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REGULARITY AND ENTROPY SOLUTIONS OF SOME ELLIPTIC EQUATIONS

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Abstract. Here, we would study regularity of solutions and existence at least one entropy solution for L^1 -data and duality solution. This result would improve some results of Laplacian differential equations.

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1. INTRODUCTION

First, we summarize some result notions of Schwartz space and tempered distributions. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is a topological vector space of all $f : \mathbb{R}^n \rightarrow \mathbb{C}$ such that $f \in C^\infty(\mathbb{R}^n)$ and $x^\alpha \partial^\beta f(x)$ is bounded. For every pair of multi-indices $\alpha, \beta \in \mathbb{N}^n$, we set

$$\|f\|_{\alpha, \beta} := \sup_x |x^\alpha \partial^\beta f|$$

which induces a family of semi-norms on $\mathcal{S}(\mathbb{R}^n)$. A tempered distribution is a continuous linear functional $T : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathbb{C}$ and $\mathcal{S}'(\mathbb{R}^n)$ is the space of all tempered distributions. $\mathcal{D}(\mathbb{R}^n)$ is the space $C_c^\infty(\mathbb{R}^n)$ endowed with the topology in which $f_n \rightarrow 0$ means that, there is a compact set K ; such that $\text{Supp} f_n \subseteq K$ ($n = 1, 2, \dots$) and for each $\alpha \in \mathbb{N}^n$, $D^\alpha f_n \rightarrow 0$ uniformly.

The Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ is the function $\widehat{f} : \mathbb{R}^n \rightarrow \mathbb{C}$ defined by

$$\widehat{f}(k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) e^{-ikx} dx.$$

It is well known that

- 1) $\widehat{\cdot} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ is continuous one to one.
- 2) $\widehat{\partial^\alpha f}(k) = (ik)^\alpha \widehat{f}(k)$.
- 3) $\widehat{(-ix)^\beta f}(k) = \partial^\beta \widehat{f}(k)$.

For regularity of Laplacian, Ma and Thompson [8], Ma [7] proved regularity, where

$f \in C[0, 1]$. Moreover, Lee and Sim [6] proved it but for $f \in L^1(0, 1)$. Recently, interior regularity have studied by many mathematician:

M. Cozzi [4] studied regularity theory of weak solutions for the second order linear elliptic differential equations $-div(A(\cdot)\nabla u) = f$ in Ω , where Ω is an open bounded subset of \mathbb{R}^n and $A = [a_{ij}]$ is $n \times n$ matrix uniformly elliptic, $a_{ij} \in C_{loc}^{0,1}(\Omega)$ and $f \in L^2(\Omega)$. In fact, it is proved that for any $\Omega' \subset\subset \Omega$, $\|u\|_{H^2(\Omega')} \leq C(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)})$. Moreover, it was shown the interior $H^{2s-\epsilon}$ regularity for weak solutions of some linear elliptic differential equations.

J. Siljander, J. M. Urbano [11] studied the Serrin-type interior regularity result.

$$u \in L_{loc}^{2+\epsilon}(\Omega_T) \Rightarrow \text{regularity}$$

for a weak solution in the energy space $L_t^\infty L_x^2$ satisfying in appropriate vorticity estimates for

$$\partial_t u + (u \cdot \nabla)u + \nabla p = 0 \text{ and } \operatorname{div} u = 0.$$

S. Gustafson and co authors [5] gave an interior regularity criteria for suitable weak solutions of the 3D Navier-Stokes equations. In fact they considered the regularity problem for a suitable weak solution $(u, p) : \Omega \times I \rightarrow \mathbb{R}^3 \times \mathbb{R}$ of three-dimensional incompressible Navier-Stokes equations (NS)

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla)u + \nabla p = f & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \times I \end{cases}$$

and proved $u \in L^\infty(Q_{z,r})$ for some $B_{x,r} \times (t-r^2, t) = Q_{z,r} \subseteq \Omega \times I$, $r > 0$.

These brand of problems have potential applications to the modeling of combustion, thermal explosions, nonlinear heat generation, gravitational equilibrium of polytropic stars, glaciology, non-Newtonian fluids, and the flow through porous media.

In this paper, we would study the regular property of

$$-\Delta_\alpha u + \lambda u = f, \tag{1.1}$$

in $\mathcal{D}'(\Omega)$, where we define $\Delta_\alpha := \alpha_1 \frac{\partial^2}{\partial x_1^2} + \dots + \alpha_n \frac{\partial^2}{\partial x_n^2}$ and $\alpha = (\alpha_1, \dots, \alpha_n)$, $(\alpha_i > 0, \forall i)$, our aim is to show $u \in C^\infty(\mathbb{R}^n)$.

2. RESULTS

Here, we use of \approx for equivalent norms.

Lemma 1. *Suppose that $m \in \mathbb{Z}$, $\lambda > 0$ and let $u, f \in \mathcal{S}'(\mathbb{R}^n)$ satisfy (1.1) as distributions. If $f \in W^{m,2}(\mathbb{R}^n)$, then $u \in W^{m+2,2}(\mathbb{R}^n)$ and there exists a constant C such that $\|u\|_{W^{m+2,2}} \leq C \|f\|_{W^{m,2}}$.*

Proof. Taking the Fourier transform of (1.1), we have

$$\mathcal{F}(-\alpha_1 u_{x_1 x_1} - \dots - \alpha_n u_{x_n x_n} + \lambda u) = \mathcal{F}(f),$$

$$(\alpha_1 \xi_1^2 + \dots + \alpha_n \xi_n^2 + \lambda) \mathcal{F}(u) = \mathcal{F}(f).$$

From theorem 5.2.3 of [3] for any $m \in \mathbb{Z}$ and $a, b > 0$ we note to the following equivalent norms:

$$\begin{aligned} W^{m,2}(\mathbb{R}^n) &:= \{u \in \mathcal{S}'(\mathbb{R}^n); \mathcal{F}^{-1}[(a + b|\xi|^2)^{\frac{m}{2}} \mathcal{F}(u)] \in L^2(\mathbb{R}^n)\} \\ \|u\|_{W^{m,2}} &\approx \|\mathcal{F}^{-1}[(a + b|\xi|^2)^{\frac{m}{2}} \mathcal{F}(u)]\|_{L^2}, \quad u \in W^{m,2}(\mathbb{R}^n). \end{aligned}$$

By using Parseval theorem

$$\|\mathcal{F}^{-1}[(a + b|\xi|^2)^{\frac{m}{2}} \mathcal{F}(u)]\|_{L^2} = \|(a + b|\xi|^2)^{\frac{m}{2}} \mathcal{F}(u)\|_{L^2}.$$

Thus,

$$\|(a + b|\xi|^2)^{\frac{m}{2}} \mathcal{F}(u)\|_{L^2} \approx \|(\alpha_1 \xi_1^2 + \dots + \alpha_n \xi_n^2 + \lambda)^{\frac{m}{2}} \mathcal{F}(u)\|_{L^2} \quad (2.1)$$

and

$$(\alpha_1 \xi_1^2 + \dots + \alpha_n \xi_n^2 + \lambda)^{\frac{m+2}{2}} \mathcal{F}(u) = (\alpha_1 \xi_1^2 + \dots + \alpha_n \xi_n^2 + \lambda)^{\frac{m}{2}} \mathcal{F}(f)$$

so the result follows from (2.1). \square

We now consider the case of a general domain Ω

Theorem 1. Suppose that $\lambda \in \mathbb{R}$ and $u, f \in \mathcal{D}'(\Omega)$ satisfy the equation (1.1) in $\mathcal{D}'(\Omega)$.

(i) If $f \in W_{loc}^{m,2}(\Omega)$ and $u \in W_{loc}^{n,2}(\Omega)$ for some $m \geq 0$ and $n \in \mathbb{Z}$, then $u \in W_{loc}^{m+2,2}(\Omega)$ and for every $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega$, there exists a constant C (depending only on m, Ω_2 and Ω_1) such that $\|u\|_{W^{m+2,2}(\Omega_2)} \leq C(\|f\|_{W^{m,2}(\Omega_1)} + \|u\|_{W^{n,2}(\Omega_1)})$.

(ii) If $f \in C^\infty(\Omega)$ and $u \in W_{loc}^{n,2}(\Omega)$ for some $n \in \mathbb{Z}$ then $u \in C^\infty(\Omega)$.

Proof. We proceed in two steps.

Step 1: Consider $M'' \subset\subset M' \subset\subset \Omega$ and $k \in \mathbb{Z}$. If $u \in W^{k,2}(M')$ and $f \in W^{k-1,2}(M')$ solve the equation (1.1) in $\mathcal{D}'(\Omega)$, thus, $u \in W^{k+1,2}(M'')$ and there exists C such that $\|u\|_{W^{k+1,2}(M'')} \leq C(\|f\|_{W^{k-1,2}(M')} + \|u\|_{W^{k,2}(M')})$. To show this, consider $\rho \in C_c^\infty(\mathbb{R}^n)$ such that $\rho \equiv 1$ on M'' and $\text{supp } \rho \subset M'$ and define $v \in \mathcal{D}'(\mathbb{R}^n)$ by $v = \rho u$, i.e.

$$(v, \varphi)_{\mathcal{D}'(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)} = (u, \rho \varphi)_{\mathcal{D}'(M'), \mathcal{D}(M')}.$$

Clearly $v \in W^{k,2}(\mathbb{R}^n)$ and $\|v\|_{W^{k,2}(\mathbb{R}^n)} \leq C \|u\|_{W^{k,2}(M')}$.

v solves the equation

$$-\Delta_\alpha v + v = T_1 + T_2 + T_3 \quad (2.2)$$

in $\mathcal{D}'(\mathbb{R}^n)$, where the distributions T_1, T_2 , and T_3 are defined by

$$(T_1, \varphi)_{\mathcal{D}'(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)} = (f + (1 - \lambda)u, \rho \varphi)_{\mathcal{D}'(M'), \mathcal{D}(M')},$$

$$(T_2, \varphi)_{\mathcal{D}'(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)} = -(u, \Delta_\alpha \rho \cdot \varphi)_{\mathcal{D}'(M'), \mathcal{D}(M')},$$

$$(T_3, \varphi)_{\mathcal{D}'(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)} = -2(u, (\alpha \cdot \nabla \rho) \varphi)_{\mathcal{D}'(M'), \mathcal{D}(M')},$$

for every $\varphi \in C_c^\infty(\mathbb{R}^n)$, since

$$\begin{aligned} & (-\Delta_\alpha v + v, \varphi)_{\mathcal{D}'(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)} = (-\alpha_1 v_{x_1 x_1} - \dots - \alpha_n v_{x_n x_n} + v, \varphi)_{\mathcal{D}'(\mathbb{R}^n), \mathcal{D}(\mathbb{R}^n)} \\ & = -(\alpha_1 [(u, \rho_{x_1 x_1} \varphi)_{\mathcal{D}'(M'), \mathcal{D}(M')} + 2(u, \rho_{x_1} \varphi)_{\mathcal{D}'(M'), \mathcal{D}(M')} + (u_{x_1 x_1}, \rho \varphi)_{\mathcal{D}'(M'), \mathcal{D}(M')}] \\ & + \dots + \alpha_n [(u, \rho_{x_n x_n} \varphi)_{\mathcal{D}'(M'), \mathcal{D}(M')} + 2(u, \rho_{x_n} \varphi)_{\mathcal{D}'(M'), \mathcal{D}(M')} + (u_{x_n x_n}, \rho \varphi)_{\mathcal{D}'(M'), \mathcal{D}(M')}] \\ & + (u, \rho \varphi)_{\mathcal{D}'(M'), \mathcal{D}(M')}) = (-\alpha_1 u_{x_1 x_1} - \dots - \alpha_n u_{x_n x_n} + \lambda u, \rho \varphi)_{\mathcal{D}'(M'), \mathcal{D}(M')} \\ & - (u, (\alpha_1 \rho_{x_1 x_1} + \dots + \alpha_n \rho_{x_n x_n}) \varphi)_{\mathcal{D}'(M'), \mathcal{D}(M')} - 2(u, (\alpha_1 \rho_{x_1} + \dots + \alpha_n \rho_{x_n}) \varphi)_{\mathcal{D}'(M'), \mathcal{D}(M')}. \end{aligned}$$

Thus $T_j \in W^{k-1,2}(\mathbb{R}^n)$ and

$$\|T_j\|_{W^{k-1,2}(\mathbb{R}^n)} \leq C(\|f\|_{W^{k-1,2}(M')} + \|u\|_{W^{k,2}(M')}),$$

for $j = 1, 2, 3$. Applying (2.1) and lemma 1, we deduce $v \in W^{k+1,2}(\mathbb{R}^n)$ and $\|v\|_{W^{k+1,2}(\mathbb{R}^n)} \leq C(\|f\|_{W^{k-1,2}(M')} + \|u\|_{W^{k,2}(M')})$.

Step 2: (Conclusion) Without loss of generality, we may assume $n = -\ell \leq 0$. Let $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega$. Consider a family $(M_j)_{0 \leq j \leq m+\ell+1}$ of open subsets of Ω , such that

$$\Omega_2 = M_{m+\ell+1} \subset\subset \dots \subset\subset M_0 \subset\subset \Omega_1$$

(one constructs easily such a family). It follows from Step 1 that $u \in W^{-\ell+1,2}(M_0)$ and can

$$\begin{aligned} \|u\|_{W^{-\ell+1,2}(M_0)} & \leq C(\|f\|_{W^{-\ell-1,2}(\Omega_1)} + \|u\|_{W^{-\ell,2}(\Omega_1)}) \\ & \leq C(\|f\|_{W^{m,2}(\Omega_1)} + \|u\|_{W^{n,2}(\Omega_1)}). \end{aligned} \quad (2.3)$$

(2.3) and lemma 1 imply that $u \in W^{-\ell+2,2}(M_1)$ and

$$\begin{aligned} \|u\|_{W^{-\ell+2,2}(M_1)} & \leq C(\|f\|_{W^{-\ell,2}(M_0)} + \|u\|_{W^{-\ell+1,2}(M_0)}) \\ & \leq C(\|f\|_{W^{m,2}(\Omega_1)} + \|u\|_{W^{n,2}(\Omega_1)}). \end{aligned} \quad (2.4)$$

Iterating the above argument, $u \in W^{m+2,2}(M_{m+\ell+1}) = W^{m+2,2}(\Omega_2)$ and that there exists C in which

$$\|u\|_{W^{m+2,2}(\Omega_2)} \leq C(\|f\|_{W^{m,2}(\Omega_1)} + \|u\|_{W^{n,2}(\Omega_1)}).$$

Hence, property (i) satisfies since Ω_1 and Ω_2 are arbitrary. Property (ii) follows from Property (i) and $C^\infty(\Omega) = \bigcap_{m \geq 0} W_{loc}^{m,2}(\Omega)$. \square

Before paying to entropy solutions, we remember some notions:

Let f, g and q be functions in $L^\infty(\Omega)$, u and v be the solutions of

$$\begin{cases} -div(A(x)\nabla u) + q(x)u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (2.5)$$

and

$$\begin{cases} -div(A^*(x)\nabla v) + q(x)v = g & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \quad (2.6)$$

respectively, where A^* is the transposed matrix of A and

$$A(x)y \cdot y \geq \alpha |y|^2, |A(x)| \leq \beta \tag{2.7}$$

for every $y \in \mathbb{R}^n, 0 < \alpha \leq \beta$. It is well known that (2.5) has a weak unique solution (Theorem 1.6.1 [1]). Since both u and v belong to $H_0^1(\Omega)$, u can be chosen as test function in the formulation of weak solution for v and vice versa. One obtains

$$\int f v = \int A(x) \nabla u \cdot \nabla v + \int q(x) u \cdot v = \int A^*(x) \nabla v \cdot \nabla u + \int q(x) v \cdot u = \int u g$$

for every $f, g \in L^\infty(\Omega)$, where u and v solve the corresponding problems with data f and g respectively. $u, v \in L^\infty(\Omega)$ (Theorem 2.3 [9]), but we remark that the two integrals are well-defined also if $f \in L^1(\Omega)$ and $u \in L^1(\Omega)$ (always maintaining the assumption that g and so v is a bounded function). This fact inspired to Guido Stampacchia the following definition of solution for (2.5) if the datum is in $L^1(\Omega)$.

Definition 1. Suppose that $f \in L^1(\Omega)$. A function $u \in L^1(\Omega)$ is called a *duality solution* with datum f if one has $\int u g = \int f v$, for every $g \in L^\infty(\Omega)$, where v is the solution of

$$\begin{cases} -div(A^*(x)\nabla v) + q(x)v = g & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

Theorem 2 (Stampacchia, theorem 3.3 of [9]). *For $f \in L^1(\Omega)$ there exists a unique duality solution with datum f . Furthermore, $u \in L^q(\Omega)$ for every $q < \frac{N}{N-2}$.*

Remark 1. In special case if

$$A = \begin{bmatrix} \alpha_1 & 0 & 0 & \dots & \dots & 0 \\ 0 & \alpha_2 & 0 & \dots & \dots & 0 \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ 0 & 0 & \dots & \dots & 0 & \alpha_n \end{bmatrix}$$

and $q(x) = \lambda$, problems (2.5) and (2.6) change to

$$\begin{cases} -\Delta_\alpha u + \lambda u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta_\alpha v + \lambda v = g & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

Respectively.

Definition 2. For $k > 0$, set

$$T_k(s) := \max\{-k, \min\{s, k\}\}$$

and

$$\tau_0^{1,2} = \{u : \Omega \rightarrow \mathbb{R} \text{ measurable} : T_k(u) \in H_0^1(\Omega), \forall k > 0\},$$

It is well known that $\nabla T_k(u) = \nabla u \chi_{\{|u| \leq k\}}$.

Lemma 2. Suppose that $u \in \tau_0^{1,2}(\Omega)$, $q \in L^\infty(\Omega)$ and (2.7) valid. Then there is $\alpha > 0$ in which

$$\alpha \int_{\Omega} |\nabla T_k(u)|^2 \leq \int_{\Omega} A(x) \nabla u \cdot \nabla T_k(u) + \int_{\Omega} q(x) u T_k(u).$$

Proof. By (2.7)

$$\begin{aligned} \int_{\Omega} A(x) \nabla u \cdot \nabla T_k(u) + \int_{\Omega} q(x) u T_k(u) &\geq \gamma \int_{\Omega} |\nabla u|^2 + \int_{\Omega} q(x) u T_k(u) \\ &\geq \gamma \int_{\Omega} |\nabla u|^2 - \int_{\Omega} |q(x)| |u| |T_k(u)|, \end{aligned}$$

since $|T_k(u)| \leq |u|$, $|q(x)| \leq b$ (almost every where) for a suitable $b > 0$ and from Poincaré inequality

$$\begin{aligned} \gamma \int_{\Omega} |\nabla u|^2 - \int_{\Omega} |q(x)| |u| |T_k(u)| &\geq \gamma \int_{\Omega} |\nabla u|^2 - \int_{\Omega} |q(x)| |u|^2 \\ &\geq \gamma \int_{\Omega} |\nabla u|^2 - b C_q \int_{\Omega} |\nabla u|^2 = \alpha \int_{\Omega} |\nabla T_k(u)|^2, \end{aligned}$$

for $\alpha := \gamma - b C_q$, where C_q is multiplier in Poincaré inequality. \square

Definition 3. Suppose that $f \in L^1(\Omega)$. A function $u \in \tau_0^{1,2}(\Omega)$ is called an entropy solution of (2.5) if

$$\int_{\Omega} A(x) \nabla u \cdot \nabla T_k(u - \varphi) + \int_{\Omega} q(x) u T_k(u - \varphi) \leq \int_{\Omega} f T_k(u - \varphi), \quad (2.8)$$

for every $k > 0$ and for every φ in $H_0^1(\Omega) \cap L^\infty(\Omega)$.

Theorem 3. Suppose that $f \in L^1(\Omega)$. Then there exists an entropy solution u for (2.5).

Proof. We do by approximation; Suppose that $f_n = T_n(f)$ and by the Lax-Miligram theorem, there exists a weak solution u_n for

$$\begin{cases} -\operatorname{div}(A(x) \nabla u_n) + q(x) u_n = f_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $k > 0$. Taking $T_k(u_n)$ as test function and using of lemma 2,

$$\begin{aligned} \alpha \int_{\Omega} |\nabla T_k(u_n)|^2 &\leq \int_{\Omega} A(x) \nabla u_n \cdot \nabla T_k(u_n) + \int_{\Omega} q(x) u_n T_k(u_n) \\ &= \int_{\Omega} f_n T_k(u_n) \leq k \|f\|_{L^1(\Omega)}. \end{aligned}$$

Therefore, $(T_k(u_n))_n$ is bounded in $H_0^1(\Omega)$ for a fixed k . This implies that there exists a function $v_k \in H_0^1(\Omega)$ such that, up to subsequences $T_k(u_n)$ converges to v_k

weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. From lemma 2, one can deduce that

$$\int_{\Omega} |\nabla(u_n - u_m)|^q \leq C_q \|f_n - f_m\|_{L^1(\Omega)}^q$$

and since $(f_n)_n$ is a Cauchy sequence in $L^1(\Omega)$, so $(u_n)_n$ is a Cauchy sequence in $W_0^{1,q}(\Omega)$ and then u_n converges strongly to a suitable $u \in W_0^{1,q}(\Omega)$. For every $q < \frac{N}{N-1}$, ∇u_n converges to ∇u almost everywhere in Ω . Thus, $T_k(u_n)$ converges strongly to $T_k(u)$ in $L^2(\Omega)$, and so $v_k = T_k(u)$. Therefore, by Fatou lemma,

$$\alpha \int_{\Omega} |\nabla T_k(u_n)|^2 \leq \liminf_{n \rightarrow +\infty} \alpha \int_{\Omega} |\nabla T_k(u_n)|^2 \leq k \|f\|_{L^1(\Omega)},$$

which implies that u belongs to $\tau_0^{1,2}$. Fix $k > 0$, φ in $H_0^1(\Omega) \cap L^\infty(\Omega)$, and $v := T_k(u_n - \varphi)$ as test function in the weak formulation of (2.5). Then

$$\int_{\Omega} A(x) \nabla u_n \cdot \nabla T_k(u_n - \varphi) + \int_{\Omega} q(x) u_n T_k(u_n - \varphi) = \int_{\Omega} f_n T_k(u_n - \varphi).$$

For the right hand side we have $T_n \rightarrow I$ as $n \rightarrow \infty$ and $f_n = T_n(f) \rightarrow f$. Thus, $f_n \rightarrow f$ point wise in $L^1(\Omega)$ and $|f_n T_k(u_n - \varphi)| \leq 2k|f|$. Lebesgue theorem implies that

$$\lim_{n \rightarrow +\infty} \int_{\Omega} f_n T_k(u_n - \varphi) = \int_{\Omega} f T_k(u - \varphi),$$

while the left hand side can be rewritten as

$$\begin{aligned} \int_{\Omega} A(x) \nabla T_k(u_n - \varphi) \cdot \nabla T_k(u_n - \varphi) + \int_{\Omega} A(x) \nabla \varphi \cdot T_k(u_n - \varphi) \\ + \int_{\Omega} q(x) u_n T_k(u_n - \varphi). \end{aligned}$$

The first term is non-negative, thus, the almost everywhere convergence of ∇u_n to ∇u follows by Fatou lemma,

$$\int_{\Omega} A(x) \nabla T_k(u - \varphi) \cdot \nabla T_k(u - \varphi) \leq \liminf_{n \rightarrow +\infty} \int_{\Omega} A(x) \nabla T_k(u_n - \varphi) \cdot \nabla T_k(u_n - \varphi).$$

For the second, since u_n converges to u in $H_0^1(\Omega)$ so $u_n - \varphi$ to $u - \varphi$ in $H_0^1(\Omega)$, then $T_k(u_n - \varphi)$ to $T_k(u - \varphi)$ in $H_0^1(\Omega)$ and since $-\nabla(A(x) \nabla \varphi) \in H_0^1(\Omega)$

$$\langle -\nabla(A(x) \nabla \varphi), T_k(u_n - \varphi) \rangle \rightarrow \langle -\nabla(A(x) \nabla \varphi), T_k(u - \varphi) \rangle$$

i.e.

$$\int_{\Omega} A(x) \nabla \varphi \cdot T_k(u - \varphi) = \lim_{n \rightarrow +\infty} \int_{\Omega} A(x) \nabla \varphi \cdot T_k(u_n - \varphi).$$

For third term since

$$|q(x) u_n T_k(u_n - \varphi)| \leq 2k \|q\|_{\infty} |u|$$

by Lebesgue dominated convergence theorem

$$\int_{\Omega} q(x)uT_k(u-\varphi) = \lim_{n \rightarrow +\infty} \int_{\Omega} q(x)u_nT_k(u_n-\varphi).$$

Then by cancelling equal terms:

$$\int_{\Omega} A(x)\nabla u \cdot \nabla T_k(u-\varphi) + \int_{\Omega} q(x)uT_k(u-\varphi) \leq \int_{\Omega} fT_k(u-\varphi),$$

so u is an entropy solution of (2.5). \square

Theorem 4. Let $f \in L^1(\Omega)$ and u be an entropy solution of (2.5) with datum f . Then u belongs to $W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$ and it is a distributional solution for (2.5).

Proof. Taking $\varphi = 0$ in (2.8)

$$\alpha \int_{\Omega} |\nabla T_k(u)|^2 \leq \int_{\Omega} A(x)\nabla u \cdot \nabla T_k(u) + \int_{\Omega} q(x)uT_k(u) = \int_{\Omega} fT_k(u) \leq k\|f\|_{L^1(\Omega)}.$$

Proof of Theorem 4.1 in [9] shows that $u \in W_0^{1,q}(\Omega)$ for every $q < \frac{N}{N-1}$. We now fix $h > 0$ and choose $\varphi = T_h(u)$ as test function in (2.8). Then

$$\int_{\Omega} A(x)\nabla u \cdot \nabla T_k(u - T_h(u)) + \int_{\Omega} q(x)uT_k(u - T_h(u)) \leq \int_{\Omega} fT_k(u - T_h(u)).$$

Moreover,

$$T_k(u - T_h(u)) = \begin{cases} u - T_h(u) & -k \leq u - T_h(u) \leq k, \\ k & k \leq u - T_h(u), \\ -k & u - T_h(u) \leq -k, \end{cases}$$

where

$$u - T_h(u) = \begin{cases} 0 & -h \leq u \leq h, \\ u - h & h \leq u, \\ u + h & u \leq -h. \end{cases}$$

Therefore, if $|u| \leq h$, then $T_k(u - T_h(u)) = 0$. Moreover, if $h - k \leq |u| \leq h + k$, then $T_k(u - T_h(u)) = u - T_h(u)$. Thus,

$$\begin{aligned} & \int_{\{h-k \leq |u| \leq h+k\}} A(x)\nabla u \cdot \nabla u + \int_{\{|u| \geq h\}} q(x)uT_k(u - T_h(u)) \\ &= \int_{\{|u| \geq h\}} fT_k(u - T_h(u)) \leq k \int_{\{|u| \geq h\}} |f|. \end{aligned}$$

Defining $A_h = \{|u| \geq h\}$, $m(A_h) \rightarrow 0$ as $h \rightarrow \infty$ (since $u \in W_0^{1,1}(\Omega)$), thus, in $L^1(\Omega)$. From $f \in L^1(\Omega)$,

$$\lim_{h \rightarrow +\infty} \int_{\{|u| \geq h\}} |f| = 0,$$

hence by recalling (2.7)

$$\lim_{h \rightarrow +\infty} \int_{\{h-k \leq |u| \leq h+k\}} |\nabla u|^2 = 0. \quad (2.9)$$

For $h > 0$, η in $C_0^1(\Omega)$ and $\varphi = T_h(u) - \eta$ as test function in the entropy formulation (2.8), where $k = \|\eta\|_{L^\infty(\Omega)}$ then

$$\int_{\Omega} A(x) \nabla u \cdot \nabla T_k(u - T_h(u) + \eta) + \int_{\Omega} q(x) u T_k(u - T_h(u) + \eta) \leq \int_{\Omega} f T_k(u - T_h(u) + \eta).$$

By Lebesgue dominated theorem and choice of k

$$\lim_{h \rightarrow +\infty} \int_{\Omega} f T_k(u - T_h(u) + \eta) = \int_{\Omega} f T_k(\eta) = \int_{\Omega} f \eta.$$

For the left hand side, using again the choice of k

$$\begin{aligned} & \int_{\{|u| \leq h\}} (A(x) \nabla u \cdot \nabla T_k(u - T_h(u) + \eta) + \int_{\{|u| \leq h\}} q(x) u T_k(u - T_h(u) + \eta)) \\ & + \int_{\{|u| \geq h\}} (A(x) \nabla u \cdot \nabla T_k(u - T_h(u) + \eta) + \int_{\{|u| \geq h\}} q(x) u T_k(u - T_h(u) + \eta)). \end{aligned}$$

Since A is bounded, so $u \in W_0^{1,1}(\Omega)$ and $\eta \in C_0^1(\Omega)$. For $\{|u| \leq h\}$ we have $T_k(u - T_h(u) + \eta) = T_k(\eta)$. Thus, by Lebesgue dominated theorem

$$\lim_{h \rightarrow +\infty} \int_{\{|u| \leq h\}} A(x) \nabla u \cdot \nabla T_k(\eta) = \lim_{h \rightarrow +\infty} \int_{\{|u| \leq h\}} A(x) \nabla u \cdot \nabla \eta = \int_{\Omega} A(x) \nabla u \cdot \nabla \eta.$$

Similarly

$$\int_{\{|u| \leq h\}} q(x) u T_k(u - T_h(u) + \eta) = \int_{\{|u| \leq h\}} q(x) u T_k(\eta) = \int_{\{|u| \leq h\}} q(x) u \eta.$$

Then

$$\lim_{h \rightarrow +\infty} \int_{\{|u| \leq h\}} q(x) u \eta = \int_{\Omega} q(x) u \eta.$$

Since

$$\{|u - T_h(u) + \eta| \leq k, |u| \geq h\} \subseteq \{h - 2k \leq |u| \leq h + 2k\}$$

by (2.7) and choice of k

$$\begin{aligned} & \int_{\{|u| \geq h\}} A(x) \nabla u \cdot \nabla T_k(u - T_h(u) + \eta) \leq \left| \int_{\{|u| \geq h\}} A(x) \nabla u \cdot \nabla T_k(u - T_h(u) + \eta) \right| \\ & \leq \int_{\{|u| \geq h\}} A(x) |\nabla u| \cdot |\nabla T_k(u - T_h(u) + \eta)| \leq \beta \int_{\{h-2k \leq |u| \leq h+2k\}} |\nabla u| (|\nabla u| + |\nabla \eta|). \end{aligned}$$

Thus by (2.9) and Hölder Inequality

$$\begin{aligned} \beta \int_{\{h-2k \leq |u| \leq h+2k\}} |\nabla u| (|\nabla u| + |\nabla \eta|) &\leq \beta \int_{\{h-2k \leq |u| \leq h+2k\}} (|\nabla u|^2 + |\nabla u| \cdot |\nabla \eta|) \\ &\leq \beta \left[\int_{\{h-2k \leq |u| \leq h+2k\}} |\nabla u|^2 \right]^{\frac{1}{2}} \cdot \left[\int_{\{h-2k \leq |u| \leq h+2k\}} |\nabla \eta|^2 \right]^{\frac{1}{2}} = 0. \end{aligned}$$

Therefore,

$$\lim_{h \rightarrow +\infty} \int_{\{|u| \geq h\}} (A(x) \nabla u \cdot \nabla T_k(u - T_h(u) + \eta)) = 0$$

and

$$|q(x)uT_k(u - T_h(u) + \eta)| \leq k \|q\|_{\infty} |u|$$

so $q(x)uT_k(u - T_h(u) + \eta) \in L^1(\Omega)$. Thus, $\lim_{h \rightarrow +\infty} \int_{\{|u| \geq h\}} k \|q\|_{\infty} |u| = 0$ and

$$\int_{\{|u| \geq h\}} q(x)uT_k(u - T_h(u) + \eta) = 0$$

Putting together the results,

$$\int_{\Omega} (A(x) \nabla u \cdot \nabla \eta + q(x)u\eta) \leq \int_{\Omega} f\eta,$$

for any $\eta \in C_0^1(\Omega)$. Exchanging η with $-\eta$ we obtain the reverse inequality so that u is a distributional solution of (2.5). \square

Finally, we would show uniqueness of entropy solution.

Theorem 5. *Let $f \in L^1(\Omega)$. Then the entropy solution of (2.5) is unique.*

Proof. We proceed in three steps.

Step 1 (An entropy solution is a duality solution): Consider g is in $L^\infty(\Omega)$ and v is a weak solution of

$$\begin{cases} -\operatorname{div}(A^*(x)\nabla v) + q(x)v = g & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

According to the Stampacchia's theorem [2], $v \in L^\infty(\Omega)$. We repeat the proof of Theorem 4. By choosing $\varphi = T_h(u) - v$ in the entropy formulation, for $h > 0$ and $k = \|v\|_{L^\infty(\Omega)}$:

$$\int_{\Omega} A(x) \nabla u \cdot \nabla T_k(u - T_h(u) + v) + \int_{\Omega} q(x)uT_k(u - T_h(u) + v) \leq \int_{\Omega} fT_k(u - T_h(u) + v).$$

Similar to theorem 4, from Lebesgue dominated theorem and choose of k ,

$$\lim_{h \rightarrow +\infty} \int_{\Omega} fT_k(u - T_h(u) + v) = \int_{\Omega} fv,$$

Moreover, the left hand side can be rewritten as

$$\int_{\{|u|\leq h\}} (A(x)\nabla u \cdot \nabla v + q(x)uv) + \int_{\{|u|\geq h\}} A(x)\nabla u \cdot \nabla T_k(u - T_h(u) + v) + \int_{\{|u|\geq h\}} q(x)uT_k(u - T_h(u) + v).$$

For the second and the third term, similar to the proof of Theorem 4 (using (2.9))

$$\lim_{h \rightarrow +\infty} \int_{\{|u|\geq h\}} A(x)\nabla u \cdot \nabla T_k(u - T_h(u) + v) + \int_{\{|u|\geq h\}} q(x)uT_k(u - T_h(u) + v) = 0,$$

and the first term can be rewritten as

$$\begin{aligned} \int_{\{|u|\leq h\}} (A(x)\nabla u \cdot \nabla v + q(x)uv) &= \int_{\Omega} (A(x)\nabla T_h(u) \cdot \nabla v + q(x)T_h(u)v) \\ &= \int_{\Omega} (A^*(x)\nabla v \cdot \nabla T_h(u) + q(x)vT_h(u)) = \int_{\Omega} gT_h(u), \end{aligned}$$

since $T_h(u) \in H_0^1(\Omega)$ can be chosen as test function in the problem solved by v . Then, by Lebesgue dominated theorem,

$$\int_{\{|u|\leq h\}} (A(x)\nabla u \cdot \nabla v + q(x)uv) = \lim_{h \rightarrow +\infty} \int_{\{|u|\leq h\}} (A(x)\nabla u \cdot \nabla v + q(x)uv) = \int_{\Omega} gu.$$

Putting together the results, we obtain

$$\int_{\Omega} gu \leq \int_{\Omega} fv.$$

Exchanging g with $-g$ (and so v with $-v$, by linearity), we obtain the reverse inequality, therefore, u is a duality solution of (2.5).

Step 2 (An entropy solution is a solution obtained by approximation (See [10])): Suppose that $f_n \in L^\infty(\Omega)$ be a sequence of functions that converges to $f \in L^1(\Omega)$, and suppose that u_n be the solution of

$$\begin{cases} -div(A^*(x)\nabla u_n) + q(x)u_n = f_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{cases}$$

$u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$, so that $\varphi = u_n$ is an admissible choice in the entropy formulation for u . Thus,

$$\int_{\Omega} A(x)\nabla u \cdot \nabla T_k(u - u_n) + \int_{\Omega} q(x)uT_k(u - u_n) \leq \int_{\Omega} fT_k(u - u_n).$$

On the other hand, $T_k(u - u_n)$ belongs to $H_0^1(\Omega)$ and so it can be chosen as test function in the weak formulation for u_n . Then

$$\int_{\Omega} A(x)\nabla u_n \cdot \nabla T_k(u - u_n) + \int_{\Omega} q(x)u_nT_k(u - u_n) = \int_{\Omega} f_nT_k(u - u_n).$$

Thus,

$$\int_{\Omega} A(x) \nabla(u - u_n) \cdot \nabla T_k(u - u_n) + \int_{\Omega} q(x)(u - u_n) T_k(u - u_n) \leq \int_{\Omega} (f - f_n) T_k(u - u_n).$$

By (2.7) and lemma 2

$$\begin{aligned} \alpha \int_{\Omega} |\nabla T_k(u - u_n)|^2 &\leq \int_{\Omega} A(x) \nabla(u - u_n) \cdot \nabla T_k(u - u_n) + \int_{\Omega} q(x)(u - u_n) T_k(u - u_n) \\ &\leq k \|(f - f_n)\|_{L^1(\Omega)}. \end{aligned}$$

approaching $n \rightarrow \infty$, $T_k(u - u_n) \rightarrow 0$ in $H_0^1(\Omega)$ and this implies that u_n converges to the entropy solution u . From solutions obtained by approximation are unique, hence, the entropy solution u is unique.

Step 3 (There exists at most an entropy solution): Here, we follow [2]. Suppose that u and v be two entropy solutions of (2.5), with the same datum f , and let $h > k > 0$. Then $\varphi = T_h(v)$ is admissible in the entropy formulation for u and $\varphi = T_h(u)$ is admissible in the entropy formulation for v . Thus,

$$\int_{\Omega} A(x) \nabla u \cdot \nabla T_k(u - T_h(v)) + \int_{\Omega} q(x) u T_k(u - T_h(v)) \leq \int_{\Omega} f T_k(u - T_h(v)),$$

and

$$\int_{\Omega} A(x) \nabla v \cdot \nabla T_k(v - T_h(u)) + \int_{\Omega} q(x) v T_k(v - T_h(u)) \leq \int_{\Omega} f T_k(v - T_h(u)).$$

Summing these two inequalities,

$$\begin{aligned} &\int_{\Omega} A(x) \nabla u \cdot \nabla T_k(u - T_h(v)) + \int_{\Omega} q(x) u T_k(u - T_h(v)) \\ &+ \int_{\Omega} A(x) \nabla v \cdot \nabla T_k(v - T_h(u)) + \int_{\Omega} q(x) v T_k(v - T_h(u)), \end{aligned}$$

in the left hand side is less than or equal to

$$\int_{\Omega} f (T_k(u - T_h(v)) + T_k(v - T_h(u)))$$

in the right hand side. From oddness of $T_k(s)$ and Lebesgue dominated theorem,

$$\lim_{h \rightarrow +\infty} \int_{\Omega} f (T_k(u - T_h(v)) + T_k(v - T_h(u))) = 0.$$

Hence,

$$\begin{aligned} &\limsup_{h \rightarrow +\infty} \int_{\Omega} A(x) \nabla u \cdot \nabla T_k(u - T_h(v)) + \int_{\Omega} q(x) u T_k(u - T_h(v)) \\ &+ \int_{\Omega} A(x) \nabla v \cdot \nabla T_k(v - T_h(u)) + \int_{\Omega} q(x) v T_k(v - T_h(u)) \leq 0. \end{aligned}$$

For the sake of simplicity we will suppose from now on that $u \geq 0$ and $v \geq 0$, since the proof turns out to be considerably simplified. We refer to [2] for the proof in the general case of changing sign solutions. We set

$$\Omega = \{u \leq h, v \leq h\} \cup \{u > h, v \leq h\} \cup \{v > h\} = E_0^h \cup F_1^h \cup F_2^h,$$

and

$$\Omega = \{v \leq h, u \leq h\} \cup \{v > h, u \leq h\} \cup \{u > h\} = E_0^h \cup F_3^h \cup F_4^h.$$

Then

$$\begin{aligned} & \int_{E_0^h} A(x) \nabla u \cdot \nabla T_k(u - T_h(v)) + \int_{E_0^h} q(x) u T_k(u - T_h(v)) \\ &= \int_{E_0^h} A(x) \nabla u \cdot \nabla T_k(u - v) + \int_{E_0^h} q(x) u T_k(u - v). \end{aligned}$$

Similarly,

$$\begin{aligned} & \int_{E_0^h} A(x) \nabla v \cdot \nabla T_k(v - T_h(u)) + \int_{E_0^h} q(x) v T_k(v - T_h(u)) \\ &= \int_{E_0^h} A(x) \nabla v \cdot \nabla T_k(v - u) + \int_{E_0^h} q(x) v T_k(v - u). \end{aligned}$$

On F_1^h ,

$$\int_{F_1^h} A(x) \nabla u \cdot \nabla T_k(u - T_h(v)) = \int_{\{u > h, v \leq h, 0 \leq u - v \leq k\}} A(x) \nabla u \cdot \nabla(u - v).$$

On $\{u > h, v \leq h, 0 \leq u - v \leq k\}$ it valid $h < u \leq h + k$ and $h - k < v \leq h$, so

$$\left| \int_{F_1^h} A(x) \nabla u \cdot \nabla T_k(u - T_h(v)) \right| \leq \beta \int_{\{h < u \leq h + k, h - k < v \leq h\}} |\nabla u| |\nabla v|.$$

By (2.9)

$$\lim_{h \rightarrow +\infty} \int_{\{h < u \leq h + k\}} |\nabla u|^2 = 0,$$

and

$$\lim_{h \rightarrow +\infty} \int_{\{h - k < v \leq h\}} |\nabla v|^2 = 0,$$

Hence, by Hölder inequality

$$\lim_{h \rightarrow +\infty} \left| \int_{F_1^h} A(x) \nabla u \cdot \nabla T_k(u - T_h(v)) \right| = 0;$$

and $|q(x) u T_k(u - T_h(v))| \leq k \|q\|_\infty |u|$, so $q(x) u T_k(u - T_h(v)) \in L^1(\Omega)$. Thus,

$$\lim_{h \rightarrow +\infty} \int_{F_1^h} k \|q\|_\infty |u| = 0,$$

hence,

$$\int_{F_1^h} q(x)uT_k(u - T_h(v)) = 0.$$

Repeating the same for F_3^h we have

$$\lim_{h \rightarrow +\infty} \left| \int_{F_3^h} A(x)\nabla v \cdot \nabla T_k(v - T_h(u)) \right| = 0.$$

and

$$\int_{F_3^h} q(x)vT_k(v - T_h(u)) = 0$$

Moreover on F_2^h ,

$$\begin{aligned} & \int_{F_2^h} (A(x)\nabla u \cdot \nabla T_k(u - T_h(v)) + q(x)uT_k(u - T_h(v))) \\ &= \int_{\{v>h, 0 \leq u < h+k\}} (A(x)\nabla u \cdot \nabla u) + q(x)uu \geq 0, \end{aligned}$$

and similarly on F_4^h ,

$$\begin{aligned} & \int_{F_4^h} (A(x)\nabla v \cdot \nabla T_k(v - T_h(u)) + q(x)vT_k(v - T_h(u))) \\ &= \int_{\{u>h, 0 \leq v < h+k\}} (A(x)\nabla v \cdot \nabla v) + q(x)vv \geq 0, \end{aligned}$$

Putting the results together,

$$\limsup_{h \rightarrow +\infty} \int_{E_0^h} (A(x)\nabla(u - v) \cdot \nabla T_k(u - v) + q(x)(u - v)T_k(u - v)) \leq 0,$$

which, by Fatou lemma, implies, from E_0^h "fills" Ω as $h \rightarrow +\infty$,

$$0 \leq \int_{\Omega} (A(x)\nabla(u - v) \cdot \nabla T_k(u - v) + q(x)(u - v)T_k(u - v)) \leq 0,$$

Using (2.7) and lemma 2 we have $\nabla T_k(u - v) \equiv 0$, thus $u = v$. □

Remark 2. In special case if

$$A = \begin{bmatrix} \alpha_1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \alpha_2 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ \cdot & & & & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & \alpha_n \end{bmatrix}$$

and $q(x) = \lambda$, problem (2.5) can be rewritten

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) + q(x)u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \implies \begin{cases} -\Delta_{\alpha}u + \lambda u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Then above theorems are satisfied for this problem.

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