other hand, we have $f(a), c_1 \in A_{x,y,\leq\alpha}$, since $\gamma(f(a)-x) = \gamma(c_1-x) = \frac{1}{2} = \alpha = \alpha\gamma(f(a)-y) = \alpha\gamma(c_1-y)$. In fact, we have $f(a), c_1 \in \operatorname{bd}(A_{x,y,\leq\alpha})$ because $f(a)$ and c_1 are limit points of $U(y,1)$, and every point $z \in U(y,1) \subset X \setminus B(x,\frac{1}{2})$ satisfies $\gamma(z-x) > \frac{1}{2} = \alpha > \alpha\gamma(z-y)$, i.e., $z \notin A_{x,y,\leq\alpha}$. The inclusions $u \in \operatorname{int}(A_{x,y,\leq\alpha})$, $\{f(a),c_1\} \subset \operatorname{bd}(A_{x,y,\leq\alpha})$, and $c_1 \in (f(a),u)$ show that $A_{x,y,\leq\alpha}$ is not convex. □

Now we are able to prove Theorems 5.28 and 5.29.

Proof of Theorem 5.28. The implication (a)⇒(c) is known from Euclidean geometry. Furthermore, the implications (c)⇒(b) and (c)⇒(d) are trivial. For (d)⇒(a), we refer to [241]. We prove the remaining implication ¬(a)⇒ ¬(b) by considering two cases.
Case 1: γ is rotund. By [241], there exist distinct points $x,y \in X$ such that $\operatorname{bsc}(x,y)$ is not convex. (See again Proposition 5.22.) Hence there exist points $a,b \in \operatorname{bsc}(x,y)$ and $c \in [a,b]$ such that $c \notin \operatorname{bsc}(x,y)$, say, $\gamma(c-x) > \gamma(c-y)$. (Otherwise we change the roles of x and y.) By continuity of γ, there exists a number $\lambda_0 > 0$ such that

$$\gamma((c+\lambda_0(x-y))-x) > \gamma((c+\lambda_0(x-y))-y). \tag{5.7}$$

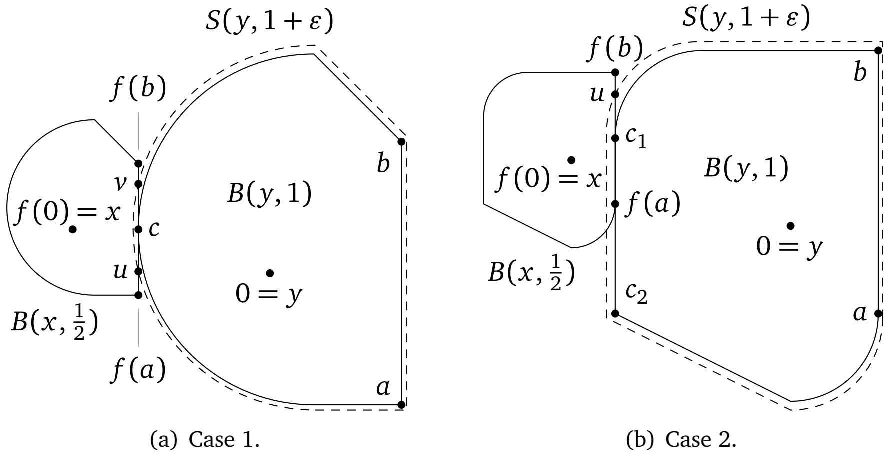


Figure 5.7. Proof of Lemma 5.41.

By Corollary 5.36, we have $\gamma((a+\lambda_0(x-y))-x) < \gamma((a+\lambda_0(x-y))-y)$ and $\gamma((b+\lambda_0(x-y))-x) < \gamma((b+\lambda_0(x-y))-y)$. It follows that

$$\alpha := \max\left\{\begin{matrix}\gamma((a+\lambda_0(x-y))-x)-\gamma((a+\lambda_0(x-y))-y),\\ \gamma((b+\lambda_0(x-y))-x)-\gamma((b+\lambda_0(x-y))-y)\end{matrix}\right\} < 0,$$

and both $a+\lambda_0(x-y)$ and $b+\lambda_0(x-y)$ belong to $H_{x,y,\leq\alpha}$. By (5.7), we have $c+\lambda_0(x-y) \notin H_{x,y,\leq\alpha}$. However, from $c \in [a,b]$ it follows that $c+\lambda_0(x-y) \in [a+\lambda_0(x-y), b+\lambda_0(x-y)]$. This implies the nonconvexity of $H_{x,y,\leq\alpha}$.

Case 2: γ is not rotund. Now $S(0,1)$ contains a non-singleton line segment $[u,v]$. Let $L := \mathrm{lin}(\{u,v\})$. The generalized Minkowski plane $(L, \gamma|_L)$ is not rotund, too, and Lemma 5.40 yields the existence of distinct points $x,y \in L$ and a number $\alpha \in (-\gamma|_L(x-y), 0)$ such that the set

$$H^L_{x,y,\leq\alpha} := \{z \in L \mid \gamma|_L(z-x) - \gamma|_L(z-y) \leq \alpha\}$$

is not convex. Then the corresponding set $H_{x,y,\leq\alpha}$ is not convex as well because $H_{x,y,\leq\alpha} \cap L = H^L_{x,y,\leq\alpha}$. □

Proof of Theorem 5.29. The classical (Euclidean) theorem on Apollonian circles shows the implication (a)⇒(b). Corollary 5.27 gives (b)⇒(c). We prove ¬(a)⇒ ¬(c) by considering two cases.

Case 1: γ is rotund. By [241], there exist distinct points $x,y \in X$ such that $\mathrm{bsc}(x,y)$ is not convex. (See again Proposition 5.22.) Hence there exist points $a,b \in \mathrm{bsc}(x,y)$ and $c \in [a,b]$ such that $c \notin \mathrm{bsc}(x,y)$, say, $\gamma(c-x) > \gamma(c-y)$. By continuity of γ, there exists a number $\lambda_0 > 0$ such that

$$\gamma((c+\lambda_0(x-y))-x) > \gamma((c+\lambda_0(x-y))-y). \tag{5.8}$$

By Corollary 5.36, we have $\gamma((a+\lambda_0(x-y))-x) < \gamma((a+\lambda_0(x-y))-y)$ and $\gamma((b+\lambda_0(x-y))-x) < \gamma((b+\lambda_0(x-y))-y)$. Thus,

$$\alpha := \max\left\{\frac{\gamma((a+\lambda_0(x-y))-x)}{\gamma((a+\lambda_0(x-y))-y)}, \frac{\gamma((b+\lambda_0(x-y))-x)}{\gamma((b+\lambda_0(x-y))-y)}\right\} < 1,$$

and both $a+\lambda_0(x-y)$ and $b+\lambda_0(x-y)$ belong to $A_{x,y,\leq\alpha}$. By (5.8), we have $c+\lambda_0(x-y) \notin A_{x,y,\leq\alpha}$. However, from $c \in [a,b]$ it follows that $c+\lambda_0(x-y) \in [a+\lambda_0(x-y), b+\lambda_0(x-y)]$. This implies the nonconvexity of $A_{x,y,\leq\alpha}$.

Case 2: γ is not rotund. Now $S(0,1)$ contains a non-singleton line segment $[u,v]$. Let $L := \mathrm{lin}(\{u,v\})$. The generalized Minkowski plane $(L, \gamma|_L)$ is not rotund, too, and Lemma 5.41 yields the existence of distinct points $x, y \in L$ and a number $\alpha \in (0,1)$ such that

$$A^L_{x,y,\leq\alpha} := \{z \in L \mid \gamma|_L(z-x) \leq \alpha\, \gamma|_L(z-y)\}$$

is not convex. Then the corresponding set $A_{x,y,\leq\alpha}$ is not convex as well because $A_{x,y,\leq\alpha} \cap L = A^L_{x,y,\leq\alpha}$. □

In the Euclidean plane, every pair of a hyperbola and an ellipse sharing their foci have orthogonal tangents at their common points, see, e.g., [109, pp. 5–6] and [244, Théorème 3]. Here we present a related result for generalized Minkowski planes (X,γ) with polygonal unit balls as an extension of [77, Theorem 9]. Namely, a hyperbola $H_{x,y,=\beta}$ and a *bifocal ellipse*

$$E_{x,y,=\alpha} := \{z \in X \mid \gamma(z-x)+\gamma(z-y) = \alpha\}$$

with common foci $x, y \in X$ intersect Birkhoff orthogonally under some mild circumstances, see Figure 5.8 for an illustration. Note that we may speak about Birkhoff orthogonality of a ray to a straight line as $x \perp_B y$ implies $\alpha x \perp_B \beta y$ for all numbers $\alpha > 0$ and $\beta \in \mathbb{R}$.

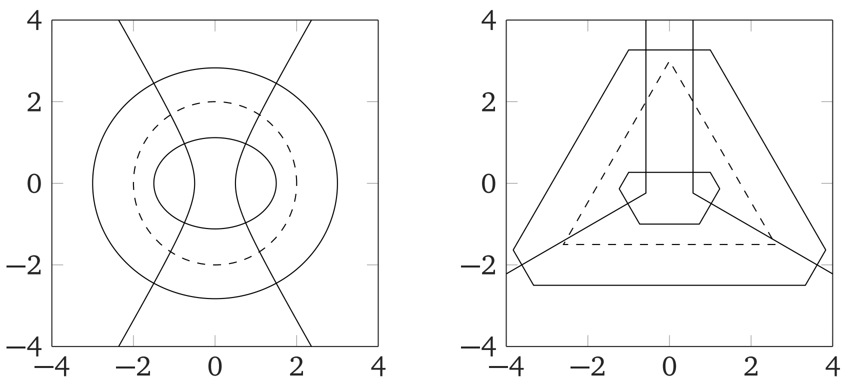


Figure 5.8. Illustration for Theorem 5.42: Confocal ellipses and hyperbolas intersect Birkhoff orthogonally: Euclidean norm (left), polygonal gauge (right). The corresponding unit balls are depicted in dashed lines.

Theorem 5.42. *Let (X,γ) be a generalized Minkowski plane such that $B(0,1)$ is a polygon. Fix distinct points $x,y\in X$ and a number $\beta\in[-\gamma(x-y),\gamma(y-x)]$. Assume that $H_{x,y,=\beta}$ does not contain any cones, and choose $\alpha>0$ large enough. Then every intersection point $x_0\in H_{x,y,=\beta}\cap E_{x,y,=\alpha}$ is the initial point of a ray $[x_0,x_0+z\rangle\subset H_{x,y,=\beta}$ which is Birkhoff orthogonal to any supporting line of $E_{x,y,=\alpha}$ at x_0.*

Proof. As before, find linear functionals $\phi_1,\dots,\phi_n\in X^*$, $\gamma^\circ(\phi_i)=1$ for $i\in\{1,\dots,n\}$ such that $B(0,1)=\bigcap_{i=1}^n(\phi_i)_{\leq 1}$ and $F_i:=(\phi_i)_{=1}\cap B(0,1)$ is a 1-face of $B(0,1)$. On each of the regions $G_{i,j}:=C(x,\phi_i)\cap C(y,\phi_j)$, the functions $\gamma(\cdot-x),\gamma(\cdot-y):X\to\mathbb{R}$ are affine functions, and so are their pointwise sum and difference. For

$$\alpha>\sup\{f(x)\mid \exists\, i,j\in\{1,\dots,n\}:x\in G_{i,j}\text{ and }G_{i,j}\text{ is bounded}\},$$

the ellipse $E_{x,y,=\alpha}$ is a subset of the union of the unbounded regions $G_{i,j}$. Let $x_0\in H_{x,y,=\beta}\cap E_{x,y,=\alpha}$ be fixed. Then x_0 is contained in some unbounded region G_{i_0,j_0}. The set G_{i_0,j_0} is unbounded only if $F_{i_0}\cap F_{j_0}\neq\emptyset$. If $F_{i_0}=F_{j_0}$, then the function $\gamma(\cdot-x)-\gamma(\cdot-y):X\to\mathbb{R}$ is constant on G_{i_0,j_0}. This constant value is clearly $\beta=\gamma(x_0-x)-\gamma(x_0-y)$ because $x_0\in G_{i_0,j_0}$. But then the cone G_{i_0,j_0} is contained in $H_{x,y,=\beta}$, which contradicts the assumptions. Thus $F_{i_0}\cap F_{j_0}=\{z\}$. Next we show that $[x_0,x_0+z\rangle\subset H_{x,y,=\beta}$, that is, the function $\gamma(\cdot-x)-\gamma(\cdot-y):X\to\mathbb{R}$ is constant on $[x_0,x_0+z\rangle$. Let $v\in[x_0,x_0+z\rangle$. Choose $w_1,w_2\in[x,x+z\rangle$, $z_1,z_2\in[y,y+z\rangle$ such that

$$\begin{aligned}
\gamma(x_0-x)&=\gamma(w_1-x), & \gamma(v-x)&=\gamma(w_2-x),\\
\gamma(x_0-y)&=\gamma(z_1-y), & \gamma(v-y)&=\gamma(z_2-y).
\end{aligned}$$

Since the restrictions of $\gamma(\cdot-x),\gamma(\cdot-y):X\to\mathbb{R}$ to $C_\gamma(x,\phi_{i_0})$ and $C_\gamma(y,\phi_{j_0})$, respectively, are affine functions, both $\{w_1,w_2,v,x_0\}\subset C_\gamma(x,\phi_{i_0})$ and $\{z_1,z_2,v,x_0\}\subset C_\gamma(y,\phi_{j_0})$ are the vertex sets of (possibly degenerate) parallelograms. Thus $v-x_0=w_2-w_1=z_2-z_1$ and

$$\begin{aligned}
\gamma(x_0-x)-\gamma(x_0-y)&=\gamma(w_1-x)-\gamma(z_1-y)\\
&=\gamma(w_1-x)+\gamma(w_2-w_1)-\gamma(z_1-y)-\gamma(z_2-z_1)\\
&=\gamma(w_2-x)-\gamma(z_2-y)\\
&=\gamma(v-x)-\gamma(v-y).
\end{aligned}$$

Therefore, we have $[x_0,x_0+z\rangle\subset H_{x,y,=\beta}$. Note that z is Birkhoff orthogonal to (the directions of) the line segments F_{i_0} and F_{j_0}, since $F_{i_0}\cap F_{j_0}=\{z\}$. The restriction of the functions $\gamma(\cdot-x),\gamma(\cdot-y):X\to\mathbb{R}$ to G_{i_0,j_0} are affine functions, thus Gâteaux differentiable. The same applies to the restriction of $\gamma(\cdot-x)+\gamma(\cdot-y):X\to\mathbb{R}$ and its Gâteaux derivative is sum of the Gâteaux derivatives of $\gamma(\cdot-x)$ and of $\gamma(\cdot-y)$. In other words, the slope of the intersection of the level sets of $\gamma(\cdot-x)+\gamma(\cdot-y)$ and G_{i_0,j_0} lies between the slopes of the line segments F_{i_0} and F_{j_0}, which are the slopes of the level sets of $\gamma(\cdot-x)+\gamma(\cdot-y)$ in the neighboring regions G_{i_0,i_0} and G_{j_0,j_0}. In particular, every straight line which supports $E_{x,y,=\alpha}$ at $x_0\in G_{i_0,j_0}$ has a slope between those of F_{i_0} and F_{j_0}. Since z is Birkhoff orthogonal to (the directions of) F_{i_0} and F_{j_0}, the ray $[x_0,x_0+z\rangle$ is Birkhoff orthogonal to every supporting line of $E_{x,y,=\alpha}$ at x_0. □

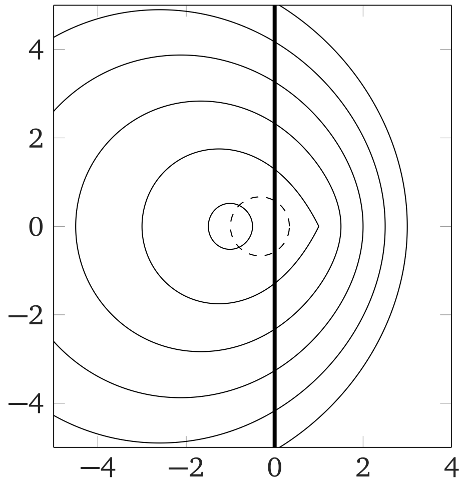


Figure 5.9. Illustration for Remark 5.43: Confocal ellipses (thin solid lines) and hyperbolas (bold solid line) need not intersect Birkhoff orthogonally for arbitrary gauges. The corresponding unit ball is depicted in dashed lines.

Remark 5.43. The assumption of $B(0,1)$ being a polygon cannot be dropped in Theorem 5.42. For instance, let $X = \mathbb{R}^2$, $\gamma : X \to \mathbb{R}$, $\gamma(\xi_1, \xi_2) := \sqrt{4\xi_1^2 + 3\xi_2^2} + \xi_1$, $x := (1,0)$, and $y := (-1,0)$ as in Figure 5.9. Then γ is smooth, i.e., it is Gâteaux differentiable at $(\xi_1, \xi_2) \neq (0,0)$ with Gâteaux derivative $\nabla f(\xi_1, \xi_2) = \left(\frac{4\xi_1}{\sqrt{4\xi_1^2+3\xi_2^2}} + 1, \frac{3\xi_2}{\sqrt{4\xi_1^2+3\xi_2^2}} \right)$. As a consequence, the functions $f := \gamma(\cdot - x) + \gamma(\cdot - y) : X \to \mathbb{R}$ and $g := \gamma(\cdot - x) - \gamma(\cdot - y) : X \to \mathbb{R}$ are Gâteaux differentiable on $X \setminus \{x, y\}$. Given a point $x_0 \in X \setminus \{x, y\}$, set $\alpha := f(x_0)$ and $\beta := g(x_0)$. In the usual manner, the Gâteaux derivatives $\nabla f(x_0)$ and $\nabla g(x_0)$ define tangent lines to $E_{x,y,=\alpha}$ and $H_{x,y,\beta}$, respectively. (In the case of $\nabla f(x_0)$ and $E_{x,y,=\alpha}$, this tangent line coincides with the supporting line of $E_{x,y,\leq\alpha} := \operatorname{co}\big(E_{x,y,=\alpha}\big)$ at x_0.) For a moment, we shall call a ray $[z, z+v\rangle$ a *half-tangent* to $H_{x,y,=\beta}$ at z, if it is contained in the tangent line to $H_{x,y,=\beta}$ at z. Unlike in Theorem 5.42, there is a hyperbola $H_{x,y,=\beta}$ such that no half-tangent at $x_0 \in H_{x,y,=\beta}$ is Birkhoff orthogonal to a supporting line of $E_{x,y,=\alpha}$ at x_0, where $\alpha = \alpha(x_0) = \gamma(x_0 - x) + \gamma(x_0 - y)$. We choose $\beta := -2$ and obtain $H_{x,y,=-2} = \{(0, \xi_2) \mid \xi_2 \in \mathbb{R}\}$. That is, at every point $x_0 \in H_{x,y,=-2}$ half-tangents of $H_{x,y,=-2}$ have direction $(0,1)$ or $(0,-1)$. By definition, the point $(0,1)$ is Birkhoff orthogonal to (η_1, η_2) if and only if $\eta_1 + \sqrt{3}\eta_2 = 0$, and $(0,-1)$ is Birkhoff orthogonal to (η_1, η_2) if and only if $\eta_1 - \sqrt{3}\eta_2 = 0$. Since $B(0,1)$ and $\{x, y\}$ are mirror-symmetric with respect to the horizontal axis, we restrict our considerations to half-tangents with direction $(0,1)$. Assume that there exists a point $x_0 = (0, \xi_2)$ such that $(0,1)$ is Birkhoff orthogonal to the supporting line of $E_{x,y,=\alpha}$ at x_0. In other words, the supporting line of $E_{x,y,=\alpha}$ at x_0 is parallel to $\{(\eta_1, \eta_2) | \eta_1 + \sqrt{3}\eta_2 = 0\}$. Furthermore, we have $\nabla f(x_0) = \left(2, \frac{6\xi_2}{\sqrt{4+3\xi_2^2}}\right)$. If the Birkhoff orthogonality assumption is true, then this derivative can be written as $\lambda(1, \sqrt{3})$ with $\lambda \in \mathbb{R}$. The equality of the first coordinates implies $\lambda = 2$, but $\frac{6\xi_2}{\sqrt{4+3\xi_2^2}} < 2\sqrt{3}$ for all $\xi_2 \in \mathbb{R}$. This contradiction completes the proof.

6 Ellipsoids and the Fermat–Torricelli problem

In his treatise on maxima and minima dating back to around 1640, Fermat poses the problem of finding a fourth point which minimizes the sum of distances to given three points, see [62, p. 153] and [145, p. 145]. Both Fermat's problem and the answer given by Torricelli around 1645 turned into an active research topic in the following centuries; see again [145, p. 145] but also [230]. In the 20th century, Fermat-type problems caught the interest of researchers in location science and discrete geometry, stimulating a change of reference from *Fermat–Torricelli problem* to *Steiner–Weber problem* in some communities. A common feature of this kind of optimization problem is that the objective function is a sum of convex functions. This contributes to the applicability of techniques of convex analysis, particularly for establishing dual descriptions of the minimizers of the objective function. In view of applications in location science, it is also reasonable to keep the aspect of adding distances to fixed points (or convex sets) in generalizations of Fermat's problem. Sets of points for which this sum of distances is (less than or) equal to a certain constant are then extending the notions of ellipses and ellipsoids. Following and extending the presentation of [122] and [123, Section 3], the present chapter is a continuation of this line of research by addressing functions of the form

$$f : X \to \mathbb{R}, \quad f(x) := \sum_{i=1}^{n} \underline{\operatorname{dist}}_{\gamma_i}(x, K_i) \tag{6.1}$$

where $\mathfrak{K} = (K_1, \ldots, K_n)$ is a collection of $n \geq 1$ convex sets $K_i \in \mathscr{C}^X$, and $\Gamma = (\gamma_1, \ldots, \gamma_n)$ is a family of gauges on X. Notable contributions to the analysis of the problem of minimizing special cases of (6.1) are due to Durier and Michelot [71], Martini et al. [155], and Mordukhovich and Nam [170, Section 4.5]. In the version of [71], each of the sets $K_i = \{p_i\}$ is a singleton but the summands of (6.1) carry additional positive multiplicative or strictly increasing convex functional weights:

$$f : X \to \mathbb{R}, \quad f(x) := \sum_{i=1}^{n} \alpha_i \gamma_i(x - p_i), \tag{6.2}$$

$$f : X \to \mathbb{R}, \quad f(x) := \sum_{i=1}^{n} f_i(\gamma_i(x - p_i)).$$

Using subdifferential calculus, the sets of minimizers of the investigated problems are linked to a subdivision of the space X which depends on $\mathfrak{K}$ and Γ, see [71, Section 3] again as well as Proposition 6.10 below. The multiplicative weights in (6.2) may be omitted since $\alpha_i \gamma_i$ is a gauge when γ is. Therefore (6.2) is a proper special case of (6.1). The authors of [155] consider (6.1) for singletons $K_i = \{p_i\}$ and coinciding norms $\gamma_1 = \ldots = \gamma_n = \|\cdot\|$, i.e.,

$$f : X \to \mathbb{R}, \quad f(x) := \sum_{i=1}^{n} \|x - p_i\| \tag{6.3}$$

Several geometric properties of the set of minimizers of (6.3) are derived in [155]. We extend some of them to our more general setting below. In the setting used in [170, Section 4.5], the gauges γ_i coincide and distance functions are termed *minimal time functions*:

$$f : X \to \mathbb{R}, \quad f(x) := \sum_{i=1}^{n} \underline{\operatorname{dist}}_{\gamma}(x, K_i).$$

Despite the proximity of the latter setting to ours, the selection of results proved in the present chapter is mainly guided by the presentation in [155] and thus different from the one in [170, Section 4.5]. As usual, resemblances and overlaps will be indicated. In the sequel, we present dual characterizations of the *Fermat–Torricelli locus*, i.e., the set $\operatorname{ft}_\Gamma(\mathfrak{K})$ of minimizers of (6.1) as well as results on the shape of the sublevel sets of (6.2) subject to the boundary structure of the unit balls of the used gauges. It turns out that the transition from norms to gauges gives rise to several new phenomena already in the special cases in which all of the sets $K_i = \{p_i\}$ are singletons or in which the gauges $\gamma_1 = \ldots = \gamma_n =: \gamma$ coincide. Accordingly, we consider $P = (p_1, \ldots, p_n) \in X^n$ or γ as input datum of (6.1) and write $\operatorname{ft}_\gamma(\mathfrak{K})$, $\operatorname{ft}_\Gamma(P)$, or $\operatorname{ft}_\gamma(P)$ for $\operatorname{ft}_\Gamma(\mathfrak{K})$, accordingly.

6.1 Classical properties of Fermat–Torricelli loci

We start the discussion of (6.1) by extending statements on the Fermat–Torricelli locus from the literature (mostly [155]) to our more general setting. This includes not only results on the existence and uniqueness of minimizers but also dual characterizations of the Fermat–Torricelli locus $\operatorname{ft}_\Gamma(\mathfrak{K})$ and consequences for its shape. Our first statement has a particular emphasis on the case where all sets K_i are affine subspaces of X.

Proposition 6.1. *Let $\mathfrak{K} = (K_1, \ldots, K_n)$ be a collection of convex sets $K_i \in \mathscr{C}^X$, and let $\Gamma = (\gamma_1, \ldots, \gamma_n)$ be a family of gauges on X.*

(a) The set $\operatorname{ft}_\Gamma(\mathfrak{K})$ is closed and convex.
(b) If one of the sets $K_1, \ldots, K_n$ is bounded, then $\operatorname{ft}_\Gamma(\mathfrak{K})$ is non-empty and bounded.
(c) If all sets $K_1, \ldots, K_n$ are affine subspaces of X, then $\operatorname{ft}_\Gamma(\mathfrak{K})$ is non-empty. Moreover, $\operatorname{ft}_\Gamma(\mathfrak{K})$ is a Minkowski sum of a closed, bounded, convex set and a (possibly trivial) linear subspace of X.

Proof. Claim (a) holds, since $\operatorname{ft}_\Gamma(\mathfrak{K})$ is a sublevel set of the bounded below, convex, and continuous function $f := \sum_{i=1}^{n} \underline{\operatorname{dist}}_{\gamma_i}(\cdot, K_i) : X \to \mathbb{R}$. Statement (b) can be proved similarly to [169, Proposition 4.1(i)]: Suppose that K_{i_0} is bounded. If $\alpha := \inf\{f(x) \mid x \in X\}$, then $f_{\leq\alpha} = \operatorname{ft}_\Gamma(\mathfrak{K})$ is contained in the sublevel set $g_{\leq\alpha}$ of $g := \underline{\operatorname{dist}}_{\gamma_{i_0}}(\cdot, K_{i_0}) : X \to \mathbb{R}$. Moreover, the continuous function f attains its infimum over the non-empty compact set $g_{\leq\alpha}$ which implies nonemptiness of $\operatorname{ft}_\Gamma(\mathfrak{K})$. For (c), we proceed by induction on $\dim(X)$. Statement (c) is a simple consequence of (b) if $\dim(X) = 1$. Now suppose that $\dim(X) \geq 2$ and, contrary to our claim, all sets K_i are affine subspaces, but either f has no minimizer in X (Case 1) or the $\mathrm{ft}_\Gamma(\mathfrak{K})$ is non-empty and cannot be represented as it is claimed under (c) (Case 2). Note that $\mathrm{ft}_\Gamma(\mathfrak{K})$ must be unbounded in the latter case. We fix a gauge γ on X. There exists a sequence $(x_k)_{k\in\mathbb{N}}$ of points $x_k \in X$ such that $f(x_k) \to \inf_{x\in X} f(x) =: \alpha$ and $\gamma(x_k) \to +\infty$. To see this in Case 1, choose any sequence $(x_k)_{k\in\mathbb{N}}$ such that $f(x_k) \to \alpha$. Then necessarily $\gamma(x_k) \to +\infty$, since otherwise $(x_k)_{k\in\mathbb{N}}$ would have a bounded subsequence converging to a minimizer of f. In Case 2, we choose any unbounded sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathrm{ft}_\Gamma(\mathfrak{K})$. By compactness of the set $S_\gamma(0,1)$, the sequence $\left(\frac{x_k}{\gamma(x_k)}\right)_{k\in\mathbb{N}}$ has a convergent subsequence whose limit is a point of $S_\gamma(0,1)$. Without loss of generality, we may assume that $\left(\frac{x_k}{\gamma(x_k)}\right)_{k\in\mathbb{N}}$ is a convergent sequence, and we denote its limit by $x_0 \in S_\gamma(0,1)$. Now $f(x_k) \to \alpha$ implies boundedness of the set $\{f(x_k) \mid k \geq 1\}$ and in turn of each of the sets $\{\underline{\mathrm{dist}}_{\gamma_i}(x_k, K_i) \mid k \geq 1\}$ for $i \in \{1,\dots,n\}$. We set $L_i := K_i - K_i$ and conclude that $\{\underline{\mathrm{dist}}_{\gamma_i}(x_k, L_i) \mid k \geq 1\}$ is bounded from above by some constant $\alpha_i > 0$. Since L_i is a linear subspace of X, we obtain

$$\underline{\mathrm{dist}}_{\gamma_i}\left(\frac{x_k}{\gamma(x_k)}, L_i\right) = \frac{1}{\gamma(x_k)}\underline{\mathrm{dist}}_{\gamma_i}(x_k, L_i) \leq \frac{\alpha_i}{\gamma(x_k)}$$

for $k \geq 1$ and $i \in \{1,\dots,n\}$. Taking the limit $k \to +\infty$, we obtain $\underline{\mathrm{dist}}_{\gamma_i}(x_0, L_i) = 0$, i.e., $x_0 \in L_i$ and

$$K_i = K_i + \lambda x_0 \quad \text{for} \quad \lambda \in \mathbb{R},\ i \in \{1,\dots,n\}. \tag{6.4}$$

Choose a linear subspace $\bar{X} \subset X$ of dimension $\dim(\bar{X}) = \dim(X) - 1$ such that

$$X = \bar{X} + \mathrm{lin}(\{x_0\}). \tag{6.5}$$

For $\bar{x} \in \bar{X}$ and $i \in \{1,\dots,n\}$, let $\bar{\gamma}_i(\bar{x}) = \inf\{\gamma_i(\bar{x} + \lambda x_0) \mid \lambda \in \mathbb{R}\}$. Then $\bar{\gamma}_1,\dots,\bar{\gamma}_n$ turn out to be a gauges on $\bar{X}$, cf. [4, Proposition (3.1)]. Indeed, one readily checks that $\bar{\gamma}_i(\bar{x}) = 0$ is equivalent to $\underline{\mathrm{dist}}_{\gamma_i^\vee}(\bar{x}, \mathrm{lin}(\{x_0\})) = 0$, i.e., to $\bar{x} \in \mathrm{lin}(\{x_0\})$. By (6.5), this implies $\bar{x} = 0$, and item (b) of Definition 1.2 is verified. Items (c) and (d) of Definition 1.2 follow easily from the respective properties of γ_i. (In particular, we have $B_{\bar{\gamma}_i}(0,1) = \bar{X} \cap (B_{\gamma_i}(0,1) + \mathrm{lin}(\{x_0\}))$.) Now, from (6.4) and (6.5), we obtain $K_i = \bar{K}_i + \mathrm{lin}(\{x_0\})$ for $i \in \{1,\dots,n\}$ where $\bar{K}_i := K_i \cap \bar{X}$. Consider

$$\bar{f} : \bar{X} \to \mathbb{R}, \quad \bar{f}(\bar{x}) := \sum_{i=1}^{n} \underline{\mathrm{dist}}_{\bar{\gamma}_i}(\bar{x}, \bar{K}_i). \tag{6.6}$$

For $\bar{x} \in \bar{X}$ and $i \in \{1,\dots,n\}$,

$$\begin{aligned}
\underline{\mathrm{dist}}_{\bar{\gamma}_i}(\bar{x}, \bar{K}_i) &= \inf\left\{\bar{\gamma}_i(\bar{x} - \bar{y}) \,\middle|\, \bar{y} \in \bar{K}_i\right\} \\
&= \inf\left\{\inf\{\gamma_i(\bar{x} + \lambda x_0 - \bar{y}) \mid \lambda \in \mathbb{R}\} \,\middle|\, \bar{y} \in \bar{K}_i\right\} \\
&= \inf\left\{\gamma_i(\bar{x} - (\bar{y} - \lambda x_0) \,\middle|\, \bar{y} \in \bar{K}_i, \lambda \in \mathbb{R}\right\} \\
&= \inf\{\gamma_i(\bar{x} - y) \mid y \in K_i\} \\
&= \underline{\mathrm{dist}}_{\gamma_i}(\bar{x}, K_i).
\end{aligned}$$

This gives the identity $\bar{f}(\bar{x}) = f(\bar{x})$ for all $\bar{x} \in \bar{X}$. Moreover, (6.4) shows that $f(x) = f(x + \lambda x_0)$ for all $x \in X$, $\lambda \in \mathbb{R}$. So $\bar{f}(\bar{x}) = f(\bar{x} + \lambda x_0)$ for $\bar{x} \in \bar{X}$ and $\lambda \in \mathbb{R}$. Now we see that $\bar{f}$ and f attain the same values. Moreover, the Fermat–Torricelli loci F and $\bar{F}$ of (6.1) and (6.6), respectively, are related via $F = \bar{F} + \operatorname{lin}(\{x_0\})$. However, the induction hypothesis states that $\bar{F} = \bar{K} + \bar{L}$, where $\bar{K} \subset \bar{X}$ is non-empty, closed, bounded, and convex and $\bar{L}$ is a linear subspace of $\bar{X}$. Then $F = \bar{K} + L$, where $L := \bar{L} + \operatorname{lin}(\{x_0\})$ is a linear subspace of X, and the proof is complete. □

Remark 6.2. (a) The set $\operatorname{ft}_\Gamma(\mathfrak{K})$ of minimizers of (6.1) may be empty. For example, consider the $n := 2$ sets $K_1 := \{(\xi_1, 0) \mid \xi_1 \in \mathbb{R}\}$ and

$$K_2 := \left\{ (\xi_1, \xi_2) \in X \,\middle|\, \xi_1 > 0, \xi_2 \geq \frac{1}{\xi_1} \right\}$$

in $X = \mathbb{R}^2$ equipped with arbitrary gauges γ_1 and γ_2.

(b) If $K_1, \ldots, K_n \subset X$ are affine subspaces, the set $\operatorname{ft}_\Gamma(\mathfrak{K})$ may be unbounded and may not be an affine subspace of X. For example, take $n := 2$, let K_1 and K_2 be parallel hyperplanes, and $\gamma_1 = \gamma_2 : X \to \mathbb{R}$ be a norm. Then $\operatorname{ft}_\Gamma(\mathfrak{K}) = \operatorname{co}(K_1 \cup K_2)$.

In order to give an optimality condition for the minimizers of (6.1) in terms of linear functionals, we compute the conjugate and the subdifferential of the distance function $\underline{\operatorname{dist}}_\gamma(\cdot, K)$, where γ is a gauge on X and $K \in \mathscr{C}^X$. For all $x \in X$, we have $\underline{\operatorname{dist}}_\gamma(x, K) = (\gamma \,\square\, \delta(\cdot, K))(x)$. From Theorem 1.25, we obtain

$$\underline{\operatorname{dist}}_\gamma(\cdot, K)^* = (\gamma \,\square\, \delta(\cdot, K))^* = \gamma^* + \delta(\cdot, K)^* = \delta(\cdot, B(0,1)^\circ) + h(\cdot, K).$$

The subdifferential of the distance function can be computed using Lemma 1.20 as follows:

$$\begin{aligned}
\partial_\varepsilon \underline{\operatorname{dist}}_\gamma(x, K) &= \left\{ \phi \in X^* \,\middle|\, \underline{\operatorname{dist}}_\gamma(\cdot, K)^*(\phi) + \underline{\operatorname{dist}}_\gamma(x, K) \leq \langle \phi \mid x \rangle + \varepsilon \right\} \\
&= \left\{ \phi \in X^* \,\middle|\, \delta(\phi, B(0,1)^\circ) + h(\phi, K) + \underline{\operatorname{dist}}_\gamma(x, K) \leq \langle \phi \mid x \rangle + \varepsilon \right\} \\
&= \left\{ \phi \in X^* \,\middle|\, \gamma^\circ(\phi) \leq 1, h(\phi, K) + \underline{\operatorname{dist}}_\gamma(x, K) \leq \langle \phi \mid x \rangle + \varepsilon \right\} \\
&= B(0,1)^\circ \cap \left\{ \phi \in X^* \,\middle|\, \underline{\operatorname{dist}}_\gamma(x, K) \leq \inf_{y \in K} \langle \phi \mid x - y \rangle + \varepsilon \right\}. \qquad (6.7)
\end{aligned}$$

where, for $x \in K$, we have

$$\begin{aligned}
\left\{ \phi \in X^* \,\middle|\, \underline{\operatorname{dist}}_\gamma(x, K) \leq \inf_{y \in K} \langle \phi \mid x - y \rangle + \varepsilon \right\} &= \left\{ \phi \in X^* \,\middle|\, 0 \leq \inf_{y \in K} \langle \phi \mid x - y \rangle + \varepsilon \right\} \\
&= \left\{ \phi \in X^* \,\middle|\, \varepsilon \geq \sup_{y \in K} \langle \phi \mid y - x \rangle \right\} \\
&= \{ \phi \in X^* \mid \varepsilon \geq \langle \phi \mid y - x \rangle \ \forall\, y \in K \} \\
&= \operatorname{nor}_\varepsilon(x, K). \qquad (6.8)
\end{aligned}$$

It follows that $\partial_\varepsilon \underline{\operatorname{dist}}_\gamma(\cdot, K)(x) = B(0,1)^\circ \cap \operatorname{nor}_\varepsilon(x, K)$ for $x \in K$, see [20, Example 16.62] for the case that γ is the Euclidean norm and $\varepsilon = 0$. This formula is a finite-dimensional special case of formula (17) in [169]. Now we are able to formulate an optimality condition for minimizers of (6.1).

Theorem 6.3. *Let $\mathfrak{K} = (K_1, \dots, K_n)$ be a collection of convex sets $K_i \in \mathscr{C}^X$, and let $\Gamma = (\gamma_1, \dots, \gamma_n)$ be a family of gauges on X. Then $x \in X$ is a point of $\mathrm{ft}_\Gamma(\mathfrak{K})$ if and only if there exist linear functionals $\phi_1, \dots, \phi_n \in X^*$ with $\gamma_i^\circ(\phi_i) \leq 1$ and $\underline{\mathrm{dist}}_{\gamma_i}(x, K_i) = \inf_{y \in K_i} \langle \phi_i \,|\, x - y \rangle$ such that $\sum_{i=1}^n \phi_i = 0$.*

Proof. As both conditions are equivalent to $0 \in \partial \left(\sum_{i=1}^n \underline{\mathrm{dist}}_{\gamma_i}(\cdot, K_i) \right)(x)$, the assertion follows from Lemma 1.19. □

Analogously to the situation of normed spaces, the characterization splits into two mutually distinct cases when the sets $K_i = \{p_i\}$ are singletons. Depending on whether or not the minimizer is one of the given points, the two cases are commonly termed *absorbed* and *floating case*, respectively, see [155, Theorem 3.1] for the corresponding result in normed spaces. Plastria discusses in [194, Section 4] the case of so-called *skewed norms*, i.e., gauges γ_i whose dual unit ball $B_{\gamma_i}(0,1)^\circ$ admits a center of symmetry.

Corollary 6.4. *Let $P = (p_1, \dots, p_n)$ be a family of mutually distinct points $p_i \in X$, and let $\Gamma = (\gamma_1, \dots, \gamma_n)$ be a collection of gauges on X.*

(a) If $x \in X \setminus \{p_1, \dots, p_n\}$, then $x \in \mathrm{ft}_\Gamma(P)$ if and only if for every $i \in \{1, \dots, n\}$, there exists a γ_i-norming functional $\phi_i \in X^*$ of $x - p_i$ such that $\sum_{i=1}^n \phi_i = 0$.
(b) A point p_j (for some $1 \leq j \leq n$) belongs to $\mathrm{ft}_\Gamma(P)$ if and only if for every $i \in \{1, \dots, n\}$, $i \neq j$, there exists a γ_i-norming functional ϕ_i of $p_j - p_i$ such that $\gamma_j^\circ \left(-\sum_{i \neq j} \phi_i \right) \leq 1$.

Note that the assumption $p_i \neq p_j$ for $i \neq j$ in Corollary 6.4 is important in the absorbed case (b) but may be omitted in the floating case (a). Fermat–Torricelli loci obey certain local stability rules regarding transformations of the input data, that is, a point p_0 minimizing (6.1) stays a minimizer of this function after careful manipulations have been applied to $\mathfrak{K}$ and Γ. First, we show that any minimizer of (6.1) may be inserted to the input data or may serve as the center of independent dilations of the input sets without losing its minimality property, see [155, Proposition 3.1] for a special case.

Proposition 6.5. *Let $K_0 = \{p_0\}$, $K_1, \dots, K_n \in \mathscr{C}^X$. Furthermore, let $\gamma_0, \gamma_1, \dots, \gamma_n$ be gauges on X and $\lambda_1, \dots, \lambda_n > 0$. Furthermore, let $\mathfrak{K} := (K_1, \dots, K_n)$, $\mathfrak{K}' := (K_0, \dots, K_n)$, $\Gamma := (\gamma_1, \dots, \gamma_n)$, and $\Gamma' := (\gamma_0, \dots, \gamma_n)$.*

(a) If $p_0 \in \mathrm{ft}_\Gamma(\mathfrak{K})$, then $p_0 \in \mathrm{ft}_{\Gamma'}(\mathfrak{K}')$.
(b) If $p_0 \in \mathrm{ft}_\Gamma(\mathfrak{K})$, then $p_0 \in \mathrm{ft}_\Gamma(p_0 + \lambda_1(K_1 - p_0), \dots, p_0 + \lambda_n(K_n - p_0))$.

Proof. For (a), consider

$$\sum_{i=0}^n \underline{\mathrm{dist}}_{\gamma_i}(p_0, K_i) = \sum_{i=1}^n \underline{\mathrm{dist}}_{\gamma_i}(p_0, K_i) \leq \sum_{i=1}^n \underline{\mathrm{dist}}_{\gamma_i}(x, K_i) \leq \sum_{i=0}^n \underline{\mathrm{dist}}_{\gamma_i}(x, K_i).$$

for all $x \in X$. Without loss of generality, we may assume that $p_0 = 0$ in (b). Then $p_0 \in \mathrm{ft}_\Gamma(\mathfrak{K})$ implies $0 \in \sum_{i=1}^n \partial \, \underline{\mathrm{dist}}_{\gamma_i}(\cdot, K_i)(0)$ by Lemma 1.19. Using (6.7), one readily checks that $\partial \, \underline{\mathrm{dist}}_{\gamma_i}(\cdot, K_i)(0) = \partial \, \underline{\mathrm{dist}}_{\gamma_i}(\cdot, \lambda_i K_i)(0)$. Now apply Lemma 1.19 again. □

Second, any of the input sets of (6.1) may be replaced by a minimizer p_0 of (6.1) without losing p_0 as a minimizer of the modified objective function. A special case of this result can be found in [155, Corollary 3.1].

Proposition 6.6. *Let $K_0 = \{p_0\}$, $K_1, \ldots, K_n \in \mathscr{C}^X$, and let $\gamma_0, \gamma_1, \ldots, \gamma_n$ be gauges on X. Furthermore, let $\mathfrak{K} := (K_1, \ldots, K_n)$, $\mathfrak{K}' := (K_0, \ldots, K_{n-1})$, $\Gamma := (\gamma_1, \ldots, \gamma_n)$, and $\Gamma' := (\gamma_0, \ldots, \gamma_{n-1})$. If $p_0 \in \mathrm{ft}_\Gamma(\mathfrak{K})$, then $p_0 \in \mathrm{ft}_{\Gamma'}(\mathfrak{K}')$.*

Proof. If $p_0 \in \mathrm{ft}_\Gamma(\mathfrak{K})$, then

$$\begin{aligned} 0 \in \sum_{i=1}^{n} \partial \, \underline{\mathrm{dist}}_{\gamma_i}(\cdot, K_i)(p_0) &\subset B_{\gamma_0}(0,1)^\circ + \sum_{i=1}^{n-1} \partial \, \underline{\mathrm{dist}}_{\gamma_i}(\cdot, K_i)(p_0) \\ &= \partial \, \underline{\mathrm{dist}}_{\gamma_0}(\cdot, K_0)(p_0) + \sum_{i=1}^{n-1} \partial \, \underline{\mathrm{dist}}_{\gamma_i}(\cdot, K_i)(p_0), \end{aligned}$$

whence $p_0 \in \mathrm{ft}_{\Gamma'}(\mathfrak{K}')$. □

When computing Fermat–Torricelli loci, "small" instances of (6.1) can be assembled if there exists a common minimizer to all of them, see [155, Corollary 3.3] for an analogous result in Minkowski spaces.

Lemma 6.7. *Let $\mathfrak{K} = (K_1, \ldots, K_n)$ be a collection of convex sets $K_i \in \mathscr{C}^X$, and let $\Gamma = (\gamma_1, \ldots, \gamma_n)$ be a family of gauges on X. Assume that there is a partition of $\{1, \ldots, n\}$ into disjoint sets $I_1, \ldots, I_m$ such that $\bigcap_{j=1}^m \mathrm{ft}_{\Gamma_{I_j}}(\mathfrak{K}_{I_j}) \neq \emptyset$ where $\Gamma_{I_j} := (\gamma_i)_{i \in I_j}$ and $\mathfrak{K}_{I_j} := (K_i)_{i \in I_j}$. Then we have $\mathrm{ft}_\Gamma(\mathfrak{K}) = \bigcap_{j=1}^m \mathrm{ft}_{\Gamma_{I_j}}(\mathfrak{K}_{I_j})$.*

The last claim is an immediate consequence of the following simple fact.

Lemma 6.8. *Let $f_1, \ldots, f_m : A \to \mathbb{R}$ be $m \geq 1$ functions defined on the same set A. If there exists a common minimizer $x \in A$ of $f_1, \ldots, f_m$, then a point $y \in A$ is a minimizer of $f := \sum_{j=1}^m f_j : A \to \mathbb{R}$ if and only if y is a common minimizer of $f_1, \ldots, f_m$.*

Proof. For all $y \in A$, we have

$$f(y) = \sum_{j=1}^{m} f_j(y) \geq \sum_{j=1}^{m} f_j(x) = f(x)$$

with equality if and only if $f_j(y) = f_j(x)$, $j \in \{1, \ldots, m\}$. This yields the claim. □

Absorbed minimizers of (6.2) are in convex position, that is, none of them is contained in the convex hull of the remaining ones. For the special case (6.3), this is stated in [155, Lemma 4.1].

Lemma 6.9. *Let $P = (p_1, \ldots, p_n) \in X^n$ and let $\Gamma = (\gamma_1, \ldots, \gamma_n)$ be a collection of gauges on X. Suppose that $p_n \in \mathrm{ft}_\Gamma(P)$. Then p_n is an exposed point of the polytope $\mathrm{co}(\{p_1, \ldots, p_n\} \cap \mathrm{ft}_\Gamma(P))$, and*

$$\left\{ \frac{x - p_n}{\gamma_n(x - p_n)} \,\middle|\, x \in \{p_1, \ldots, p_n\} \cap \mathrm{ft}_\Gamma(P), x \neq p_n \right\}$$

is a subset of an exposed face of the unit ball of γ_n.

Proof. By Corollary 6.4, there exist γ_i-norming functionals ϕ_i of $p_n - p_i$ for each $i \in \{1, \ldots, n-1\}$ such that $\gamma_n^\circ\left(-\sum_{i=1}^{n-1} \phi_i\right) \leq 1$. Then, for any $x \in \{p_1, \ldots, p_n\} \cap \mathrm{ft}_\Gamma(P)$ with $x \neq p_n$, we have

$$\begin{aligned}
\sum_{i=1}^{n} \gamma_i(x-p_i) &= \sum_{i=1}^{n-1} \gamma_i(p_n - p_i) \\
&= \sum_{i=1}^{n-1} \langle \phi_i \,|\, p_n - p_i \rangle \\
&= \sum_{i=1}^{n-1} \langle \phi_i \,|\, x - p_i \rangle - \sum_{i=1}^{n-1} \langle \phi_i \,|\, x - p_n \rangle \\
&\leq \sum_{i=1}^{n-1} \gamma_i^\circ(\phi_i)\gamma_i(x-p_i) + \gamma_n^\circ\left(-\sum_{i=1}^{n-1} \phi_i\right)\gamma_n(x-p_n) \\
&\leq \sum_{i=1}^{n-1} \gamma_i(x-p_i) + \gamma_n(x-p_n) \\
&= \sum_{i=1}^{n} \gamma_i(x-p_i).
\end{aligned}$$

It follows that $\phi := -\sum_{i=1}^{n-1} \phi_i$ is a γ_n-norming functional of $x - p_n$. In other words, the point $\frac{x-p_n}{\gamma_n(x-p_n)}$ is contained in the exposed face $\phi_{=1} \cap B_{\gamma_n}(0,1)$ of the unit ball $B_{\gamma_n}(0,1)$. Furthermore, we have $\langle \phi \,|\, x \rangle = \gamma_n(x-p_n) + \langle \phi \,|\, p_n \rangle$ for all $x \in \{p_1, \ldots, p_n\} \cap \mathrm{ft}_\Gamma(P)$ with $x \neq p_n$, i.e., there is a number $\alpha \in \mathbb{R}$ such that the hyperplane $\phi_{=\alpha}$ strictly separates $\{p_n\}$ and $(\{p_1, \ldots, p_n\} \cap \mathrm{ft}_\Gamma(P)) \setminus \{p_n\}$. Thus p_n is an exposed point of $\mathrm{co}(\{p_1, \ldots, p_n\} \cap \mathrm{ft}_\Gamma(P))$. □

Durier and Michelot [71, Definition 3.1] refer to intersections $\bigcap_{i=1}^{n} C_{\gamma_i}(p_i, \phi_i)$ for given n-element sets $\{p_1, \ldots, p_n\} \subset X$ and $\{\phi_1, \ldots, \phi_n\} \subset B_\gamma(0,1)^\circ$ as *elementary convex sets*. Trivially, the restriction of (6.2) to an elementary convex set is an affine function. In [71, Theorem 3.1], Durier and Michelot prove that $\mathrm{ft}_\Gamma(P)$ is a bounded elementary convex set with $\sum_{i=1}^{n} \phi_i = 0$. Conversely, it is shown that an elementary convex set with $\sum_{i=1}^{n} \phi_i = 0$ is bounded and coincides with $\mathrm{ft}_\Gamma(P)$. This is equivalent to Corollary 6.4. Martini et al. [155, Theorem 3.2] establish a version in normed spaces given the prior knowledge of a floating minimizer, i.e., a point $x_0 \in \mathrm{ft}_\Gamma(P) \setminus \{p_1, \ldots, p_n\}$ for $P = (p_1, \ldots, p_n) \in X^n$ and $\gamma : X \to \mathbb{R}$ a norm. The next result shows that the proof of [155, Theorem 3.2] is independent of the symmetry of the norm.

Proposition 6.10. *Let $P = (p_1, \ldots, p_n) \in X^n$ and let $\Gamma = (\gamma_1, \ldots, \gamma_n)$ be a collection of gauges on X. Suppose that we are given a point $x_0 \in \mathrm{ft}_\Gamma(P) \setminus \{p_1, \ldots, p_n\}$. For each $i \in \{1, \ldots, n\}$, let ϕ_i be a γ_i-norming functional of $x_0 - p_i$ such that $\sum_{i=1}^{n} \phi_i = 0$. Then*

$$\mathrm{ft}_\Gamma(P) = \bigcap_{i=1}^{n} C_{\gamma_i}(p_i, \phi_i).$$

Proof. From Definition 3.5, we know that $x \in \bigcap_{i=1}^{n} C_{\gamma_i}(p_i, \phi_i)$ if and only if $\langle \phi_i \,|\, x - p_i \rangle = \gamma_i(x - p_i)$ for all $i \in \{1, \ldots, n\}$. Thus, if $x \notin \{p_1, \ldots, p_n\}$, we have $x \in \bigcap_{i=1}^{n} C_{\gamma_i}(p_i, \phi_i)$ if and only if ϕ_i

is a γ_i-norming functional of $x-p_i$ for each $i \in \{1,\dots,n\}$. This yields $x \in \mathrm{ft}_\Gamma(P)$ by Corollary 6.4 and the assumption $\sum_{i=1}^n \phi_i = 0$. On the other hand, if $x = p_j$ for some $j \in \{1,\dots,n\}$, then $x \neq p_i$ for all $i \neq j$, and $x \in \bigcap_{i=1}^n C_{\gamma_i}(p_i,\phi_i)$ implies that the linear functional ϕ_i is a γ_i-norming functional of $x - p_i$ for all $i \neq j$. Taking Corollary 6.4 and $\gamma_j^\circ\left(-\sum_{i\neq j}\phi_i\right) = \gamma_j^\circ(\phi_j) = 1$ into account, this shows that $x \in \mathrm{ft}_\Gamma(P)$. Thus $\bigcap_{i=1}^n C_{\gamma_i}(p_i,\phi_i) \subset \mathrm{ft}_\gamma(P)$. Conversely, if $x \in \mathrm{ft}_\Gamma(P)$, then

$$\begin{aligned}\sum_{i=1}^n \langle \phi_i \,|\, x-p_i\rangle &= \sum_{i=1}^n \langle \phi_i \,|\, x-x_0+x_0-p_i\rangle\\ &= \sum_{i=1}^n \langle \phi_i \,|\, x-x_0\rangle + \sum_{i=1}^n \langle \phi_i \,|\, x_0-p_i\rangle\\ &= \sum_{i=1}^n \gamma_j(x_0-p_i)\\ &= \sum_{i=1}^n \gamma_j(x-p_i),\end{aligned}$$

and hence, each Cauchy–Schwarz inequality $\langle \phi_i \,|\, x-p_i\rangle \leq \gamma_j(x-p_i)$ must hold as an equality. In other words, the linear functional ϕ_i is a γ_i-norming functional for $x - p_i$, i.e., $x \in \bigcap_{i=1}^n C_{\gamma_i}(p_i,\phi_i)$. □

As a special case of [71, Theorem 3.1] and a generalization of [155, Corollary 3.2], we obtain also

Corollary 6.11. *Let $P = (p_1,\dots,p_n) \in X^n$ and let $\Gamma = (\gamma_1,\dots,\gamma_n)$ be a collection of gauges on X. If all exposed faces of the unit balls $B_{\gamma_i}(0,1)$ of a generalized Minkowski space (X,γ) are polytopes, $i \in \{1,\dots,n\}$, then $\mathrm{ft}_\Gamma(P)$ is a convex polytope that may have empty interior. In particular, this applies if X is two-dimensional or if $B_{\gamma_i}(0,1)$ is a polytope for all $i \in \{1,\dots,n\}$.*

Since the set $\mathrm{ft}_\Gamma(P)$ of minimizers of (6.2) is an intersection of cones $C_{\gamma_i}(p_i,\phi_i)$, this set reduces to a singleton if the cones are non-collinear rays. Rotundity of all involved gauges is a sufficient condition for this. The necessity of rotundity can be invalidated in generalized Minkowski spaces by a counterexample such as Example 6.20 below. We give a proof of an extension of [179, Corollary 3.14] analogously to [155, Theorem 3.3].

Proposition 6.12. *Let $P = (p_1,\dots,p_n)$ be a family of non-collinear points $p_i \in X$, and let $\Gamma = (\gamma_1,\dots,\gamma_n)$ be a collection of rotund gauges on X. Then $\mathrm{ft}_\Gamma(P)$ is a singleton.*

Proof. Assume that there are points $x,y \in \mathrm{ft}_\Gamma(P)$, $x \neq y$. By convexity of $\mathrm{ft}_\Gamma(P)$, see Proposition 6.1, we have $[x,y] \subset \mathrm{ft}_\Gamma(P)$. Since P is finite, we may assume $x,y \notin \{p_1,\dots,p_n\}$. Thus, by Corollary 6.4, there exist γ_i-norming functionals ϕ_i of $x-p_i$ for each $i \in \{1,\dots,n\}$ such that $\sum_{i=1}^n \phi_i = 0$. We have

$$\sum_{i=1}^n \gamma_i(x-p_i) = \sum_{i=1}^n \langle \phi_i \,|\, x-p_i\rangle$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \langle \phi_i \,|\, x - y \rangle + \sum_{i=1}^{n} \langle \phi_i \,|\, y - p_i \rangle \\
&\le \sum_{i=1}^{n} \gamma_i(y - p_i) \\
&= \sum_{i=1}^{n} \gamma_i(x - p_i).
\end{aligned}
$$

It follows that $\langle \phi_i \,|\, y - p_i \rangle = \gamma_i(y - p_i)$ for each $i \in \{1, \dots, n\}$ or, in other words, that ϕ_i is also a γ_i-norming functional of $y - p_i$. Since P is non-collinear, there is a number $i \in \{1, \dots, n\}$ such that x, y, and p_i are not collinear. Hence $\frac{x-p_i}{\gamma_i(x-p_i)}$ and $\frac{y-p_i}{\gamma_i(y-p_i)}$ are distinct unit vectors with a common γ_i-norming functional, i.e., distinct members of an exposed face of $B_{\gamma_i}(0,1)$. This is a contradiction to the rotundity assumption, see Theorem 2.23. □

Rotundity of the involved gauges also guarantees the uniqueness of minimizers of (6.2) restricted to straight lines.

Lemma 6.13. *Let $P = (p_1, \dots, p_n)$ be a family of mutually distinct points $p_i \in X$, and let $\Gamma = (\gamma_1, \dots, \gamma_n)$ be a collection of rotund gauges on X. If $L \subset X$ is a straight line satisfying $\{p_1, \dots, p_n\} \not\subset L$, then the restriction of $f := \sum_{i=1}^{n} \gamma_i(\cdot - p_i) : X \to \mathbb{R}$ to L is strictly convex in the sense of Remark 2.24.*

Proof. Assume that $f|_L$ is not strictly convex. Then there exist points $x, y \in L$, $x \neq y$, such that $f\left(\frac{x+y}{2}\right) \ge \frac{f(x)+f(y)}{2}$, which is equivalent to

$$
\sum_{i=1}^{n} \frac{\gamma_i((x-p_i)+(y-p_i))}{2} \ge \sum_{i=1}^{n} \frac{\gamma_i(x-p_i)+\gamma_i(y-p_i)}{2}.
$$

Consequently, the triangle inequalities $\gamma_i((x-p_i)+(y-p_i)) \le \gamma_i(x-p_i)+\gamma_i(y-p_i)$ hold as equalities for $i \in \{1, \dots, n\}$. Using Theorem 2.23, the points $x - p_i$ and $y - p_i$ are linearly dependent, and $p_i \in \operatorname{aff}(\{x, y\}) = L$. This gives $\{p_1, \dots, p_n\} \subset L$, a contradiction. □

The investigation of the function (6.1) can be restricted to a non-empty closed convex set $K_0 \subset X$. This gives rise to the problem of finding the infimum

$$
\inf_{x \in K_0} \sum_{i=1}^{n} \underline{\operatorname{dist}}_{\gamma_i}(x, K_i). \tag{6.9}
$$

where $K_1, \dots, K_n \subset X$ are non-empty closed convex sets and $\gamma_1, \dots, \gamma_n$ are gauges on X again. Problem (6.9) is usually termed *generalized Heron problem* in the literature, see [85, 171, 172]. The usage of extended real-valued functions in convex analysis and convex optimization makes it possible to incorporate the feasible set K_0 from (6.9) into its objective function and apply the methods of unrestricted convex optimization instead. Accordingly, the existence of minimizers of the function

$$
f := \delta(\cdot, K_0) + \sum_{i=1}^{n} \underline{\operatorname{dist}}_{\gamma_i}(\cdot, K_i) : X \to \overline{\mathbb{R}},
$$

can be shown as in [171, Proposition 3.1] if one of the sets $K_0, \dots, K_n$ is bounded. As before, a dual characterization of the minimizers of (6.1) can be derived from Lemma 1.19. An analogous result is true in possibly infinite-dimensional Banach spaces, see [171, Theorem 3.2].

Theorem 6.14. *Let $K_0, \dots, K_n \in \mathscr{C}^X$, and let $\gamma_1, \dots, \gamma_n$ be gauges on X. A point $x \in K_0$ is a minimizer of the function $f := \delta(\cdot, K_0) + \sum_{i=1}^n \underline{\operatorname{dist}}_{\gamma_i}(\cdot, K_i) : X \to \overline{\mathbb{R}}$ if and only if there exist linear functionals $\phi_0 \in \operatorname{nor}(x, K_0)$ and $\phi_i \in X^*$ satisfying $\gamma_i^\circ(\phi_i) \leq 1$ and $\underline{\operatorname{dist}}_{\gamma_i}(x, K_i) = \inf_{y \in K_i} \langle \phi_i \,|\, x - y \rangle$, $i \in \{1, \dots, n\}$, such that $\sum_{i=0}^n \phi_i = 0$.*

Proof. Both conditions are equivalent to $0 \in \partial f(x)$, see (6.7), (6.8), and Theorem 1.23. □

6.2 Nonclassical properties of Fermat–Torricelli loci

In this section, we present new phenomena in the investigation of Fermat–Torricelli loci for gauges. A key role is played by so-called metrically defined segments which are conceptually based in the work [166] by Menger in which he considers a kind of betweenness relation in metric spaces. In a Minkowski space $(X, \|\cdot\|)$, the *d-segment* between two points $x, y \in X$ is defined as

$$[x, y]_d := \{z \in X \mid \|x - z\| + \|z - y\| = \|x - y\|\},$$

cf. [32, Chapter II]. (Here, the symbol d is not a variable but a reference to the term “metric”.) Clearly, one has $[x, y]_d = [y, x]_d$. The triangle inequality shows that $\operatorname{ft}_\gamma(x, y) = [x, y]_d$, see [155, p. 290], and there are several other connections between the function defined in (6.3) and the notion of d-segments, see [155, Proposition 3.3, Corollaries 3.3 and 3.5, and Theorem 4.1]. In generalized Minkowski spaces (X, γ), we give the analogous definition

$$[x, y]_\gamma := \{z \in X \mid \gamma(x - z) + \gamma(z - y) = \gamma(x - y)\}$$

but now we cannot expect $[x, y]_\gamma = [y, x]_\gamma$ in general. However, metrically defined segments are still convex sets, and they can be constructed using homothetic images of the unit sphere and its reflection about the origin.

Lemma 6.15. *Let (X, γ) be a generalized Minkowski space and $x, y \in X$.*

(a) The set $[x, y]_\gamma$ is closed and convex.
(b) We have $[x, y] \subset [x, y]_\gamma$.
(c) We have $[x, y]_\gamma = \bigcup_{\lambda \in [0,1]} (x - S(0, \lambda\gamma(x - y))) \cap (y + S(0, (1 - \lambda)\gamma(x - y)))$.

Proof. Convexity of $[x, y]_\gamma$ is a consequence of the positive homogeneity of γ and of the triangle inequality. Furthermore, the continuity of γ implies the closedness of $[x, y]_\gamma$. From $x, y \in [x, y]_\gamma$ and the convexity of $[x, y]_\gamma$, we obtain $[x, y] \subset [x, y]_\gamma$. Finally, observe that

$$\begin{aligned}
[x, y]_\gamma &= \{z \in X \mid \gamma(x - z) + \gamma(z - y) = \gamma(x - y)\} \\
&= \bigcup_{\lambda \in [0,1]} \left\{ z \in X \;\middle|\; \begin{array}{c} \gamma(x - z) = \lambda\gamma(x - y), \\ \lambda\gamma(x - y) + \gamma(z - y) = \gamma(x - y) \end{array} \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{\lambda\in[0,1]} \left\{ z \in X \,\middle|\, \begin{matrix} \gamma(x-z) = \lambda\gamma(x-y), \\ \gamma(z-y) = (1-\lambda)\gamma(x-y) \end{matrix} \right\} \\
&= \bigcup_{\lambda\in[0,1]} \left\{ z \in X \,\middle|\, \begin{matrix} x-z \in S(0,\lambda\gamma(x-y)), \\ z-y \in S(0,(1-\lambda)\gamma(x-y)) \end{matrix} \right\} \\
&= \bigcup_{\lambda\in[0,1]} \left\{ z \in X \,\middle|\, \begin{matrix} z \in x - S(0,\lambda\gamma(x-y)), \\ z \in y + S(0,(1-\lambda)\gamma(x-y)) \end{matrix} \right\} \\
&= \bigcup_{\lambda\in[0,1]} (x - S(0,\lambda\gamma(x-y))) \cap (y + S(0,(1-\lambda)\gamma(x-y))).
\end{aligned}
$$

This completes the proof. □

Using the above-mentioned equation $\mathrm{ft}_\gamma(x,y) = [x,y]_d$ from [155, p. 290], we know now that

$$\{x,y\} \subset [x,y] \subset [x,y]_\gamma = [x,y]_d = \mathrm{ft}_\gamma(x,y) \tag{6.10}$$

for all points $x,y \in X$, provided that γ is a norm. The situation is different if γ is not a norm.

Proposition 6.16. *Let (X,γ) be a generalized Minkowski space such that γ is not a norm. Then there exists a point $x_0 \in X \setminus \{0\}$ such that* $\mathrm{ft}_\gamma(x_0,0) = \{0\}$.

Proof. By compactness of $S(0,1)$, there exists a point $x_0 \in -S(0,1)$ such that

$$\gamma(x_0) = \max\{\gamma(-x) \mid x \in S(0,1)\} = \max\left\{ \frac{\gamma(-x)}{\gamma(x)} \,\middle|\, x \in X \setminus \{0\} \right\}.$$

Since γ is not a norm, we have

$$\gamma(x_0) > \gamma(-x_0) = 1 \tag{6.11}$$

and

$$\gamma(x) \geq \frac{\gamma(-x)}{\gamma(x_0)} \quad \text{for all } x \in X. \tag{6.12}$$

The set $\mathrm{ft}_\gamma(x_0,0)$ consists of all minimizers of the function $f := \gamma(\cdot - x_0) + \gamma(\cdot - 0) : X \to \mathbb{R}$. In order to show that $\mathrm{ft}_\gamma(x_0,0) = \{0\}$, it is enough to prove that $f(x) > f(0)$ for all $x \in X \setminus \{0\}$. For an arbitrary point $x \neq 0$, we use (6.11), (6.12), and the triangle inequality to obtain

$$
\begin{aligned}
f(x) &= \gamma(x-x_0) + \gamma(x-0) \\
&\geq \frac{1}{\gamma(x_0)}\gamma(x_0 - x) + \gamma(x) \\
&\geq \frac{1}{\gamma(x_0)}\Big(\gamma(x_0) - \gamma(x)\Big) + \gamma(x) \\
&= 1 + \left(1 - \frac{1}{\gamma(x_0)}\right)\gamma(x) \\
&> 1
\end{aligned}
$$

$$= \gamma(-x_0)$$
$$= f(0),$$

and the proof is complete. □

Remark 6.17. The above proof together with a dilation argument shows the following: If (X,γ) is a generalized Minkowski space and if a point $x_0 \in X \setminus \{0\}$ satisfies

$$\frac{\gamma(x_0)}{\gamma(-x_0)} = \max\left\{ \frac{\gamma(-x)}{\gamma(x)} \,\middle|\, x \in X \setminus \{0\} \right\} > 1,$$

then $\mathrm{ft}_\gamma(x_0,0) = \{0\}$.

We obtain several characterizations of norms among arbitrary gauges.

Corollary 6.18. *Let (X,γ) be a generalized Minkowski space. The following statements are equivalent.*

(a) The gauge γ is a norm.
(b) For any two distinct points $x, y \in X$, we have $\mathrm{card}(\mathrm{ft}_\gamma(x,y)) > 1$.
(c) For any two distinct points $x, y \in X$, we have $\mathrm{card}(\mathrm{ft}_\gamma(x,y)) = \infty$.
(d) For any two distinct points $x, y \in X$, we have $\mathrm{ft}_\gamma(x,y) \setminus \{x\} \neq \emptyset$.
(e) For any two distinct points $x, y \in X$, we have $\mathrm{ft}_\gamma(x,y) \setminus \{x,y\} \neq \emptyset$.
(f) For any two distinct points $x, y \in X$, we have $x \in \mathrm{ft}_\gamma(x,y)$.
(g) For any two distinct points $x, y \in X$, we have $\{x,y\} \subset \mathrm{ft}_\gamma(x,y)$.
(h) For any two distinct points $x, y \in X$, we have $[x,y] \subset \mathrm{ft}_\gamma(x,y)$.
(i) For any two distinct points $x, y \in X$, we have $\mathrm{ri}([x,y]) \cap \mathrm{ft}_\gamma(x,y) \neq \emptyset$.
(j) For any two distinct points $x, y \in X$, we have $[x,y]_\gamma \subset \mathrm{ft}_\gamma(x,y)$.
(k) For any two distinct points $x, y \in X$, we have $\mathrm{ri}([x,y]_\gamma) \cap \mathrm{ft}_\gamma(x,y) \neq \emptyset$.
(l) For any two distinct points $x, y \in X$, we have $[x,y]_\gamma = \mathrm{ft}_\gamma(x,y)$.

Proof. We know from (6.10) that (a) implies the other conditions. Proposition 6.16 shows that each of the other conditions implies (a). However, we give details for the implication (k)⇒(a), since it is less obvious. We assume that (k) is satisfied, whereas (a) fails. Then Proposition 6.16 yields the existence of a point $x_0 \in X \setminus \{0\}$ such that $\mathrm{ft}_\gamma(x_0,0) = \{0\}$, and (k) yields $0 \in \mathrm{ri}([x_0,0]_\gamma)$. Since $[x_0,0] \subset [x_0,0]_\gamma$, cf. Lemma 6.15(b), the origin 0 must be an interior point of the set $[x_0,0]_\gamma \cap \mathrm{aff}(\{x_0,0\})$ with respect to the subspace topology in $\mathrm{aff}(\{x_0,0\}) = \mathrm{lin}(\{x_0\})$. Hence, there exists a number $\varepsilon > 0$ such that $-\varepsilon x_0 \in [x_0,0]_\gamma \cap \mathrm{aff}(\{x_0,0\}) \subset [x_0,0]_\gamma$. The inclusion $-\varepsilon x_0 \in [x_0,0]_\gamma$ yields

$$\gamma(x_0 - (-\varepsilon x_0)) + \gamma(-\varepsilon x_0 - 0) = \gamma(x_0 - 0).$$

We obtain the contradiction $(1+\varepsilon)\gamma(x_0) + \gamma(-\varepsilon x_0) = \gamma(x_0)$, and the proof is complete. □

Example 6.19. Take $X = \mathbb{R}^2$ and consider the gauge $\gamma : X \to \mathbb{R}$,

$$\gamma(\xi_1,\xi_2) := \max\left\{ \frac{1}{2}\xi_1, -\xi_1 - \xi_2, -\xi_1 + \xi_2, -\frac{1}{2}\xi_1 - \xi_2, -\frac{1}{2}\xi_1 + \xi_2 \right\}.$$

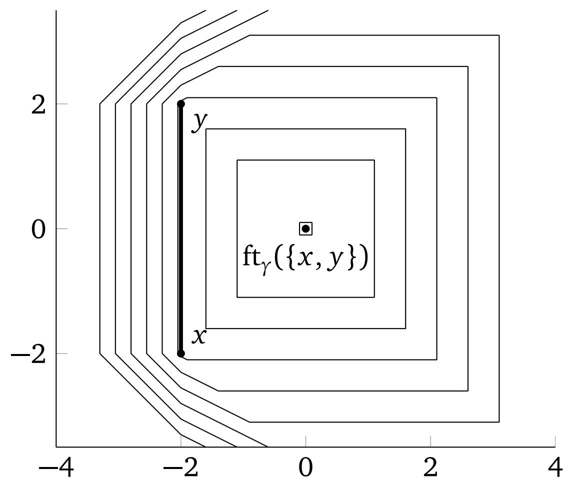


Figure 6.1. The Fermat–Torricelli locus $\operatorname{ft}_\gamma(x,y)$ of two points x and y need not belong to the $[x,y]_\gamma \cap [y,x]_\gamma$ (bold line). Level sets of the function $\gamma(\cdot - x) + \gamma(\cdot - y) : X \to \mathbb{R}$ are depicted in thin lines.

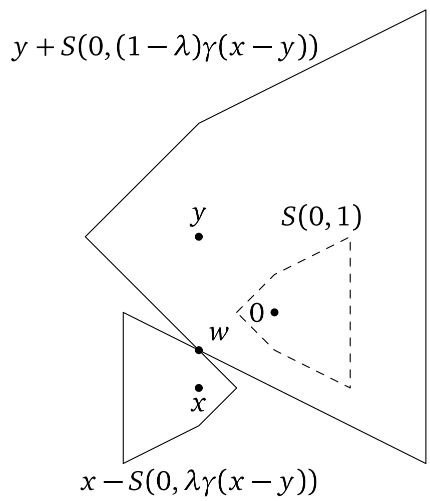


Figure 6.2. Constructing a point $w \in [x,y]_\gamma$. The unit ball is depicted in dashed lines.

For $x := (-2,-2)$ and $y := (-2,2)$, we have $\operatorname{ft}_\gamma(x,y) = \{(0,0)\}$, see Figure 6.1. Moreover, we obtain $[x,y]_\gamma = [y,x]_\gamma = [x,y]$ as illustrated in Figure 6.2. This can be shown with the help of Lemma 6.15(c). This example shows that the Fermat–Torricelli locus has the following two properties that are known to be impossible in normed spaces.

(i) There are a generalized Minkowski plane (X,γ) and finitely many points $p_1,\ldots,p_n \in X$ such that $\operatorname{co}(\{p_1,\ldots,p_n\}) \cap \operatorname{ft}_\gamma(p_1,\ldots,p_n) = \emptyset$, cf. [155, Theorem 3.4] for the classical setting.

(ii) There are a generalized Minkowski plane (X,γ) and points $x,y\in X$ such that $\mathrm{ft}_\gamma(x,y)\not\subset [x,y]_\gamma$.

Example 6.20. Rotundity of norms can be characterized by means of the Fermat–Torricelli locus: the norm of a Minkowski space $(X,\|\cdot\|)$ is rotund if and only if $\mathrm{ft}_{\|\cdot\|}(P)$ is a singleton for every non-collinear finite set $P\subset X$, see [155, Theorem 3.3]. For generalized Minkowski spaces, this condition remains necessary for rotundity of gauges, see Proposition 6.12, but the sufficiency fails. For instance, take $X=\mathbb{R}^2$ and consider the gauge $\gamma : X\to\mathbb{R}$,

$$\gamma(\xi_1,\xi_2):=\begin{cases}|\xi_1|+|\xi_2| & \text{if } \xi_1,\xi_2>0,\\ \sqrt{\xi_1^2+\xi_2^2} & \text{else.}\end{cases}$$

Assume that there is a family $P=(p_1,\dots,p_n)\in X^n$ of non-collinear points p_i such that $\mathrm{ft}_\gamma(P)$ is not a singleton. Then, by convexity of $\mathrm{ft}_\gamma(P)$, there is a point $x\in\mathrm{ft}_\gamma(P)\setminus\{p_1,\dots,p_n\}$, and, by Corollary 6.4 and Proposition 6.10, there are γ_i-norming functionals ϕ_i of $x-p_i$ for $i\in\{1,\dots,n\}$ such that $\sum_{i=1}^n\phi_i=0$ and $\mathrm{ft}_\gamma(P)$ is the intersection of the cones $C_{\gamma_i}(p_i,\phi_i)$. Note that every linear functional ϕ_i belongs to

$$S_{\gamma^\circ}(0,1)=\left\{(\cos(\alpha),\sin(\alpha))\,\middle|\,\frac{\pi}{2}<\alpha<2\pi\right\}\\ \cup\{(1,\beta)\mid 0\le\beta\le 1\}\cup\{(\beta,1)\mid 0\le\beta\le 1\},$$

and $C_{\gamma_i}(p_i,\phi_i)$ is a ray if and only if $\phi_i\neq(1,1)$. Suppose that $\phi_i\neq(1,1)$ for $i\in\{1,\dots,k\}$ and $\phi_i=(1,1)$ for $i\in\{k+1,\dots,n\}$. Since $\mathrm{card}(\mathrm{ft}_\gamma(P))>1$, the rays $C_{\gamma_i}(p_i,\phi_i)$, $i\in\{1,\dots,k\}$, are contained parallel straight lines. Hence, we have the following cases:

Case 1: $\phi_1,\dots,\phi_k\in\{\pm(\cos(\alpha_0),\sin(\alpha_0))\}$ with fixed $\alpha_0\in\left(\frac{\pi}{2},\pi\right)$ (rays with negative slope),
Case 2: $\phi_1,\dots,\phi_k\in\{(1,\beta)\mid 0\le\beta<1\}\cup\{(-1,0)\}$ (horizontal rays),
Case 3: $\phi_1,\dots,\phi_k\in\{(\beta,1)\mid 0\le\beta<1\}\cup\{(0,-1)\}$ (vertical rays),
Case 4: $\phi_1=\ldots=\phi_k=(\cos(\alpha_0),\sin(\alpha_0))$ with fixed $\alpha_0\in\left(\pi,\frac{3\pi}{2}\right)$ (rays with positive slope).

Since $\{p_1,\dots,p_n\}$ is not collinear, we obtain $k<n$. The equation $\sum_{i=1}^n\phi_i=0$ yields

$$(\eta_1,\eta_2):=\sum_{i=1}^k\phi_i=-\sum_{i=k+1}^n\phi_i=(n-k)(-1,-1).$$

This is impossible in Case 1, because then (η_1,η_2) is an integer multiple of $(\cos(\alpha_0),\sin(\alpha_0))$. In this case, (η_1,η_2) is either zero or its coordinates have different signs. In Case 2 (Case 3), we obtain a contradiction, since then $\eta_2\ge 0$ ($\eta_1\ge 0$). Finally, Case 4 gives $(\eta_1,\eta_2)=k(\cos(\alpha_0),\sin(\alpha_0))$, the equality $\eta_1=-(n-k)=\eta_2$ implies $\alpha_0=\frac{5\pi}{4}$, and we obtain a contradiction from $-(n-k)=\eta_1=-k\frac{\sqrt{2}}{2}$, since $\sqrt{2}$ is irrational. Summarizing, there is another property of the Fermat–Torricelli locus which is impossible in Minkowski spaces.

(iii) There is a non-rotund generalized Minkowski space (X,γ) such that $\mathrm{card}(\mathrm{ft}_\gamma(P))=1$ for any family $P=(p_1,\dots,p_n)\in X^n$ of non-collinear points p_i.

The last example shows that a characterization of rotundity of norms does not extend to arbitrary gauges. However, there is a characterization in terms of metrically defined segments.

Proposition 6.21. *A generalized Minkowski space (X,γ) has a rotund gauge if and only if $[x,y]_\gamma = [x,y]$ for all $x,y \in X$.*

Proof. By Lemmas 2.22 and 6.15(b), the condition $[x,y]_\gamma \neq [x,y]$ is equivalent to $[x,y]_\gamma \not\subset [x,y]$, which is in turn equivalent to

$$\exists z \in X \setminus [x,y] : \left[\frac{x-z}{\gamma(x-z)}, \frac{z-y}{\gamma(z-y)} \right] \subset S(0,1). \tag{6.13}$$

Now (6.13) shows that γ is not rotund because $\left[\frac{x-z}{\gamma(x-z)}, \frac{z-y}{\gamma(z-y)} \right]$ is not a singleton, since this would imply

$$z = \frac{\gamma(z-y)}{\gamma(x-z)+\gamma(z-y)} x + \frac{\gamma(x-z)}{\gamma(x-z)+\gamma(z-y)} y \in [x,y].$$

Conversely, if γ is not rotund, there exists a linearly independent subset $\{z_1, z_2\} \in X$ such that $[z_1, z_2] \subset S(0,1)$. This yields (6.13) for $x := z_1 + z_2$, $y := 0$, and $z =:= z_1$. □

6.3 Boundary structure of ellipsoids

A usual Euclidean ellipsoid is the collection of all points of Euclidean space whose sum of distances to two fixed points is (less than or) equal to a given constant. This polysemy is caused by incorporating or omitting the phrase in parentheses—a decision which depends on whether the objects of our investigation shall be convex bodies or just their boundary surfaces. In any case, the two defining fixed points are called *foci* due to fact that light emitted at one of them is focused at the other one after having been reflected at the surface of the ellipsoid. For extending the notion of ellipsoids, one may apply the sum-of-distances principle to $n \geq 1$ fixed points which are again called foci [159]. Early contributions to the study of such curves are due to Descartes [64, p. 324] who proposed to investigate the case $n = 4$, and von Tschirnhaus [235, p. 93] who extended the gardener's string construction of ellipses to some multifocal cases. The string construction has been addressed later again by Maxwell in [159–162]. For the most part, the literature is restricted to the investigation of the planar case, i.e., to the corresponding generalization of ellipses. Note that these curves have been termed variously, reaching from *isodapanes* [81], *Tschirnhaussche Eikurven* [225, 226], *multifocal ellipses* [75, 193], *polyellipses* [165], and *n-ellipses* [184, 209] to *egglipses* [203]. Topics of interest regarding multifocal ellipses in the Euclidean plane include for instance their regularity and curvature [165], which is in some sense a continuation of Descartes's interest in construction of tangents, and their qualification as approximations for simple closed curves [75]. In [143, pp. 174–183], points for which a linear combination of the distances from two foci is fixed, form a *Cartesian oval*. Zamfirescu [244] alters the nature of the foci by investigated generalized ellipsoids whose foci are convex sets in Euclidean space. In present section, we interpret the (sub)level sets of the objective function of (6.2) as multifocal ellipsoids in generalized Minkowski spaces. This incorporates the presence of multiple foci, which is common to the mentioned references, but also extensions to vector spaces of dimension > 2 and a change of distance measurement to gauges. (Note that Groß and Strempel [95] investigate multifocal ellipsoids in normed spaces.) In particular, we investigate bisectors of chords of a multifocal ellipsoid which do or do not intersect the convex hull of the foci in Theorem 6.25, and we give a characterization of rotundity in terms of bifocal ellipsoids in Theorem 6.28. But first, we start with an observation concerning Figure 6.1 which illustrates sublevel sets $f_{\leq\alpha}$ of the function $f = \gamma(\cdot - p_1) + \gamma(\cdot - p_2) : X \to \mathbb{R}$ for two points $p_1, p_2 \in X = \mathbb{R}^2$. Every extreme point x_0 of a sublevel set $f_{\leq\alpha}$ is of the form $x_0 = p_i + \lambda w$, where $i \in \{1,2\}$, $\lambda \in [0,+\infty)$, and w is an extreme point of $B(0,1)$. This turns out to be a special case of a more general phenomenon.

Proposition 6.22. *Let* $\dim(X) \geq 2$, $n \geq 1$, *and* $p_1,\ldots,p_n \in X$. *Furthermore, let* $\gamma_1,\ldots,\gamma_n$ *be gauges on* X, $f := \sum_{i=1}^n \gamma_i(\cdot - p_i) : X \to \mathbb{R}$, *and* $\alpha \in \mathbb{R}$. *Then every point* $x_0 \in \operatorname{ext}_{\dim(X)-2}(f_{\leq\alpha})$ *can be expressed as* $x_0 = p_i + \lambda w$ *for suitable choices of* $i \in \{1,\ldots,n\}$, $\lambda \in [0,\alpha]$, *and* $w \in \operatorname{ext}_{\dim(X)-2}(B_{\gamma_i}(0,1))$.

Proof. The claim is trivial if $x_0 = p_i$ for some $i \in \{1,\ldots,n\}$, because then $x_0 = p_i + 0w$ for all points $w \in \operatorname{ext}_{\dim(X)-2}(B_{\gamma_i}(0,1))$. Hence, we may assume that $x_0 \neq p_i$ for $i \in \{1,\ldots,n\}$. Putting $\lambda_i := \gamma_i(x_0 - p_i) > 0$, we obtain

$$x_0 = p_i + \lambda_i w_i \text{ with } \lambda_i > 0,\ w_i \in S_{\gamma_i}(0,1) \text{ for } i \in \{1,\ldots,n\},\ \sum_{i=1}^n \lambda_i = \alpha. \tag{6.14}$$

It remains to show that $w_i \in \operatorname{ext}_{\dim(X)-2}(B_{\gamma_i}(0,1))$ for some $i \in \{1,\ldots,n\}$. Suppose that this is not the case. Then, for all $i \in \{1,\ldots,n\}$, we have $w_i \in \operatorname{ri}(F_i)$ for some exposed $(\dim(X)-1)$-face F_i of $B_{\gamma_i}(0,1)$, according to Lemma 1.13. Denoting the corresponding supporting hyperplane by $H_i := \operatorname{aff}(F_i)$, we have

$$w_i \in \operatorname{ri}(F_i) = \operatorname{ri}(H_i \cap S_{\gamma_i}(0,1)). \tag{6.15}$$

Since F_i is of dimension $\dim(X)-1$, the set $V_i := \operatorname{cone}(\operatorname{ri}(H_i \cap S_{\gamma_i}(0,1)))$ is open in X. Formulas (6.14) and (6.15) and the linearity of the restriction if γ_i to V_i show that $p_i + V_i$ is an open neighborhood of x_0 and that the restricted function $f_i|_{p_i+V_i} = \gamma_i(\cdot - p_i)|_{p_i+V_i}$ is affine. Now it follows that $K := (p_1+V_1)\cap\ldots\cap(p_n+V_n)$ is an open neighborhood of x_0 and that $f|_V = \left(\sum_{i=1}^n f_i|_{p_i+V_i}\right)\Big|_V$ is an affine function. However, boundary points of a sublevel set of an affine function are never $(\dim(X)-2)$-extreme. Thus $x_0 \notin \operatorname{ext}_{\dim(X)-2}(f_{\leq\alpha})$. This contradiction completes the proof. □

For the case $\dim(X) = 2$, Proposition 6.22 states that every extreme point x_0 of $f_{\leq\alpha}$ is of the form $x_0 = p_i + \lambda w$ for some $i \in \{1,\ldots,n\}$, $\lambda \in [0,\alpha]$, and an extreme point w of $B_{\gamma_i}(0,1)$. This fails in general for spaces X of dimensions $\dim(X) > 2$, as it is illustrated by the following example.

Example 6.23. Let $X = \mathbb{R}^3$, $n := 3$, $p_1 := (1,0,-1)$, $p_2 := (0,-1,1)$, $p_3 := (-1,1,0)$, $\gamma_1 = \gamma_2 = \gamma_3 := \|\cdot\|_1$, and let $\alpha := 6$. For $f := \sum_{i=1}^n \gamma_i(\cdot - p_i) : X \to \mathbb{R}$, we then have $f_{\leq 6} = \{(0,0,0)\}$ because $f(0,0,0) = 6$ and, for arbitrary $(\xi_1,\xi_2,\xi_3) \neq (0,0,0)$ with $\max\{|\xi_1|,|\xi_2|,|\xi_3|\} \leq 1$, we have

$$\begin{aligned} f(\xi_1,\xi_2,\xi_3) = {} & (|\xi_1 - 1| + |\xi_2| + |\xi_3 + 1|) + (|\xi_1| + |\xi_2 + 1| + |\xi_3 - 1|) \\ & + (|\xi_1 + 1| + |\xi_2 - 1| + |\xi_3|) \end{aligned}$$

$$
\begin{aligned}
&= (|\xi_1 - 1| + |\xi_1| + |\xi_1 + 1|) + (|\xi_2 - 1| + |\xi_2| + |\xi_2 + 1|) \\
&\quad + (|\xi_3 - 1| + |\xi_3| + |\xi_3 + 1|) \\
&= ((1 - |\xi_1|) + |\xi_1| + (1 + |\xi_1|)) + ((1 - |\xi_2|) + |\xi_2| + (1 + |\xi_2|)) \\
&\quad + ((1 - |\xi_3|) + |\xi_3| + (1 + |\xi_3|)) \\
&= 6 + \|(\xi_1, \xi_2, \xi_3)\|_1 > 6.
\end{aligned}
$$

Hence $x_0 := (0, 0, 0)$ is an extreme point of $f_{\leq 6}$. But one readily checks that x_0 does not admit a representation $x_0 = p_i + \lambda w$ with $i \in \{1, 2, 3\}$, $\lambda \geq 0$, and an extreme point w of $B_{\gamma_i}(0, 1)$.

One might expect that Proposition 6.22 (at least if $\dim(X) = 2$) can be extended to the case of non-empty compact, convex sets $K_1, \ldots, K_n \subset X = \mathbb{R}^2$ instead of $p_1, \ldots, p_n$ as in Theorem 6.3 in the following sense: Every extreme point x_0 of $f_{\leq \alpha}$ can be expressed as $x_0 = p + \lambda w$ with an extreme point p of K_i for suitable $i \in \{1, \ldots, n\}$, a number $\lambda \geq 0$, and an extreme point w of $-B_{\gamma_i}(0, 1)$. But this is not the case, as the following example shows.

Example 6.24. Consider $X = \mathbb{R}^2$, $n := 4$,

$$
\begin{aligned}
K_1 &:= [(-4, -1), (4, -1)], & K_2 &:= [(-4, 1), (4, 1)], \\
K_3 &:= [(-1, -4), (-1, 4)], & K_4 &:= [(1, -4), (1, 4)],
\end{aligned}
$$

and let $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 := \|\cdot\|_\infty$ and $\alpha := 4$. Then the extreme points of of the sublevel set $f_{\leq 4} = [-1, 1]^2$ of the function $f := \sum_{i=1}^{n} \gamma_i(\cdot - p_i) : X \to \mathbb{R}$ do not admit the above representation.

Szőkefalvi-Nagy [225] shows that the outer normal at any smooth point of a multifocal ellipsoid in Euclidean space meets the convex hull of the foci. For generalized Minkowski spaces, we emulate the concept of perpendicularity at boundary points using bisectors of chords.

Theorem 6.25. *Let (X, γ) be a generalized Minkowski space and $p_1, \ldots, p_n, x, y \in X$. If x and y are points of the same level set of the function $f := \sum_{i=1}^{n} \gamma(\cdot - p_i) : X \to \mathbb{R}$, then the intersection of the bisector $\operatorname{bsc}_{\gamma^\vee}(x, y)$ and the polytope $\operatorname{co}(\{p_1, \ldots, p_n\})$ is non-empty.*

Proof. Since x and y belong to the same level set of f, we have

$$
\sum_{i=1}^{n} \gamma(x - p_i) = f(x) = f(y) = \sum_{i=1}^{n} \gamma(y - p_i).
$$

Then there exist numbers $i, j \in \{1, \ldots, n\}$ such that $\gamma(x - p_i) \leq \gamma(y - p_i)$ and $\gamma(x - p_j) \geq \gamma(y - p_j)$. The intermediate value theorem yields the existence of a point $z \in [p_i, p_j] \subset \operatorname{co}(\{p_1, \ldots, p_n\})$ such that $\gamma(x - z) = \gamma(y - z)$, i.e., $z \in \operatorname{bsc}_{\gamma^\vee}(x, y)$. □

Corollary 6.26. *Let $(X, \|\cdot\|)$ be a Minkowski space and $p_1, \ldots, p_n, x, y \in X$. If x and y are points of the same level set of the function $f := \sum_{i=1}^{n} \|\cdot - p_i\| : X \to \mathbb{R}$, then $\operatorname{bsc}_\gamma(x, y) \cap \operatorname{co}(\{p_1, \ldots, p_n\}) \neq \emptyset$.*

The last claim is not true for arbitrary gauges, as the following example demonstrates.

Example 6.27. For $X = \mathbb{R}^2$, a gauge γ is uniquely determined by its unit sphere

$$S(0,1) := \{(\xi_1, \xi_2) \in X \mid \xi_1^2 + (\xi_2 - 4)^2 = 5^2\}.$$

Then $x := (0,1)$ and $y := (0,7)$ belong to the level set $f_{=2}$ of the function $f := \gamma(\cdot - p_1) + \gamma(\cdot - p_2) : X \to \mathbb{R}$ with $p_1 := (-4,0)$ and $p_2 := (4,0)$. On the other hand,

$$\mathrm{bsc}_\gamma(x,y) = \{z \in X \mid \gamma(z-x) = \gamma(z-y)\} = \bigcup_{\lambda>0} (S(x,\lambda) \cap S(y,\lambda))$$

consists of points from intersections of pairs of distinct Euclidean circles of the same radius whose Euclidean centers are of the form $(0, \xi_2)$ with $\xi_2 \geq 1$. This shows that the second coordinates of all points from $\mathrm{bsc}_\gamma(x,y)$ are larger than 1. Thus

$$\mathrm{bsc}_\gamma(x,y) \cap \mathrm{co}(\{p_1,p_2\}) = \mathrm{bsc}_\gamma(x,y) \cap [p_1,p_2] = \emptyset,$$

see Figure 6.3.

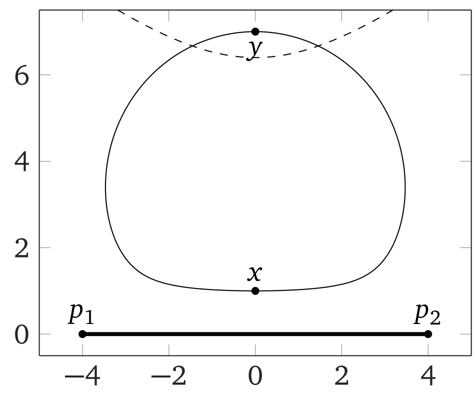


Figure 6.3. Illustration of Example 6.27: The bisector (dashed line) of two points of the same ellipsoid (thin solid line) need not intersect the convex hull (bold solid line) of the foci of the ellipsoid.

Complementing the results on rotundity in terms of the Fermat–Torricelli locus and metrical segments, see Propositions 6.12 and 6.21 again, we give a characterization in terms of ellipses and ellipsoids.

Theorem 6.28. *Let (X,γ) be a generalized Minkowski plane. The following statements are equivalent.*

(a) The gauge γ is not rotund.

(b) There exist points $p_1, \dots, p_n \in X$ such that some level set of the function $f := \sum_{i=1}^n \gamma(\cdot - p_i) : X \to \mathbb{R}$ contains a non-singleton line segment $[z_1, z_2]$ satisfying $\{p_1, \dots, p_n\} \not\subset \langle z_1, z_2 \rangle$.

(c) For any family of $n \geq 1$ points $p_1, \dots, p_n \in X$, there exists a number $\alpha_0 \in \mathbb{R}$ such that for every $\alpha > \alpha_0$, the level set $f_{=\alpha}$ of the function $f := \sum_{i=1}^n \gamma(\cdot - p_i) : X \to \mathbb{R}$, contains a non-singleton line segment.

Proof. For (a)⇒(c), assume that γ is not rotund. Then there exists a linear functional $\phi \in X^*$ with $\gamma^\circ(\phi) = 1$ such that $\phi_{=1} \cap S(0,1)$ is not a singleton. Then $\bigcap_{i=1}^n C_\gamma(p_i, \phi)$ is a cone with non-empty interior, and the restriction of f to this cone is an affine function. To see (c)⇒(b), consider $n := 1$, $p_1 := 0$ and choose a number $\alpha > \alpha_0$. Then $f_{=\alpha}$ contains a non-singleton line segment $[z_1, z_2]$. Since $f_{=\alpha} = S(p_1, \alpha)$, we have $p_1 \notin \langle z_1, z_2 \rangle$. It remains to show the implication (b)⇒(a). By (b), there are distinct points $z_1, z_2 \in X$ and a number $\alpha > 0$ such that $[z_1, z_2] \subset f_{=\alpha}$ and $\{p_1, \ldots, p_n\} \not\subset \langle z_1, z_2 \rangle$. After suitably shortening the line segment, we may assume that $z_1, z_2 \notin \{p_1, \ldots, p_n\}$. For $\lambda \in [0,1]$, we obtain $f(\lambda z_1 + (1-\lambda) z_2) = \alpha = \lambda f(z_1) + (1-\lambda) f(z_2)$. Due to the triangle inequality, we have $\gamma(\lambda z_1 + (1-\lambda) z_2 - p_i) = \lambda \gamma(z_1 - p_i) + (1-\lambda)\gamma(z_2 - p_i)$ for $i \in \{1, \ldots, n\}$. Taking Lemma 2.22 into account, we obtain $\left[\frac{z_1 - p_i}{\gamma(z_1 - p_i)}, \frac{z_2 - p_i}{\gamma(z_2 - p_i)}\right] \subset S(0,1)$ for $i \in \{1, \ldots, n\}$. But we cannot have $\frac{z_1 - p_i}{\gamma(z_1 - p_i)} = \frac{z_2 - p_i}{\gamma(z_2 - p_i)}$ for all $i \in \{1, \ldots, n\}$, since otherwise $z_1 - p_i$ and $z_2 - p_i$ would be linearly dependent and $\{p_1, \ldots, p_n\} \in \langle z_1, z_2 \rangle$. Therefore, at least one of the line segments $\left[\frac{z_1 - p_i}{\gamma(z_1 - p_i)}, \frac{z_2 - p_i}{\gamma(z_2 - p_i)}\right]$ is not a singleton and γ is not rotund. □

The above observation gives rise to a characterization of rotundity by a property of bifocal ellipsoids.

Corollary 6.29. *The generalized Minkowski space (X, γ) is not rotund if and only if there exist distinct points $p_1, p_2 \in X$ such that some level set of the function $f := \gamma(\cdot - p_1) + \gamma(\cdot - p_2) : X \to \mathbb{R}$ contains a non-singleton line segment $[z_1, z_2]$ satisfying $\langle p_1, p_2 \rangle \neq \langle z_1, z_2 \rangle$.*

Proof. If γ is not rotund, then there exists a two-dimensional subspace $L \subset X$ such that $\gamma|_L$ is not rotund. We apply (a)⇒(c) from Theorem 6.28. Then there exist two points $p_1, p_2 \in L$ such that the corresponding function f has several level sets that contain non-singleton line segments. Since the restriction $f|_{\langle p_1, p_2 \rangle}$ is convex, at most one level set of f contains a non-singleton line segment of $\langle p_1, p_2 \rangle$. Thus, there is another level set satisfying the claim of the corollary. The proof of the converse implication follows the lines of the proof of (b)⇒(a) from Theorem 6.28. □

7 Cassini sets

The multiplicative combination of the Euclidean distances to two fixed points in the plane yields a class of curves which are called Cassini curves or Cassini ovals, honoring the Italian astronomer Giovanni Domenico Cassini. According to his son Jacques Cassini (also an astronomer), he proposed these curves as trajectories for the relative motion of Earth and Sun, see [47, pp. 149–151]. Despite the failure of Cassini's proposal, his curves have applications in science and engineering [135], e.g., in biology [67] and optics [102, 150]. Various mathematical aspects of Cassini curves have been elaborated mostly in the 20th century. For instance, Cassini curves are special planar sections of tori [143, p. 126]. Topics like tangents, normals, curvature, and rectification of Cassini curves are covered in [143, pp. 208–214]. The most prominent member of the Cassini curves, the lemniscate of Bernoulli, became eponymous for a frequently pursued generalization of Cassini curves: polynomial lemniscates. These are the multifocal analogs in the Euclidean plane, that is, sets of the form

$$\left\{ x \in \mathbb{R}^2 \;\middle|\; \prod_{i=1}^{n} \|x - p_i\|_2 = \alpha \right\},$$

where $p_1, \dots, p_n \in \mathbb{R}^2$ are given points (the *foci*), $\alpha \in \mathbb{R}$ is a given number, and $\|\cdot\|_2$ is the Euclidean norm. The defining equation of these curves can be readily transformed into a polynomial one. Contributions to their investigation regarding singularities, number and convexity of components, curvature, orthogonal trajectories can be found in [211, 212, 222, 224, 236, 237]. Due to the large variety of shapes that polynomial lemniscates and their analogs in higher dimensions can take, these curves and surfaces are attractive also for the approximation of curves [138, 200, 201] or computer-aided geometric design [13, 187, 193]. While the multifocality approach has already been extended to three-dimensional Euclidean space by Szőkefalvi-Nagy [225, 226], the study of non-Euclidean generalizations of Cassini curves is much younger. The concept of bifocal Cassini curves has been translated to the setting of the Minkowskian space-time plane [215]. However, the multifocal extension to finite-dimensional normed spaces proposed in [95] and the bifocal version in two-dimensional normed spaces in [157] are closer to the topic of the present chapter, in which we follow the presentation of [123, Section 5]. More precisely, we consider level sets $f_{=\alpha}$ and sublevel sets $f_{\leq\alpha}$ of functions of the form

$$f := \prod_{i=1}^{n} \gamma_i(\cdot - p_i) : X \to \mathbb{R}$$

where $\gamma_1, \dots, \gamma_n : X \to \mathbb{R}$ are gauges, $p_1, \dots, p_n \in X$ are fixed points (which we shall call *foci*), and $\alpha \in \mathbb{R}$ is a number. In the following two sections, we study these *Cassini level and sublevel sets* in view of their starshapedness and their number of connected components.

7.1 Starshapedness

First, we show that Cassini sublevel sets are star-shaped for large parameters (thus connected) and that they split into finitely many star-shaped components for small parameters. The former claim is a consequence of coercivity, which under a convexity assumption coincides with the boundedness of the set of minimizers, see also [202, Corollary 8.7.1].

Lemma 7.1. *Let (X,γ) be a generalized Minkowski space, and let $f : X \to \mathbb{R}$ be a convex function such that the set of minimizers of f is non-empty and bounded. Then the following statements are true.*

(a) For all $\alpha \in \mathbb{R}$, the sublevel set $f_{\leq\alpha}$ is bounded.
(b) There exists a number $\mu \geq 0$ such that $f(\lambda_1 x) < f(\lambda_2 x)$ whenever $x \in X \setminus B(0,\mu)$ and $1 \leq \lambda_1 < \lambda_2$.

Proof. For (a), assume that $f_{\leq\alpha_0}$ is unbounded for some number $\alpha_0 \in \mathbb{R}$, and (without loss of generality) that 0 is a minimizer of f. Then there are points $x_i \in f_{\leq\alpha_0}$ such that $\gamma(x_i) \geq i$ for $i \in \mathbb{N}$. By compactness of $S(0,1)$, we may assume that $\left(\frac{x_i}{\gamma(x_i)}\right)_{i\in\mathbb{N}}$ is a convergent sequence whose limit shall be denoted by $x_0 \in S(0,1)$. Then the ray $\{\lambda x_0 \mid \lambda \geq 1\}$ is a subset of $f_{\leq f(0)}$, i.e., the set of minimizers of f, because

$$\begin{aligned}
f(\lambda x_0) &= \lim_{i\to+\infty} f\left(\lambda\frac{x_i}{\gamma(x_i)}\right)\\
&= \lim_{i\to+\infty}\left(f\left(\left(\frac{\lambda}{\gamma(x_i)}x_i\right)+\left(1-\frac{\lambda}{\gamma(x_i)}\right)0\right)\right)\\
&\leq \lim_{i\to+\infty}\left(\frac{\lambda}{\gamma(x_i)}f(x_i)+\left(1-\frac{\lambda}{\gamma(x_i)}\right)f(0)\right)\\
&\leq \lim_{i\to+\infty}\frac{\lambda}{\gamma(x_i)}\alpha_0+\lim_{i\to+\infty}\left(1-\frac{\lambda}{\gamma(x_i)}\right)f(0)\\
&= f(0).
\end{aligned}$$

To see (b), assume that, contrary to our claim, there are points $y_i \in X$ and numbers $\lambda_{1,i}, \lambda_{2,i} \in \mathbb{R}$ for all $i \in \mathbb{N}$ such that $\gamma(y_i) > i$, $1 \leq \lambda_{1,i} < \lambda_{2,i}$, and $f(\lambda_{1,i}y_i) \geq f(\lambda_{2,i}y_i)$ for all $i \in \mathbb{N}$. Then the function

$$\mathbb{R} \ni \lambda \mapsto f\left(\lambda\frac{y_i}{\gamma(y_i)}\right)$$

is non-increasing on the interval $(-\infty, \lambda_{1,i}\gamma(y_i)] \supset (-\infty, i]$ because it is a convex function. As above, we may assume that $\left(\frac{y_i}{\gamma(y_i)}\right)_{i\in\mathbb{N}}$ is a convergent sequence whose limit shall be denoted by $y_0 \in S(0,1)$. The convex function

$$\mathbb{R} \ni \lambda \mapsto f(\lambda y_0)$$

is also non-increasing since, for $\lambda_1 < \lambda_2$, we have

$$f(\lambda_1 y_0) - f(\lambda_2 y_0) = \lim_{i\to+\infty}\underbrace{\left(f\left(\lambda_1\frac{y_i}{\gamma(y_i)}\right)-f\left(\lambda_2\frac{y_i}{\gamma(y_i)}\right)\right)}_{\geq 0 \text{ for } \lambda_2 < i} \geq 0.$$

Therefore $f(\lambda y_0) \leq f(0y_0) = f(0)$ for $\lambda \geq 0$, which implies that $f_{\leq f(0)}$ is unbounded. This contradicts (a). $\square$

From this, we are able to derive a result on the starshapedness of sublevel sets of the pointwise product of coercive functions.

Lemma 7.2. *Let (X,γ) be a generalized Minkowski space, and let $f_1,\ldots,f_n : X \to \mathbb{R}$ be convex and coercive functions. Define $f := \prod_{i=1}^n f_i : X \to \mathbb{R}$. Then there exists a number $\alpha_0 \in \mathbb{R}$ such that the sublevel set $f_{\leq\alpha}$ is star-shaped with respect to $x_0 = 0$ for every number $\alpha \geq \alpha_0$.*

Proof. By Lemma 7.1, there exist numbers $\mu_1,\mu_2 > 0$ such that $\bigcup_{i=1}^n (f_i)_{\leq 0} \subset B_\gamma(0,\mu_1)$ and the function $\mathbb{R} \ni \lambda \mapsto f_i(\lambda x)$, is monotonically increasing on $[1,\infty)$ for all $i \in \{1,\ldots,n\}$ and $x \in X \setminus B_\gamma(0,\mu_2)$. Define $\mu_0 := \max\{\mu_1,\mu_2\}$ and $\alpha_0 := \max\{f(x) \mid x \in B_\gamma(0,\mu_0)\}$. Let $\alpha \geq \alpha_0$. We will show that for all $x \in f_{\leq\alpha}$ and $\lambda \in [0,1]$, we have $\lambda x \in f_{\leq\alpha}$. First, assume that $\lambda x \in B_\gamma(0,\mu_0)$. Then $\lambda x \in B_\gamma(0,\mu_0) \subset f_{\leq\alpha_0} \subset f_{\leq\alpha}$. Else, we have $\lambda x \in X \setminus B_\gamma(0,\mu_0)$. By choice of μ_0, we have $f_i(\lambda x) > 0$ and $f_i(\lambda x) = f_i(1\lambda x) \leq f_i(\frac{1}{\lambda}\lambda x) = f_i(x)$ for all $i \in \{1,\ldots,n\}$. It follows that

$$f(\lambda x) = \prod_{i=1}^n f_i(\lambda x) \leq \prod_{i=1}^n f_i(x) = f(x) \leq \alpha,$$

and hence $\lambda x \in f_{\leq\alpha}$. This completes the proof. $\square$

Applied to the pointwise product of gauges, Lemma 7.2 yields the starshapedness of n-focal Cassini sublevel sets.

Theorem 7.3. *Let (X,γ) be a generalized Minkowski space and $p_1,\ldots,p_n,x_0 \in X$. Furthermore, let $\gamma_1,\ldots,\gamma_n : X \to \mathbb{R}$ be gauges and define $f := \prod_{i=1}^n \gamma_i(\cdot - p_i) : X \to \mathbb{R}$. Then there exists a number $\alpha_0 \in \mathbb{R}$ such that the sublevel set $f_{\leq\alpha}$ is star-shaped with respect to x_0 for all $\alpha \geq \alpha_0$.*

Proof. Apply Lemma 7.2 to the function $g := \prod_{i=1}^n \gamma_i(\cdot + x_0 - p_i) : X \to \mathbb{R}$ and obtain the starshapedness of its sublevel sets $g_{\leq\alpha}$ with respect to the point 0 for sufficiently large $\alpha \in \mathbb{R}$. Observe that $f = g(\cdot - x_0)$, and thus $f_{\leq\alpha} = g_{\leq\alpha} + x_0$. $\square$

In normed spaces, bifocal Cassini sublevel sets can be decomposed into two star-shaped sets and, therefore, have at most two connected components, cf. [157, Theorem 8].

Lemma 7.4. *Let $(X,\|\cdot\|)$ be a Minkowski space, $p_1,p_2 \in X$, and $f := \|\cdot - p_1\|\,\|\cdot - p_2\| : X \to \mathbb{R}$. Furthermore, let $V_{p_1,P}$ and $V_{p_2,P}$ be the Voronoi cells associated to $P = \{p_1,p_2\}$ as in Corollary 5.27. Then, for $\alpha \geq 0$ and $i \in \{1,2\}$, the set $f_{\leq\alpha} \cap V_{p_i,P}$ is star-shaped with respect to p_i.*

Proof. Without loss of generality, let $i = 1$. Furthermore, let $x \in V_{p_1,P}$ and $\lambda \in [0,1]$. We have

$$\begin{aligned}
f(\lambda x + (1-\lambda)p_1) &= \|\lambda x + (1-\lambda)p_1 - p_1\|\,\|\lambda x + (1-\lambda)p_1 - p_2\| \\
&= \lambda\,\|x - p_1\|\,\|x - p_2 - (1-\lambda)(x - p_1)\| \\
&\leq \lambda\,\|x - p_1\|\,(\|x - p_2\| + (1-\lambda)\,\|x - p_1\|) \\
&\leq \lambda\,\|x - p_1\|\,(\|x - p_2\| + (1-\lambda)\,\|x - p_2\|) \\
&= \lambda(2-\lambda)\,\|x - p_1\|\,\|x - p_2\|
\end{aligned}$$

$$\leq f(x).$$

Thus, we have $\lambda x + (1-\lambda)p_1 \in f_{\leq f(x)}$. □

Lemma 7.2 shows that Cassini sublevel sets are star-shaped for large levels. For small levels like zero, this is clearly wrong because those Cassini sublevel sets are not even connected. However, connectedness does not imply starshapedness, see [157, Remark 15]. The next theorem refers to starshapedness of n-focal Cassini sublevel sets for small levels.

Theorem 7.5. *Let (X,γ) be a generalized Minkowski space, $p_1,\ldots,p_n \in X$, and let $\gamma_1,\ldots,\gamma_n : X \to \mathbb{R}$ be gauges. Define $f := \prod_{i=1}^n \gamma_i(\cdot - p_i) : X \to \mathbb{R}$ and*

$$\mu_i := \frac{1}{n}\min\left\{ \frac{\gamma_i(z)}{\gamma_j(-z)}\gamma_j(p_i-p_j) \,\middle|\, z \in X\setminus\{0\}, j \in \{1,\ldots,n\}\setminus\{i\} \right\}$$

for $i \in \{1,\ldots,n\}$. Then, for every $\alpha \geq 0$ and $i \in \{1,\ldots,n\}$, the set $f_{\leq\alpha} \cap B_{\gamma_i}(p_i,\mu_i)$ is star-shaped with respect to p_i, and the open balls $U_{\gamma_i}(p_i,\mu_i)$, $i \in \{1,\ldots,n\}$, are mutually disjoint. Moreover, if $\alpha \in \left[0, \prod_{i=1}^n \mu_i\right)$, then $f_{\leq\alpha} \subset \bigcup_{i=1}^n U_{\gamma_i}(p_i,\mu_i)$, so that $f_{\leq\alpha}$ splits into exactly n star-shaped connected components $f_{\leq\alpha} \cap U_{\gamma_i}(p_i,\mu_i)$, $i \in \{1,\ldots,n\}$.

Note that μ_i is well-defined and positive, because for every $j \in \{1,\ldots,n\}\setminus\{i\}$,

$$\inf\left\{ \frac{\gamma_i(z)}{\gamma_j(-z)} \,\middle|\, z \in X\setminus\{0\} \right\} = \inf\left\{ \gamma_i(z) \,\middle|\, z \in S_{\gamma_j^\vee}(0,1) \right\} > 0$$

is the infimum of a continuous function relative to the compact set $S_{\gamma_j^\vee}(0,1)$ on which the function is bounded below by a positive number.

Proof. Step 1: The set $f_{\leq\alpha} \cap B_{\gamma_i}(p_i,\mu_i)$ is star-shaped with respect to p_i. It is sufficient to show that $f(\lambda x + (1-\lambda)p_i) < f(x)$ for all $x \in B_{\gamma_i}(p_i,\mu_i)\setminus\{p_i\}$ and $\lambda \in (0,1)$. Fix $x \in B_{\gamma_i}(p_i,\mu_i)\setminus\{p_i\}$, $\lambda \in (0,1)$, and $j \in \{1,\ldots,n\}\setminus\{i\}$. The inclusion $x \in B_{\gamma_i}(p_i,\mu_i)$ implies

$$\gamma_i(x-p_i) \leq \mu_i \leq \frac{1}{n}\frac{\gamma_i(x-p_i)}{\gamma_j(p_i-x)}\gamma_j(p_i-p_j).$$

We estimate

$$\begin{aligned}
&\gamma_j((\lambda x + (1-\lambda)p_i) - p_j) \\
&= \gamma_j((1-\lambda)(p_i - x) + (x-p_j)) \\
&\leq (1-\lambda)\gamma_j(p_i-x) + \gamma_j(x-p_j) \\
&= \left(\lambda^{-\frac{1}{n-1}} - 1\right)\left(\lambda^{\frac{1}{n-1}} + \lambda^{\frac{2}{n-1}} + \ldots + \lambda^{\frac{n-1}{n-1}}\right)\gamma_j(p_i-x) + \gamma_j(x-p_j) \\
&< \left(\lambda^{-\frac{1}{n-1}} - 1\right)(n-1)\gamma_j(p_i-x) + \gamma_j(x-p_j) \\
&= \left(\lambda^{-\frac{1}{n-1}} - 1\right)\left(n\frac{\gamma_j(p_i-x)}{\gamma_i(x-p_i)}\gamma_i(x-p_i) - \gamma_j(p_i-x)\right) + \gamma_j(x-p_j)
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\lambda^{-\frac{1}{n-1}}-1\right)\left(n\frac{\gamma_j(p_i-x)}{\gamma_i(x-p_i)}\mu_i-\gamma_j(p_i-x)\right)+\gamma_j(x-p_j)\\
&\leq \left(\lambda^{-\frac{1}{n-1}}-1\right)\left(\gamma_j(p_i-p_j)-\gamma_j(p_i-x)\right)+\gamma_j(x-p_j)\\
&\leq \left(\lambda^{-\frac{1}{n-1}}-1\right)\gamma_j(x-p_j)+\gamma_j(x-p_j)\\
&= \lambda^{-\frac{1}{n-1}}\gamma_j(x-p_j)
\end{aligned}$$

and conclude

$$\begin{aligned}
&f(\lambda x+(1-\lambda)p_i)\\
&=\gamma_i((\lambda x+(1-\lambda)p_i)-p_i)\prod_{j\in\{1,\dots,n\}\setminus\{i\}}\gamma_j((\lambda x+(1-\lambda)p_i)-p_j)\\
&<\lambda\gamma_i(x-p_i)\prod_{j\in\{1,\dots,n\}\setminus\{i\}}\lambda^{-\frac{1}{n-1}}\gamma_j(x-p_j)\\
&=\lambda\lambda^{-1}\prod_{j=1}^{n}\gamma_j(x-p_j)\\
&=f(x).
\end{aligned}$$

This finishes Step 1.
Step 2: We have $U_{\gamma_i}(p_i,\mu_i)\cap U_{\gamma_j}(p_j,\mu_j)=\emptyset$ *for* $1\leq i<j\leq n$. Assume that, contrary to our claim, there exists a point $x\in U_{\gamma_i}(p_i,\mu_i)\cap U_{\gamma_j}(p_j,\mu_j)$, i.e., $\gamma_i(x-p_i)<\mu_i$ and $\gamma_j(x-p_j)<\mu_j$. Note that $x\neq p_j$ because otherwise we would have

$$\gamma_i(p_j-p_i)<\mu_i\leq\frac{1}{n}\frac{\gamma_i(p_j-p_i)}{\gamma_j(p_i-p_j)}\gamma_j(p_i-p_j)=\frac{1}{n}\gamma_i(p_j-p_i),$$

which is impossible. We obtain

$$\begin{aligned}
&\gamma_i(p_j-p_i)\\
&\leq\gamma_i(p_j-x)+\gamma_i(x-p_i)\\
&=\gamma_j(x-p_j)\frac{\gamma_i(p_j-x)}{\gamma_j(x-p_j)}+\gamma_i(x-p_i)\\
&<\mu_j\frac{\gamma_i(p_j-x)}{\gamma_j(x-p_j)}+\mu_i\\
&\leq\left(\frac{1}{n}\frac{\gamma_j(x-p_j)}{\gamma_i(p_j-x)}\gamma_i(p_j-p_i)\right)\frac{\gamma_i(p_j-x)}{\gamma_j(x-p_j)}+\frac{1}{n}\frac{\gamma_i(p_j-p_i)}{\gamma_j(p_i-p_j)}\gamma_j(p_i-p_j)\\
&=\frac{2}{n}\gamma_i(p_j-p_i).
\end{aligned}$$

This contradiction completes Step 2.
Step 3: For $\alpha\in\left[0,\prod_{i=1}^n\mu_i\right)$, *we have* $f_{\leq\alpha}\subset\bigcup_{i=1}^n U_{\gamma_i}(p_i,\mu_i)$. Choose a point $x\in X$ with $x\notin\bigcup_{i=1}^n U_{\gamma_i}(p_i,\mu_i)$. Then $\gamma_i(x-p_i)\geq\mu_i$ for all $i\in\{1,\dots,n\}$, and thus $f(x)\geq\prod_{i=1}^n\mu_i>\alpha$ and $x\notin f_{\leq\alpha}$. This completes the proof. □

In particular, the last part of Theorem 7.5 shows that for small levels $\alpha \geq 0$, Cassini sublevel sets consist of as many components as there are foci. Another result on unions of balls containing Cassini (sub)level sets in the Euclidean plane can be found in [222, Section 4]. The following example illustrates a bifocal situation of Theorem 7.5 for a specific gauge.

Example 7.6. Consider $\gamma : X \to \mathbb{R}$ on $X = \mathbb{R}^2$ defined by

$$\gamma(\xi_1, \xi_2) = \max\left\{-\xi_1, \frac{2}{3}\xi_1, -\frac{2}{3}\xi_2, -2\xi_1 + 2\xi_2, 2\xi_1 + 2\xi_2\right\}.$$

Figure 7.1 depicts level sets of the function $f := \gamma(\cdot - p_1)\gamma(\cdot - p_2) : X \to \mathbb{R}$ where $p_1 := (-1, 0)$ and $p_2 := (1, 0)$. We obtain $\mu_1 = \mu_2 = \frac{1}{3}$, and Theorem 7.5 yields a decomposition of $f_{\leq\alpha}$ into $n := 2$ star-shaped components if $f_{\leq\alpha} \subset U(p_1, \mu_1) \cup U(p_2, \mu_2)$. This leads to the sharp inequality $\alpha < \frac{96}{169}$ here, the latter number being the value of f at its local minimizer $(\frac{5}{13}, -\frac{12}{13})$.

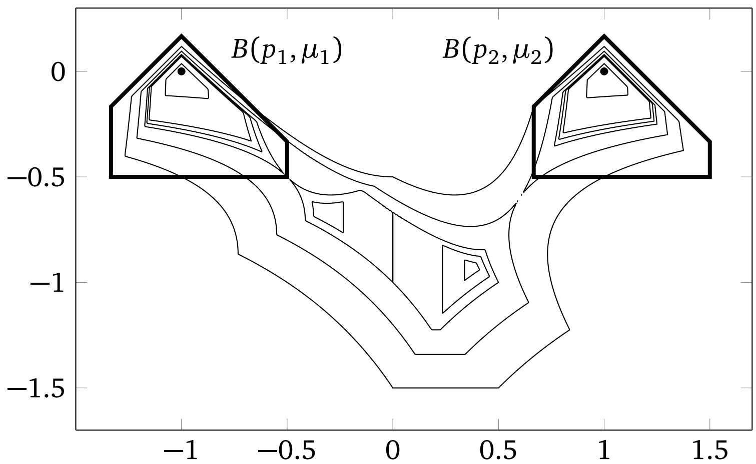


Figure 7.1. Illustration of Theorem 7.5: Sublevel sets of $f := \gamma(\cdot - p_1)\gamma(\cdot - p_2) : X \to \mathbb{R}$ which are contained in $U(p_1, \mu_1) \cup U(p_2, \mu_2)$ consist of two star-shaped components. Here $p_1 := (-1, 0)$, $p_2 := (1, 0)$, and $\mu_1 = \mu_2 = \frac{1}{3}$.

Lemma 7.4 and Theorem 7.5 partially answer the following problem.

Problem 7.7. *Let (X, γ) be a generalized Minkowski space and $p_1, \ldots, p_n, x \in X$. Does there exist a local minimizer $x_0 \in X$ of the function $f := \prod_{i=1}^{n} \gamma(\cdot - p_i) : X \to \mathbb{R}$ such that $[x, x_0] \subset f_{\leq f(x)}$?*

Rotundity of gauges impacts the objects and concepts of our interest in all preceding chapters. This includes best approximations (Proposition 2.25), circumcenters (Theorem 3.8), ball convexity (Proposition 4.16), hyperboloids (Lemma 5.40), apollonoids (Lemma 5.41), and ellipsoids (Theorem 6.28). Following this principle, let us consider Cassini level sets in non-rotund generalized Minkowski planes.

Proposition 7.8. *Let (X, γ) be a non-rotund generalized Minkowski plane and $p_1, \ldots, p_n \in X$. Then there is a level set of the function $f := \prod_{i=1}^{n} \gamma(\cdot - p_i) : X \to \mathbb{R}$ which contains a non-singleton line segment.*

Proof. If γ is not rotund, there exists a linear functional $\phi \in X^*$ with $\gamma^\circ(\phi) = 1$ such that $S(0,1) \cap \phi_{=1}$ is not a singleton. In particular, the restriction of the function $\gamma(\cdot - p_i) : X \to \mathbb{R}$ to sets of the form $\left(\bigcap_{i=1}^n C_\gamma(p_i,\phi)\right) \cap \phi_{=\alpha}$ with $\alpha \in \mathbb{R}$ is constant. Thus, the same applies to the function f. □

7.2 Connectedness

Apart from the cases discussed in Section 7.1, the connectedness structure of Cassini (sub)level sets can be surprisingly exotic, even in the bifocal planar case.

Example 7.9. Consider the increasing concave function

$$g : [-2,4] \to [0,4], \quad g(\alpha) := \begin{cases} \alpha + 2 & \text{if } \alpha \in [-2,1], \\ 4\sqrt{\alpha} - \alpha & \text{if } \alpha \in [1,4]. \end{cases}$$

Let F be a closed set satisfying $\{1,4\} \subset F \subset [1,4]$, define

$$B_F := \operatorname{co}(\{(\alpha, \pm g(\alpha)) \in \mathbb{R}^2 \mid \alpha \in [-2,1]\} \cup F\}),$$

and let $\gamma^F : \mathbb{R}^2 \to \mathbb{R}$ be the gauge with unit ball B_F. For $p_1 := (0,4)$ and $p_2 := (0,-4)$, consider the function $f^F := \gamma^F(\cdot - p_1)\gamma^F(\cdot - p_2) : X \to \mathbb{R}$. Then $f^F_{\leq 1} \cap ((2,\infty) \times \mathbb{R}) = f^F_{=1} \cap ((2,\infty) \times \mathbb{R}) = \{(2\sqrt{\alpha}, \pm(4 - 2\sqrt{\alpha})) \mid \alpha \in F\}$. In particular, $f^F_{=1} \cap ((2,\infty) \times \mathbb{R})$ consists of local minimizers of f^F. Figure 7.2 illustrates an example. There are some interesting special cases of this construction:

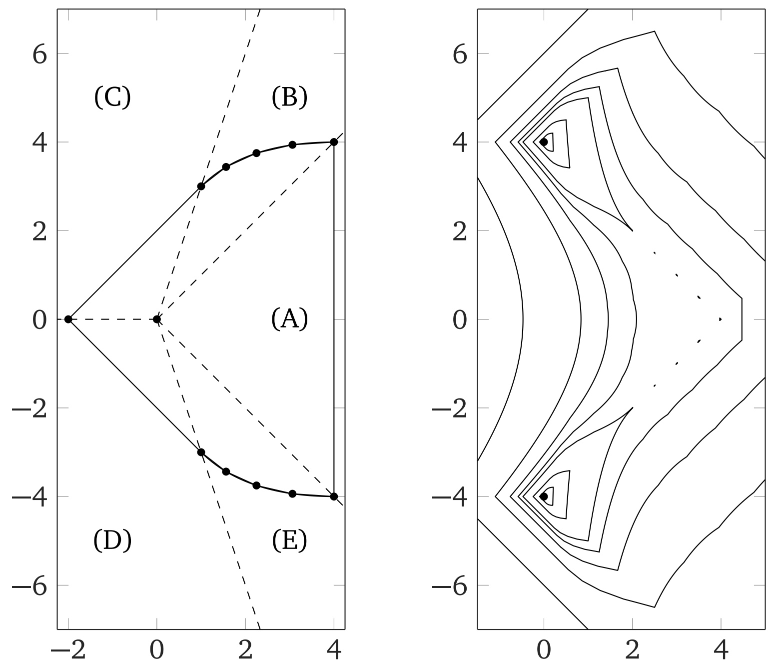
(a) For $F := [1,4]$, the set $f^F_{\leq 1}$ is connected but not star-shaped.
(b) F may have uncountably many connected components. For example, this is the case when F is the classical Cantor set scaled to $[1,4]$. Then $f^F_{\leq 1} \cap ((2,\infty) \times \mathbb{R})$ splits into uncountably many connected components, which are singletons.

Proof. Step 1: We consider $F := [1,4]$. We shall write γ for $\gamma^{[1,4]}$. Let $(\xi_1,\xi_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Depending on the position of $\frac{(\xi_1,\xi_2)}{\gamma(\xi_1,\xi_2)}$ in $S_\gamma(0,1)$, we obtain

$$\gamma(\xi_1,\xi_2) = \begin{cases} \frac{\xi_1}{4} & \text{if } -\xi_1 \leq \xi_2 \leq \xi_1 & \text{(type A)}, \\ \frac{(\xi_1+\xi_2)^2}{16\xi_1} & \text{if } \xi_1 \leq \xi_2 \leq 3\xi_1 & \text{(type B)}, \\ \frac{-\xi_1+\xi_2}{2} & \text{if } \max\{0, 3\xi_1\} \leq \xi_2 & \text{(type C)}, \\ \frac{-\xi_1-\xi_2}{2} & \text{if } \xi_2 \leq \min\{0, -3\xi_1\} & \text{(type D)}, \\ \frac{(\xi_1-\xi_2)^2}{16\xi_1} & \text{if } -3\xi_1 \leq \xi_2 \leq -\xi_1 & \text{(type E)}, \end{cases}$$

see Figure 7.2(a). We have to show that the function $f := f^{[1,4]}$ satisfies

$$f(\xi_1,\xi_2) \begin{cases} > 1 & \text{if } \xi_1 > 2 \text{ and } |\xi_2| \neq 4 - \xi_1, \\ = 1 & \text{if } \xi_1 > 2 \text{ and } |\xi_2| = 4 - \xi_1. \end{cases}$$



(a) The sets B_F and $B_{[1,4]}$. The reader should notice that the two sets differ only slightly.

(b) Level sets of the function f^F.

Figure 7.2. Illustration of Example 7.9 with $F := \left\{1, \frac{25}{16}, \frac{9}{4}, \frac{49}{16}, 4\right\}$.

Both $B_{[1,4]}$ and $\{p_1, p_2\}$ are mirror-symmetric with respect to the straight line $\{(\xi_1, 0) \mid \xi_1 \in \mathbb{R}\}$. Thus, we may assume that $\xi_2 \geq 0$. Fix $(\xi_1, \xi_2) \in (2, \infty) \times [0, \infty)$. The following cases cover the whole situation.

Case 1: $\xi_2 \geq 4$. We have $\gamma((\xi_1, \xi_2) - p_1) = \gamma(\xi_1, \xi_2 - 4) > \frac{1}{2}$ because $\xi_1 > 2$, and we have $\gamma((\xi_1, \xi_2) - p_2) = \gamma(\xi_1, \xi_2 + 4) \geq 2$ because $\xi_2 + 4 \geq 8$. Thus $f(\xi_1, \xi_2) > 1$.

Case 2: $\xi_2 < \xi_1 - 4$. Then $\xi_1 > 4$ and $f(\xi_1, \xi_2) = \gamma(\xi_1, \xi_2 - 4)\gamma(\xi_1, \xi_2 + 4) = \frac{\xi_1}{4}\frac{\xi_1}{4} > 1$ (both lengths are of type A).

Case 3: $3\xi_1 - 4 < \xi_2 < 4$. Now $\gamma(\xi_1, \xi_2 - 4) = \frac{\xi_1}{4}$ (type A) and $\gamma(\xi_1, \xi_2 + 4) = \frac{-\xi_1 + (\xi_2 + 4)}{2}$ (type C). Hence, $f(\xi_1, \xi_2) = \frac{\xi_1(-\xi_1 + \xi_2 + 4)}{8} > \frac{\xi_1(-\xi_1 + (3\xi_1 - 4) + 4)}{8} = \frac{\xi_1^2}{4} > 1$.

Case 4: $\xi_1 + \xi_2 < 4$. In this case $\gamma(\xi_1, \xi_2 - 4) = \frac{(\xi_1 - (\xi_2 - 4))^2}{16\xi_1}$ (type E) and $\gamma(\xi_1, \xi_2 + 4) = \frac{(\xi_1 + (\xi_2 + 4))^2}{16\xi_1}$ (type B). Thus,

$$f(\xi_1, \xi_2) = \left(1 + \frac{(\xi_1 + \xi_2 - 4)(\xi_1 - \xi_2 - 4)}{16\xi_1}\right)^2 > 1,$$

because $\xi_1 + \xi_2 - 4 < 0$ as well as $\xi_1 - \xi_2 - 4 < 0$.

Case 5: $\xi_1 - 4 \leq \xi_2 \leq 3\xi_1 - 4$, $\xi_1 + \xi_2 \geq 4$ *and* $\xi_2 < 4$. We obtain $\gamma(\xi_1, \xi_2 - 4) = \frac{\xi_1}{4}$ (type A)

and $\gamma(\xi_1,\xi_2+4)=\frac{(\xi_1+(\xi_2+4))^2}{16\xi_1}$ (type B). This gives $f(\xi_1,\xi_2)=\frac{(\xi_1+\xi_2+4)^2}{64}$. Hence $f(\xi_1,\xi_2)=1$ if $\xi_1+\xi_2=4$, and $f(\xi_1,\xi_2)>1$ if $\xi_1+\xi_2>4$.
Step 2: We consider arbitrary closed sets F with $\{1,4\}\subset F\subset[1,4]$. The inclusion $F\subset[1,4]$ implies $B_F\subset B_{[1,4]}$, $\gamma^F\geq\gamma^{[1,4]}$, and $f^F\geq f^{[1,4]}=f$. Hence, by Step 1, it is sufficient to show that for all $\alpha\in[1,4]$, we have $f^F(2\sqrt{\alpha},\pm(4-2\sqrt{\alpha}))=f(2\sqrt{\alpha},\pm(4-2\sqrt{\alpha}))$ if and only if $\alpha\in F$. By symmetry, it remains prove that $f^F(2\sqrt{\alpha},4-2\sqrt{\alpha})=f(2\sqrt{\alpha},4-2\sqrt{\alpha})$ if and only if $\alpha\in F$. As in Step 1, we obtain $\gamma^F(2\sqrt{\alpha},(4-2\sqrt{\alpha})-4)=\gamma(2\sqrt{\alpha},(4-2\sqrt{\alpha})-4)=\frac{2\sqrt{\alpha}}{4}=\frac{\sqrt{\alpha}}{2}$ (type A), and our claim is equivalent to $\gamma^F(2\sqrt{\alpha},(4-2\sqrt{\alpha})+4)=\gamma(2\sqrt{\alpha},(4-2\sqrt{\alpha})+4)$ if and only if $\alpha\in F$. The last equation is satisfied if and only if the ray

$$[(0,0),(2\sqrt{\alpha},(4-2\sqrt{\alpha})+4)\rangle=[(0,0),(\sqrt{\alpha},4-\sqrt{\alpha})\rangle$$

meets the boundaries of B_F and of $B_{[1,4]}$ at the same points. The intersection point with $\operatorname{bd}(B_{[1,4]})$ is $(\alpha,4\sqrt{\alpha}-\alpha)=(\alpha,g(\alpha))$ (see Step 1, type B). The same intersection is obtained with $\operatorname{bd}(B_F)$ if and only if $(\alpha,g(\alpha))\in\operatorname{bd}(B_F)$. By definition of B_F, this is equivalent to $\alpha\in F$. This finishes the proof. □

This qualitatively new behavior of Cassini sets for gauges (compared to the geometry of Cassini curves studied in [157]) gives rise to further problems.

Problem 7.10. *Let (X,γ) be a generalized Minkowski space, $p_1,\dots,p_n\in X$, and define $f:=\prod_{i=1}^n\gamma(\cdot-p_i):X\to\mathbb{R}$.*

(a) How large can level sets $f_{=\alpha}$ be (nowhere dense, Lebesgue measure zero, Hausdorff dimension)?
(b) How small can the sets

$$\{\alpha\in\mathbb{R}\mid f_{=\alpha}\text{ has infinitely many connected components}\},$$
$$\{\alpha\in\mathbb{R}\mid f_{=\alpha}\text{ is not a disjoint union of finitely many closed curves}\}\ (\text{if } d=2),$$
$$\{\alpha\in\mathbb{R}\mid f_{=\alpha}\text{ is not a union of finitely many closed curves}\}\ (\text{if } d=2)$$

be (countable, Lebesgue measure zero, meager)?

Of course, one may consider these problems for special cases, such as for $d=2$ or $n=2$.

In the bifocal case, level sets $f_{=\alpha}$ are nowhere dense, as the following theorem shows.

Theorem 7.11. *Let (X,γ) be a generalized Minkowski space, $p_1,p_2\in X$, $p_1\neq p_2$, and $f:=\gamma(\cdot-p_1)\gamma(\cdot-p_2):X\to\mathbb{R}$. Then, for each $\alpha\geq0$, the interior of the level set $f_{=\alpha}$ is empty.*

Proof. If $\alpha=0$, then $f_{=\alpha}=\{p_1,p_2\}$ and the claim is trivial. Let $x\in X\setminus\{p_1,p_2\}$ be an arbitrary point.
Case 1: Every neighborhood of x contains points of $X\setminus(B(p_1,\gamma(x-p_1))\cup B(p_2,\gamma(x-p_2)))$. That is, in every neighborhood of x, there is a point y with $f(y)=\gamma(y-p_1)\gamma(y-p_2)>\gamma(x-p_1)\gamma(x-p_2)=f(x)=:\alpha$. Therefore x is not an interior point of $f_{=\alpha}$.
Case 2: There is a neighborhood $U=U(x,\varepsilon)$ of x which does not contain any points of $X\setminus(B(p_1,\gamma(x-p_1))\cup B(p_2,\gamma(x-p_2)))$. In this case, we have $U\subset B(p_1,\gamma(x-p_1))\cup B(p_2,\gamma(x-p_2))$.

For $i \in \{1,2\}$, let H_i^+ be a closed half-space supporting $B(p_i, \gamma(x-p_i))$ at x. Then U is contained in the union $H_1^+ \cup H_2^+$ of half-spaces whose bounding hyperplanes meet at $x \in U$. It follows that $\mathrm{bd}(H_1^+) = \mathrm{bd}(H_2^+)$ is the unique supporting hyperplane of $B(p_i, \gamma(x-p_i))$ at x, and $H_i^+ \cap U = B(p_i, \gamma(x-p_i)) \cap U$ for $i \in \{1,2\}$. Hence x lies in the relative interiors of respective $(d-1)$-faces of these balls. Then the restrictions of $\gamma(\cdot - p_1)$ and $\gamma(\cdot - p_2)$ to every sufficiently small neighborhood V of x are non-constant affine functions, whose product f cannot be constant on V. $\square$

8 Outlook

Because of the stronger interest in norms, the theory of vector spaces equipped with a general gauge is less well developed and the existing literature is widely scattered. The present thesis is an attempt to present several facets of the theory of generalized Minkowski spaces in a consistent manner. A collection of future research directions is entailed by the selection of topics and by the nature of the transition from Minkowski spaces to generalized Minkowski spaces.

Not all techniques from Minkowski geometry work without the symmetry of the unit ball. In this case, we first have to generalize or alter the methods we use before we might be able to extend the results. Nonetheless, some of the techniques are independent of symmetry because they are a matter of general convexity. Sometimes, this results in bifurcations in the sense that coinciding mathematical objects in Minkowski spaces may be different from each other when we extend their definitions to generalized Minkowski spaces. These situations can be used to give characterizations of (special classes of) norms among gauges or, equivalently, central symmetry of (special) convex bodies. Also, there need not exist a unique way of extending mathematical objects from Minkowski geometry to generalized Minkowski spaces, or this ambiguity is inherited from the translation of Euclidean concepts to Minkowski spaces. Then, one may study the properties of alternative definitions for their own sake or derive characterizations of special generalized Minkowski spaces by drawing comparisons between the alternatives.

For instance, we may consider the binary relation on (X,γ) defined by $\gamma(x) \leq \gamma(x+\lambda y)+\varepsilon\gamma(x)$ for all $\lambda \in \mathbb{R}$ as an alternation of the notion of ε-Birkhoff orthogonality introduced in Chapter 2. In contrast to Definition 2.1, this definition in the spirit of [68] is left-homogeneous. Chmieliński [51] discusses the two approximate orthogonality relations on a Minkowski space $(X,\|\cdot\|)$ which are defined via $\|x+\lambda y\|^2 \geq \|x\|^2 - 2\varepsilon\|x\|\|\lambda y\|$ and $\|x+\lambda y\| \geq \sqrt{1-\varepsilon^2}\|x\|$ for all $\lambda \in \mathbb{R}$, respectively, the latter one being a reparametrization of Dragomir's condition in [68]. Both relations are left-homogeneous and right-homogeneous in Minkowski spaces. In inner-product spaces, the condition $\|x+\lambda y\| \geq \sqrt{1-\varepsilon^2}\|x\|$ for all $\lambda \in \mathbb{R}$ is equivalent to $|\langle y \,|\, x\rangle| \leq \varepsilon\|x\|\|y\|$. Due to the close relationship between orthogonality and the Cauchy–Schwarz inequality (see Lemma 2.17), the relaxed inequality $|\langle y \,|\, x\rangle| \leq \varepsilon\|x\|\|y\|$ itself might serve as a definition of approximate orthogonality. This approach has to be modified in generalized Minkowski spaces as the Cauchy–Schwarz inequality is not a statement about pairs of elements of the same vector space there.

Semi-inner products, including the superior and the inferior ones, are intertwined with Birkhoff orthogonality in Minkowski spaces, see [51] and [70, Chapters 8–11]. Is there a reasonable extension of semi-inner products to generalized Minkowski spaces which plays the same role for Birkhoff orthogonality in this setting? In the interplay between orthogonality types in Minkowski spaces, metric projections onto linear subspaces, the radial projection onto the unit ball, and, of course, related characterizations of special classes of Banach spaces, also several constants and moduli which describe the geometry of the underlying space take part. Notable examples are the James constant [99, 158], the Dunkl–Williams constant [168], the rectangular constant [19, 21, 87, 132, 133], and the Schäffer–Thele constant which also coincides with the bias and the metric projection bounds of Smith, Baronti, and Franchetti [63]. (Note that the rectangular constant and the Schäffer–Thele constant are special values of the rectangular modulus introduced in [210].) As a single instance, Cobzaș discusses extensions of the moduli of uniform rotundity and of uniform smoothness in [56, Section 2.4.7].

Propositions 3.2(c), 3.25(c), 3.16(c), and 3.32(c) in Chapter 3 refer to lower bounds for the circumradius, the diameter, and to upper bounds for the inradius and the minimum width of the Minkowski sum of two sets in terms of the sum of the respective quantities for the single sets. Reverse inequalities exist only in the case of the circumradius and the diameter. Optimal constants in these inequalities, however, seem to be unknown. Furthermore, having notions of diameter and minimum width, the investigation of diametrical maximal bodies [175, 176], constant width bodies [152, Section 2], and reduced bodies [141] in generalized Minkowski spaces is enabled. First results in this direction are derived in [35, 36]. As mentioned in Section 3.3, orthogonal projections which appear in the Euclidean definitions of some successive radii are replaced by the computation of radii with respect to certain cylinders already in Minkowski geometry. If we replace orthogonal projections by sets of best approximations instead, can we still prove basic properties like the monotonicity with respect to the dimension of the participating linear subspaces?

In the context of Chapter 4, one may also study abstract convexity notions defined by spindles, i.e., sets $\mathrm{bh}(\{x,y\},B,1)$ for $x,y \in X$, as a replacement for line segments, cf. [140]. A set $K \subset X$ is then called *spindle convex* if, for all $x,y \in K$, we have $\mathrm{bh}(\{x,y\},B,1) \subset K$. This gives rise to the concept of the *spindle convex hull* of a subset of $\mathbb{R}^n$. Note that spindle convex sets are not necessarily closed, in contrast to b-convex sets. Closed sets turn out to be spindle convex if and only if they are b-convex, provided the underlying Minkowski space is Euclidean or two-dimensional or its unit ball is (an affine image of) a cube, see [26, Corollary 3.4] as well as [140, Corollaries 3.13 and 3.15]. The space $(\mathbb{R}^3, \|\cdot\|_1)$ from Example 4.13 shows that closed spindle convex sets need not be b-convex in general, see also [140, Example 3.1].

In Section 3.1 and Chapter 6, we address scalar convex optimization problems which model single-facility location problems. As such, characterizations of the uniqueness of their solutions in terms of the geometry of the unit ball are interesting. Rotundity is the desired property for both problems in Minkowski spaces. The situation is different for generalized Minkowski spaces where, in both cases, rotundity remains a sufficient condition but not a necessary one, see Remark 3.13 and Example 6.20. Theorem 3.8 provides a characterization of of the uniqueness of circumcenters in terms of the boundary structure of the unit ball. Is there an analogous characterization of the uniqueness of solutions of the Fermat–Torricelli problem? Sufficient conditions for both problems are also discussed in [179, Section 3]. Alternatively, one may ask for characterizations of the situations in which the Fermat–Torricelli locus is a bounded set, an affine subspace, or a convex set of a certain dimension. Under the assumption of uniqueness of circumcenters, we present in [119] a construction of the circumcenter of a finite subset of a Minkowski plane. This construction is based on triangle classes introduced in [7, 8] which have not been extended to generalized Minkowski planes yet.

Apart from the need of determining the utility of dual characterizations of the Fermat–Torricelli locus given in Corollary 6.4 for the design of numerical algorithms, one may also enter D.C. programming by considering weighted minsum location problems with positive and non-positive weights. Such problems are treated mostly numerically [12, 48, 182]. However, the geometric study of such problems would give a unified framework of the hyperboloids and the ellipsoids studied in Section 5.4 and Chapter 6, respectively. On the other hand, a common thread to the ellipsoids and Cassini sublevel sets studied in Chapters 6 and 7 is their definition by average distances from the given foci. Here, future investigations might involve sets defined by other means in place of the arithmetic and the geometric ones.

Finally, of course, one may also consider other settings like infinite-dimensional vector spaces equipped with gauges, Hadamard spaces, i.e., complete metric spaces with non-positive curvature and unique geodesics, and thus the applicability of some methods of convex analysis [16], or Riemannian and Finsler manifolds [231].

Theses

(1) A generalized Minkowski space is a pair (X, γ) consisting of a finite-dimensional real vector space X and a gauge $\gamma : X \to \mathbb{R}$, i.e., a non-negative, positively homogeneous, and subadditive function which vanishes only at 0. We generalize metrically defined sets and relations in these spaces and address them by means of convex geometry and convex analysis.

(2) We introduce ε-Birkhoff orthogonality in a generalized Minkowski space (X, γ) as a binary relation on X, writing $x \perp_B^\varepsilon y$ if $\gamma(x) \leq \gamma(x + \lambda y) + \varepsilon$ for all $\lambda \in \mathbb{R}$. For $\varepsilon = 0$, the geometry of ε-Birkhoff orthogonality coincides with the one of supporting hyperplanes of balls. In particular, it is shown that $x \perp_B^0 y$ if and only if there exists $\phi \in \partial\gamma(x)$ such that $\langle \phi \,|\, y \rangle = 0$, where $\partial\gamma(x)$ is the set of unit outer normals of the ball centered at 0 with radius $\gamma(x)$.

(3) We complement the existing results on the circumradius $R(K, C)$, inradius $r(K, C)$, diameter $D(K, C)$, and minimum width $\Delta(K, C)$ of a convex body K with respect to another convex body C, the latter of which may be the unit ball of a generalized Minkowski space. For instance, we show that the set of circumcenters of K with respect to C is a convex set with empty interior.

(4) We introduce eight series of successive radii in generalized Minkowski spaces which interpolate between circumradius, inradius, diameter, and minimum width in a way resembling their counterparts from normed spaces. For instance, if $R_\pi^j(K, C)$ is the supremum of the values $R(K, C + L)$, where L traverses the $(\dim(X) - j)$-dimensional linear subspaces of X, then $\frac{1}{2}D(K, C) = R_\pi^1(K, C) \leq \ldots \leq R_\pi^{\dim(X)}(K, C) = R(K, C)$.

(5) We complement the existing results on ball convexity with respect to a convex body, which in our case is the unit ball of a generalized Minkowski space. A non-empty bounded subset $K \subset X$ is b-convex body if it coincides with its ball hull, i.e., the intersection of balls of radius 1 containing K. For instance, we show that if $K_1 \subset K_2 \subset \ldots$ is an increasing sequence of b-convex bodies, such that the ball hull of its union has dimension 0, 1, $\dim(X) - 1$, or $\dim(X)$, then the closure of the union is a b-convex body.

(6) Isosceles orthogonality in a generalized Minkowski space (X, γ) is introduced as a binary relation on X, writing $y \perp_I x$ if $\gamma(y + x) = \gamma(y - x)$. This generalization of Euclidean orthogonality resembles the geometry of perpendicular bisectors. In particular, it is shown that the set $\{\alpha \in \mathbb{R} \,|\, (\alpha x + y) \perp_I x\}$ is non-empty, closed, bounded, and convex whenever $x \neq 0$, but $\{\alpha \in \mathbb{R} \,|\, x \perp_I (\alpha x + y)\}$ may be empty.

(7) Using additive and multiplicative perturbations of isosceles orthogonality, we introduce hyperboloids $H_{x,y,\leq\alpha}$ and apollonoids $A_{x,y,\leq\alpha}$ in generalized Minkowski spaces (X, γ) as counterparts of hyperbolas and Apollonian circles in the Euclidean plane. We show that

the gauge γ is a norm induced by an inner product if and only if $H_{x,y,\leq\alpha}$ is convex for all $\alpha < 0$ if and only if $A_{x,y,\leq\alpha}$ is convex for all $\alpha \in (0,1)$.

(8) We address functions of the form $f := \sum_{i=1}^{n} \underline{\mathrm{dist}}_{\gamma_i}(x, K_i) : X \to \mathbb{R}$ where $\mathfrak{K} = (K_1, \ldots, K_n)$ is a collection of $n \geq 1$ non-empty closed convex sets $K_i \subset X$, and $\Gamma = (\gamma_1, \ldots, \gamma_n)$ is a family of gauges on X. Among others, we show that the set $\mathrm{ft}_\Gamma(\mathfrak{K})$ of minimizers of f has the following property: If $p_0 \in \mathrm{ft}_\Gamma(\mathfrak{K})$, then $p_0 \in \mathrm{ft}_\Gamma(p_0 + \lambda_1(K_1 - p_0), \ldots, p_0 + \lambda_n(K_n - p_0))$.

(9) Given two points $x, y \in X$, we also show that the set of points z with $\gamma(x-z)+\gamma(z-y) = \gamma(x-y)$ and the set of minimizers of the function $\gamma(\cdot - x) + \gamma(\cdot - y) : X \to \mathbb{R}$ may be disjoint. Furthermore, the $(\dim(X)-2)$-extreme points of the sublevel sets of the function $\sum_{i=1}^{n} \gamma_i(\cdot - p_i)$ are determined by the points $p_i \in X$ and the $(\dim(X)-2)$-extreme points of the unit balls of the gauges $\gamma_i : X \to \mathbb{R}$.

(10) In the spirit of classical Cassini curves, we study sets which are defined by a product of distances to a finite number of foci regarding their starshapedness and connectedness. For instance, we show that there is a generalized Minkowski space (X, γ), points $p_1, p_2 \in X$, and a number $\alpha \in \mathbb{R}$ such that the set $\{x \in X \mid \gamma(x-p_1)\gamma(x-p_2) \leq \alpha\}$ has uncountably many connected components.

(11) Rotundity is a recurring topic in generalized Minkowski geometry.

- Rotundity of γ implies the uniqueness of circumcenters of all convex bodies with respect to the unit ball of γ.
- In a two-dimensional generalized Minkowski space (X, γ), rotundity of the gauge is necessary for $(x+y) \perp_B^0 z$ for all $x, y, z \in X$ with $x \perp_B^0 z$ and $y \perp_B^0 z$.
- The gauge γ is rotund if and only if every b-convex body with circumradius ≥ 1 is a closed ball of radius 1 if and only if every b-convex body is rotund.
- In a two-dimensional generalized Minkowski space (X, γ), rotundity of the gauge is equivalent to the emptiness of the interiors of bisectors $\{y \in X \mid y \perp_I x\}$, $x \in X$.
- For functions $\sum_{i=1}^{n} \gamma_i(\cdot - p_i) : X \to \mathbb{R}$, where $\gamma_1, \ldots, \gamma_n : X \to \mathbb{R}$ are gauges, and $p_1, \ldots, p_n \in X$, rotundity of the involved gauges implies uniqueness of minimizers.
- Nonrotundity implies the existence of level sets containing non-singleton line segments for some functions, such as $\prod_{i=1}^{n} \gamma(\cdot - p_i)$, where γ is a gauge, and $p_1, \ldots, p_n \in X$.

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Curriculum vitae

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[1] *The Elzinga–Hearn algorithm in strictly convex planes*. Sachsen-anhaltischer Geometrietag & Friends, Otto von Guericke University Magdeburg, Germany, December 2013.
[2] *The Elzinga–Hearn algorithm in strictly convex planes*. 15th Southeastern Germany Workshop on Combinatorics, Graph Theory, and Algorithms, Hochschule für Technik und Wirtschaft Dresden, Germany, June 2014.
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[10] *Hunting for reduced polytopes*. Convex, Discrete, and Integral Geometry, Banach Conference Center Będlewo, Poland, June 2017.
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