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Positive solutions of fractional differential equation nonlocal boundary value problems

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Abstract
In this paper, we study the existence and uniqueness of positive solutions for a class of higher-order nonlocal fractional differential equations with Riemann-Stieltjes integral boundary conditions. We firstly convert the problem to an equivalent integral equation, and then by applying a fixed point theorem of a sum operator, the existence and uniqueness of positive solutions is established. Furthermore, an iterative scheme to approximate the solution is constructed and an example is given to illuminate the application of the main results.

Keywords: fractional differential equation; boundary value problem; fixed point theorem; mixed monotone operators; Riemann-Stieltjes integral

1 Introduction

In this paper, we are interested in the existence and uniqueness of positive solutions for a fractional differential equation nonlocal boundary value problem (BVP for short):

$$\begin{cases} -D_{0+}^{\alpha}x(t) = f(t,x(t),x(t)) + g(t,x(t)), & 0 < t < 1, n-1 < \alpha \leq n, n \geq 3, \\ x^{(k)}(0) = 0, \quad 0 \leq k \leq n-2, & x(1) = \int_0^1 x(s)\,dA(s), \end{cases} \tag{1}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative, $f : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ and $g : [0,1] \times [0,+\infty) \to [0,+\infty)$ are continuous, $\int_0^1 x(s)\,dA(s)$ denotes the Riemann-Stieltjes integral of x with respect to A, $A : [0,1] \to R$ is a function of bounded variation and dA can be a signed measure. Here we also recall that the idea using a Riemann-Stieltjes integral with a signed measures is due to Webb and Infante in [1, 2], which can cover the multi-point boundary conditions and the integral boundary conditions in a single framework as special cases.

Differential equations have recently been proved to be a valuable tool in modeling many phenomena arising from various fields of science and engineering. In consequence, the subject of differential equations has received much attention and many results on boundary value problems of differential equations have been reported. In particular, since many phenomena arising in a variety of different areas of applied mathematics and physics, such as heat conduction, polymer rheology, chemistry physics, fluid flows and electrical networks can be reduced to nonlocal Riemann-Stieltjes integral boundary value problems, a lot of work has been carried out to deal with the existence of solutions of nonlocal boundary value problems by using techniques of functional analysis (see [3–13]). In [14], the

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existence and uniqueness of positive solutions for the following nonlocal BVP:

$$\begin{cases} -x''(t) = f(t, x(t)) - k^2 x, & 0 < t < 1, \\ x(0) = 0, & x(1) = \int_0^1 x(s)\,dA(s), \end{cases}$$

is investigated by using the monotone iterative technique, where dA is allowed to be a signed measure. The same problem was studied by Webb and Zima [15] and the existence of multiple positive solutions under suitable conditions on $f(t, x)$ was established.

On the other hand, fractional differential operator is nonlocal and thus fractional differential equations serve as an excellent tool for the description of hereditary properties of various materials and processes and many physical phenomena in natural sciences and engineering, such as earthquake, traffic flow, measurement of viscoelastic material properties, electrodynamics of a complex medium, polymer rheology (see [16–24]). Recently, Ahmad and Nieto [25] discussed the nonlinear Dirichlet boundary value problems of sequential fractional integro-differential equations in the sense of the Caputo fractional derivative, and the existence results are established by means of some standard tools of fixed point theory. Some special cases of the BVP (1) were also studied, for example, Salm [26] studied the case of multi-point boundary vale problems when $x(1) = \sum_{i=1}^{m-2} \zeta_i x(\eta_i)$ and $g(t, x(t)) \equiv 0$, and Zhang and Han [27] considered a singular $(n-1, n)$ conjugate-type fractional differential equation

$$\begin{cases} -D_{0+}^{\alpha} x(t) = f(t, x(t)), & 0 < t < 1, \alpha \in (n-1, n], \alpha \geq 2, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, & x(1) = \int_0^1 x(s)\,dA(s), \end{cases}$$

and the existence and uniqueness of the positive solutions was obtained provided that $f(t, x)$ satisfies some growth conditions.

Motivated by the work mentioned above, we focus on the existence and uniqueness of positive solutions for the nonlocal BVP (1) based on a fixed point theorem of a sum operator. Our work presented in this work has the following new features. Firstly, the existence and uniqueness of positive solutions are obtained, which possess a nice estimate, *i.e.*, there exist $\lambda > \mu > 0$ such that $\mu t^{\alpha-1} \leq x^*(t) \leq \lambda t^{\alpha-1}$; secondly, the boundary conditions are nonlocal which involve the Riemann-Stieltjes integral of x with respect to A, moreover, dA can be a signed measure, this implies that it can cover the multi-point and integral boundary value problems as special cases; thirdly, we also construct an iterative sequence to approximate the positive solution.

The rest of this paper is organized as follows. In Section 2, we recall some definitions and facts. In Section 3, the main results are discussed by using the properties of the Green function and a fixed point theorem of a sum operator. Finally, in Section 4, an illustrative example is also presented.

2 Preliminaries

We use the following notations in this paper:

$$\Lambda = \int_0^1 t^{\alpha-1}\,dA(t), \qquad \Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t}\,dt.$$

Now we begin this section with some preliminaries of cone and fractional calculus. Recall that a non-empty closed convex set $P \subset E$ is called a cone if it satisfies

(i) $x \in P$, $\lambda \geq 0 \Rightarrow \lambda x \in P$, and
(ii) $x \in P$, $-x \in P \Rightarrow x = \theta$,

where $(E, \|\cdot\|)$ is a real Banach space with partially ordered by a cone $P \subset E$, *i.e.*, $x \leq y$ if and only if $y - x \in P$. Cone P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, and N is called the normal constant. If $x, y \in E$, the set $[x, y] = \{z \in E \mid x \leq z \leq y\}$ is called the order interval, and denote $x \sim y$ if there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (*i.e.*, $h \geq \theta$ and $h \neq \theta$), let $P_h = \{x \in E \mid x \sim h\}$.

We say that an operator $A : E \to E$ is increasing (decreasing) if $x \leq y$ implies $Ax \leq Ay$ ($Ax \geq Ay$).

Definition 2.1 ([28]) Let $x : [a, \infty) \to R$ and $\alpha > 0$ with $\alpha \in R$. Then the Riemann-Liouville fractional integral is defined to be

$$I_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s)\, ds,$$

whenever the right side is defined. Similarly, $\alpha > 0$ with $\alpha \in R$, we define the αth Riemann-Liouville fractional derivative to be

$$D_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{(n)} \int_a^t (t-s)^{n-\alpha-1} x(s)\, ds,$$

where $n \in N$ is the unique positive integer satisfying $n - 1 \leq \alpha < n$ and $t > a$.

Proposition 2.1 ([28]) *The equality*

$$D_{0+}^{\alpha} I_{0+}^{\alpha} f(x) = f(x), \quad \alpha > 0,$$

holds for $f \in L^1(a, b)$.

In [29], the authors obtained the following results.

Lemma 2.1 ([29]) *Given* $y \in C[0,1]$. *Then the BVP*:

$$\begin{cases} D_{0+}^{\alpha} x(t) + y(t) = 0, & 0 < t < 1, n-1 < \alpha \leq n, n \geq 3, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, & x(1) = 0, \end{cases} \tag{2}$$

has a unique solution

$$x(t) = \int_0^1 G(t,s) y(s)\, ds,$$

where

$$G(t,s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \\ \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \end{cases}$$

is the Green function of BVP (2).

Lemma 2.2 ([29]) *The Green function $G(t,s)$ satisfies the following properties*:

(1) $G(t,s) > 0$, *for all* $t,s \in (0,1)$;

(2)

$$(1-t)t^{\alpha-1}s(1-s)^{\alpha-1} \leq \Gamma(\alpha)G(t,s) \leq (\alpha-1)(1-t)t^{\alpha-1}, \quad \text{for } t,s \in [0,1].$$

The following lemmas are obtained by Zhang and Han [27].

Lemma 2.3 ([27]) *Given $y \in L^1[0,1]$. Then the BVP*:

$$\begin{cases} D_{0+}^{\alpha}x(t) + y(t) = 0, \quad 0 < t < 1, n-1 < \alpha \leq n, n \geq 3, \\ x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \qquad x(1) = \int_0^1 x(s)\,dA(s), \end{cases} \tag{3}$$

has a unique solution

$$x(t) = \int_0^1 H(t,s)y(s)\,ds,$$

where $H(t,s)$ is the Green function of BVP (3) *and is given by*

$$H(t,s) = \frac{t^{\alpha-1}}{1-\Lambda}\mathscr{G}_A(s) + G(t,s), \quad \mathscr{G}_A(s) = \int_0^1 G(t,s)\,dA(t). \tag{4}$$

Lemma 2.4 *Let $0 \leq \Lambda < 1$ and $\mathscr{G}_A(s) \geq 0$ for $s \in [0,1]$. Then the Green function defined by* (4) *satisfies the following properties*:

(1) $H(t,s) > 0$, *for all* $t,s \in (0,1)$;

(2)

$$\frac{1}{1-\Lambda}t^{\alpha-1}\mathscr{G}_A(s) \leq H(t,s) \leq \left(\frac{\|\mathscr{G}_A(s)\|}{1-\Lambda} + \frac{1}{\Gamma(\alpha-1)}\right)t^{\alpha-1}, \quad \text{for } t,s \in [0,1].$$

We recall the following lemmas and definitions which are important to prove our main results.

Definition 2.2 ([30]) An operator $A: E \to E$ is said to be positive homogeneous if it satisfies $A(tx) = tAx$, $\forall t > 0$, $x \in E$. An operator $A: P \to P$ is said to be sub-homogeneous if it satisfies

$$A(tx) \geq tAx, \quad \forall t \in (0,1), x \in P.$$

Definition 2.3 ([30]) Let r be a real number with $0 \leq r < 1$. An operator $A: P \to P$ is said to be r-concave if it satisfies

$$A(tx) \geq t^r Ax, \quad \forall t \in (0,1), x \in P.$$

Lemma 2.5 ([31]) *Let $h > \theta$ and $\beta \in (0,1)$, $A: P \times P \to P$ is a mixed monotone operator satisfying*

$$A\bigl(tx, t^{-1}y\bigr) \geq t^{\beta}A(x,y), \quad \forall t \in (0,1), x,y \in P, \tag{5}$$

and $B : P \to P$ *is an increasing sub-homogeneous operator. Assume that*

(i) *there is a* $h_0 \in P_h$ *such that* $A(h_0, h_0) \in P_h$ *and* $Bh_0 \in P_h$;

(ii) *there exists a constant* $\delta_0 > 0$ *such that* $A(x, y) \geq \delta_0 Bx$, $\forall x, y \in P$.

Then

(1) $A : P_h \times P_h \to P_h$, $B : P_h \to P_h$;

(2) *there exist* $u_0, v_0 \in P_h$ *and* $\gamma \in (0, 1)$ *such that*

$$\gamma v_0 \leq u_0 < v_0, \qquad u_0 \leq A(u_0, v_0) + Bu_0 \leq A(v_0, u_0) + Bv_0 \leq v_0;$$

(3) *the operator equation* $A(x, x) + Bx = x$ *has a unique solution* x^* *in* P_h;

(4) *for any initial values* $x_0, y_0 \in P_h$, *constructing successively sequences*

$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1, 2, \ldots,$$

then $x_n \to x^*$ *and* $y_n \to x^*$ *as* $n \to \infty$.

Lemma 2.6 ([32]) *Let* $A : P \to P$ *be an increasing* γ*-concave operator and* $B : P \to P$ *is an increasing sub-homogeneous operator. Assume that*

(i) *there exists a* $h > \theta$ *such that* $Ah \in P_h$ *and* $Bh \in P_h$;

(ii) *there exists a constant* $\delta_0 > 0$ *such that* $Ax \geq \delta_0 Bx$, $\forall x \in P$.

Then the operator equation $Ax + Bx = x$ *has a unique solution* x^* *in* P_h. *Moreover, for any initial value* $y_0 \in P_h$, *constructing successively sequences* $y_n = Ay_{n-1} + By_{n-1}$, $n = 1, 2, \ldots$, *then* $y_n \to x^*$ *as* $n \to \infty$.

Remark 2.1 If operator $B \equiv 0$, Lemma 2.5, and Lemma 2.6 still hold.

Lemma 2.7 ([31]) *Let* $h > \theta$ *and* $\alpha \in (0, 1)$. $A : P \times P \to P$ *is a mixed monotone operator and satisfies*

$$A\bigl(tx, t^{-1}y\bigr) \geq tA(x, y), \quad \forall t \in (0, 1), x, y \in P. \tag{6}$$

$B : P \to P$ *is an increasing* α*-concave operator. Assume that*

(i) *there is a* $h_0 \in P_h$ *such that* $A(h_0, h_0) \in P_h$ *and* $Bh_0 \in P_h$;

(ii) *there exists a constant* $\delta_0 > 0$ *such that* $A(x, y) \leq \delta_0 Bx$, $\forall x, y \in P$.

Then

(1) $A : P_h \times P_h \to P_h$ *and* $B : P_h \to P_h$;

(2) *there exist* $u_0, v_0 \in P_h$ *and* $\gamma \in (0, 1)$ *such that*

$$\gamma v_0 \leq u_0 < v_0, \qquad u_0 \leq A(u_0, v_0) + Bu_0 \leq A(v_0, u_0) + Bv_0 \leq v_0;$$

(3) *the operator equation* $A(x, x) + Bx = x$ *has a unique solution* x^* *in* P_h;

(4) *for any initial values* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \qquad y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ *and* $y_n \to x^*$ *as* $n \to \infty$.

Lemma 2.8 *Assume that* $f[0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ *and* $g : [0,1] \times [0,\infty) \to [0,\infty)$ *are continuous. Then the BVP* (1) *has a unique solution*

$$x(t) = \int_0^1 H(t,s)\big[f\big(s,x(s),x(s)\big) + g\big(s,x(s)\big)\big]\,ds,$$

where $H(t,s)$ *is defined by* (4).

Proof By using similar method to Lemma 2.3 and standard arguments, we can show the conclusion. □

3 Main results

The basic space used in this paper is the space $C[0,1]$, it is a Banach space if it is endowed with the norm $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$ for any $x \in C[0,1]$, and E can equip with a partial order $x,y \in C[0,1]$, $x \le y \Longleftrightarrow x(t) \le y(t)$ for $t \in [0,1]$. Let $P = \{x \in C[0,1] \mid x(t) \ge 0, t \in [0,1]\}$. Clear P is a normal cone in $C[0,1]$ and the normality constant is 1.

First, we give the existence and uniqueness of positive solutions to the BVP (1).

Theorem 3.1 *Assume that*

(H$_1$) *A is a function of bounded variation such that* $\mathscr{G}_A(s) \ge 0$ *for* $s \in [0,1]$ *and* $\Lambda \in [0,1)$;

(H$_2$) $f(t,x,y) : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ *is continuous and increasing in x and y decreasing, and there exists a constant* $\gamma \in (0,1)$ *such that*

$$f\big(t,\lambda x,\lambda^{-1}y\big) \ge \lambda^{\gamma} f(t,x,y), \quad \forall t \in [0,1], x,y \in [0,+\infty);$$

(H$_3$) $g(t,x) : [0,1] \times [0,+\infty) \to [0,+\infty)$ *is continuous and increasing in* $x \in [0,+\infty)$, $g(t,\lambda x) \ge \lambda g(t,x)$ *for* $\lambda \in (0,1)$, $(t,x) \in [0,1] \times [0,+\infty)$, *and* $g(t,0) \not\equiv 0$;

(H$_4$) *there exists a constant* $\delta_0 > 0$ *such that* $f(t,x,y) \ge \delta_0 g(t,x)$, $t \in [0,1]$, $x,y \ge 0$.

Then the BVP (1) *has a unique positive solution* x^* *in* P_h, *where* $h(t) = t^{\alpha-1}$, $t \in [0,1]$. *And for any initial value* $x_0,y_0 \in P_h$, *constructing successively the sequences*

$$x_n(t) = \int_0^1 H(t,s)\big[f\big(s,x_{n-1}(s),y_{n-1}(s)\big) + g\big(s,x_{n-1}(s)\big)\big]\,ds, \quad n = 1,2,\ldots,$$

$$y_n(t) = \int_0^1 H(t,s)\big[f\big(s,y_{n-1}(s),x_{n-1}(s)\big) + g\big(s,y_{n-1}(s)\big)\big]\,ds, \quad n = 1,2,\ldots,$$

we have $x_n(t) \to x^*(t)$ *and* $y_n(t) \to x^*(t)$ *as* $n \to \infty$.

Proof Applying Lemma 2.8, BVP (1) is equivalent to the integral equation

$$x(t) = \int_0^1 H(t,s)\big[f\big(s,x(s),x(s)\big) + g\big(s,x(s)\big)\big]\,ds.$$

Let $A : P \times P \to E$ be the operator defined by

$$A(x,y)(t) = \int_0^1 H(t,s) f\big(s,x(s),y(s)\big)\,ds,$$

and $B : P \to E$ be the operator defined by

$$(Bx)(t) = \int_0^1 H(t,s)g\bigl(s,x(s)\bigr)\,ds.$$

It is simple to show that x is the solution of BVP (1) if and only if x solves the operator equation $x = A(x,x) + Bx$. From (H_1) and (H_2) we know that $A : P \times P \to P$ and $B : P \to P$. We shall prove the theorem through two steps.

Step 1. We assert that A is a mixed monotone operator and satisfies (5) and B is an increasing sub-homogeneous operator. In fact, for x_i, $y_i \in P$, $i = 1,2$ with $x_1 \geq x_2$, $y_1 \leq y_2$, we know that $x_1(t) \geq x_2(t)$ and $y_1(t) \leq y_2(t)$ for all $t \in [0,1]$. It follows from (H_1), (H_2), and Lemma 2.4 that

$$\begin{aligned} A(x_1,y_1)(t) &= \int_0^1 H(t,s)f\bigl(s,x_1(s),y_1(s)\bigr)\,ds \\ &\geq \int_0^1 H(t,s)f\bigl(s,x_2(s),y_2(s)\bigr)\,ds \\ &= A(x_2,y_2)(t), \end{aligned} \tag{7}$$

which implies that $A(x_1,y_1) \geq A(x_2,y_2)$. Similar to the argument of (7), we get $Bx_1 \geq Bx_2$. For any $\lambda \in (0,1)$ and $x,y \in P$, together with (H_2), we obtain

$$\begin{aligned} A\bigl(\lambda x,\lambda^{-1}y\bigr)(t) &= \int_0^1 H(t,s)f\bigl(s,\lambda x(s),\lambda^{-1}y(s)\bigr)\,ds \\ &\geq \lambda^{\gamma}\int_0^1 H(t,s)f\bigl(s,x(s),y(s)\bigr)\,ds \\ &= \lambda^{\gamma}A(x,y)(t). \end{aligned}$$

This means $A(\lambda x,\lambda^{-1}y) \geq \lambda^{\gamma}A(x,y)$ holds for $\lambda \in (0,1)$, $x,y \in P$. Therefore the operator A satisfies (5). Also, for any $\lambda \in (0,1)$, $x \in P$, by (H_3), we get

$$B(\lambda x)(t) = \int_0^1 H(t,s)g\bigl(s,\lambda x(s)\bigr)\,ds \geq \lambda\int_0^1 H(t,s)g\bigl(s,x(s)\bigr)\,ds = \lambda Bx(t),$$

that is, $B(\lambda x) \geq \lambda B(x)$ for any $\lambda \in (0,1)$, $x \in P$. Hence the operator B is sub-homogeneous.

Step 2. Now we verify that conditions (i) and (ii) of Lemma 2.5. First, we prove that $A(h,h) \in P_h$ and $Bh \in P_h$. It is enough to address the following conclusions:

(a) there exist $a_1,a_2 > 0$ such that $a_2h(t) \leq A_1(h,h)(t) \leq a_1h(t)$, $t \in [0,1]$;

(b) there exist $b_1,b_2 > 0$ such that $b_2h(t) \leq B_1h(t) \leq b_1h(t)$, $t \in [0,1]$.

Let

$$\begin{aligned} a_1 &= \int_0^1 \biggl(\frac{\|\mathcal{G}_A(s)\|}{1-\Lambda} + \frac{1}{\Gamma(\alpha-1)}\biggr)f(s,1,0)\,ds, \\ a_2 &= \frac{1}{1-\Lambda}\int_0^1 \mathcal{G}_A(s)f(s,0,1)\,ds. \end{aligned}$$

It follows from (H_2) and Lemma 2.4 that, for any $t \in [0,1]$,

$$\begin{aligned} A(h,h)(t) &= \int_0^1 H(t,s)f\big(s,h(s),h(s)\big)\,ds \\ &\leq \int_0^1 \left(\frac{\|\mathscr{G}_A(s)\|}{1-\Lambda} + \frac{1}{\Gamma(\alpha-1)}\right)t^{\alpha-1}f(s,1,0)\,ds \\ &= a_1h(t) \end{aligned}$$

and

$$A(h,h)(t) \geq \frac{1}{1-\Lambda}t^{\alpha-1}\int_0^1 \mathscr{G}_A(s)f(s,0,1)\,ds = a_2h(t).$$

According to (H_2)-(H_4), we get

$$f(s,1,0) \geq f(s,0,1) \geq \delta_0 g(s,0) \geq 0.$$

Due to $g(t,0) \not\equiv 0$, we obtain

$$\int_0^1 f(s,1,0)\,ds \geq \int_0^1 f(s,0,1)\,ds \geq \delta_0 \int_0^1 g(s,0)\,ds > 0,$$

and in consequence, $a_1 > 0$ and $a_2 > 0$. Thus, $a_2h(t) \leq A(h,h)(t) \leq a_1h(t)$, $t \in [0,1]$, and hence we get (a). An argument similar to the one used in (a) shows that (b) holds with

$$b_1 = \left(\frac{\|\mathscr{G}_A(s)\|}{1-\Lambda} + \frac{1}{\Gamma(\alpha-1)}\right)\int_0^1 g(s,1)\,ds, \qquad b_2 = \frac{1}{1-\Lambda}\int_0^1 \mathscr{G}_A(s)g(s,0)\,ds.$$

Hence the condition (i) of Lemma 2.5 is proved. It remains to show that the condition (ii) of Lemma 2.5 is satisfied. For $x \in P$, and for any $t \in [0,1]$, taking (H_4) into consideration, we get

$$A(x,y)(t) = \int_0^1 H(t,s)f\big(s,x(s),y(s)\big)\,ds \geq \delta_0 \int_0^1 H(t,s)g\big(s,x(s)\big)\,ds = \delta_0 Bx(t),$$

in other words, $A(x,y) \geq \delta_0 Bx$, $\forall x \in P$. Therefore, an application of Lemma 2.5 implies: the operator equation $x = A(x,x) + Bx$ has a unique positive solution $x^*(t)$ in P_h. Consequently, BVP (1) has a unique positive solution $x^*(t)$ in P_h. Moreover, for any initial value $x_0, y_0 \in P_h$, constructing successively the sequence

$$\begin{aligned} x_n(t) &= \int_0^1 H(t,s)\big[f\big(s,x_{n-1}(s),y_{n-1}(s)\big) + g\big(s,x_{n-1}(s)\big)\big]\,ds, \quad n = 1,2,\ldots, \\ y_n(t) &= \int_0^1 H(t,s)\big[f\big(s,y_{n-1}(s),x_{n-1}(s)\big) + g\big(s,y_{n-1}(s)\big)\big]\,ds, \quad n = 1,2,\ldots, \end{aligned}$$

we have $x_n(t) \to x^*(t)$, $y_n(t) \to x^*(t)$ as $n \to \infty$. □

Remark 3.1 In Theorem 3.1, we cannot only obtain the existence of unique positive solution, but also construct an iterative sequence for approximate the unique positive solution for any initial value in P_h. Moreover, the estimate of unique positive solution is derived with $\mu t^{\alpha-1} \leq x^*(t) \leq \lambda t^{\alpha-1}$ for some $\lambda > \mu > 0$. Thus the property of the unique positive solution is more clear.

If $g(t,x(t)) \equiv 0$, from Remark 2.1, we have the following corollary.

Corollary 3.1 *Assume that* (H_1) *and* (H_2) *hold. If* $f(t,0) \not\equiv 0$.
Then the problem

$$\begin{cases} -D^{\alpha}_{0+}x(t) = f(t,x(t),x(t)), & 0 < t < 1, n-1 < \alpha \leq n, n \geq 3, \\ x^{(k)}(0) = 0, \quad 0 \leq k \leq n-2, & x(1) = \int_0^1 x(s)\,dA(s), \end{cases}$$

has a unique positive solution x^* *in* P_h, *where* $h(t) = t^{\alpha-1}$, $t \in [0,1]$. *Moreover, constructing successively the sequence*

$$x_n(t) = \int_0^1 H(t,s)f\bigl(s,x_{n-1}(s),y_{n-1}(s)\bigr)\,ds, \quad n = 1,2,\ldots,$$

$$y_n(t) = \int_0^1 H(t,s)f\bigl(s,y_{n-1}(s),x_{n-1}(s)\bigr)\,ds, \quad n = 1,2,\ldots,$$

for any initial value $x_0, y_0 \in P_h$, *we have* $x_n(t) \to x^*(t)$, $y_n(t) \to x^*(t)$ *as* $n \to \infty$.

If the nonlinear term $f(t,x,x)$ is replaced by $f(t,x)$, we can get the following results.

Theorem 3.2 *Assume that* (H_1) *and* (H_3) *hold and*

(H_5) $f(t,x) : [0,1] \times [0,+\infty) \to [0,+\infty)$ *is continuous and increasing with respect to the second argument, and there exists a constant* $\gamma \in (0,1)$ *such that* $f(t,\lambda x) \geq \lambda^{\gamma} f(t,x)$, $\forall t \in [0,1]$, $\lambda \in (0,1)$, $x \in [0,\infty)$;

(H_6) *there exists a constant* $\delta_0 > 0$ *such that* $f(t,x) \geq \delta_0 g(t,x)$ *for* $t \in [0,1]$, $x \geq 0$.

Then the BVP

$$\begin{cases} -D^{\alpha}_{0+}x(t) = f(t,x(t)) + g(t,x(t)), & 0 < t < 1, n-1 < \alpha \leq n, n \geq 3, \\ x^{k}(0) = 0, \quad 0 \leq k \leq n-2, & x(1) = \int_0^1 x(s)\,dA(s), \end{cases} \tag{8}$$

has a unique positive solution x^* *in* P_h, *where* $h(t) = t^{\alpha-1}$, $t \in [0,1]$. *Moreover, for any initial value* $y_0 \in P_h$, *constructing successively the sequences*

$$y_n(t) = \int_0^1 H(t,s)\bigl[f\bigl(s,y_{n-1}(s)\bigr) + g\bigl(s,y_{n-1}(s)\bigr)\bigr]\,ds, \quad n = 1,2,\ldots,$$

we have $y_n(t) \to x^*(t)$ *as* $n \to \infty$.

Proof Applying Lemma 2.3, BVP (8) is equivalent to the integral formulation given by

$$x(t) = \int_0^1 H(t,s)\bigl[f\bigl(s,x(s)\bigr) + g\bigl(s,x(s)\bigr)\bigr]\,ds.$$

Let $A : P \to E$ be the operator defined by

$$(Ax)(t) = \int_0^1 H(t,s)f\bigl(s,x(s)\bigr)\,ds$$

and $B : P \to E$ be the operator defined by

$$(Bx)(t) = \int_0^1 H(t,s)g\bigl(s,x(s)\bigr)\,ds.$$

It is simple to show that x^* is the solution of BVP (1) if and only if x^* solves the operator equation $x = Ax + Bx$. Similar to the proof of Theorem 3.1, we know A is an increasing γ-concave operator and B is an increasing sub-homogeneous operator.

Take

$$\begin{aligned}
a_1 &= \int_0^1 \biggl(\frac{\|\mathscr{G}_A(s)\|}{1-\Lambda} + \frac{1}{\Gamma(\alpha-1)}\biggr)f(s,1)\,ds,\\
a_2 &= \frac{1}{1-\Lambda}\int_0^1 \mathscr{G}_A(s)f(s,0)\,ds,\\
b_1 &= \int_0^1 \biggl(\frac{\|\mathscr{G}_A(s)\|}{1-\Lambda} + \frac{1}{\Gamma(\alpha-1)}\biggr)g(s,1)\,ds,\\
b_2 &= \frac{1}{1-\Lambda}\int_0^1 \mathscr{G}_A(s)g(s,0)\,ds.
\end{aligned}$$

Combining the proof of Theorem 3.1 with (H_3), (H_5), (H_6), and Lemma 2.4, the conditions (i) and (ii) of Lemma 2.6 are satisfied. Therefore, an application of Lemma 2.6 implies: the operator equation $x = Ax + Bx$ has a unique positive solution x^* in P_h. Consequently, BVP (8) has a unique positive solution x^* in P_h. Moreover, constructing successively the sequence

$$\begin{aligned}
y_n(t) &= Ay_{n-1}(t) + By_{n-1}(t)\\
&= \int_0^1 H(t,s)\bigl[f\bigl(s,y_{n-1}(s)\bigr) + g\bigl(s,y_{n-1}(s)\bigr)\bigr]\,ds, \quad n = 1,2,\ldots,
\end{aligned}$$

for any initial value $y_0 \in P_h$, we have $y_n(t) \to x^*(t)$ as $n \to \infty$. □

Corollary 3.2 *Assume that* (H_1) *and* (H_5) *hold.*

Then the problem

$$\begin{cases} -D_{0+}^{\alpha}x(t) = f(t,x(t)), & 0 < t < 1, n-1 < \alpha \le n, n \ge 3,\\ x^{(k)}(0) = 0, \quad 0 \le k \le n-2, & x(1) = \int_0^1 x(s)\,dA(s), \end{cases}$$

has a unique positive solution x^* in P_h, where $h(t) = t^{\alpha-1}$, $t \in [0,1]$. Moreover, for any initial value $y_0 \in P_h$, constructing successively the sequences

$$y_n(t) = \int_0^1 H(t,s)f\bigl(s,y_{n-1}(s)\bigr)\,ds, \quad n = 1,2,\ldots,$$

we have $y_n(t) \to x^*(t)$ as $n \to \infty$.

From the proof of Theorem 3.1 and using Lemma 2.7, we can prove the following conclusion.

Theorem 3.3 *Assume that* (H_1) *holds and*

(H_7) $f(t,x,y):[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ *is continuous and increasing in* $x\in[0,+\infty)$ *for fixed* $t\in[0,1]$, $y\in[0,+\infty)$ *decreasing in* $y\in[0,+\infty)$ *for fixed* $t\in[0,1]$, $x\in[0,+\infty)$, *and* $f(t,\lambda x,\lambda^{-1}y)\geq\lambda f(t,x,y)$, $\forall t\in[0,1]$, $x,y\in[0,+\infty)$;

(H_8) $g(t,x):[0,1]\times[0,+\infty)\to[0,+\infty)$ *is continuous and increasing in* $x\in[0,+\infty)$ *for fixed* $t\in[0,1]$, *and there exists a constant* $\gamma\in(0,1)$ *such that* $g(t,\lambda x)\geq\lambda^{\gamma}g(t,x)$ *for* $\lambda\in(0,1)$, $t\in[0,1]$, $u\in[0,+\infty)$ *and* $g(t,0)\not\equiv 0$;

(H_9) *there exists a constant* $\delta_0>0$ *such that* $f(t,x,y)\leq\delta_0 g(t,x)$, $t\in[0,1]$, $x,y\geq 0$.

Then BVP (1) *has a unique positive solution* x^* *in* P_h *and for any* $x_0,y_0\in P_h$, *constructing successively the sequences*

$$x_n(t)=\int_0^1 G(t,s)\big[f\big(s,x_{n-1}(s),y_{n-1}(s)\big)+g\big(s,x_{n-1}(s)\big)\big]\,ds,\quad n=1,2,\ldots,$$

$$y_n(t)=\int_0^1 G(t,s)\big[f\big(s,y_{n-1}(s),x_{n-1}(s)\big)+g\big(s,y_{n-1}(s)\big)\big]\,ds,\quad n=1,2,\ldots,$$

we have $x_n(t)\to x^*(t)$ *and* $y_n(t)\to x^*(t)$ *as* $n\to\infty$.

If the nonlocal boundary condition $x(1)=\int_0^1 x(s)\,dA(s)$ replace by local boundary condition $u(1)=0$, we can obtain the following results.

Corollary 3.3 *Assume that* (H_2)-(H_4) *hold. Then the problem*

$$\begin{cases}-D_{0+}^{\alpha}x(t)=f(t,x(t),x(t))+g(t,x(t)), & 0<t<1, n-1<\alpha\leq n, n\geq 3,\\ x^{(k)}(0)=0,\quad 0\leq k\leq n-2, & x(1)=0,\end{cases}$$

has a unique positive solution x^* *in* P_h, *where* $h(t)=(1-t)t^{\alpha-1}$, $t\in[0,1]$. *Moreover, for any initial value* $x_0,y_0\in P_h$, *constructing successively the sequences*

$$x_n(t)=\int_0^1 G(t,s)\big[f\big(s,x_{n-1}(s),y_{n-1}(s)\big)+g\big(s,x_{n-1}(s)\big)\big]\,ds,\quad n=1,2,\ldots,$$

$$y_n(t)=\int_0^1 G(t,s)\big[f\big(s,y_{n-1}(s),x_{n-1}(s)\big)+g\big(s,y_{n-1}(s)\big)\big]\,ds,\quad n=1,2,\ldots,$$

we have $x_n(t)\to x^*(t)$ *and* $y_n(t)\to x^*(t)$ *as* $n\to\infty$.

Corollary 3.4 *Assume that* (H_2) *holds. Then the problem*

$$\begin{cases}-D_{0+}^{\alpha}x(t)=f(t,x(t),x(t)), & 0<t<1, n-1<\alpha\leq n, n\geq 3,\\ x^{(k)}(0)=0,\quad 0\leq k\leq n-2, & x(1)=0,\end{cases}$$

has a unique positive solution x^* *in* P_h, *where* $h(t)=(1-t)t^{\alpha-1}$, $t\in[0,1]$. *Moreover, for any initial value* $x_0,y_0\in P_h$, *constructing successively the sequences*

$$x_n(t)=\int_0^1 G(t,s)f\big(s,x_{n-1}(s),y_{n-1}(s)\big),\quad n=1,2,\ldots,$$

$$y_n(t) = \int_0^1 G(t,s)f\big(s, y_{n-1}(s), x_{n-1}(s)\big)\,ds, \quad n = 1, 2, \ldots,$$

we have $x_n(t) \to x^*(t)$ *and* $y_n(t) \to x^*(t)$ *as* $n \to \infty$.

Corollary 3.5 *Assume that* (H_3), (H_5), *and* (H_6) *hold. Then the problem*

$$\begin{cases} -D_{0+}^{\alpha} x(t) = f(t, x(t)) + g(t, x(t)), & 0 < t < 1, n-1 < \alpha \le n, n \ge 3, \\ x^{(k)}(0) = 0, \quad 0 \le k \le n-2, & x(1) = 0, \end{cases}$$

has a unique positive solution x^* *in* P_h, *where* $h(t) = (1-t)t^{\alpha-1}$, $t \in [0,1]$. *Moreover, for any initial value* $y_0 \in P_h$, *constructing successively the sequences*

$$y_n(t) = \int_0^1 G(t,s)\big[f\big(s, y_{n-1}(s)\big) + g\big(s, y_{n-1}(s)\big)\big]\,ds, \quad n = 1, 2, \ldots,$$

we have $y_n(t) \to x^*(t)$ *as* $n \to \infty$.

Corollary 3.6 *Assume that* (H_5) *holds. Then the problem*

$$\begin{cases} -D_{0+}^{\alpha} x(t) = f(t, x(t)), & 0 < t < 1, n-1 < \alpha \le n, n \ge 3, \\ x^{(k)}(0) = 0, \quad 0 \le k \le n-2, & x(1) = 0, \end{cases}$$

has a unique positive solution x^* *in* P_h, *where* $h(t) = (1-t)t^{\alpha-1}$, $t \in [0,1]$. *Moreover, for any initial value* $y_0 \in P_h$, *constructing successively the sequences*

$$y_n(t) = \int_0^1 G(t,s)\big[f\big(s, y_{n-1}(s)\big) + g\big(s, y_{n-1}(s)\big)\big]\,ds, \quad n = 1, 2, \ldots,$$

we have $y_n(t) \to x^*(t)$ *as* $n \to \infty$.

4 Example

Consider the following boundary value problem:

$$\begin{cases} -D_{0+}^{\frac{5}{2}} x = x^{\frac{1}{2}} + y^{-\frac{1}{2}} + \arctan x + t^2 + t^3 + \frac{\pi}{2}, & 0 < t < 1, \\ x(0) = x'(0) = 0, \quad x(1) = 2x(\frac{1}{2}) - x(\frac{3}{4}). \end{cases} \tag{9}$$

In this case, $\alpha = \frac{5}{2}$. Problem (9) can be regard as a boundary value problem of form (1) with

$$f(t,x) = x^{\frac{1}{2}} + y^{-\frac{1}{2}} + t^2 + \frac{\pi}{2} \tag{10}$$

and

$$g(t,x) = \arctan x + t^3. \tag{11}$$

Now we verify that conditions (H_1)-(H_4) are satisfied. By a simple computation, we have

$$G(t,s) = \begin{cases} \frac{[t(1-s)]^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}, & 0 \le t \le s \le 1, \\ \frac{[t(1-s)]^{\frac{3}{2}} - (t-s)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}, & 0 \le s \le t \le 1, \end{cases}$$

and

$$\mathscr{G}_A(s) = \begin{cases} \frac{2[\frac{1}{2}(1-s)]^{\frac{3}{2}}-(\frac{1}{2}-s)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} - \frac{2[\frac{3}{4}(1-s)]^{\frac{3}{2}}-(\frac{3}{4}-s)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}, & 0 \le s < \frac{1}{2}, \\ \frac{2[\frac{1}{2}(1-s)]^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} - \frac{2[\frac{3}{4}(1-s)]^{\frac{3}{2}}-(\frac{3}{4}-s)^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}, & \frac{1}{2} \le s \le \frac{3}{4}, \\ \frac{2[\frac{1}{2}(1-s)]^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} - \frac{[\frac{3}{4}(1-s)]^{\frac{3}{2}}}{\Gamma(\frac{5}{2})}, & \frac{3}{4} < s \le 1. \end{cases}$$

Then $\Lambda \approx 0.0576$ and $\mathscr{G}_A(s) \ge 0$ for all $s \in [0,1]$. This implies that (H_1) holds. From (10) and (11) we have f and g are continuous and increasing in $x \in [0,\infty)$ for fixed $t \in [0,1]$. Moreover, for any $\lambda \in (0,1)$, $t \in [0,1]$, $x \in (0,\infty)$, we get $\arctan(\lambda x) \ge \lambda \arctan x$. Therefore

$$f(t,\lambda x,\lambda^{-1}y) = \lambda^{\frac{1}{2}}x^{\frac{1}{2}} + \lambda^{\frac{1}{2}}y^{-\frac{1}{2}} + t^2 + \frac{\pi}{2} \ge \lambda^{\frac{1}{2}}\left(x^{\frac{1}{2}} + y^{-\frac{1}{2}} + t^2 + \frac{\pi}{2}\right) = \lambda^{\gamma} f(t,x,y)$$

and

$$g(t,\lambda x) = \arctan(\lambda x) + t^3 \ge \lambda\left(\arctan x + t^3\right) = \lambda g(t,x),$$

where $\gamma = \frac{1}{2}$. Thus (H_2) and (H_3) are proved and $g(t,0) = t^3 \not\equiv 0$. It remains to show that (H_4) holds. Take $\delta_0 \in (0,1]$, and we obtain

$$f(t,x,y) = x^{\frac{1}{2}} + y^{-\frac{1}{2}} + t^2 + \frac{\pi}{2} \ge t^2 + \frac{\pi}{2} \ge t^3 + \arctan x \ge \delta_0\left(t^3 + \arctan x\right) = \delta_0 g(t,x).$$

Therefore, all of the conditions in Theorem 3.1 are satisfied. By using Theorem 3.1, we know that the BVP (9) has a unique positive solution in P_h with $h(t) = t^{\frac{3}{2}}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors typed, read, and approved the final manuscript.

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