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Der **Dannie-Heineman-Preis 2013** wurde Emmanuel Jean Candès, Stanford/USA, verliehen. Herr Candès hat als einer der Architekten des Compressive Sensing Prinzips die Brücke zwischen Grundlagenforschung und der vielfältigen praktischen Nutzung dieser Theorie hergestellt und hierdurch die Entwicklung der mathematischen Statistik, der angewandten Mathematik und angrenzender Gebiete in jüngster Zeit maßgeblich geprägt.

Emmanuel J. Candès

A Short Tour of Compressive Sensing

Emmanuel J. Candès, Professor of Mathematics, of Statistics and of Electrical Engineering (by courtesy) Stanford University, Dannie-Heineman-Preisträger 2013

Ladies and gentlemen,
I am truly honored and happy to have been selected to receive the Dannie Heineman Prize 2013, and would like to offer my warmest thanks to the Academy. I am especially grateful to the members of the selection committee for proposing my name.

It is a real privilege for me to be addressing such a distinguished audience, and speak in a city of science where some of my scientific heroes, at one time or another, have lived and worked. One of these is of course Carl Friedrich Gauss, who more than 150 years after his death, still is the ultimate inspiration and model for any mathematical scientist in the world. David Hilbert, Werner Heisenberg are other “Göttingen heroes” of mine and in truth, I have learned much about their accomplishments as an undergraduate studying mathematics and physics in France, long long time ago. Speaking of early years, some of you may have heard of a French singer named Barbara. I was and am still very fond of her. She wrote a beautiful song called “Göttingen” – there are both a French and a German version – which paints a poetic picture of this city, and which I adored. So I suppose that because of the scientific history attached to this place and the romantic view I had of this city, the name “Göttingen” always fascinated me. The couple of days I spent here far from shattered this mindset.

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My goal in this lecture is to give you a little idea of my work and I will take you through a short tour of compressive sensing (CS). While this subject can be highly mathematical, this tour will be non-technical and I will essentially present my story through pictures and diagrams.

From my point of view, the story of compressive sensing really begins with a surprising experiment in the area of medical imaging. To tell the truth, I have always found biomedical imaging techniques both imaginative and daring, and because this lecture is designed for a broad audience, it is probably best to start with a very brief history of such methods. Centuries ago, before World War I, to understand anatomy, one had to perform dissections. Simply put, we could not image internal organs, and had to open up the human body just to see what it is made of and visualize traumas. The discovery of X-rays by Wilhelm Röntgen in 1895 – Röntgen received the very first Physics Nobel Prize for this discovery – changed all of this. It was indeed quickly realized that X-rays had great medical applications since they could be used to view a non-uniform medium such as the human body (this is the basis of radiography). The limitation of radiography, however, is that this technique only provides a two-dimensional impression of structures inside the human body. We cannot really distinguish what is far from what is close, what is in front from what is behind. Expressed differently, it is not really possible to recover this three-dimensional universe as we only see its shadow. Two fairly contemporary breakthroughs radically changed the situation: the first is the discovery of computed tomography (CT) in 1972 by Hounsfield and Cormack, and the second, that of magnetic resonance imaging (MRI) by Lauterbur and Mansfield in 1972. Both these discoveries profoundly altered medical diagnostics and were each eventually recognized by the Nobel Prize in Medicine. In a nutshell, CT and MR scans probe the human body by measuring its response to an excitation and, hence, provide indirect measurements about biological tissues. A CT scan measures the absorption of energy along the path of X-rays, which are shot from various locations and at various angles. This process produces a large series of two-dimensional radiographic images from which it is possible to generate an image of the 3-dimensional human body. Mathematically, there is a relationship between the collected data and the object of interest known as the Radon transform, named after the mathematician Johannes Radon (1887–1956) who first proposed its study. Inverting this transformation yields the image we are looking for. MRI is an even more revolutionary non-invasive technique, which uses the quantum properties of matter. An MR scan excites the nucleus of atoms by means of a magnetic field. These nuclei have a magnetic spin, and will respond to this excitation, and it precisely is this response that gets recorded. As in CT, there is a mathematical transformation, which relates the object we wish to infer and the data we collect. In this case, after performing a few approximations, this mathematical transformation is given by the very well-known Fourier transform, introduced 150 years earlier by Jean-Baptiste Joseph Fourier as a tool for understanding heat transfer or thermal conduction. To say the least, it is rather spectacular to find the same transformation arising naturally in a completely different context! Applying the inverse Fourier transform to MR data produces an MR image of body tissues.

A major issue with MRI is that data collection is *inherently* slow. Long acquisition times are especially problematic in pediatrics because children have difficulties staying still and cannot hold their breadth. The consequence is that images are always blurred and often cannot be used for medical diagnostic. A current solution would shut down a child's breathing for a couple minutes by deepening the anesthesia; this is obviously risky and costly. Away from pediatrics, faster imaging would decrease scan time, decrease image artifacts, increase spatial and temporal resolution, increase coverage; in short, enable a host of new applications. Now long scan times are caused by incompressible relaxation times (this why the data collection process is inherently slow) so that faster acquisition truly means a reduced data rate.

A few years ago, I was fortunate to be contacted by radiologists from the University of Wisconsin, who were trying to speed things up. The idea was to reconstruct a 512×512 pixel image from about 22×512 data points, i.e. Fourier samples. We need to do the math here: we have about 262,000 unknown pixel values and roughly 11,200 equations (this means that we are missing about 95% of the minimum number of equations we would need). How can we possibly solve such a heavily underdetermined system? If Gauss were to listen to us, he would likely be scornful. Looking at Figure 1, this may explain why the naive method radiologists were using gives disastrous results, compare the reconstruction in (b) with the original picture in (a). The surprise is the reconstruction in (c) obtained by solving a simple convex optimization problem: among all the feasible solutions to the underdetermined system, find that whose sum of gradient magnitudes is minimal, see [4] for details. This method, which does not make any parametric assumption about the unknown image we wish to reconstruct, is perfect; that is to say, it perfectly recovers the input image with no error whatsoever. The same 'miracle' occurs if we substitute the image in Figure 1 (a) with other images of this type.

How can this be? Together with Justin Romberg and Terence Tao, we set out to mathematically explain this curious phenomenon. Suppose we have a signal $x[t]$, $t = 0, 1, \dots, n-1$, with possibly complex-valued amplitudes and let $\hat{x}$ be the discrete Fourier transform (DFT) of x defined by

$$\hat{x}[\omega] = \sum_{t=0}^{n-1} x[t] e^{-i2\pi\omega t/n}, \quad \omega = 0, 1, \dots, n-1.$$

We do not have the time to acquire all the Fourier coefficients so we only sample m of them by sampling frequencies ω uniformly at random. This leads to an underdetermined system of the form $y = Ax$, where y is the vector of Fourier samples at the observed frequencies and A is the $m \times n$ matrix whose rows are correspondingly sampled from the DFT matrix. To recover, we simply find among all solutions that with minimum ℓ_1

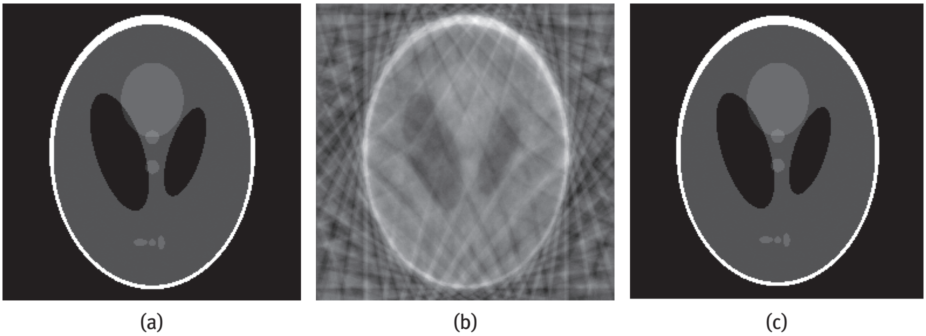

Fig. 1. (a) The Logan-Shepp phantom test image. (b) Minimum energy reconstruction obtained by setting unobserved Fourier coefficients to zero. (c) Reconstruction obtained by minimizing the sum of gradient magnitudes. (Figure reproduced from [4].)

norm:

$$\begin{array}{ll} \text{minimize} & \|x\|_{\ell_1} = \sum_{t=0}^{n-1} |x[t]| \\ \text{subject to} & Ax = y \end{array}$$

(with real data, this can be recast as a linear program). In other words, we just minimize the sum of the amplitudes of the signal. The main result in [4] proves that on the order of k log n samples suffice for this method to achieve perfect recovery almost always, see Figure 2 for an illustrative example (also showing the role played by the ℓ_1 norm versus the usual ℓ_2 norm). This phenomenon is not limited to Fourier sampling. It is now well understood that ℓ_1 minimization recovers exactly sparse solutions to underdetermined systems of equations $Ax = y$ provided that the rows of A are not sparse and diverse [3]. It is also established that recovery is accurate – although not exact – in case of approximate sparsity.

Early papers such as [5] and [2] extended the initial discovery in [4] – I would like to note that in addition to having made seminal contributions, Donoho coined the term 'compressed sensing' – triggering a massive literature so that by now CS is a rich and well-developed mathematical theory. In fact, "CS built on, and helped make coherent, ideas that had been applied or developed in particular scientific contexts, such as geophysical imaging and theoretical computer science, and even in mathematics itself (e.g., geometric functional analysis)."[1] In particular, the use of the ℓ_1 norm is not new. On the practical side, thanks to the tireless work of several teams around the world, this discovery is changing MRI. Instead of having to sedate a sick toddler for a couple of minutes, CS scans can be made in just 15 seconds – the length of a single breath. Figure 3 and its caption, retrieved from http://www.eecs.berkeley.edu/Research/Projects/Data/106899.html, will give the rea-

1 The quote is taken from [7].

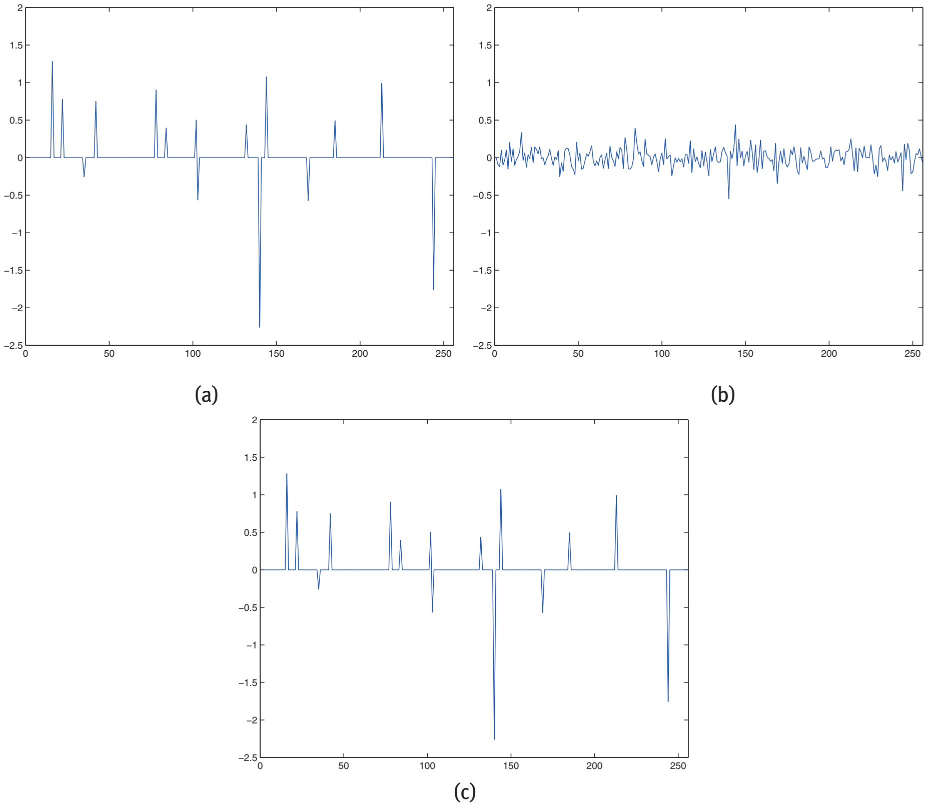

Fig. 2. Recovery of a sparse signal of length $n = 256$ from $m = 32$ complex-valued Fourier samples. (a) Sparse signal (15 spikes). (b) Minimum ℓ_2 reconstruction. (c) Minimum ℓ_1 reconstruction is perfect!

der a sense of recent progress in this field, please see the accompanying paper [6] as well as [8] for further contemporary developments.

Finally, compressive sensing has broader implications than accelerating MR scans, and touches on the very nature of signal acquisition. Consider that while a digital camera records millions of pixel values stored in a very large data file, it is however often possible to compress these data 10 or even 100 fold without much distortion. (In fact, a digital camera begins to throw away most of the measured bits as soon as the shutter closes. This is a little disturbing although how would we know which bits to discard a priori?) I quote from David Brady [1]: "one can regard the possibility of digital compression as a failure of sensor design. If it is possible to compress measured data, one might argue that too many measurements were taken." The reason why data compression works is that information in an image is often redundant – it is not white noise. Against this background, CS asserts that if you know that the

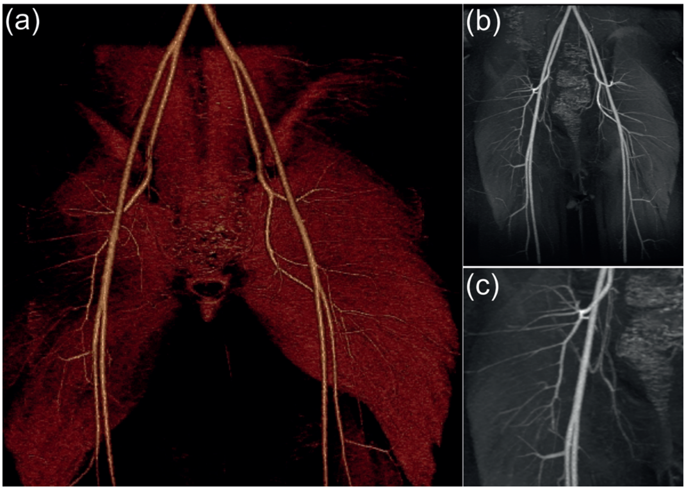

Fig. 3. Example of image quality: Submilimeter resolution, 8-fold accelerated acquisition of a first pass contrast MR angiography with compressed sensing of a 6 years old patient. Pediatric patients have smaller vessels and faster circulation than adults and require much faster imaging. (a) Volume rendering (b) Maximum intensity projection (MIP) and (c) Zoomed MIP showing extraordinary level of details. The data was acquired within 16 seconds compared to 2 min that are required for Nyquist sampling and was reconstructed with our parallel implementation in less than 2 min. At that temporal resolution there is no venus contamination in the image.

scene we are interested in photographing is information-sparse, then it is possible to condense it in just a few measurements, each measurement being a randomly weighted sum of all the pixel values in the scene, just as in an MR scan. To recover the scene and look at it, simply solve an ℓ_1 minimization problem as to find the simplest object consistent with the measured data. Just as Shannon sampling theory has informed sensor design, the possibility of fast acquisition techniques of approximately sparse signals offers new trade-offs and perspectives for sensor design. Such techniques begin to be applied in microscopy, astronomy, and electronics for radio-frequency sensing.

Literatur

[1] D. J. Brady. *Optical imaging and spectroscopy*. Wiley Online Library, 2009.

[2] E. Candès and T. Tao. Near-optimal signal recovery from random projections: Universal encoding strategies? *IEEE Trans. Inform. Theory*, 52(12):5406–5425, 2006.

[3] E. J. Candès and Y. Plan. A probabilistic and RIPless theory of compressed sensing. *IEEE Trans. Inform. Theory*, 57(11):7235–7254, 2011.

[4] E. J. Candès, J. Romberg, and T. Tao. Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information. *IEEE Trans. Inform. Theory*, 52(2):489–509, 2006.

[5] D. Donoho. Compressed sensing. *IEEE Trans. Inform. Theory*, 52(4):1289–1306, 2006.

[6] M. Murphy, K. Keutzer, S. Vasanawala, and M. Lustig. Clinically feasible reconstruction time for l1-spirit parallel imaging and compressed sensing mri. *18th Annual ISMRM Scientific Meeting and Exhibition*, 2010, 2010.

[7] Committee on the Mathematical Sciences in 2025; Board on Mathematical Sciences, Their Applications; Division on Engineering, and Physical Sciences; National Research Council. *Fueling Innovation and Discovery: The Mathematical Sciences in the 21st Century*. The National Academies Press, 2012.

[8] T. Zhang, S. Chowdhury, M. Lustig, R. Barth, M. Alley, T. Grafendorfer, P. Calderon, F. Robb, J. Pauly, and S. Vasanawala. Clinical performance of contrast enhanced abdominal pediatric mri with fast combined parallel imaging compressed sensing reconstruction. *Journal of Magnetic Resonance Imaging*, 2013.