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ON α_β-CONTRACTIVE AND ADMISSIBLE MAPPINGS

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Abstract. In this paper, we introduce the concept of a pair (f,h) called upper class of type II and α_β-contractive mappings. We obtain that all the corresponding established results of Hussain et al. [7] are immediately consequences of our main result. Our main result generalizes and modifies several existing results in literature. Also, an example is given to support the main result.

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1. Introduction and preliminaries

It is well known that Banach contraction principle [2] is one of the most interesting and useful result in nonlinear analysis and whole mathematics in general. This famous result has also very large applications in various fields such as engineering, economic, computer sciences, and many others. This theorem has been extended and generalized by various authors (see, e.g., [7–9, 11, 12]).

In an attempt to generalize this significant principle, many researchers have extended the following result in certain directions.

Theorem 1 ([3–6, 8]). *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping. Assume that there exists a function $\beta : [0,\infty) \to [0,1]$ such that, for any bounded sequence $\{t_n\}$ of positive real numbers, $\beta(t_n) \to 1$ implies $t_n \to 0$ and $d(Tx,Ty) \le \beta(d(x,y))d(x,y)$ for all $x,y \in X$. Then T has a unique fixed point.*

In this paper we introduce a concept of pair (f,h) which we denote as upper class of type II and α_β-contractive mappings to show the theorems in [7] are immediately consequences of our main approach.

Now, we introduce some definitions and new notations which will be used in the sequel.

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Definition 1. We say that $h:[0,+\infty)^3 \to \mathbb{R}$ is a function of subclass of type II if it is continuous and

$$x, y \geq 1 \Rightarrow h(1,1,z) \leq h(x,y,z)$$

Example 1. Define $h: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

(a) $h(x,y,z) = (z+l)^{xy}, l > 1$;
(b) $h(x,y,z) = (xy+l)^z, l > 1$;
(c) $h(x,y,z) = z$;
(d) $h(x,y,z) = x^m y^n z^p$, $m,n,p \in \mathbb{N}$;
(e) $h(x,y,z) = \frac{x^m + x^n y^p + y^q}{3} z^k$, $m,n,p,q,k \in \mathbb{N}$

for all $x,y,z \in \mathbb{R}^+$. Then h is a function of subclass of type II.

Definition 2. Suppose that $f:[0,+\infty)^2 \to \mathbb{R}$ and $h:[0,+\infty)^3 \to \mathbb{R}$. The pair (f,h) is called upper class of type II if f is a continuous function, h a subclass of type II with

$$0 \leq s \leq 1 \Rightarrow f(s,t) \leq f(1,t),$$

$$\text{and } h(1,1,z) \leq f(s,t) \Rightarrow z \leq st.$$

Example 2. Define $h: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $\mathcal{F}: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

(a) $h(x,y,z) = (z+l)^{xy}$, $l > 1$, $\mathcal{F}(s,t) = st + l$;
(b) $h(x,y,z) = (xy+l)^z$, $l > 1$, $\mathcal{F}(s,t) = (1+l)^{st}$;
(c) $h(x,y,z) = z$, $F(s,t) = st$;
(d) $h(x,y,z) = x^m y^n z^p$, $m,n,p \in \mathbb{N}$, $\mathcal{F}(s,t) = s^p t^p$
(e) $h(x,y,z) = \frac{x^m + x^n y^p + y^q}{3} z^k$, $m,n,p,q,k \in \mathbb{N}$, $\mathcal{F}(s,t) = s^k t^k$

for all $x,y,z,s,t \in \mathbb{R}^+$. Then the pair $(\mathcal{F},h)$ is an upper class of type II.

Definition 3. Let (X,d) be a metric space and $T: X \to X$ be a mapping. A nonempty subset F of X is called invariant under T if $Tx \in F$, for every $x \in F$.

Definition 4. Let $T: X \to X$ be a mapping, F a nonempty subset of X which is invariant under T and $\alpha: F \times F \to [0,+\infty)$. We say that T is an α_F-admissible mapping if $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$, for all $x, y \in F$.

Remark 1. A mapping T is called an α-admissible mapping (see [12]) if we take $F = X$ in Definition 4.

Definition 5. A function $\psi:[0,+\infty) \to [0,+\infty)$ is called altering distance function if the following properties are satisfied:

(1) ψ is continuous and non-decreasing;
(2) $\psi^{-1}(\{0\}) = 0$.

We denote Ψ the set of all altering distance functions.

The following result will be used in the sequel.

Lemma 1 ([1, 9–11]). *Suppose (X,d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n,x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that the following sequences tend to ε when $k \to \infty$:*

$$d(x_{m(k)},x_{n(k)}),\ d(x_{m(k)-1},x_{n(k)+1}),\ d(x_{m(k)-1},x_{n(k)}),\\ d(x_{m(k)-1},x_{n(k)-1}),\ d(x_{m(k)+1},x_{n(k)+1}).$$

2. Main results

In this section by using the new concept we consider, discuss, improve and generalize the main results from [7]. It is worth mentioning here that in our approach the implication $\beta(t_n) \to 1 \Rightarrow t_n \to 0$ holds also for unbounded sequences (t_n). Therefore, our new results generalize the recent results of [7] in several directions.

Definition 6. Let (X,d) be a metric space, F a nonempty subset of X, $T : X \to X$ and $\alpha : F \times F \to [0,+\infty)$. A mapping T is said to be α_β-contractive mapping if there exists a $\beta : [0,+\infty) \to [0,\ 1)$ with the property that $t_n \to 0$ whenever $\beta(t_n) \to 1$ as well as for all $x,y \in F$, the following condition holds:

$$h(\alpha(x,Tx),\alpha(y,Ty),\psi(d(Tx,Ty))) \le f(\beta(d(x,y)),\psi(d(x,y))), \tag{2.1}$$

where the pair (f,h) is a upper class of type II and $\psi \in \Psi$.

Theorem 2. *Let (X,d) be a complete metric space and F be a nonempty closed subset of X. Suppose that $T : X \to X$ is an α_F-admissible mapping and F is invariant under T. Further assume that T is an α_β-admissible contractive mapping. Suppose that there exists $x_0 \in F$ such that $\alpha(x_0,Tx_0) \ge 1$ and one of the following conditions holds,*
(a) T is continuous.
(b) if $\{x_n\}$ is a sequence in F such that $x_n \to z$, $\alpha(x_n,x_{n+1}) \ge 1$, for all n, then $\alpha(z,Tz) \ge 1$.
Then T has a fixed point.

Proof. Let $x_0 \in F$ such that $\alpha(x_0,Tx_0) \ge 1$. Now, we construct a sequence $\{x_n\}$ in F by $x_n = Tx_{n-1}$, for $n \ge 1$, such that $\alpha(x_n,x_{n+1}) = \alpha(x_n,Tx_n) \ge 1$. Substituting $x = x_{n-1}$ and $y = x_n$ in (2.1), we obtain

$$\begin{aligned} h(1,1,\psi(d(x_n,x_{n+1}))) &\le h(\alpha(x_{n-1},x_n),\alpha(x_n,x_{n+1}),\psi(d(x_n,x_{n+1})))\\ &\le f(\beta(d(x_{n-1},x_n)),\psi(d(x_{n-1},x_n)))\end{aligned}$$

which implies that

$$\begin{aligned}\psi(d(x_n,x_{n+1})) &\le \beta(d(x_{n-1},x_n))\psi(d(x_{n-1},x_n))\\ &\le \psi(d(x_{n-1},x_n)).\end{aligned} \tag{2.2}$$

As $\psi \in \Psi$, we have

$$d(x_n,x_{n+1}) \le d(x_{n-1},x_n), \tag{2.3}$$

for every $n \in \mathbb{N}$. Therefore, $\{d(x_n, x_{n+1})\}$ is a decreasing sequence, so there exists some $r \geq 0$, such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r. \tag{2.4}$$

Further from (2.2), we have

$$\frac{\psi(d(x_n, x_{n+1}))}{\psi(d(x_{n-1}, x_n))} \leq \beta(d(x_{n-1}, x_n)) \leq 1.$$

Letting $n \to \infty$ in the above inequality, we have $\lim_{n\to\infty} \beta(d(x_{n-1}, x_n)) = 1$, and this implies that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.5}$$

Now, we will show that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence.

By Lemma 1, there exists $\delta > 0$ for which we can find subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \delta. \tag{2.6}$$

Setting $x = x_{m_k-1}$ and $y = x_{n_k-1}$ in (2.1), we obtain

$$\begin{aligned} h\left(1,1,\psi\left(d(x_{n_k}, x_{m_k})\right)\right) &\leq h\left(\alpha\left(x_{m_k-1}, x_{m_k}\right), \alpha\left(x_{n_k-1}, x_{n_k}\right), \psi\left(d(x_{n_k}, x_{m_k})\right)\right) \\ &\leq f(\beta\left(d\left(x_{m_k-1}, x_{n_k-1}\right)\right), \psi\left(d\left(x_{m_k-1}, x_{n_k-1}\right)\right)), \end{aligned}$$

i.e. $\psi\left(d(x_{n_k}, x_{m_k})\right) \leq \beta\left(d\left(x_{m_k-1}, x_{n_k-1}\right)\right)\psi\left(d\left(x_{m_k-1}, x_{n_k-1}\right)\right)$, which implies that

$$\frac{\psi\left(d(x_{n_k}, x_{m_k})\right)}{\psi\left(d\left(x_{m_k-1}, x_{n_k-1}\right)\right)} \leq \beta\left(d\left(x_{m_k-1}, x_{n_k-1}\right)\right) \leq 1. \tag{2.7}$$

Letting $k \to \infty$ and using (2.6) and (2.7), we obtain

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = 0 \leq \delta, \tag{2.8}$$

which is a contradiction.

This shows that $\{x_n\}$ is a Cauchy sequence and hence it is convergent in the complete set F. Hence $x_n \to z \in F$ as $n \to \infty$.

First, we suppose that T is continuous. Therefore, we have

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T \lim_{n\to\infty} x_n = Tz.$$

Next, we suppose that condition (b) holds. Therefore, $\alpha(z, Tz) = 1$. Now, by (2.1), we have

$$\begin{aligned} h(1,1,\psi(d(Tz, x_{n+1}))) &\leq h(\alpha(z, Tz), \alpha(x_n, Tx_n), \psi(d(Tz, x_{n+1}))) \\ &\leq f(\beta(d(z, x_n)), \psi(d(z, x_n))), \end{aligned}$$

which implies that

$$\psi(d(Tz, x_{n+1})) \leq \beta(d(z, x_n))\psi(d(z, x_n)).$$

Taking $n \to \infty$ and using the properties of ψ and β, we have $d(Tz,z) = 0$, that is, $z = Tz$. □

3. SOME CONSEQUENCES OF THE MAIN RESULT

If $h(x,y,z) = (z+l)^{xy}$, $l > 1$, $f(x,y) = xy + l$, $\psi(t) = t$, and $F = X$ in Theorem 2, we have Theorem 4 of [7].

Corollary 1. *Let (X,d) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Assume that there exists a function $\beta : [0,\infty) \to [0,1)$ with the property that $t_n \to 0$ whenever $\beta(t_n) \to 1$, such that*

$$(d(Tx,Ty)+l)^{\alpha(x,Tx)\alpha(y,Ty)} \leq \beta(d(x,y))\, d(x,y) + l$$

for all $x,\ y \in X$. Suppose that either
(a) T is continuous, or
(b) if $\{x_n\}$ is a sequence in X such that $x_n \to z$, $\alpha(x_n, x_{n+1}) \geq 1$ for all n, then $\alpha(z,Tz) \geq 1$.
If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.

If $h(x,y,z) = (xy+l)^z$, $l > 1$, $f(x,y) = (1+m)^{xy}$, $m = 1$, $\psi(t) = t$, and $F = X$ in Theorem 2, we have Theorem 6 of [7].

Corollary 2. *Let (X,d) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Assume that there exists a function $\beta : [0,\infty) \to [0,\ 1)$ with the property that $t_n \to 0$ whenever $\beta(t_n) \to 1$, such that*

$$(\alpha(x,Tx)\,\alpha(y,Ty)+l)^{d(Tx,Ty)} \leq 2^{\beta(d(x,y))d(x,y)}$$

for all $x,\ y \in X$. Suppose that either
(a) T is continuous, or
(b) if $\{x_n\}$ is a sequence in X such that $x_n \to z$, $\alpha(x_n, x_{n+1}) \geq 1$ for all n, then $\alpha(z,Tz) \geq 1$
If there exists $x_0 \in X$ such that $\alpha(x_0,\ Tx_0) \geq 1$, then T has a fixed point.

If $h(x,y,z) = xyz$, $f(x,y) = xy$, $\psi(t) = t$, and $F = X$ in Theorem 2, we have Theorem 8 of [7].

Corollary 3. *Let (X,d) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Assume that there exists a function $\beta : [0,\infty) \to [0,\ 1)$ with the property that $t_n \to 0$ whenever $\beta(t_n) \to 1$, such that*

$$\alpha(x,Tx)\alpha(y,Ty)d(Tx,Ty) \leq \beta(d(x,y))d(x,y)$$

for all $x,\ y \in X$. Suppose that either
(a) T is continuous, or
(b) if $\{x_n\}$ is a sequence in X such that $x_n \to z$, $\alpha(x_n, x_{n+1}) \geq 1$ for all n, then $\alpha(z,Tz) \geq 1$.
If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.

Remark 2. Let $h(x,y,z) = xyz$, $f(x,y) = xy$, $\psi(t) = t$, $\alpha(x,y) = 1$, $F = X$ for all $x,y,z \in X$ and $t > 0$. Then we get Theorem 1.

Example 3. Let $X = [1,\infty)$ be endowed with a usual metric $d(x,y) = |x-y|$ for all $x,y \in X$ and $T : X \to X$ be defined by

$$T(x) = \begin{cases} \frac{x+14}{8} & 1 \le x \le 4 \\ \frac{x^2}{4}, & x > 4 \end{cases}$$

Define the function α, β, and ψ given by

$$\alpha(x,y) = \begin{cases} 1, & x,y \in [1,4] \\ 0, & \text{otherwise} \end{cases}$$

$$\beta(t) = \frac{1}{1+t}, \ \psi(t) = t$$

Then T is α-admissible and we obtain $1 \le y \le x \le 4$

$$\begin{aligned} \alpha(x,Tx)\alpha(y,Ty)d(Tx,Ty) &= d\left(\frac{x+14}{8}, \frac{y+14}{8}\right) \\ &= \frac{1}{8}|x-y| \\ &\le \beta(d(x,y))\psi(d(x,y)) \end{aligned}$$

Hence, T satisfies all the assumptions of Theorem 2 with $h(x,y,z) = xyz$ and $f(s,t) = st$ and thus it has a fixed point (which is $x = 2$).

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