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A GENERALIZATION OF NIL-CLEAN RINGS

ALEKSANDRA KOSTIĆ, ZORAN Z. PETROVIĆ, ZORAN S. PUCANOVIĆ,
AND MAJA ROSLAVCEV

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Abstract. The conditions that allow an element of an associative, unital, not necessarily commutative ring R, to be represented as a sum of (commuting) idempotents and one nilpotent element are analyzed. Some applications to group rings are also presented.

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1. INTRODUCTION

An element a in an associative unital ring R is called clean if it can be represented as a sum $a = e + u$, where e is an idempotent element and u is a unit. This notion was introduced by Nicholson in [8]. If one can find such elements e and u such that $a = e + u$ and $eu = ue$, the element a is called strongly clean. The ring R itself is called (strongly) clean if every element in R is (strongly) clean.

Many families of clean rings were investigated in previous decades. In recent years, a particular attention has been paid to the nil-clean rings and its relatives. A nil-clean ring (see [5]) is a ring in which every element is nil-clean, which means that every element can be written as a sum of an idempotent element and a nilpotent one. Analogously, we have a notion of strongly nil-clean elements (and rings). For some of the results concerning this class of rings and some of the related classes of rings, the reader may wish to consult also [1, 3, 7, 10].

A class of strongly 2-nil-clean rings was introduced in [4]. Namely, an element a in a ring R is called strongly 2-nil-clean if it can be represented in the form $a = e + f + n$, where e and f are idempotents, n is a nilpotent element and they all commute with each other.

In this paper we analyze elements of a ring which can be written as a sum of finitely many idempotents and one nilpotent element which are pairwise commutative. If the number of idempotents which appear in this sum is s, we call these elements strongly

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s-nil-clean. It turns out that if every element in a ring is strongly s-nil-clean for some s, this ring has finite characteristic and every element in this ring is strongly $(p-1)$-nil-clean, where p is the largest prime dividing the characteristic of this ring (see Theorem 1). These rings are naturally called strongly $(p-1)$-nil-clean rings and they are all strongly clean (see Corollary 3 and the discussion preceding it). There are many examples of strongly $(p-1)$-nil-clean rings. Theorem 2 shows that in every commutative ring R of finite characteristic k, elements which are strongly s-nil-clean for some s, form a subring which is $(p-1)$-nil clean (where p is the largest prime dividing k) and if this ring contains idempotents or nilpotents not belonging to $\mathbb{Z}_k$ (which is necessarily contained in R) we have a non-trivial example of such a ring. Proposition 5 provides examples of finite commutative local rings which are $(p-1)$-nil-clean.

The plan of the paper is as follows. In Section 2, we analyze sums of idempotents and one nilpotent element and derive our main criteria for strongly $(p-1)$-nil-clean elements in a ring. Section 3 deals with some structure theorems. It is shown that in analyzing strongly $(p-1)$-nil-clean rings, we may reduce this analysis to the case when p is a nilpotent element in a ring under investigation. We also show in this section that strongly $(p-1)$-nil-clean rings are strongly π-regular and, consequently, strongly clean.

Section 4 deals with the investigation of group rings RG where R is a $(p-1)$-nil-clean commutative ring and G is a group. For example, we show that, for the ring RG to be strongly $(p-1)$-nil-clean, when the characteristic of R is of the form p^s, for a prime integer p, it is necessary that the order of any element of a group G is of the form dp^k, for some $d \mid (p-1)$ and $k \geq 0$ (see Lemma 3). In the case of a commutative group G this condition is also sufficient (see Theorem 4).

We emphasize that we work in associative, unital rings which need not be commutative. When a ring is commutative, we drop the adjective "strongly" since it is unnecessary. We use the same symbol k to denote the integer k and to denote the ring element $k1_R$. It will always be clear what we mean. Finally, we denote the field with p elements by $\mathbb{Z}_p$, the Jacobson radical by $J(R)$ and the set of nilpotent elements by $N(R)$.

2. Basic results

A ring in which every element is a sum of certain number of idempotents and one nilpotent element, that commute with each other, is a generalization of strongly nil-clean rings and strongly 2-nil-clean rings. In view of this, we introduce the following definition.

Definition 1. An element a of a ring R is s-nil-clean if it can be written in the following form:

$$a = e_1 + \cdots + e_s + n, \tag{2.1}$$

where elements $e_1,\dots,e_s$ are idempotents and n is nilpotent. If an element a can be written in the form (2.1) so that elements in this sum are pairwise commutative, we say that this element is strongly s-nil-clean. If every element in R is (strongly) s-nil clean, we say that R is a (strongly) s-nil-clean ring.

Of course, if a is (strongly) s-nil-clean and $s<t$, a is also t-nil-clean — we simply add $t-s$ zeroes to the presentation of a as a (strongly) s-nil-clean element.

Remark 1. It is clear that, if $f:R\to S$ is a ring homomorphism and an element $a\in R$ is (strongly) s-nil-clean, then $f(a)\in S$ is (strongly) s-nil-clean. Similarly, an element $a=(a_1,\dots,a_l)\in R_1\times\cdots\times R_l$ is (strongly) s-nil-clean iff a_i is (strongly) s-nil-clean for all i. So, homomorphic images and finite direct products of (strongly) s-nil-clean rings are itself (strongly) s-nil-clean. Also, a subring of a strongly s-nil-clean ring has the same property, as we shall see later. However, this does not hold for s-nil-clean rings. Namely, if we have a ring which is s-nil-clean, but it is not strongly s-nil-clean, it is enough to take an element a which is not strongly s-nil-clean and look at the subring generated by this element. This subring is commutative, so it cannot be s-nil-clean – if it were, this element would also be strongly s-nil-clean.

We begin our analysis with a useful result concerning sum of several idempotents and one nilpotent element.

Proposition 1. *Let R be a ring and suppose that element $a\in R$ is strongly s-nil clean. Then $a(a-1)\cdots(a-s)$ is nilpotent.*

Proof. Let $a=e_1+e_2+\cdots+e_s+n$, where $e_1,e_2,\dots,e_s$ are idempotents and n is nilpotent that commute with each other. Observe that

$$1=((1-e_1)+e_1)((1-e_2)+e_2)\cdots((1-e_s)+e_s).$$

After multiplication, we get a sum of products of the form

$$e_{i_1}\cdots e_{i_k}(1-e_{j_1})\cdots(1-e_{j_{s-k}}),$$

where $1\le k\le s$, $i_1<\cdots<i_k$, $j_1<\cdots<j_{s-k}$ and $\{i_1,\dots,i_k,j_1,\dots,j_{s-k}\}=\{1,\dots,s\}$. Next, we get that

$$\begin{aligned}&(k-a)e_{i_1}\cdots e_{i_k}(1-e_{j_1})\cdots(1-e_{j_{s-k}})\\&=((1-e_{i_1})+\cdots+(1-e_{i_k})-e_{j_1}-\cdots-e_{j_{s-k}}-n)e_{i_1}\cdots e_{i_k}(1-e_{j_1})\cdots(1-e_{j_{s-k}})\\&=-ne_{i_1}\cdots e_{i_k}(1-e_{j_1})\cdots(1-e_{j_{s-k}}).\end{aligned}$$

This follows from the fact that $(1-e)e=e-e^2=0$, for an idempotent e. Since n is nilpotent, so is this product. Thus, when 1 is multiplied by $a(a-1)\cdots(a-s)$, we get a sum of nilpotent elements that commute with each other. Therefore, $a(a-1)\cdots(a-s)$ is nilpotent. □

The following corollary is simple, but important.

Corollary 1. *a) If a ring R is such that -1 is strongly s-nil-clean for some $s \geq 1$, then this ring has finite characteristic.*
b) If $\mathrm{char}(R) = k$*, then -1 is strongly $(p-1)$-nil-clean, where p is the largest prime dividing k.*

Proof. a) From the previous proposition we conclude that $(-1)(-2)\cdots(-(s+1)) = (-1)^{s+1}(s+1)!$ is nilpotent, so $((s+1)!)^m = 0$ for some $m \geq 1$ and the characteristic of the ring R is not 0.
b) It is enough to show that -1 is $(p-1)$-nil-clean in the ring $\mathbb{Z}_k$, which is contained in R. If $k = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$ is the prime factorization of k, where $p_1 < \cdots < p_l = p$, then $\mathbb{Z}_k \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_l^{\alpha_l}}$. Since $-1 \mapsto (-1, \ldots, -1)$ under this isomorphism, this reduces the proof to show that -1 is $(p-1)$-nil-clean in $\mathbb{Z}_{p_i^{\alpha_i}}$. This is clear since $-1 = p_i^{\alpha_i} - 1 = \underbrace{1 + \cdots + 1}_{p_i - 1} + p_i(p_i^{\alpha_i - 1} - 1)$, and $p_i(p_i^{\alpha_i-1} - 1)$ is nilpotent in $\mathbb{Z}_{p_i^{\alpha_i}}$. □

Example 1. The ring $\mathbb{Z}_k$ is (strongly) $(p-1)$-nil-clean, where p is the largest prime integer dividing k. Namely, using the notation from the previous corollary, this reduces to show that $\mathbb{Z}_{p_i{}^{\alpha_i}}$ is $(p-1)$-nil-clean. Since any element in $\mathbb{Z}_{p_i^{\alpha_i}}$ can be written in the form $\underbrace{1 + \cdots + 1}_{s} + p_i t$, for some $s \in \{0, \ldots, p_i - 1\}$ and $t \in \{0, \ldots, p_i^{\alpha_i - 1} - 1\}$, and since p_i is nilpotent in $\mathbb{Z}_{p_i^{\alpha_i}}$, we are done.

Lemma 1. *If $k =$* $\mathrm{char}(R)$ *$= p_1^{\alpha_1} \cdots p_l^{\alpha_l}$ is the prime factorization of the characteristic of the ring R, then $R \cong R_1 \times \cdots \times R_l$, where $R_i = R/p_i^{\alpha_i} R$. In particular,* $\mathrm{char}(R_i) = p_i^{\alpha_i}$.

Proof. This follows easily from the Chinese remainder theorem taking into account the fact that elements $p_i^{\alpha_i}$ are central. □

Since the products of the form $a(a-1)\cdots(a-s)$ are important for our investigation, we introduce the symbol $(a)_k := a(a-1)\cdots(a-(k-1))$ (falling factorial, as is known in combinatorics), where k is a positive integer.
We have the following corollary.

Corollary 2. *Let R be a ring. Suppose that element $a \in R$ is strongly s-nil clean and $k (< s)$ is nilpotent. Then $(a)_k$ is a nilpotent element.*

Proof. Clearly

$$(a)_s = a(a-1)\cdots(a-(s-1)) = a^{t_0}(a-1)^{t_1}\cdots(a-(k-1))^{t_{k-1}} + kq(a),$$

for some non-negative integers t_i, such that $\sum_{i=0}^{k-1} t_i = s$ and polynomial $q(X) \in \mathbb{Z}[X]$. Taking into account that k is nilpotent, the result follows. □

In order to see what the fact that $(a)_s$ is nilpotent implies, we start with a simple lemma.

Lemma 2. *Let p be a prime integer and $r,m \geq 1$ arbitrary positive integers. Then the element x in the ring $\mathbb{Z}_{p^r}[X]/\langle X^m(X-1)^m \cdots (X-(p-1))^m \rangle$ is (strongly) $(p-1)$-nil-clean, where x is the class of X in this quotient ring.*

Proof. Ideals $\langle X-i \rangle$ and $\langle X-j \rangle$ are coprime in $\mathbb{Z}_{p^r}[X]$, for all $0 \leq i < j < p$, since $(X-i)-(X-j) = j-i$ and $j-i$ is invertible in the ring $\mathbb{Z}_{p^r}[X]$. So, $\langle (X-i)^m \rangle, \langle (X-j)^m \rangle$ are coprime as well. By applying the Chinese remainder theorem we obtain the isomorphism

$$\mathbb{Z}_{p^r}[X]/\langle X^m \cdots (X-(p-1))^m \rangle \cong \mathbb{Z}_{p^r}[X]/\langle X^m \rangle \times \cdots \times \mathbb{Z}_{p^r}[X]/\langle (X-(p-1))^m \rangle,$$

such that $x \mapsto (x,\ldots,x)$. Thus, it is enough to show that x has the desired presentation in all factors and since $x = \underbrace{1+\cdots+1}_{i} + (x-i)$ in the factor $\mathbb{Z}_{p^r}[X]/\langle (X-i)^m \rangle$, this is true. □

Proposition 2. *Let R be a ring. Suppose that the element p is nilpotent, where p is a prime integer, and let $a \in R$ be such that $(a)_p$ is nilpotent. Then a is strongly $(p-1)$-nil-clean.*

Proof. Consider the homomorphism $f : \mathbb{Z}[X] \to R$, given by $f(X) = a$. An immediate consequence of the fact that $(a(a-1)\cdots(a-(p-1)))^m = 0$ and that $p^r = 0$ in R, for some $m,r \geq 1$, is the existence of an induced homomorphism

$$\bar{f} : \mathbb{Z}_{p^r}[X]/\langle X^m(X-1)^m \cdots (X-(p-1))^m \rangle \to R, \text{ such that } x \mapsto a.$$

Since x is strongly $(p-1)$-nil-clean, so is its image a. □

Proposition 3. *Let R be a ring of characteristic $k(>0)$. If p is the largest prime dividing k and $a \in R$ is such that $(a)_s$ is nilpotent for some $s \geq 1$, then a is strongly $(p-1)$-nil-clean.*

Proof. Under isomorphism $R \cong R_1 \times \cdots \times R_l$, where $R_i = R/p_i^{\alpha_i}R$ and $k = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$, implied by Lemma 1, the element a goes to $(a_1,\ldots,a_l)$. Also, a is strongly $(p-1)$-nil-clean iff a_i is such for all i. However, from the fact that $(a)_s$ is nilpotent, it follows that $(a_i)_s \in R_i$ is nilpotent for all i. From this, it easily follows that $(a_i)_{p_i}$ is also nilpotent for all i. Namely, if $s < p_i$, this follows since $(a_i)_{p_i} = (a_i)_s(a_i-s)_{p_i-s}$ and if $s > p_i$, it follows from Corollary 2. Since p_i is nilpotent in R_i, Proposition 2 gives that $a_i \in R_i$ is strongly (p_i-1)-nil-clean for all i. From the fact that $p_i \leq p$ for all i, it follows that these elements a_i are all $(p-1)$-nil-clean and so is a. □

Theorem 1. *Let R be a ring such that every element $a \in R$ is strongly s-nil-clean for some s. Then R has finite characteristic and R is strongly $(p-1)$-nil-clean, where p is the largest prime dividing* $\operatorname{char}(R)$.

Proof. Corollary 1 shows that R has finite characteristic and from Proposition 1 and Proposition 3 it follows that every element is strongly $(p-1)$-nil-clean, where p is the largest prime dividing this characteristic. □

Theorem 1 shows that one needs only to investigate strongly $(p-1)$-nil-clean rings, where p is a prime integer. For example, the class of all strongly 3-nil-clean rings is the same as the class of strongly 2-nil-clean rings, and the class of strongly 9-nil-clean-rings is the same as the class of strongly 6-nil-clean rings. Namely, if a ring is, say, strongly 9-nil-clean, then $(a)_{10}$ is nilpotent for all $a \in R$. So, this is true for $a = 10$. Consequently, $10! = (10)_{10}$ is nilpotent, so $\operatorname{char}(R) \mid (10!)^m$, for some $m \geq 1$. We conclude that the largest prime dividing $\operatorname{char}(R)$ is at most 7 (it may be smaller), so our ring is strongly 6-nil-clean.

Proposition 4. *A subring of a strongly $(p-1)$-nil-clean ring is also strongly $(p-1)$-nil-clean.*

Proof. Let S be a subring of a strongly $(p-1)$-nil-clean ring R and let $a \in S$. Since $a \in R$, a is strongly $(p-1)$-nil-clean, and according to Proposition 1 element $(a)_p$ is nilpotent. R is of finite characteristic, say k, which means that $\operatorname{char}(S) = k$, with p being the largest prime dividing k. When we apply Proposition 3, we get that a is strongly $(p-1)$-nil-clean in S. □

Theorem 2. *Let R be a commutative ring of finite characteristic k and p the largest prime dividing k. Let $S = \{a \in R : a \text{ is } s\text{-nil-clean for some } s\}$. Then S is the largest subring of R which is $(p-1)$-nil-clean.*

Proof. Since $k = 0$ in R, we have: $-1 = \underbrace{1 + \cdots + 1}_{k-1}$, so -1 is $(k-1)$-nil-clean and $-1 \in S$. Also, if $a, b \in S$, then $a = e_1 + \cdots + e_s + n$, $b = f_1 + \cdots + f_t + n'$ and we get $ab = \sum_{i,j} e_i f_j + N$, where N is nilpotent and all $e_i f_j$ are idempotents (R is a commutative ring). So, $ab \in S$. Similarly, $a + b = e_1 + \cdots + e_s + f_1 + \cdots + f_t + n + n' \in S$. Finally, since $a - b = a + (-1)b$ we conclude that $a - b \in S$ as well. So, S is a ring in which every element is s-nil-clean for some s. From Theorem 1, we conclude that S is actually $(p-1)$-nil-clean ring. It is clear that S is the largest such subring. □

Remark 2. From the Theorem 2 it is clear that, in order to show that a commutative ring of finite characteristic k is $(p-1)$-nil-clean, it is enough to check only its ring generators over $\mathbb{Z}_k$. For example, the ring $\mathbb{Z}_{p^r}[X]/\langle X^m (X-1)^m \cdots (X-(p-1))^m \rangle$, appearing in Lemma 2, is $(p-1)$-nil-clean, since it is generated by x and this element is $(p-1)$-nil-clean.

The following proposition provides us with a lot of examples of commutative $(p-1)$-nil-clean rings.

Proposition 5. *Let R be a finite commutative local ring, M its maximal ideal and $R/M \cong \mathbb{Z}_p$. Then R is $(p-1)$-nil-clean.*

Proof. We know that every element in M is nilpotent. If $x \in R$, then $x + M = s + M$, for some $s \in \{0, \dots, p-1\}$. So, $x = \underbrace{1 + \cdots + 1}_{s} + m$, where m is nilpotent. □

Remark 3. In the case of non-commutative rings, the set of all elements which are strongly s-nil-clean for some s do not necessarily form a subring. Such examples will be given in Section 3 and Section 4. However, a simple application of Zorn's lemma shows that there exist maximal subrings which are strongly $(p-1)$-nil-clean.

The following proposition gives another characterization of strongly $(p-1)$-nil-clean elements.

Proposition 6. *Let R be a ring of finite characteristic k, p the largest prime dividing k and $a \in R$. The following conditions are equivalent.*

(1) *$(a)_p$ is nilpotent.*
(2) *a is strongly $(p-1)$-nil-clean.*
(3) *$a = b + n$, where $b \in R$ is such that $(b)_p = 0$, n is nilpotent and $bn = nb$.*

Proof. (1) $\Longrightarrow$ (2). This is contained in Proposition 3.
(2) $\Longrightarrow$ (3). Let $a = e_1 + \cdots + e_{p-1} + n$ be a $(p-1)$-nil-clean decomposition of a. Take $b := e_1 + \cdots + e_{p-1}$. The proof of Proposition 1 shows that $(b)_p = 0$.
(3) $\Longrightarrow$ (1). Assume that $a = b + n$. So, we have

$$(a)_p = (b+n)((b-1)+n)\cdots((b-p+1)+n).$$

Therefore, since n and b commute, $(a)_p = (b)_p + nq(n,b) = nq(n,b)$, for some polynomial $q(X,Y) \in \mathbb{Z}[X,Y]$. Since n is nilpotent, so is $(a)_p$. □

3. Structure theorems

The purpose of this section is to discuss the structure of (strongly) $(p-1)$-nil-clean rings, for prime number p.

The following proposition sums up the discussion from the previous section.

Proposition 7. *Suppose that* $\operatorname{char}(R) = k = p_1^{\alpha_1} \cdots p_l^{\alpha_l}$, *where $p_1 < \cdots < p_l = p$. Then R is strongly $(p-1)$-nil-clean if and only if R_i is strongly $(p_i - 1)$-nil-clean, where $R_i = R/p_i^{\alpha_i} R$ and $1 \le i \le l$.*

This shows that in investigation of strongly $(p-1)$-nil-clean rings, for p prime, we can reduce our analysis to the case when p is nilpotent (equivalently, when the characteristic of the ring is a power of a prime).

Let us recall that a ring R is called strongly π-regular if for every element $a \in R$ there exists $n \ge 1$ and $x \in R$ such that $a^n = a^{n+1}x$.

Theorem 3. *Every strongly $(p-1)$-nil-clean ring is strongly π-regular.*

Proof. It is enough to consider the case when p is a nilpotent element. Then $(p-1)!$ is invertible. Let $a \in R$. Since $(a)_p$ is nilpotent, we have that $((a)_p)^s = 0$ for some s. But

$$0 = ((a)_p)^s = (a(a-1)\cdots(a-(p-1)))^s = a^s((p-1)!)^s + a^{s+1}y,$$

for some $y \in R$. Since $(p-1)!$ is invertible, we get that $a^s = a^{s+1}x$, for some $x \in R$ and we are done. □

It is well known that strongly π-regular ring is strongly clean (see [2, Proposition 2.6], [9, Theorem 1], [5, Corollary 2.4]). Also, Jacobson radical of a strongly π-regular ring is nil and commutative strongly π-regular rings have Krull dimension 0 (see [2]). So, we have the following corollary.

Corollary 3. *Every strongly $(p-1)$-nil-clean ring is strongly clean.*

The following proposition is rather useful.

Proposition 8. *Let R be a ring, $a \in R$ and let p be a nilpotent element, where p is prime. Then $a^p - a$ is nilpotent if and only if $(a)_p$ is nilpotent.*

Proof. It is well-known that $X^p - X = (X)_p$ in $\mathbb{Z}_p[X]$. So, $a^p - a - (a)_p = pr(a)$, for some polynomial $r(X) \in \mathbb{Z}[X]$. From this fact, the proof follows immediately. □

For future reference, we formulate the following corollary which directly follows from Proposition 6 and Proposition 8.

Corollary 4. *Let R be a ring. If p is nilpotent, then R is a strongly $(p-1)$-nil-clean ring if and only if $a^p - a$ is nilpotent for every $a \in R$.*

Let us proceed with some of the special properties of $(p-1)$-nil-clean rings.

Proposition 9. *Let R be a ring and let I be any nil ideal of R. Then R is $(p-1)$-nil-clean if and only if R/I is $(p-1)$-nil-clean.*

Proof. ($\Longrightarrow$) As observed before, this is trivial since R/I is a homomorphic image of R.
($\Longleftarrow$) Suppose that R/I is $(p-1)$-nil-clean. Let $x \in R$. Then $x + I$ is $(p-1)$-nil-clean. Thus, $x + I = (x_1 + I) + (x_2 + I) + \cdots + (x_{p-1} + I) + (y + I)$, where $x_i + I$ are idempotents, $1 \le i \le p-1$, and $y + I$ is nilpotent. It is well known that idempotents lift modulo nil ideals (see, e.g. [6, Theorem 21.28]) so there are idempotents e_i such that $x_i + I = e_i + I$. So, $x - e_1 - e_2 - \cdots - e_{p-1} - y \in I$, i.e., $x = e_1 + e_2 + \cdots + e_{p-1} + y + n$, for some $n \in I$. Element $y + n$ is nilpotent. Indeed, since $y + I$ is nilpotent, $y^k \in I$ for some $k \in \mathbb{N}$. Every element different from y^k in the sum one gets in the expansion of $(y+n)^k$, is in I and we can conclude that $(y+n)^k \in I$, so $y + n$ is nilpotent. Therefore R is $(p-1)$-nil-clean. □

An analogous result holds for the strongly $(p-1)$-nil-clean rings.

Proposition 10. *Let R be a ring and let I be any nil ideal of R. Then R is strongly $(p-1)$-nil-clean if and only if R/I is strongly $(p-1)$-nil-clean.*

Proof. ($\Longrightarrow$) Again, this is trivial since R/I is a homomorphic image of R.
($\Longleftarrow$) Let $a \in R$. Since R/I is strongly $(p-1)$-nil-clean, one has $((a+I)_p)^k = I$, for some $k \in \mathbb{N}$. Consequently, $((a)_p)^k \in I$. As I is nil ideal, $((a)_p)^k \in N(R)$. So, $(a)_p$ is nilpotent. Since $(p)_p$ is also nilpotent, the characteristic k of R is finite. The characteristic l of R/I has the property that p is the largest prime dividing this characteristic, but this also holds for k. Namely, $l \in I$ and therefore l is nilpotent in R. So we have that $k \mid l^s$ for some s, and also $l \mid k$. It follows that the sets of primes dividing k and l are the same. Now the result follows from Proposition 6. □

The following corollary follows directly from the fact that $J(R)$ is nil for a strongly $(p-1)$-nil-clean ring and Proposition 10.

Corollary 5. *A ring R is strongly $(p-1)$-nil-clean if and only if $J(R)$ is nil and $R/J(R)$ is strongly $(p-1)$-nil-clean.*

4. Group rings

Let us recall the notion of a group ring. Let G be a group, written multiplicatively, and let R be a commutative ring. The group ring of G over R, denoted by RG, is a free R-module with generating set G, i.e.:

$$RG = \bigoplus_{g \in G} Rg.$$

So, elements of RG are formal finite sums of the form $\sum_i r_i g_i$, with $r_i \in R$, $g_i \in G$, while the multiplication is implied by multiplication in G. The identity of this ring is $1_R e$, where 1_R is the identity in R and e is the neutral element of G. We denote the identity simply by 1.

Our main interest here is focused on strongly $(p-1)$-nil-clean group rings RG. It is obvious that if RG is strongly $(p-1)$-nil-clean, so is R. Since we assume that the coefficient ring R is commutative, we refrain from using adjective “strongly” when referring to R, we use it only for RG when appropriate. We begin by discussing rings R, such that $\mathrm{char}(R)$ is a power of a prime.

Lemma 3. *Let R be a $(p-1)$-nil-clean ring such that* $\mathrm{char}(R) = p^s$, *for a prime p and some $s \geq 1$ and let G be a group. For the list of conditions:*

(1) *RG is strongly $(p-1)$-nil-clean;*
(2) *For each $g \in G$, the element $g^{p-1} - 1$ is nilpotent;*
(3) *For each $g \in G$, there exists $k \geq 0$ and $d \mid (p-1)$ such that $\omega(g) = dp^k$,*

the following holds: (1) $\Longrightarrow$ (2) *and* (2) $\Longleftrightarrow$ (3). *Here, $\omega(g)$ denotes the order of g in G.*

Proof. (1) $\implies$ (2). Let $g \in G$. Since RG is strongly $(p-1)$-nil-clean, from Corollary 4 it follows that $g^p - g$ is nilpotent. Since g is invertible, we get that $g^{p-1}-1$ is nilpotent.
(3) $\implies$ (2). Let $g \in G$. Then $\omega(g) = dp^k$, where $k \geq 0$ and $d \mid (p-1)$. Let $p-1 = ds$. Since $\omega(g^d) = p^k$ and $\gcd(s, p^k) = 1$, we have that $\omega(g^{p-1}) = \omega((g^d)^s) = p^k$. Therefore,

$$(g^{p-1}-1)^{p^k} = \underbrace{(g^{p-1})^{p^k}-1}_{0} + p\cdot u, \text{ for some } u \in RG.$$

Since p is nilpotent, the element $g^{p-1}-1$ is nilpotent as well.
(2) $\implies$ (3). Let $g \in G$. The order of g cannot be infinite — in that case, it would not be possible for $g^{p-1}-1$ to be nilpotent. Namely, the element $g^{(p-1)s}$ in the sum $(g^{p-1}-1)^s = g^{(p-1)s} + \cdots + (-1)^s$ cannot be cancelled out.

So, let us suppose that $\omega(g) = tp^k$, for some $k \geq 0$ and t such that $p \nmid t$ and $\gcd(t, p-1) = d \neq t$. Also, let $t = dt_1$, $p-1 = dz_1$ and $h = g^{p-1}$. Since $\omega(g^d) = t_1 p^k$ and $\gcd(z_1, t_1 p^k) = 1$, it follows that

$$\omega(h) = \omega(g^{p-1}) = \omega((g^d)^{z_1}) = t_1 p^k.$$

Since $h-1$ is nilpotent, $h^{p^k}-1$ is nilpotent as well. Let $h_1 = h^{p^k}$. Then $\omega(h_1) = t_1$. Let us focus on the polynomial $f(X) = (X-1)^{t_1} - (X^{t_1}-1)$, which is clearly divisible by $X-1$:

$$f(X) = (X-1)((X-1)^{t_1-1} - (X^{t_1-1} + \cdots + X + 1)) = (X-1)f_1.$$

This follows from the fact that $t_1 \neq 1$ (which also implies that $h_1 - 1 \neq 0$). We can see that

$$f(X) = (X-1)(-t_1 + (X-1)q(X)),$$

for some polynomial $q \in \mathbb{Z}[X]$, since $f_1(1) = -t_1$. We can conclude that

$$f(h_1) = (h_1-1)^{t_1} - \underbrace{(h_1^{t_1}-1)}_{0} = (h_1-1)(-t_1 + (h_1-1)q(h_1)).$$

We know that $h_1 - 1$ is nilpotent. As $p \nmid t_1$ and p is nilpotent, element t_1 is invertible in R. So

$$(h_1-1)^{t_1} = u(h_1-1),$$

for an invertible $u \in RG$. So

$$(h_1-1)((h_1-1)^{t_1-1} - u) = 0,$$

and since $(h_1-1)^{t_1-1} - u$ is invertible, we have that $h_1 - 1 = 0$. That is a contradiction.

It is easy to check that the proof of 2 $\iff$ 3 is valid, although shorter, even for $p = 2$. $\square$

In the previous lemma, G was an arbitrary group. If we add commutativity, we actually get equivalence (1) $\iff$ (2).

Theorem 4. *Let R be a $(p-1)$-nil-clean ring such that* $\operatorname{char}(R) = p^s$, *for a prime p and some $s \geq 1$ and let G be an Abelian group. Then RG is $(p-1)$-nil-clean iff $g^{p-1}-1$ is nilpotent for every $g \in G$.*

Proof. We only need to prove the "if" part. It follows directly from the Remark following Theorem 2 since elements of the group G form a generating set for RG over R. □

Example 2. The previous theorem does not hold for non-commutative groups. Let us take $R = \mathbb{Z}_5$ and $G = \mathbb{D}_4$, the dihedral group of order 8 generated by elements s and r such that $s^2 = 1 = r^4$, $sr = r^3 s$. In this group, $g^4 = 1$ for all $g \in G$, so the condition that $g^4 - 1$ is nilpotent is trivially satisfied. However, direct computation shows that

$$\begin{aligned}(s+sr)_5 &= 2s + 2r + 3sr^2 + 3r^3\\ ((s+sr)_5)^8 &= 3 + 2r^2\\ (3+2r^2)^2 &= 3 + 2r^2,\end{aligned}$$

so $(s+sr)_5$ is not nilpotent and the group ring $\mathbb{Z}_5\mathbb{D}_4$ is not strongly 4-nil-clean.

Let us concentrate now on the general case.

Proposition 11. *Let R be a $(p-1)$-nil-clean ring and* $\operatorname{char}(R) = p_1^{\alpha_1}\cdots p_l^{\alpha_l}$, *so that $l > 1$, $p_1 < \cdots < p_l = p$.*

(1) *If G is an elementary Abelian 2-group, then RG is strongly $(p-1)$-nil-clean.*
(2) *If G is an elementary Abelian group in which every element has order q and $q \mid \gcd(p_1 - 1, \ldots, p_l - 1)$, then RG is strongly $(p-1)$-nil-clean.*

Proof. (1) As we know, $R \cong R_1 \times \cdots \times R_l$, where p_i is nilpotent in R_i, and consequently $RG \cong R_1G \times \cdots \times R_lG$. So, all the rings R_i are strongly $(p-1)$-nil-clean, and since p_i is nilpotent in R_i, R_i is actually strongly $(p_i - 1)$-nil-clean. We will use Theorem 4. Since $2 \mid (p_i - 1)$ for all $i \geq 2$ and $g^2 = 1$ for all $g \in G$, we have that $g^{p_i - 1} - 1 = 0$ for all $g \in G$. If $p_1 > 2$, the same holds for p_1. If $p_1 = 2$, then $(g-1)^2 = g^2 - 2g + 1 = 2(1-g)$. Since 2 is nilpotent in R_1G, $g-1$ is also nilpotent in R_1G. In this case also, from G being Abelian, we can conclude that R_iG is $(p_i - 1)$-nil-clean. Therefore, RG is $(p-1)$-nil-clean.
(2) Similarly, from $g^q = 1$, and $q \mid \gcd(p_1 - 1, \ldots, p_l - 1)$, we conclude that $g^{p_i - 1} - 1 = 0$ for all $g \in G$, and proceed as in (1). □

Theorem 5. *Let* $\operatorname{char}(R) = p_1^{\alpha_1}\cdots p_l^{\alpha_l}$, *where $l > 1$, $p_1 < \cdots < p_l = p$ and let G be a group. Suppose that RG is strongly $(p-1)$-nil-clean ring.*

(1) *For all $g \in G$ the following holds: $\omega(g) \mid \gcd(p_2 - 1, \ldots, p_l - 1)$ and $\omega(g) = d_1 p_1^s$, such that $d_1 \mid (p_1 - 1)$ and $s \geq 0$.*

(2) *If there exists $h \in G$ such that $\omega(h) \geq p_1$, then $p_1 \mid (p_i - 1)$, for $2 \leq i \leq l$.*
(3) *If for all $g \in G$, $\omega(g) < p_1$, then $\omega(g) \mid \gcd(p_1 - 1, \dots, p_l - 1)$.*
(4) *If $p_1 = 2$, then every element in G has order 2^s for some $s \geq 0$.*
(5) *If $p_1 = 2$ and $p_i \equiv 3 \pmod 4$ for at least one $i \geq 2$, then G is an elementary Abelian 2-group.*

Proof. (1) As before, $RG \cong R_1 G \times \cdots \times R_l G$, where p_i is nilpotent in $R_i G$. Lemma 3 shows that $g^{p_i - 1} - 1$ is nilpotent in $R_i G$. It follows that $\omega(g) = d_i p_i^{k_i}$ for some $k_i \geq 0$ and $d_i \mid (p_i - 1)$. So,

$$d_1 p_1^{k_1} = d_2 p_2^{k_2} = \cdots = d_l p_l^{k_l}.$$

Since $d_1 < p_1$, it is clear that $d_1 p_1^{k_1}$ cannot be divisible by any prime greater than p_1. So $k_i = 0$ for $i \geq 2$. Therefore,

$$\omega(g) = d_1 p_1^{k_1} = d_2 = \cdots = d_l, \tag{4.1}$$

for all $g \in G$, where $d_i \mid (p_i - 1)$. Hence, $\omega(g) \mid (p_2 - 1), \dots, \omega(g) \mid (p_l - 1)$, and we are done.
(2) If $h \in G$ is such that $\omega(h) \geq p_1$, we have $\omega(h) = d_1 p_1^{k_1} = d_2 = \cdots = d_l$, where $k_1 \geq 1$. Since $d_i \mid (p_i - 1)$, it follows that $p_1 \mid (p_i - 1)$.
(3) Under this assumption, we get that $k_1 = 0$ in 4.1, hence for all $g \in G$

$$\omega(g) = d_1 = d_2 = \cdots = d_l.$$

We conclude that $\omega(g) \mid \gcd(p_1 - 1, \dots, p_l - 1)$.
(4) The fourth assertion follows easily. Namely, in this case $p_1 = 2$, so $\omega(g) = d_1 2^{k_1}$, where $d_1 \mid (2 - 1)$. So, $\omega(g) = 2^{k_1}$.
(5) It is enough to show that there are no elements of order 4 in G. If it were, then for an element $g \in G$, we would have equalities

$$4 = d_2 = \cdots = d_l,$$

where $d_i \mid (p_i - 1)$, for $2 \leq i \leq l$. This would imply that $4 \mid (p_i - 1)$, that is $p_i \equiv 1 \pmod 4$, for $2 \leq i \leq l$, which is a contradiction. Hence, we can conclude that G is an elementary Abelian 2-group. □

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Authors’ addresses

Aleksandra Kostić
University of Belgrade, Faculty of Mathematics, Studentski trg 16, Belgrade, Serbia
E-mail address: `alex@matf.bg.ac.rs`

Zoran Z. Petrović
University of Belgrade, Faculty of Mathematics, Studentski trg 16, Belgrade, Serbia
E-mail address: `zoranp@matf.bg.ac.rs`

Zoran S. Pucanović
University of Belgrade, Faculty of Civil Engineering, Bulevar kralja Aleksandra 73, Belgrade, Serbia
E-mail address: `pucanovic@grf.bg.ac.rs`

Maja Roslavcev
University of Belgrade, Faculty of Mathematics, Studentski trg 16, Belgrade, Serbia
E-mail address: `roslavcev@matf.bg.ac.rs`