DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume , Number **0**,

doi:

pp. **000–000**

ROBUSTNESS OF DYNAMICALLY GRADIENT MULTIVALUED DYNAMICAL SYSTEMS

Rubén Caballero

Centro de Investigación Operativa
Universidad Miguel Hernández de Elche
03202-Elche, Alicante, SPAIN

Alexandre N. Carvalho

Instituto de Ciências Matemáticas e de Computaçao
Universidade de São Paulo
Campus de São Carlos, Caixa Postal 668, 13560-970 São Carlos, SP BRAZIL

Pedro Marín-Rubio

Departamento de Ecuaciones Diferenciales y Análisis Numérico
Universidad de Sevilla
c/Tarfia s/n, 41012-Sevilla, SPAIN

José Valero

Centro de Investigación Operativa
Universidad Miguel Hernández de Elche
03202-Elche, Alicante, SPAIN

To Professor Valery Melnik, in Memoriam

Abstract. In this paper we study the robustness of dynamically gradient multivalued semiflows. As an application, we describe the dynamical properties of a family of Chafee-Infante problems approximating a differential inclusion studied in [3], proving that the weak solutions of these problems generate a dynamically gradient multivalued semiflow with respect to suitable Morse sets.

2010 *Mathematics Subject Classification.* 37B25, 35B40, 35B41, 35K55,37L30, 58C06.

Key words and phrases. Attractors; reaction-diffusion equations; stability; dynamically gradient multivalued semiflows, Morse decomposition, set-valued dynamical systems.

The authors of this work have been partially funded by the following projects: R. Caballero is a fellow of Programa de FPU del Ministerio de Educación, Cultura y Deporte, reference FPU15/03080; P. Marín-Rubio was supported by projects PHB2010-0002-PC (Ministerio de Educación-DGPU), MTM2015-63723-P (MINECO/FEDER, EU) and P12-FQM-1492 (Junta de Andalucía); and J.Valero by projects MTM2015-63723-P, MTM2016-74921-P (MINECO/FEDER, EU) and P12-FQM-1492 (Junta de Andalucía).

1. **Introduction.** One of the main goals of the theory of dynamical systems is to characterize the structure of global attractors. It is possible to find a wide literature about this problem for semigroups; however, it has been recently when new results in this direction for multivalued dynamical systems have been proved [3], [13], [14].

In this sense, the theory of Morse decomposition plays an important role. In fact, the existence of a Lyapunov function, the property of being a dynamically gradient semiflow and the existence of a Morse decomposition are shown to be equivalent for multivalued dynamical systems in [9].

In this work we show under suitable assumptions that a dynamically gradient multivalued semiflow is stable under perturbations, that is, the family of perturbed multivalued semiflows remains dynamically gradient.

For a fixed dynamically gradient multivalued semiflow with a global attractor we also analyze the rearrangement of a pairwise disjoint finite family of isolated weakly invariant sets, included in the attractor, in such a way that the dynamically gradient property is satisfied in the stronger sense of [16].

These results extend previous ones in the single-valued framework in [7, 1, 2] to the case where uniqueness of solution does not hold. Additionally, it is worth saying that the multivalued semiflows here are not supposed to be general dynamical systems as in [16], where a robustness theorem for Morse decompositions of multivalued dynamical systems is also proved under a suitable continuity assumption.

We also apply this general robustness theorem in order to show that a family of Chafee-Infante problems approximating a differential inclusion is dynamically gradient if it is close enough to the original problem.

This paper is organized as follows.

Firstly, we introduce in Section 2 basic concepts and properties related to fixed points, complete trajectories and global attractors. In this way we are able to present in Section 3 the main result about robustness of dynamically gradient multivalued semiflows. Further, in Section 4 we prove a theorem which allows us to reorder the family of weakly invariants sets, thus establishing an equivalent definition of dynamically gradient families.

Afterwards, we consider a Chafee-Infante problem in Section 5, where the equivalence of weak and strong solutions is established. Once the set of fixed points is analyzed, we consider a family of Chafee-Infante equations, approximating the differential inclusion tackled in [3]. We check that this family of Chafee-Infante equations verifies the hypotheses of the robustness theorem in order to obtain, therefore, that the multivalued semiflows generated by the solutions of the approximating problems are dynamically gradient if this family is close enough to the original one.

2. **Preliminaries.** Consider a metric space (X, d) and a family of functions $\mathcal{R} \subset \mathcal{C}(\mathbb{R}_+; X)$. Denote by $P(X)$ the class of nonempty subsets of X. Then, define the multivalued map $G : \mathbb{R}_+ \times X \to P(X)$ associated with the family $\mathcal{R}$ as follows

$$G(t, u_0) = \{u(t) : u(\cdot) \in \mathcal{R}, u(0) = u_0\}. \tag{1}$$

In this abstract setting, the multivalued map G is expected to satisfy some properties that fit in the framework of multivalued dynamical systems. The first concept is given now, although a more axiomatic construction will be provided below.

Definition 1. A multivalued map $G : \mathbb{R}_+ \times X \to P(X)$ is a multivalued semiflow (or m-semiflow) if $G(0, x) = x$ for all $x \in X$ and $G(t + s, x) \subset G(t, G(s, x))$ for all

$t, s \geq 0$ and $x \in X$.
If the above is not only an inclusion, but an equality, it is said that the m-semiflow is strict.

In order to obtain a detailed characterization of the internal structure of a global attractor, we introduce an axiomatic set of properties on the set $\mathcal{R}$ (see [4] and [13]).

The set of axiomatic properties that we will deal with is the following.

(K1) For any $x \in X$ there exists at least one element $\varphi \in \mathcal{R}$ such that $\varphi(0) = x$.
(K2) $\varphi_\tau(\cdot) := \varphi(\cdot + \tau) \in \mathcal{R}$ for any $\tau \geq 0$ and $\varphi \in \mathcal{R}$ (translation property).
(K3) Let $\varphi_1, \varphi_2 \in \mathcal{R}$ be such that $\varphi_2(0) = \varphi_1(s)$ for some $s > 0$. Then, the function φ defined by

$$\varphi(t) = \begin{cases} \varphi_1(t) & 0 \leq t \leq s, \\ \varphi_2(t-s) & s \leq t, \end{cases}$$

belongs to $\mathcal{R}$ (concatenation property).
(K4) For any sequence $\{\varphi^n\} \subset \mathcal{R}$ such that $\varphi^n(0) \to x_0$ in X, there exist a subsequence $\{\varphi^{n_k}\}$ and $\varphi \in \mathcal{R}$ such that $\varphi^{n_k}(t) \to \varphi(t)$ for all $t \geq 0$.

It is immediate to observe [6, Proposition 2] or [15, Lemma 9] that $\mathcal{R}$ fulfilling (K1) and (K2) gives rise to an m-semiflow G through (1), and if besides (K3) holds, then this m-semiflow is strict. In such a case, a global bounded attractor, supposing that it exists, is strictly invariant [19, Remark 8].

From now on (K1)-(K2) are always satisfied and G will be the multivalued semiflow associated to $\mathcal{R}$.

Once a multivalued semiflow is defined, we recall the concepts of invariance and global attractor, with evident differences with respect to the single-valued case.

Definition 2. A map $\gamma : \mathbb{R} \to X$ is called a complete trajectory of $\mathcal{R}$ (resp. of G) if $\gamma(\cdot + h) \mid_{[0,\infty)} \in \mathcal{R}$ for all $h \in \mathbb{R}$ (resp. if $\gamma(t+s) \in G(t, \gamma(s))$ for all $s \in \mathbb{R}$ and $t \geq 0$).

A point $z \in X$ is a fixed point of $\mathcal{R}$ (resp. of G) if $\varphi(\cdot) \equiv z \in \mathcal{R}$ (resp. $z \in G(t, z)$ for all $t \geq 0$).

Definition 3. A set $B \subset X$ is said to be negatively invariant if $B \subset G(t, B)$ for all $t \geq 0$, and strictly invariant (or, simply, invariant) if the above relation is not only an inclusion but an equality.

The set B is said to be weakly invariant if for any $x \in B$ there exists a complete trajectory γ of $\mathcal{R}$ contained in B such that $\gamma(0) = x$. We observe that weak invariance implies negative invariance.

Definition 4. A set $\mathcal{A} \subset X$ is called a global attractor for an m-semiflow if it is negatively semi-invariant and it attracts all attainable sets through the m-semiflow starting in bounded subsets, i.e., $dist_X(G(t, B), \mathcal{A}) \to 0$ as $t \to \infty$, where $dist_X(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b)$.

Remark 1. A global attractor for an m-semiflow does not have to be unique, nor a bounded set (see [24] for a non-trivial example of an unbounded non-locally compact attractor). However, if a global attractor is bounded and closed, it is minimal among all closed sets that attract bounded sets [19]. In particular, a bounded and closed global attractor is unique.

Several properties concerning fixed points, complete trajectories and global attractors are summarized in the following results [13].

Lemma 1. *Let (K1)-(K2) be satisfied. Then every fixed point (resp. complete trajectory) of $\mathcal{R}$ is also a fixed point (resp. complete trajectory) of G.*

If $\mathcal{R}$ fulfills (K1)-(K4), then the fixed points of $\mathcal{R}$ and G coincide. Besides, a map $\gamma : \mathbb{R} \to X$ is a complete trajectory of $\mathcal{R}$ if and only if it is continuous and a complete trajectory of G.

The standard well-known result in the single-valued case for describing the attractor as the union of bounded complete trajectories reads in the multivalued case as follows.

Theorem 1. *Consider $\mathcal{R}$ satisfying (K1) and (K2), and either (K3) or (K4). Assume also that G possesses a compact global attractor $\mathcal{A}$. Then*

$$\mathcal{A} = \{\gamma(0) : \gamma \in \mathbb{K}\} = \cup_{t\in\mathbb{R}}\{\gamma(t) : \gamma \in \mathbb{K}\}, \tag{2}$$

where $\mathbb{K}$ denotes the set of all bounded complete trajectories in $\mathcal{R}$.

Now we recall the definitions of some important sets in the literature of dynamical systems. Let $B \subset X$ and let $\varphi \in \mathcal{R}$. We define the $\omega-$limit sets $\omega(B)$ and $\omega(\varphi)$ as follows:

$$\begin{aligned} \omega(B) =& \{y \in X : \text{ there are sequences } t_n \to \infty, y_n \in G(t_n, B) \text{ such that } y_n \to y\}, \\ \omega(\varphi) =& \{y \in X : \text{ there is a sequence } t_n \to \infty \text{ such that } \varphi(t_n) \to y\}. \end{aligned}$$

If γ is a complete trajectory of $\mathcal{R}$, then the $\alpha-$limit set is defined by

$$\alpha(\gamma) = \{y \in X : \text{ there is a sequence } t_n \to -\infty \text{ such that } \gamma(t_n) \to y\}.$$

Some useful properties of these sets [4, Lemma 3.4] are summarized in the following lemma.

Lemma 2. *Assume that $(K1), (K2)$ and $(K4)$ hold. Let G be asymptotically compact, that is, every sequence $y_n \in G(t_n, B)$, where $t_n \to \infty$ and $B \subset X$ is bounded, is relatively compact. Then:*

1. *For any non-empty bounded set $B, \omega(B)$ is non-empty, compact, weakly invariant and*
$$dist_X(G(t, B), \omega(B)) \to 0, \text{ as } t \to +\infty.$$
2. *For any $\varphi \in \mathcal{R}$, $\omega(\varphi)$ is non-empty, compact, weakly invariant and*
$$dist_X(\varphi(t), \omega(\varphi)) \to 0, \text{ as } t \to +\infty.$$
3. *For any $\gamma \in \mathbb{K}$, $\alpha(\gamma)$ is non-empty, compact, weakly invariant and*
$$dist_X(\gamma(t), \alpha(\gamma)) \to 0, \text{ as } t \to -\infty.$$

In order to give a more detailed description of the internal structure of the attractor under special cases, additional concepts are required.

Definition 5. Consider the m-semiflow G assoociated to $\mathcal{R}$.

1. We say that $\mathcal{S} = \{\Xi_1, \ldots, \Xi_n\}$ is a family of isolated weakly invariant sets if there exists $\delta > 0$ such that $\mathcal{O}_\delta(\Xi_i) \cap \mathcal{O}_\delta(\Xi_j) = \emptyset$ for $1 \le i < j \le n$, and each Ξ_i is the maximal weakly invariant subset in $\mathcal{O}_\delta(\Xi_i) := \{x \in X : dist_X(x, \Xi_i) < \delta\}$.

2. For an m-semiflow G on (X,d) with a global attractor $\mathcal{A}$ and a finite number of weakly invariant sets $\mathcal{S}$, a homoclinic orbit in $\mathcal{A}$ is a collection $\{\Xi_{p(1)},\ldots,\Xi_{p(k)}\}$ $\subset \mathcal{S}$ and a collection of complete trajectories $\{\gamma_i\}_{1\le i\le k}$ of $\mathcal{R}$ in $\mathcal{A}$ such that (putting $p(k+1):=p(1)$)

$$\lim_{t\to-\infty} dist_X(\gamma_i(t),\Xi_{p(i)})=0, \lim_{t\to\infty} dist_X(\gamma_i(t),\Xi_{p(i+1)})=0,\ 1\le i\le k,$$

   and

$$\text{for each } i \text{ there exists } t_i\in\mathbb{R} \text{ such that } \gamma_i(t_i)\notin \Xi_{p(i)}\cup\Xi_{p(i+1)}.$$

3. We say that the m-semiflow G on (X,d) with the global attractor $\mathcal{A}$ is dynamically gradient if the following two properties hold:
   (G1) there exists a finite family $\mathcal{S}=\{\Xi_1,\ldots,\Xi_n\}$ of isolated weakly invariant sets in $\mathcal{A}$ with the property that any bounded complete trajectory γ of $\mathcal{R}$ in $\mathcal{A}$ satisfies

$$\lim_{t\to-\infty} dist_X(\gamma(t),\Xi_i)=0,\ \lim_{t\to\infty} dist_X(\gamma(t),\Xi_j)=0$$

   for some $1\le i,j\le n$;
   (G2) $\mathcal{S}$ does not contain homoclinic orbits.

Remark 2. The last definition generalizes the concept of dynamically gradient semigroups (see [7], where they are called gradient-like semigroups) to the multivalued case. Observe that the above definitions are concerned with weakly invariant families, which need not to be unitary sets. This is to deal with the more general concept of generalized gradient-like semigroups [7], in contrast with gradient-like semigroups (when the invariant sets are unitary).

Now, we introduce the concept of unstable manifold, that will allow us to describe more precisely the structure of a global attractor of a dynamically gradient m-semiflow.

Definition 6. The unstable manifold of a set Ξ is

$$W^u(\Xi)=\{u_0\in X:\ \text{there exists complete trajectory } \gamma \text{ of } \mathcal{R} \text{ such that } \gamma(0)=u_0 \text{ and } \lim_{t\to-\infty} dist_X(\gamma(t),\Xi)=0\}.$$

Now the following result, relating the global attractor with unstable manifolds, is standard. The first statement is straightforward to see. The second one, supposing that the global attractor is compact, follows directly from the structure described in Theorem 1 and the definition of dynamically gradient semiflows.

Lemma 3. *Consider a family $\mathcal{R}\subset\mathcal{C}(\mathbb{R}_+;X)$ satisfying (K1) and (K2). Suppose that the associated m-semiflow has a global attractor $\mathcal{A}$. Then, for any bounded set $\Xi\subset X, W^u(\Xi)\subset\bar{\mathcal{A}}$.*

Moreover, assume that $\mathcal{R}$ satisfies either (K3) or (K4), and that the global attractor $\mathcal{A}$ is compact. Suppose also that the associated m-semiflow G defined in (1) is dynamically gradient. Then

$$\mathcal{A}=\bigcup_{i=1}^{n} W^u(\Xi_i). \tag{3}$$

3. **Robustness of dynamically gradient m-semiflows.** Our first main goal is to prove that a dynamically gradient multivalued semiflow is stable under suitable perturbations, that is, a family of perturbed multivalued semiflows remains dynamically gradient if it is close enough to the original semiflow, generalizing the corresponding result in the single-valued case [7]. This is rigorously formulated in the following theorem.

Theorem 2. *Let η be a parameter in [0,1], $\mathcal{R}_\eta \subset \mathcal{C}(\mathbb{R}_+; X)$ fulfill (K1), (K2), (K3) and (K4), and let G_η be the corresponding m-semiflow on X having the global compact attractor $\mathcal{A}_\eta$. Assume that*

(H1) $\overline{\bigcup_{\eta\in[0,1]} \mathcal{A}_\eta}$ *is compact.*

(H2) G_0 *is a dynamically gradient m-semiflow with finitely many isolated weakly invariant sets* $\mathcal{S}^0 = \{\Xi_1^0, \dots, \Xi_n^0\}$.

(H3) $\mathcal{A}_\eta$ *has a finite number of isolated weakly invariant sets* $\mathcal{S}_\eta = \{\Xi_1^\eta, \dots, \Xi_n^\eta\}$, $\eta \in [0,1]$, *which satisfy*

$$\lim_{\eta\to 0} \sup_{1\le i\le n} dist_X(\Xi_i^\eta, \Xi_i^0) = 0.$$

(H4) *Any sequence* $\{\gamma_\eta\}$ *with* $\gamma_\eta \in \mathcal{R}_\eta$ *such that* $\{\gamma_\eta(0)\}$ *converges for* $\eta \to 0^+$ *possesses a subsequence* $\{\gamma_{\eta'}\}$ *that converges uniformly in bounded intervals of* $[0,\infty)$ *to* $\gamma \in \mathcal{R}_0$.

(H5) *There exists* $\overline{\eta} > 0$ *and neighborhoods* V_i *of* Ξ_i^0 *such that* Ξ_i^η *is the maximal weakly invariant set for* G_η *in* V_i *for any* $i = 1, \dots, n$ *and for each* $0 < \eta \le \overline{\eta}$.

Then there exists $\eta_0 > 0$ such that for all $\eta \le \eta_0$, $\{G_\eta\}$ is a dynamically gradient m-semiflow. In particular, the structure of $\mathcal{A}_\eta$ is analogous to that given in (3).

Proof. Observe that assumption (H5) concerning certain neighborhood V_i of Ξ_i^0 involves a hyperbolicity condition of G_0 w.r.t. each Ξ_i^0, and as far as (H3) is also assumed, there exist $\{\eta(V_i)\}_{i=1,\dots,n}$ such that $\Xi_i^\eta \subset V_i$ for all $\eta \le \eta(V_i)$. W.l.o.g. assume that $\delta > 0$ is such that $\{x \in X : dist_X(x, \Xi_i^0) \le \delta\} \subset V_i$ for all $i = 1, \dots, n$.

By Theorem 1, we have that $\mathcal{A}_\eta$ is composed by all the orbits of bounded complete trajectories of $\mathcal{R}_\eta$, $\mathbb{K}_\eta$.

We are going to prove by contradiction arguments that there exists $\eta_0 \in (0,1]$ such that $\{G_\eta\}_{\eta\le\eta_0}$ is dynamically gradient.

Step 1: There exists $\eta_0 > 0$ such that for all $\eta < \eta_0$, any bounded complete trajectory ξ_η of $\mathcal{R}_\eta$ satisfies that there exist $i \in \{1, \dots, n\}$ and t_0 such that for all $t \ge t_0$, $dist_X(\xi_\eta(t), \Xi_i^0) \le \delta$.

After proving the above claim, we consider the sets $B_\eta := \{\xi_\eta(s) : s \ge t_0\} \subset A = \{y : dist_X(y, \Xi_i^0) \le \delta\}$ and $\omega(\xi_\eta)$. It follows that $\omega(\xi_\eta) \subset A$, since $dist_X(\xi_\eta(t), \omega(\xi_\eta))$ goes to 0 as $t \to +\infty$. On the other hand, by Lemma 2 $\omega(\xi_\eta)$ is a weakly invariant set of G_η contained in V_i. By assumption (H5) we have that $\omega(\xi_\eta) \subset \Xi_i^\eta$, whence the 'forward part' of property (G1) of a dynamically gradient m-semiflow will follow immediately.

We prove this Step 1 by contradiction. Suppose it does not hold. Then, there exist a sequence $\eta_k \to 0$ (as $k \to \infty$) and bounded complete trajectories ξ_k of $\mathcal{R}_{\eta_k}$ (therefore, from $\mathcal{A}_{\eta_k}$) such that

$$\sup_{t\ge t_0} dist_X(\xi_k(t), \mathcal{S}^0) > \delta \ \forall t_0 \in \mathbb{R}. \tag{4}$$

The set $\{\xi_k(0)\} \subset \overline{\cup_{\eta\in[0,1]}\mathcal{A}_\eta}$ is relatively compact from assumption (H1). So, there exists a converging subsequence (relabeled the same) in X. From (H4), there exist a subsequence (relabeled the same, again) and $\xi_0 \in \mathcal{R}_0$, such that $\{\xi_k|_{[0,\infty)}\}$ converges to ξ_0 in bounded intervals of $[0,\infty)$. Actually, if we argue similarly not for time 0, but now for times -1, $-2, \dots$, and use a diagonal argument, we have that $\xi_0 = \gamma_0|_{[0,\infty)}$ where $\gamma_0 \in \mathbb{K}_0$, and the convergence of (a subsequence of) $\{\xi_k\}$ toward γ_0 holds uniformly in bounded intervals $[a,b]$ of $\mathbb{R}$.

Since G_0 is dynamically gradient, there exists $i \in \{1,\dots,n\}$ such that

$$dist_X(\gamma_0(t), \Xi_i^0) \to 0 \text{ as } t \to \infty.$$

Therefore, for all $r \in \mathbb{N}$, there exist t_r and k_r such that $dist_X(\xi_k(t_r), \Xi_i^0) < 1/r$ for all $k \geq k_r$. Indeed, this is done as follows: $dist_X(\gamma_0(s), \Xi_i^0) < 1/r$ for all $s \geq t_r$ (for some t_r, w.l.o.g. $t_r \geq r > 1/\delta$); now, combining this with the uniform convergence on $[0, t_r]$ of ξ_k toward γ_0, the existence of k_r follows.

However, from (4), there exists $t_r' > t_r$ such that $dist_X(\xi_{k_r}(t), \Xi_i^0) < \delta$ for all $t \in [t_r, t_r')$ and $dist_X(\xi_{k_r}(t_r'), \Xi_i^0) = \delta$.

Now we distinguish two cases and we will arrive to the same conclusion in both of them.

Case (1a): Suppose that $t_r' - t_r \to \infty$ as $r \to \infty$ (at least for a certain subsequence).

Since $\{\xi_{k_r}(t_r')\}$ is also relatively compact (by (H1), again), and $\xi^1_{k_r}(\cdot) = \xi_{k_r}(t_r' + \cdot)$ is a bounded complete trajectory of $\mathcal{R}_{k_r}$, from (H4) we deduce that a subsequence (relabeled the same) is converging on bounded time-intervals of $[0,\infty)$, i.e. $\gamma_1(t) := \lim_{r\to\infty} \xi_{k_r}(t + t_r')$ holds for certain $\gamma_1 \in \mathcal{R}_0$. Moreover, as before, a diagonal argument, using not t_r' above, but $t_r' - 1$, $t_r' - 2, \dots$ implies that γ_1 can be extended to the whole real line (the function will still be denoted the same; and the convergence holds in bounded time-intervals of $\mathbb{R}$), in particular, by (H1) and (H4), $\gamma_1 \in \mathbb{K}_0$.

Moreover, by its construction, we have that $dist_X(\gamma_1(t), \Xi_i^0) \leq \delta$ for all $t \leq 0$. By Lemma 2 we have that the α-limit set $\alpha(\gamma_1)$ is weakly invariant.

As long as Ξ_i^0 is the biggest weakly invariant set contained in V_i, we deduce that $dist_X(\gamma_1(\tau), \Xi_i^0) \to 0$ when $\tau \to -\infty$.

On the other hand, from (G1) and (G2) we have that $dist_X(\gamma_1(t), \Xi_j^0) \to 0$ as $t \to \infty$ for $j \neq i$.

Case (1b): Suppose that there exists $C > 0$ such that $|t_r' - t_r| \leq C$ as $r \to \infty$. (W.l.o.g. we assume that $t_r' - t_r \to t_*$.)

Recall that $dist_X(\xi_{k_r}(t_r), \Xi_i^0) < 1/r$. By [9, Lemma 19] Ξ_i^0 is closed, so, up to a subsequence $\xi_{k_r}(t_r) \to y \in \Xi_i^0$. Denote $\xi^1_{k_r}(\cdot) = \xi_{k_r}(\cdot + t_r)$. From (H4), there exist a subsequence $\{\xi^1_{k_r}\}$ and $\xi^1 \in \mathcal{R}_0$ with $\xi^1(0) = y$ such that $\xi^1_{k_r}$ converge towards ξ^1 uniformly in bounded intervals of $[0,\infty)$. In particular, $\xi^1_{k_r}(t_r' - t_r) \to \xi^1(t_*)$, so that $dist_X(\xi^1(t_*), \Xi_i^0) \geq \delta$.

Since Ξ_i^0 is weakly invariant, there exists $\gamma \in \mathbb{K}_0$ with $\gamma(0) = \xi^1(0)$ and $\gamma(t) \in \Xi_i^0$ for all $t \in \mathbb{R}$. By (K3) consider the concatenation

$$\gamma_1(t) := \begin{cases} \gamma(t), & \text{if } t \leq 0, \\ \xi^1(t), & \text{if } t \geq 0. \end{cases}$$

Then by (G1)-(G2) it follows that $dist_X(\gamma_1(t), \Xi_j^0) \to 0$ as $t \to \infty$ with $j \neq i$. This is exactly the same conclusion we arrived in Case (1a).

Reasoning now with the subsequence $\{\xi^1_{k_r}\}$, and proceeding as above, we obtain the existence of $\gamma_2 \in \mathbb{K}_0$ such that $dist_X(\gamma_2(t), \Xi^0_j) \to 0$ as $t \to -\infty$ and $dist_X(\gamma_2(t), \Xi^0_p) \to 0$ as $t \to \infty$, with $p \notin \{i, j\}$.

Thus, in a finite number of steps we arrive to a contradiction, since G_0 satisfies (G2). Therefore, (4) is absurd, and Step 1 is proved.

Step 2: There exists $\eta_1 > 0$ such that for all $\eta < \eta_1$, any bounded complete trajectory ξ_η of $\mathcal{R}_\eta$ satisfies that there exist $j \in \{1, \dots, n\}$ and t_1 such that $dist_X(\xi_\eta(t), \Xi^0_j) \leq \delta$ for all $t \leq t_1$.

The above claim can be proved analogously as before, and since for any bounded complete trajectory $\xi_\eta \in \mathbb{K}_\eta$, by Lemma 2, $\alpha(\xi_\eta)$ is weakly invariant for G_η, and contained in some V_j, the 'backward part' of property (G1) of a dynamically gradient m-semiflow will follow immediately.

Hence, for all suitable small η, $\{G_\eta(t) : t \geq 0\}$ satisfies (G1).

Step 3: There exists $\eta_2 > 0$ such that $\{G_\eta\}_{\eta \leq \eta_2}$ satisfies (G2).

If not, there exist a sequence $\eta_k \to 0$, with G_{η_k} having an homoclinic structure. We may suppose that the number of elements of weakly invariant subsets connected on each homoclinic chain in $\mathcal{S}_{\eta_k}$ is the same. Moreover, by assumption (H3) each $\Xi^{\eta_k}_j$ is contained in V_j for η_k small enough and w.l.o.g. the order in the route of the homoclinics visiting the V_j sets is the same.

Therefore, for $k \geq k_0$ there exist a sequence of subsets $\Xi^{\eta_k}_{p(1)}, \dots \Xi^{\eta_k}_{p(l)}$ in $\mathcal{S}_{\eta_k}$ (with $p(l+1) = p(1)$), and a sequence of complete trajectories $\{\{\xi^k_i\}^l_{i=1}\}_k$, each collection of l elements in the corresponding attractor $\mathcal{A}_{\eta_k}$, with

$$\lim_{t\to-\infty} dist_X(\xi^k_i(t), \Xi^{\eta_k}_{p(i)}) = 0, \ \lim_{t\to\infty} dist_X(\xi^k_i(t), \Xi^{\eta_k}_{p(i+1)}) = 0, 1 \leq i \leq l.$$

If we argue now as in the proof of (G1), we may construct a homoclinic structure of G_0, getting a contradiction with the fact that the m-semiflow G_0 is dynamically gradient. □

Remark 3. The above result also applies to the particular case of a dynamically gradient m-semiflow when the weakly invariant families of the original and perturbed problems are reduced to unitary sets (Remark 2 and [7, Theorem 1.5]).

4. **An equivalent definition of dynamically gradient families.** We will give an equivalent definition of dynamically gradient families. For proving the main result in this section we will need a stronger condition than $(K4)$. Namely, we shall consider the following stronger condition:

$(\overline{K}4)$ For any sequence $\{\varphi^n\} \subset \mathcal{R}$ such that $\varphi^n(0) \to x_0$ in X, there exists a subsequence $\{\varphi^n\}$ and $\varphi \in \mathcal{R}$ such that φ^n converges to φ uniformly in bounded subsets of $[0, \infty)$.

As before, let $\mathcal{A}$ be the global attractor of the m-semiflow G associated to $\mathcal{R}$.

Remark 4. We have seen that the property of being dynamically gradient for a disjoint family of isolated weakly invariant sets $\mathcal{S} = \{\Xi_1, \dots, \Xi_n\} \subset \mathcal{A}$ is stable under perturbations. We observe that in the paper [16] a slightly different definition was used for dynamically gradients families. Namely, instead of conditions $(G1)$-$(G2)$ it is assumed that any bounded complete trajectory $\gamma(\cdot)$ of $\mathcal{R}$ in $\mathcal{A}$ satisfies one of the following properties:

1. $\{\gamma(t) : t \in \mathbb{R}\} \subset \Xi_i$ for some i.

2. There are $i < j$ for which
$$\gamma(t) \underset{t\to\infty}{\rightarrow} \Xi_i,\ \gamma(t) \underset{t\to-\infty}{\rightarrow} \Xi_j.$$

These assumptions are clearly stronger than $(G1)$-$(G2)$ and imply that the sets Ξ_j are ordered. Our aim is to show that when $\mathcal{S}$ is a disjoint family of isolated weakly invariant sets, these conditions are equivalent. For this we will need to introduce the concept of local attractor and its repeller and study their properties.

We say that $A \subset \mathcal{A}$ is a local attractor in $\mathcal{A}$ if for some $\varepsilon > 0$ we have that $\omega(\mathcal{O}_\varepsilon(A) \cap \mathcal{A}) = A$. Let A be a local attractor in $\mathcal{A}$. Then its repeller A^* is defined by
$$A^* = \{x \in \mathcal{A} : \omega(x) \backslash A \neq \emptyset\}.$$

Some properties about local attractors and its repeller as well as the proof of the following three lemmas can be found in [9].

Lemma 4. *Assume that* $(K1) - (K4)$ *hold and that a global compact attractor* $\mathcal{A}$ *exists. Then a local attractor* A *is invariant.*

Remark 5. Although in [9] the stronger assumption $(\overline{K}4)$ is assumed, the proof is valid for just $(K4)$.

Lemma 5. *Assume that* $(K1)$-$(K3)$, $(\overline{K}4)$ *hold and that a global compact attractor* $\mathcal{A}$ *exists. Then the repeller* A^* *of a local attractor* $A \subset \mathcal{A}$ *is weakly invariant and compact.*

Lemma 6. *Assume that* $(K1)$-$(K3)$, $(\overline{K}4)$ *hold and that a global compact attractor* $\mathcal{A}$ *exists. Let us consider the sequences* $x_k \in \mathcal{A}$, $t_k \to +\infty$ *and* $\varphi_k(\cdot) \in \mathcal{R}$ *such that* $\varphi_k(0) = x_k$. *Then from the sequence of maps* $\xi_k(\cdot) : [-t_k, +\infty) \to \mathcal{A}$ *defined by*
$$\xi_k(t) = \varphi_k(t + t_k)$$
one can extract a subsequence converging to some $\psi(\cdot) \in \mathbb{K}$ *uniformly on bounded subsets of* $\mathbb{R}$.

In order to prove the equivalent definition of dynamically gradient families, we have to ensure the existence of one local attractor in a family of isolated weakly invariant sets.

Lemma 7. *Assume that* $(K1)$-$(K3)$, $(\overline{K}4)$ *hold and that a global compact attractor* $\mathcal{A}$ *exists. Let* $\mathcal{S} = \{\Xi_1, \ldots, \Xi_n\} \subset \mathcal{A}$ *be a disjoint family of isolated weakly invariant sets. If* G *is dynamically gradient with respect to* $\mathcal{S}$, *then one of the sets* Ξ_j *is a local attractor in* $\mathcal{A}$.

Proof. Let $\delta_0 > 0$ be such that $\mathcal{O}_{\delta_0}(\Xi_i) \cap \mathcal{O}_{\delta_0}(\Xi_j) = \varnothing$ if $i \neq j$ and Ξ_j be the maximal weakly invariant set in $\mathcal{O}_{\delta_0}(\Xi_j)$ for all j. First we will prove the existence of $j \in \{1, ..., n\}$ such that for all $\delta \in (0, \delta_0)$ there exists $\delta' \in (0, \delta)$ satisfying
$$\cup_{t\geq 0} G(t, \mathcal{O}_{\delta'}(\Xi_j) \cap \mathcal{A}) \subset \mathcal{O}_\delta(\Xi_j). \tag{5}$$
If not, there would exist $0 < \delta < \delta_0$ and for each j sequences $t_k^j \in \mathbb{R}^+$, $x_k^j \in \mathcal{A}$, $\varphi_k^j \in \mathcal{R}$ with $\varphi_k^j(0) = x_k^j$ such that
$$d(x_k^j, \Xi_j) < \frac{1}{k},$$
$$d(\varphi_k^j(t_k^j), \Xi_j) = \delta,$$
$$d(\varphi_k^j(t), \Xi_j) < \delta \text{ for all } t \in [0, t_k^j).$$

We have to consider two cases: $t_k^j \to +\infty$ or $t_k^j \leq C$.

Let $t_k^j \to +\infty$. We define the sequence

$$\psi_k^j(t) = \varphi_k^j(t + t_k^j) \text{ for } t \in [-t_k^j, \infty).$$

By Lemma 6 we obtain the existence of a complete trajectory of $\mathcal{R}$, $\psi^j(\cdot)$, such that a subsequence of ψ_k^j satisfies $\psi_k^j(t) \to \psi^j(t)$ for every $t \in \mathbb{R}$. Hence, $d(\psi^j(t), \Xi_j) \leq \delta < \delta_0$ for all $t \leq 0$. Therefore, as $\psi^j \in \mathbb{K}$, condition $(G1)$ implies that $d(\psi^j(t), \Xi_j) \to 0$ as $t \to -\infty$. On the other hand, since $d(\psi^j(0), \Xi_j) = \delta$, conditions $(G1) - (G2)$ imply that $d(\psi^j(t), \Xi_i) \to 0$ as $t \to +\infty$, where $i \neq j$.

Let now $t_k^j \leq C$. We can assume that $t_k^j \to t^j$. By $(\overline{K}4)$ we obtain the existence of $\varphi^j \in \mathcal{R}$ such that φ_k^j converges to φ^j uniformly on bounded sets of $[0, \infty)$. It is clear then that $d(\varphi^j(t^j), \Xi_j) = \delta$. As $\varphi^j(0) \in \Xi_j$ and Ξ_j is weakly invariant, there exists a complete trajectory of $\mathcal{R}$, $\psi_j^-(\cdot)$, such that $\psi_j^-(0) = \varphi^j(0)$ and $\psi_j^-(t) \in \Xi_j$ for all $t \leq 0$. Concatenating ψ_j^- and φ^j we define

$$\psi^j(t) = \begin{cases} \psi_j^-(t) \text{ if } t \leq 0, \\ \varphi^j(t) \text{ if } t \geq 0, \end{cases}$$

which is a complete trajectory by $(K3)$. Again, conditions $(G1) - (G2)$ imply that $d(\psi^j(t), \Xi_i) \to 0$ as $t \to +\infty$, where $i \neq j$.

We have obtained then a connection from Ξ_j to a different Ξ_i. Since this is true for any Ξ_j, we would obtain a homoclinic structure, which contradicts $(G2)$. Therefore, (5) holds for some j. It follows that

$$\omega(\mathcal{O}_{\delta'}(\Xi_j) \cap \mathcal{A}) \subset \overline{\mathcal{O}_\delta(\Xi_j)} \subset \mathcal{O}_{\delta_0}(\Xi_j).$$

Since $\omega(\mathcal{O}_{\delta'}(\Xi_j) \cap \mathcal{A})$ is weakly invariant, we obtain that $\omega(\mathcal{O}_{\delta'}(\Xi_j) \cap \mathcal{A}) \subset \Xi_j$. But $\Xi_j \subset G(t, \Xi_j) \subset G(t, \mathcal{O}_{\delta'}(\Xi_j) \cap \mathcal{A})$ for any $t \geq 0$ implies the converse inclusion, so that $\Xi_j = \omega(\mathcal{O}_{\delta'}(\Xi_j) \cap \mathcal{A})$. Thus, Ξ_j is a local attractor in $\mathcal{A}$. □

Now we prove the main result of this section which allows us to establish the equivalent definition of dynamically gradient families.

Theorem 3. *Assume that (K1)-(K3), $(\overline{K}4)$ hold and that a global compact attractor $\mathcal{A}$ exists. Let $\mathcal{S} = \{\Xi_1, \ldots, \Xi_n\} \subset \mathcal{A}$ be a disjoint family of isolated weakly invariant sets. Then G is dynamically gradient with respect to $\mathcal{S}$ in the sense of Definition 5 if and only if $\mathcal{S}$ can be reordered in such a way that any bounded complete trajectory $\gamma(\cdot)$ satisfies one of the following properties:*

1. *$\{\gamma(t) : t \in \mathbb{R}\} \subset \Xi_i$ for some i.*
2. *There are $i < j$ for which*

$$\gamma(t) \underset{t\to\infty}{\to} \Xi_i, \ \gamma(t) \underset{t\to-\infty}{\to} \Xi_j.$$

Proof. It is obvious that conditions 1-2 imply that G is dynamically gradient. We shall prove the converse.

By Lemma 7 one of the sets Ξ_i is a local attractor. After reordering the sets, we can say that Ξ_1 is the local attractor. Let

$$\Xi_1^* = \{x \in \mathcal{A} : \omega(x) \backslash \Xi_1 \neq \varnothing\}$$

be its repeller, which is weakly invariant by Lemma 5. Since Ξ_j are closed (cf. [9, Lemma 19]), weakly invariant and disjoint, we obtain that $\Xi_j \subset \Xi_1^*$ for $j \geq 2$.

We will consider only the dynamics inside the repeller Ξ_1^*, that is, we define the following set:

$$\mathcal{R}_1 = \{\varphi \in \mathcal{R} : \varphi(t) \in \Xi_1^* \ \forall t \geq 0\}.$$

Since Ξ_1^* is weakly invariant, $\mathcal{R}_1$ satisfies $(K1)$. Further, let $\varphi_\tau(\cdot) = \varphi(\cdot + \tau)$, where $\varphi \in \mathcal{R}_1$ and $\tau \geq 0$. Then it is clear that $\varphi_\tau(t) \in \mathcal{R}_1$ for all $t \geq 0$, and then $(K2)$ holds. If $\varphi_1(\cdot),\ \varphi_2(\cdot) \in \mathcal{R}_1$, it follows by $(K3)$ that the concatenation belongs also to $\mathcal{R}_1$. Finally, if $\varphi_n(0) \to \varphi_0$ with $\varphi_n(0) \in \Xi_1^*$ and $\varphi_n(\cdot) \in \mathcal{R}_1$, then $\varphi_0 \in \Xi_1^*$ (as Ξ_1^* is closed) and by $(\overline{K}4)$ passing to a subsequence $\varphi_n(t_n) \to \varphi(t)$, for $t_n \to t \geq 0$, where $\varphi \in \mathcal{R}$. Again, the closedness of Ξ_1^* implies that $\varphi \in \mathcal{R}_1$. Hence, $(\overline{K}4)$ also holds. We can define then the multivalued semiflow $G_1 : \mathbb{R}^+ \times \Xi_1^* \to P(\Xi_1^*)$:

$$G_1(t,x) = \{y \in \Xi_1^* : y = \varphi(t) \text{ for some } \varphi \in \mathcal{R}_1,\ \varphi(0) = x\},$$

which is strict by $(K3)$. This definition is equivalent to the following one:

$$\overline{G}_1(t,x) = G(t,x) \cap \Xi_1^* \text{ for } x \in \Xi_1^*.$$

Indeed, $G_1(t,x) \subset \overline{G}_1(t,x)$ is obvious. Conversely, let $y \in \overline{G}_1(t,x)$. Then, $y = \varphi(t),\ \varphi(\cdot) \in \mathcal{R}$, and $y \in \Xi_1^*$. We state that $\varphi(s) \in \Xi_1^*$ for all $0 \leq s \leq t$. Assume by contradiction that $\varphi(s) \notin \Xi_1^*$ for $0 < s < t$. Therefore, $\omega(\varphi(s)) \subset \Xi_1$. But then by $(K3)$,

$$G(T,y) \subset G(T, G(t-s, \varphi(s))) \subset G(T+t-s, \varphi(s)) \to \Xi_1 \text{ as } T \to \infty,$$

which is a contradiction with $y \in \Xi_1^*$. Using again $(K3)$ one can define a function $\psi(\cdot) \in \mathcal{R}_1$ such that $\psi(0) = y$, so that $y \in G_1(t,x)$.

It is clear that G_1 possesses a global compact attractor, which is the union of all bounded complete trajectories of $\mathcal{R}_1$, and that G_1 is dynamically gradient with respect to $\{\Xi_2, \ldots, \Xi_n\}$. Then, again by Lemma 7 we can reorder the sets in such a way that Ξ_2 is a local attractor in Ξ_1^*. Let $\Xi_{2,1}^*$ be the repeller of Ξ_2 in Ξ_1^*. Then we restrict as before the dynamics to the set $\Xi_{2,1}^*$ and so on. Hence, we have reordered the sets Ξ_j in such a way that Ξ_1 is a local attractor and Ξ_j is a local attractor for the dynamics restricted to the repeller of the previous local attractor $\Xi_{j-1,j-2}^*$ for $j \geq 2$, and $\Xi_i \subset \Xi_{j-1,j-2}^*$ if $i \geq j$, where $\Xi_{1,0}^* = \Xi_1^*$.

Now, if $\gamma(\cdot)$ is a bounded complete trajectory such that

$$\gamma(t) \underset{t\to\infty}{\to} \Xi_i,\ \gamma(t) \underset{t\to-\infty}{\to} \Xi_j,$$

then we shall prove that $i \leq j$. Moreover, if $\gamma(\cdot)$ is not completely contained in some Ξ_k, then $i < j$.

If $i = 1$, then it is clear that $j \geq 1$. Also, if there exists $\gamma(t_0) \notin \Xi_1$, then $j > 1$, as Ξ_1 is a local attractor.

Let $i = 2$. Then $\gamma(t) \in \Xi_1^*$ for all $t \in \mathbb{R}$, and then $\gamma(t) \underset{t\to-\infty}{\to} \Xi_1$ is forbidden. Hence, $j \geq 2$. Again, if there exists $\gamma(t_0) \notin \Xi_2$, then the fact that Ξ_2 is a local attractor in Ξ_1^* implies that $j > 2$.

Further, note that if $i \geq 3$, then $\gamma(t) \in \Xi_1^*$ for all $t \in \mathbb{R}$. Also, by induction, it follows that $\gamma(t) \in \Xi_{k,k-1}^*$ for all $t \in \mathbb{R}$ and $2 \leq k \leq i-1$. Indeed, let $\gamma(t) \in \Xi_{k-1,k-2}^*$ for all $t \in \mathbb{R}$ with $2 \leq k \leq i-1$. Then $\gamma(t) \underset{t\to\infty}{\to} \Xi_i$ implies clearly that $\gamma(t) \in \Xi_{k,k-1}^*$ for all $t \in \mathbb{R}$. In particular, $\gamma(t) \in \Xi_{i-1,i-2}^*$ for all $t \in \mathbb{R}$. Hence, $\Xi_j \in \Xi_{i-1,i-2}^*$, so that $j \geq i$. Finally, if there exists $\gamma(t_0) \notin \Xi_i$, then $j > i$ as Ξ_i is a local attractor in $\Xi_{i-1,i-2}^*$. □

To finish this section, we recall that the disjoint family of isolated weakly invariant sets $\mathcal{S} = \{\Xi_1, \ldots, \Xi_n\} \subset \mathcal{A}$ is a Morse decomposition of the global compact attractor $\mathcal{A}$ if there is a sequence of local attractors $\emptyset = A_0 \subset A_1 \subset \ldots \subset A_n = \mathcal{A}$ such that for every $k \in \{1, \ldots, n\}$ it holds

$$\Xi_k = A_k \cap A_{k-1}^*.$$

It is well known [16] that for general dynamical systems conditions 1-2 in Theorem 3 are equivalent to the fact that $\mathcal{S}$ generates a Morse decomposition. This fact can be proved also under conditions (K1)-(K3), $(\overline{K}4)$ [9].

Thus, Theorem 3 implies that under conditions $(K1)$-$(K3)$,$(\overline{K}4)$ the family $\mathcal{S}$ generates a Morse decomposition if and only if G is dynamically gradient.

5. **Application to a reaction-diffusion equation.**

We will consider the Chafee-Infante problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = f(u),\ t > 0,\ x \in (0,1)\,, \\ u(t,0) = 0,\ u(t,1) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{6}$$

where f satisfies

(A1) $f \in C(\mathbb{R})$;
(A2) $f(0) = 0$;
(A3) $f'(0) > 0$ exists and is finite;
(A4) f is strictly concave if $u > 0$ and strictly convex if $u < 0$;
(A5) Growth condition:

$$|f(u)| \leq C_1 + C_2 |u|^{p-1},$$

where $p \geq 2, C_1, C_2 > 0$;

(A6) Dissipation condition:

(a) If $p > 2$:

$$f(u)u \leq C_3 - C_4 |u|^p, \quad C_3, C_4 > 0.$$

(b) If $p = 2$:

$$\limsup_{u \to \pm\infty} \frac{f(u)}{u} \leq 0.$$

Remark 6. Note that as a consequence of condition (A6)(b), we have that $f(u)u \leq (\lambda_1 - C_5)u^2 + C_6$, where $C_5, C_6 > 0$ and $\lambda_1 = \pi^2$ is the first eigenvalue of the operator $-\frac{\partial^2 u}{\partial x^2}$ with Dirichlet boundary conditions.

Let $\Omega = (0,1)$ and $1/p + 1/q = 1$. Denote by $(\cdot,\cdot)$ and $\|\cdot\|_{L^2}$ the scalar product and norm in $L^2(\Omega)$, by $\|\cdot\|_{H_0^1}$ the norm in $H_0^1(\Omega)$ associated to the scalar product of gradients in $L^2(\Omega)$ thanks to Poincaré's inequality. As usual, let $H^{-1}(\Omega)$ be the dual space to $H_0^1(\Omega)$. Denote by $\langle \cdot,\cdot \rangle$ pairing between the space $L^p(\Omega) \cap H_0^1(\Omega)$ and its dual $L^q(\Omega) \cap H^{-1}(\Omega)$.

Definition 7. The function $u(\cdot) \in C([0,T], L^2(\Omega))$ is called a strong solution of (6) on $[0,T]$ if:

1. $u(0) = u_0$;
2. $u(\cdot)$ is absolutely continuous on compact subsets of $(0,T)$;

3. $u(t) \in H^2(\Omega) \cap H_0^1(\Omega)$, $f(u(t)) \in L^2(\Omega)$ for a.e. $t \in (0,T)$ and
$$\frac{du(t)}{dt} - \Delta u = f(u(t)), \text{ a.e. } t \in (0,T);$$
where the equality is understood in the sense of the space $L^2(\Omega)$.

Definition 8. The function $u(\cdot) \in C([0,T], L^2(\Omega))$ is called a weak solution of (6) on $[0,T]$ if:

1. $u \in L^\infty(0,T;L^2(\Omega))$;
2. $u \in L^2(0,T;H_0^1(\Omega)) \cap L^p(0,T;L^p(\Omega))$;
3. The equality in (6) is understood in the weak sense, i.e.
$$\frac{d}{dt}\langle u(t), v\rangle - \langle \Delta u, v\rangle = \langle f(u(t)), v\rangle, \ \forall v \in H_0^1(\Omega) \cap L^p(\Omega),$$
where the equality is understood in the sense of distributions.

Let us make some comments on the natural relation among the above two definitions. Let $u(\cdot)$ be a strong solution such that $f(u(\cdot)) \in L^2(0,T;L^2(\Omega))$. In view of [3, Proposition 2.2] we have that $u \in L^2(0,T;H_0^1(\Omega))$, so $\Delta u \in L^2(0,T;H^{-1}(\Omega))$ and then $\frac{du}{dt} \in L^2(0,T;H^{-1}(\Omega))$. Hence, by [20, Lemma 7.4] we get
$$\langle \frac{du}{dt}, v\rangle - \langle \Delta u, v\rangle = \langle f(u(t)), v\rangle, \ \forall v \in H_0^1(\Omega).$$
Using [22, p.250] we obtain
$$\frac{d}{dt}\langle u, v\rangle - \langle \Delta u, v\rangle = \langle f(u(t)), v\rangle, \ \forall v \in H_0^1(\Omega),$$
so point 3 of Definition 8 is satisfied.

Finally, if $p > 2$ by condition (A6)(a) we have
$$|u(t,x)|^p \le \frac{C_3}{C_4} - \frac{f(u(t,x))u(t,x)}{C_4}$$
Thus, $f(u)u \in L^1((0,T)\times\Omega)$ implies that $u \in L^p((0,T)\times\Omega) = L^p(0,T;L^p(\Omega))$. Hence, $u(\cdot)$ is a weak solution as well.

In view of [8, p.283], for any $u_0 \in L^2(\Omega)$ there exists at least one weak solution. Moreover, if $f(u(\cdot)) \in L^2(0,T;L^2(\Omega))$, then putting $g(\cdot) = f(u(\cdot))$ we obtain by [5, p.189] that the problem
$$\begin{cases} \dfrac{dv}{dt} - \Delta v = g(t), \\ v(0) = u_0, \end{cases}$$
possesses a unique strong solution $v(\cdot)$. Since this problem has also a unique weak solution $\tilde{v}(\cdot)$ and the strong solution is a weak solution as well, then $v(\cdot) = \tilde{v}(\cdot) = u(\cdot)$. Hence $u(\cdot)$ is also a strong solution of problem (6).

Therefore, we have checked that the sets of weak and strong solutions satisfying $f(u(\cdot)) \in L^2(0,T;L^2(\Omega))$ coincide.

5.1. **Stationary points.** We now focus on the properties of the stationary points. To this end, we have followed the classic procedure from [11] and [12]. Moreover, we have also taken some ideas from [18].

Properties $(K1)-(K4)$ are satisfied (cf. [13]). In view of $(K3)$ every weak solution can be extended for any $t \ge 0$, that is, to a globally defined one. Let $\mathcal{R} \subset C([0,\infty), L^2(\Omega))$ be the set of all globally defined weak solutions of problem (6) and let G be the associated multivalued semiflow (see Section 2). It is shown in [13, Lemma 12] that v is a fixed point of $\mathcal{R}$ (equivalently, of G) if and only if $v \in H_0^1(\Omega)$ and

$$\frac{\partial^2 v}{\partial x^2} + f(v) = 0, \text{ in } H^{-1}(\Omega). \tag{7}$$

The inclusion $H_0^1(\Omega) \subset L^\infty(\Omega)$ implies that $f(v) \in L^\infty(\Omega)$, so that $v \in H^2(\Omega) \cap H_0^1(\Omega)$. Therefore, $v(\cdot)$ is a strong solution as well.

Let consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(s) = \int_0^s f(r)\mathrm{d}r, \;\; s \in \mathbb{R}.$$

We define

$$a_- = inf\{s < 0 : \;\text{sgn}\; f(x) = \;\text{sgn}\; x, \; \forall x; s < x < 0\}$$

and

$$a_+ = sup\{s > 0 : \;\text{sgn}\; f(x) = \;\text{sgn}\; x, \; \forall x; 0 < x < s\}.$$

If follows from conditions (A2) and (A3) of f that $-\infty \leq a_- < 0 < a_+ \leq +\infty$. Since f is positive on $(0, a_+)$ and negative on $(a_-, 0)$, we have that F is strictly increasing on $[0, a_+)$, strictly decreasing on $(a_-, 0]$ and $F(0) = 0$. We consider $E_+, E_- \in [0, \infty]$ defined by

$$E_+ = \lim_{s \to a_+} F(s),$$
$$E_- = \lim_{s \to a_-} F(s).$$

Then, F has the inverse functions $U_+ : [0, E_+) \to [0, a_+)$, $U_- : [0, E_-) \to (a_-, 0]$.

We also define the following functions with domains $(0, E_+)$ and $(0, E_-)$, respectively, with values on $[0, \infty)$:

$$\tau_+(E) = \int_0^{U_+(E)} (E - F(u))^{-1/2} \,\mathrm{d}u, \;\; 0 < E < E_+,$$

$$\tau_-(E) = \int_{U_-(E)}^0 (E - F(u))^{-1/2} \,\mathrm{d}u, \;\; 0 < E < E_-.$$

Let us consider $v_0 \in \mathbb{R}$ and a solution u of

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + f(u) = 0, \\ u(0) = 0, u'(0) = v_0. \end{cases} \tag{8}$$

Note that the solution of the problem (8) is unique, since f is convex for $u < 0$ and concave for $u > 0$, so it is Lipschitz on compact intervals (see [28, p.4] or [10, p.8]).

If we define $E = v_0^2/2$, then:

$$\frac{(u'(x))^2}{2} + F(u(x)) = E.$$

On the other hand, the functions τ_+, τ_- evaluated in $E = v_0^2/2$ give us $\sqrt{2}$ the x-time necessary to go from the initial condition $u(0) = 0$, with initial velocity $v_0, -v_0$ respectively, to the point where $u'(T_+(E)) = 0$. Indeed, $u(x)$ satisfies $\frac{(u'(x))^2}{2} + F(u(x)) = E$, so $\frac{dx}{du} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{E - F(u)}}$. Since $u'(T_+(E)) = 0$ for $u = U_+(E)$, then

$$\sqrt{2} \int_0^{T_+(E)} 1 \;\, \mathrm{d}x = \int_0^{U_+(E)} \frac{1}{\sqrt{E - F(u)}} \mathrm{d}u = \tau_+(E).$$

By symmetry with respect to the u axis, the $x-$time it takes for $u(x)$ to go from $(U^+(E),0)$ to $(0,-v_0)$ is $T_+(E)$. By this way, if $2T_+(E)=1$, that is, $\tau^+(E)=\frac{1}{\sqrt{2}}$, then $u(\cdot)$ is a solution satisfying the boundary conditions $u(0)=u(1)=0$. Applying a similar reasoning for $\tau^-(E)$, we obtain that u satisfies the boundary conditions if, and only if, E satisfies for some $k\in\mathbb{N}$ only one of the following conditions:

$$k\tau_+(E)+(k-1)\tau_-(E)=\frac{1}{\sqrt{2}}, \tag{9}$$

$$k\tau_-(E)+(k-1)\tau_+(E)=\frac{1}{\sqrt{2}}, \tag{10}$$

$$k\tau_+(E)+k\tau_-(E)=\frac{1}{\sqrt{2}}. \tag{11}$$

Remark 7. Note that if E satisfies (9) or (10) for a certain k, then u has $2k$ zeros and if E satisfies (11), then u has $2k+1$ zeros. Our goal is to solve these equations for E as a function of $f'(0)$. To this end, we study the properties of $\tau_\pm$.

In order to obtain solutions of the equations (9), (10) and (11) it is necessary to make a change of variable for the functions $\tau_\pm$. Given $E\in(0,E_\pm)$, we put

$$Ey^2=F(u),\ 0\le y\le 1,\ 0\le u\le U_+(E)$$

and

$$Ey^2=F(u),\ -1\le y\le 0,\ U_-(E)\le u\le 0.$$

Hence, $du=(2yE/f(u))dy$ and $E-F(u)=E(1-y^2)$. By this change, we obtain

$$\tau_+(E)=2\sqrt{E}\int_0^1(1-y^2)^{-1/2}\frac{y}{f(u)}dy, 0<E<E_+;\ u=U_+(Ey^2), 0\le y\le 1;$$

$$\tau_-(E)=2\sqrt{E}\int_{-1}^0(1-y^2)^{-1/2}\frac{y}{f(u)}dy, 0<E<E_-; u=U_-(Ey^2), -1\le y\le 0.$$

The next results show some properties of these functions.

Theorem 4. *The functions $\tau_\pm$ satisfy*

$$\lim_{E\to 0^+}\tau_\pm(E)=\frac{\pi}{(2f'(0))^{1/2}}.$$

Proof. Since $f'(0)>0$ and $f(0)=0$, given $\varepsilon\in(0,1)$, there exists $\delta>0$ such that

$$\begin{gathered} f'(0)(1-\varepsilon)u\le f(u)\le f'(0)(1+\varepsilon)u,\quad 0\le u\le\delta.\\ \frac{1}{f'(0)(1+\varepsilon)}\le\frac{u}{f(u)}\le\frac{1}{f'(0)(1-\varepsilon)},\ 0\le u\le\delta. \end{gathered} \tag{12}$$

Moreover, as $U_+(E)$ is continuous at 0, given $\delta>0$, there exists $\eta>0$ such that for $0<E\le\eta$, $U_+(E)\le\delta$. Now, if we integrate (12) between 0 and u we obtain the following inequality

$$\frac{f'(0)}{2}(1-\varepsilon)u^2\le F(u)\le\frac{f'(0)}{2}(1+\varepsilon)u^2,\ 0\le u\le\delta.$$

Using the change of variable $Ey^2=F(u)$, we have

$$\left(\frac{f'(0)(1-\varepsilon)}{2E}\right)^{1/2}u\le y\le\left(\frac{f'(0)(1+\varepsilon)}{2E}\right)^{1/2}u,\quad\text{for}\ \ 0<E\le\eta,\ 0\le y\le 1.$$

Dividing the previous expression by $f(u)$ and using (12) we obtain

$$\left(\frac{1-\varepsilon}{2Ef'(0)(1+\varepsilon)^2}\right)^{1/2} \leq \frac{y}{f(u)} \leq \left(\frac{1+\varepsilon}{2Ef'(0)(1-\varepsilon)^2}\right)^{1/2}, \text{ for } \ 0 < E \leq \eta, \ 0 \leq y \leq 1.$$

Now if we multiply by $2\sqrt{E}(1-y^2)^{-\frac{1}{2}}$ and integrate from 0 to 1, we get

$$\pi\left(\frac{1-\varepsilon}{2f'(0)(1+\varepsilon)^2}\right)^{1/2} \leq \tau_+(E) \leq \pi\left(\frac{1+\varepsilon}{2f'(0)(1-\varepsilon)^2}\right)^{1/2}, \text{ for } \ 0 < E \leq \eta.$$

Finally, taking $\varepsilon \to 0$, the theorem follows. The proof for τ_- is analogous. □

Theorem 5. *The functions $\tau_\pm$ are strictly increasing on their domains.*

Proof. Let consider the expression of τ_+ and $0 < E_1 < E_2 < E_+$. Then,

$$\tau_+(E_2) - \tau_+(E_1) = \int_0^1 \frac{2y}{\sqrt{1-y^2}} \left[\frac{\sqrt{E_2}}{f(U^+(E_2 y^2))} - \frac{\sqrt{E_1}}{f(U^+(E_1 y^2))}\right] dy.$$

From [10, p.8] we have that the function f is differentiable almost everywhere in $\mathbb{R}$, so $\alpha(E) = \dfrac{\sqrt{E}}{f(U^+(Ey^2))}$ is differentiable as well. Hence,

$$\alpha'(E) = \frac{f^2(U^+(Ey^2)) - 2y^2 E f'(U^+(Ey^2))}{2\sqrt{E} f^3(U^+(Ey^2))}.$$

Recall the change of variable $F(u) = Ey^2$. Consider the numerator of α', that is, $\beta(u) = f^2(u) - 2F(u)f'(u)$. Then we obtain

$$\beta(u) = 2\int_0^u f(s)(f'(s) - f'(u))ds, \ 0 < s < u.$$

Since f is strictly concave, if $s < u$, then $f'(s) > f'(u)$ (cf. [28, p.5]). As a result, $\beta(u) > 0$.

In order to finish the proof rigorously, we have to justify the previous calculations. Indeed, from [10, p.5], we have that the function f is absolutely continuous and from [5, p.16], $f' \in L^1_{loc}$. Therefore, $\alpha' \in L^1_{loc}$ and $\alpha' > 0$ a.e., which implies that $\alpha(E)$ is strictly increasing and the proof is finished.

The claim for $\tau_-(E)$ follows analogously. □

Theorem 6. *The functions $\tau_\pm$ satisfy*

$$\lim_{E \to E^\pm} \tau_\pm(E) = \infty$$

Then, $\tau_\pm : (0, E^\pm) \to \left(\dfrac{\pi}{(2f'(0))^{1/2}}, \infty\right)$.

Proof. **Case $a_+ < \infty$.** Then, we have $f(a_+) = 0$ and $\bar{u}(x) = a_+$ is a constant solution to the problem $\frac{\partial^2 u}{\partial x^2} + f(u) = 0$. Let us consider $E_+ = F(a_+)$ and the solution u to this problem satisfying the conditions $u(0) = 0, u'(0) = v_0, E = \frac{1}{2}v_0^2$. As a_+ is a constant solution, by uniqueness $\tau_+(E^+) = \infty$. Therefore, given $T > 0$, there exists $\delta > 0$ such that if $E > E_+ - \delta$, then $\tau_+(E) > T$, which follows from the continuity of u with respect to its initial conditions.

Case $a_+ = \infty$. Note that if $p > 2$, then $a_+ < \infty$. Therefore, $p = 2$. In this case, $f(u) > 0$ for all $u \in (0, \infty)$. From condition (A5), there exist $\alpha, \beta > 0$ such that $f(u) \leq \alpha + \beta u$. For $u > 0$ we have

$$\frac{f(u)}{u^2} \leq \frac{\alpha}{u^2} + \frac{\beta}{u}.$$

Hence, $f(u)/u^2 \to 0$, as $u \to \infty$.

On the other hand, $\int_0^u f(s)ds \leq \int_0^u (\alpha + \beta s)\, ds$. Thus, we have $F(u) \leq \alpha u + \beta u^2/2$ and

$$0 \leq \frac{F(u)}{u^3} \leq \frac{\alpha}{u^2} + \frac{\beta}{2}\frac{1}{u}.$$

Hence, $F(u)/u^3 \to 0$, as $u \to \infty$.

We claim that $\lim\limits_{u \to 0^+} f(u)/u^2 = \infty$. Indeed, since $f'(0)$ exists, for any $\varepsilon \in (0, f'(0))$, there exists $\delta > 0$ such that $|f'(0) - f(u)/u| < \varepsilon$, for any $|u| < \delta$. Thus, dividing by u^2, we obtain

$$\frac{u(f'(0) - \varepsilon)}{u^2} < \frac{f(u)}{u^2} < \frac{u(\varepsilon + f'(0))}{u^2}$$

and the result follows.

Since $f(u)/u^2 \to 0$, as $u \to \infty$, and $f(u)/u^2 \to \infty$, as $u \to 0^+$, for any $\varepsilon > 0$, there exists a first value $u_0 \in (0, \infty)$ where $f(u_0)/u_0^2 = \varepsilon$. Hence,

$$\frac{f(u)}{u^2} > \varepsilon,\ 0 < u < u_0.$$

From the above expression, we have $\int_0^u f(s)ds > \int_0^u \varepsilon s^2 ds$ and $\varepsilon u^3/3 < F(u)$. Then, $F(u)/u^3 > \varepsilon/3$, if $0 < u \leq u_0$. Since $F(u)/u^3 \to 0$, as $u \to \infty$, we deduce that there exists a first $\overline{u} > u_0$ such that $F(\overline{u})/\overline{u}^3 = \varepsilon/3$. Hence, we have

$$\frac{F(u)}{u^3} > \frac{\varepsilon}{3},\ 0 < u < \overline{u},$$

with $F(\overline{u}) = \frac{\varepsilon}{3}\overline{u}^3$.

Now, computing τ_+ in $\overline{E} = F(\overline{u})$, we have

$$\begin{aligned}
\tau_+(\overline{E}) &= \int_0^{U^+(\overline{E})} \frac{1}{\sqrt{\overline{E} - F(u)}} du = \int_0^{\overline{u}} \frac{1}{\sqrt{\frac{\varepsilon}{3}\overline{u}^3 - F(u)}} du \\
&\geq \int_0^{\overline{u}} \frac{1}{\sqrt{\frac{\varepsilon}{3}\overline{u}^3 - \frac{\varepsilon}{3}u^3}} du = \frac{\sqrt{3}}{\sqrt{\varepsilon}} \int_0^{\overline{u}} \frac{1}{\sqrt{\overline{u}^3 - u^3}} du \\
&= \frac{\sqrt{3}}{\sqrt{\varepsilon}} \int_0^1 \frac{\overline{u}}{\sqrt{\overline{u}^3 - \overline{u}^3 t^3}} dt = \frac{\sqrt{3}}{\sqrt{\varepsilon}} \frac{\overline{u}}{\sqrt{\overline{u}^3}} \int_0^1 \left(1 - t^3\right)^{-\frac{1}{2}} dt \\
&= \frac{\sqrt{3}}{\sqrt{\varepsilon}} \frac{\overline{u}}{\sqrt{\overline{u}^3}} \frac{1}{3} \int_0^1 s^{\frac{1}{3}-1} (1 - s)^{\frac{1}{2}-1} ds \\
&= \frac{1}{\overline{u}^{\frac{1}{2}}} \frac{1}{\sqrt{\varepsilon}} \frac{\sqrt{3}}{3} \mathcal{B}\left(\frac{1}{2}, \frac{1}{3}\right).
\end{aligned}$$

Recall that $\varepsilon \overline{u}^3 = 3F(\overline{u})$. Then,

$$\varepsilon \overline{u} = 3\frac{F(\overline{u})}{\overline{u}^2}.$$

Taking $\varepsilon \to 0$, by construction $\overline{u} \to \infty$. Therefore, from condition (A6)(b) we have that $\lim_{u\to\infty} f(u)/u \leq 0$, so the last expression tends to 0 and $\tau_+(\overline{E}) \to \infty$. □

Theorem 7. *Consider*

$$\lambda_n = n^2\pi^2.$$

Then, for each $n \geq 1$, there exist two continuous functions $E_n^\pm : [\lambda_n, \infty) \to [0, E_\pm)$ with the following properties:

1. *For each integer $k \geq 1$ and for $f'(0) \in [\lambda_{2k-1}, \infty)$ the only solution of the equation (9) (resp. 10) is the value $E_{2k-1}^+(f'(0))$ (resp. $E_{2k-1}^-(f'(0))$);*
2. *For each integer $k \geq 1$ and for $f'(0) \in [\lambda_{2k}, \infty)$ the only solution of the equation (11) is the value $E_{2k}^-(f'(0)) = E_{2k}^+(f'(0)) = E_{2k}$;*
3. *For each integer $n \geq 1$, $E_n^\pm(f'(0)) = 0$, if $f'(0) = \lambda_n$.*

Proof. Let be $n \geq 1$. If n is odd, then $n = 2k - 1$ for $k \geq 1$. First, we prove that we can define the function

$$E_n^\pm : [\lambda_n, \infty) \longrightarrow [0, E_\pm)$$

by putting $E_n^\pm(f'(0)) = E$, where E satisfies $k\tau_\pm(E) + (k-1)\tau_\mp(E) = 1/\sqrt{2}$.

Consider the function

$$h_\pm^n : (0, E_\pm) \longrightarrow (n\pi/\sqrt{2f'(0)}, \infty),$$

defined by $h_\pm^n(E) := k\tau_\pm(E) + (k-1)\tau_\mp(E)$. If $f'(0) > \lambda_n$ then, as $h_\pm$ is a strictly increasing function, there exists a unique $E_{2k-1}^\pm \in (0, E_\pm)$ such that $h_\pm^n(E_{2k-1}^\pm) = 1/\sqrt{2}$.

Since $h_\pm$ has inverse, $E_{2k-1}^\pm = (h_\pm^n)^{-1}(1/\sqrt{2})$ is the solution of the expressions (9) and (10). Moreover, $E_{2k-1}^\pm(\lambda_n) = 0$ by construction.

Second, if n is even, then $n = 2k$ for $k \geq 1$. As before, we consider $h_\pm^n(E) := k\tau_\pm(E) + k\tau_\mp(E)$. Since it is an increasing function, for $f'(0) > \lambda_n$, there exists a unique $E_{2k} \in (0, E_\pm)$ such that $h_\pm^n(E_{2k}) = 1/\sqrt{2}$. Analogously, we obtain the solution of the expression (11), $E_{2k}^\pm = (h_\pm^n)^{-1}(1/\sqrt{2})$, and $E_{2k-1}^\pm(\lambda_n) = 0$. □

Theorem 8. *For each $n \geq 1$ and $f'(0) \in [\lambda_n, \infty)$, the equation (7) has two new more solutions $v_n^\pm$ with the following properties:*

1. *$a_- < u_n^\pm(x) < a_+$ for all $x \in [0, 1]$;*
2. *If $f'(0) = \lambda_n$, then $v_n^\pm = 0$;*
3. *For $f'(0) \in (\lambda_n, \infty)$, $v_n^\pm$ has $n + 1$ zeros in $[0, 1]$. Denoting these zeros by $x_q^\pm$, $q = 0, 1, \dots, n$ with $0 = x_0^\pm < x_1^\pm < x_2^\pm < \dots < x_n^\pm = 1$, we have $(-1)^q v_n^+(x) > 0$ for $x_q^+ < x < x_{q+1}^+, q = 0, 1, \dots, n-1$ and $(-1)^q v_n^-(x) < 0$ for $x_q^- < x < x_{q+1}^-, q = 0, 1, \dots, n-1$. Also, $v_n^+ = -v_n^-$, if f is odd;*

Proof. The first point follows from $F(u_n^\pm(x)) \leq E < E_\pm$.

The second point follows from the third one of Theorem 7. Indeed, for each $n \geq 1$ and $f'(0) \in [\lambda_n, \infty)$ we have the values $E_n^\pm(f'(0))$ by the above theorem. Also, we have a solution of the equation (7) which is denoted by $v_n^\pm$. If $f'(0) = \lambda_n$, then $E_n^\pm(\lambda_n) = 0$ and $v_0 = 0$, so $v_n^\pm = 0$.

The third point follows by Remark 7. If f is odd, then $-U^-(E) = U^+(E)$, $\tau_+(E) = \tau_-(E)$, so we have $v_n^+ = -v_n^-$. □

Corollary 1. *If $n^2\pi^2 < f'(0) \leq (n+1)^2\pi^2, n \in \mathbb{N}$, then there are $2n+1$ fixed points: 0, $v_1^{\pm}$, ..., $v_n^{\pm}$, where $v_j^{\pm}$ possesses $j+1$ zeros in $[0,1]$.*

5.2. **Approximations.** From now on, we shall consider the following family of Chafee-Infante equations

$$\left\{ \begin{array}{c} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = f_\varepsilon(u),\ t>0,\ x\in(0,1)\,, \\ u(t,0)=0,\ u(t,1)=0, \\ u(0,x)=u_0(x), \end{array} \right. \tag{13}$$

where $\varepsilon \in (0,1]$ is a small parameter and f_ε satisfies

$(\widetilde{A1})$ $f_\varepsilon \in C(\mathbb{R})$ and is non-decreasing;
$(\widetilde{A2})$ $f_\varepsilon(0) = 0$;
$(\widetilde{A3})$ $f_\varepsilon'(0) > 0$ exists, is finite, monotone in ε and $f_\varepsilon'(0) \to \infty$, as $\varepsilon \to 0^+$;
$(\widetilde{A4})$ f_ε is strictly concave if $u > 0$ and strictly convex if $u < 0$;
$(\widetilde{A5})$ $-1 < f_\varepsilon(s) < 1$, for all s, and

$$|f_\varepsilon(s) - H_0(s)| < \varepsilon, \quad \text{if } |s| > \varepsilon, \tag{14}$$

where

$$H_0(u) = \left\{ \begin{array}{ll} -1, & \text{if} \quad u<0, \\ [-1,1]\,, & \text{if} \quad u=0, \\ 1, & \text{if} \quad u>0, \end{array} \right.$$

is the Heaviside function.

Conditions (A1)-(A6) are satisfied with $p = 2$, so problem (13) is a particular case of (6).

Our aim now is to prove that for ε sufficiently small the multivalued semiflow G_ε generated by the weak solutions of problem (13) is dynamically gradient. Problem (13) is an approximation of the following problem, governed by a differential inclusion

$$\left\{ \begin{array}{c} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} \in H_0(u),\ \text{on } \Omega\times(0,T), \\ u|_{\partial\Omega} = 0, \\ u(0,x)=u_0(x). \end{array} \right. \tag{15}$$

We say that the function $u \in C([0,T], L^2(\Omega))$ is a strong solution of (15) if

1. $u(0) = u_0$;
2. $u(\cdot)$ is absolutely continuous on $(0,T)$ and $u(t) \in H^2(\Omega) \cap H_0^1(\Omega)$ for a.e. $t \in (0,T)$;
3. There exists a function $g(\cdot)$ such that $g(t) \in L^2(\Omega)$, a.e. on $(0,T)$, $g(t,x) \in H_0(u(t,x))$, for a.e. $(t,x) \in (0,T) \times \Omega$, and
$$\frac{du}{dt} - \frac{\partial^2 u}{\partial x^2} - g(t) = 0,\ \text{a.e. } t \in (0,T).$$

In this case we put $\mathcal{R}$ as the set of all strong solutions such that the map g belongs to $L^2(0,T;L^2(\Omega))$. Conditions (K1)-(K4) are satisfied (cf. [9]) and the map $G : \mathbb{R}_+ \times L^2(\Omega) \to P(L^2(\Omega))$ defined by (1) is a strict multivalued semiflow possessing a global compact attractor $\mathcal{A}_0$ (cf. [25]) in $L^2(\Omega)$, which is connected (cf. [26]). The structure of this attractor is studied in [3]. It is shown that there exists an infinite (but countable) number of fixed points

$$v_0 = 0, v_1^+, v_1^-, \ldots, v_n^+, v_n^-, \ldots,$$

and that $\mathcal{A}_0$ consists of these fixed points and all bounded complete trajectories $\psi(\cdot)$, which always connect two fixed points, that is,

$$\begin{aligned}\psi(t) &\to z_1 \text{ as } t \to \infty,\\ \psi(t) &\to z_2 \text{ as } t \to -\infty,\end{aligned} \tag{16}$$

where $z_i = 0, z_i = v_n^+$ or $z_i = v_n^-$ for some $n \geq 1$. Moreover, if ψ is not a fixed point, then either $z_2 = 0$ and $z_1 = v_n^\pm$, for some $n \geq 1$, or $z_2 = v_k^\pm$, $z_1 = v_n^\pm$ with $k > n$.

We fix some $N_0 \in \mathbb{N}$. Denote

$$Z_{N_0} = \left(\cup_{k \geq N_0} \{v_k^\pm\}\right) \cup \{v_0\}$$

and define the sets

$$\Xi_k^0 = \{v_k^+, v_k^-\}, \quad 1 \leq k \leq N_0 - 1,$$

$$\Xi_{N_0}^0 = \left\{ \begin{array}{c} y : \exists \psi \in \mathbb{K} \text{ such that (16) holds with } z_j \in Z_{N_0}, \\ j = 1, 2 \text{ and } y = \psi(t) \text{ for some } t \in \mathbb{R} \end{array} \right\},$$

where $\mathbb{K}$ stands for the set of all bounded complete trajectories. We note that set $\Xi_{N_0}^0$ contains the fixed points in Z_{N_0} and all bounded complete trajectories connecting them.

Remark 8. It is known [9] that the family $\mathcal{M} = \{\Xi_1^0, \ldots, \Xi_{N_0}^0\}$ is a disjoint family of isolated weakly invariant sets and that G_0 is dynamically gradient with respect to $\mathcal{M}$ in the sense of Remark 4. Hence, G_0 is dynamically gradient with respect to $\mathcal{M}$ in the sense of Definition 5.

Now our purpose is to adapt some lemmas from [3, p.2979] to problem (13). In view of Theorems 7 and 8 and the third condition on f_ε, there exists a sequence $\overline{\varepsilon}_k \to 0$, as $k \to \infty$, such that for every $\varepsilon \in (\overline{\varepsilon}_k, \overline{\varepsilon}_{k+1}]$ and any $k \geq 1$ problem (13) has exactly $2k+1$ fixed points $\{v_0^\varepsilon = 0, \{v_{\varepsilon,j}^+\}_{j=1}^k\}$ such that for each $1 \leq n \leq k$ $v_{\varepsilon,n}^\pm$ has $n+1$ zeros in $[0,1]$.

Let us consider a sequence $\{\varepsilon_m\}$ converging to zero.

Lemma 8. *Let $n \in \mathbb{N}$ be fixed. Then, $v_{\varepsilon_m,n}^+$ (resp. $v_{\varepsilon_m,n}^-$) do not converge to 0 in $H_0^1(0,1)$ as $\varepsilon_m \to 0$.*

Proof. Suppose that $v_{\varepsilon_m,n}^+ \to 0$ in $H_0^1(0,1)$. Then $v_{\varepsilon_m,n}^+ \to 0$ in $C([0,1])$. By Remark 7, $v_{\varepsilon_m,n}^+$ has a unique maximum in $a \in (0, x_1^+)$ and by the properties of τ_+ described before $a = \frac{x_1^+}{2}$. We may assume that x_1^+ does not converge to 0. Let $x_0(\varepsilon_m)$ be the first point where $v_{\varepsilon_m,n}^+(x_0) = \varepsilon_m$ or $x_0 = a$ if such a point does not exist. We claim that $x_0(\varepsilon_m) \to 0$, as $\varepsilon_m \to 0$. It is clear that $\partial^2 v_{\varepsilon_m,n}^+/\partial x^2 = -f_{\varepsilon_m}(v_{\varepsilon_m,n}^+) < 0$ in $(0, x_1^+)$, and then

$$\frac{v_{\varepsilon_m,n}^+(x_0)}{x_0} x \leq v_{\varepsilon_m,n}^+(x) \leq \varepsilon_m, \quad \forall x \in [0, x_0], \tag{17}$$

by concavity. Hence, integrating first on (s,a) and then on $(0,x)$ with $x \leq x_0$, we have

$$\frac{d}{dx} v_{\varepsilon_m,n}^+(s) = \int_s^a f_{\varepsilon_m}(v_{\varepsilon_m,n}^+(\tau)) d\tau, \tag{18}$$

$$v_{\varepsilon_m,n}^+(x) = \int_0^x \int_{x_0}^a f_{\varepsilon_m}(v_{\varepsilon_m,n}^+(\tau)) d\tau ds + \int_0^x \int_s^{x_0} f_{\varepsilon_m}(v_{\varepsilon_m,n}^+(\tau)) d\tau ds.$$

Since $f_\varepsilon(u)$ is concave, we have that $f_\varepsilon(u)/u \geq f_\varepsilon(\varepsilon)/\varepsilon$, $\forall\ 0 < u \leq \varepsilon$. Moreover, by assumption $(\widetilde{A5})$ of f_ε we get $f_\varepsilon(u) \geq \frac{1-\varepsilon}{\varepsilon}u$, for all $0 < u \leq \varepsilon$. Hence, using (17) we have

$$v^+_{\varepsilon_m,n}(x) \geq \int_0^x \int_s^{x_0} \frac{1-\varepsilon_m}{\varepsilon_m} v^+_{\varepsilon_m,n}(\tau) d\tau ds \geq \frac{1-\varepsilon_m}{\varepsilon_m} \frac{v^+_{\varepsilon_m,n}(x_0)}{x_0} \int_0^x \int_s^{x_0} \tau d\tau ds.$$

Thus,

$$1 \geq \frac{1-\varepsilon_m}{\varepsilon_m}\left(\frac{xx_0}{2} - \frac{x^3}{6x_0}\right),$$

so it follows that $x_0 \to 0$, as $\varepsilon_m \to 0$.

Let $\delta_1 < 0 < \delta_2$ be such that $x_0(\varepsilon_m) \leq \delta_1 < \delta_2 \leq a(\varepsilon_m)$. Since $v^+_{\varepsilon_m,n}(x) \geq \varepsilon_m$ $\forall x \in [x_0, a]$, if we intregate (18) over (δ_1, x) with $\delta_1 < x \leq \delta_2$, we have

$$v^+_{\varepsilon_m,n}(x) - v^+_{\varepsilon_m,n}(\delta_1) = \int_{\delta_1}^x \int_s^a f(v^+_{\varepsilon_m,n}(\tau)) d\tau ds \geq (1-\varepsilon_m) \int_{\delta_1}^x \int_s^a d\tau ds,$$

which implies a contradiction if $v^+_{\varepsilon_m,n} \to 0$ in $C([0,1])$.

The proof is similar for $v^-_{\varepsilon_m,n}$. □

Lemma 9. *$v^+_{\varepsilon_m,k}$ (resp. $v^-_{\varepsilon_m,k}$) converges to v^+_k (resp. v^-_k) in $H^1_0(\Omega)$ as $m \to \infty$ for any $k \geq 1$.*

Proof. It is easy to see that $v^+_{\varepsilon_m k}$ is bounded in $H^2(\Omega) \cap H^1_0(\Omega)$, so $v^+_{\varepsilon_m k} \to v$ strongly in $H^1_0(\Omega)$ and $C([0,1])$ up to a subsequence. The proof will be finished if we prove that $v = v^+_k$. We observe that since in such a case every subsequence would have the same limit, the whole sequence would converge to v^+_k

It is clear that the functions $g_{\varepsilon_m} = f_{\varepsilon_n}(v^+_{\varepsilon_m k})$ are bounded in $L^\infty(0,1)$.

Passing to a subsequence we can then assume that g_{ε_n} converges to some g weakly in $L^2(0,1)$. It is clear that $-(\partial^2 v/\partial x^2) = g$ and v is a fixed point if we prove the inclusion $g(x) \in H_0(v(x))$ for a.e. $x \in (0,1)$. By Masur's theorem [29, p.120] there exist $z_m \in V_m = conv(\cup_{k \geq m}^\infty g_{\varepsilon_k})$ such that $z_m \to g$, as $m \to \infty$, strongly in $L^2(0,1)$. Taking a subsequence we have $z_m(x) \to g(x)$, a.e. in $(0,1)$. Since $z_m \in V_m$, we get $z_m = \sum_{i=1}^{N_m} \lambda_i g_{\varepsilon_{k_i}}$, where $\lambda_i \in [0,1]$, $\sum_{i=1}^{N_m} \lambda_i = 1$ and $k_i \geq m$, for all i.

Now (14) implies that $|g_{\varepsilon_k}(x) - H_0(v(x))| \to 0$, as $k \to \infty$, for a.e. x. Indeed, if $v(x) = 0$, then $g_{\varepsilon_k}(x) \in [-1,1] = H_0(v(x))$. If $v(x) > 0$, then $|g_{\varepsilon_k}(x) - H_0(v(x))| = |f_{\varepsilon_k}(v_{\varepsilon_k}(x)) - 1| \to 0$, as $k \to \infty$. If $v(x) < 0$, we apply a similar argument.

Thus, for any $\delta > 0$ and a.e. x there exists $m(x,\delta)$ such that $g_{\varepsilon_k}(x) \subset [a(x) - \delta, b(x) + \delta]$, for all $k \geq m$, where $[a(x), b(x)] = H_0(v(x))$. Hence, $z_m(x) \subset [a(x) - \delta, b(x) + \delta]$, as well. Passing to the limit we obtain $g(x) \in [a(x), b(x)]$, a.e. on $(0,1)$.

To conclude the proof, we have to prove that $v = v^+_k$. By Lemma 8 $v \neq 0$. Hence, as $v^+_{\varepsilon_m k}(x) > 0$ for all $x \in (0, x^+_1(\varepsilon_m))$, $v = v^+_n$ for some $n \in \mathbb{N}$. Since v^+_n has $n+1$ zeros, the convergence $v^+_{\varepsilon_m k} \to v^+_n$ implies that $v^+_{\varepsilon_m k}$ has $n+1$ zeros for $m \geq N$. But $v^+_{\varepsilon_m k}$ possesses $k+1$ zeros. Thus, $k = n$.

For the sequence $v^-_{\varepsilon_m k}$ the proof is analogous. □

Lemma 10. *Let $\varepsilon_m \to 0$, $k_m \to \infty$ as $m \to \infty$. Then $v^+_{\varepsilon_m,k_m}$ (resp. $v^-_{\varepsilon_m,k_m}$) converges to 0 as $m \to \infty$.*

Proof. In the same way as in the proof of Lemma 9 we obtain that up to a subsequence $v^+_{\varepsilon_m,k_m} \to v$ in $H^1_0(\Omega)$ and $C([0,1])$, where v is a fixed point of problem (15). We will prove that $v = 0$ by contradiction. If not, then $v = v^\pm_n$ for some

$n \in \mathbb{N}$. However, since $v_n^{\pm}$ has exactly $n+1$ zeros and $v^+_{\varepsilon_m,k_m} \to v$ in $C([0,1])$, we have that $v^+_{\varepsilon_m,k_m}$ has $n+1$ zeros for any $m \geq M$ with M big enough. This contradicts the fact that $v^+_{\varepsilon_m,k_m}$ possesses k_m+1 zeros and $k_m \to \infty$. As the limit is 0 for every converging subsequence, the whole sequence converges to 0.

For the sequence $v^-_{\varepsilon_m k}$ the proof is analogous. □

Once we have described the preliminary properties, we are now ready to check that (13) satisfies the conditions given in Theorem 2 for certain families $\mathcal{M}_\varepsilon$. We recall that [27, Theorem 10] guarantees the existence of the global compact invariant attractors $\mathcal{A}_\varepsilon$, where each $\mathcal{A}_\varepsilon$ is the union of all bounded complete trajectories.

Let us check assumptions (H1)-(H5) of Theorem 2.

As we have seen before, condition (H2) follows from Remark 8. Therefore, we prove now condition (H1).

Multiplying the equation in (13) by u, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} + \|u\|^2_{H^1_0} &\leq \int_\Omega |u|dx \\ &\leq \frac{1}{2}\|u\|^2_{H^1_0} + C, \end{aligned} \tag{19}$$

where we have used Poincaré's inequality. Denoting by λ_1 the first eigenvalue of the operator $-\Delta$ in $H^1_0(\Omega)$, we have

$$\frac{d}{dt}\|u\|^2_{L^2} \leq -\lambda_1 \|u\|^2_{L^2} + K.$$

Gronwall's lemma gives

$$\|u(t)\|^2_{L^2} \leq e^{-\lambda_1 t}\|u(0)\|^2_{L^2} + \frac{K}{\lambda_1}, \quad t \geq 0. \tag{20}$$

Integrating (19) over $(t, t+r)$ with $r > 0$ we have

$$\|u(t+r)\|^2_{L^2} + \int_t^{t+r} \|u\|^2_{H^1_0} ds \leq \|u(t)\|^2_{L^2} + rK$$

Then by (20),

$$\int_t^{t+r} \|u\|^2_{H^1_0} ds \leq \|u(0)\|^2_{L^2} e^{-\lambda_1 t} + \left(\frac{1}{\lambda_1} + r\right) K. \tag{21}$$

On the other hand, multiplying (13) by $-\Delta u$ and using Young's inequality we obtain

$$\frac{d}{dt}\|u\|^2_{H^1_0} + 2\|\Delta u\|^2_{L^2} \leq \|f_\varepsilon(u)\|^2_{L^2} + \|\Delta u\|^2_{L^2} \tag{22}$$

Since $f_\varepsilon(u(\cdot)) \in L^2(0,T;L^2(\Omega)), \forall T > 0$, we obtain by [5, p.189] that

$$u \in L^\infty(\eta, T; H^1_0(\Omega)),$$

$$\frac{du}{dt} \in L^2(\eta, T; L^2(\Omega)), \quad \forall\ 0 < \eta < T.$$

This regularity guarantees that the equality

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{H^1_0} = \langle \frac{du}{dt}, -\Delta u \rangle, \text{ for a.e. } t, \tag{23}$$

is correct [21, p.102]. Then

$$\frac{d}{dt}\|u\|^2_{H^1_0} \leq \overline{K} + \|u\|^2_{H^1_0}.$$

We apply the uniform Gronwall lemma [22, p. 91] with $y(s) = \|u(s)\|^2_{H^1_0}$, $g(s) = 1$ and $w(s) = \overline{K}$. Also, using (21) we obtain

$$\|u(t+r)\|^2_{H^1_0} \leq \left(\frac{\|u(0)\|^2_{L^2} e^{-\lambda_1 t} + (\frac{1}{\lambda_1} + r)K}{r} + \overline{K}r \right) e^r. \tag{24}$$

It follows from (20) that $\|y\|_{L^2} \leq \frac{K}{\lambda_1}$ for any $y \in \mathcal{A}_\varepsilon$, $0 < \varepsilon \leq 1$. Hence, $\cup_{0<\varepsilon\leq 1}\mathcal{A}_\varepsilon$ is bounded in $L^2(\Omega)$. Since $\mathcal{A}_\varepsilon \subset G_\varepsilon(t, \mathcal{A}_\varepsilon)$ for any $t \geq 0$, for any $y \in \mathcal{A}_\varepsilon$ there exists $z \in \mathcal{A}_\varepsilon$ such that $y \in G_\varepsilon(1, z)$. Then using (24) with $r = 1$ and $t = 0$ we obtain that

$$\|y\|^2_{H^1_0} \leq \left(\|z\|^2_{L^2} + \left(\frac{1}{\lambda_1} + 1\right) K + \overline{K} \right) e,$$

so $\cup_{0<\varepsilon\leq 1}\mathcal{A}_\varepsilon$ is bounded in $H^1_0(\Omega)$. The compact embedding $H^1_0(\Omega) \subset L^2(\Omega)$ implies that $\cup_{0<\varepsilon\leq 1}\mathcal{A}_\varepsilon$ is relatively compact in $L^2(\Omega)$. As the global attractor A_0 of the differential inclusion (15) is compact, the set $\overline{\cup_{0\leq\varepsilon\leq 1}\mathcal{A}_\varepsilon}$ is compact in $L^2(\Omega)$.

In order to establish that (13) satisfies the rest of conditions given in Theorem 2, we need to proof two previous results related to the convergence of solutions of the approximations and the connections between fixed points.

Theorem 9. *If $u_{\varepsilon_n 0} \to u_0$ in $L^2(\Omega)$ as $\varepsilon_n \to 0$, then for any sequence of solutions of (13) $u_{\varepsilon_n}(\cdot)$ with $u_{\varepsilon_n}(0) = u_{\varepsilon_n 0}$ there exists a subsequence of ε_n such that u_{ε_n} converges to some strong solution u of (15) in the space $C([0,T], L^2(\Omega))$, for any $T > 0$.*

Proof. We define $g_n(t) = f_{\varepsilon_n}(u_{\varepsilon_n}(t))$ and $u_n(t) = u_{\varepsilon_n}(t)$. From (20) we have that $\|u_n(t)\|_{L^2} \leq C_0$, for all $t \geq 0$, so that $\|g_n(t)\|_{L^2} \leq C_1$, for a.e. $t \geq 0$. Hence, there exists a subsequence such that $u_n \to u$ weakly in $L^2(0,T;L^2(\Omega))$. It follows from (22) and $\|g_n(t)\|_{L^2} \leq C_1$ that $\int_r^T \|\Delta u\|^2_{L^2} ds \leq C_1^2(T-r) + \|u_n(r)\|^2_{H^1_0}$. Using (24) we obtain that $\int_r^T \|\Delta u_n\|^2_{L^2} ds \leq C(r)$. Hence, $\frac{du_n}{dt}$ is bounded in $L^2(r,T;L^2(\Omega))$ for any $0 < r < T$, so passing to a subsequence $\frac{du_n}{dt} \to \frac{du}{dt}$ weakly in $L^2(r,T;L^2(\Omega))$.

Moreover, Ascoli-Arzelà theorem implies that for any fixed $r > 0$ we have $u_n \to u$ in $C([r,T], L^2(\Omega))$ and u is absolutely continuous on $[r,T]$.

Also, g_n converges to some $g \in L^\infty(0,T;L^2(\Omega))$ weakly star in $L^\infty(0,T;L^2(\Omega))$ and weakly in $L^2(0,T;L^2(\Omega))$. On the other hand, since $-\Delta u_n = -\frac{du_n}{dt} + g_n$, $-\Delta u_n$ converges to $l(t) = -(\frac{du}{dt}) + g$ weakly in $L^2(r,T;L^2(\Omega))$. Hence, we find at once that u satisfies

$$\frac{du}{dt} - \Delta u(t) = g(t), \text{ a.e. on } (0,T).$$

We need to prove that $u(\cdot)$ is a strong solution of (15). Now, we show that $g(t) \in H_0(u(t))$, a.e. in $(0,T)$. For this, we shall prove first that for a.e. $x \in \Omega$ and $s \in (0,T)$

$$|g_n(s,x) - H_0(u(s,x))| \to 0, \text{ as } n \to \infty.$$

Indeed, if $u(s,x) = 0$, then $g_n(s,x) = f_{\varepsilon_n}(u_n(s,x)) = 0 \in [-1,1] = H_0(u(s,x))$, for all n, so that the result is evident. If $u(s,x) < 0$, then

$$|g_n(s,x) - H_0(u(s,x))| = |f_{\varepsilon_n}(u_n(s,x)) + 1| \to 0, \text{ as } n \to \infty.$$

Finally, if $u(s,x) > 0$, then

$$|g_n(s,x) - f_0(u(s,x))| = |f_{\varepsilon_n}(u_n(s,x)) - 1| \to 0, \text{ as } n \to \infty.$$

Now, by [23, Proposition 1.1] we have that for a.e. $t \in (0,T)$

$$g(t) \in \bigcap_{n\geq 0} \overline{co} \bigcup_{k\geq n} g_k(t).$$

Then $g(t) = \lim_{n\to\infty} y_n(t)$ strongly in $L^2(\Omega)$, where

$$y_n(t) = \sum_{i=1}^{M} \lambda_i g_{k_i}(t), \sum_{i=1}^{M} \lambda_i = 1, k_i \geq n.$$

We note that for any $t \in [0,T]$ and a.e. $x \in \Omega$ we can find $n(\varepsilon, x, t)$ such that if $k \geq n$, then $|g_k(t,x) - H_0(u(t,x))| \leq \varepsilon$. Therefore,

$$|y_n(t,x) - H_0(u(t,x))| \leq \sum_{i=1}^{M} \lambda_i |g_{k_i}(t,x) - H_0(u(t,x))| \leq \varepsilon.$$

Hence, since we can assume that for a.e. $(t,x) \in (0,T)\times\Omega, y_n(t,x) \to g(t,x)$, it follows that $g(t,x) \in H_0(u(t,x))$.

It remains to check that u is continuous as $t \to 0^+$. Let $\hat{u}$ be the unique solution of

$$\begin{cases} \dfrac{du}{dt} - \Delta u = 0, \\ u|_{\partial\Omega} = 0, \\ u(0) = u_0, \end{cases}$$

and let $v_n(t) = u_n(t) - \hat{u}(t)$. Multiplying by v_n the equation

$$\frac{dv_n}{dt} - \Delta v_n = f_{\varepsilon_n}(u_n),$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\|v_n\|_{L^2}^2 + \|v_n\|_{H_0^1}^2 \leq (f_{\varepsilon_n}(u_n(t)), v_n) \tag{25}$$

$$\leq \frac{1}{2}\|f_{\varepsilon_n}(u_n)\|_{L^2}^2 + \frac{1}{2}\|v_n\|_{L^2}^2, \tag{26}$$

so that

$$\|v_n(t)\|_{L^2}^2 \leq \|v_n(0)\|_{L^2}^2 + Kt.$$

Hence, $\|u(t) - \hat{u}(t)\|_{L^2}^2 = \lim_{n\to\infty} \|v_n(t)\|_{L^2}^2 \leq Kt$, for $t > 0$, and

$$\|u(t) - u_0\|_{L^2} \leq \|u(t) - \hat{u}(t)\|_{L^2} + \|\hat{u}(t) - u_0\|_{L^2} < \delta,$$

as soon as $t < \varepsilon(\delta)$. Therefore, $u(\cdot)$ is a strong solution.

Finally, if $t_n \to 0$, then

$$\begin{aligned} \|u_n(t_n) - u_0\|_{L^2} &\leq \|v_n(t_n)\|_{L^2} + \|\widehat{u}(t_n) - u_0\|_{L^2} \\ &\leq \sqrt{\|v_n(0)\|_{L^2}^2 + Kt_n} + \|\widehat{u}(t_n) - u_0\|_{L^2} \to 0. \end{aligned}$$

Hence, $u_n \to u$ in $C([0,T], L^2(\Omega))$. By a diagonal argument we obtain that the result is true for every $T > 0$. □

As a consequence of the last theorem, condition (H4) follows.

Remark 9. Let be $u_{\varepsilon_n}(\cdot)$ a bounded complete trajectory of (13). Fix $T > 0$. Since $\bigcup_{0<\varepsilon\leq\varepsilon_0} \mathcal{A}_\varepsilon$ is precompact in $L^2(\Omega)$, $u_{\varepsilon_n}(-T) \to y$ in L^2 up to a subsequence. Theorem 9 implies that u_{ε_n} converges in $C([0,T], L^2(\Omega))$ to some solution u of (15). If we choose successive subsequences for $-2T, -3T, \ldots,$ and apply the standard diagonal procedure, we obtain that a subsequence u_{ε_n} converges to a complete trajectory u of (15) in $C([-T,T],L^2(\Omega))$ for any $T>0$. Since $\cup_{0<\varepsilon\leq 1}\mathcal{A}_\varepsilon$ is bounded in $L^2(\Omega)$ (in fact in $H_0^1(\Omega)$), it is clear that u is a bounded complete trajectory of problem (15).

Now, we need to prove a previous lemma to obtain the convergence of solutions of the approximations in the space $C([0,T],H_0^1)$.

Lemma 11. *Any sequence $\xi_n\in A_{\varepsilon_n}$ with $\varepsilon_n\to 0$ is relatively compact in $H_0^1(\Omega)$.*

Proof. There exists a bounded complete trajectory ψ_{ε_n} of (13) with $\psi_{\varepsilon_n}(0)=\xi_n$. Denote $u_n(\cdot)=\psi_{\varepsilon_n}(-T+\cdot)$ and choose some $T>0$. Then $\xi_n=u_n(T)$, $u_n(0)=\psi_{\varepsilon_n}(-T)$. In view of Remark 9 up to a subsequence $u_n\to u$ in $C([0,T],L^2(\Omega))$, where u is a strong solution of (15). On top of that, by (24) and the argument in the proof of Theorem 9 we obtain that for $r>0$,

$$\begin{aligned} u_n &\to u \text{ weakly star in } L^\infty(r,T;H_0^1(\Omega)),\\ \frac{du_n}{dt} &\to \frac{du}{dt} \text{ weakly in } L^2(r,T;L^2(\Omega)),\\ u_n &\to u \text{ weakly in } L^2(r,T;H^2(\Omega)).\end{aligned}$$

Therefore, by the Compactness Theorem [17, p.58] we have

$$\begin{aligned} u_n &\to u \text{ strongly in } L^2(r,T,H_0^1(\Omega)),\\ u_n(t) &\to u\,(t) \ \text{ in } H_0^1(\Omega) \text{ for a.a. } t\in(r,T).\end{aligned}$$

In addition, by standard results [21, p.102] we have that $u_n,u\in C([r,T],H_0^1(\Omega))$.

Multiplying (13) by $\frac{du_n}{dt}$ and using (23), we obtain

$$\left\|\frac{du_n}{dt}\right\|_{L^2}^2+\frac{d}{dt}\|u_n\|_{H_0^1}^2\leq\|f_\varepsilon(u_n)\|_{L^2}^2.$$

Thus,

$$\|u_n(t)\|_{H_0^1}^2\leq\|u_n(s)\|_{H_0^1}^2+C(t-s),\ C>0,\ t\geq s\geq r.$$

The same inequality is valid for the limit function $u(\cdot)$. Hence, the functions $J_n(t)=\|u_n(t)\|_{H_0^1}^2-Ct$, $J(t)=\|u(t)\|_{H_0^1}^2-Ct$, are continuous and non-increasing in $[r,T]$. Moreover, $J_n(t)\to J(t)$ for a.e. $t\in(r,T)$. Take $r<t_m<T$ such that $t_m\to T$ and $J_n(t_m)\to J(t_m)$ for all m. Then

$$J_n(T)-J(T)\leq J_n(t_m)-J(T)\leq|J_n(t_m)-J(t_m)|+|J(t_m)-J(T)|.$$

For any $\varepsilon>0$ there exist $m(\varepsilon)$ and $N(\varepsilon)$ such that $J_n(T)-J(T)\leq\varepsilon$ if $n\geq N$. Then $\limsup J_n(T)\leq J(T)$, so $\limsup\|u_n(T)\|_{H_0^1}^2\leq\|u(T)\|_{H_0^1}^2$. As $u_n(T)\to u(T)$ weakly in H_0^1 implies $\liminf\|u_n(T)\|_{H_0^1}^2\geq\|u(T)\|_{H_0^1}^2$, we obtain

$$\|u_n(T)\|_{H_0^1}^2\to\|u(T)\|_{H_0^1}^2,$$

so that $u_n(T)\to u(T)$ strongly in $H_0^1(\Omega)$. Hence, the result follows. □

Corollary 2. *If $u_{\varepsilon 0}\to u_0$ in $L^2(\Omega)$, where $u_{\varepsilon 0}\in\mathcal{A}_\varepsilon$, $u_0\in\mathcal{A}_0$, then for any $T>0$ there exists a subsequence ε_n such that u_{ε_n} converges to some strong solution u of (15) in $C([0,T],H_0^1(\Omega))$.*

Proof. We know from Theorem 9 that there exists a subsequence such that u_{ε_n} converges to some strong solution u of (15) in $C([0,T],L^2(\Omega))$. Then the statement follows from the invariance of $\mathcal{A}_\varepsilon$ and Lemma 11. □

Remark 10. Let $u_{\varepsilon_n}(\cdot)$ be a bounded complete trajectory of (13). Fix $T>0$. By Lemma 11 $u_{\varepsilon_n}(-T)\to y$ in $H_0^1(\Omega)$ up to a subsequence. Corollary 2 implies then that u_{ε_n} converges in $C([0,T],H_0^1(\Omega))$ to some solution u of (15). If we choose successive subsequences for $-2T,-3T\ldots$ and apply the standard diagonal procedure we obtain that a subsequence u_{ε_n} converges to a complete trajectory u of (15) in $C([-T,T],H_0^1(\Omega))$ for any $T>0$. By Remark 9 this trajectory is bounded.

Lemma 12. *$dist_{H_0^1}(\mathcal{A}_\varepsilon,\mathcal{A}_0)\to 0$, as $\varepsilon\to 0$.*

Proof. By contradiction let there exist $\delta>0$ and a sequence $y_{\varepsilon_n}\in\mathcal{A}_{\varepsilon_n}$ such that

$$dist_{H_0^1}(y_{\varepsilon_n},\mathcal{A}_0)>\delta.$$

Hence, as $y_{\varepsilon_n}=u_{\varepsilon_n}(0)$, where u_{ε_n} is a bounded complete trajectory of problem (13), using Remark 10 we obtain that up to a sequence u_{ε_n} converges to a bounded complete trajectory u of the problem (15) in the spaces $C([-T,T],H_0^1(\Omega))$ for every $T>0$. Thus, $u(t)\in\mathcal{A}_0$ for any $t\in\mathbb{R}$. We infer then that

$$y_{\varepsilon_n}=u_{\varepsilon_n}(0)\to u(0)\in\mathcal{A}_0,$$

which is a contradiction. □

We choose some $\delta>0$ such that

$$\mathcal{O}_\delta(\Xi_i^0)\cap\mathcal{O}_\delta(\Xi_j^0)=\emptyset \text{ if } i\neq j$$

and Ξ_i^0 are maximal weakly invariant.

For problem (13) let us define the sets

$$M_i^\varepsilon=\{v_{\varepsilon,i}^+,v_{\varepsilon,i}^-\} \text{ for } 1\le i<N_0,$$

$$Z_{N_0}^\varepsilon=\left(\cup_{k\ge N_0}\{v_{\varepsilon,k}^\pm\}\right)\cup\{0\},$$

$$M_{N_0}^\varepsilon=\left\{\begin{array}{c} y:\exists\psi\in\mathbb{K}^\varepsilon \text{ such that (16) holds with } z_j\in Z_{N_0}^\varepsilon,\\ j=1,2 \text{ and } y=\psi(t) \text{ for some } t\in\mathbb{R}\end{array}\right\},$$

where $\mathbb{K}^\varepsilon$ is the set of all bounded complete trajectories of (13).

In view of Lemma 9 we have

$$dist_{H_0^1}(M_i^\varepsilon,\Xi_i^0)\to 0, \text{ as } \varepsilon\to 0, \quad 1\le i<N_0$$

Lemma 13. *$dist_{H_0^1}(M_{N_0}^\varepsilon,\Xi_{N_0}^0)\to 0$, as $\varepsilon\to 0$.*

Proof. Suppose the opposite, that is, there exists $\delta>0$ and a sequence $y_{\varepsilon_k}\in M_{N_0}^{\varepsilon_k}$ such that

$$dist_{H_0^1}(y_{\varepsilon_k},\Xi_{N_0}^0)>\delta \text{ for all } k. \tag{27}$$

Let ξ_{ε_k} be a sequence of bounded complete trajectories of problem (13) such that $\xi_{\varepsilon_k}(0)=y_{\varepsilon_k}$ and

$$\xi_{\varepsilon_k}(t)\to z_{-1}^k \text{ as } t\to-\infty,$$

$$\xi_{\varepsilon_k}(t)\to z_0^k \text{ as } t\to\infty,$$

where $z_{-1}^k,z_0^k\in Z_{N_0}^{\varepsilon_k}$. By Lemmas 9 and 10, passing to a subsequence we have that

$$z_i^k\to z_i\in Z_{N_0}, i=-1,0.$$

By Remark 10 we obtain that up to a subsequence ξ_{ε_k} converges to a complete trajectory ψ_0 of problem (15) in the spaces $C([-T,T],H_0^1(\Omega))$ for every $T>0$, so

$y_{\varepsilon_k} \to \psi_0(0)$ in $H_0^1(\Omega)$. Thus, either ψ_0 is equal to a fixed point $\overline{z}_0 \neq 0$ or there exist two fixed points of problem (15), denoted by $\overline{z}_{-1}, \overline{z}_0$ such that

$$E(\overline{z}_{-1}) > E(\overline{z}_0),$$
$$\psi_0(t) \to \overline{z}_{-1} \text{ as } t \to -\infty,$$
$$\psi_0(t) \to \overline{z}_0 \text{ as } t \to \infty.$$

If $\overline{z}_0 = z_0$, then $\overline{z}_{-1}, \overline{z}_0 \in Z_{N_0}$, which means that $\psi_0(0) \in \Xi^0_{N_0}$. This would imply a contradiction with (27). Therefore, we assume that $\overline{z}_0 \neq z_0$. Also, it is clear that $\overline{z}_0 = v_m^{\pm} \neq 0$, for some $m \in \mathbb{N}$.

Let $r_0 > 0$ be such that $\mathcal{O}_{r_0}(\overline{z}_0) \cap \mathcal{O}_{r_0}(z_0) \neq \emptyset$ and $\mathcal{O}_{2r_0}(\overline{z}_0)$ does not contain any other fixed point of problem (15). The previous convergences imply that for each $r \leq r_0$ there exist a moment of time t_r and k_r such that $\xi_{\varepsilon_k}(t_r) \in \mathcal{O}_r(\overline{z}_0)$ for all $k \geq k_r$. On the other hand, since $\xi_{\varepsilon_k}(t) \to z_0^k$, as $t \to \infty$, and $z_0^k \to z_0$, there exists $t_r' > t_r$ such that

$$\xi_{\varepsilon_{k_r}}(t) \in \mathcal{O}_{r_0}(\overline{z}_0) \text{ for all } t \in [t_r, t_r'),$$
$$\|\xi_{\varepsilon_{k_r}}(t_r') - \overline{z}_0\|_{L^2} = r_0.$$

Let us consider two cases: 1) $t_r' - t_r \to \infty; 2) |t_r' - t_r| \leq C$. We begin with the first case. We define the sequence of bounded complete trajectories of problem (13) given by

$$\xi^1_{k_r}(t) = \xi_{\varepsilon_{k_r}}(t + t_r').$$

By Remark 10 we can extract a subsequence of this sequence converging to a bounded complete trajectory ψ_1 of problem (15). Since $t_r' - t_r \to \infty$, we obtain that $\psi_1(t) \in \mathcal{O}_{r_0}(\overline{z}_0)$ for all $t \leq 0$. Since $\mathcal{O}_{2r_0}(\overline{z}_0)$ does not contain any other fixed point of problem (15), it follows that $\psi_1(t) \to \overline{z}_0$ as $t \to -\infty$. But $\|\psi_1(0) - \overline{z}_0\|_{L^2} = r_0$, so ψ_1 is not a fixed point. Therefore, $\psi_1(t) \to \overline{z}_1$ as $t \to \infty$, where $\overline{z}_1$ is a fixed point such that $E(\overline{z}_1) < E(\overline{z}_0)$.

In the second case we define the sequence

$$\xi^1_{k_r}(t) = \xi_{\varepsilon_{k_r}}(t + t_r).$$

Passing to a subsequence we have that

$$\xi^1_{k_r}(0) \to \overline{z}_0,$$
$$t_r' - t_r \to t'.$$

As $\xi^1_{k_r}$ converges to a solution ξ^1 of problem (15) uniformly in bounded subsets from $[0, \infty)$ such that $\xi^1(0) = \overline{z}_0$, $\xi^1_{k_r}(t_r' - t_r) \to \xi^1(t')$, so that $\|\xi^1(t') - \overline{z}_0\|_{L^2} = r_0$. We put

$$\psi_1(t) = \begin{cases} \overline{z}_0 \text{ if } t \leq 0, \\ \xi^1(t) \text{ if } t \geq 0. \end{cases}$$

Then ψ_1 is a bounded complete trajectory of problem (15) such that $\psi_1(t) \to \overline{z}_1$ as $t \to \infty$, where $\overline{z}_1$ is a fixed point satisfying $E(\overline{z}_1) < E(\overline{z}_0)$.

Now, if $\overline{z}_1 = z_0$, then we have the chain of connections

$$\psi_0(t) \to \overline{z}_{-1} \text{ as } t \to -\infty, \psi_0(t) \to \overline{z}_0 \text{ as } t \to +\infty,$$
$$\psi_1(t) \to \overline{z}_0 \text{ as } t \to -\infty, \psi_1(t) \to \overline{z}_1 \text{ as } t \to +\infty,$$

which implies that $\overline{z}_{-1}, \overline{z}_0, \overline{z}_1 \in Z_n$, an then $\psi_0(0) \in \Xi^0_n$. This would imply a contradiction with (27).

However, if $\overline{z}_1 \neq \overline{z}_0$, then we proceed in the same way and obtain a new connection from the point $\overline{z}_1$ to another fixed point with less energy. Since the number of fixed points with energy less than or equal to $E(\overline{z}_0)$ is finite, we will finally obtain a chain of connections of the form

$$\begin{aligned}
&\psi_0(t) \to \overline{z}_{-1} \text{ as } t \to -\infty,\ \psi_0(t) \to \overline{z}_0 \text{ as } t \to +\infty,\\
&\psi_1(t) \to \overline{z}_0 \text{ as } t \to -\infty,\ \psi_1(t) \to \overline{z}_1 \text{ as } t \to +\infty,\\
&\qquad\vdots\\
&\psi_n(t) \to \overline{z}_{m-1} \text{ as } t \to -\infty,\ \psi_n(t) \to \overline{z}_m = z_0 \text{ as } t \to +\infty.
\end{aligned}$$

And again, this implies a contradiction with (27). □

These convergences imply the existence of ε_0 such that if $\varepsilon \leq \varepsilon_0$, then

$$M_i^\varepsilon \subset \mathcal{O}_\delta(\Xi_i^0) \text{ for any } 1 \leq i \leq N_0.$$

Further, let

$$\Xi_i^\varepsilon = \left\{ \begin{array}{c} y : \exists \psi \in \mathbb{K}^\varepsilon \text{ such that } \psi(0) = y \\ \text{and } \psi(t) \in \mathcal{O}_\delta(\Xi_i^0) \text{ for all } t \in \mathbb{R} \end{array} \right\}.$$

These sets are clearly maximal weakly invariant for G_ε in $\mathcal{O}_\delta(\Xi_i^0)$, so condition (H5) is satisfied for $V_i = \mathcal{O}_\delta(\Xi_i^0)$. As a consequence of Lemmas 9, 13, Remark 9 and the definition of δ we have

$$dist_{L^2}(\Xi_i^\varepsilon, \Xi_i^0) \to 0, \text{ as } \varepsilon \to 0, \text{ for } 1 \leq i \leq N_0.$$

Therefore, condition (H3) is satisfied.

We also get by Remark 10 and the definition of δ that

$$dist_{H_0^1}(\Xi_i^\varepsilon, \Xi_i^0) \to 0, \text{ as } \varepsilon \to 0, \text{ for } 1 \leq i \leq N_0.$$

Moreover, $\mathcal{M}^\varepsilon = \{\Xi_1^\varepsilon, \ldots, \Xi_{N_0}^\varepsilon\}$ is a disjoint family of isolated weakly invariant sets.

Applying Theorem 2 we obtain the following result.

Theorem 10. *There exists $\varepsilon_1 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_1$ the multivalued semiflow G_ε is dynamically gradient with respect to the family $\mathcal{M}^\varepsilon$.*

Acknowledgments. This paper is dedicated to the memory of Professor Valery Melnik, on the tenth anniversary of his passing away, with our deepest respect and sorrow.

We would like to thank the referees for their careful reading and suggestions that led us to improve the paper.

References

[1] E. R. Aragão-Costa, T. Caraballo, A. N. Carvalho and J. A. Langa, Stability of gradient semigroups under perturbations, *Nonlinearity*, **24** (2011), 2099-2117.

[2] E. R. Aragão-Costa, T. Caraballo, A. N. Carvalho and J. A. Langa, Non-autonomous Morse-decomposition and Lyapunov functions for gradient-like processes. *Transactions of the American Mathematical Society (10)*, **365** (2013), 5277-5312.

[3] J. M. Arrieta, A. Rodríguez-Bernal and J. Valero, Dynamics of a reaction-diffusion equation with a discontinuous nonlinearity, *International Journal of Bifurcation and Chaos (10)*, **16** (2006), 2965-2984.

[4] J. M. Ball, Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equations, *Journal of Nonlinear Science*, **7** (1997), 475-502.

[5] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Editura Academiei, Bucuresti, 1976.

[6] T. Caraballo, P. Marín-Rubio and J. Robinson, A comparison between two theories for multi-valued semiflows and their asymptotic behaviour, *Set-Valued Analysis*, **11** (2003), 297-322.

[7] A. N. Carvalho and J. A. Langa, An extension of the concept of gradient semigroups which is stable under perturbation, *J. Differential Equations*, **246** (2009), 2646-2668.
[8] V. V. Chepyzhov and M. I. Vishik, *Attractors for Equations of Mathematical Physics*, American Mathematical Society, Providence, 2002.
[9] H. B. da Costa and J. Valero, Morse decompositions and Lyapunov functions for dynamically gradient multivalued semiflows, *Nonlinear Dyn.*, **84** (2016), 19-34.
[10] P. Gruber, *Convex and Discrete Geometry*, Springer, 2007.
[11] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer, Berlin, 2007.
[12] D. Henry, Some infinite-dimensional Morse-Smale systems defined by parabolic partial differential equations, *J. Diff. Eqs.*, **59** (2007), 165-205.
[13] O. V. Kapustyan, P. O. Kasyanov and J. Valero, Structure and regularity of the global attractor of a reacction-diffusion equation with non-smooth nonlinear term, *Discrete Continuous Dynamical Systems,* **32** (2014), 4155-4182.
[14] O.V. Kapustyan, P.O. Kasyanov, J. Valero, Structure of the global attractor for weak solutions of a reaction-diffusion equation, *Applied Mathematics & Information Sciences,* **9** (2015), 2257-2264.
[15] O. V. Kapustyan, V. Pankov and J. Valero, On global attractors of multivalued semiflows generated by the 3D Bénard system, *Set-Valued and Variational Analysis*, **20** (2012), 445-465.
[16] D. Li, Morse decompositions for general dynamical systems and differential inclusions with applications to control systems, *SIAM Journal on Control and Optimization*, **46** (2007), 35-60.
[17] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Gauthier-Villar, Paris, 1969.
[18] S. Mazzini, *Atratores para o problema de Chafee-Infante*, PhD-thesis, Universidade de São Paulo, 1997.
[19] V. S. Melnik and J. Valero, On attractors of multi-valued semi-flows and differential inclusions, *Set-Valued Analysis*, **6** (1998), 83-111.
[20] J. C. Robinson, *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabilic PDEs and the Theory of Global Attractors*, Cambridge University Press, Cambridge, UK, 2001.
[21] G. R. Sell and Y. You, *Dynamics of evolutionary equations*, Springer, 2002.
[22] R. Temam, *Navier-Stokes Equations*, North-Holland, Amsterdam-New York, 1977.
[23] A. Tolstonogov, On solutions of evolution inclusions I, *Siberian Math. J.*, **33** (1992), 500-511.
[24] J. Valero, On locally compact attractors of dynamical systems, *J. Math. Anal. Appl.*, **237**, (1999), 43-54.
[25] J. Valero, Attractors of parabolic equations without uniqueness, *J. Dynamics Differential Equations*, **13** (2001), 711-744.
[26] J. Valero, On the Kneser property for some parabolic problems, *Topology Appl.,* **153** (2005), 975-989.
[27] J. Valero and A. V. Kapustyan, On the connectedness and symptotic behaviour of solutions of reaction-diffusion systems, *J. Math. Anal. Appl.*, **323** (2006), 614-633.
[28] A. Wayne and D. Varberg, *Convex functions*, Academic Press, Elsevier, 1973.
[29] K. Yosida, *Functinoal Analysis*, Springer-Verlag, Berlin, 1965.

E-mail address, Rubén Caballero: ruben.caballero@umh.es
E-mail address, Alexandre N. Carvalho: andcarva@icmc.usp.br
E-mail address, Pedro Marín-Rubio: pmr@us.es
E-mail address, José Valero: jvalero@umh.es