COMMON FIXED POINT THEOREMS WITH AN APPLICATION IN DYNAMIC PROGRAMMING

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Abstract. Two common fixed point theorems for a class of contractive mappings are proved in metric spaces. As an application, the existence and uniqueness of solution for a functional equation arising in dynamic programming is given. The results presented in this paper generalize some known results in the literature.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, unless otherwise stated, (X, d) denotes a metric space. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, and for each $t \in \mathbb{R}$, $[t]$ denote the greatest integer not exceeding t. Let ω and $\mathbb{N}$ denote the sets of all nonnegative integers and positive integers, respectively, and

$$\Phi = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is upper semi-continuous and nondecreasing and } \phi(t) < t$$
$$\text{for } t > 0\}.$$

Let f be a self mapping of (X, d). For $x, y \in X$ and $A, B \subseteq X$, define

$$O(x, f) = \{f^n x : n \in \omega\}, \quad O(x, y, f) = O(x, f) \cup O(y, f),$$
$$C_f = \{g : g \text{ is a self mapping of } X \text{ and } gf = fg\},$$
$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}, \quad \delta(A) = \delta(A, A).$$

It is easy to see that $\{f^n : n \in \omega\} \subseteq C_f$.

Many authors studied the existence and uniqueness of fixed point and common fixed point for several classes of contractive mappings and families of contractive mappings in metrics spaces, and they used a few fixed point and common fixed point theorems to establish the existence and uniqueness of solution and common solutions for some kinds of functional equations and systems of functional equations arising

from dynamic programming, for example, see [1–25] and the references therein. Jungck [9] proved some fixed point theorems for C_f in metric spaces. Ohta and Nikaido [21] established two fixed point theorems for contractive mappings which satisfy

$$d(f^k x, f^k y) \leq \alpha \delta(O(x, y, f)), \quad \forall x, y \in X, \tag{1.1}$$

where $\alpha \in [0, 1)$.

As suggested in Bellman and Lee [1], the basic form of the functional equations in dynamic programming is

$$f(x) = \max_{y \in S} H(x, y, f(T(x, y))), \quad \forall x \in D, \tag{1.2}$$

where x and y denote the state and decision vectors, respectively, T denotes the transformation of the process and $f(x)$ denotes the optimal return function with the initial state x.

Motivated by the results in [1–25], in this paper we extend the classes of mappings (1.1) and functional equations (1.2) considered by Ohta and Nikaido [21] and Bellman and Lee [1], respectively, to the following more general classes of contractive mappings and functional equations:

$$d(f^p x, f^q y) \leq \varphi(\delta(\cup_{g \in C_f} g O(x, y, f))), \quad \forall x, y \in X, \tag{1.3}$$

where $\varphi \in \Phi$ and p, q are some positive integers, and

$$f(x) = \operatorname*{opt}_{y \in D} \{u(x, y) + H(x, y, f(T(x, y)))\}, \quad \forall x \in S, \tag{1.4}$$

where $u : S \times D \to \mathbb{R}$, $T : S \times D \to S$ and $H : S \times D \times \mathbb{R} \to \mathbb{R}$, the opt denotes max or min. Under certain conditions we study the existence and uniqueness of fixed point, common fixed point and solution for the contractive mapping (1.3), the family of mappings C_f and the functional equation (1.4), respectively, and establish the convergence and error estimates of Picard iterations with respect to the fixed point of the contractive mapping (1.3) and the solution of the functional equation (1.4), respectively. The results presented in this paper extend and improve some known results in [7, 21].

Let us recall the following notation, definitions and lemmas.

Definition 1 ([2])**.** Let (X, d) be a metric space. A mapping $f : X \to X$ is said to have diminishing orbital diameters in X if

$$\lim_{n \to \infty} \delta(O(f^n x, f)) < \delta(O(x, f)) \quad \text{for all } x \in X \text{ with } \delta(O(x, f)) > 0.$$

Definition 2 ([6])**.** Let (X, d) be a metric space, $A \subseteq X$ and $A_n \subseteq X$ for $n \in \mathbb{N}$. The sequence $\{A_n\}_{n \in \mathbb{N}}$ is said to converge to A if

(1) each point $a \in A$ is the limit of some convergent sequence $\{a_n : a_n \in A_n$ for each $n \in \mathbb{N}\}$;
(2) for arbitrary $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $A_n \subseteq A_\epsilon$ for $n > k$, where A_ϵ is the union of all open spheres with centers in A and radius ϵ.

Lemma 1 ([24]). *Let $\varphi \in \Phi$ and φ^n denote the composition of φ with itself n times. Then for every $t > 0$, $\varphi(t) < t$ if and only if $\lim_{n \to \infty} \varphi^n(t) = 0$.*

Lemma 2 ([20]). *Let f a continuous self mapping of a metric space (X, d) such that*

(a) *f has a unique fixed point $u \in X$;*

(b) *$\lim_{n \to \infty} f^n x = u$ for all $x \in X$;*

(c) *there exists an open neighborhood G of u with the property that given any open set V containing u there exists $k \in \mathbb{N}$ such that $f^n G \subseteq V$ for all $n > k$.*

Then for any $r \in (0, 1)$, there exists a metric d', topologically equivalent to d, such that $d'(fx, fy) \le rd'(x, y)$ for all $x, y \in X$.

2. COMMON FIXED POINT THEOREMS FOR C_f

Our main results are as follows.

Theorem 1. *Let (X, d) be a bounded complete metric space and $f : X \to X$ satisfy (1.3). Assume that f is continuous. Then*

(i) *f has diminishing orbital diameters in X;*

(ii) *f has a unique fixed point $u \in X$, which is also a unique common fixed point of C_f;*

(iii) *$d(f^n x, u) \le \varphi^{[n/s]}(\delta(X))$ for all $x \in X$ and $n \in \mathbb{N}$, where $s = \max\{p, q\}$;*

(iv) *$\lim_{n \to \infty} f^n x = u$ for all $x \in X$ and $\{f^n X\}_{n \in \mathbb{N}}$ converges to $\{u\}$;*

(v) *for any $r \in (0, 1)$, there exists a metric d', topological equivalent to d, such that $d'(fx, fy) \le rd'(x, y)$ for all $x, y \in X$.*

Proof. We may assume, without loss of generality, $p \ge q$. For any $x, y \in X$ and $n \ge p$, it follows from (1.3) that

$$
\begin{aligned}
d(f^n x, f^n y) &= d(f^p f^{n-p} x, f^q f^{n-q} y) \\
&\le \varphi(\delta(\cup_{g \in C_f} gO(f^{n-p} x, f^{n-q} y, f))) \\
&= \varphi(\delta(\cup_{g \in C_f} O(f^{n-p} gx, f^{n-q} gy, f))) \\
&\le \varphi(\delta(O(f^{n-p} X \cup f^{n-q} X))) \\
&= \varphi(\delta(f^{n-p} X)),
\end{aligned}
$$

which implies that

$$
\delta(f^n X) \le \varphi(\delta(f^{n-p} X)), \quad \forall n \ge p. \tag{2.1}
$$

It follows from (2.1) and Lemma 1 that

$$
\delta(f^{kp} X) \le \varphi(\delta(f^{(k-1)p} X)) \le \cdots \le \varphi^k(\delta(X)) \to 0 \quad \text{as } k \to \infty. \tag{2.2}
$$

Because

$$
X \supseteq fX \supseteq f^2 X \supseteq \cdots \supseteq f^n X \supseteq f^{n+1} X \supseteq \cdots, \quad \forall n \in \omega,
$$

by (2.2) we infer that

$$\delta(f^n X) \to 0 \quad \text{as } n \to \infty. \tag{2.3}$$

In view of (2.3), we get that

$$d(f^n x, f^{n+h} x) \leq \delta(O(f^n x, f)) \leq \delta(f^n X), \quad \forall x \in X, \forall n, h \in \mathbb{N}. \tag{2.4}$$

It follows from (2.3) and (2.4) that f has diminishing orbital diameters and for each $x \in X$, $\{f^n x\}_{n \in \mathbb{N}}$ is a Cauchy sequence. Thus $\{f^n x\}_{n \in \mathbb{N}}$ converges to some point $u \in X$ by completeness of X.

Since f is continuous, it follows that $fu = u$. Suppose that f has a second fixed point $v \in X$. From (2.3) we have

$$d(u, v) \leq \delta(f^n X) \to 0 \quad \text{as } n \to \infty,$$

which means that $u = v$, that is, f has a unique fixed point u. Let $g \in C_f$. Note that $gu = gfu = fgu$. It follows that gu is a fixed point of f. Therefore $gu = u$. It follows from $f \in C_f$ that u is a unique common fixed point of C_f. Using (2.1) and (2.3) we conclude that for $x \in X$

$$\begin{aligned}
d(f^n x, u) &\leq \delta(f^n X, \{u\}) \leq \delta(f^n X) \\
&\leq \varphi^{[n/p]}(\{\delta(f^i X) : 0 \leq i < p\}) \\
&= \varphi^{[n/p]}(\delta(X)), \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{2.5}$$

Given $\epsilon > 0$. (2.3) ensures that there exists $k \in \mathbb{N}$ such that $\delta(f^n X) < \frac{\epsilon}{2}$ for $n > k$. Consequently, by (2.4) we deduce that $f^n X \subseteq B(u, \epsilon) = \{x \in X : d(u, x) < \epsilon\}$ for $n > k$. Thus $\{f^n X\}_{n \in \mathbb{N}}$ converges to $\{u\}$.

To show (v), it suffices to show that (c) of Lemma 2 holds. Take $G = X$. For any open set V containing u there exists $\epsilon > 0$ with $B(u, \epsilon) = \{x \in X : d(u, x) < \epsilon\} \subseteq V$. It follows from what we have just proved that $f^n G \subseteq B(u, \epsilon) \subseteq V$ for $n > k$. Hence (c) is satisfied. This completes the proof. □

Remark 1. The following example shows that the condition that f be continuous when $p, q > 1$ is necessary in Theorem 1.

Example 1. Let $X = [0, 1]$ with the usual metric. Define a discontinuous mapping $f : X \to X$ by $f0 = 1$ and $fx = \frac{x}{2}$ for $x \in (0, 1]$. Take $p = 2, q = 3$ and $\varphi(t) = \frac{t}{2}$ for $t \geq 0$. It is easy to check that the conditions of Theorem 1 are satisfied except for the continuity assumption. f however has no fixed point in X.

Remark 2. The following example reveals that the boundedness of X is necessary in Theorem 1.

Example 2. Let $X = [1, +\infty)$ with the usual metric. Define a mapping $f : X \to X$ by $fx = 2x$ for $x \in X$. Set $p = 2, q = 3$ and $\varphi(t) = \frac{t}{2}$ for $t \geq 0$. It is easily proved that the conditions of Theorem 1 are satisfied except for the boundedness assumption. f however has no fixed point in X.

Replacing the boundedness of X by the boundedness of fX, as in the proof of Theorem 1, we have

Theorem 2. *Let (X,d) be a complete metric space and $f : X \to X$ be a continuous mapping such that fX is bounded and (1.3) holds. Then* (i)*,* (ii)*,* (iv) *and* (v) *in Theorem 1 and the following*
(iii)$'$ $d(f^n x, u) \leq \varphi^{[n/s]}(\delta(fX))$ *for all $x \in X, n \in \mathbb{N}$*
hold.

Remark 3. Theorems 1 and 2 extend, improve and unify the corresponding results in [7, 21].

3. AN APPLICATION IN DYNAMIC PROGRAMMING

Throughout this section, we assume that X, Y are Banach spaces, $S \subseteq X$ is the state space, $D \subseteq Y$ is the decision space, $B(S)$ denotes the set of all bounded real-valued functions on S and $d(f,g) = \sup\{|f(x) - g(x)| : x \in S\}$ for $f, g \in B(S)$. Clearly $(B(S), d)$ is a complete metric space.

Theorem 3. *Suppose that the following conditions are satisfied:*

(a) *u and H are bounded;*
(b) *$|H(x, y, g(t)) - H(x, y, h(t))| \leq \varphi(\delta(\cup_{m \in C_A} mO(g, h, A)))$ for all $(x, y) \in S \times D, g, h \in B(S)$ and $t \in S$, where $\varphi \in \Phi$ and the mapping $A : B(S) \to B(S)$ defined by*

$$Ag(x) = \operatorname*{opt}_{y \in D} \{u(x, y) + H(x, y, g(T(x, y)))\}, \quad \forall (x, g) \in S \times B(S) \qquad (3.1)$$

satisfies
(c) *For any sequence $\{h_n\}_{n \in \mathbb{N}} \subseteq B(S)$ and $h \in B(S)$,*

$$\lim_{n \to \infty} \sup_{x \in S} |h_n(x) - h(x)| = 0 \quad \Longrightarrow \quad \lim_{n \to \infty} \sup_{x \in S} |Ah_n(x) - Ah(x)| = 0.$$

Then

(i) *A has diminishing orbital diameters in $B(S)$;*
(ii) *the functional equation (1.4) possesses a unique solution $v \in B(S)$, which is both a unique fixed point of A and a unique common fixed point of C_A;*
(iii) *$d(A^n x, v) \leq \varphi^{[n/s]}(\delta(AB(S)))$ for all $x \in B(S), n \in \mathbb{N}$;*
(iv) *$\lim_{n \to \infty} A^n x = v$ for all $x \in B(S)$ and $\{A^n B(S)\}_{n \in \mathbb{N}}$ converges to $\{v\}$;*
(v) *for any $r \in (0, 1)$, there exists a metric d', topological equivalent to d, such that $d'(Ax, Ay) \leq rd'(x, y)$ for all $x, y \in B(S)$.*

Proof. It follows from (a), (c) and (3.1) that $AB(S)$ is bounded and A is continuous. We assume that without loss of generality opt $=$ inf. For any $\varepsilon > 0, x \in S$ and $h, g \in B(S)$, there exist $y, z \in D$ such that

$$Ag(x) > u(x, y) + H(x, y, g(T(x, y))) - \varepsilon, \qquad (3.2)$$

$$Ah(x) > u(x,z) + H(x,z,h(T(x,z))) - \varepsilon. \tag{3.3}$$

Also we have

$$Ag(x) \le u(x,z) + H(x,z,g(T(x,z))), \tag{3.4}$$

$$Ah(x) \le u(x,y) + H(x,y,h(T(x,y))). \tag{3.5}$$

From (3.2), (3.5) and (b) we infer that

$$Ag(x) - Ah(x) > H(x,y,g(T(x,y))) - H(x,y,h(T(x,y))) - \varepsilon$$
$$\ge -\varphi(\delta(\cup_{m \in C_A} mO(g,h,A))) - \varepsilon.$$

Similarly, from (3.3) and (3.4) and (b) we know that

$$Ag(x) - Ah(x) < H(x,z,g(T(x,z))) - H(x,z,h(T(x,z))) + \varepsilon$$
$$\le \varphi(\delta(\cup_{m \in C_A} mO(g,h,A))) + \varepsilon.$$

It is easy to see that

$$d(Ag,Ah) = \sup_{x \in S} |Ag(x) - Ah(x)|$$
$$\le \varphi(\delta(\cup_{m \in C_A} mO(g,h,A))) + \varepsilon.$$

Letting ε tend to zero, we have

$$d(Ag,Ah) \le \varphi(\delta(\cup_{m \in C_A} mO(g,h,A))), \quad \forall g,h \in B(S).$$

Thus Theorem 3 follows from Theorem 2 with $p = q = 1$. Particularly, the unique fixed point $v \in B(S)$ of A is a unique solution of the functional equation (1.4) in $B(S)$. This completes the proof. $\qquad \square$

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