# EQUILIBRIUM PROBLEMS ON RIEMANNIAN MANIFOLDS WITH APPLICATIONS 

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#### Abstract

We study the equilibrium problem on general Riemannian manifolds. The results on existence of solutions and on the convex structure of the solution set are established. Our approach consists in relating the equilibrium problem to a suitable variational inequality problem on Riemannian manifolds, and is completely different from previous ones on this topic in the literature. As applications, the corresponding results for the mixed variational inequality and the Nash equilibrium are obtained. Moreover, we formulate and analyze the convergence of the proximal point algorithm for the equilibrium problem. In particular, correct proofs are provided for the results claimed in J. Math. Anal. Appl. 388, 61-77, 2012 (i.e., Theorems 3.5 and 4.9 there) regarding the existence of the mixed variational inequality and the domain of the resolvent for the equilibrium problem on Hadamard manifolds.


Key words. Riemannian manifold, equilibrium problem, variational inequality problem, proximal point algorithm

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1. Introduction. Let $X$ be a metric space, $Q \subseteq X$ a nonempty subset and $F: X \times$ $X \rightarrow(-\infty,+\infty]$ a bifunction. The equilibrium problem (introduced by Blum and Oettli in (6), abbreviated as EP, associated to the pair $(F, Q)$ is to find a point $\bar{x} \in Q$ such that

$$
\begin{equation*}
F(\bar{x}, y) \geq 0 \quad \text { for any } y \in Q \tag{1.1}
\end{equation*}
$$

As shown in [6, 36], EP contains, as special cases, optimization problems, complementarity problems, fixed point problems, variational inequalities and problems of Nash equilibria; and it has been broadly applied in many areas, such as economics, image reconstruction, transportation, network, and elasticity. In recent years, EP has been studied extensively, including the issues regarding existence of solutions and iterative algorithms for finding solutions; see e.g., [5, 6, 12, 15, 21].

Since the classical existence results in EPs work for the case when $Q$ is a convex set and the bifunction $F$ is convex in the second variable, some authors focused on exploiting

[^0]the possible existence without the convexity assumption; see e.g., 3, 4, 9, 20. One useful approach used in [23] and [24] is to embed the underlying nonconvex and/or nonsmooth Nash/Nash-type equilibrium problems into a suitable Riemannain manifold $M$ to study the existence and the location problems of the Nash/Nash-type equilibrium points. In particular, Kristály established in [23], the existence results for Nash equilibrium points associated to strategy sets $\left\{Q_{i} \subseteq M_{i}\right\}_{i \in I}$ and loss-functions $\left\{f_{i}\right\}_{i \in I}(I:=\{1,2, \ldots, n\})$ under the following assumption:
$\left(\mathrm{A}_{\mathrm{K}}\right)$ : each $Q_{i}$ is a compact and geodesic convex set of $M_{i}$ for all $i \in I$
(see item (e) in Definition 2.3 for the notion of the geodesic convexity). This class of approaches has also been used extensively in many optimization problems since some nonconvex and/or nonsmooth problems of the constrained optimization in $\mathbb{R}^{n}$ can be reduced to convex and/or smooth unconstrained optimization problems on appropriate Riemannian manifolds; see, for examples, [14, 30, 31, 40, 42]. More about optimization techniques and notions in Riemannian manifolds can be found in [1, 2, 8, 18, 26, 27, 28, 43, 45] and the bibliographies therein.

For the equilibrium problem (1.1) on a Riemannian manifold $M$, Colao et. al, by generalizing the KKM lemma to a Hadamard manifold, established an existence result (i.e., [11, Theorem 3.2]) for solutions of EP under the following assumptions:
( $\mathrm{A}_{\mathrm{C}}-1$ ): $M$ is a Hadamard manifold and $Q$ is closed and convex; ( $\mathrm{A}_{\mathrm{C}}-2$ ): the set $\{y \in Q: F(x, y) \leq 0\}$ is convex for any $x \in Q$
(which was extensively studied in 47 for the generalized vector equilibrium problem). This existence result was applied there to solve the following problems:
(P1). the existence problem of solutions for mixed variational inequality problems;
(P2). the well-definedness of the resolvent and the proximal point algorithm for solving EP;
(P3). the existence problem of fixed points for set-valued mappings;
(P4). the existence problem of solutions for Nash equilibrium problems.
However, the applications to problems (P1)-(P3) above rely heavily on the following claim:

$$
\begin{equation*}
\text { the function } y \mapsto\left\langle u_{x}, \exp _{x}^{-1} y\right\rangle \text { is quasi-convex, } \tag{1.2}
\end{equation*}
$$

where $x \in M$ and $u_{x} \in T_{x} M$; see the proofs for Theorems 3.5, 3.10 and 4.9 in [11. Unfortunately, unlike in the linear space setting, claim (1.2) is not true in general as pointed out in [42, Theorem 2.1, p. 299] or [25]. Note that, for any $x \in M$, the function defined by (1.2) is convex at $x$ (see Definition 2.4 (i)); this motivates us to introduce the new notion of the point-wise (weak) convexity for a bifunction on general manifolds (see Definition 3.1 (c)).

Our main purpose in the present paper is to develop a new approach (based on the new notion and the work in [28]) to study the issue on the existence and structure of solutions for equilibrium problems on general Riemannian manifolds, which, in particular, covers problems (1), (2) and (4) as special cases. In our approach, rather than assumptions ( $\mathrm{A}_{\mathrm{K}}$ ) or ( $\left.\mathrm{A}_{\mathrm{C}}-1\right)-\left(\mathrm{A}_{\mathrm{C}}-2\right)$, we make the following ones on the involved $Q$ and $F$ :

- $Q$ is a closed and weakly convex subset of Riemannian manifold $M$;
- $F$ is point-wise weakly convex on $Q$.

The technique used in the present paper for proving the main results is completely different from the ones used in [11, 47, 23. Actually, our technique here is mainly focused on establishing the equivalence between the EP and a suitable variational inequality problem; and then apply the corresponding results in [28] for the variational inequality problem to study the existence of solutions and the convex structure of the solution set of the EP .

As applications to problems (P1), (P2) and (P4), we obtained some results on the existence of solutions and convexity of the solution sets for mixed variational inequality problems and Nash equilibrium problems (see Theorem 5.1 and 5.2), as well as the convergence of the proximal point algorithm for solving EP. In particular, the existence result for mixed variational inequality problems and the well-definedness results of the resolvent for solving EP on Hadamard manifolds provide correct proofs for the corresponding ones [11, Theorem 3.5 and 4.9] (see the explanations before Corollaries 4.5 and 5.2 in Section 4 and 5, respectively); while the existence result for the Nash equilibrium on general manifolds relaxes the geodesic convexity assumption made on $\left\{Q_{i}\right\}$ in [23, Theorem 1.1] to the weaker one that each $Q_{i}$ is weakly convex. It is worthwhile to notice that the geodesic convexity assumption for $\left\{Q_{i}\right\}$ in 23 prevents its application to some special but important Riemannian manifolds, such as compact Stiefel manifolds $\operatorname{St}(p, n)$ and Grassmann manifolds $\operatorname{Grass}(p, n)(p<n)$, in which there is no geodesic convex subset; see Remark 5.1 in Section 5. Moreover, to our best knowledge, the convex structure results on the solution set for mixed variational inequality problems and Nash equilibrium problems are new even in Hadamard manifold settings.

The paper is organized as follows. In the next section, we introduce some basic notions and notations on Riemannian manifolds, some properties about the (weakly) convex function and the results about the VIP in [28] which will be used in our approach. In section 3, we establish the existence and the uniqueness result of the solution and the convexity of the solution set of the EP on general Riemannian manifolds. Following these, the formulation of the proximal point algorithm for the equilibrium problem on general Riemnannian manifolds is given and the convergence property about the algorithm is analyzed in section 4 . The last section is devoted to the applications to the Nash equilibrium problem and the mixed variational inequality problem.

## 2. Notations and preliminary results.

2.1. Background of Riemannian manifolds. The notations used in the present paper are standard; and the readers are referred to some textbooks for more details, for example, [13, 39, 42].

Let $M$ be a connected $n$-dimensional Riemannian manifold with the Levi-Civita connection $\nabla$ on $M$. Let $x \in M$, and let $T_{x} M$ stand for the tangent space at $x$ to $M$ endowed with the scalar product $\langle,\rangle_{x}$ and the associated norm $\|\cdot\|_{x}$, where the subscript $x$ is sometimes
omitted. Thus the tangent bundle, denoted by $T M$, is defined by

$$
T M:=\left\{(x, v): x \in M, v \in T_{x} M\right\} .
$$

Fix $y \in M$, and let $\gamma:[0,1] \rightarrow M$ be a piecewise smooth curve joining $x$ to $y$. Then, the arc-length of $\gamma$ is defined by $l(\gamma):=\int_{0}^{1}\|\dot{\gamma}(t)\| d t$, while the Riemannian distance from $x$ to $y$ is defined by $\mathrm{d}(x, y):=\inf _{\gamma} l(\gamma)$, where the infimum is taken over all piecewise smooth curves $\gamma:[0,1] \rightarrow M$ joining $x$ to $y$. We use $\mathbf{B}(x, r)$ and $\overline{\mathbf{B}(x, r)}$ to denote, respectively, the open metric ball and the closed metric ball at $x$ with radius $r$, that is,

$$
\mathbf{B}(x, r):=\{y \in M: \mathrm{d}(x, y)<r\} \quad \text { and } \quad \overline{\mathbf{B}(x, r)}:=\{y \in M: \mathrm{d}(x, y) \leq r\} .
$$

A vector field $V$ is said to be parallel along $\gamma$ if $\nabla_{\dot{\gamma}} V=0$. In particular, for a smooth curve $\gamma$, if $\dot{\gamma}$ is parallel along itself, then $\gamma$ is called a geodesic, that is, a smooth curve $\gamma$ is a geodesic if an only if $\nabla_{\dot{\gamma}} \dot{\gamma}=0$. A geodesic $\gamma:[0,1] \rightarrow M$ joining $x$ to $y$ is minimal if its arc-length equals its Riemannian distance between $x$ and $y$. By the Hopf-Rinow theorem [13], if $M$ is complete, then $(M, \mathrm{~d})$ is a complete metric space, and there is at least one minimal geodesic joining $x$ to $y$. One of the important structures on $M$ is the exponential $\operatorname{map} \exp _{x}: T_{x} M \rightarrow M$, which is defined at $x \in M$ by $\exp _{x} v=\gamma_{v}(1, x)$ for each $v \in T_{x} M$, where $\gamma_{v}(\cdot, x)$ is the geodesic starting at $x$ with velocity $v$. Then, $\exp _{x} t v=\gamma_{v}(t, x)$ for each real number $t$. Another useful tool is the parallel transport $P_{\gamma,,,}$ on the tangent bundle $T M$ along a geodesic $\gamma$, which is defined by

$$
P_{\gamma, \gamma(b), \gamma(a)}(v)=V(\gamma(b)) \quad \text { for any } a, b \in \mathbb{R} \text { and } v \in T_{\gamma(a)} M
$$

where $V$ is the unique vector field satisfying $V(\gamma(a))=v$ and $\nabla_{\dot{\gamma}(t)} V=0$ for all $t$. Then, for any $a, b \in \mathbb{R}, P_{\gamma, \gamma(b), \gamma(a)}$ is an isometry from $T_{\gamma(a)} M$ to $T_{\gamma(b)} M$. We will write $P_{y, x}$ instead of $P_{\gamma, y, x}$ in the case when $\gamma$ is a minimal geodesic joining $x$ to $y$ and no confusion arises. The following lemma can be checked easily.

Lemma 2.1. Let $x_{0} \in M$ and $\left\{x_{k}\right\} \subset M$ be such that $\lim _{k \rightarrow \infty} x_{k}=x_{0}$. Let $u_{0}, v_{0} \in$ $T_{x_{0}} M$ and let $\left\{u_{k}\right\},\left\{v_{k}\right\}$ be sequences with each $u_{k}, v_{k} \in T_{x_{k}} M$ such that $u_{k} \rightarrow u_{0}$ and $v_{k} \rightarrow v_{0}$. Then

$$
\exp _{x_{k}} u_{k} \rightarrow \exp _{x_{0}} u_{0} \quad \text { and } \quad\left\langle u_{k}, v_{k}\right\rangle \rightarrow\left\langle u_{0}, v_{0}\right\rangle .
$$

The following result is known in any textbook about Riemannain geometry, see e.g., 13 , Corollary, p. 73] or 39, Exercise 5, p. 39].

Lemma 2.2. Let $\gamma:[a, b] \rightarrow M$ be a piecewise differentiable curve. If $l(\gamma)=\mathrm{d}(\gamma(a), \gamma(b))$, then $\gamma$ is a geodesic joining $\gamma(a)$ and $\gamma(b)$.

Consider a set $Q \subseteq M$ and $x, y \in Q$. The set of all geodesics $\gamma:[0,1] \rightarrow M$ with $\gamma(0)=x$ and $\gamma(1)=y$ satisfying $\gamma([0,1]) \subseteq Q$ is denoted by $\Gamma_{x y}^{Q}$, that is

$$
\Gamma_{x y}^{Q}:=\left\{\gamma:[0,1] \rightarrow Q: \gamma(0)=x, \gamma(1)=y \text { and } \nabla_{\dot{\gamma}} \dot{\gamma}=0\right\}
$$

In particular, we write $\Gamma_{x y}$ for $\Gamma_{x y}^{M}$, and $\Gamma_{x y}$ is nonempty for all $x, y \in M$ provided that $M$ is complete. Furthermore, for a subset $\Gamma_{0} \subseteq \Gamma_{x y}$, we use min $-\Gamma_{0}$ to denote the subset of $\Gamma_{0}$
consisting of all minimal geodesics in $\Gamma_{0}$. Thus $\gamma_{x y} \in \min -\Gamma_{0}$ means that $\gamma_{x y} \in \Gamma_{0}$ and $\gamma_{x y}$ is minimal.

Recall that a Hadamard manifold is a complete simply connected $m$-dimensional Riemannian manifold with nonpositive sectional curvatures. In a Hadamard manifold, the geodesic between any two points is unique and the exponential map at each point of $M$ is a global diffeomorphism; see, e.g., [39, Theorem 4.1, p. 221]. Thus min $-\Gamma_{x y}$ coincides with $\Gamma_{x y}$ in a Hadamard manifold for any $x, y \in M$.
2.2. Convex analysis on Riemmanian manifolds. Definition 2.3 below presents the notions of the convexity for subsets in $M$, where item (e) is known in [23], and see e.g., [29, 43 for the others. As usual, we use $\bar{C}$ to stand for the closure of a subset $C \subseteq M$.

Definition 2.3. Let $Q \subseteq M$ be a nonempty set. The set $Q$ is said to be
(a) weakly convex if, for any $x, y \in Q$, there is a minimal geodesic of $M$ joining $x$ to $y$ and it is in $Q$;
(b) strongly convex if, for any $x, y \in Q$, the minimal geodesic in $M$ joining $x$ to $y$ is unique and lies in $Q$;
(c) locally convex if, for any $x \in \bar{Q}$, there is a positive $\varepsilon>0$ such that $Q \cap \mathbf{B}(x, \varepsilon)$ is strongly convex;
(d) r-convex if, for any $x, y \in Q$ with $\mathrm{d}(x, y) \leq r$, the minimal geodesic in $M$ joining $x$ to $y$ is unique and lies in $Q$;
(e) geodesic convex if, for any $x, y \in Q$, the geodesic in $M$ joining $x$ to $y$ is unique and lies in $Q$.

REmARK 2.1. (a) The following implications are obvious:

$$
\text { geodesic convexity } \Rightarrow \text { strong convexity } \Rightarrow r \text {-convexity/weak convexity } \Rightarrow \text { local convexity. }
$$

(b) The intersection of a weakly convex set and a strongly convex set is strongly convex.
(c) All convexities (except the local convexity) in a Hadamard manifold coincide and are simply called the convexity.

Recall that the convexity radius at $x$ is defined by

$$
r_{x}:=\sup \left\{r>0: \quad \begin{array}{l}
\text { each ball in } \mathbf{B}(x, r) \text { is strongly convex }  \tag{2.1}\\
\text { and each geodesic in } \mathbf{B}(x, r) \text { is minimal }
\end{array}\right\}
$$

Then $r_{x}$ is well defined and positive, and $r_{x}=+\infty$ for each $x \in M$ in the case when $M$ is a Hadamard manifold. Moreover, for any compact subset $Q \subseteq M$, we have that

$$
\begin{equation*}
r_{Q}:=\inf \left\{r_{x}: x \in Q\right\}>0 \tag{2.2}
\end{equation*}
$$

see [39, Theorem 5.3, p. 169] or [29, Lemma 3.1].

Consider now an extended real-valued function $f: M \rightarrow \overline{\mathbb{R}}:=(-\infty, \infty]$ and let $\mathcal{D}(f)$ denote its domain, that is, $\mathcal{D}(f):=\{x \in M: f(x) \neq \infty\}$. Write $\Gamma_{x y}^{f}:=\Gamma_{x y}^{\mathcal{D}(f)}$ for simplicity, that is $\Gamma_{x y}^{f}$ stands for the subset consisting of all $\gamma_{x y} \in \Gamma_{x y}$ such that $\left.\gamma([0,1]]\right) \subseteq \mathcal{D}(f)$. In the following definition, we introduce the notions of the convexity for functions, where item (c) is known in [27, 28].

DEFINITION 2.4. Let $f: M \rightarrow \overline{\mathbb{R}}$ be a proper function with a weakly convex domain $\mathcal{D}(f)$, and let $x \in \mathcal{D}(f)$. Then, $f$ is said to be
(a) convex (resp. strictly convex) at $x$ if, for any $y \in \mathcal{D}(f) \backslash\{x\}$ and any geodesic $\gamma_{x y} \in \Gamma_{x y}^{f}$ the composition $f \circ \gamma_{x y}:[0,1] \rightarrow \mathbb{R}$ is convex (resp. strictly convex) on $(0,1)$ :

$$
\begin{equation*}
f \circ \gamma_{x y}(t) \leq(\text { resp. }<)(1-t) f(x)+t f(y) \quad \text { for all } t \in(0,1) \tag{2.3}
\end{equation*}
$$

(b) weakly convex (resp. weakly strictly convex) at $x$ if, for any $y \in \mathcal{D}(f)$ there exists $\gamma_{x y} \in \min -\Gamma_{x y}^{f}$ such that (2.3) holds;
(c) weakly convex (resp. convex, strictly convex, weakly strictly convex) if so is it at each $x \in \mathcal{D}(f)$.

Clearly, for a proper function $f$ on $M$, the convexity implies the weak convexity, and the strict convexity implies the convexity.

Let $f: M \rightarrow \overline{\mathbb{R}}$ be proper and weakly convex at $x \in \mathcal{D}(f)$. The directional derivative in direction $u \in T_{x} M$ and the subdifferential of $f$ at $x$ are, respectively, defined by

$$
f^{\prime}(x ; u):=\lim _{t \rightarrow 0^{+}} \frac{f\left(\exp _{x} t u\right)-f(x)}{t}
$$

and

$$
\partial f(x):=\left\{v \in T_{x} M:\langle v, u\rangle \leq f^{\prime}(x ; u) \text { for any } u \in T_{x} M\right\} .
$$

Then, by [27, Proposition 3.8(iii)], the following relationship holds between $\partial f(x)$ and $c l f^{\prime}(x ; \cdot)$, the lower semi-continuous hull of $f^{\prime}(x ; \cdot)$ :

$$
\begin{equation*}
\operatorname{cl} f^{\prime}(x ; u)=\sup \{\langle u, v\rangle: v \in \partial f(x)\} \quad \text { for any } u \in T_{x} M \tag{2.4}
\end{equation*}
$$

Lemma 2.5. Let $f: M \rightarrow \overline{\mathbb{R}}$ be proper with a weakly convex domain $\mathcal{D}(f)$. Let $x \in \mathcal{D}(f)$ and $v \in T_{x} M$.
(i) If $f$ is weakly convex (resp. weakly strictly convex) at $x$, then $v \in \partial f(x)$ if and only if, for some or any constant $r>0$, and for any $y \in \mathcal{D}(f) \cap \mathbf{B}(x, r)$, there exists a geodesic $\gamma_{x y} \in \min -\Gamma_{x y}^{f}$ such that

$$
\begin{equation*}
f(y) \geq(\text { resp } .>) f(x)+\left\langle v, \dot{\gamma}_{x y}(0)\right\rangle . \tag{2.5}
\end{equation*}
$$

(ii) If $f$ is convex (resp. strictly convex) at $x$, then $v \in \partial f(x)$ if and only if, for some or any constant $r>0$, the inequality (2.5) holds for any $y \in \mathcal{D}(f) \cap \mathbf{B}(x, r)$ and any $\gamma_{x y} \in \min -\Gamma_{x y}^{f}$.

Proof. We only prove assertion (i) (as the proof for assertion (ii) is similar). To do this, suppose that $f$ is weakly convex (resp. weakly strictly convex) at $x$. It suffices to verify that the following statements are equivalent:
(a) $v \in \partial f(x)$.
(b) For any $r>0$ and any $y \in \mathcal{D}(f) \cap \mathbf{B}(x, r)$, there exists a geodesic $\gamma_{x y} \in \min -\Gamma_{x y}^{f}$ such that (2.5) holds.
(c) There is some $r>0$ such that for any $y \in \mathcal{D}(f) \cap \mathbf{B}(x, r)$, there exists a geodesic $\gamma_{x y} \in \min -\Gamma_{x y}^{f}$ such that (2.5) holds.

We shall complete the proof by showing the implications (a) $\Rightarrow(\mathrm{b}) \Rightarrow(\mathrm{c}) \Rightarrow(\mathrm{a})$. To do this, assume (a). Then $v \in \partial f(x)$, and by definition, we have that

$$
\begin{equation*}
\langle v, u\rangle \leq f^{\prime}(x ; u) \quad \text { for any } u \in T_{x} M \tag{2.6}
\end{equation*}
$$

Let $r>0$ and $y \in \mathcal{D}(f) \cap \mathbf{B}(x, r)$ be arbitrary. Noting that $f$ is weak convex (resp. weakly strictly convex) at $x$, there exists $\gamma_{x y} \in \min -\Gamma_{x y}^{f}$ such that the composite $f \circ \gamma_{x y}:[0,1] \rightarrow \mathbb{R}$ is convex (resp. strictly convex) on $[0,1]$. Therefore,

$$
f^{\prime}\left(x ; \dot{\gamma}_{x y}(0)\right)=\inf _{t>0} \frac{f\left(\exp _{x} t \dot{\gamma}_{x y}(0)\right)-f(x)}{t} \leq(r e s p .<) f(y)-f(x)
$$

This, together with (2.6), yields that

$$
\left\langle v, \dot{\gamma}_{x y}(0)\right\rangle \leq f^{\prime}\left(x ; \dot{\gamma}_{x y}(0)\right) \leq(\text { resp. }<) f(y)-f(x)
$$

Hence (b) holds, and the implication $(\mathrm{a}) \Rightarrow(\mathrm{b})$ is checked. Noting that the implication (b) $\Rightarrow(\mathrm{c})$ is evident, it remains to show the implication $(\mathrm{c}) \Rightarrow(\mathrm{a})$. To this end, assume (c). Then one can choose $r>0$ and $\gamma_{x y} \in \min -\Gamma_{x y}^{f}$ for any $y \in \mathcal{D}(f) \cap \mathbf{B}(x, r)$ such that (2.5) holds. Without loss of generality, one could assume that $r \leq r_{x}$. Let $u \in T_{x} M \backslash\{0\}$ be arbitrary, and set $s_{0}:=\frac{r}{\|u\|}$. Then, for any $s \in\left(0, s_{0}\right), y(s):=\exp _{x}(s u) \in \mathbf{B}(x, r) \subseteq \mathbf{B}\left(x, r_{x}\right)$. It follows that the geodesic $\gamma_{x y(s)}$ joining $x$ and $y(s)$ is unique. Therefore, $\gamma_{x y(s)}$ is of the form:

$$
\gamma_{x y(s)}(t)=\exp _{x}(t(s u)) \quad \text { for each } t \in[0,1]
$$

and if $y(s) \in \mathcal{D}(f)$, then (2.5) holds with $y(s)$ in place of $y$ :

$$
\langle v, s u\rangle=\left\langle v, \dot{\gamma}_{x y(s)}(0)\right\rangle \leq f(y(s))-f(x)=f\left(\exp _{x}(s u)\right)-f(x) \quad \text { for any } s \in\left(0, s_{0}\right)
$$

Note that the above inequality holds trivially if $y(s) \notin \mathcal{D}(f)$. Then, by definition, we get that $\langle v, u\rangle \leq f^{\prime}(x ; u)$, and so (a) holds as $u \in T_{x} M$ is arbitrary. Thus, the implication (c) $\Rightarrow(\mathrm{a})$ is shown and the proof is complete.

Fix $\bar{x} \in \mathcal{D}(f)$ and recall that $f$ is center Lipschitz continuous at $\bar{x}$ if there exits a neighborhood $U$ of $\bar{x}$ and a constant $L$ such that

$$
|f(x)-f(\bar{x})| \leq L \mathrm{~d}(x, \bar{x}) \quad \text { for any } x \in U
$$

The center Lipschitz constant $L_{\bar{x}}^{f}$ at $\bar{x}$ is defined to be the minimum of all $L$ such that above inequality holds for some neighborhood $U$ of $\bar{x}$. Then it is clear that

$$
L_{\bar{x}}^{f}=\lim _{\delta \rightarrow 0+} \sup \left\{\frac{|f(x)-f(\bar{x})|}{\mathrm{d}(x, \bar{x})}: 0<\mathrm{d}(x, \bar{x}) \leq \delta\right\}
$$

The following properties about the subdifferential of a (weakly) convex function can be found in [28, Proposition 6.2] except (2.7) by definition.

Lemma 2.6. Let $f: M \rightarrow \overline{\mathbb{R}}$ be a proper function. Then the following assertions hold:
(i) If is weakly convex, then $f$ is continuous on $\operatorname{int} \mathcal{D}(f)$.
(ii) If $\bar{x} \in \operatorname{int} \mathcal{D}(f)$ and $f$ is weakly convex at $\bar{x}$, then $\partial f(\bar{x})$ is a nonempty, compact and convex set satisfying

$$
\begin{equation*}
\|v\| \leq L_{\bar{x}}^{f} \quad \text { for any } v \in \partial f(\bar{x}) . \tag{2.7}
\end{equation*}
$$

The following lemma, which provides some sufficient conditions ensuring the sum rule of subdifferential, was proved in [27, Proposition 4.3].

Lemma 2.7. Let $f, g: M \rightarrow \overline{\mathbb{R}}$ be proper functions such that $f, g$ and $f+g$ are weakly convex at $x \in \operatorname{int} \mathcal{D}(f) \cap \mathcal{D}(g)$. Then the following sum rule for the subdifferential holds:

$$
\partial(f+g)(x)=\partial f(x)+\partial g(x)
$$

2.3. VIP: existence and convexity properties of solution sets. Let $Q \subseteq M$ be a nonempty subset and let $A: Q \rightrightarrows T M$ be a set-valued vector field defined on $Q$, that is, $A(x) \subseteq T_{x} M$ is nonempty for each $x \in Q$. Consider the following variational inequality problem (VIP for short) associated to the pair $(A, Q)$ : To find a point $\bar{x} \in Q$ such that

$$
\begin{equation*}
\exists \bar{v} \in A(\bar{x}) \text { s.t. }\left\langle\bar{v}, \dot{\gamma}_{\bar{x} y}(0)\right\rangle \geq 0 \text { for any } y \in Q \text { and } \gamma_{\bar{x} y} \in \Gamma_{\bar{x} y}^{Q} . \tag{2.8}
\end{equation*}
$$

Any point $\bar{x} \in Q$ satisfying (2.8) is called a solution of VIP, and the set of all solutions is denoted by $\operatorname{VIP}(A, Q)$.

Variational inequality problem (2.8) was first introduced in [49], for single-valued vector fields on Hadamard manifolds, and extended respectively in [29] and [28] for single-valued vector fields and multivalued vector fields on general Riemannian manifolds. As we have mentioned previously, our approach to solve the EP is founded strongly on some existence results about the VIP, which are taken from [28. For this purpose, we recall some notions in the following definition; see, e.g, [26, 28].

DEfinition 2.8. Let $Q \subseteq M$ be a subset and $A: Q \rightrightarrows T M$ be a set-valued vector field on $Q$. $A$ is said to be
(a) upper semi-continuous (usc for short) at $x_{0}$, if, for any open set $U$ satisfying $A\left(x_{0}\right) \subseteq$ $U \subseteq T_{x_{0}} M$, there exists an open neighborhood $U\left(x_{0}\right)$ of $x_{0}$ such that $P_{x_{0}, x} A(x) \subseteq U$ for any $x \in U\left(x_{0}\right) \cap Q$;
(b) upper Kuratowski semi-continuous (uKsc for short) at $x_{0}$ if, for any sequences $\left\{x_{k}\right\} \subset Q$ and $\left\{u_{k}\right\} \subset T M$ with each $u_{k} \in A\left(x_{k}\right)$, relations $\lim _{k \rightarrow \infty} x_{k}=x_{0} \in Q$ and $\lim _{k \rightarrow \infty} u_{k}=u_{0}$ imply $u_{0} \in A\left(x_{0}\right)$;
(c) usc (resp., uKsc) on $Q$ if it is usc (resp., uKsc) at each $x \in Q$.

By definition, it is evident that upper semi-continuity implies upper Kuratowski semicontinuity. In the following example, we provide two set-valued vector fields which are useful for our study in next sections of the present paper.

Example 2.1. Fix a point $y \in M$, and define vector fields $\operatorname{Exp}_{(.)}^{-1} y: \rightrightarrows T M$ and $\exp _{(\cdot)}^{-1} y: \rightrightarrows T M$ respectively by

$$
\operatorname{Exp}_{x}^{-1} y:=\left\{u \in T_{x} M: \exp _{x} u=y\right\} \quad \text { for each } x \in M,
$$

and

$$
\begin{equation*}
\exp _{x}^{-1} y:=\left\{u \in \operatorname{Exp}_{x}^{-1} y:\|u\|=\mathrm{d}(x, y)\right\} \quad \text { for each } x \in M . \tag{2.9}
\end{equation*}
$$

Then one can check easily by definition and Lemma 2.1 that $\operatorname{Exp}_{(\cdot)}^{-1} y$ is $u K s c$ on $M$ and $\exp _{(.)}^{-1} y$ is usc on $M$.

Recall from [28] that a point $o \in Q$ is called a weak pole of Q if for each $x \in Q, \min -\Gamma_{o x}$ is a singleton and $\min -\Gamma_{o x} \subseteq Q$. Clearly, any subset with a weak pole is connected. The notions of the monotonicity in the following definition are well known; see for example [11, 28.

Definition 2.9. Let $Q \subseteq M$ be a subset and $A: Q \rightrightarrows T M$ be a set-valued vector field. The vector field $A$ is said to be
(a) monotone on $Q$ if, for any $x, y \in Q$ and $\gamma_{x y} \in \Gamma_{x y}^{Q}$ the following inequality holds:

$$
\left\langle v_{x}, \dot{\gamma}_{x y}(0)\right\rangle-\left\langle v_{y}, \dot{\gamma}_{x y}(1)\right\rangle \leq 0 \quad \text { for any } v_{x} \in A(x), v_{y} \in A(y) ;
$$

(b) strictly monotone on $Q$ if it is monotone and, for any $x, y \in Q$ with $x \neq y$ and $\gamma_{x y} \in \Gamma_{x y}^{Q}$ the following inequality holds:

$$
\left\langle v_{x}, \dot{\gamma}_{x y}(0)\right\rangle-\left\langle v_{y}, \dot{\gamma}_{x y}(1)\right\rangle<0 \quad \text { for any } v_{x} \in A(x), v_{y} \in A(y) .
$$

Let $Q \subseteq M$ be a closed connected and locally convex set. By [39, p. 170], there exists a connected (embedded) $k$-dimensional totally geodesic sub-manifold $N$ of $M$ such that $Q=\bar{N}$. Following [28], the set $\operatorname{int}_{R} Q:=N$ is called the relative interior of $Q$. Moreover, as in [28], we say that a closed locally convex set $Q$ has the BCC (bounded convex cover) property if there exists $o \in Q$ such that, for any $R \geq 0$, there exists a weakly convex compact subset of $M$ containing $Q \cap \mathbf{B}(o, R)$.

Remark 2.2. We remark that the notion of the BCC property defined above is a litter stronger than that defined in [28, Definition 3.9], where it is required that the compact subset containing $Q \cap \mathbf{B}(o, R)$ is "locally convex" rather than "weakly convex". From its proof, one sees that the BCC property assumption defined in [28, Definition 3.9] seems insufficient for [28, Theorem 3.10], while the stronger version of the BCC property defined here is sufficient.

For the remainder, we use $\mathcal{V}(Q)$ to denote the set of all uKsc set-valued vector fields $A$ such that $A(x)$ is compact and convex for each $x \in Q$.

Proposition 2.10 below extends the corresponding existence result in [28, Theorem 3.10] (see the explanation made in Remark 2.3). The proof of Proposition 2.10 is similar to that for [28, Theorem 3.10], and is kept here for completeness.

Proposition 2.10. Let $Q \subseteq M$ be a closed locally convex subset with a weak pole $o \in \operatorname{int}_{R} Q$ and $A \in \mathcal{V}(Q)$. Then $\operatorname{VIP}(A, Q) \neq \emptyset$ provided one of the following assumptions holds:
(a) $Q$ is compact;
(b) $Q$ has the BCC property and there exists a compact subset $L \subseteq M$ such that

$$
\begin{equation*}
x \in Q \backslash L \Rightarrow\left[\forall v \in A(x), \exists y \in Q \cap L, \gamma_{x y} \in \min -\Gamma_{x y}^{Q} \text { s.t. }\left\langle v, \dot{\gamma}_{x y}(0)\right\rangle<0\right] . \tag{2.10}
\end{equation*}
$$

Proof. It was known in [28, Theorem 3.6] in the case when $Q$ is compact. Below we assume that assumption (b) holds. Then, there exists a compact subset $L$ such that (2.10) holds. Then there exist $R>0$ and a weakly convex and compact subset $K_{R}$ of $M$ such that $L \subset \mathbf{B}(o, R)$ and $Q \cap \mathbf{B}(o, R) \subseteq K_{R}$. Write $Q_{R}:=Q \cap \overline{\mathbf{B}(o, R)}$, and $\hat{Q}_{R}:=Q \cap K_{R}$ for saving the print space. Then

$$
Q \cap L \subseteq Q \cap \mathbf{B}(o, R) \subseteq Q_{R} \subseteq \hat{Q}_{R}
$$

Thus, by (2.10), one checks that

$$
\begin{equation*}
\operatorname{VIP}\left(A, \hat{Q}_{R}\right) \subseteq \operatorname{VIP}\left(A, Q_{R}\right) \subseteq Q \cap L \subseteq \mathbf{B}(o, R) \tag{2.11}
\end{equation*}
$$

Moreover, since $o \in \operatorname{int}_{R} Q$ is a weak pole of $Q$ (and so the minimal geodesic $\gamma_{o x}$ joining $o$ to $x$ is unique) and $K_{R}$ is weakly convex, one can check by definition that $o$ is a weak pole of $\hat{Q}_{R}$ and $o \in \operatorname{int}_{R} \hat{Q}_{R}$ (noting that $Q \cap \mathbf{B}(o, R) \subseteq K_{R}$ ). Thus [28, Theorem 3.6] is applied (with $\hat{Q}_{R}$ in place of $\left.A\right)$ to get that $\operatorname{VIP}\left(A, \hat{Q}_{R}\right) \neq \emptyset$. In view of (2.11), $\emptyset \neq \operatorname{VIP}\left(A, Q_{R}\right) \subseteq \mathbf{B}(o, R)$, and it follows from [28, Proposition 3.2] that

$$
\operatorname{VIP}\left(A, Q_{R}\right)=\operatorname{VIP}\left(A, Q_{R}\right) \cap \mathbf{B}(o, R) \subseteq \operatorname{VIP}(A, Q)
$$

and so $\operatorname{VIP}(A, Q) \neq \emptyset$, completing the proof. $\square$
REMARK 2.3. Let $Q \subseteq M$ be a locally convex subset with a weak pole $o \in \operatorname{int}_{R} Q$. Recall from [28] that the vector field $A$ satisfies the coerciveness condition on $Q$ if

$$
\sup _{v_{o} \in A(o), v_{x} \in A(x)} \frac{\left\langle v_{x}, \dot{\gamma}_{x o}(0)\right\rangle-\left\langle v_{o}, \dot{\gamma}_{x o}(1)\right\rangle}{\mathrm{d}(o, x)} \rightarrow-\infty \quad \text { as } \mathrm{d}(o, x) \rightarrow+\infty \text { for } x \in Q \text {. }
$$

Then one checks directly by definition that the coerciveness condition for A implies that there exists a compact subset $L \subseteq M$ such that (2.10) in (b) of Proposition 2.10 holds (noting that $A(o)$ is compact). However, the converse is not true, in general, even in the Euclidean space setting. To see this, one may consider the simple mapping $A$ on $Q:=\mathbb{R}$ defined by $A(x):=[-1,1]$ if $x=0$ and $A(x):=\operatorname{sign}(x)$ otherwise. Thus Proposition 2.10 is an extension of the corresponding existence result in [28, Theorem 3.10].

As usual, we set $D_{\kappa}:=\frac{\pi}{\sqrt{\kappa}}$ if $\kappa>0$ and $D_{\kappa}:=+\infty$ if $\kappa \leq 0$ (see e.g., [28, 39]). The following proposition lists some results on the structure of the solution set $\operatorname{VIP}(A, Q)$, which are known in [28, Theorems 3.13, 4.6 and 4.8], respectively.

Proposition 2.11. Suppose that $A \in \mathcal{V}(Q)$ is monotone on $Q \subseteq M$ and $\operatorname{VIP}(A, Q) \neq \emptyset$. Then the following assertion holds:
(i) If $Q$ is a locally convex subset, then the solution set $\operatorname{VIP}(A, Q)$ is locally convex.
(ii) If $A$ is strictly monotone on $Q$, then $\operatorname{VIP}(A, Q)$ is a singleton.
(iii) If $M$ is of the sectional curvatures bounded from above by some $\kappa \in[0,+\infty)$ and $Q$ is a $D_{\kappa}$-convex subset, then the solution set $\operatorname{VIP}(A, Q)$ is $D_{\kappa}$-convex. .
3. Equilibrium problem. Throughout the whole section, we always assume that

- $Q \subseteq M$ is a nonempty closed and locally convex subset;
- $F: M \times M \rightarrow \overline{\mathbb{R}}$ is a proper bifunction with $0 \leq F(x, x)<+\infty$ for any $x \in Q$.

The domain $\mathcal{D}(F)$ of $F$ is defined by

$$
\mathcal{D}(F):=\{(x, y) \in M \times M:-\infty<F(x, y)<+\infty\} .
$$

Recall that the EP associated to the pair $(F, Q)$ is to find a point $\bar{x} \in Q$ such that $F(\bar{x}, y) \geq 0$ for any $y \in Q$. Any point $\bar{x} \in Q$ satisfying (1.1) is called a solution of EP (1.1), and the set of all solutions is denoted by $\operatorname{EP}(F, Q)$.
3.1. Properties of bifunctions. In the following definition we introduce some monotonicity and convexity notions for bifuctions on Riemannian manifolds. In particular, the corresponding ones of items (a) and (b) in linear spaces are refereed to, for example, 10, 22; while item (c) as far as we know are new and plays a key role in our study in the present paper.

Definition 3.1. The bifunction $F$ is said to be
(a) monotone on $Q \times Q$ if $F(x, y)+F(y, x) \leq 0$ for any $(x, y) \in Q \times Q$;
(b) strictly monotone on $Q \times Q$ if $F(x, y)+F(y, x)<0$ for any $(x, y) \in Q \times Q$ with $x \neq y$ and

$$
\begin{equation*}
F(x, x)=0 \quad \text { for any } x \in Q ; \tag{3.1}
\end{equation*}
$$

(c) point-wise weakly convex (resp. point-wise convex) on $Q$ if, for any $x \in Q$, the function $F(x, \cdot): M \rightarrow \overline{\mathbb{R}}$ is weakly convex (resp. convex) at $x$.

Note that if $F$ is monotone on $Q \times Q$, then (3.1) holds (as $F(x, x) \geq 0$ for any $x \in Q$ by assumption).

Let $V: Q \rightrightarrows T M$ be a vector field. Associated to $V$, we define the bifunction $G_{V}$ : $M \times M \rightarrow \overline{\mathbb{R}}$ by

$$
\begin{equation*}
G_{V}(x, y):=\sup _{u \in V(x), v \in \exp _{x}^{-1} y}\langle u, v\rangle \quad \text { for any }(x, y) \in M \times M, \tag{3.2}
\end{equation*}
$$

where for any $(x, y) \in M \times M, \exp _{x}^{-1} y$ is defined by (2.9), and we adopt the the convention that $\sup \emptyset=+\infty$. Proposition 3.2 below provides some properties of the bifunctions $G_{V}$ that will be used in the sequel. As usual, for a subset $Z$ of $T_{x} M$, we use $\overline{\operatorname{co}} Z$ to denotes the closed and convex hull of the set $Z$ in $T_{x} M$.

Proposition 3.2. Suppose $V(x) \subseteq T_{x} M$ is nonempty for each $x \in Q$, and let $G_{V}$ be defined by (3.2). Then the following assertions hold:
(i) If $V(x)$ is compact-valued, then

$$
\begin{equation*}
\mathcal{D}\left(G_{V}\right)=Q \times M \quad \text { and } \quad G_{V}(x, x)=0 \quad \text { for any } x \in Q \tag{3.3}
\end{equation*}
$$

(ii) $G_{V}(x, \cdot) \circ \gamma_{x y}$ is convex on $[0,1]$ for any $x, y \in Q$ and any geodesic $\gamma_{x y} \in \min -\Gamma_{x y}$.
(iii) If $G: Q \times Q \rightarrow \overline{\mathbb{R}}$ is point-wise weakly convex on $Q$, then so is $G_{V}+G$.
(iv) $\partial G_{V}(x, \cdot)(x)=\overline{\operatorname{co}} V(x)$ for any $x \in Q$.
(v) If $V(x)$ is compact-valued and $V$ usc on $Q$, then the function $x \mapsto G_{V}(x, y)$ is usc on $Q$ for each $y \in Q$.

Proof. Assertion (i) is clear by definition. To show assertion (ii), fix $x, y \in Q$ and let $\gamma_{x y} \in \min -\Gamma_{x y}$ and $y_{t}:=\gamma_{x y}(t)$. Then we have that

$$
\begin{equation*}
\exp _{x}^{-1} y_{t} \subseteq t \exp _{x}^{-1} y \quad \text { for each } t \in(0,1) \tag{3.4}
\end{equation*}
$$

Indeed, let $v_{t} \in \exp _{x}^{-1} y_{t}$ with some $t \in(0,1)$. Then, $\left\|v_{t}\right\|=\mathrm{d}\left(x, y_{t}\right)=t \mathrm{~d}(x, y)$ and $\exp _{x} v_{t}=y_{t}$. Define a curve $\beta:[0,1] \rightarrow M$ by

$$
\beta(s):= \begin{cases}\exp _{x} \frac{s}{t} v_{t}, & s \in[0, t] \\ \gamma_{x y}(s), & s \in(t, 1]\end{cases}
$$

Then $l(\beta)=\left\|v_{t}\right\|+\mathrm{d}\left(y_{t}, y\right)=\mathrm{d}(x, y)$. This means that $\beta \in \min -\Gamma_{x y}$ thanks to Lemma 2.2. Therefore, $\frac{1}{t} v_{t} \in \exp _{x}^{-1} y$ by definition because $\dot{\beta}(0)=\frac{1}{t} v_{t}$; hence (3.4) holds. Thus

$$
G_{V}\left(x, \gamma_{x y}(t)\right)=\sup _{u \in V(x), v_{t} \in \exp _{x}^{-1} y_{t}}\left\langle u, v_{t}\right\rangle \leq t \sup _{u \in V(x), v \in \exp _{x}^{-1} y}\langle u, v\rangle=t G(x, y) .
$$

This shows that $G_{V}\left(x, \gamma_{x y}(\cdot)\right)$ is convex on [0, 1] (noting that $G_{V}(x, x)=0$ ), and assertion (ii) is shown as $\gamma_{x y} \in \min -\Gamma_{x y}$ is arbitrary.

Assertion (iii) follows immediately from assertion (ii). Now, we verify assertion (iv). To proceed, let $x \in Q$ and $\xi \in T_{x} M$. Then for any $t>0$ small enough, one has that $\exp _{x}^{-1} \exp _{x} t \xi=\{t \xi\}$. Thus noting that $G_{V}(x, x)=0$, we have by definition that

$$
\begin{equation*}
G_{V}(x, \cdot)^{\prime}(x ; \xi)=\lim _{t \rightarrow 0^{+}} \frac{\sup _{u \in V(x)}\langle u, t \xi\rangle}{t}=\sup _{u \in V(x)}\langle u, \xi\rangle=\sup _{u \in \overline{\operatorname{co}} V(x)}\langle u, \xi\rangle . \tag{3.5}
\end{equation*}
$$

This, together with (2.4), implies that

$$
\sup _{u \in \overline{\operatorname{co}} V(x)}\langle u, \xi\rangle \geq \sup _{u \in \partial G_{V}(x, \cdot)(x)}\langle u, \xi\rangle \quad \text { for any } \xi \in T_{x} M
$$

and so $\partial G_{V}(x, \cdot)(x) \subseteq \overline{c o} V(x)$ by [37, Corollary 13.1.1, p.113]. Moreover, by (3.5), one have by definition that

$$
\partial G_{V}(x, \cdot)(x)=\partial G_{V}(x, \cdot \cdot)^{\prime}(x ; 0) \supseteq \overline{\operatorname{co}} V(x)
$$

Thus assertion (iv) is shown.
It remains to show assertion (v). To this end, fix $y \in Q$. Let $\varepsilon>0, x \in Q$ and let $\left\{x_{n}\right\} \subseteq Q$ be such that $\lim _{n \rightarrow \infty} x_{n}=x$. Since $V$ and $\exp _{(.)}^{-1} y$ are usc at $x$ (see Lemma 2.1), there is $K \in \mathbb{N}$ such that

$$
\begin{equation*}
P_{x, x_{n}} V\left(x_{n}\right) \subseteq \mathbf{B}(V(x), \varepsilon) \quad \text { and } \quad P_{x, x_{n}} \exp _{x_{n}}^{-1} y \subseteq \mathbf{B}\left(\exp _{x}^{-1} y, \varepsilon\right) \quad \text { for each } n \geq K . \tag{3.6}
\end{equation*}
$$

Set $R:=\max \{|V(x)|, \mathrm{d}(x, y)\}\left(\right.$ where $|V(x)|:=\max _{v \in V(x)}\{\|v\|\}<+\infty$ as $V(x)$ is compact), and, without loss of generality, assume that $\varepsilon<R$. Then, it follows from (3.6) that, for any $n>K$,

$$
\sup _{v_{n} \in V\left(x_{n}\right), u_{n} \in \exp _{x_{n}^{-1}}^{-1} y}\left\langle v_{n}, u_{n}\right\rangle \leq \sup _{v \in \mathbf{B}(V(x), \varepsilon), u \in \mathbf{B}\left(\exp _{x}^{-1} y, \varepsilon\right)}\langle v, u\rangle \leq \sup _{v \in V(x), u \in \exp _{x}^{-1} y}\langle v, u\rangle+3 \varepsilon R,
$$

and so $\varlimsup_{n \rightarrow \infty} G_{V}\left(x_{n}, y\right) \leq G_{V}(x, y)+3 R \varepsilon$. Thus, assertion (v) holds as $\varepsilon>0$ is arbitrary and the proof is complete.
3.2. Relationship between VIP and EP. For the remainder of the paper, we will make use of the following hypotheses for the bifunction $F$, where, as usual, we use $\delta_{C}(\cdot)$ to denote the indicator function of the nonempty subset $C$ defined by $\delta_{C}(x):=0$ if $x \in C$ and $+\infty$ otherwise:
(H1) $F$ is point-wise weakly convex on $Q$ and, $x \in \operatorname{int} \mathcal{D}(F(x, \cdot))$ for each $x \in Q$.
(H2) $F+\delta_{Q \times Q}$ is point-wise weakly convex on $Q$.
(H3) For any $y \in Q$, the function $x \mapsto F(x, y)$ is usc on $Q$.
(H4) The function $x \mapsto F(x, x)$ is lower semi-continuous (lsc for short) on $Q$.
Remark 3.1. We remark that the latter part of hypothesis (H1) is particularly satisfied if $Q \times Q \subseteq \operatorname{int} \mathcal{D}(F)$. The first part of hypothesis (H1) and hypothesis (H2) are satisfied in the case when $Q$ is weakly convex and $F(x, \cdot)$ is (weakly) convex for any $x \in Q$, which, together with Hypothesis (H3) are standard assumption for the EP (see, e.g, [10, [11, 12, 20, 47]); while hypothesis $(\mathrm{H} 4)$ is particularly satisfied if $F(x, x)=0$ for any $x \in Q$ (which was used in [10, 12, 20]).

Note that, by definition, the following implication holds:

$$
\begin{equation*}
(\mathrm{H} 2) \Longrightarrow Q \text { is weakly convex. } \tag{3.7}
\end{equation*}
$$

Associated to the pair $(F, Q)$, we define the set-valued vector field $A_{F}: Q \rightrightarrows T M$ by

$$
\begin{equation*}
A_{F}(x):=\partial F(x, \cdot)(x) \quad \text { for any } x \in Q \tag{3.8}
\end{equation*}
$$

Then the following proposition is clear from Lemma 2.6 (ii).

Proposition 3.3. Suppose that $F$ satisfies (H1). Then, the set-valued vector field $A_{F}$ is well-defined, compact convex-valued on $Q$ and satisfies

$$
\max _{v \in A_{F}(x)}\|v\| \leq L_{x}^{F} \quad \text { for each } x \in Q
$$

where $L_{x}^{F}$ stands for the center Lipschitz constant of $F(x, \cdot)$ at $x$.
The following proposition establishes the relationship between the EP associated to the pair $(F, Q)$ and the VIP associated to the pair $\left(A_{F}, Q\right)$.

Proposition 3.4. Suppose that $F$ satisfies (H1) and (H2). Then

$$
\begin{equation*}
\operatorname{VIP}\left(A_{F}, Q\right) \subseteq \operatorname{EP}(F, Q) \tag{3.9}
\end{equation*}
$$

and the equality holds if (3.1) is additionally assumed.
Proof. Let $\bar{x} \in Q$ and note that $F$ satisfies (H1) and (H2). Then, by implication (3.7), $Q$ is weakly convex, and then the same argument for proving [28, Proposition 6.4] (with $F(\bar{x}, \cdot), Q$ in place of $f, A$ there) works for the following equivalence:

$$
\begin{equation*}
\bar{x} \in \operatorname{VIP}\left(A_{F}, Q\right) \Longleftrightarrow[F(\bar{x}, y) \geq F(\bar{x}, \bar{x}) \text { for any } y \in Q] . \tag{3.10}
\end{equation*}
$$

Thus (3.9) follows from the assumption that $F(\bar{x}, \bar{x}) \geq 0$; while the converse inclusion of (3.9) holds trivially by (3.10) if (3.1) is additionally assumed. The proof is complete.

Proposition 3.5. Suppose that $F$ satisfies (H1). Then the following assertions hold:
(i) If $F$ is monotone (resp. strictly monotone) on $Q \times Q$, then so is $A_{F}$ on $Q$.
(ii) If $F$ satisfies $(\mathrm{H} 3)$ and $(\mathrm{H} 4)$, then $A_{F}$ is uKsc on $Q$; hence $A_{F} \in \mathcal{V}(Q)$.

Proof. (i). Suppose that $F$ is monotone on $Q \times Q$. Let $x, y \in Q, u_{x} \in A_{F}(x), u_{y} \in A_{F}(y)$ and let $\gamma_{x y} \in \Gamma_{x y}^{Q}$. We have to show

$$
\begin{equation*}
\left\langle u_{x}, \dot{\gamma}_{x y}(0)\right\rangle-\left\langle u_{y}, \dot{\gamma}_{x y}(1)\right\rangle \leq 0 \tag{3.11}
\end{equation*}
$$

To do this, subdivide $\gamma_{x y}$ into $n$ subsegments with the equal length determined by the consecutive points

$$
x=x_{0}<x_{1}<\ldots<x_{n-1}<x_{n}=y
$$

such that

$$
\mathrm{d}\left(x_{i-1}, x_{i}\right)=\frac{l\left(\gamma_{x y}\right)}{n} \leq \bar{r} . \quad i=1,2, \ldots, n
$$

where $\bar{r}:=\min \left\{r_{z}: z \in \gamma_{x y}[0,1]\right\}>0$ by (2.2). Thus, for each $i=1,2, \ldots, n, \exp _{x_{i-1}}^{-1} x_{i}$ is a singleton, and

$$
\begin{equation*}
\min -\Gamma_{x_{i-1} x_{i}}=\left\{\gamma_{x_{i-1} x_{i}}\right\} \quad \text { with } \quad \gamma_{x_{i-1} x_{i}}(\cdot):=\exp _{x_{i-1}}(\cdot)\left(\exp _{x_{i-1}}^{-1} x_{i}\right) \tag{3.12}
\end{equation*}
$$

Moreover, we have that

$$
\begin{equation*}
\exp _{x_{0}}^{-1} x_{1}=\frac{1}{n} \dot{\gamma}_{x y}(0), \quad \exp _{x_{n}}^{-1} x_{n-1}=-\frac{1}{n} \dot{\gamma}_{x y}(1) \tag{3.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\exp _{x_{i}}^{-1} x_{i+1}+\exp _{x_{i}}^{-1} x_{i-1}=0 \quad \text { for each } i=1,2, \ldots, n-1 \tag{3.14}
\end{equation*}
$$

To proceed, set $u_{0}:=u_{x}, u_{n}:=u_{y}$ and take $u_{i} \in A_{F}\left(x_{i}\right)$ for each $i=1,2, \ldots, n-1$. Now fix $i=1,2, \ldots, n$. Then, by assumption (H1), Lemma 2.5(i) is applicable, and thus, thanks to (3.12), we have that

$$
F\left(x_{i-1}, x_{i}\right) \geq\left\langle u_{i-1}, \exp _{x_{i-1}}^{-1} x_{i}\right\rangle \quad \text { and } \quad F\left(x_{i}, x_{i-1}\right) \geq\left\langle u_{i}, \exp _{x_{i}}^{-1} x_{i-1}\right\rangle
$$

as $F\left(x_{i}, x_{i}\right) \geq 0$. This, together with the monotonicity of $F$, implies that $\left\langle u_{i-1}, \exp _{x_{i-1}}^{-1} x_{i}\right\rangle+$ $\left\langle u_{i}, \exp _{x_{i}}^{-1} x_{i-1}\right\rangle \leq 0$; hence,

$$
\begin{equation*}
\sum_{i=1}^{n}\left(\left\langle u_{i-1}, \exp _{x_{i-1}}^{-1} x_{i}\right\rangle+\left\langle u_{i}, \exp _{x_{i}}^{-1} x_{i-1}\right\rangle\right) \leq 0 . \tag{3.15}
\end{equation*}
$$

Since by (3.14), $\left.\left\langle u_{i}, \exp _{x_{i}}^{-1} x_{i-1}\right\rangle+\left\langle u_{i}, \exp _{x_{i}}^{-1} x_{i+1}\right\rangle\right)=0$ for each $i=1,2, \ldots, n$, and since

$$
\begin{aligned}
& \left\langle u_{0}, \exp _{x_{0}}^{-1} x_{1}\right\rangle+\sum_{i=1}^{n-1}\left(\left\langle u_{i}, \exp _{x_{i}}^{-1} x_{i-1}\right\rangle+\left\langle u_{i}, \exp _{x_{i}}^{-1} x_{i+1}\right\rangle\right)+\left\langle u_{n}, \exp _{x_{n}}^{-1} x_{n-1}\right\rangle \\
& =\sum_{i=1}^{n}\left(\left\langle u_{i-1}, \exp _{x_{i-1}}^{-1} x_{i}\right\rangle+\left\langle u_{i}, \exp _{x_{i}}^{-1} x_{i-1}\right\rangle\right)
\end{aligned}
$$

it follows from (3.15) that $\left\langle u_{0}, \exp _{x_{0}}^{-1} x_{1}\right\rangle+\left\langle u_{n}, \exp _{x_{n}}^{-1} x_{n-1}\right\rangle \leq 0$. Thus (3.11) is seen to hold by (3.13), and the proof for assertion (i) is complete.
(ii). Let $x_{0} \in Q$ and let $\left\{x_{k}\right\} \subset Q,\left\{u_{k}\right\} \subset T M$ with each $u_{k} \in A_{F}\left(x_{k}\right)$ such that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} x_{k}=x_{0} \quad \text { and } \quad \lim _{k \rightarrow \infty} P_{x_{0}, x_{k}} u_{k}=u_{0} \tag{3.16}
\end{equation*}
$$

It suffices to show $u_{0} \in A_{F}\left(x_{0}\right)$. To do this, write $r_{\mathbf{B}}:=r_{\mathbf{B}\left(x_{0}, r_{x_{0}}\right)}>0$ (see (2.2)). Without loss of generality, we may assume that $x_{k} \in \mathbf{B}\left(x_{0}, \frac{r_{\mathrm{B}}}{2}\right)$ for all $k$. Let $y \in \mathbf{B}\left(x_{0}, \frac{r_{\mathrm{B}}}{2}\right)$. Then, for each $k$,

$$
\mathrm{d}\left(x_{k}, y\right) \leq \mathrm{d}\left(x_{0}, y\right)+\mathrm{d}\left(x_{0}, x_{k}\right) \leq r_{\mathbf{B}}
$$

and so $\exp _{x_{k}}^{-1} y$ is a singleton. It immediately follows from (3.16) and Lemma 2.1 that

$$
\lim _{k \rightarrow \infty}\left\langle u_{k}, \exp _{x_{k}}^{-1} y\right\rangle=\left\langle u_{0}, \exp _{x_{0}}^{-1} y\right\rangle
$$

Now, suppose hypotheses (H3) and (H4) are satisfied. Then,

$$
\begin{equation*}
F\left(x_{0}, y\right) \geq \varlimsup_{k \rightarrow \infty} F\left(x_{k}, y\right) \quad \text { and } \quad F\left(x_{0}, x_{0}\right) \leq \underline{\lim }_{k \rightarrow \infty} F\left(x_{k}, x_{k}\right) \tag{3.17}
\end{equation*}
$$

Recalling that each $u_{k} \in A_{F}\left(x_{k}\right)=\partial F\left(x_{k}, \cdot\right)\left(x_{k}\right)$, we get by (H1) that

$$
F\left(x_{k}, y\right) \geq F\left(x_{k}, x_{k}\right)+\left\langle u_{k}, \exp _{x_{k}}^{-1} y\right\rangle \quad \text { for each } k ;
$$

hence

$$
\begin{aligned}
\varlimsup_{k \rightarrow \infty} F\left(x_{k}, y\right) & \geq \varliminf_{k \rightarrow \infty}\left(F\left(x_{k}, x_{k}\right)+\lim _{k \rightarrow \infty}\left\langle u_{k}, \exp _{x_{k}}^{-1} y\right\rangle\right) \\
& =\underline{\lim }_{k \rightarrow \infty} F\left(x_{k}, x_{k}\right)+\left\langle u_{0}, \exp _{x_{0}}^{-1} y\right\rangle
\end{aligned}
$$

This, together with (3.17), yields that

$$
F\left(x_{0}, y\right) \geq F\left(x_{0}, x_{0}\right)+\left\langle u_{0}, \exp _{x_{0}}^{-1} y\right\rangle
$$

Thus, Lemma 2.5 is applicable to concluding that $u_{0} \in \partial F\left(x_{0}, \cdot\right)\left(x_{0}\right)=A_{F}\left(x_{0}\right)$ as $y \in$ $\mathbf{B}\left(x_{0}, \frac{r_{\mathbf{B}}}{2}\right)$ is arbitrary, and the upper Kuratowski semi-continuity of $A_{F}$ is proved. Furthermore, by Proposition 3.3. $A_{F}(x)$ is nonempty, compact and convex for each $x \in Q$. Hence $A_{F} \in \mathcal{V}(Q)$. Thus the proof is complete.
3.3. Existence and convexity properties of the solution set. Let $F: M \times M \rightarrow \overline{\mathbb{R}}$ and $Q \subseteq M$ satisfy the conditions assumed at the beginning of the present section. We have the following existence result on the solution of EP associated to the pair $(F, Q)$.

Theorem 3.6. Suppose that $Q$ contains a weak pole $o \in \operatorname{int}_{R} Q$ and that $F$ satisfies (H1)(H4). Then $\operatorname{EP}(F, Q) \neq \emptyset$ provided that $Q$ is compact, or assumptions (b) in Proposition 2.10 is satisfied with $A_{F}$ in place of $A$.

Proof. By hypotheses (H1) and (H2), we see from Proposition 3.4 that

$$
\begin{equation*}
\operatorname{VIP}\left(A_{F}, Q\right) \subseteq \operatorname{EP}(F, Q) \tag{3.18}
\end{equation*}
$$

Moreover, by hypotheses (H3) and (H4), we get by Proposition 3.5 (ii) that $A_{F} \in \mathcal{V}(Q)$. Thus, by assumption, Proposition 2.10 is applicable to getting that

$$
\operatorname{VIP}\left(A_{F}, Q\right) \neq \emptyset
$$

The result follows immediately from (3.18) and the proof is complete. $\square$
Remark 3.2. Assumption (b) in Proposition 2.10 is satisfied with $A_{F}$ in place of $A$ if $Q$ has the BBC and one of the following assumptions holds (in particular, assumption (b2) was used by Colao et al in [11]):
(b1) $A_{F}$ satisfies the coerciveness condition on $Q$.
(b2) There exists a compact set $L \subseteq M$ such that

$$
\begin{equation*}
x \in Q \backslash L \Rightarrow[\exists y \in Q \cap L \text { s.t. } F(x, y)<0] . \tag{3.19}
\end{equation*}
$$

In fact, it is clear from Remark 2.3 in the case of (b1). To check this for the case of (b2), let $L \subseteq M, x \in Q \backslash L$ and let $y \in Q \cap L$ be given by (3.19) such that $F(x, y)<0$. Then,

$$
\begin{equation*}
F(x, y)-F(x, x) \leq F(x, y)<0 \tag{3.20}
\end{equation*}
$$

By assumption (H2) and the definition of $A_{F}$, we see that for any $v \in A_{F}(x)$, there exists a minimal geodesic $\gamma_{x y} \in \min -\Gamma_{x y}^{Q}$ such that $F(x, y) \geq F(x, x)+\left\langle v, \dot{\gamma}_{x y}(0)\right\rangle$. This, together with (3.20), implies that $\left\langle v, \dot{\gamma}_{x y}(0)\right\rangle<0$. Hence condition (2.10) is satisfied as $x \in Q \backslash L$ is arbitrary, and the proof is complete.

The following theorem provides the convexity properties of the solution set $\mathrm{EP}(F, Q)$, which is a direct consequence of Propositions 3.43 .5 and 2.11 (noting by (3.7) that $Q$ is weakly convex).

Theorem 3.7. Suppose that $F$ satisfies $(\mathrm{H} 1)-(\mathrm{H} 3)$ and $\mathrm{EP}(F, Q) \neq \emptyset$. Suppose further that $A_{F}$ is monotone on $Q$ with (3.1) (e.g., $F$ is monotone on $Q \times Q$ ). Then the following assertions hold:
(i) The solution set $\operatorname{EP}(F, Q)$ is locally convex.
(ii) If $A_{F}$ is strictly monotone on $Q$ (e.g., $F$ is strictly monotone on $Q \times Q$ ), then $\mathrm{EP}(F, Q)$ is a singleton.
(iii) If $M$ is of the sectional curvatures bounded above by $\kappa>0$, then $\operatorname{EP}(F, Q)$ is $D_{\kappa}$-convex.

In particular, in the case when $M$ is a Hadamard manifold, hypothesis (H1) implies (H2), and every convex subset has both weak poles and the BCC property. Thus the following corollary is immediate from Theorems 3.6 and 3.7 .

Corollary 3.8. Let $M$ be a Hadamard manifold. Suppose that $F$ satisfies (H1) and (H3). Then the following assertions hold:
(i) If $(\mathrm{H} 4)$ holds, then $\operatorname{EP}(F, Q) \neq \emptyset$ provided that $Q$ is compact, or one of (b1) and (b2) in Remark 3.2 holds.
(ii) If $F$ is monotone on $Q \times Q$ with $\operatorname{EP}(F, Q) \neq \emptyset$, then $\operatorname{EP}(F, Q)$ is convex.

REmARK 3.3. Assertion (ii) in Corollary 3.8 seems new even in the Hadamard manifold setting; while assertion (i) was established in [11, Theorem 3.2] under the following assumptions:
(c1) there exists a compact set $L \subseteq M$ and $y_{0} \in Q \cap L$ such that $F\left(x, y_{0}\right)<0 \forall x \in Q \backslash L$;
(c2) the set $\{y \in Q: F(x, y)<0\}$ is convex for each $x \in Q$.
Clearly assumption (c1) implies our assumption (b2) in Remark 3.2. Moreover, as will be seen in the application to the proximal point algorithm in the next section and to the mixed variational inequalities in Subsection 5.2, assumption (c2) is not satisfied, in general (thus [11, Theorem 3.2] is not applicable); while Corollary 3.8 is applicable because our assumptions (H1) and (H2) presented here are satisfied there.
4. Resolvent and proximal point algorithm for EP. As in the previous section, we always assume that $F: M \times M \rightarrow \overline{\mathbb{R}}$ and $Q \subseteq M$ satisfy the conditions assumed at the beginning of Section 3. Recall the equilibrium problem is defines by (1.1) and its solution set is denoted by $\operatorname{EP}(F, Q)$. The aim of this section is to introduce the resolvent and the proximal point algorithm for EP (1.1) on general manifolds and show convergence of this algorithm. The applications of the proximal point method to solve many different problems in the Riemannian context could be fond in e.g., [26, 28, 46, 48,

Fix $z \in M$ and define the bifunction $G_{z}: M \times M \rightarrow \overline{\mathbb{R}}$ by

$$
G_{z}(x, y):=\sup _{u \in \exp _{x}^{-1} z, v \in \exp _{x}^{-1} y}\langle-u, v\rangle_{x} \quad \text { for any }(x, y) \in M \times M
$$

In the following definition, we extend the notion of the resolvent defined in 11, definition 4.6] for the bifunction $F$ on Hadamard manifolds to the general manifold setting. Let $\lambda>0$.

DEFInition 4.1. The resolvent $J_{\lambda}^{F}: M \rightrightarrows Q$ of $F$ is defined by

$$
\begin{equation*}
J_{\lambda}^{F}(z):=\operatorname{EP}\left(F_{\lambda, z}, Q\right) \quad \text { for any } z \in M \tag{4.1}
\end{equation*}
$$

where the bifunction $F_{\lambda, z}: M \times M \rightarrow \overline{\mathbb{R}}$ is defined as

$$
F_{\lambda, z}(x, y):=\lambda F(x, y)+G_{z}(x, y) \quad \text { for any }(x, y) \in M \times M
$$

For the remainder, we always assume that $M$ is of the sectional curvature bounded above by $\kappa \geq 0$. Recall that $D_{\kappa}=\frac{\pi}{\sqrt{\kappa}}$ if $\kappa>0$ and $D_{\kappa}=+\infty$ if $\kappa=0$. Then, for any $z \in M, \mathbf{B}\left(z, \frac{D_{\kappa}}{4}\right)$ is strongly convex (see, e.g., [39, p. 169]), and so $\exp _{(\cdot)}^{-1} z$ is a singleton on $\mathbf{B}\left(z, \frac{D_{\kappa}}{4}\right)$.

Recall that $A_{F}$ is the set valued vector field associated to the bifunction $F$ (see (3.8)). Following [28], the resolvent $J_{\lambda}^{A_{F}}: M \rightrightarrows Q$ of $A_{F}$ is defined by

$$
\begin{equation*}
J_{\lambda}^{A_{F}}(z):=\operatorname{VIP}\left(A_{\lambda, z}^{F}, Q\right) \quad \text { for any } z \in M \tag{4.2}
\end{equation*}
$$

where $A_{\lambda, z}^{F}: M \rightrightarrows T M$ is defined by

$$
A_{\lambda, z}^{F}(x):=\lambda A_{F}(x)-E_{z}^{Q}(x) \quad \text { for any } x \in Q
$$

with the set-valued vector field $E_{z}^{Q}: M \rightrightarrows T M$ defined by

$$
E_{z}^{Q}(x):=\left\{u \in \exp _{x}^{-1} z: \exp _{x} t u \in Q \quad \forall t \in[0,1]\right\} \quad \text { for any } x \in Q
$$

The following theorem provides an estimate for the domain of the resolvent $J_{\lambda}^{F}$. Recall that $L_{z}^{F}$ denotes the center Lipschitz constant of $F(z, \cdot)$ at $z \in M$. Set

$$
\begin{equation*}
\mathcal{D}_{\lambda}^{F}:=\left\{z \in Q: \lambda L_{z}^{F}<\frac{D_{\kappa}}{4}\right\} \tag{4.3}
\end{equation*}
$$

THEOREM 4.2. Suppose that $F$ satisfies hypotheses (H1)-(H3) and is monotone on $Q \times Q$. Then,
(i) $\lambda \mathrm{d}\left(0, A_{F}(z)\right)<\frac{D_{\kappa}}{4}$ for each $z \in \mathcal{D}_{\lambda}^{F}$.
(ii) $\mathcal{D}_{\lambda}^{F} \subseteq \mathcal{D}\left(J_{\lambda}^{F}\right)$.
(iii) $J_{\lambda}^{F}(z) \cap \mathbf{B}\left(z, \frac{D_{\kappa}}{4}\right)=J_{\lambda}^{A_{F}}(z)$ is a singleton for each $z \in \mathcal{D}_{\lambda}^{F}$.

Proof. Note that $Q$ is weakly convex by implication (3.7). Moreover, by assumptions made for $F$, one sees by Propositions 3.3 and 3.5 that $A_{F} \in \mathcal{V}(Q)$ is monotone, and

$$
\mathrm{d}\left(0, A_{F}(z)\right) \leq L_{z}^{F} \quad \text { for each } z \in Q
$$

Thus assertion (i) follows from the definition of $\mathcal{D}_{\lambda}^{F}$ in (4.3). Below we show assertions (ii) and (iii). To do this, let $z \in \mathcal{D}_{\lambda}^{F}$. Then, $\lambda \mathrm{d}\left(0, A_{F}(z)\right)<\frac{D_{\kappa}}{4}$ by (i), and it follows from [28, Lemma 4.3 and Corollary 5.4] (applied to $A_{F}, Q$ in place of $V, A$ there) that

$$
\begin{equation*}
J_{\lambda}^{A_{F}}(z) \text { is a singleton and } J_{\lambda}^{A_{F}}(z) \subseteq \mathbf{B}\left(z, \frac{D_{\kappa}}{4}\right) \tag{4.4}
\end{equation*}
$$

Moreover, thanks to hypotheses (H1) and (H2), we have by Proposition 3.2 (i) and (iii) that the bifunction $F_{\lambda, z}=\lambda F+G_{z}$ satisfies hypotheses (H1) and (H2) (with $F_{\lambda, z}$ in place of $F$ ).

Furthermore, for any $x \in Q, F(x, x)=0$ by the monotonicity of $F$ and $G_{z}(x, x)=0$ by of Proposition 3.2 (i); hence $F_{\lambda, z}(x, x)=0$. Thus one can apply Proposition 3.4 to get that

$$
\begin{equation*}
\operatorname{EP}\left(F_{\lambda, z}, Q\right)=\operatorname{VIP}\left(A_{F_{\lambda, z}}, Q\right) \tag{4.5}
\end{equation*}
$$

Noting that $\mathcal{D}\left(G_{z}(x, \cdot)\right)=M$ by Proposition 3.2 (i), we see from Lemma 2.7 and Proposition 3.2 (iv) that, for any $x \in Q$,

$$
\begin{equation*}
A_{F_{\lambda, z}}(x):=\partial\left(\lambda F(x, \cdot)+G_{z}(x, \cdot)\right)(x)=\lambda A_{F}(x)-\overline{\operatorname{co}} E_{z}^{Q}(x) \tag{4.6}
\end{equation*}
$$

Hence $A_{\lambda, z}^{F}(x) \subseteq A_{F_{\lambda, z}}(x)$ for any $x \in Q$, and then $\operatorname{VIP}\left(A_{\lambda, z}^{F}, Q\right) \subseteq \operatorname{VIP}\left(A_{F_{\lambda, z}}, Q\right)$. By defintion (see (4.1) and (4.2)) and (4.5), it follows that

$$
\begin{equation*}
J_{\lambda}^{A_{F}}(z)=\operatorname{VIP}\left(A_{\lambda, z}^{F}, Q\right) \subseteq \operatorname{VIP}\left(A_{F_{\lambda, z}}, Q\right)=\operatorname{EP}\left(F_{\lambda, z}, Q\right)=J_{\lambda}^{F}(z) \tag{4.7}
\end{equation*}
$$

In light of (4.4), we see that $J_{\lambda}^{F}(z) \neq \emptyset$, and so assertion (ii) holds as $z \in \mathcal{D}_{\lambda}^{F}$ is arbitrary. To show assertion (iii), note that $A_{\lambda, z}^{F}(x)=A_{F_{\lambda, z}}(x)$ if $\mathrm{d}(x, z)<D_{\kappa}$ by (4.6). It follows from (4.7) that

$$
J_{\lambda}^{A_{F}}(z) \cap \mathbf{B}\left(z, \frac{D_{\kappa}}{4}\right)=J_{\lambda}^{F}(z) \cap \mathbf{B}\left(z, \frac{D_{\kappa}}{4}\right)
$$

This, together with (4.4), implies that $J_{\lambda}^{F}(z) \cap \mathbf{B}\left(z, \frac{D_{\kappa}}{4}\right)=J_{\lambda}^{A_{F}}(z)$ is a singleton, and so assertion (iii) holds. The proof is complete.

The following theorem provides sufficient conditions for $\mathcal{D}\left(J_{\lambda}^{F}\right)=M$. In particular, in the Hadamard manifold setting, this result was claimed in [11, Theorem 4.9] under the additional assumption (c1) in Remark 3.3 but the proof presented there is not correct.

THEOREM 4.3. Suppose that $F$ satisfies hypotheses (H1)-(H3) and is monotone on $Q \times Q$. Then, $\mathcal{D}\left(J_{\lambda}^{F}\right)=M$ provided that one of the following assumptions holds:
(a) $Q$ is compact and contains a weak pole $o \in \operatorname{int}_{R} Q$;
(b) $M$ is a Hadamard manifold.

Proof. Let $z \in M$. Then by the assumptions made for $F$ and Proposition 3.2 (i), (iii) and (v), one can checks easily that the bifunction $F_{\lambda, z}=\lambda F+G_{z}$ satisfies (H1)-(H4). To complete the proof, it suffices to verify that $J_{\lambda}^{F}(z) \neq \emptyset$, which is true by Theorem 3.6 in case (a). Thus we only consider case (b). To do this, we assume that $M$ is a Hadamard manifold. Then, for any $x, y \in M, \exp _{x}^{-1} y$ is a singleton and $F_{\lambda, z}(x, y)$ is reduced to

$$
F_{\lambda, z}(x, y):=\lambda F(x, y)-\left\langle\exp _{x}^{-1} z, \exp _{x}^{-1} y\right\rangle
$$

Recalling from [11, (2.7)] that

$$
\left\langle\exp _{x}^{-1} z, \exp _{x}^{-1} y\right\rangle+\left\langle\exp _{y}^{-1} w, \exp _{y}^{-1} x\right\rangle \geq \mathrm{d}^{2}(x, y) \quad \text { for any } x, y \in M
$$

and that $F$ is monotone on $Q \times Q$, we get that

$$
\begin{equation*}
F_{\lambda, z}(x, y)+F_{\lambda, z}(y, x) \leq-\mathrm{d}^{2}(x, y) \quad \text { for any }(x, y) \in Q \times Q \tag{4.8}
\end{equation*}
$$

Below we show that $F_{\lambda, z}$ satisfies (b2) in Remark 3.2, there is a compact subset $L \subseteq M$ such that

$$
\begin{equation*}
x \in Q \backslash L \Rightarrow\left[\exists y \in Q \cap L \text { s.t. } F_{\lambda, z}(x, y)<0\right] . \tag{4.9}
\end{equation*}
$$

Granting this, we get $J_{\lambda}^{F}(z)=\operatorname{EP}\left(F_{\lambda, z}, Q\right) \neq \emptyset$ by Corollary 3.8, and the proof is complete. To show (4.9), take $y \in Q$ and set $R:=L_{y}^{F_{\lambda, z}}$. Then $R<+\infty$ as $F_{\lambda, z}$ satisfies (H1), and $L:=\overline{\mathbf{B}(y, R)}$ is as desired. To show this, let $x \in Q \backslash L$ and $v \in A_{F_{\lambda, z}}(y)$. Then, $\mathrm{d}(x, y)>R$, and $\|v\| \leq R$ by Proposition 3.3. Therefore, we have that

$$
F_{\lambda, z}(y, x) \geq F_{\lambda, z}(y, y)+\left\langle v, \exp _{y}^{-1} x\right\rangle \geq-R \mathrm{~d}(x, y)
$$

(noting that $F_{\lambda, z}(y, y)=0$ ). This, together with (4.8), implies

$$
F_{\lambda, z}(x, y) \leq-\mathrm{d}^{2}(x, y)-F_{\lambda, z}(y, x) \leq(R-\mathrm{d}(x, y)) \mathrm{d}(x, y)<0
$$

Thus, (4.9) is shown, and the proof is complete.
To define the proximal point algorithm for solving EP (1.1), let $x_{0} \in Q$ and $\left\{\lambda_{k}\right\} \subset$ $(0,+\infty)$. Thus the proximal point algorithm can be formulated as follows.

Algorithm $\mathbf{P}$ Letting $k=1,2, \ldots$ and having $x_{k}$, choose $x_{k+1}$ such that

$$
x_{k+1} \in J_{\lambda_{k}}^{F}\left(x_{k}\right) \cap \mathbf{B}\left(x_{k}, \frac{D_{\kappa}}{4}\right) .
$$

Clearly, in the case when $M$ is a Hadamard manifold, Algorithm $\mathbf{P}$ is reduced to the one defined in [11]:

$$
x_{k+1} \in J_{\lambda_{k}}^{F}\left(x_{k}\right) \quad \text { for each } k \in \mathbb{N} .
$$

The convergence result of Algorithm $\mathbf{P}$ is as follows.
THEOREM 4.4. Suppose that $F$ satisfies hypotheses (H1)-(H3) and is monotone on $Q \times Q$ with $\operatorname{EP}(F, Q) \neq \emptyset$. Let $x_{0} \in Q$ and $\left\{\lambda_{k}\right\} \subset(0, \infty)$ be such that

$$
\begin{gather*}
\mathrm{d}\left(x_{0}, \operatorname{EP}(F, Q)\right)<\frac{D_{\kappa}}{8}  \tag{4.10}\\
\Sigma_{k=0}^{\infty} \lambda_{k}^{2}=\infty \quad \text { and } \quad \lambda_{k} L_{x_{k}}^{F}<\frac{D_{\kappa}}{4} \quad \text { for all } k \in \mathbb{N} . \tag{4.11}
\end{gather*}
$$

Then, Algorithm $\mathbf{P}$ is well-defined, and converges to a point in $\operatorname{EP}(F, Q)$.
Proof. Recall that $A_{F}: Q \rightrightarrows T M$ is defined by (3.8). By assumption, Propositions 3.4 and 3.5 are applicable; hence $A_{F}$ is monotone, $A_{F} \in \mathcal{V}(Q)$, and

$$
\begin{equation*}
\operatorname{VIP}\left(A_{F}, Q\right)=\operatorname{EP}(F, Q) \tag{4.12}
\end{equation*}
$$

(noting that (3.1) hold by the monotonicity assumption). Then, thanks to (4.10), one sees that

$$
\begin{equation*}
\mathrm{d}\left(x_{0}, \operatorname{VIP}\left(A_{F}, Q\right)\right)<\frac{D_{\kappa}}{8} \tag{4.13}
\end{equation*}
$$

Let $\left\{\tilde{x}_{k}\right\}$ be a sequence generated by the following proximal algorithm with initial point $\tilde{x}_{0}:=x_{0}$, which was introduced in [28] for finding a point in $\operatorname{VIP}\left(A_{F}, Q\right)$ :

$$
\begin{equation*}
\tilde{x}_{k+1} \in J_{\lambda_{k}}^{A}\left(\tilde{x}_{k}\right) \quad \text { for each } k \in \mathbb{N} \tag{4.14}
\end{equation*}
$$

In view of the second assumption in (4.11), and applying Theorem4.2 (iii), we can check inductively that Algorithm $\mathbf{P}$ is well-defined and that the generated sequence $\left\{x_{k}\right\}$ coincides with $\left\{\tilde{x}_{k}\right\}$ and satisfies

$$
\lambda_{k} \mathrm{~d}\left(0, A_{F}\left(\tilde{x}_{k}\right)\right)<\frac{D_{\kappa}}{4} \quad \text { for each } k \in \mathbb{N}
$$

This, together with the first assumption in (4.11) and (4.13), implies that [28, Corollary 5.8] (with $A_{F}, Q$ in place of $V, A$ ) is applicable, and the sequence $\left\{\tilde{x}_{k}\right\}$ and so $\left\{x_{k}\right\}$ converges to a point in $\operatorname{VIP}\left(A_{F}, Q\right)$. Thus the conclusion follows immediately from (4.12), and the proof is complete.

In the special case when $M$ is a Hadamard manifold, assumption (4.10) and the second one in (4.11) are satisfied automatically. Therefore the following corollary is direct from Theorem 4.4, which was claimed in [11, Theorem 4.9, 4.10] (for constant parameters $\lambda_{k} \equiv$ $\lambda>0)$ but with an incorrect proof there as we explained in Section 1).

Corollary 4.5. Suppose that $M$ is a Hadamard manifold, and that $F$ satisfies hypotheses $(\mathrm{H} 1)-(\mathrm{H} 3)$ and is monotone on $Q \times Q$ with $\operatorname{EP}(F, Q) \neq \emptyset$. Let $\left\{\lambda_{k}\right\} \subset(0, \infty)$ be such that $\Sigma_{k=0}^{\infty} \lambda_{k}^{2}=\infty$. Then, Algorithm $\mathbf{P}$ is well-defined, and converges to a solution in $\operatorname{EP}(F, Q)$.
5. Applications. This section is devoted to two applications of the results regarding the solution set of the EP in the previous sections: One is to the Nash equilibrium and the other to the mixed variational inequality.
5.1. Nash equilibrium. We consider the Nash equilibrium problem (NEP for short) on Riemannian manifolds in this subsection, which is formulated as follow. Let $I=$ $\{1,2, \ldots, m\}$ be a finite index set which denotes the set of players, and let $\left(M_{i}, \mathrm{~d}_{i}\right), i \in I$, be a Riemannian manifold. For each $i \in I$, let $Q_{i} \subseteq M_{i}$ be the strategy set of the $i$-th player, and $f_{i}: M \rightarrow \overline{\mathbb{R}}$ be his loss-function, where $M:=M_{1} \times M_{2} \times \cdots \times M_{m}$ is the product manifold with the standard Riemannian product metric. The Nash equilibrium problem associated to $Q:=Q_{1} \times Q_{2} \times \cdots \times Q_{m} \subseteq M$ and $\left\{f_{i}\right\}_{i \in I}$ consists of finding a point $\bar{x}=\left(\bar{x}_{i}\right) \in Q$ such that

$$
\begin{equation*}
f_{i}(\bar{x})=\min _{y_{i} \in Q_{i}} f_{i}\left(\bar{x}_{1}, \ldots, \bar{x}_{i-1}, y_{i}, \bar{x}_{i+1}, \ldots, \bar{x}_{m}\right) \quad \text { for each } i \in I \tag{5.1}
\end{equation*}
$$

Any point $\bar{x} \in Q$ satisfying (5.1) is called a Nash equilibrium point of the NEP, and we denote the set of all Nash equilibrium points by $\operatorname{NEP}\left(\left\{f_{i}\right\}_{i \in I}, Q\right)$.

The most well-known existence results for the classical NEP in the linear space setting is due to Nash [33, 34], where it is assumed that each $Q_{i}$ is compact and convex and each $f_{i}$ is (quasi)convex in the $i$-th variable. Further extensions and applications of Nash's original work could be founded in [16, 32, 35, 41] and references therein. Kristály seems the first
one to consider the existence and localization of NEP in the framework of Riemannian manifolds; see [23]. Recently, Kristály used in [24] a variational approach to analyzed the NEP with nonconvex strategy sets and nonconvex / nonsmooth payoff functions in Hadamard manifolds.

To proceed, we assume for the whole subsection that
$\left(\mathrm{H}_{\mathrm{N}}-\mathbf{a}\right) Q_{i}$ is closed and weakly convex in $M_{i}$ and $Q \subseteq \operatorname{int} \bigcap_{i \in I} \mathcal{D}\left(f_{i}\right) ;$
$\left(\mathrm{H}_{\mathrm{N}}-\mathbf{b}\right)$ for each $i \in I, f_{i}$ is continuous on $Q$;
$\left(\mathrm{H}_{\mathrm{N}}-\mathbf{c}\right)$ for each $i \in I, f_{i}$ and $f_{i}+\delta_{Q}$ are weakly convex in the $i$-th variable.
To apply our results in the previous sections, we, following [11] and 38], reformulate NEP (5.1) as an EP as follows. Let $r:=\left(r_{i}\right) \in \mathbb{R}_{++}^{m}:=\left\{\left(r_{i}\right) \in \mathbb{R}^{m}:\right.$ each $\left.r_{i}>0\right\}$, and define the bifunction $F_{r}: M \times M \rightarrow \mathbb{R}$ as the weighted positive sum of the functions $\left\{f_{i}\right\}$ :

$$
F_{r}(x, y):=\sum_{i \in I} r_{i}\left(f_{i}\left(x_{1}, \ldots, x_{i-1}, y_{i}, x_{i+1}, \ldots, x_{n}\right)-f_{i}(x)\right)
$$

for any $x=\left(x_{i}\right)_{i \in I}, y=\left(y_{i}\right)_{i \in I} \in M$. Then, it is easy to check that

$$
\begin{equation*}
\operatorname{EP}\left(F_{r}, Q\right)=\operatorname{NEP}\left(\left\{f_{i}\right\}_{i \in I}, Q\right) \tag{5.2}
\end{equation*}
$$

In the spirit of the idea in [38] for the NEP in the Euclidean space setting, we introduce the pseudosubgradient mapping $g_{r}: M \rightarrow T M$ for functions $\left\{f_{i}\right\}$ in the Riemannian manifold setting, which is defined by

$$
g_{r}(x):=\left(r_{1} \partial_{1} f_{1}(x), r_{2} \partial_{2} f_{2}(x), \ldots, r_{m} \partial_{m} f_{m}(x)\right) \quad \text { for each } x \in M
$$

where, for each $i \in I$ and $x \in M, \partial_{i} f_{i}(x)$ stands for the subdifferential of the function $f_{i}\left(x_{1}, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_{m}\right)$ at $x_{i}$, that is

$$
\partial_{i} f_{i}(x):=\partial f_{i}\left(x_{1}, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_{m}\right)\left(x_{i}\right)
$$

By definition, we check that

$$
\begin{equation*}
A_{F_{r}}(x):=\partial F_{r}(x, \cdot)(x)=g_{r}(x) \quad \text { for any } x \in Q \tag{5.3}
\end{equation*}
$$

The main theorem in this subsection is as follows, which provides the results on the existence, the uniqueness and the convexity of the Nash equilibrium point.

THEOREM 5.1. Let $r \in \mathbb{R}_{++}^{m}$. Then the following assertions hold:
(i) Suppose that $Q$ contains a weak pole $o \in \operatorname{int}_{R} Q$. Then $\operatorname{NEP}\left(\left\{f_{i}\right\}_{i \in I}, Q\right) \neq \emptyset$ provided that $Q$ is compact, or $Q$ has the BCC property and that there exists a compact subset $L \subseteq M$ such that

$$
\begin{equation*}
x \in Q \backslash L \Rightarrow\left[\forall v \in g_{r}(x), \exists y \in Q \cap L, \gamma_{x y} \in \min -\Gamma_{x y}^{Q} \text { s.t. }\left\langle v, \dot{\gamma}_{x y}(0)\right\rangle<0\right] . \tag{5.4}
\end{equation*}
$$

(ii) If $g_{r}$ is strictly monotone on $Q$, then $\operatorname{NEP}\left(\left\{f_{i}\right\}_{i \in I}, Q\right)$ is at most a singleton.
(iii) If $g_{r}$ is monotone on $Q$ and $\operatorname{NEP}\left(\left\{f_{i}\right\}_{i \in I}, Q\right) \neq \emptyset$, then $\operatorname{NEP}\left(\left\{f_{i}\right\}_{i \in I}, Q\right)$ is locally convex, and $\operatorname{NEP}\left(\left\{f_{i}\right\}_{i \in I}, Q\right)$ is $D_{\kappa}$-convex if $M$ is additionally assumed to be of the sectional curvatures bounded above by some $\kappa \geq 0$.

Proof. In view of (5.2), (5.3), (5.4) and thanks to Theorems 3.6 and 3.7 (applied to $F_{r}$ in place of $F$ ), it suffices to show that $F_{r}$ satisfies hypotheses (H1)-(H4) made in Section 3. Note that (H3) follows trivially from assumption $\left(\mathrm{H}_{\mathrm{N}}-\mathrm{b}\right)$; while ( H 4 ) is clear as $F_{r}(x, x)=0$ for any $x \in Q$. Thus we only need to show that $F_{r}$ satisfies hypotheses (H1) and (H2). To do this, let $x=\left(x_{i}\right)_{i \in I} \in Q$, and write

$$
\mathcal{D}_{i}:=\mathcal{D}\left(f_{i}\left(x_{1}, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_{m}\right)\right) \quad \text { for each } i \in I
$$

Then

$$
\begin{equation*}
\mathcal{D}\left(F_{r}(x, \cdot)\right)=\mathcal{D}_{1} \times \ldots \times \mathcal{D}_{i} \times \ldots \times \mathcal{D}_{m} \tag{5.5}
\end{equation*}
$$

By assumption $\left(\mathrm{H}_{\mathrm{N}^{-}} \mathrm{a}\right)$, each $Q_{i} \subseteq \operatorname{int} \mathcal{D}_{i}$ and so

$$
x \in Q \subseteq \operatorname{int} \mathcal{D}\left(F_{r}(x, \cdot)\right)
$$

Furthermore, in light of assumption $\left(\mathrm{H}_{\mathrm{N}}-\mathrm{c}\right)$, one sees that each $\mathcal{D}_{i}$ is weakly convex in $M_{i}$. This, together with (5.5), implies that $\mathcal{D}\left(F_{r}(x, \cdot)\right)$ is weakly convex in $M$. We claim that $F_{r}(x, \cdot)$ and $F_{r}(x, \cdot)+\delta_{Q \times Q}(x, \cdot)$ are weakly convex in $M$. Granting this, (H1) and (H2) are checked. In fact, let $y=\left(y_{i}\right), z=\left(z_{i}\right) \in \mathcal{D}\left(F_{r}(x, \cdot)\right)$. Then, by assumption $\left(\mathrm{H}_{\mathrm{N}^{-}} \mathrm{c}\right)$, for each $i \in I$, there is a geodesic $\gamma_{i} \in \min -\Gamma_{z_{i} y_{i}}^{\mathcal{D}_{i}}$ such that

$$
\begin{equation*}
f_{i}\left(x_{1}, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_{m}\right) \circ \gamma_{i} \text { is convex on }[0,1] \tag{5.6}
\end{equation*}
$$

Define $\gamma_{z y}[0,1] \rightarrow M$ by $\gamma_{z y}(t):=\left(\gamma_{1}(t), \gamma_{2}(t), \ldots, \gamma_{m}(t)\right)$ for each $t \in[0,1]$. Then, $\gamma_{z y} \in$ $\min -\Gamma_{z y}^{F_{r}(x, \cdot)}$ (see, e.g., [7]), and

$$
F_{r}(x, \cdot) \circ \gamma_{z y}=\sum_{i \in I} f_{i}\left(x_{1}, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_{m}\right) \circ \gamma_{i}
$$

This means that $F_{r}(x, \cdot) \circ \gamma_{z y}$ is clearly convex thanks to (5.6), and so $F_{r}(x, \cdot)$ is weakly convex in $M$. Similarly, one can checks that $F_{r}(x, \cdot)+\delta_{Q \times Q}(x, \cdot)$ is also weakly convex in $M$. Thus the claim stands, and the proof is complete.

REMARK 5.1. Assertion (i) extends the corresponding one in [23, Theorem 1.1], which was proved under the assumption that each $Q_{i}$ is compact and geodesic convex. It is worthy remarking that the geodesic convexity assumption for $Q_{i}$ prevents its application to some special but important Riemannian manifolds, such as compact Stiefel manifolds $\operatorname{St}(p, n)$ and Grassmann manifolds $\operatorname{Grass}(p, n)(p<n)$, in which there is no geodesic convex subset (see [1, p. 104 (5.27)]).

Example 5.1 below provides the case where our existence result of Theorem 5.1 is applicable but not [23, Theorem 1.1]. Note also that the NEP in Example[5.1]is originally defined on the Euclidean space, and the corresponding existing results in the Euclidean space setting (see, e.g., [16, 32, 33, 34]), to the best our knowledge, are nor applicable because the set $Q_{2}$ involved is not convex in the usual sense.

Example 5.1. Consider the Nash equilibrium problem (5.1) with the associated $Q:=$ $Q_{1} \times Q_{2} \subseteq \mathbb{R} \times \mathbb{R}^{3}$ and $\left\{f_{i}\right\}_{i=1,2}$ defined respectively as follows:

$$
Q_{1}:=[-1,1], \quad Q_{2}:=\left\{\left(t_{1}, t_{2}, t_{3}\right): t_{1}^{2}+t_{2}^{2}+t_{3}^{2}=1, t_{1}>0,\left|t_{2}\right| \leq \frac{1}{2}, t_{3}>0\right\}
$$

$f_{1}\left(x_{1}, x_{2}\right)=\left(x_{1}-t_{3}\right)^{2} \quad$ and $\quad f_{2}\left(x_{1}, x_{2}\right)=\arccos t_{1} \quad$ for any $x_{1} \in \mathbb{R}, x_{2}=\left(t_{1}, t_{2}, t_{3}\right) \in \mathbb{R}^{3}$.
Clearly $Q_{2} \subset \mathbb{R}^{3}$ is not convex, and so the existence results in the Euclidean space setting are not applicable.

Below, we shall consider the problem on the Riemannian manifold $M:=\mathbb{R} \times \mathbb{S}^{2}$, where

$$
\mathbb{S}^{2}:=\left\{\left(t_{1}, t_{2}, t_{3}\right) \in \mathbb{R}^{3} \mid t_{1}^{2}+t_{2}^{2}+t_{3}^{2}=1\right\}
$$

is the 2 -dimensional unit sphere. Denote $\mathbf{x}:=(0,0,1), \mathbf{y}:=(0,0,-1)$, and consider system of coordinates $\Phi:(0, \pi) \times[0,2 \pi] \subset \mathbb{R}^{2} \rightarrow \mathbb{S}^{2} \backslash\{\mathbf{x}, \mathbf{y}\}$ around $x \in \mathbb{S}^{2} \backslash\{\mathbf{x}, \mathbf{y}\}$ defined by

$$
\Phi(\theta, \varphi):=(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)^{T} \quad \text { for each }(\theta \varphi) \in(0, \pi) \times[0,2 \pi] .
$$

Then the Riemannian metric on $\mathbb{S}^{2} \backslash\{\mathbf{x}, \mathbf{y}\}$ is given by

$$
g_{11}=1, \quad g_{12}=0, \quad g_{22}=\sin ^{2} \theta \quad \text { for each } \theta \in(0, \pi) \quad \text { and } \varphi \in[0,2 \pi]
$$

and the geodesics of $\mathbb{S}^{2} \backslash\{\mathbf{x}, \mathbf{y}\}$ are great circles or semicircles; see [42, p. 84] for more details.

Restricting $f_{1}$ and $f_{2}$ to $M=\mathbb{R} \times \mathbb{S}^{2}$, one can check by definition that assumptions $\left(\mathrm{H}_{\mathrm{N}}-a\right)-\left(\mathrm{H}_{\mathrm{N}}-c\right)$ are satisfied (noting that $\arccos t_{1}=\mathrm{d}\left(x_{2}, z_{0}\right)$ for each $x_{2}=\left(t_{1}, t_{2}, t_{3}\right) \in \mathbb{S}^{2}$, where $z_{0}:=(1,0,0)$ ), and that $Q \subset M$ is compact and has a weak pole in int $Q$. Thus, Theorem 5.1 is applicable and guarantees $\operatorname{NEP}\left(\left\{f_{1}, f_{2}\right\}, Q_{1} \times Q_{2}\right) \neq \emptyset$. Indeed, by a simple calculation, we see that $\operatorname{NEP}\left(\left\{f_{1}, f_{2}\right\}, Q_{1} \times Q_{2}\right)=\{(1,(1,0,0))\}$. However, the existence result in [23, Theorem 1.1] is not applicable because there is no geodesic subset on $\mathbb{S}^{2}$.

As explained before (see the paragraph right before Corollary 3.8), the following corollary is a direct consequence of Theorem 5.1. In particular, assertion (i) was proved in [11, Theorem 3.12] with each $Q_{i}$ being compact; while assertion (ii) is new even in the Hadamard manifold setting.

Corollary 5.2. Suppose that each $M_{i}$ is a Hadamard manifold. Then, the following assertions hold:
(i) The solution set $\operatorname{NEP}\left(\left\{f_{i}\right\}_{i \in I}, Q\right) \neq \emptyset$ provided $Q$ is compact, or there exists a compact subset $L \subseteq M$ such that (5.4) holds for some $r \in \mathbb{R}_{++}^{m}$.
(ii) If there exists some $r \in \mathbb{R}_{++}^{m}$ such that $g_{r}$ is monotone on $Q$, then $\operatorname{NEP}\left(\left\{f_{i}\right\}_{i \in I}, Q\right)$ is convex.

REmARK 5.2. In view of (5.3), one checks by Remark 3.2 (applied to $g_{r}, F_{r}$ in place of $\left.A_{F}, F\right)$ that a compact subset $L$ exists such that (5.4) holds provided one of the following assumptions holds:
(a) $g_{r}$ satisfies the coerciveness condition on $Q$;
(b) there exists a compact set $L \subseteq M$ such that

$$
x \in Q \backslash L \Rightarrow\left[\exists y \in Q \cap L \text { s.t. } F_{r}(x, y)<0\right] .
$$

5.2. Mixed variational inequalities. Let $Q \subset M$ be a nonempty closed subset. Given a vector field $V: Q \rightarrow T M$ and a real-valued function $f: M \rightarrow \overline{\mathbb{R}}$. The mixed variational inequality problem (MVIP for short) associated to $V$ and $f$ is to find $\bar{x} \in Q$, called a solution of the MVIP, such that

$$
\begin{equation*}
\left\langle V(\bar{x}), \dot{\gamma}_{\bar{x} y}(0)\right\rangle+f(y)-f(\bar{x}) \geq 0 \quad \text { for any } y \in Q, \gamma_{\bar{x} y} \in \Gamma_{\bar{x} y}^{Q} \tag{5.7}
\end{equation*}
$$

The set of all solutions of MVIP (5.7) is denoted by $\operatorname{MVIP}(V, f, Q)$. The MVIP has been studied extensively in the linear space setting; see, e.g., [17, 19, 44; and it seems that [11] is the first paper to explore the MVIP in the Hadamard manifold setting, where only the existence issue of the solution for the MVIP is concerned with.

To reformulate the MVIP as an EP considered in the previous sections, we define $F$ : $M \times M \rightarrow(-\infty,+\infty]$ as follows:

$$
\begin{equation*}
F(x, y):=\sup _{u \in \exp _{x}^{-1} y}\langle V(x), u\rangle_{x}+f(y)-f(x) \quad \text { for any }(x, y) \in M \times M \tag{5.8}
\end{equation*}
$$

where $\exp _{x}^{-1} y$ is defined by (2.9) and we adopt the convention that $a-(+\infty)=+\infty$ for any $a \in \mathbb{R}$.

Proposition 5.3. Let $F: M \times M \rightarrow(-\infty,+\infty]$ be defined by (5.8). Suppose that $f$ is convex. Then we have

$$
\begin{equation*}
\operatorname{MVIP}(V, f, Q)=\operatorname{EP}(F, Q) \tag{5.9}
\end{equation*}
$$

Proof. It is evident that $\operatorname{MVIP}(V, f, Q) \subseteq \mathrm{EP}(F, Q)$. To show the converse inclusion, let $\bar{x} \in \operatorname{EP}(F, Q)$ and it suffices to prove that (5.7) holds. To this end, let $y \in Q$ and $\gamma_{\bar{x} y} \in \Gamma_{\bar{x} y}^{Q}$. We have to show that

$$
\begin{equation*}
\left\langle V(\bar{x}), \dot{\gamma}_{\bar{x} y}(0)\right\rangle+f(y)-f(\bar{x}) \geq 0 \tag{5.10}
\end{equation*}
$$

Take $\bar{t} \in(0,1]$ such that $\mathrm{d}\left(\bar{x}, \gamma_{\bar{x} y}(\bar{t})\right) \leq r_{\bar{x}}$ (note that $r_{\bar{x}}>0$ by (2.1)). Denote $\bar{y}:=\gamma_{\bar{x} y}(\bar{t})$. Then $\bar{y} \in Q$ and $\Gamma_{\bar{x} \bar{y}}^{Q}=\left\{\gamma_{\bar{x} \bar{y}}\right\}$ is a singleton, where $\gamma_{\bar{x} \bar{y}}:[0,1] \rightarrow M$ is defined by

$$
\gamma_{\bar{x} \bar{y}}(s):=\gamma_{\bar{x} y}(\bar{t} s) \quad \text { for any } s \in[0,1] .
$$

Then $\dot{\gamma}_{\bar{x} \bar{y}}(0)=\bar{t} \dot{\gamma}_{\bar{x} y}(0)$. In view of $\bar{x} \in \operatorname{EP}(F, Q)$ and $\bar{y} \in Q$, we see that

$$
\left\langle V(\bar{x}), \dot{\gamma}_{\bar{x} \bar{y}}(0)\right\rangle+f(\bar{y})-f(\bar{x})=\left\langle V(\bar{x}), \bar{t} \dot{\gamma}_{\bar{x} y}(0)\right\rangle+f(\bar{y})-f(\bar{x}) \geq 0
$$

Noting that $\frac{f(\bar{y})-f(\bar{x})}{t} \leq f(y)-f(\bar{x})$ by the convexity of $f \circ \gamma_{\bar{x} y}$ (as $f$ is convex), we conclude that (5.10) holds, which completes the proof.

We assume in the present subsection that
$\left(\mathrm{H}_{\mathrm{M}}-\mathrm{a}\right) f$ is convex and $Q \subseteq \operatorname{int} \mathcal{D}(f)$ is closed weakly convex.
$\left(\mathrm{H}_{\mathrm{M}}-\mathrm{b}\right) V$ is continuous on $Q$.

The following theorem gives the existence, the uniqueness and the convexity property about the solution set $\operatorname{MVIP}(V, f, Q)$.

Theorem 5.4. The following assertions hold:
(i) Suppose that $Q$ contains a weak pole $o \in \operatorname{int}_{R} Q$. Then $\operatorname{MVIP}(V, f, Q) \neq \emptyset$ provided that $Q$ is compact, or $Q$ has the $B C C$ property and there exists a compact subset $L \subseteq M$ such that

$$
\begin{equation*}
x \in Q \backslash L \Rightarrow\left[\forall v \in \partial f(x), \exists y \in Q \cap L, \gamma_{x y} \in \min -\Gamma_{x y}^{Q} \text { s.t. }\left\langle V(x)+v, \dot{\gamma}_{x y}(0)\right\rangle<0\right] . \tag{5.11}
\end{equation*}
$$

(ii) If $V+\partial f$ is strictly monotone on $Q$, then $\operatorname{MVIP}(V, f, Q)$ is at most a singleton.
(iii) If $V+\partial f$ is monotone on $Q$ and $\operatorname{MVIP}(V, f, Q) \neq \emptyset$, then $\operatorname{MVIP}(V, f, Q)$ is locally convex, and is $D_{\kappa}$-convex if $M$ is additionally assumed to be of the sectional curvatures bounded above by some $\kappa \geq 0$.

Proof. We first show that $F$ satisfies hypotheses (H1)-(H4) made in Section 3. To do this, let $G_{V}: M \times M \rightarrow \mathbb{R}$ be defined by (3.2), and let $G: M \times M \rightarrow \mathbb{R}$ be defined by

$$
\begin{equation*}
G(x, y):=f(y)-f(x) \quad \text { for any }(x, y) \in Q \times M \tag{5.12}
\end{equation*}
$$

Then $F=G_{V}+G$. Noting by assumption ( $\mathrm{H}_{\mathrm{M}}-\mathrm{a}$ ) that both $G$ and $G+\delta_{Q \times Q}$ are point-wise weakly convex on $Q$, we see from Proposition 3.2(iii) that $F=G_{V}+G$ and $F+\delta_{Q \times Q}=$ $G_{V}+\left(G+\delta_{Q \times Q}\right)$ are point-wise weakly convex on $Q$. This particularly means that $F$ satisfies (H2). To show (H1) and (H4), recalling (3.3) in Proposition 3.2(i), one checks that

$$
\mathcal{D}(F(x, \cdot))=\mathcal{D}(G(x, \cdot)) \bigcap \mathcal{D}\left(G_{V}(x, \cdot)\right)=\mathcal{D}(f) \quad \text { for any } x \in Q
$$

In view of assumption $\left(\mathrm{H}_{\mathrm{M}}-\mathrm{a}\right)$, ( H 1 ) is checked; while $(\mathrm{H} 4)$ is trivial since, by (3.3), $F(x, x)=$ 0 for any $x \in Q$. Thus it remains to check (H3). Since by assumption $\left(\mathrm{H}_{\mathrm{M}}-\mathrm{a}\right)$, the function $x \mapsto G(x, y)$ is continuous on $Q$ (see Lemma [2.6 (i)). In view of assumption $\left(\mathrm{H}_{\mathrm{M}}-\mathrm{b}\right)$, Proposition 3.2(v) is applicable to getting that $x \mapsto G_{V}(x, y)$ is usc on $Q$ and so is $F$. Thus, (H3) is checked. Next, we check that

$$
\begin{equation*}
A_{F}(x)=V(x)+\partial f(x) \quad \text { for each } x \in Q \tag{5.13}
\end{equation*}
$$

where $A_{F}$ is defined by (3.8). Granting this, one verifies that conditions of Theorems 3.6 and 3.7 are satisfied, and then assertions (i)-(iii) follow by (5.9) (which is valid by assumption $\left(\mathrm{H}_{\mathrm{M}}-\mathrm{a}\right)$ ). To show (5.13), let $x \in Q$. Then $\partial G_{V}(x, \cdot)(x)=V(x)$ by Proposition 3.2(iv). Thus, by assumption $\left(\mathrm{H}_{\mathrm{M}}-\mathrm{a}\right)$, one applies Lemma 2.7 to obtain (5.13) and the proof is complete.

With a similar argument that we did for Corollary 5.2, but using Theorem 5.4 in place of Theorem 5.1, we have the following corollary. In particular, assertion (i) was claimed in [11, Theorem 3.5] with its proof being incorrect; while assertion (ii) is new even in the Hadamard manifold setting.

Corollary 5.5. Suppose that $M$ is a Hadamard manifold. Then, the following assertions hold:
(i) The solution set $\operatorname{MVIP}(V, f, Q) \neq \emptyset$ provided $Q$ is compact, or there exists a compact subset $L \subseteq M$ such that (5.11) holds.
(ii) If $V+\partial f$ is monotone on $Q$ with $\operatorname{MVIP}(V, f, Q) \neq \emptyset$, then $\operatorname{MVIP}(V, f, Q)$ is convex.

REMARK 5.3. Under one of the following assumptions, a compact subset $L \subseteq M$ exists such that (5.11) holds:
(a) $V$ satisfies the coerciveness condition on $Q$;
(b) $\partial f$ satisfies the coerciveness condition on $Q$ and $V$ is monotone on $Q$.

Indeed, in view of assumption ( $\mathrm{H}_{\mathrm{M}}-a$ ), we see that $\partial f$ is monotone on $Q$ by definition of the subdifferential of $f$. Assuming (b) or (c), it is easy to verify by definition that $V+\partial f$ satisfies the coerciveness condition on $Q$. Thus, one checks that (5.11) is satisfied as we have explained in Remark 2.3 with $V+\partial f$ in place of $A$.

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