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# Elliptic boundary value problems with measurable coefficients and explosive boundary conditions

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# Contents

# Introduction

This Thesis has been carried out under the supervision of Prof. Lucio Boccardo and Doct. Tommaso Leonori. The resulting work follows two different directions, although it always refers to elliptic boundary value problems. The first one concerns existence and regularity results for a wide class of operators in divergence form with discontinuous coefficients. The second one focuses on the qualitative behaviour of large solutions, namely solutions that blows up to infinity at the boundary of the domain, to semilinear elliptic problems.

## Existence and regularity results

In this part of the Thesis we consider three different classes of elliptic boundary value problems in divergence form with measurable coefficients. The initial question that guided our study is the same, even if it has brought to different type of results. The question is

*which are the less restrictive assumptions on the coefficients that preserve some good properties of a given problem?*

With good properties we mean existence and regularity of a reasonable solution; we stress that we do not deal with uniqueness issues. Let us now give a description of our results. In this section $\Omega$ is a bounded open set of $\mathbb{R}^N$, with $N > 2$.

**1.** We start considering a problem with a first order term in divergence form, called *convection term*. In order to avoid technicalities we present its linear form.

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) = -\mathrm{div}\,(uE(x)) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the measurable function $A(x)$ satisfies for $0 < \alpha < \beta$

$$\alpha \leq A(x) \leq \beta, \tag{2}$$

the vector field $E(x)$ belongs to $\left(L^N(\Omega)\right)^N$ and the function $f(x)$ belongs to a suitable Lebesgue o Lorentz space to be precised. If $f \in L^{(2^*)'}(\Omega)$ we can consider the *weak formulation* of (1), namely

$$u \in W_0^{1,2}(\Omega) \ : \ \int_\Omega A(x)\nabla u\nabla\phi = \int_\Omega uE(x)\nabla\phi + \int_\Omega f(x)\phi \quad \forall\, \phi \in W_0^{1,2}(\Omega). \tag{3}$$

Note that the assumption

$$E(x) \in \left(L^N(\Omega)\right)^N \tag{4}$$

is natural for the lower order term of (3) to be well defined, indeed

$$vE(x)\nabla\phi \in L^1(\Omega) \quad \forall\, v, \phi \in W_0^{1,2}(\Omega). \tag{5}$$

Since we are also interested in solutions of (1) outside the energy space, let us set $f \in L^1(\Omega)$ and introduce the *distributional formulation* of (1).

$$u \in W_0^{1,1,}(\Omega) \ : \ \int_\Omega A(x)\nabla u \nabla\phi = \int_\Omega uE(x)\nabla\phi + \int_\Omega f(x)\phi \quad \forall\, \phi \in C_0^1(\Omega). \tag{6}$$

Notice again that, assuming (4), $vE(x) \in L^1(\Omega)$ for all $v \in W_0^{1,1,}(\Omega)$. Let us stress anyway that (4) is not the sharp condition to consider weak or distributional solutions to problem (1), as we remark later on (see (14) below).
The difficulty of (1) lies in the non coercivity of the convection term, as it can be seen with the following heuristic argument. If $u \in W_0^{1,2}(\Omega)$ solves (3), we obtain that

$$\alpha\|u\|^2_{W_0^{1,2}(\Omega)} \le \frac{\|E\|_{L^N(\Omega)}}{\mathcal{S}_2}\|u\|^2_{W_0^{1,2}(\Omega)} + \|f\|_{L^{(2^*)'}(\Omega)}\|u\|_{L^{2^*}(\Omega)};$$

where $\mathcal{S}_2$ is the Sobolev constant relative to $W_0^{1,2}$. Thus, if the value of $\|E\|_{L^N(\Omega)}$ is large, it seems that the presence of the convection term obstructs the achievement of the standard a priori estimates. Problems like (1) are widely studied in the classical literature. We refer to [85], [65] and [88], where (1) is solved with some additional hypothesis on $E(x)$ than (4), as smallness conditions on the $L^N(\Omega)$-norm,

$$\|E\|_{L^N(\Omega)} < \mathcal{S}_2\alpha, \tag{7}$$

or sign conditions on the distributional divergence of $E(x)$,

$$\int_\Omega E(x)\nabla\phi \ge 0 \quad \forall\, \phi \in C_0^1(\Omega). \tag{8}$$

Alternatively, to restore the lack of coercivity, one can add an *absorption term* in the left hand side of (1) (see for instance [85] or the more recent [54].
One naturally wonders if such assumptions are necessary. The negative answer is given in [52] and [26] where it is proved the following result.

**Theorem 0.1** ([52], [26]). *Let us assume* (2)*, $E \in \left(L^N(\Omega)\right)^N$ and that $f \in L^m(\Omega)$ with $1 < m < \frac{N}{2}$. Then*
*(i) if $(2^*)' \le m < \frac{N}{2}$ there exists $u \in L^{m^{**}}(\Omega) \cap W_0^{1,2}(\Omega)$ solution of* (3)*;*
*(ii) if $1 < m < (2^*)'$ there exists $u \in W_0^{1,m^*}(\Omega)$ solution of* (6)*.*

Thus, not only problem (1) is solvable in $W_0^{1,2}(\Omega)$ for any vector field $E$ satisfying (4) (no matter the value of its norm), but also the same regularity result of the case $E \equiv 0$ (see [34]) is recovered, even for distributional solutions with data outside the dual space. Let us also mention [5] and [51] for similar results but with more restrictive assumptions on the summability of $E(x)$.
We stress that, even if Theorem 0.1 is stated for a linear problem, in [52], [51] and [27] a more general non linear versions of (1) is treated (see below for more details). Moreover [52] and [51] consider an equation with both convection and drift (see (15) below) first order terms, assuming a smallness condition on at least one of them. We do not treat these two lower order terms together and the reason is explained shortly.

Let us briefly describe the methods used in [26] and [52] to deal with problem (1). The strategy of the first paper hings on the following *log*-estimate

$$\int_\Omega |\nabla \log(1+|u|)|^2 \le \frac{1}{\alpha^2}\int_\Omega |E|^2 + \frac{2}{\alpha}\int_\Omega |f|. \tag{9}$$

Despite it gives poor information on the summability of $\nabla u$, it requires just $|E| \in L^2(\Omega)$ and $f \in L^1(\Omega)$. Such an estimate bypasses the non coercivity of the problem, since provides a preliminary information on the measure of the super level sets of $u$. This, together with the strategy of power test functions developed in [34], allows us

to prove Theorem 0.1 (see Subsection 2.1.1 below).
On the other hand, in [52] (see also [51]) the authors take advantage of the symmetrization technique introduced in [86]: the main idea is to deduce a differential inequality for the decreasing rearrangement of $u$ (see Chapter 1 for a brief introduction on this subject), that produces a comparison with the rearrangement of the solution of a suitable symmetrized problem. Since the solution of the symmetrized problem is explicit one recovers the a priori estimate for $u$ and,, in turn the energy estimate for the gradient.

Our main contribution about problem (1) (and its nonlinear counterpart) is to complete the relation between the regularity of $f$ and $u$ in the framework of Marcinkiewicz (and more generically Lorentz) spaces. In the case $E \equiv 0$ this is done in [25] for $f$ in Marcinkiewicz spaces (see also [63]) and in [4] for data in Lorentz spaces. The presence of the convection term totally prevents us to adapt the technique of [25], as detailed explained at the beginning of Section 2.1. On the other hand, using the symmetrization technique of [86] and [52] and inspired by [4], we obtain pointwise estimates for both $\overline{u}$ and $\overline{\nabla u}$, respectively the rearrangement of $u$ and $\nabla u$. In turn such estimates allow us to prove the following result (see Theorem 2.7).

**Theorem 0.2.** *Assume (2), $|E| \in L^N(\Omega)$ and $f \in M^m(\Omega)$ with $1 < m < \frac{N}{2}$. Hence there exists $u$ solution of (6). Moreover*

- *if $1 < m < (2^*)'$, then $u \in M^{m^{**}}(\Omega)$ and $|\nabla u| \in M^{m^*}(\Omega)$;*
- *if $(2^*)' < m < \frac{N}{2}$, then $u \in W_0^{1,2}(\Omega) \cap M^{m^{**}}(\Omega)$.*

We stress that the more interesting (and difficult) part of Theorem 0.2 is the first one, where the regularity of the gradient increases with the regularity of the datum. We have also to notice that unfortunately our approach is not sharp enough to cover $m = (2^*)'$. This borderline case has been recently solved by [76] if $E \equiv 0$, using non standard (nonlinear) potential arguments.
The estimate obtained for $\overline{u}$ is

$$\overline{u}(t) \le \frac{\mathcal{C}_1}{t^\gamma} \int_t^{|\Omega|} s^{\frac{2}{N}-1+\gamma} \tilde{f}(s)ds, \quad \text{for} \quad t \in (0, |\Omega|), \tag{10}$$

for any $\gamma < \frac{1}{2m^{**}}$, with $\mathcal{C}_1 = \mathcal{C}_1(\alpha, N, E, m, \gamma)$ a positive constant and

$$\tilde{f}(s) = \frac{1}{s} \int_0^s \overline{f}(t)dt.$$

As said, estimates of this type are already known in the literature (see [20] and [52]) and they are particularly well designed to prove the membership of $u$ to Marcinkiewicz (or Lorentz) spaces. In order to better understand (10) let us set

$$v(x) = \frac{\mathcal{C}_1}{(\omega_N |x|^N)^\gamma} \int_{\omega_N |x|^N}^{|\Omega|} s^{\frac{2}{N}-1+\gamma} \tilde{f}(s)ds \quad \text{with} \quad \omega_N = |B_1|$$

and notice that it solves

$$\begin{cases} -\Delta v = \mathcal{C}_2 \text{div}\left(v \dfrac{x}{|x|^2}\right) + \mathcal{C}_3 \tilde{f}(\omega_n |x|^N) & \text{in } B_\Omega, \\ \qquad v = 0 & \text{on } \partial B_\Omega, \end{cases} \tag{11}$$

where $B_\Omega$ is the ball centered at the origin sucht that $|B_\Omega| = |\Omega|$ and $\mathcal{C}_2, \mathcal{C}_3$ positive constants. Thus (10) reads as $\overline{u}(t) \le \overline{v}(t)$, namely the already mentioned comparison between the rearrangements of the solutions of the original problem and the symmetrized one.
Let us focus now on the estimate for $\overline{|\nabla u|}$, the rearrangement of the gradient. This part is more involved and represents the main novelty of the Theorem. Indeed in the literature there are no results on the Marcinkiewicz (or Lorentz) regularity for the gradient of the solution of problem (1).

Notice that we cannot hope to derive any information on the regularity of $|\nabla u_n|$ (10). This is because the symmetrization processes transforms an elliptic operator with measurable coefficients into a more regular one: the problem solved by $v(x)$ involves exactly the Laplacian as principal operator and hence $|\nabla v|$ is much more regular than $|\nabla u|$. We already said that it seems not possible to follows [25] and hence we developed an alternative approach similar to the one proposed in [4]. We provide the following pointwise estimate for $s \in (0, |\Omega|)$

$$\frac{1}{s}\int_0^s \overline{|\nabla u_n|} \leq \mathcal{C}_4 \left[ \frac{1}{s}\int_0^s (\overline{v}(t)D^{\frac{1}{2}}(t) + \tilde{f}t^{\frac{1}{N}})dt + \left( \frac{1}{s}\int_s^{|\Omega|} (\overline{v}(t)^2 D(t) + \tilde{f}^2 t^{\frac{2}{N}})dt \right)^{\frac{1}{2}} \right] \tag{12}$$

where $D(t) \in L^{\frac{N}{2}}(0, |\Omega|)$ is the so called pseudo-rearrangement of $|E|^2$ with respect to $u$ (see [57] and Lemma 1.5 in Chapter 1 for the definition of pseudo-rearrangement). The key observation in the achievement of (10) is that (see the proof of Lemma 2.6) for $s \in (0, |\Omega|)$

$$\int_0^s \overline{|\nabla u|} d\tau \leq \int_{\{|u|>\overline{u}(s)\}} |\nabla u| dx + \left( s \int_{\{|u|\leq \overline{u}(s)\}} |\nabla u|^2 dx \right)^{\frac{1}{2}}. \tag{13}$$

This information, coupled with (10) and with the differential inequality satisfied by $\int_{\{|u|>\overline{u}(s)\}} |\nabla u|^2 dx$ (see [4]), allows us to obtain (12). The achievement of the estimates (10) and (12) is the core of the proof of Theorem 0.2.

As a matter of fact, in dealing with problem (1), one can consider a slightly more general assumption than (4). Indeed in [20] and [28] problem (1) is treated assuming $|E| \in M^N(\Omega)$, with smallness condition on $\|E\|_{M^N(\Omega)}$. Following the previously outlined strategy, we prove (see Theorem 2.10) that the same results of Theorem 0.2 continues to hold if $E$ is such that

$$E = \mathcal{F} + \mathcal{E} \quad \text{with} \quad \mathcal{F} \in (L^\infty(\Omega))^N \quad \text{and} \quad \overline{\mathcal{E}}(s) \leq \frac{B}{s^{\frac{1}{N}}} \quad \text{with} \quad B < \alpha\omega_N^{\frac{1}{N}} \frac{N-2m}{m}. \tag{14}$$

Notice that the lower order term in (3) and (6) is well defined under assumption (14) thanks to the sharp Sobolev Embedding in Lorentz spaces (see [87] and reference therein)

$$W_0^{1,q}(\Omega) \subset L^{q^*,q}(\Omega) \quad \text{with} \quad 1 \leq q < \infty.$$

Of course if $E$ satisfies (4), it also satisfies (14). The previous assumption, up to the addition of a whichever bounded vector filed, prescribes a threshold on the $M^N(\Omega)$-norm of $E$. It seems that this smallness condition is sharp and cannot be weakened. It is also interesting to note that (14) is less and less restrictive as $m$ tends to 1.

The structure of (10) and (12) suggests to consider not only Marcikiewcz but also more general Lorentz data (see Section 1.1 for the formal definition). Moreover the techniques used to prove Theorem 0.2 do not require essentially the linearity of the operator. Inspired by the existing literature (see as an example [20], [52] and [27]), it is thus natural to extend our results to more general nonlinear problems like

$$\begin{cases} -\text{div}\left(A(x)|\nabla u|^{p-2}\nabla u\right) = -\text{div}\left(u|u|^{p-2}E(x)\right) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

where $1 < p < N$, $A(x)$ satisfies (2), $E$ belong to $\left(L^{\frac{N}{p-1}}(\Omega)\right)^N$ and $f$ belongs to a suitable Lorentz space to be defined in the sequel. One can indeed assume the $p$-version of (14) for $E(x)$ and consider a more general Leray-Lions principal operator (see Section 4.1 for more details). It is well known (see [17]) that, if $p$ is small, some additional difficulties may arise in dealing with (15) even for $E \equiv 0$. Roughly speaking this is because the gradient of the expected solution might not be an integrable function. Here we avoid the treatment of this

situation, considering always distributional solutions whose gradient is at least in $L^1(\Omega)$. The interested reader is referred to [28] for entropy formulations of problems with convection first order term.
For the complete results concerning problem (15) see Theorems 4.1 and 4.2. Wishing to give a schematic overview of them, we have the following result

**Theorem 0.3.** *Let $1<p<N$, assume $E\in\left(L^{\frac{N}{p-1}}(\Omega)\right)^N$ and $f\in L^1(\Omega)$. Then*

- *if $f\in L^{m,q}(\Omega)$ with $\max\{1,\frac{N}{N(p-1)+1}\}<m<(p^*)'$ and $0<q\leq+\infty$, then there exists $u$ distributional solution of (15) with $|\nabla u|\in L^{(p-1)m^*,(p-1)q}(\Omega)$;*

- *if $p>2-\frac{1}{N}$ and $f\in\mathbb{L}^{1,q}(\Omega)$ with $0<q\leq+\infty$, then there exist $u$ distributional solution of (15) with $|\nabla u|\in L^{\frac{(p-1)N}{N-1},(p-1)q}(\Omega)$;*

- *if $p>2-\frac{1}{N}$ and $f\in L^1(\Omega)$, then there exist $u$ distributional solution of (15) with $|\nabla u|\in L^{(p-1)m^*,\infty}(\Omega)$;*

- *if $p=2-\frac{1}{N}$ and $f\in\mathbb{L}^{1,q}(\Omega)$ with $0<q\leq\frac{1}{p-1}=\frac{N}{N-1}$, then there exists $u$ distributional solution of (15) with $|\nabla u|\in L^{1,(p-1)q}(\Omega)$;*

- *if $p<2-\frac{1}{N}$ and $f\in L^{m,q}(\Omega)$ with $m=\frac{N}{N(p-1)+1}$ and $0<q\leq\frac{1}{p-1}$, then there exists $u$ distributional solution of (15) with $|\nabla u|\in L^{1,(p-1)q}(\Omega)$.*

For the definition of the Lorentz Spaces $L^{m,q}(\Omega)$ and $\mathbb{L}^{1,q}(\Omega)$ see Chapter 1 below.
Also in this nonlinear framework, we recover exactly the same relationship between the regularity of $f$ and $\nabla u$ proved in [4], [34] and [76] without the convection term, namely $E\equiv 0$. Let us notice that we cover also the more difficult case $p=2-\frac{1}{N}$ and $f\in\mathbb{L}^{1,q}(\Omega)$ with $0<q\leq\frac{1}{p-1}=\frac{N}{N-1}$ (see [35] in the case $E\equiv 0$). On the other hand we stress again that the borderline value $m=(p^*)'$ remains open (see [76] for the case $E\equiv 0$).

**2.** Let us focus now on

$$\begin{cases}-\mathrm{div}(A(x)\nabla w)=\nabla w\,E(x)+f(x) & \text{in } \Omega,\\ w=0 & \text{on } \partial\Omega,\end{cases}\tag{16}$$

with the measurable function $A(x)$ satisfying (2), $E\in\left(L^N(\Omega)\right)^N$ and $f$ that belongs to a Lorentz space to be define later. The first order term in the equation above is also called *drift* term. In this linear setting (16) is (at least formally) the dual problem of (1) and one can use a duality approach to recover existence and regularity results (see [3], [54], [29], [31]). Anyway here we treat problem (16) independently from (1), following the same spirit and aims of the previous *convection* case. For $f(x)$ belonging to $L^{(2^*)'}(\Omega)$ or $L^m(\Omega)$ with $1<m$, we consider weak and distributional formulations of (16) respectively, namely

$$w\in W_0^{1,2}(\Omega)\ :\ \int_\Omega A(x)\nabla w\nabla\phi=\int_\Omega \nabla wE(x)\phi+\int_\Omega f(x)\phi\quad\forall\,\phi\in W_0^{1,2}(\Omega)\tag{17}$$

and

$$w\in W_0^{1,r}(\Omega)\ :\ \int_\Omega A(x)\nabla w\nabla\phi=\int_\Omega \nabla wE(x)\phi+\int_\Omega f(x)\phi\quad\forall\,\phi\in C_0^1(\Omega),\tag{18}$$

with $r>\frac{N}{N-1}$. Notice that we have to impose that $w\in W_0^{1,r}(\Omega)$ in order to have the lower order term of (18) well defined. Similarly to the convection term, also the drift term makes the operator of (16) not coercive, unless an additional smallness assumption on the $L^N(\Omega)$ norm of $E(x)$ is assumed. Once again it is proved that such assumption is unnecessary for the existence of a weak solution. For the next result we refer mainly to [20] and the already cited [52] (see also [5], [50] and [51]).

**Theorem 0.4** ([20],[52]). *Let us assume* (2), $E \in \left(L^N(\Omega)\right)^N$ *and that* $f \in L^{(2^*)'}(\Omega)$. *Hence there exists* $w$ *solution of* (17).

In [20] the authors obtain energy estimates for (17) by means of a *slice method* that is based on continuity properties of some modified distribution function of $w$ (see Proposition 3.1 and Lemma 3.2). We also cite [21] where the slice method is used to treat (16) with measure data. As in the case of a convection lower order term, Theorem 0.4 (ad its nonlinear counterpart) is proved in [52] by means of symmetrization techniques.

The first original result that we present for problem (16) uses the slice method of [20] and the power test functions of [25] to generalize Theorem 0.4 as follows (see also [50] for the same result with more restrictive assumptions on (16)).

**Theorem 0.5.** *Let us assume* (2), $E \in \left(L^N(\Omega)\right)^N$ *and that* $f \in L^m(\Omega)$ *with* $1 < m < \frac{N}{2}$.
*(i) If* $(2^*)' \leq m < \frac{N}{2}$ *there exists* $w \in L^{m^{**}}(\Omega) \cap W_0^{1,2}(\Omega)$ *solution of* (17).
*(ii) If* $1 < m < (2^*)'$ *there exists* $w \in W_0^{1,m^*}(\Omega)$ *solution of* (18).

For the proof of this result see Section 3.1.1.

The next step is to adapt the technique developed for problem (1) to recover Marcinkiewicz and Lorentz regularity results also for (16). We are able to obtain the following pointwise estimates for $\overline{w}$ and $\overline{|\nabla w|}$, the decreasing rearrangements of $w$ and $\nabla w$. We have

$$\overline{w}(\tau) \leq \mathcal{C}_5 \int_\tau^{|\Omega|} t^{\frac{2}{N}-2+\gamma} \int_0^t \overline{f}(s) s^{-\gamma} ds dt \tag{19}$$

and

$$\frac{1}{\tau}\int_0^\tau \overline{|\nabla w|} \leq \mathcal{C}_6 \left[ \frac{1}{\tau}\int_0^\tau s^{\frac{1}{N}-1+\gamma} \left( \int_0^s t^{-\gamma}\overline{f}(t)dt \right) ds + \left( \frac{1}{\tau}\int_\tau^{|\Omega|} s^{2\left(\frac{1}{N}-1+\gamma\right)} \left( \int_0^s t^{-\gamma}\overline{f}(t)dt \right)^2 ds \right)^{\frac{1}{2}} \right], \tag{20}$$

where $\gamma = \frac{1}{2m'}$. All the considerations on the comparison with the rearrangement of the symmetrized problem hold true also for (19) and the starting point in obtaining estimate (20) is always (13), the literature to which we refer it is the same too.
The relative existence and regularity result is the following one (see Theorem 3.10).

**Theorem 0.6.** *Let us assume* (2), $E \in \left(L^N(\Omega)\right)^N$ *and* $f \in M^m(\Omega)$ *with* $1 < m < \frac{N}{2}$. *Then there exists* $w$ *solution of* (18). *Moreover*

- *if* $1 < m < (2^*)'$, *then* $w \in M^{m^{**}}(\Omega)$ *and* $|\nabla w| \in M^{m^*}(\Omega)$,
- *if* $(2^*)' < m < \frac{N}{2}$, *then* $w \in W_0^{1,2}(\Omega) \cap M^{m^{**}}(\Omega)$.

Of course one can also treat the nonlinear version of problem (16) with $f$ in a Lorentz space and $E$ in the Macinkiewicz space of order $N$ (see [19] and [69]). We refer to Section 4.2 for the precise statements of the results and the relative proofs. Here we just report the assumption equivalent to (14) for $E(x)$.

$$E = \mathcal{F} + \mathcal{E} \quad \text{with} \quad \mathcal{F} \in (L^\infty(\Omega))^N \quad \text{and} \quad \overline{\mathcal{E}}(s) \leq \frac{B}{s^{\frac{1}{N}}} \quad \text{with} \quad B < \alpha\omega_N^{\frac{1}{N}} N\frac{m-1}{m}.$$

It is immediate to note that this assumption becomes more and more restrictive as $m$ approaches 1. This is not just a technical inconvenient and prevent us to treat the case $f$ in a Lorentz space with first coefficient 1. Indeed for such type of data the expected regularity of the gradient is too low to have the drift term of (16) well defined (see [50] and [21]).

After studying problem (1) and (16) separately, one natural question is why to not consider the convection and the drift term at once. This is what is actually done in [85], [88] and [52] but always imposing some additional constraints, as smallness assumption on the $L^N$ norm of at least one of the coefficients or divergence free assumption like (8). One may wonder if, also in this case, these are just technical assumptions, or rather the presence of the two first order term represents a genuine obstruction to the solvability of problems like

$$\begin{cases} -\text{div}(A(x)\nabla u) = -\text{div}\,(uE(x)) + \nabla u B(x) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{21}$$

Let us observe that, assuming $E(x)$ and $B(x)$ equal and regular, say $C^1(\Omega)$, problem (21) becomes

$$\begin{cases} -\text{div}(A(x)\nabla u) = g(x)u + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

with $g(x) = -\text{div}(E(x))$, that of course is not solvable for a general $g(x)$. Thus the presence of the two lower order term involves some spectral issues and we decided to not treat it.

**3.** Let us finally consider the following general elliptic nonlinear problem in divergence form

$$\begin{cases} -\text{div}(a(x,u,\nabla u)) + b(x,u,\nabla u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{22}$$

The datum $f$ belongs to some suitable Lebesgue space to be defined later and the Carathéodory functions $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^{\mathbb{N}}$ and $b : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy the standard structural assumptions of a Leray-Lions operator, namely for $1 < p < \infty$, $0 < \alpha \leq \beta$ and $0 \leq \beta_0$,

$$\begin{rcases} |a(x,s,\xi)| \leq \beta\left(|s|^{p-1} + |\xi|^{p-1}\right) \\ |b(x,s,\xi)| \leq \beta_0\left(|s|^{p-1} + |\xi|^{p-1}\right) \end{rcases} \text{growth conditions,}$$

$$\begin{rcases} a(x,s,\xi)\xi \geq \alpha|\xi|^2 \\ b(x,s,\xi)s \geq 0 \end{rcases} \text{coercive conditions,}$$

$$(a(x,s,\xi) - a(x,s,\xi^*))\,(\xi - \xi^*) > 0, \quad \xi \neq \xi^* \quad \text{monotonicity condition.}$$

In the seminal paper [66], it is proved that the operator

$$\begin{aligned} \mathcal{A} \ : \ W_0^{1,p}(\Omega) \to \ & W^{-1,p'}(\Omega) \\ u \to \ & -\text{div}(a(x,u,\nabla u)) + a_0(x,u,\nabla u) \end{aligned}$$

is well defined, coercive and monotone; hence (22) admits a (unique) weak solution for any $f \in W^{-1,p'}(\Omega)$. To keep in mind a concrete example one can set, for instance,

$$a(x,u,\nabla u) = A(x)|\nabla u|^{p-1}\nabla u \quad \text{and} \quad b(x,u,\nabla u) = B(x)\left(u|u|^{p-2} + \frac{u}{1+|u|}|\nabla u|^{p-1}\right),$$

with $A(x)$ satisfying (2) and $|B(x)| \leq \beta_0$ in $\Omega$. Such a result can be generalized in two directions. The first generalization concerns data outside the the dual space. The main difficulty is to find proper notions of solutions

that assure uniqueness, that fails for distributional solutions (see [84] for Serrin Counterexample), and give sense to problem (22) for small values of $p$. Indeed for $p$ close to 1 the gradient of the expected solution is not in general an integrable function. These two issues are treated at once in [17] and [36], where the notion of entropy solution is introduced and problems like (22) are uniquely solved for $L^1(\Omega)$ data or even for a more general class of measures (see also [48] for the equivalent notion of renormalized solution).
The second direction focuses on more general growth conditions on $a(x, u, \nabla u)$ and $b(x, u, \nabla u)$, that make the resulting differential operator not anymore well defined between $W_0^{1,p}(\Omega)$ and its dual. Consider the following $p$-growth with respect to the gradient for the lower order term

$$b(x, u, \nabla u) = D(x)u|\nabla u|^p, \tag{23}$$

with $D(x) \in L^\infty(\Omega)$. The literature concerning this type of first order terms, with *natural* growth and sign condition, is broad and it is well know that the presence of (23) gives rise to regularizing effects (see for example [34], [37], [40], the monograph [33] and reference therein). We also quote [43], [38], [46] (see also [78]) for purely semilinear lower order term with sing condition.
As far as the principal part of the operator is concerned, the authors of [68] and [80] propose a polynomial growth with respect to the $u$-variable of the type

$$a(x, u, \nabla u) = A(x)(1 + |u|^r)|\nabla u|^{p-2}\nabla u, \quad \text{with} \quad r > 0$$

and the measurable function $A(x)$ satisfying (2). The difficulty here is that a priori there is no reason for the function $a(x, u, \nabla u)$ to belong to $\left(L^1(\Omega)\right)^N$, namely the term $-\mathrm{div}(a(x, u, \nabla u))$ might not even have a distributional sense.

Note at this point that all the previous structural assumptions imply that both $a(x, u, \nabla u)$ and $b(x, u, \nabla u)$ are bounded with respect to the $x$ variable. Our aim is to pass from this $L^\infty$-setting to a general $L^1$-setting, namely we want to consider problem like (22) with a $x$-dependence expressed through $L^1(\Omega)$ coefficients.

For the sake of simplicity let us focus at first on the following linear model

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{24}$$

Assuming the ellipticity condition (2) and that $f \in L^{(2^*)'}(\Omega) \subset W^{-1,2}(\Omega)$, it is straightforward (Lax-Milgram Theorem) to prove that there exists a weak solution of (24), namely

$$u \in W_0^{1,2}(\Omega) \ : \ \int_\Omega A(x)\nabla u \nabla \phi = \int f\phi \quad \forall \phi \in W_0^{1,2}(\Omega). \tag{25}$$

Our first aim is to generalize this result assuming, instead of (2), that for $\alpha > 0$

$$\alpha \le A(x), \quad A \in L^1(\Omega). \tag{26}$$

The first step is to give an appropriate notion of solution to our problem, since the first term of (25) is not well defined for the unbounded coefficients $A(x)$. Hence we define

$$X_0^2(\Omega) := \left\{ \varphi \in W_0^{1,2}(\Omega) \quad \text{such that} \quad \int_\Omega A(x)|\nabla \varphi|^2 < \infty \right\}$$

Notice that $C_0^1(\Omega) \subset X_0^2(\Omega)$. We say that $u$ is a solution of (24) if

$$u \in X_0^2(\Omega) \ : \ \int_\Omega A(x)\nabla u \nabla \phi = \int f\phi \quad \forall \phi \in X_0^2(\Omega). \tag{27}$$

Through an approximation procedure it is not difficult, in this linear setting, to prove the following result (see Theorem 5.1).

**Theorem 0.7.** *If $f$ belongs to $L^{(2^*)'}(\Omega)$ and $A(x)$ satisfies* (26) *there exists a solution of* (27).

Indeed this result is already known in the literature and it was proven by Trudinger in [88] for a complete linear elliptic operator (see also [6] for related results obtained via simmetrization techniques and the more recent [32]). Let us point out that the method used in [88] is essentially linear and relies on a weighted functional framework. On the contrary our approach is more direct and it is based on an intrinsic approximation procedure that does not require the use of weighed Sobolev spaces. Indeed in the proof we do not assume a priory that the solution satisfies $A(x)|\nabla u|^2 \in L^1(\Omega)$. Rather this information follows from the equation as a sort of *regularizing effect*. The advantage of such a strategy is that it can be adapted to deal with more general non linear problems like

$$\begin{cases} -\mathrm{div}\big(A(x)(1+|u|^r)|\nabla u|^{p-2}\nabla u\big) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{28}$$

with $r \geq 0$, $1 < p < N$, $A(x)$ now satisfying (26) and $f$ belonging to some Lebesgue space to be defined later (see assumptions 5.17 for a more general non linear operator in the spirit of [68]). In order to state the following existence result, let us define

$$X_0^p(\Omega) := \left\{ \varphi \in W_0^{1,p}(\Omega) \quad \text{such that} \quad \int_\Omega A(x)|\nabla\varphi|^p < \infty \right\}.$$

**Theorem 0.8.** *Assume that $1 < p < \infty$, $r \geq 0$ and* (26)*. Then, if $f \in L^{(p^*)'}(\Omega)$, there exists a solution $u$ of* (28) *in the following weak sense*

$$u \in X_0^p(\Omega), \quad A(x)|u|^{rp'}|\nabla u|^p \in L^1(\Omega) \quad \textit{and}$$
$$\int_\Omega A(x)(1+|u|^r)|\nabla u|^{p-2}\nabla u \nabla\phi = \int f\phi \quad \forall \phi \in X_0^p(\Omega).$$

*If $f \in L^1(\Omega)$ there exists a solution $u$ of* (28) *in the following entropy sense*

$$\forall\, k > 0 \;\; T_k(u) \in X_0^p(\Omega), \quad A(x)|u|^{rp'}|\nabla T_k(u)|^p \in L^1(\Omega) \quad \textit{and}$$
$$\int_\Omega A(x)(1+|u|^r)|\nabla u|^{p-2}\nabla u \nabla T_k(u-\phi) = \int f T_k(u-\phi) \quad \forall \phi \in X_0^p(\Omega) \cap L^\infty(\Omega),$$

*where $T_k$ denotes the truncation at level $k$.*

The core of the proof is the same of the one of Theorem 0.7, i.e. we obtain suitable a priori estimates for a sequence of approximating solution. The main difficulties here are given by the nonlinearity of the operator and they require to adapt some technical tools to this $L^1$ setting, as the Minty Lemma (see [40]) and the almost everywhere convergence of the gradients (see [24]).
Finally we use the techniques developed for solving (25) and (28) for studying problems with semilinear or quasilinear lower order terms in the following form (see respectively Theorems 5.2 and 5.4)

$$b(x, u, \nabla u) = B(x)|u|^{q-2}u \quad (q > 1) \quad \text{or} \quad b(x, u, \nabla u) = D(x)u|\nabla u|^p$$

with

$$\delta A(x) \leq B(x) \quad \text{and} \quad \tau A(x) \leq D(x) \leq \sigma A(x) \quad \text{for} \quad \delta, \tau, \sigma > 0.$$

The presence of these $L^1(\Omega)$-lower order terms with sign condition gives rise to regularizing effects in the same spirit of [46] and [34].

# Qualitative behaviour of large solutions

This second part of the Thesis is devoted to the study of semilinear elliptic problems with explosive boundary conditions; more precisely we are interested in the qualitative behaviour of solutions of problem

$$\begin{cases} -\Delta u + g(u) = f & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{29}$$

where $\Omega$ is a bounded smooth domain of $\mathbb{R}^N$, with $N \geq 1$, $g \in C^1(\mathbb{R})$ is such that

$$g(a) > 0 \quad \text{for some} \quad a \in \mathbb{R} \quad \text{and} \quad g'(s) > 0 \quad \text{for every} \quad s \in \mathbb{R}, \tag{30}$$

and $f$ is a Lipschitz continuous function. Here solutions are meant in the classical sense, i.e. $C^2(\Omega)$ functions which satisfy the differential equation above pointwise and such that

$$\lim_{x \to \partial\Omega} u(x) = +\infty.$$

In the literature solutions that blow-up at the boundary of the domain are known as *large solutions*. Looking naively at (29) one naturally wonders under which assumptions on $g$ the existence of a large solution is assured, if the monotonicity assumption on $g$ implies uniqueness of solution and how such a solution behaves near the boundary.

In the seminal works by Keller and Osserman (see [62] and [79]) it is proved that the necessary and sufficient condition for the existence of a large solution for problem (29) is the following:

$$\exists\, t_0 \in [-\infty, +\infty) \;:\; \psi(t) := \int_t^\infty \frac{ds}{\sqrt{2G(s)}} < \infty \quad \text{for} \quad t > t_0, \quad \text{where} \quad G'(s) = g(s). \tag{31}$$

This growth condition at infinity, known as Keller-Osserman condition, arises solving the one dimensional problem

$$-\phi'' + g(\phi) = 0, \quad s > 0 \quad \text{and} \quad \lim_{s \to 0^+} \phi(s) = +\infty. \tag{32}$$

We stress that, in fact, $\phi(s) = \psi^{-1}(s)$ solves problem (32). We refer the interested reader to [55] (see also the references cited therein) for existence issues with no monotonicity assumptions on $g$.

Uniqueness is not a trivial task in the sense that it is not known if the monotonicity of $g$ is a sufficient condition for it; we refer to [74], where it is proved that if $g$ is convex then (29) admits a unique large solution, and to [58] (see also [12]), where it is shown that assumptions of the type

$$\frac{g(t)}{t^q} \quad \text{increasing for } t \gg 1 \text{ and some } q > 1$$

imply uniqueness of large solution. It is worthy to mention that the special case $g(s) = |s|^{p-1}s$ with $p > 1$ satisfies the latter condition.

Let us point out now that the function $\phi$ defined in (32) is strongly related with the boundary behaviour of solutions of (29). In [11] and [12] it has been proved that the behaviour of $u$ is, in some sense, *one dimensional* near the boundary, i.e. it holds that

$$\lim_{d(x) \to 0} \frac{\psi(u(x))}{d(x)} = 1 \quad \text{where } d(x) = \text{dist}(x, \partial\Omega).$$

Moreover if $g$ is such that

$$\liminf_{t \to \infty} \frac{\psi(\beta t)}{\psi(t)} > 1, \quad \forall\, \beta \in (0, 1), \tag{33}$$

then

$$\Big| u(x) - \phi(d(x)) \Big| = o(\phi(d(x))) \quad \text{as } d(x) \to 0, \tag{34}$$

namely the first order term in the asymptotic of $u$ near the boundary only depends on the corresponding ODE (32) and in particular is not affected by the geometry of the domain. In [14] the authors improve (34); assuming in addition to (33) that

$$\frac{G(s)}{s^2} \quad \text{is strictly increasing for large s} \quad \text{and} \quad \limsup_{\beta \to 1, s \to 0} \frac{\phi'(\beta s)}{\phi'(s)} < \infty$$

they prove that

$$|u(x)-\phi(d(x))|\leq c\phi(d(x))d(x)\quad \text{as } d(x)\to 0,$$

where the positive constant $c$ depends on the mean curvature of the boundary of $\Omega$. After this first clue, the influence of the geometry of $\partial\Omega$ in the expansion of $u$ has been studied in [67] and [49] under different assumptions on $g$. The most general result in this direction has been proved in [15]; in order to state it we need to define

$$J(s):=\frac{N-1}{2}\int_0^s \Gamma(\phi(t))dt,\quad \text{where}\quad \Gamma(t):=\frac{\int_0^t\sqrt{2G(s)}ds}{G(t)}$$

and to assume that

$$\lim_{\delta\to 0}\frac{B(\phi(\delta(1+o(1))))}{B(\phi(\delta))}=1\quad\text{and}\quad \limsup_{t\to\infty}B(t)\Gamma(t)<\infty, \tag{35}$$

where

$$B(s):=\frac{d}{dt}\sqrt{2G(s)}=\frac{g(s)}{\sqrt{2G(s)}}.$$

Assuming (35), together with (31) and (33), it follows that

$$\left|u(x)-\phi\big[d(x)-H(x)J(d(x))\big]\right|\leq \phi(d(x)o(d(x))\quad \text{as } d(x)\to 0, \tag{36}$$

where $H$ is a smooth function whose restriction to $\partial\Omega$ coincides with the mean curvature of the domain; moreover it is worth stressing that (35) implies

$$J(d(x))=O(d^2(x)).$$

The relation above, together with (36), tells us that the second order contribution to the explosion of $u$ is affected by the geometry of the domain through the mean curvature of $\partial\Omega$.
More recently in [47] (see also [22]), by means of an interesting application of the contraction theorem, all the singular terms of the asymptotic of $u$ have been implicitly calculated in the special case $\Omega=B$.

For power type nonlinearities it is also possible to obtain the first asymptotic of the gradient of the solution by means of scaling arguments. In particular in [11] and [13] (see also [81]) it is proved that if

$$\lim_{s\to\infty}\frac{g(s)}{s^p}=1\quad\text{for some}\quad p>1,$$

it holds true that

$$\left|\frac{\partial u(x)}{\partial\nu}-\frac{\partial\phi(d(x))}{\partial\nu}\right|+\left|\frac{\partial u(x)}{\partial\tau}\right|\leq o(\phi'(d(x)))\qquad \text{as } d(x)\to 0, \tag{37}$$

where $\nu$ is the unit normal to $\partial\Omega$ (recall that $\nu(\bar{x})=-\nabla d(\bar{x})$ for $\bar{x}\in\partial\Omega$) and $\tau\in\mathbb{S}^{N-1}$ is such that $\tau(\bar{x})\cdot\nu(\bar{x})=0$ for every $\bar{x}\in\partial\Omega$. However, a general result for the second order term in the expansion of $\nabla u$ in the same spirit of (36) it is not available in the literature (see anyway [10] for a partial result in convex domains).

Our aim is to complete the picture of the asymptotic behaviour of the gradient of solutions of problem (38) in the case $g(s)=|s|^{p-1}s$, with $p>1$ and $\Omega$ smooth enough. Thus the problem we deal with is

$$\begin{cases}-\Delta u+|u|^{p-1}u=f & \text{in }\Omega,\\ u=+\infty & \text{on }\partial\Omega,\end{cases} \tag{38}$$

where $f\in W^{1,\infty}(\Omega)$. It is easy to verify that with such a choice of $g$, assumptions (31)–(35) are satisfied. It is also worth to recall that in this case problem (38) has a unique large solution and that the function $\phi$ defined in (32) has the following explicit form

$$\phi(s)=\frac{\sigma_0}{s^\alpha}\quad\text{with}\quad \alpha=\frac{2}{p-1}\quad\text{and}\quad \sigma_0=[\alpha(\alpha+1)]^{\frac{1}{p-1}}. \tag{39}$$

The result that we present in this paper will describe not only the second order behaviour of the gradient of the large solution of (38), but also the complete asymptotic expansion of all the singular terms of $u$ and $\nabla u$, for every arbitrary sufficiently smooth domain and every $p > 1$. As a byproduct of this expansion we will be able to provide the expected second order asymptotic for the normal and tangential components of $\nabla u$ with respect to $\partial\Omega$. Indeed we will prove

$$\begin{aligned} \lim_{x\to\bar{x}} \left[ d^\alpha(x)\frac{\partial u(x)}{\partial \nu} - \alpha\sigma_0 d(x)\right] &= c(\alpha, N)H(\bar{x}) \\ \lim_{x\to\bar{x}} d^\alpha(x)\frac{\partial u(x)}{\partial \tau} &= 0 \end{aligned} \quad \text{uniformly with respect to } \bar{x}\in\partial\Omega. \tag{40}$$

where $c(\alpha, N)$ is a precise constant that depends only on $\alpha$ and $N$ (see Corollary 6.4 for more details). More in general we will be able to prove (see Theorem 6.3 for the precise statement) that there exists a unique explicit function $S$, sum of $[\alpha] + 1$ singular terms where $\alpha$ is as in (39), such that

$$z := u - S \in W^{1,\infty}(\Omega).$$

Let us say that the formula above expresses the leitmotiv of the paper, that is try to find an explicit simple corrector function that describes the explosive behaviour of $u$.

Actually our method allows us to prove that the function $z$ satisfies the following boundary condition

$$z(\bar{x}) = 0 \quad \text{and} \quad |\nabla z(\bar{x})| = 0 \quad \forall\, \bar{x}\in\partial\Omega.$$

See Theorem 6.3 for more details.

Finally we consider a more general class of nonlinearities that will be easily treated with an extension of our method.

## Plan of the Thesis

Let us briefly describe the plan of the Thesis.

**Chapter 1.** In this first Chapter we introduce useful notations and tools about the theory of decreasing rearrangements and others preliminary results related with Chapters 2-4.

**Chapter 2.** Here we deal with problems with a convection first order term with data in Lebesgue or Marcikiewicz spaces. We present the known results for the proof of the existence and regularity of a solution with Lebesgue data and explain why it cannot be adapted to the Marcinkiewicz framework. Hence we introduce and develop our strategy, based on pointwise estimates of the rearrangement of both the solution and its gradient, to solve the issue of regularity for $f \in M^m(\Omega)$.

**Chapter 3.** It is the *twin* Chapter of the previous one. We adapt the aforementioned strategy to prove the Marcinkiewic regularity of problems with drift first order term. For the sake of completeness we also provide an alternative, *rearrangements free*, approach for data in Lebesgue space.

**Chapter 4.** In Chapter 4 we generalize the results obtained in Chapters 2 and 3 to more general nonlinear operators.

**Chapter 5.** In this Chapter we focus on problems with $L^1$ coefficients. We define suitable notions of energy and entropy solutions in the $L^1$ setting and prove the existence of such solution. Moreover we analyze the interplay between the principal part of the operator and the lower order terms and the regularizing effect that can be

obtained from this interaction.
All its content can be found in the article [44].

**Chapter 6.** This final Chapter is devoted to the study of the asymptotic behaviour of large solutions to a class of semilinear problems. In particular we give a precise description of all the singular terms in the asymptotic expansion of the gradient of the solution.
All its content can be found in the article [45].

# Chapter 1

# Preliminary results

In this Chapter we collect some preliminaries about problems with convection or drift lower order term. In Section 1.1 we give the main definitions and tools about the theory of *decreasing rearrangement* introduced in the seminal paper [86]. In Section 1.2 we prove the *almost everywhere convergence of the gradients* for suitable sequence of approximation solutions, generalizing to our setting the result by [24].

## 1.1 Rearrangements and relevant functions spaces

For any measurable function $v : \Omega \to \mathbb{R}$ we define the *distribution function* of $v$ as

$$A(t) := |\{x \in \Omega \,:\, |v(x)| > t\,\}| \quad \text{for } t \geq 0,$$

and the *decreasing rearrangement* of $v$ as

$$\overline{v}(s) := \inf\{t \geq 0 \,:\, A(t) < s\} \quad \text{for } s \in [0, |\Omega|].$$

By construction it follows that

$$|\{x \in \Omega \,:\, |v(x)| > t\}| = |\{s \in \mathbb{R} \,:\, \overline{v}(s) > t\}|, \tag{1.1}$$

namely the function and its decreasing rearrangement are equimeasurable. We define also the *maximal function* associated to $\overline{v}$, namely

$$\tilde{v}(s) = \frac{1}{s}\int_0^s \overline{v}(t)dt.$$

Notice that, since $\overline{v}(s)$ is non increasing, it follows that $\overline{v}(s) \leq \tilde{v}(s)$ for any $s \in [0, |\Omega|]$.
By definition $A(t)$ is right continuous non increasing, while $\overline{v}(s)$ is left continuous non increasing. Thus both functions are almost everywhere differentiable in $(0, |\Omega|)$.
For a more detailed treatment of $A(t)$ and $\overline{v}(s)$ we refer to [77] and [60]. We recall here the following property of decreasing rearrangements.

**Proposition 1.1.** *For $n \in \mathbb{N}$, let $v, v_n : \Omega \to \mathbb{R}$ be measurable functions such that*

$$|v(x)| \leq \liminf_{n\to\infty} |v_n(x)| \quad a.e.\ x \in \Omega.$$

*Hence*

$$\overline{v}(s) \leq \liminf_{n\to\infty} \overline{v}_n(s) \quad a.e.\ s \in (0, \Omega).$$

*Proof.* For the proof see [60] Proposition 1.4.5. □

Let us state and prove the following Propositions.

**Proposition 1.2.** *For almost every $s \in (0, |\Omega|)$*

$$A'(\overline{v}(s)) \leq 1 \quad \text{and if } \overline{v}'(s) \neq 0 \quad A'(\overline{v}(s)) = \frac{1}{\overline{v}'(s)}. \tag{1.2}$$

*Proof.* Let us consider all the values $s_i$ with $i \in \mathbb{N}$ such that the set

$$B_i = \{t \in (0, |\Omega|) \ : \ |\overline{v}(t)| = \overline{v}(s_i)\}$$

has a strictly positive measure. By constriction every $B_i$ is an half-open proper interval on which $v(s)$ is constant and, since $\overline{v}(s)$ is not increasing, $\overline{B}_i \cap \overline{B}_j = \emptyset$ for $i \neq j$ (this assures us that the $B_i$ are indeed countable). Moreover $\cup_{i\in\mathbb{N}}\overline{B}_i$ is closed and

$$A'(\overline{v}(s)) = 0 \quad \forall \, a.e. \ \ s \in \cup_{i\in\mathbb{N}}\overline{B}_i.$$

On the other hand setting $K = (0, |\Omega|) \setminus \cup_{i\in\mathbb{N}}\overline{B}_i$ we have that

$$\forall \, s \in K, \quad |\{|\overline{v}(t)| = \overline{v}(s)\}| = 0 \quad \text{hence} \quad A(\overline{v}(s)) = s.$$

Since both $\overline{v}(s)$ and $A(s)$ are almost $a.e$ differentiable in $(0, |\Omega|)$ and, since for $a.e.$ $s \in K$ it holds true that $\overline{v}'(s) \neq 0$, we have finished. □

Let us state and prove the following useful Lemma (see Lemma 9 of [77]).

**Lemma 1.3.** *For every measurable function $v : \Omega \to \mathbb{R}$, there exists a set valued map $s \to \Omega(s) \subset \Omega$ such that*

$$\begin{cases} |\Omega(s)| = s \quad \textit{for any } s \in [0, |\Omega|], \\ \Omega(s_1) \subset \Omega(s_2) \quad \textit{whenever} \quad s_1 < s_2, \\ \Omega(s) = \{|v| > \overline{v}(s)\} \quad \textit{if} \quad |\{|v| = \overline{v}(s)\}| = 0. \end{cases} \tag{1.3}$$

**Remark 1.4.** *When we use Lemma 1.3 with $v \equiv u_n$ or $w_n$ (see (1.15) and (1.16) below for the definition of $u_n$ and $w_n$) the associated set functions are denoted with $\Omega_n(s)$. When we use Lemma 1.3 with $v \equiv \nabla u_n$ or $\nabla w_n$ the associated set function is denoted with $\widetilde{\Omega}_n(s)$.*

*Proof.* By construction $v(x)$ and $\overline{v}(s)$ are equimeasurable thus

$$|\{|v(x)| > \overline{v}(s)\}| = |\{|\overline{v}(\tau)| > \overline{v}(s)\}| \leq s \leq |\{|\overline{v}(\tau)| \geq \overline{v}(s)\}| = |\{|v(x)| \geq \overline{v}(s)\}|.$$

Since the Lebesgue measure is not atomic there exists $\Omega(s)$ such that

$$\{|v(x)| > \overline{v}(s)\} \subset \Omega(s) \subset \{|v(x)| \geq \overline{v}(s)\} \quad \text{and} \quad |\Omega(s)| = s. \tag{1.4}$$

Of course if $|\{|v| = \overline{v}(s)\}| = 0$, then $\Omega(s) = \{|v(x)| > \overline{v}(s)\}$.

□

In the next Lemma we define the pseudo rearrangement of a function $g \in L^1(\Omega)$ with respect to a measurable function $v(x)$ (see [6] and [57]).

**Lemma 1.5.** *Let $v : \Omega \to \mathbb{R}$ a measurable function, $0 \leq g(x) \in L^1(\Omega)$ and $\Omega(s)$ the set valued function associated to $v(x)$ defined in (1.3). Then*

$$D(s) := \frac{d}{ds} \int_{\Omega(s)} g(x) dx, \quad s \in (0, |\Omega|) \tag{1.5}$$

*is well defined and moreover*

$$i) \quad \int_0^t D(s) ds = \int_{\Omega(t)} g(x) dx \leq \int_0^t \overline{g}(s) ds, \quad t \in (0, |\Omega|) \tag{1.6}$$

$$ii) \quad D(A(k))(-A'(k)) = -\frac{d}{dk} \int_{\{|u_n|>k\}} g(x) dx, \quad k > 0. \tag{1.7}$$

*Proof.* Note now that the function defined for $s \in (0, |\Omega|)$ as

$$s \to \int_{\Omega(s)} g(x)dx$$

is absolutely continuous in $(0, |\Omega|)$. Thus it is almost everywhere differentiable and, denoting by $D(s)$ its derivative, (1.6) holds true. Reading equation (1.6) for every $s$ such that $s = A(k)$ it follows

$$\int_0^{A(k)} D(s)ds = \int_{\Omega(A(k))} g(x)dx = \int_{\{|v|>k\}} g(x)dx,$$

where we have used that $\Omega(A(k)) = \{|v| > \overline{v}(A(k))\} = \{|v| > k\}$. Differentiating with respect to $k$ the previous identity we get (1.7). □

The following Lemma assures that the pseudo rearrangement of $g$ has the same summability of $g$ (see [6]).

**Lemma 1.6.** *Assume that $g \in L^r(\Omega)$ with $1 \le r \le \infty$. Then the function $D(s)$ defined in (1.5) belongs to $L^r((0, |\Omega|)$ and $\|D\|_{L^r(0,|\Omega|)} \le \|g\|_{L^r(\Omega)}$.*
*Moreover if we assume that $g \in M^s(\Omega)$ with $1 < s < \infty$, then $D$ belongs to $M^s(0, |\Omega|)$.*

*Proof.* **Case $g \in L^r(\Omega)$.** We follow the proof given in [6] Lemma 2.2. Let us divide the interval $(0, |\Omega|)$ into $i \in \mathbb{N}$ disjoint intervals of the type $(s_{j-1}, s_j)$, for $j = 1, \cdots, i$ ,of equal measure $|\Omega|/i$. Let us consider the restriction of $g(x)$ on the set $\Omega(s_j) \setminus \Omega(s_{j-1})$ and take its decreasing rearrangement in the interval $(s_{j-1}, s_j)$. Repeating this for any $j = 1, \cdots, i$ we define a function (up to a zero measure set) on $(0, |\Omega|)$. Clearly this function depends on $i$ and so we call it $D_i(s)$. We stress that by construction the decreasing rearrangement of $D_i(s)$ coincides with the decreasing rearrangement of $g(x)$, thus for any measurable $\omega \subset (0, |\Omega|)$

$$\int_\omega D_i^r(s)ds \le \int_0^{|\omega|} \overline{g}^r(s)ds. \tag{1.8}$$

Hence the sequence $\{D_i^r(s)\}$ is equi-integrable and there exists a function $X \in L^r(0, |\Omega|)$ such that

$$D_i \rightharpoonup X \quad \text{in } L^r(0, |\Omega|) \quad \text{as} \quad i \to \infty.$$

The proof is concluded if we show that $X \equiv D$. Let us define the function

$$\Phi_i(s) := \int_0^s \big(D_i(t) - D(t)\big)dt$$

and notice that $\Phi_i(0) = \Phi_i(|\Omega|) = 0$. Thus for any $\varphi(s) \in C^1(0, |\Omega|)$ it results

$$\int_0^{|\Omega|} \big(D_i(s) - D(s)\big)\varphi(s)ds$$
$$= -\int_0^{|\Omega|} \left[\int_0^s \big(D_i(t) - D(t)\big)dt\right] d\varphi(s) \le \|\Phi_i\|_{L^\infty(0,|\Omega|)}\|\varphi'\|_{L^\infty(0,|\Omega|)}|\Omega|. \tag{1.9}$$

By construction $\Phi_i(s_j) = 0$ for any $j = 1 \cdots i$, since

$$\int_0^{s_j} D_i(t)dt = \sum_{l=1}^{j} \int_{s_{l-1}}^{s_l} D_i(t)dt = \sum_{l=1}^{j} \int_{\Omega(s_l)/\Omega(s_{l-1})} g(x)dx$$
$$= \int_{\Omega(s_j)} g(x)dx = \int_0^{s_j} D(t)dt.$$

Hence if $s_{j-1} \le s \le s_j$ we have that

$$\Phi_i(s) = \int_{s_{j-1}}^{s} \big(D_i(t) - D(t)\big)dt.$$

Recalling (1.6) we deduce

$$-\int_0^{|\Omega|/i} \overline{\varphi}(t)dt \le -\int_{s_{j-1}}^{s} D(t)dt \le \int_{s_{j-1}}^{s} \big(D_i(t) - D(t)\big)dt \le \int_{s_{j-1}}^{s} D_i(t)dt \le \int_0^{|\Omega|/i} \overline{\varphi}(t)dt,$$

that implies the following estimate

$$|\Phi_i(s)| \le \int_0^{|\Omega|/i} \overline{\varphi}(t)dt.$$

Hence the right hand side of (1.9) goes to 0 as $i$ diverges and

$$\lim_{i\to\infty} \int_0^{|\Omega|} \big(D_i(s) - D(s)\big)\varphi(s)ds = 0, \quad \forall\, \varphi \in C^1(0, |\Omega|).$$

Since we already know that $D_i(s)$ admits $X(s)$ as weak limit in $L^r(0, |\Omega|)$, it follows that $X(s) \equiv D(s)$ and we conclude the proof.

**Case** $g \in M^s(\Omega)$**.** As in the previous step we can construct a sequence $\{D_i\}$ such that $\overline{D}_i(s) = \overline{g}(s)$ for $s \in (0, |\Omega|)$ and

$$\lim_{i\to\infty} \int_0^{|\Omega|} D_i\phi = \int_0^{|\Omega|} D\phi \quad \forall\, \phi \in L^\infty(\Omega).$$

Take $\phi_A = \chi_A$ with $A \subset (0, |\Omega|)$ and $|A| = s$. We deduce that

$$\int_0^{|\Omega|} D\phi_A \le \int_0^s \overline{g} \text{ and taking the sup with respect to } A \quad \int_0^s \overline{D} \le \int_0^s \overline{g}.$$

Thus

$$\overline{D}(s) \le \frac{1}{s}\int_0^s \overline{D} \le \frac{1}{s}\int_0^s \overline{g} \le \|g\|_{M^s(\Omega)} \frac{r}{r-1} s^{-\frac{1}{r}}.$$

□

Moreover, coupling the Fleming-Rishel coarea formula and the isoperimetric inequality, we obtain the following proposition (see [86]).

**Proposition 1.7.** *For any* $v \in W_0^{1,p}(\Omega)$ *and for any* $s \in \mathbb{R}$

$$\sigma_N \le A(s)^{\frac{1}{N}-1}\big(-A'(s)\big)^{\frac{1}{p'}} \left(-\frac{d}{ds}\int_{\{|v|>s\}} |\nabla v|^p\right)^{\frac{1}{p}}, \tag{1.10}$$

*where* $\sigma_N = N\omega_N^{\frac{1}{N}}$ *and* $\omega_N$ *is the volume of the unitary ball in dimension* $N$*.*

*Proof.* See pages 711 and 712 of [86]. □

## Lorentz Spaces

Let us give now the definition of Lorentz spaces. For $1 \leq m < \infty$ and $0 < q \leq \infty$ we says that a measurable functions $f : \Omega \to \mathbb{R}$ belongs to the Lorentz space $L^{m,q}(\Omega)$ if the quantity

$$\|f\|_{L^{m,q}(\Omega)} = \begin{cases} \left(\int_0^\infty t^{\frac{q}{m}} \overline{f}(t)^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } q < \infty, \\ \sup_{t\in(0,\infty)} t^{\frac{1}{m}} \overline{f}(t) & \text{if } q = \infty, \end{cases}$$

is finite. We recall that $L^{m,m}(\Omega) = L^m(\Omega)$ and that

$$L^{m,q}(\Omega) \subset L^{m,r}(\Omega) \quad \text{for any} \quad 0 < q < r \leq \infty.$$

Usually the space $L^{m,\infty}(\Omega)$, with $1 \leq m < \infty$ is called Marcinkiewicz space of order $m$ and we denote it by $M^m(\Omega)$.
If we replace $\overline{f}$ with $\tilde{f}$, we define another space $L^{(m,q)}(\Omega)$ given by

$$\lceil f \rceil_{L^{(m,q)}(\Omega)} = \begin{cases} \left(\int_0^\infty t^{\frac{q}{m}} \tilde{f}(t)^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } q < \infty \\ \sup_{t\in(0,\infty)} t^{\frac{1}{m}} \tilde{f}(t) & \text{if } q = \infty. \end{cases}$$

Since

$$\|f\|_{L^{m,q}(\Omega)} \leq \lceil f \rceil_{L^{(m,q)}(\Omega)} \leq m' \|f\|_{L^{m,q}(\Omega)}, \tag{1.11}$$

it results that $\|\cdot\|_{L^{m,q}(\Omega)}$ and $\lceil \cdot \rceil_{L^{(m,q)}(\Omega)}$ are equivalent if $m > 1$ and $L^{m,q}(\Omega) \equiv L^{(m,q)}(\Omega)$. Anyway in the borderline case $m = 1$ the space $L^{(1,q)}(\Omega)$ is rather unsatisfactory since for $q < \infty$ it contains only the zero function. This is because by definition $\tilde{f}(s) \approx \frac{1}{s}$ for $s > |\Omega|$. Hence, following [18], we define $\mathbb{L}^{1,q}(\Omega)$ as the set of measurable function $f$ such that

$$\|f\|_{\mathbb{L}^{1,q}(\Omega)} = \begin{cases} \left(\int_0^{|\Omega|} t^q \tilde{f}(t)^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } q < \infty \\ \sup_{t\in(0,|\Omega|)} t\tilde{f}(t) & \text{if } q = \infty, \end{cases}$$

is finite. Even if the modification is refers only to the domain of integration, we stress that

$$\mathbb{L}^{1,1}(\Omega) \subset L^{1,1}(\Omega) = L^1(\Omega), \tag{1.12}$$

and indeed the inclusion is strict as the following Lemma (see [18]) shows.

**Lemma 1.8.** *A measurable function $f$ belongs to $\mathbb{L}^{1,1}(\Omega)$ if and only if*

$$\int |f| \log(1 + |f|) < \infty.$$

*Proof.* For the proof we refer to [18]. □

The Lorentz spaces $L^{m,q}(\Omega)$ and $\mathbb{L}^{1,q}(\Omega)$ arise quite naturally in the study of elliptic PDE through rearrangement techniques.
The next Lemma (see [4]) is used to establish the membership to Lorentz spaces of some integral quantities.

**Lemma 1.9.** *Let $r : (0, +\infty) \to (0, +\infty)$ be a decreasing function and let us define for $\beta \geq 0$ and $\delta \neq 1$*

$$R_\delta(t) := \begin{cases} \int_0^t s^\beta r(s)ds & \text{if} \quad \delta < 1 \\ \int_t^{+\infty} s^\beta r(s)ds & \text{if} \quad \delta > 1. \end{cases} \tag{1.13}$$

*Then for every $\lambda > 0$ it follows that*

$$\int_0^\infty \left(\frac{R_\delta(t)}{t}\right)^\lambda t^{\delta\lambda} \frac{dt}{t} \leq C(\beta, \delta, \lambda) \int_0^\infty r(t)^\lambda t^{\lambda(\beta+\delta)} \frac{dt}{t}.$$

*Proof.* For the proof see [4] Lemma 2.1. □

## 1.2 Others useful results

The existence and regularity results of Chapters 2-4 are based on an approximation procedure. Let us introduce the following families of approximating problems.

Let $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^\mathbb{N}$ be a Carathéodory function that satisfies the following Leray-Lions assumptions, i.e. there exist $1 < p < \infty$ and $0 < \alpha, \beta$ such that

$$\begin{aligned} &\alpha|\xi|^p \leq \ a(x,\xi)\xi, \\ &|a(x,\xi)| \leq \ \beta|\xi|^{p-1}, \\ &[a(x,\xi) - a(x,\xi^*)][\xi-\xi^*] > \ 0, \quad \text{if} \quad \xi \neq \xi^*. \end{aligned} \tag{1.14}$$

With no modifications we can also consider the case $a(x, u, \nabla u)$ assuming suitable growth conditions with respect to the $u$ variable.
Assume moreover that $E \in \left(L^1(\Omega)\right)^N$ and $f \in L^1(\Omega)$. Thanks to [66], for any $n \in \mathbb{N}$ we infer the existence of $u_n \in W_0^{1,2}(\Omega)$ and $w_n \in W_0^{1,2}(\Omega)$ that solve respectively

$$\int_\Omega a(x,\nabla u_n)\nabla\phi = \int_\Omega \frac{|u_n|^{p-2}u_n}{1+\frac{1}{n}|u_n|^{p-1}}E_n(x)\nabla\phi + \int_\Omega f_n(x)\phi \quad \forall\, \phi \in C_0^1(\Omega) \tag{1.15}$$

and

$$\int_\Omega a(x,\nabla w_n)\varphi = \int_\Omega E_n(x)\frac{|\nabla w_n|^{p-2}\nabla w_n}{1+\frac{1}{n}|\nabla w_n|^{p-1}}\varphi + \int_\Omega f_n(x)\varphi \quad \forall\, \varphi \in W_0^{1,2}(\Omega), \tag{1.16}$$

where $E_n(x)$ and $f_n(x)$ are the truncation at level $n \in \mathbb{N}$ of $E(x)$ and $f(x)$. The general strategy of the existence and regularity results of Chapters 2-4 can be resumed as follows:

- a priori estimates for the sequences $\{u_n\}$ and $\{w_n\}$ in suitable spaces;
- existence of a converging subsequence;
- passage to the limit in (1.15) and (1.16) as $n \to \infty$.

In this Section we prove the almost everywhere convergence of the gradients of $\{u_n\}$ and $\{w_n\}$.

**Lemma 1.10.** *Let $\{u_n\}$ be the sequence of* approximating *solutions of* (1.15)*. Assume $f \in L^1(\Omega)$, $|E| \in L^p(\Omega)'$ and moreover that there exists $u \in W_0^{1,s}(\Omega)$ with $s \geq 1$ such that up to a subsequence $u_n \rightharpoonup u$ in $W_0^{1,s}(\Omega)$. Hence, up to a further subsequence,*

$$\nabla u_n \to \nabla u \quad a.e. \quad in\ \Omega. \tag{1.17}$$

*Proof.* We follow the approach of [24]. Taking $T_k(u_n)$ as test function in (1.15) and using Young inequality it follows that for any $\epsilon > 0$

$$\alpha\int_\Omega |\nabla T_k(u_n)|^p \leq C_\epsilon k^p\int_\Omega |E|^{p'} + \epsilon\int_\Omega |\nabla T_k(u_n)|^p + k\int_\Omega |f|,$$

with $C_\epsilon = \epsilon^{-\frac{1}{p-1}}$. Thanks to the previous estimate we deduce that for every $k > 0$

$$|\nabla T_k(u)| \in L^p(\Omega) \quad \text{and} \quad T_k(u_n) \to T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega). \tag{1.18}$$

In order to prove (1.17) let us define for $k > 0$ fixed

$$I_n^k(x) = [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u))]\nabla(T_k(u_n) - T_k(u))$$

and consider, for $0 < \theta < 1$ and $0 < h < k$,

$$\int_\Omega I_n^k(x)^\theta dx = \int_{\{|T_k(u_n)-T_k(u)|>h\}} I_n^k(x)^\theta dx + \int_{\{|T_k(u_n)-T_k(u)|\le h\}} I_n^k(x)^\theta dx$$

$$\le \left(\int_\Omega I_n^k(x)dx\right)^\theta |\{|T_k(u_n) - T_k(u)| > h\}|^{1-\theta} + \left(\int_{\{|T_k(u_n)-T_k(u)|\le h\}} I_n^k(x)dx\right)^\theta |\Omega|^{1-\theta}.$$

Note that, for every fixed $h$, the first term in the right hand side above goes to zero as $n \to \infty$ because of (1.18) and thanks to the convergence in measure of $T_k(u_n)$. We claim that also the second term converge to zero taking the limit at first with respect to $n \to \infty$ and then with respect to $h \to 0$. Once this claim is proved, it follows that

$$\lim_{n\to\infty} \int_\Omega I_n^k(x)^\theta dx = 0,$$

from which we deduce, like in [24], that $\nabla T_k(u_n)$ almost everywhere converges to $\nabla T_k(u)$ for every $k > 0$. An this is enough to infer (1.17) as in [80].
In order to prove the claim let us take $T_h(u_n - T_k(u))$, with $0 < h < k$, as a test function in (1.15). After simple manipulations we obtain that

$$-\int_{\{|u_n-T_k(u)|<h\}} a(x, \nabla G_k(u_n))\nabla T_k(u) + \int_\Omega a(x, \nabla T_k(u_n))\nabla T_h(T_k(u_n) - T_k(u))$$

$$\le h\int |f| + \int_\Omega \frac{|u_n|^{p-2}u_n}{1 + \frac{1}{n}|u_n|^{p-1}} E_n(x)\nabla T_h(u_n - T_k(u))$$

and also that

$$0 \le \int_{\{|T_k(u_n)-T_k(u)|\le h\}} I_n^k(x)dx = \int_\Omega \big[a(x, \nabla T_k(u_n)) - a(x, \nabla T_k(u))\big]\nabla T_h(T_k(u_n) - T_k(u))$$

$$\le h\int |f| + \int_\Omega \frac{|u_n|^{p-2}u_n}{1 + \frac{1}{n}|u_n|^{p-1}} E_n(x)\nabla T_h(u_n - T_k(u))$$

$$+ \int_{\{|u_n|>k\}\cap\{|u_n-T_k(u)|<h\}} a(x, \nabla u_n)\nabla T_k(u) - \int_\Omega a(x, \nabla T_k(u))\nabla T_h(T_k(u_n) - T_k(u)).$$

Noticing that $\{|u_n - T_k(u)| < h\} \subset \{|u_n| \le h + k\} \subset \{|u_n| \le 2k\}$, that the sequence $\{|a(x, \nabla T_{2k}(u_n))|\}$ is bounded in $L^{p'}(\Omega)$ and recalling (1.18), we can pass to the limit with respect to $n \to \infty$ into the previous inequality and obtain

$$\limsup_{n\to\infty} \int_{\{|T_k(u_n)-T_k(u)|\le h\}} I_n^k(x)dx \le \int_\Omega |u|^{p-2}u\, E\nabla T_h(G_k(u)) + h\int |f|$$

$$+ \int_{\{k<|u|<k+h\}} \Psi_k \nabla T_k(u).$$

where $\Psi_k \in \left(L^{p'}(\Omega)\right)^N$ is the weak limit of $a(x, \nabla T_{2k}(u_n))$. Letting $h \to 0$ we prove the claim and conclude the proof of the Lemma. □

**Lemma 1.11.** *Let $\{w_n\}$ be the sequence of approximating solution of (1.16). Assume $f \in L^1(\Omega)$, $|E| \in M^N(\Omega)$ and moreover that there exists $w \in W_0^{1,s}(\Omega)$ with $s > \frac{(p-1)N}{N-1}$ such that up to a subsequence $w_n \rightharpoonup w$ in $W_0^{1,s}(\Omega)$. Hence, up to a further subsequence,*

$$\nabla w_n \to \nabla w \quad a.e. \quad in\,\Omega. \tag{1.19}$$

*Proof.* By hypothesis the sequence $\{|\nabla w_n|^{p-1}\}$ is bounded in $L^r(\Omega)$ with $r = \frac{s}{p-1} > \frac{N}{N-1}$ and moreover $r' < N$. Hence taking $T_k(w_n)$ as a test function in (1.16), we obtain

$$\alpha \int_\Omega |\nabla T_k(w_n)|^p \leq k \left[\int_\Omega |f| + \int_\Omega |E_n(x)||\nabla w_n|^{p-1}\right]$$
$$\leq k \left[\|f\|_{L^1(\Omega)} + \|E\|_{L^{r'}(\Omega)} \||\nabla w_n|^{p-1}\|_{L^r(\Omega)}\right],$$

that implies

$$T_k(w_n) \rightharpoonup T_k(w) \quad \text{in} \quad W_0^{1,p}(\Omega) \quad \text{for any} \quad k > 0.$$

Notice that we are in the same situation of Lemma 1.10 above. Thus we conclude the proof if we show that

$$\limsup_{n\to\infty} \int_\Omega \big[a(x, \nabla T_k(w_n)) - a(x, \nabla T_k(w))\big] \nabla T_h(w_n - T_k(w)) = 0.$$

As before let us thus choose $T_h(w_n - T_k(w))$, with $0 < h < k$, as test function in (1.16). Manipulating the resulting equation, we obtain

$$\int_\Omega \big[a(x, \nabla T_k(w_n)) - a(x, \nabla T_k(w))\big] \nabla T_h(w_n - T_k(w))$$
$$\leq h \left[\|f\|_{L^1(\Omega)} + \|E\|_{L^{r'}(\Omega)} \||\nabla w_n|^{p-1}\|_{L^r(\Omega)}\right] + \int_{\{|w_n|>k\}\cap\{|w_n - T_k(w)|<h\}} a(x, \nabla w_n) \nabla T_k(w).$$
$$- \int_\Omega a(x, \nabla T_k(w)) \nabla T_h(w_n - T_k(w)).$$

Noticing that $\{|w_n - T_k(w)| < h\} \subset \{|w_n| \leq h + k\} \subset \{|w_n| \leq 2k\}$ we can pas to the limit with respect to $n \to \infty$ and obtain

$$\limsup_{n\to\infty} \int_\Omega a(x, \nabla T_k(w_n)) \nabla T_h(w_n - T_k(w)) \leq Ch + \int_{\{k<|w|<k+h\}} \Psi_k \nabla T_k(w),$$

where $\Psi_k \in \left(L^{p'}(\Omega)\right)^N$ is the weak limit of $a(x, \nabla T_{2k}(w_n))$. Letting $h \to 0$ we conclude the proof of the Lemma. □

**Lemma 1.12.** *Given the function $\lambda,\ \gamma,\ \varphi, \rho$ defined in $(0, +\infty)$, suppose that $\lambda,\ \gamma \geq 0$ and that $\lambda\gamma,\ \lambda\varphi$ and $\lambda\rho$ belong to $L^1(0, \infty)$. If for almost every $t \geq 0$ we have*

$$\varphi(t) \leq \rho(t) + \gamma(t) \int_t^{+\infty} \lambda(\tau)\varphi(\tau) d\tau,$$

*then for almost every $t \geq 0$*

$$\varphi(t) \leq \rho(t) + \gamma(t) \int_t^{+\infty} \rho(t)\lambda(\tau) e^{\int_t^\tau \lambda(s)\gamma(s) ds} d\tau.$$

*Proof.* See [7] Lemma 6.1. □

Often in this Thesis we consider a sequence of approximating solutions, say $\{u_n\}$, that hopefully shall converge to the expected solution $u$ of a certain problem. To assure this convergence, we need some compactness property. The first step for it is to obtain a bound for $\{u_n\}$ in some suitable space. This requires a massive use of *absolute* constants $C$, i.e. constant that may depend on whichever datum of the problem ($\Omega$, $E$, $f$, etc.) but that can not depend on $u_n$ neither on $u$. To avoid proliferation of sub indexes, unless otherwise explicitly specified, the value of the constant $C$ can be updated inequality after inequality even in the same proof. We definitively are not interested on sharp bounds or constants.

# Chapter 2

# Convection lower order term

This chapter is devoted to the study of existence and summability properties of solutions of

$$\begin{cases} -\text{div}\,(A(x)\nabla u) = -\text{div}\,(u\,E(x)) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega$ is a bounded open set of $\mathbb{R}^N$ with $N > 2$, $A(x)$ is a measurable matrix that satisfies for $\alpha, \beta > 0$

$$\alpha \leq A(x)\xi \cdot \xi, \quad |A(x)| \leq \beta, \quad \text{a.e. } x \in \Omega, \quad \forall\, \xi \in \mathbb{R}^N\,, \tag{2.2}$$

and the vector field $E(x)$ and the function $f(x)$ belong to suitable Lebesgue o Marcinkievicz spaces to be specified in the sequel. Problem (2.1) has to be meant in the following weak form

$$u \in W_0^{1,1}(\Omega)\ :\ \int_\Omega A(x)\nabla u \nabla\phi = \int_\Omega uE(x)\nabla\phi + \int_\Omega f(x)\phi \quad \forall\, \phi \in C_0^1(\Omega). \tag{2.3}$$

Focusing on the convection term in (2.3), we easily note that it is well defined if

$$E \in \left(L^N(\Omega)\right)^N \quad \text{or more generally} \quad E \in \left(M^N(\Omega)\right)^N.$$

Indeed, on one hand, the classical Sobolev embedding gives that

$$\int_\Omega |u||E(x)|dx \leq \|E\|_{L^N(\Omega)} \|u\|_{L^{\frac{N}{N-1}}(\Omega)} < +\infty$$

and, on the other one, the sharp Sobolev embedding in Lorentz spaces (see for example [87] and reference therein) assures also that

$$\begin{aligned} \int_\Omega |u||E(x)|dx &\leq \int_0^{|\Omega|} \overline{u}(t)\,\overline{E}(t)dt \\ &\leq \|E\|_{L^{N,\infty}(\Omega)} \int_0^{|\Omega|} \frac{\overline{u}}{t^{\frac{1}{N}-1}}\frac{dt}{t} = \|E\|_{L^{N,\infty}(\Omega)} \|u\|_{L^{\frac{N}{N-1},1}(\Omega)} < +\infty. \end{aligned}$$

Thus the aim of the next two sections is to analyze the cases of $E(x)$ in the Lebesgue and Marcinkiewicz spaces of order $N$ and, for both cases, we consider $f(x)$ belonging to $L^m(\Omega)$ or $M^m(\Omega)$ with $m > 1$.
To be more precise the existence and regularity results we are interested in are summarized in the following table.

| | $f \in L^m(\Omega)$ | | $f \in M^m(\Omega)$ | |
|---|---|---|---|---|
| | $1 < m < (2^*)'$ | $(2^*)' \leq m < N/2$ | $1 < m < (2^*)'$ | $(2^*)' < m < N/2$ |
| $u$ | $L^{m^{**}}(\Omega)$ | $L^{m^{**}}(\Omega)$ | $M^{m^{**}}(\Omega)$ | $M^{m^{**}}(\Omega)$ |
| $\nabla u$ | $L^{m^*}(\Omega)$ | $L^2(\Omega)$ | $M^{m^*}(\Omega)$ | $L^2(\Omega)$ |

Table 1

The way to read the previous scheme is that, given a datum $f$ in $L^m(\Omega)$ or $M^m(\Omega)$, then there exist a solution $u$ of (2.3) such that $u$ and $\nabla u$ belong to the relative Lebesge or Marcinkiewicz spaces. It is worth to stress that the results gathered in Table 1 are true for any $|E| \in L^N(\Omega)$ (see Section 1.1), while a if $|E| \in M^N(\Omega)$ some control on the size of $\overline{E}(s)$ as $s \to 0^+$ is required (see Section 2.2).
The original contributions of this chapter concern the second half of Table 1, while the first two column are already know in the literature (see [52][27][28]). Anyway in order to give a complete overview of the problem, we provide the proof for the all the type of data.

In order to study problem (2.3) it is useful to consider the following approximating problem

$$\int_\Omega A(x)\nabla u_n \nabla\phi = \int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|} E_n \nabla\phi + \int_\Omega f_n\phi \quad \forall\, \phi \in W_0^{1,2}(\Omega), \tag{2.4}$$

whose existence of a solution $u_n \in W_0^{1,2}(\Omega)$ is assured by Shauder's fixed point theorem.

## 2.1 Convection term in $L^N(\Omega)$

In this section we treat (2.1) assuming that

$$E : \Omega \to \mathbb{R}^N \quad \text{belongs to} \quad \left(L^N(\Omega)\right)^N. \tag{2.5}$$

We will prove that the relationship between the Lebesgue and Marcinkiewicz regularity of the datum and the solution is the same as in the case $E \equiv 0$.
As far as the the Lebesgue regularity is concerned (the first two column of the Table 1) we present here the approach provided by [26]. On the other hand the proof of the Marcinkiewicz regularity is one of the original results of this Thesis and it is obtained by means the symmetrization techniques introduced in [86], [52] and [4].

In order to better understand the nature of the problem, we believe that, before the statements and proof of the results, it is useful to present some preliminary arguments that stress the differences in dealing with $L^m(\Omega)$ or $M^m(\Omega)$ data with $1 < m < \frac{N}{2}$. Indeed in one case we look for integral estimates of the form

$$\int_\Omega |u_n|^{m^{**}} \leq C \quad \text{or} \quad \int_\Omega |\nabla u_n|^{m^*} \leq C, \tag{2.6}$$

for some absolute constant $C$ that does not depend on $n$. On the other one we need estimates like

$$k^{m^{**}}|\{|u_n| > k\}| \leq \mathcal{C} \quad \text{or} \quad k^{m^*}|\{|\nabla u_n| > k\}| \leq \mathcal{C}, \tag{2.7}$$

for some other absolute constant $\mathcal{C}$.
At first glance the achievement of (2.6) seems more direct and it is natural to try to adapt the technique of [34], where the Lebesgue regularity of problem (2.1) is treated in the case $E \equiv 0$ to. The main idea of [34] is to consider power like test functions and to dominate the right hand side by the principal part of the operator.

To give an example, let us consider the case $f \in L^m(\Omega)$, with $(2^*)' \le m < \frac{N}{2}$, set $\gamma = \frac{m^{**}}{2^*}$ and assume for the moment that

$$\|E\|_{L^N(\Omega)} < \frac{\alpha \mathcal{S}_2}{\gamma}. \tag{2.8}$$

where $\mathcal{S}_2$ is the sobolev constant relative to $W_0^{1,2}(\Omega)$. Taking $\phi(x) = \frac{1}{2\gamma-1}|u_n|^{2\gamma-2}u_n$ as a test function ( it is admissible because $2\gamma - 2 > 0$ in the considered range of the parameter $m$) in (2.4) and Using Hölder and Sobolev inequalities, we get

$$\alpha \int_\Omega |\nabla u_n|^2 |u_n|^{2\gamma-2} \le \gamma \frac{\|E\|_{L^N(\Omega)}}{\mathcal{S}_2} \int_\Omega |\nabla u_n|^2 |u_n|^{2\gamma-2} + \frac{\|f\|_{L^m(\Omega)}}{2\gamma-1} \left( \int_\Omega |u_n|^{(2\gamma-1)m'} \right)^{\frac{1}{m'}}. \tag{2.9}$$

Thanks to the definition of $\gamma$, assumption (2.8) and using once more Sobolev inequality, we deduce that there exist an absolute constant $C$ that does not depend on $n$ such that

$$\int_\Omega |u_n|^{m^{**}} \le C.$$

Similarly it is possible to treat the case $1 \le m < (2^*)'$ and obtain the relative bounds for $\{|\nabla u_n|\}$ in $L^{m^*}(\Omega)$. Once that such estimates are obtained, one deduces that, up to a subsequence, $\{u_n\}$ weakly converges in $W_0^{1,m^*}(\Omega)$ or $W_0^{1,2}(\Omega)$, depending on the value of $m$, to the expected solution of problem (2.3). This weak convergence is enough to pass to the limit in (2.4) (the principal part of the operator is linear) and prove the existence of a solution of (2.3) with the expected regularity.

Thus, under the smallness condition (2.8), one recovers the Lebesgue regularity of (2.1) as a slight generalization of the approach of [85] and [34] ($E \equiv 0$), based on the direct use of suitable test functions. We stress that this is due to the fact that, under the smallness condition (2.8), the lower order term is absorbed into the principal part of the operator.

As we shall see, things change dealing with Marcinkiewicz regularity. Even in the case $E \equiv 0$ estimates (2.7) are more subtle to obtain than (2.6). However in [85] and [25] (see also [63] for a slightly different approach) such estimates are obtained via test functions methods. In particular the strategy of [25] (that improves the one by [85]) consists in obtaining an estimate of the type

$$\int_\Omega |G_k(u_n)| \le C |\{|u_n| > k\}|^{1-\frac{1}{m^{**}}}, \tag{2.10}$$

where $G_k(s) = s - T_k(s)$ and $T_k(s) = \max\{-k, \min\{s, k\}\}$. From (2.10) it is possible to recover a bound in $M^{m^{**}}(\Omega)$ for the sequence $\{u_n\}$ taking advantage of the following relationship (see [85] and [33])

$$-\frac{d}{dk}\left( \int_\Omega |G_k(u_n)| \right) = -\frac{d}{dk}\left( \int_{\{|u_n|>k\}} (|u_n| - k) \right) = |\{|u_n| > k\}|.$$

Let us try, hence, to obtain the Marcinkiewicz regularity for (2.1) following [25] in the case of smallness condition on the norm of $E$. Let us assume (2.8), that $f \in M^m(\Omega)$ with $(2^*)' < m < \frac{N}{2}$ and set $\gamma = \frac{r^{**}}{2^*}$ with $(2^*)' < r < m$; being $2\gamma - 2 > 0$ we can take $|G_k(u_n)|^{2\gamma-2}G_k(u_n)$ as a test function in (2.4). Let us focus at first on the behaviour of the lower order term. It follows that

$$(2\gamma - 1) \int_\Omega |u_n||E||\nabla G_k(u_n)||G_k(u_n)|^{2\gamma-2}$$

$$\le k \int_\Omega |E||\nabla u_n||G_k(u_n)|^{2\gamma-2} + \int_\Omega |G_k(u_n)|^\gamma |E||\nabla G_k(u_n)|^2 |G_k(u_n)|^{\gamma-1}$$

$$\leq \frac{k^2}{\epsilon}\int_\Omega |E|^2|G_k(u_n)|^{2\gamma-2} + \left(\epsilon + \frac{\gamma\|E\|_{L^N(\Omega)}}{\mathcal{S}_2}\right)\int_\Omega |\nabla u_n|^2|G_k(u_n)|^{2\gamma-2}.$$

Thanks to (2.8) we can choose $\epsilon$ small enough to absorb the second term in the right hand side above into the principal part of the operator, but there is no hope to get rid of the first one, no matter the value of the $L^N(\Omega)$ norm of $E$. Finally we obtain the estimate

$$\left(\int_\Omega |G_k(u_n)|^{r^{**}}\right)^{\frac{2}{2^*}} \leq C\left[k^2\int_\Omega |E|^2|G_k(u_n)|^{2\gamma-2} + \|f\|_{L^r(\Omega)}\left(\int_\Omega |G_k(u_n)|^{r^{**}}\right)^{\frac{1}{r'}}\right]. \tag{2.11}$$

The point is that it is not clear how to manipulate the inequality above in order to obtain something similar to (2.10)[1]. Thus it seem that, even with the smallness assumption (2.8), the arguments used in [25] and [63] cannot be adapted to our framework (it is possible to show that also the approach of [63] does not work).

To overcome this problem let us recall that estimates (2.7) can be expressed equivalently trough the rearrangement of $u_n$ and $\nabla u_n$, namely

$$\overline{u}_n(t) \leq \frac{C}{t^{\frac{1}{m^{**}}}} \quad \text{and} \quad \overline{|\nabla u_n|}(t) \leq \frac{C}{t^{\frac{1}{m^*}}}.$$

This strongly suggests to use the theory of rearrangement introduced by Talenti in [86]. As said in the introduction, we use some ideas of [52] and [4] to obtain pointwise estimates for the decreasing rearrangement of $u_n$ and $\nabla u_n$.

In the following two subsections we address the question of existence and regularity for a solution of (2.3) for datum in Lebesgue or Marcinkiewicz spaces and $E \in \left(L^N(\Omega)\right)^N$ without smallness condition on the norm.

### 2.1.1 Data in Lebesgue spaces

To get rid of (2.8) we present here the approach of [26]. Without any control on the norm of the convection term, the problem exhibit its non-coercive character and it is not possible to absorb the lower order term as we have done in (2.9). The preliminary arguments that we presented at the beginning of the Chapter suggest, as natural way to overcome this obstacle, to use powers of $G_k(u_n)$ (instead of powers of $u_n$) as a test function in (2.4) and try to make the quantity

$$\int_{|u_n|>k} |E|^N$$

small enough for $k$ large. Thus we need to show that the measure of the sup-level set of $u_n$ is uniformly small (with respect to $n$) as $k$ increase. The key idea of [26] is to prove this uniform smallness by means of the next Lemma.

**Lemma 2.1** (Lemma 4.1 of [26]). *Assume* (2.2), $E \in L^2(\Omega)$ *and* $f \in L^1(\Omega)$. *Hence for every* $n \in \mathbb{N}$ *the solution of* (2.4) *satisfy*

$$\left(\int_\Omega |\log(|u_n|+1)|^{2^*}\right)^{\frac{2}{2^*}} \leq \frac{1}{\mathcal{S}_2^2\alpha^2}\int_\Omega |E|^2 + \frac{2}{\mathcal{S}_2^2\alpha}\int_\Omega |f|. \tag{2.12}$$

*Proof.* Using $\frac{u_n}{1+|u_n|}$ as test function in (2.4) one obtains

$$\alpha\int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^2} \leq \int_\Omega \frac{|u_n|}{1+|u_n|}|E|\frac{|\nabla u_n|}{1+|u_n|} + \int_\Omega |f|\frac{|u_n|}{1+|u_n|}$$
$$\leq \int_\Omega |E|\frac{|\nabla u_n|}{1+|u_n|} + \int_\Omega |f|,$$

from which we deduce (2.12) by means of Young and Sobolev Inequalities. □

[1] In the case $E \equiv 0$ one deduce (2.10) from (2.11) by means of Sobolev and Hölder inequalities

**Remark 2.2.** *Thanks to Sobolev and Chebyshev inequalities, estimate* (2.12) *implies that*

$$\left|\{x\in\Omega : |u_n(x)|>k\}\right|^{\frac{2}{2^*}} \le \frac{1}{|\log(1+k)|^2}\left[\frac{1}{S^2\alpha^2}\int_\Omega |E|^2 + \frac{2}{S^2\alpha}\int_\Omega |f|\right]$$

*and, due to the uniform continuity of the Lebesgue integral with respect to the domain of integration, we can infer that*

$$\forall\,\epsilon>0\ \exists\,k_0=k_0(\epsilon)>0\ :\ \int_{|u_n|>k}|E|^N<\epsilon\ \ \forall\,k>k_0. \tag{2.13}$$

The next Theorem provide the Lebesgue regularity of the solution of (2.1) in function of the Lebesgue regularity of the datum.

**Proposition 2.3** (Theorems 5.5 and 7.2 of [26]). *Assume* (2.2), (2.5) *and* $f\in L^m(\Omega)$ *with* $1<m<\frac{N}{2}$. *Hence there exists* $u$ *solution of* (2.3). *Moreover*

- *if* $1<m<(2^*)'$, *then* $u\in W_0^{1,m^*}(\Omega)$,
- *if* $(2^*)'\le m<\frac{N}{2}$, *then* $u\in W_0^{1,2}(\Omega)\cap L^{m^{**}}(\Omega)$.

*Sketch of the proof.* **Case** $(2^*)'\le m<\frac{N}{2}$. Setting $\gamma=\frac{m^{**}}{2^*}$ it results that $\frac{|G_k(u_n)|^{2\gamma-2}G_k(u_n)}{2\gamma-1}$ is an admissible test function for (2.4), because in this case $2\gamma-2>0$. Using it in (2.4) we get the following inequality

$$\alpha\int_\Omega |\nabla G_k(u_n)|^2|G_k(u_n)|^{2\gamma-2}\le k\int_\Omega |E||\nabla G_k(u_n)||G_k(u_n)|^{2\gamma-2}$$

$$+\int_\Omega |E||\nabla G_k(u_n)||G_k(u_n)|^{2\gamma-1}+\frac{\|f\|_{L^m(\Omega)}}{2\gamma-1}\left(\int_\Omega |G_k(u_n)|^{(2\gamma-1)m'}\right)^{\frac{1}{m'}},$$

that, thanks to Young and Hölder Inequalities, becomes

$$\alpha\int_\Omega |\nabla G_k(u_n)|^2|G_k(u_n)|^{2\gamma-2}\le\frac{k^2}{\alpha}\int_\Omega |G_k(u_n)|^{2\gamma-2}|E|^2+\frac{\alpha}{4}\int_\Omega |\nabla G_k(u_n)|^2|G_k(u_n)|^{2\gamma-2}$$

$$+\frac{\gamma}{\mathcal{S}_2}\left(\int_{\{|u_n|>k\}}|E|^N\right)^{\frac{1}{N}}\int_\Omega |\nabla G_k(u_n)|^2|G_k(u_n)|^{2\gamma-2}+\frac{\|f\|_{L^m(\Omega)}}{2\gamma-1}\left(\int_\Omega |G_k(u_n)|^{(2\gamma-1)m'}\right)^{\frac{1}{m'}}.$$

Thus, using once more Hölder Inequality in the first term in the right hand side above, choosing $k_0$ in (2.13) so that $\frac{\gamma}{\mathcal{S}_2}\left(\int_{\{|u_n|>k\}}|E|^N\right)^{\frac{1}{N}}\le\frac{\alpha}{4}$ and recalling the definition of $\gamma$, we conclude that

$$C_{\alpha,\gamma}\left(\int_\Omega |G_{k_0}(u_n)|^{m^{**}}\right)^{\frac{2}{2^*}}\le\frac{\alpha}{2}\int_\Omega |\nabla G_{k_0}(u_n)|^2|G_{k_0}(u_n)|^{2\gamma-2}$$

$$\le\frac{k_0^2\|E\|^2_{L^N(\Omega)}}{\alpha}\left|\{|u_n|>k_0\}\right|^{\frac{2}{m^{**}}}\left(\int_\Omega |G_{k_0}(u_n)|^{m^{**}}\right)^{\frac{2}{2^*}-\frac{2}{m^{**}}}+\frac{\|f\|_{L^m(\Omega)}}{2\gamma-1}\left(\int_\Omega |G_{k_0}(u_n)|^{m^{**}}\right)^{\frac{1}{m'}}.$$

Thus we finally obtain that

$$\left(\int_\Omega |G_{k_0}(u_n)|^{m^{**}}\right)^{\frac{2}{2^*}}\le C\left[\left(\int_\Omega |G_{k_0}(u_n)|^{m^{**}}\right)^{\frac{2}{2^*}-\frac{2}{m^{**}}}+\left(\int_\Omega |G_{k_0}(u_n)|^{(2\gamma-1)m'}\right)^{\frac{1}{m'}}\right],$$

where $C=C(\alpha,m,\Omega,k_0,f)$. Since $\frac{2}{2^*}$ is larger then $\frac{1}{m'}$ if $m<\frac{N}{2}$, we deduce that

$$\int_\Omega |G_{k_0}(u_n)|^{m^{**}}\le C.$$

This in turn implies that the sequence $\{u_n\}$ is bounded in $L^{m^{**}}(\Omega)$. In order to obtain the $L^2(\Omega)$-bound for the sequence $\{|\nabla u_n|\}$ it is enough to take $T_{k_0}(u_n)$ and $G_{k_0}(u_n)$ in (2.4). We respectively obtain

$$\frac{\alpha}{2}\int_\Omega |\nabla T_{k_0}(u_n)|^2 \le \frac{k_0^2}{2\alpha}\int_\Omega |E|^2 + k_0\int_\Omega |f| \quad \text{and} \tag{2.14}$$
$$\frac{\alpha}{2}\int_\Omega |\nabla G_{k_0}(u_n)|^2 \le \frac{2k_0^2}{\alpha}\int_\Omega |E|^2 + \frac{2}{\mathcal{S}_2^2}\|f\|^2_{L^{(2^*)'}(\Omega)},$$

namely the sequence $\{|\nabla u_n|\}$ is bounded in $L^2(\Omega)$. Thus up to a subsequence $u_n$ converges weakly in $W_0^{1,2}(\Omega)$ to some $u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$.
Such a weak convergence it is enough to pass to the limit in the linear principal part of the operator of (2.4). As far as the convection term is concerned, we couple the fact that $u_n \to u$ $a.e.$ in $\Omega$ with the Lebesgue Theorem and conclude that the limit function $u$ is indeed a solution of (2.1).
**Case** $1 < m < (2^*)'$. In this range of the parameter $m$, the correct test function to consider is $\frac{(|G_k(u_n)|+1)^{2\gamma-1}-1}{2\gamma-1}\text{sign}(u_n)$ with $\gamma = \frac{m^{**}}{2^*}$. Notice that $\frac{1}{2} < 2\gamma-1 < 1$ but $2\gamma - 2 < 0$. We get

$$\alpha\int_\Omega \frac{|\nabla G_k(u_n)|^2}{(|G_k(u_n)|+1)^{2-2\gamma}} \le k\int_\Omega |E|\frac{|\nabla G_k(u_n)|}{(|G_k(u_n)|+1)^{1-\gamma}}$$
$$+\int_\Omega |G_k(u_n)|^\gamma |E|\frac{|\nabla G_k(u_n)|}{(|G_k(u_n)|+1)^{2-2\gamma}} + \frac{\|f\|_{L^m(\Omega)}}{2\gamma-1}\left(\int_\Omega (|G_k(u_n)|+1)^{(2\gamma-1)m'}\right)^{\frac{1}{m'}},$$

where we have used the inequality $\frac{|s|}{(1+|G_k(s)|)^{1-\gamma}} \le k + |G_k(s)|^\gamma$. Using Sobolev, Young and Hölder Inequalities and recalling the definition of $\gamma$, it results

$$C_{\alpha,\gamma}\left(\int_\Omega (|G_k(u_n)|+1)^{m^{**}} - 1\right)^{\frac{2}{2^*}} \le \frac{\alpha}{2}\int_\Omega \frac{|\nabla G_k(u_n)|^2}{(1+|G_k(u_n)|)^{2-2\gamma}}$$
$$\le \frac{k^2}{\alpha}\int_\Omega |E|^2 + \frac{1}{\alpha}\left(\int_{\{|u_n|>k\}} |E|^N\right)^{\frac{2}{N}}\left(\int_\Omega (|G_k(u_n)|+1)^{m^{**}}\right)^{\frac{2}{2^*}}$$
$$+\frac{\|f\|_{L^m(\Omega)}}{2\gamma-1}\left(\int_\Omega (|G_k(u_n)|+1)^{m^{**}}\right)^{\frac{1}{m'}}. \tag{2.15}$$

Taking advantage once more of (2.13), we select $k_0$ large enough in order to infer that

$$\left(\int_\Omega (|G_{k_0}(u_n)|+1)^{m^{**}}\right)^{\frac{2}{2^*}} \le C\left[1 + \left(\int_\Omega (|G_{k_0}(u_n)|+1)^{m^{**}}\right)^{\frac{1}{m'}}\right]$$

where the absolute constant $C = C(\alpha, m, \Omega, k_0, f)$ does not depend on $n$. This estimate and (2.15) imply that

$$\{u_n\} \quad \text{and} \quad \left\{\frac{|\nabla G_{k_0}(u_n)|}{(1+|G_{k_0}(u_n)|)^{1-\gamma}}\right\}$$

are bounded in $L^{m^{**}}(\Omega)$ and $L^2(\Omega)$ respectively. In order to recover the boundedness of $\{|\nabla u_n|\}$ in $L^{m^*}(\Omega)$, let us notice at firs that, taking $T_{k_0}(u_n)$ as test function in (2.4), we obtain as before (2.14); complementary we have that

$$\int_\Omega |\nabla G_{k_0}(u_n)|^{m^*} = \int_\Omega |\nabla G_{k_0}(u_n)|^{m^*}\frac{(1+|G_{k_0}(u_n)|)^{m^*(1-\gamma)}}{(1+|G_{k_0}(u_n)|)^{m^*(1-\gamma)}}$$
$$\le \left(\int_\Omega \frac{|\nabla G_{k_0}(u_n)|^2}{(1+|G_{k_0}(u_n)|)^{2-2\gamma}}\right)^{\frac{m^*}{2}}\left(\int_\Omega (1+G_k(u_n))^{m^{**}}\right)^{1-\frac{m^*}{2}},$$

and the right hand side above is bounded thanks to (2.15). Thus there exists $u \in W_0^{1,m^*}(\Omega)$ such that, up to a subsequence, $u_n$ weakly converge to $u$ in $W_0^{1,m^*}(\Omega)$. Reasoning as in the previous case we prove that such a $u$ is a solution of (2.1). □

### 2.1.2 Data in Marcinkiewicz spaces

In this section we study the Marcinkiewicz regularity of problem (2.3). As we already said our strategy consists in constructing uniform pointwise estimates for the rearrangement of $u_n$ and $\nabla u_n$. While there is a wide literature concerning estimates of the rearrangement of solutions of problems like (2.1) (see as examples [20] and [52] and references therein), the approach that we follows to achieve the estimate for the gradient is new.

**Lemma 2.4.** *For any $n \in \mathbb{N}$, let $u_n$ be the solution of (2.4) and denote with $\overline{u}_n$ its decreasing rearrangement. It follows that for $\gamma = \frac{1}{2m^{**}}$ there exist $C = C(\alpha, N, \|E\|_{L^N(\Omega)}, \gamma)$ such that*

$$\overline{u}_n(t) \le \overline{v}(s) := \frac{C}{t^\gamma}\int_t^{|\Omega|} s^{\frac{2}{N}+\gamma-1}\tilde{f}(s)ds. \tag{2.16}$$

**Remark 2.5.** *As already said in the Introduction, it results that $v(x) = \overline{v}(\omega_N|x|^N)$ solves the symmetrized problem (11). Thus Lemma 2.4 gives a pointwise uniform estimate of $\overline{u}_n$ trough the rearrangement of a suitable symmetric problem.*

*Proof.* As in [52] let us take $\frac{T_h(G_k(u_n))}{h}$ with $h > 0$ and $k \ge 0$ as test function in (2.4). Using (2.2) we get

$$\frac{\alpha}{h}\int_{\{k<|u_n|<k+h\}} |\nabla u_n|^2 \le \int_{\{|u_n|>k\}} |f| + \frac{(k+h)}{h}\int_{\{k<|u_n|<k+h\}} |E||\nabla u_n|. \tag{2.17}$$

Applying Hölder inequality to the last integral in the right hand side above and letting $h$ go to zero, we obtain

$$-\frac{d}{dk}\int_{\{|u_n|>k\}} |\nabla u_n|^2 \le \frac{\int_{\{|u_n|>k\}}|f|}{\alpha} + \frac{k}{\alpha}\left(-\frac{d}{dk}\int_{\{|u_n|>k\}} |\nabla u_n|^2\right)^{\frac{1}{2}}\left(-\frac{d}{dk}\int_{\{|u_n|>k\}} |E|^2\right)^{\frac{1}{2}}.$$

Let us set for any $n \in \mathbb{N}$ and $k > 0$

$$A_n(k) = |\{|u_n| > k\}|,$$

namely $A_n(k)$ is the distribution function of $u_n$. Consider moreover $D_n(s)$, with $s \in (0, |\Omega|)$, the pseudo rearrangement of $|E|^2$ with respect to $u_n$ (see (1.5) for the definition). Thanks to Lemma 1.5 we have that

$$\forall\, k > 0 \quad D(A_n(k))(-A_n'(k)) = -\frac{d}{dk}\int_{\{|u_n|>k\}} |E|^2. \tag{2.18}$$

Moreover Lemma 1.6 assures that $\|D_n\|_{L^{\frac{N}{2}}(0,\Omega)} \le \|E\|^2_{L^N(\Omega)}$. Hence using (1.10) it follows that

$$\left(-\frac{d}{dk}\int_{\{|u_n|>k\}} |\nabla u_n|^2\right)^{\frac{1}{2}} \le \frac{A_n(k)^{\frac{1}{N}-1}}{\alpha\sigma_N}\int_{\{|u_n|>k\}} |f|\big(-A_n'(k)\big)^{\frac{1}{2}} + \frac{k}{\alpha}D(A_n(k))^{\frac{1}{2}}(-A_n'(k))^{\frac{1}{2}}, \tag{2.19}$$

that can be rewritten, using once more (1.10), as

$$1 \le \frac{A_n(k)^{2(\frac{1}{N}-1)}}{\alpha\sigma_N^2}\int_{\{|u_n|>k\}} |f|\big(-A_n'(k)\big) + \frac{k}{\alpha\sigma_N}D(A_n(k))^{\frac{1}{2}}A_n(k)^{\frac{1}{N}-1}(-A_n'(k)).$$

Collecting the term $-A_n'(k)$, we get

$$1 \le \left[\frac{A_n(k)^{2(\frac{1}{N}-1)}}{\alpha\sigma_N^2}\int_0^{A_n(k)} \overline{f} + \frac{k}{\alpha\sigma_N}D_n(A_n(k))^{\frac{1}{2}}A_n(k)^{\frac{1}{N}-1}\right]\big(-A_n'(k)\big).$$

Thanks to the definition of decreasing rearrangement and using Proposition 1.2 in Chapter 1, it results

$$-\frac{d}{ds}\overline{u}_n(s) \leq \frac{1}{\alpha\sigma_N^2}s^{2(\frac{1}{N}-1)}\left(\int_0^s \bar{f}(\tau)d\tau\right) + \frac{1}{\alpha\sigma_N}D_n(s)^{\frac{1}{2}}s^{\frac{1}{N}-1}\overline{u}_n(s).$$

Defining for fixed $t \in (0, |\Omega|)$ the auxiliary function

$$R_n(s) = e^{\frac{1}{\alpha\sigma_N}\int_t^s D_n(\tau)^{\frac{1}{2}}\tau^{\frac{1}{N}-1}d\tau},$$

we finally obtain that

$$-\frac{d}{ds}\big(R(s)\overline{u}_n(s)\big) \leq \frac{1}{\alpha\sigma_N^2}R_n(s)s^{2(\frac{1}{N}-1)}\left(\int_0^s \bar{f}(\tau)d\tau\right). \tag{2.20}$$

Notice that in (2.20) the presence of the lower order term is hidden inside the function $R_n(s)$. In order to use (2.20) to infer that $\{u_n\}$ is bounded in some Marcinkiewicz space, we have to estimate $R_n(s)$ in a convenient way. By means of Young inequality and recalling that $\|D_n\|_{L^{\frac{N}{2}}(0,\Omega)} \leq \|E\|^2_{L^N(\Omega)}$ (see Lemma 1.6), we obtain that

$$\int_t^s D_n(\tau)^{\frac{1}{2}}\tau^{\frac{1}{N}-1}d\tau \leq C_\gamma\|E\|_{L^N(\Omega)} + \gamma\int_t^s \tau^{-1}d\tau \qquad \text{with } \gamma = \frac{1}{2m^{**}}. \tag{2.21}$$

Hence the function $R_n(s)$ satisfies the following inequality

$$R_n(s) \leq C_{\gamma,\|E\|_{L^N(\Omega)}}\left(\frac{s}{t}\right)^\gamma.$$

Thus (2.20) becomes

$$-\frac{d}{ds}\big(R(s)\overline{u}_n(s)\big) \leq \frac{C_{\gamma,\|E\|_{L^N(\Omega)}}}{\alpha\sigma_N^2 t^\gamma}s^{2(\frac{1}{N}-1)+\gamma}\left(\int_0^s \bar{f}(\tau)d\tau\right).$$

Integrating between $t$ and $|\Omega|$ and recalling that by definition of both $\overline{u}_n(|\Omega|) = 0$ and $R(t) = 1$, we get

$$\overline{u}_n(t) = -R(|\Omega|)\overline{u}_n(|\Omega|) + R(t)\overline{u}_n(t) \leq \frac{C_{\gamma,\|E\|_{L^N(\Omega)}}}{\alpha\sigma_N^2 t^\gamma}\int_t^{|\Omega|} s^{2(\frac{1}{N}-1)+\gamma}\left(\int_0^s \bar{f}(\tau)d\tau\right)ds.$$

□

Let us now give a pointwise estimate for the rearrangement of $|\nabla u_n|$. We recall that for any measurable function $h(x)$ it follows that $\overline{h}(s) \leq \frac{1}{s}\int_0^s \overline{h}(t)dt$.

**Lemma 2.6.** *Let $\overline{|\nabla u_n|}$ be the decreasing rearrangement of $|\nabla u_n|$. There exists a positive constant $\mathcal{C} = \mathcal{C}(N, \alpha, \|E\|_{L^N(\Omega)}, \epsilon)$ such that*

$$\frac{1}{s}\int_0^s \overline{|\nabla u_n|} \leq \mathcal{C}\left[\frac{1}{s}\int_0^s \left(\overline{v}(t)D_n^{\frac{1}{2}}(t) + \tilde{f}t^{\frac{1}{N}}\right)dt + \left(\frac{1}{s}\int_s^{|\Omega|}\left(\overline{v}(t)^2 D_n(t) + \tilde{f}^2 t^{\frac{2}{N}}\right)dt\right)^{\frac{1}{2}}\right], \tag{2.22}$$

*where $\overline{v}(s)$ and $D_n$ are the functions introduced respectively in (2.16) and (2.18).*

*Proof.* Taking advantage of Lemma 1.3 and recall Remark 1.4, it follows that

$$\int_0^s \overline{|\nabla u_n|}d\tau = \int_{\widetilde{\Omega}_n(s)} |\nabla u_n|dx$$

$$= \int_{\widetilde{\Omega}_n(s)\cap\{|u_n|>\overline{u}_n(s)\}} |\nabla u_n|dx + \int_{\widetilde{\Omega}_n(s)\cap\{|u_n|\leq\overline{u}_n(s)\}} |\nabla u_n|dx$$

$$\leq \int_{\{|u_n|>\overline{u}_n(s)\}} |\nabla u_n| dx + \left( \int_{\{|u_n|\leq \overline{u}_n(s)\}} |\nabla u_n|^2 dx \right)^{\frac{1}{2}} |\widetilde{\Omega}_n(s)|^{\frac{1}{2}} \leq I_1(s) + I_2^{\frac{1}{2}}(s) s^{\frac{1}{2}}.$$

In order to estimate $I_2$ notice that the functions $k \to \int_{\{|u_n|>k\}} |\nabla u_n|$ and $s \to \overline{u}_n(s)$ are absolutely continuous and hence (see [4])

$$\frac{d}{ds} \int_{\{|u_n|>\overline{u}_n(s)\}} |\nabla u_n|^2 = \frac{d}{dk} \int_{\{|u_n|>k\}} |\nabla u_n|^2 \Big|_{k=\overline{u}_n(s)} \frac{d}{ds} \overline{u}_n(s). \tag{2.23}$$

Thus we infer from (2.19) that

$$\frac{d}{ds} \int_{\{|u_n|>\overline{u}_n(s)\}} |\nabla u_n|^2 \leq C \left( \overline{u}_n(s)^2 D(s) + s^{\frac{2}{N}} \tilde{f}(s)^2 \right).$$

Integrating between $s$ and $|\Omega|$, we get

$$I_2 = \int_{\{|u_n|\leq \overline{u}_n(s)\}} |\nabla u_n|^2 = \int_{\Omega} |\nabla u_n|^2 - \int_{\{|u_n|>\overline{u}_n(s)\}} |\nabla u_n|^2$$
$$\leq C \left[ \int_s^{|\Omega|} \overline{u}_n(t)^2 D(t) + t^{\frac{2}{N}} \tilde{f}(t)^2 dt \right].$$

As far as $I_1$ is concerned let us notice that

$$\int_{\{\overline{u}_n(s)\leq |u_n|<\overline{u}_n(s+h)\}} |\nabla u_n| \leq \left( \int_{\{\overline{u}_n(s)\leq |u_n|<\overline{u}_n(s+h)\}} |\nabla u_n|^2 \right)^{\frac{1}{2}} |\{\overline{u}_n(s) \leq |u_n| < \overline{u}_n(s+h)\}|^{\frac{1}{2}}. \tag{2.24}$$

Taking the limit as $h \to 0$ and noticing that Proposition 1.2 implies $|\{|u_n| > \overline{u}_n(s)\}|' \leq 1$, we obtain

$$\frac{d}{ds} \int_{\{|u_n|>\overline{u}_n(s)\}} |\nabla u_n| \leq \left( \frac{d}{ds} \int_{\{|u_n|>\overline{u}_n(s)\}} |\nabla u_n|^2 \right)^{\frac{1}{2}} \leq C \left( \overline{u}_n(s) D(s)^{\frac{1}{2}} + \tilde{f}(s) s^{\frac{1}{N}} \right).$$

Hence we have the following estimate for $I_1$

$$I_1 \leq C \int_0^s \left( \overline{u}_n(t) D(t)^{\frac{1}{2}} + \tilde{f}(t) t^{\frac{1}{N}} \right) dt,$$

Putting together the obtained information for $I_1$ and $I_2$ and recalling (2.16), we prove (2.22). □

Now we are in the position to state and prove the main result of this chapter.

**Theorem 2.7.** *Assume* (2.2), (2.5) *and* $f \in M^m(\Omega)$ *with* $1 < m < \frac{N}{2}$. *Hence there exists* $u$ *solution of* (2.3). *Moreover*

- *if* $1 < m < (2^*)'$, *then* $u \in M^{m^{**}}(\Omega)$ *and* $|\nabla u| \in M^{m^*}(\Omega)$.
- *if* $(2^*)' < m < \frac{N}{2}$, *then* $u \in W_0^{1,2}(\Omega) \cap M^{m^{**}}(\Omega)$

*Proof.* **Case** $1 < m < (2^*)'$**.** From (2.16) it follows that

$$\overline{u}_n(t) \leq \overline{v}(t) \leq \frac{C\|f\|_{M^m(\Omega)}}{t^\gamma} \int_t^{|\Omega|} s^{\frac{2}{N}+\gamma-1-\frac{1}{m}} ds \leq C\|f\|_{M^m(\Omega)} t^{-\frac{1}{m^{**}}} \tag{2.25}$$

where the last inequality follows from the choice $\gamma = \frac{1}{2m^{**}} < \frac{1}{m^{**}}$ and the fact that $m < \frac{N}{2}$. From estimates (2.22) and (2.25) and taking advantage of Lemma 1.6, we deduce that

$$\overline{|\nabla u_n|} \le C\left[\frac{1}{s}\int_0^s (\overline{v}(t)D_n^{\frac{1}{2}}(t) + \tilde{f}t^{\frac{1}{N}})dt + \left(\frac{1}{s}\int_s^{|\Omega|}(\overline{v}(t)D_n(t) + \tilde{f}^2 t^{\frac{2}{N}})dt\right)^{\frac{1}{2}}\right] \le$$

$$\le \frac{C}{s}\left[\left(\int_0^s D_n^{\frac{N}{2}}(t)dt\right)^{\frac{1}{N}}\left(\int_0^s \overline{v}(t)^{\frac{N}{N-1}}dt\right)^{\frac{N-1}{N}} + \int_0^s t^{\frac{1}{N}-\frac{1}{m}}dt\right]$$

$$+\frac{C}{s^{\frac{1}{2}}}\left[\left(\int_s^{|\Omega|} D_n^{\frac{N}{2}}(t)dt\right)^{\frac{2}{N}}\left(\int_s^{|\Omega|}\overline{v}(t)^{\frac{2N}{N-2}}dt\right)^{\frac{N-2}{N}} + \int_s^{|\Omega|} t^{\frac{2}{N}-\frac{2}{m}}dt\right]^{\frac{1}{2}} \le C\|f\|_{L^{m,\infty}(\Omega)}t^{-\frac{1}{m^*}}.$$

From this estimate we infer that there exists $u \in W_0^{1,r}(\Omega)$, with $1 < r < \frac{Nm}{N-m}$, such that, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{in} \quad W_0^{1,r}(\Omega).$$

This weak convergence is enough to pass to the limit in the left hand side of (2.4) for any $\varphi \in C_0^1(\Omega)$. In order to handle the lower order term, notice that for any measurable $\omega \subset \Omega$ it follows that

$$\int_\omega |u_n||E_n| \le C\int_0^{|\omega|}\frac{\overline{v}(t)}{t^{\frac{1}{N}}}dt \le C\|f\|_{L^1(\Omega)}\int_0^{|\omega|} t^{-\frac{N-p}{N}-\frac{1}{N}} \le C|\omega|^{\frac{1}{N}},$$

where we used (2.25). Namely the sequence

$$\left\{\int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|}E_n(x)\right\}$$

is equi-integrable. This together with the $a.e.$ convergence of $u_n$ allows us to take advantage of Vitali Theorem and prove that

$$\int_\Omega A(x)\nabla u\nabla\phi = \int_\Omega uE(x)\nabla\phi + \int_\Omega f(x)\phi \quad \forall\, \phi \in C_0^1(\Omega).$$

Moreover thanks to Proposition 1.1

$$u \in M^{m^{**}}(\Omega) \quad \text{and} \quad |\nabla u| \in M^{m^*}(\Omega).$$

**Case** $(2^*)' < m < \frac{N}{2}$**.** Taking $u_n$ as a test function in (2.4) and thanks to Young inequality we obtain

$$\frac{\alpha}{2}\int_\Omega |\nabla u_n| \le C\int_\Omega |u_n|^2|E|^2 + \int_\Omega |f||u_n|$$

$$\le C\int_0^{|\Omega|}(\overline{v}_n^2\overline{E}^2 + \overline{f}\overline{v}_n) \le C\int_0^{|\Omega|} t^{-\frac{2}{m^*}}dt \le C$$

where we have used Hardy Inequality and (2.25) (that holds true for $1 < m < \frac{N}{2}$). Thus there exists a function $u \in W_0^{1,2}(\Omega)$ such that, up to a subsequence $u_n \rightharpoonup u$ in $W_0^{1,2}(\Omega)$. As in the previous case we can prove that $u$ is indeed a solution of (2.4). Moreover thanks to Proposition 1.1

$$\overline{u}(t) \le C\|f\|_{M^m(\Omega)}t^{-\frac{1}{m^{**}}}.$$

□

## 2.2 Convection term in $M^N(\Omega)$

We treat now problem (2.1) assuming that

$$E:\Omega\to\mathbb{R}^N \quad \text{belongs to} \quad \left(M^N(\Omega)\right)^N. \tag{2.26}$$

An easy example to bear in mind is $E(x)=B\frac{x}{|x|^2}$. As already mentioned in the Introduction, this assumption is reasonable thanks to the inequality

$$\mathcal{H}\|v\|_{L^{2^*,2}(\Omega)}\le\|\nabla v\|_{L^2(\Omega)} \quad \forall\, v\in W_0^{1,2}(\Omega), \quad \text{with} \quad \mathcal{H}=\omega_N^{\frac{1}{N}}\frac{N-2}{2}, \tag{2.27}$$

proved in [2]. In some sense this section can be seen as a technical refinement of the previous one. The important difference is that if we want to preserve the relationship between regularity of the data and regularity of the solution sketched in Table 1, we need to impose some restriction on the $M^m(\Omega)$ norm of $E$ or, more precisely, on the size of $\overline{E}$ in a neighborhood of zero (see Comment 2.11).

### 2.2.1 Data in Lebesgue spaces

Let us start with the following result.

**Theorem 2.8.** *Assume (2.2), $f\in L^m(\Omega)$, with $1<m<\frac{N}{2}$, and that*

$$E\in M^N(\Omega) \quad \textit{with} \quad \|E\|_{M^N(\Omega)}<\alpha\omega_N^{\frac{1}{N}}\frac{N-2m}{m}. \tag{2.28}$$

*Hence there exists $u$ solution of (2.3). Moreover*

- *if $1<m<(2^*)'$, then $u\in W_0^{1,m^*}(\Omega)$;*
- *if $(2^*)'\le m<\frac{N}{2}$, then $u\in W_0^{1,2}(\Omega)\cap L^{m^{**}}(\Omega)$*

**Comment 2.9.** *A particular case of Theorem 2.8 is $0\in\Omega$ and*

$$E(x)=B\frac{x}{|x|^2} \quad \textit{with} \quad B<\alpha\frac{N-2m}{m}.$$

*Such case has been treated in [28], that inspired our approach.*

*Proof.* **Case** $1<m<(2^*)'$. Take $\phi_\epsilon(u_n)=(\epsilon+|u_n|)^{2\gamma-1}-\epsilon^{2\gamma-1}\text{sign}(u_n)$, with $\gamma=\frac{m^{**}}{2^*}$ and $\epsilon>0$, as a test function in (2.4). We get

$$\alpha\int_\Omega\frac{|\nabla u_n|^2}{(\epsilon+|u_n|)^{2-2\gamma}}\le\int_\Omega(\epsilon+|u_n|)^\gamma|E|\frac{|\nabla u_n|}{(\epsilon+|u_n|)^{1-\gamma}}+\frac{\|f\|_{L^m(\Omega)}}{2\gamma-1}\left(\int_\Omega(\epsilon+|u_n|)^{(2\gamma-1)m'}\right)^{\frac{1}{m'}} \tag{2.29}$$

We set now

$$I_\epsilon=\frac{\left(\int_\Omega(\epsilon+|u_n|)^{2\gamma}|E|^2\right)^{\frac12}}{\left(\int_\Omega[(\epsilon+|u_n|)^\gamma-\epsilon^\gamma]^2|E|^2\right)^{\frac12}}.$$

Since $1\le I_\epsilon\le 1+\epsilon^\gamma\|E\|_{L^2(\Omega)}$, it follows that $\lim_{\epsilon\to0}I_\epsilon=1$ uniformly with respect to $n$. The the lower order term of (2.29) becomes

$$\int_\Omega(\epsilon+|u_n|)^\gamma|E|\frac{|\nabla u_n|}{(\epsilon+|u_n|)^{1-\gamma}}\le I_\epsilon\left(\int_\Omega[(\epsilon+|u_n|)^\gamma-\epsilon^\gamma]^2|E|^2\right)^{\frac12}\left(\int_\Omega\frac{|\nabla u_n|^2}{(\epsilon+|u_n|)^{2-2\gamma}}\right)^{\frac12}.$$

At this point set $z_{\epsilon,\gamma} := (\epsilon + |u_n|)^\gamma - \epsilon^\gamma$ and notice that, thanks to (2.27), it follows

$$\left(\int_\Omega z_{\epsilon,\gamma}^2 |E|^2\right)^{\frac{1}{2}} \le \|E\|_{M^N(\Omega)} \left(\int_0^{|\Omega|} \overline{z}_{\epsilon,\gamma}^2(t) t^{\frac{2}{2^*}} \frac{dt}{t}\right)^{\frac{1}{2}}$$

$$\le \frac{\|E\|_{M^N(\Omega)}}{\mathcal{H}} \left(\int_\Omega |\nabla z_{\epsilon,\gamma}|^2\right)^{\frac{1}{2}} = \frac{\gamma \|E\|_{M^N(\Omega)}}{\mathcal{H}} \left(\int_\Omega \frac{|\nabla u_n|^2}{(\epsilon + |u_n|)^{2-2\gamma}}\right)^{\frac{1}{2}}.$$

Thus the estimate of the convection term becomes

$$\int_\Omega (\epsilon + |u_n|)^\gamma |E| \frac{|\nabla u_n|}{(\epsilon + |u_n|)^{1-\gamma}} \le I_\epsilon \frac{\gamma}{\mathcal{H}} \|E\|_{M^N(\Omega)} \int_\Omega \frac{|\nabla u_n|^2}{(\epsilon + |u_n|)^{2-2\gamma}}.$$

Since assumption (2.28) assures that $\alpha > \frac{\gamma}{\mathcal{H}} \|E\|_{M^N(\Omega)}$ and $I_\epsilon \to 1$ as $\epsilon$ goes to 0, we can absorb the lower order term of (2.29) in the principal part taking $\epsilon$ small enough. Then using Sobolev inequality and letting $\epsilon \to 0$ we obtain that the sequence $\{u_n\}$ is bounded in $L^{m^{**}}(\Omega)$. This also implies that

$$\left\{ \frac{|\nabla u_n|^2}{(1 + |u_n|)^{2-2\gamma}} \right\} \quad \text{is bounded in } L^2(\Omega),$$

from which we deduce that $\{|\nabla u_n|\}$ is bounded in $L^{m^*}(\Omega)$. Thus there exists $u \in W_0^{1,m^*}(\Omega)$ such that, up to a sub sequence, $u_n$ weakly converges to $u$ in $W_0^{1,m^*}(\Omega)$. Noticing that for any measurable $A \subset \Omega$

$$\int_A |uE| \le C \int_0^{|A|} t^{-\frac{1}{m^*}} \le C|A|^{1-\frac{1}{m^*}},$$

we can pass to the limit in (2.4) and conclude that $u \in W_0^{1,m^*}(\Omega)$ is a solution of (2.1).

**Case** $(2^*)' \le m < \frac{N}{2}$**.** We just sketch the main differences with respect to the previous case. Taking $u_n$ as a test function in (2.4), it results

$$\alpha \int_\Omega |\nabla u_n|^2 \le \int_\Omega |u_n||E||\nabla u_n| + \int_\Omega |f||u_n|$$

$$\le \|E\|_{M^N(\Omega)} \frac{1}{\omega_N^{\frac{1}{N}}} \frac{2}{N} \int_\Omega |\nabla u_n| + \mathcal{S}_2 \|f\|_{L^{(2^*)'}(\Omega)} \left(\int_\Omega |\nabla u_n|^2\right)^{\frac{1}{2}},$$

that gives the bound of $\{u_n\}$ in $W_0^{1,2}(\Omega)$. Moreover taking $|u_n|^{2\gamma-2}u_n$ with $\gamma = \frac{m^{**}}{2^*}$ as a test function in (2.4) (now $2\gamma - 2 > 0$), it follows as in the previous case that

$$\{u_n\} \quad \text{is bounded in} \quad L^{m^{**}}(\Omega).$$

The existence of a solution in $W_0^{1,2}(\Omega)$ is straightforward. □

Theorem 2.8 is somehow unsatisfactory because, even if it deals with $E \in \left(M^N(\Omega)\right)^N$, it does not generalize Theorem 2.3: there exist $E$ in $\left(L^N(\Omega)\right)^N$ that do not satisfy (2.8). This problem is solved in the next subsection via symmetrization techniques.

### 2.2.2 Data in Marcinkiewicz spaces

**Theorem 2.10.** *Assume (2.2), $f \in M^m(\Omega)$ with $1 < m < \frac{N}{2}$ and $E \in \left(M^N(\Omega)\right)^N$ such that*

$$E = \mathcal{F} + \mathcal{E} \quad \text{with} \quad \mathcal{F} \in (L^\infty(\Omega))^N \quad \text{and} \quad \overline{\mathcal{E}}(s) \le \frac{B}{s^{\frac{1}{N}}} \quad \text{with} \quad B < \alpha\omega_N^{\frac{1}{N}} \frac{N-2m}{m}. \tag{2.30}$$

*Then there exists $u$ solution of (2.3). Moreover*
*(i) if $1 < m < (2^*)'$, then $|u| \in M^{m^{**}}(\Omega)$ and $|\nabla u| \in M^{m^*}$;*
*(ii) if $(2^*)' < m < \frac{N}{2}$, then $u \in W_0^{1,2}(\Omega) \cap M^{m^{**}}(\Omega)$.*

**Remark 2.11.** *We split in two the contribution of the $E$ because the real obstruction in the achievement of the required estimates is not exactly the value of the Marcinkiewicz norm of $E$ but rather the size of its singular component near zero. And we control it by means of the constant $B$ in (2.30). This is essential in the estimates for $\overline{u}_n$ (2.34) and (2.35) below. Notice moreover that any $E \in \left(L^N(\Omega)\right)^N$ satisfies (2.30).*

*Proof.* The fact that $|E| \in M^N(\Omega)$ require some additional technicalities but the general strategy is the same of the one followed in the Subsection 2.1.2; here we sketch the main differences. The proof is split into the following steps.
**Step 1.** Pointwise estimate for $\overline{u}_n$ and $\overline{|\nabla u_n|}$.
**Step 2.** A priori estimate for $\overline{u}_n$ and $\overline{|\nabla u_n|}$.
**Step 3.** Existence and regularity for $1 < m < (2^*)'$.
**Step 4.** Existence and regularity for $(2^*)' < m < \frac{N}{2}$.

**Step 1.** Setting $\mathcal{E}_n = T_n(\mathcal{E})$, the family of approximating problems that we consider in this case is

$$\int_\Omega A(x)\nabla u_n \nabla\phi = \int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|}(\mathcal{F}+\mathcal{E}_n)\nabla\phi + \int_\Omega f_n\phi \quad \forall\, \phi \in W_0^{1,2}(\Omega), \tag{2.31}$$

whose existence of a solution $u_n$ for any $n \in \mathbb{N}$ is assured by Shauder's fix point theorem. Taking $\frac{T_h(G_k(u_n))}{h}$, with $h, k > 0$, as a test function in (2.31), we obtain, as in the proof of Lemma 2.4, that

$$-\frac{d}{ds}\overline{u}_n \le \frac{1}{\alpha\sigma_N^2} s^{2\left(\frac{1}{N}-1\right)} \int_0^s \overline{f}(\tau)d\tau + \frac{1}{\alpha\sigma_N}\left((D_{1,n}(s))^{\frac{1}{2}} + (D_{2,n}(s))^{\frac{1}{2}}\right) s^{\frac{1}{N}-1}\overline{u}_n,$$

where $D_{1,n}$ and $D_{2,n}$ are given by Lemma 1.5 and

$$D_{1,n}(s) := \frac{d}{ds}\int_{\Omega_n(s)} |\mathcal{F}(x)|^2 dx \quad \text{and} \quad D_{2,n}(s) := \frac{d}{ds}\int_{\Omega_n(s)} |\mathcal{E}(x)|^2 dx.$$

As in Lemma 2.4 we define for fixed $t \in (0, |\Omega|)$ the auxiliary function

$$R_n(s) = e^{\frac{1}{\alpha\sigma_N}\int_t^s \left(D_{1,n}(\tau)^{\frac{1}{2}} + D_{2,n}(\tau)^{\frac{1}{2}}\right)\tau^{\frac{1}{N}-1}d\tau},$$

in order to obtain that

$$-\frac{d}{ds}\left(R_n(s)\overline{u}_n(s)\right) \le C_1 R_n(s) s^{2\left(\frac{1}{N}-1\right)} \left(\int_0^s \bar{f}(\tau)d\tau\right). \tag{2.32}$$

To estimate the function $R_n(s)$ notice that by constriction

$$\frac{1}{\alpha\sigma_N}\int_t^s D_{1,n}(\tau)^{\frac{1}{2}}\tau^{\frac{1}{N}-1}d\tau \le \|\mathcal{F}\|_{L^\infty(\Omega)}\frac{N}{\alpha\sigma_N}|\Omega|^N$$

and

$$\frac{1}{\alpha\sigma_N}\int_t^s D_{2,n}(\tau)^{\frac{1}{2}}\tau^{\frac{1}{N}-1}d\tau \le \frac{1}{2\alpha\sigma_N B}\int_t^s D_{2,n}(\tau)\tau^{\frac{2}{N}-1}d\tau + \frac{B}{2\alpha\sigma_N}\int_t^s \frac{1}{\tau}d\tau \tag{2.33}$$

$$\leq \frac{1}{2\alpha\sigma_N B}\left[s^{\frac{2}{N}-1}\int_0^s \overline{\mathcal{E}}^2 - t^{\frac{2}{N}-1}\int_0^t \overline{\mathcal{E}}^2 - \frac{2-N}{N}\int_t^s \tau^{\frac{2}{N}-2}\int_0^\tau \overline{\mathcal{E}}^2 d\tau\right] + \frac{B}{2\alpha\sigma_N}\log\left(\frac{s}{t}\right)$$
$$\leq \frac{NB}{2\alpha\sigma_N(N-2)} + \frac{B}{\alpha\sigma_N}\log\left(\frac{s}{t}\right),$$

where we have used Young Inequality, integration by parts and assumption (2.30). Hence, integrating (2.32) between $t$ and $\Omega$ and setting $\gamma = \frac{B}{\alpha\sigma_N}$, we get

$$\overline{u}_n(t) = -R(|\Omega|)\overline{u}_n(|\Omega|) + R(t)\overline{u}_n(t) \leq \overline{v}(t) := \frac{C}{t^\gamma}\int_t^{|\Omega|} s^{\frac{2}{N}-1+\gamma}\tilde{f}(s)ds. \tag{2.34}$$

The pointwise estimate for the rearrangement of the gradient is obtained as in Lemma 2.6 with the following preliminary estimate

$$\int_0^s \overline{|\nabla u_n|}d\tau \leq \int_{\{|u_n|>\overline{u}_n(s)\}} |\nabla u_n|dx + \left(\int_{\{|u_n|\leq\overline{u}_n(s)\}} |\nabla u_n|^2 dx\right)^{\frac{1}{2}} s^{\frac{1}{2}},$$

from which one deduces that

$$\frac{1}{s}\int_0^s \overline{|\nabla u_n|} \leq \mathcal{C}\left[\frac{1}{s}\int_0^s \left(\overline{v}(t)D_n^{\frac{1}{2}}(t) + \tilde{f}t^{\frac{1}{N}}\right)dt + \left(\frac{1}{s}\int_s^{|\Omega|}\left(\overline{v}(t)^2 D_n(t) + \tilde{f}^2 t^{\frac{2}{N}}\right)dt\right)^{\frac{1}{2}}\right], \tag{2.35}$$

where, in order to have a more compact notation, we set

$$D_n(s) = \frac{d}{ds}\int_{\Omega_n(s)} |\mathcal{F}+\mathcal{E}_n|^2 dx = \frac{d}{ds}\int_{\Omega_n(s)} |E(x)|^2 dx.$$

**Step 2.** From (2.34), it follows

$$\overline{u}_n(t) \leq \frac{C}{t^\gamma}\int_t^{|\Omega|} s^{\frac{2}{N}-1+\gamma-\frac{1}{m}}ds \leq Cs^{-\frac{1}{m^{**}}}, \tag{2.36}$$

where we have used that by definition $\gamma < \frac{1}{m^{**}}$. The achievement of the estimate for $\{|\nabla u_n|\}$ is more technical. Notice at first that, integrating by part and using Lemma 1.5, we get

$$\int_0^s t^{-\frac{1}{m^{**}}} D_n^{\frac{1}{2}}(t)dt \leq \left(\int_0^s t^{-\frac{1}{m^{**}}} D_n(t)dt\right)^{\frac{1}{2}}\left(\int_0^s t^{-\frac{1}{m^{**}}}dt\right)^{\frac{1}{2}}$$
$$\leq \left(s^{-\frac{1}{m^{**}}}\int_0^s \overline{E}^2(\tau)d\tau + \frac{1}{m^{**}}\int_0^s t^{-1-\frac{1}{m^{**}}}\int_0^t \overline{E}^2(\tau)d\tau dt\right)^{\frac{1}{2}} s^{\frac{1}{2}-\frac{1}{2m^{**}}}$$
$$= \left(Cs^{1-\frac{2}{N}-\frac{1}{m^{**}}}\right)^{\frac{1}{2}} s^{\frac{1}{2}-\frac{1}{2m^{**}}} = Cs^{1-\frac{1}{m^*}},$$

and that

$$\int_s^{|\Omega|} t^{-\frac{2}{m^{**}}} D_n(t)dt \leq |\Omega|^{-\frac{2}{m^{**}}}\int_0^{|\Omega|}\overline{E}^2(\tau)d\tau + \frac{2}{m^{**}}\int_s^{|\Omega|} t^{-1-\frac{2}{m^{**}}}\int_0^t \overline{E}^2(\tau)d\tau dt$$
$$= Cs^{1-\frac{2}{m^*}} - \frac{m^*}{2-m^*}|\Omega|^{1-\frac{2}{m^*}} \leq Cs^{1-\frac{2}{m^*}}.$$

Thanks to (2.36) and to these two pieces of information, estimate (2.35) becomes

$$\frac{1}{s}\int_0^s \overline{|\nabla u_n|} \leq \mathcal{C}\left[\frac{1}{s}\int_0^s \left(t^{-\frac{1}{m^{**}}} D_n^{\frac{1}{2}}(t) + t^{-\frac{1}{m^*}}\right)dt + \left(\frac{1}{s}\int_s^{|\Omega|}\left(t^{-\frac{2}{m^{**}}} D_n(t) + t^{-\frac{2}{m^*}}\right)dt\right)^{\frac{1}{2}}\right]$$

$$\leq Ct^{-\frac{1}{m^*}}.$$

**Step 3.** Thanks to the previous steps we deduce, as in the proof of Theorem 2.7, the existence of a candidate solution $u \in W_0^{1,r}(\Omega)$ with $1 < r < \frac{Nm}{N-m}$ such that up to a subsequence

$$u_n \rightharpoonup u \quad \text{in} \quad W_0^{1,r}(\Omega),$$

and

$$\frac{u_n}{1+\frac{1}{n}|u_n|}(\mathcal{F}(x)+\mathcal{E}_n(x)) \to uE(x) \quad \text{in} \quad L^1(\Omega).$$

This is enough to pass to the limit in (2.31) and conclude that $u$ is a solution of (2.1). Moreover thanks to the almost everywhere convergence of $\{|\nabla u_n|\}$ provided by Lemma 1.10 we conclude that

$$|u| \in M^{m^{**}}(\Omega) \quad \text{and} \quad |\nabla u| \in M^{m^*}(\Omega).$$

This weak convergence is enough to pass to the limit in the left hand side of (2.4) for any $\varphi \in C_0^1(\Omega)$. In order to handle the lower order term, notice that for every $A \subset \Omega$ it follows that

$$\int_A |u_n||E_n| \leq C\int_0^{|A|} \frac{\overline{v}(t)}{t^{\frac{1}{N}}}dt \leq C\|f\|_{L^1(\Omega)} \int_0^{|A|} t^{-\frac{N-p}{N}-\frac{1}{N}} \leq C|A|^{\frac{1}{N}},$$

where we used (2.25). Namely the sequence

$$\left\{\int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|}E_n(x)\right\}$$

is equi-integrable. This together with the $a.e.$ convergence of $u_n$ allows us to take advantage of Vitali Theorem and prove that

$$\int_\Omega A(x)\nabla u\nabla\phi = \int_\Omega uE(x)\nabla\phi + \int_\Omega f(x)\phi \quad \forall\, \phi \in C_0^1(\Omega).$$

To conclude we still have to prove that

$$\|u\|_{M^{m^{**}}(\Omega)} + \|\nabla u\|_{M^{m^*}(\Omega)} \leq C\|f\|_{M^m(\Omega)}.$$

To this aim we use the almost everywhere convergence of $\{|\nabla w_n|\}$ and the argument of Theorem 2.7 to infer that $u$ itself satisfies (2.34) and (2.35).

**Step 4.** Choosing $u_n$ as a test function in (2.4) and Using Hölder's inequality we get

$$\alpha\int_\Omega |\nabla u_n|^2 \leq \left(\int_\Omega |E|^2|u|^2\right)^{\frac{1}{2}}\left(\int_\Omega |\nabla u_n|^2\right)^{\frac{1}{2}} + \frac{1}{\mathcal{S}_2}\|f\|_{L^{(2^*)'}(\Omega)}\left(\int_\Omega |\nabla u_n|^2\right)^{\frac{1}{2}}$$

Moreover thanks to (2.36) it results that $\{u_n\}$ is bounded in $L^q(\Omega)$ for $2^* < q < m^{**}$. Thus

$$\int_0^{|\Omega|} t^{-\frac{N}{2}}\overline{v}^2(t)dt \leq \left(\int_0^{|\Omega|} \overline{v}^q(t)dt\right)^{\frac{2}{q}}\left(\int_0^{|\Omega|} t^{-\frac{2q}{N(q-2)}}\right)^{\frac{q-2}{q}} \leq C$$

since $1 - \frac{2q}{N(q-2)} > 0$. Hence

$$\|\nabla u_n\|_{L^2(\Omega)} \leq \|E\|_{M^N(\Omega)}\left(\int_0^{|\Omega|} t^{-\frac{N}{2}}\overline{v}^2(t)dt\right)^{\frac{1}{2}} + \frac{1}{\mathcal{S}_2}\|f\|_{L^{(2^*)'}(\Omega)} \leq C$$

At this point we conclude as in the previous step that there exists a function $u \in W_0^{1,2}(\Omega)$ solution of (2.3) and that belongs to $M^{m^{**}}(\Omega)$. □

# Chapter 3

# Drift lower order term

In this chapter we consider the following problem

$$\begin{cases} -\text{div}\,(A(x)\nabla w) = E(x)\nabla w + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where as before $\Omega$ is a bounded open set of $\mathbb{R}^N$ with $N > 2$, $A(x)$ is a measurable matrix that satisfies the standard condition (2.2) and as before

$$E \in \left(L^N(\Omega)\right)^N \quad \text{or} \quad E \in \left(M^N(\Omega)\right)^N \quad \text{and} \quad f \in L^m(\Omega) \quad \text{or} \quad f \in M^m(\Omega),$$

for some $m > 1$. The weak formulation of (3.1) is

$$w \in W_0^{1,r}(\Omega) \; : \; \int_\Omega A(x)\nabla w \nabla \phi = \int_\Omega E(x)\nabla w\, \phi + \int_\Omega f(x)\phi \quad \forall\, \phi \in C_0^1(\Omega). \tag{3.2}$$

where $r > \frac{N}{N-1}$. Notice that, while in (2.3) one can in principle consider $W_0^{1,1}(\Omega)$ solutions, here an higher regularity is required for the drift term of (3.2) to be well defined. As in Chapter 2, our aim is to provide existence and regularity results for problem (3.1), under borderline assumptions on the summability of $E$ so that the relationship between the summability of the datum and the solution continues to be expressed by Table 1. Also in this drift case it results that for every $|E| \in L^N(\Omega)$ we recover for problem (3.2) the same results of the case $E \equiv 0$. On the other hand, if $|E|$ belongs to $M^N(\Omega)$ a smallness condition of its size is required. Our main references are [42], [50], [52] and [69] (see the Introduction for a more detailed discussion).

Our starting point is once again to built a sequence of approximating solutions and thereafter to attain bounds in suitable spaces for such a sequence. Consider hence

$$w_n \in W_0^{1,2}(\Omega) \; : \; \int_\Omega A(x)\nabla w_n \nabla \phi = \int_\Omega E_n(x)\frac{\nabla w_n}{1 + \frac{1}{n}|\nabla w_n|}\phi + \int_\Omega f_n \phi \quad \forall \phi \in C_0^1(\Omega), \tag{3.3}$$

that admits a solution for every $n \in \mathbb{N}$ thanks to the Shauder's fix point Theorem.

## 3.1 Drift term in $L^N(\Omega)$

As pointed out in Section 2.1, it is possible to prove the Lebesgue regularity of problem (3.1) with small condition on the $\left(L^N(\Omega)\right)^N$ norm of $E(x)$ by means of a direct generalization of [34].
To get rid of such a smallness condition, we couple the method of power-like test functions of [34] with a *slice technique* originally introduced in [42]. We stress that such a slice technique does not involve estimates like

(2.12). Rather priori estimates are obtaind through an iterative procedure made of a finite number of steps.
To treat the case of Marcinkiewicz data we adapt the strategy developed in Subsection 2.1.2 to obtain a pointwise estimate for $\overline{w}_n$ and $\overline{|\nabla w_n|}$.
To our knowledge, both the Lebesgue and Marcinkiewicz regularity results for problem (3.1), with $E \in \left(L^N(\Omega)\right)^N$ without smallness assumption, are new.

### 3.1.1 Data in Lebesgue spaces

Here we consider problem (3.1) assuming

$$|E| \in L^N(\Omega) \quad \text{and} \quad f \in L^m(\Omega) \quad \text{with} \quad 1 < m < \frac{N}{2}.$$

As already said we use a slice technique introduced at first in [42] to deal with existence of $W_0^{1,2}(\Omega)$ solution of (3.1). The main idea is to divide $\Omega$ in regions in which the corresponding $L^N$-norm of $|E|$ is small enough. This partition, together with the use of power test functions, allows us to achieve the expected bounds for $\{w_n\}$ and $\{|\nabla w_n|\}$ through an iterative procedure.

We need some preliminary results. Let us define for $k < h$ e $v \in W_0^{1,2}(\Omega)$

$$A_{t,l}(v) := \{x \in \Omega \ : \ t < |v(x)| < l, \ |\nabla v(x)| \neq 0\}.$$

The following Proposition provides an important property of $A_{t,l}(v)$.

**Proposition 3.1** (See [42]). *For any $v \in W_0^{1,2}(\Omega)$ and $0 < h \leq \infty$ the function $k \to |A_{k,h}(v)|$ is continuous in $0 \leq k \leq h$.*

*Proof. Right continuity.* Let $\{k_n\}_{n\in\mathbb{N}}$ be a decreasing sequence converging to $k \in [0, h)$. It follows that

$$A_{k,h}(v) = \bigcup_{n\in\mathbb{N}} A_{k_n,h}(v)$$

and that

$$|A_{k,h}(v)| = |\bigcup_{n\in\mathbb{N}} A_{k_n,h}(v)| = \lim_{n\to\infty} |A_{k_n,h}(v)|.$$

Thus we can infer the continuity from the right

$$\lim_{j\to k^+} |A_{j,h}(v)| = |A_{k,h}(v)|.$$

*Left continuity.* Let us now consider an increasing sequence $\{k_n\}_{n\in\mathbb{N}} \nearrow k$ with $k \in (0, h]$. We have that

$$\bigcap_{n\in\mathbb{N}} A_{k_n,h}(v) = A_{k,h}(v) \cup \{x \in \Omega \ : \ |v(x)| = k, \ \nabla u(x) \neq 0\}.$$

Thanks to Stampacchia's Theorem it follows that $|\{x \in \Omega \ : \ |v(x)| = k, \ \nabla u(x) \neq 0\}| = 0$, thus

$$|A_{k,h}(v)| = |\bigcap_{n\in\mathbb{N}} A_{k_n,h}(v)| = \lim_{n\to\infty} |A_{k_n,h}(v)|,$$

and we recover also the continuity from the left

$$\lim_{j\to k^-} |A_{j,h}(v)| = |A_{k,h}(v)|.$$

□

Thanks to Proposition 3.1 we can prove the next Lemma.

**Lemma 3.2** (See [42]). *Assume that* $v \in W_0^{1,2}(\Omega)$, $g \in L^1(\Omega)$, $\eta > 0$ *and define*

$$J = \frac{\int_{|\nabla v|\neq 0} |g|}{\eta} \quad \text{and} \quad I = \begin{cases} J & \text{if } J \in \mathbb{N} \\ [J] & \text{if } J \notin \mathbb{N} \end{cases}.$$

*Hence there exist* $0 = k_{I+1} < \cdots < k_0 = +\infty$ *such that*

$$\int_{A_{k_{j+1},k_j}(v)} |g| = \eta \ \text{ for } j = 0, \cdots, I-1, \quad \int_{A_{k_{I+1},k_I}(w_n)} |E|^N \leq \eta.$$

*Proof.* If $I = 0$ we take $k_0 = +\infty$ and $k_1 = 0$ and we are done. If not take a sequence of $I + 2$ real numbers such that $0 = k_{I+1} < \cdots < k_0 = +\infty$. Of course

$$\int_{|\nabla v|\neq 0} |g| = \sum_{j=0}^{I_n} \int_{A_{k_{j+1},k_j}(v)} |g|.$$

Thus thanks to the continuity of the function $k \to |A_{k,h}(v)|$ proved in Lemma 3.1 and the definition of $I$ we infer that the numbers $k_j$, with $j = 0, \cdots, I+1$ can be chosen with the required property. □

Now we are in the position to state and prove two Lemmas that give us the required bounds for $\{w_n\}$ and $\{|\nabla w_n|\}$.

**Lemma 3.3.** *Let us assume* (2.2) *that* $E \in L^N(\Omega)$ *and* $f \in L^m(\Omega)$ *with* $1 < m \leq \frac{2N}{N+2}$ *and consider the solutions of the family of approximating problems* (3.3). *Hence there exists a constant* $C = C(\alpha, E, m, N)$ *such that*

$$\|\nabla w_n\|_{L^m(\Omega)^*} \leq C\|f\|_{L^m(\Omega)}.$$

*Proof.* Take $\gamma := \frac{m^{**}}{2^*}$ (in this case $\frac{1}{2} < \gamma \leq 1$) and define

$$J_n = \frac{\int_{\{|\nabla w_n|\neq 0\}} |E|^N}{\left(\alpha \mathcal{S}_{m^*} \frac{2\gamma-1}{2^{\frac{2}{m^*}}}\right)^N} \quad \text{and} \quad I_n = \begin{cases} J_n & \text{if } J_n \in \mathbb{N} \\ [J_n] & \text{if } J_n \notin \mathbb{N} \end{cases} \tag{3.4}$$

$$J = \frac{\int_\Omega |E|^N}{\left(\alpha \mathcal{S}_{m^*} \frac{2\gamma-1}{2^{\frac{2}{m^*}}}\right)^N} \quad \text{and} \quad I = \begin{cases} J & \text{if } J \in \mathbb{N} \\ [J] & \text{if } J \notin \mathbb{N} \end{cases}$$

where $\mathcal{S}_{m^*}$ is the Sobolev constant relative to $W_0^{1,m^*}(\Omega)$. Note by definition that $I_n \leq I$. We divide the proof in the following steps:
**Step 1.** Case $I_n = 0$.
**Step 2.** Case $I_n = 1$.
**Step 3.** Case $I_n \geq 2$.
**Step 4.** Conclusions.
**Step 1.** In this case

$$\left(\int_{|\nabla w_n|>0} |E|^N\right)^{\frac{1}{N}} < \alpha \mathcal{S}_{m^*} \frac{2\gamma - 1}{2^{\frac{2}{m^*}}}. \tag{3.5}$$

Take $\phi_\epsilon(w_n) = \left[(\epsilon + |w_n|)^{2\gamma-1} - \epsilon^{2\gamma-1}\right] \text{sign}(w_n)$ as test function in (3.3). We get

$$\alpha(2\gamma - 1) \int_\Omega \frac{|\nabla w_n|^2}{(\epsilon + |w_n|)^{2-2\gamma}} \leq \int_\Omega |E||\nabla w_n|\phi_\epsilon(w_n) + \int_\Omega |f|\phi_\epsilon(w_n)$$

$$\leq \left[\left(\int_{\nabla w_n\neq 0}|E|^N\right)^{\frac{1}{N}}\|\nabla w_n\|_{L^{m^*}(\Omega)}+\|f\|_{L^m(\Omega)}\right]\|\phi_\epsilon(w_n)\|_{L^{m'}(\Omega)},$$

where we have used Hölder Inequality and the fact that $\frac{1}{N}+\frac{1}{m^*}+\frac{1}{m'}=1$. Moreover

$$\begin{aligned}\int_\Omega|\nabla w_n|^{m^*}&=\int_\Omega\frac{|\nabla w_n|^{m^*}}{(\epsilon+|w_n|)^{(1-\gamma)m^*}}(\epsilon+|w_n|)^{(1-\gamma)m^*}\\&\leq\left(\int_\Omega\frac{|\nabla w_n|^2}{(\epsilon+|w_n|)^{2(1-\gamma)}}\right)^{\frac{m^*}{2}}\left(\int_\Omega(\epsilon+|w_n|)^{\frac{(1-\gamma)2m^*}{2-m^*}}\right)^{\frac{2-m^*}{2}}\\&\leq C_{\alpha,\gamma}^{\frac{m^*}{2}}\left[\left(\int_{\nabla w_n\neq 0}|E|^N\right)^{\frac{1}{N}}\|\nabla w_n\|_{L^{m^*}(\Omega)}+\|f\|_{L^m(\Omega)}\right]^{\frac{m^*}{2}}\\&\qquad\times\|\phi_\epsilon(w_n)\|_{L^{m'}(\Omega)}^{\frac{m^*}{2}}\left(\int_\Omega(\epsilon+|w_n|)^{\frac{(1-\gamma)2m^*}{2-m^*}}\right)^{\frac{2-m^*}{2}},\end{aligned}$$

where $C_{\alpha,\gamma}=[\alpha(2\gamma-1)]^{-1}$. Taking the limit with respecto to $\epsilon\to 0$ (by means of Lebesgue Theorem), recalling the definition of $\phi_\epsilon$ and that $(2\gamma-1)m'=\frac{(1-\gamma)2m^*}{2-m^*}=m^{**}$, it results

$$\begin{aligned}\int_\Omega|\nabla w_n|^{m^*}&\leq C_{\alpha,\gamma}^{\frac{m^*}{2}}\left[\left(\int_{\nabla w_n\neq 0}|E|^N\right)^{\frac{1}{N}}\|\nabla w_n\|_{L^{m^*}(\Omega)}+\|f\|_{L^m(\Omega)}\right]^{\frac{m^*}{2}}\left(\int_\Omega|w_n|^{m^{**}}\right)^{1-\frac{m^*}{2m}}\\&\leq\left(\frac{C_{\alpha,\gamma}}{\mathcal{S}_{m^*}}\right)^{\frac{m^*}{2}}\left[\left(\int_{\nabla w_n\neq 0}|E|^N\right)^{\frac{1}{N}}\|\nabla w_n\|_{L^{m^*}(\Omega)}+\|f\|_{L^m(\Omega)}\right]^{\frac{m^*}{2}}\left(\int_\Omega|\nabla w_n|^{m^*}\right)^{\frac{1}{2}},\end{aligned}$$

where in the last inequality we have used the Sobolev embedding for the space $W_0^{1,m^*}(\Omega)$. Since

$$\left(\frac{C_{\alpha,\gamma}}{\mathcal{S}_{m^*}}\left(\int_{\nabla w_n\neq 0}|E|^N\right)^{\frac{1}{N}}\right)^{\frac{m^*}{2}}\leq\frac{1}{2},$$

we conclude that

$$\|\nabla w_n\|_{L^{m^*}(\Omega)}\leq C_1\|f\|_{L^m(\Omega)},\tag{3.6}$$

where $C_1=\frac{2^{\frac{2}{m^*}}}{\alpha(2\gamma-1)\mathcal{S}_{m^*}}$.

**Step 2.** Thanks to Lemma 3.2 (applied with $I_n$ given by (3.12)) there exist $0<k_{1,n}<+\infty$ such that

$$\int_{A_{k_{1,n},\infty}(w_n)}|E|^N=\left(\alpha\mathcal{S}_{m^*}\frac{2\gamma-1}{2^{\frac{2}{m^*}}}\right)^N,\quad\int_{A_{0,k_{1,n}}(w_n)}|E|^N\leq\left(\alpha\mathcal{S}_{m^*}\frac{2\gamma-1}{2^{\frac{2}{m^*}}}\right)^N.\tag{3.7}$$

Now we separately recover uniform $W_0^{1,m^{**}}(\Omega)$ estimates for $G_{k_{1,n}}(w_n)$ and $T_{k_{1,n}}(w_n)$ taking advantage of (3.7). Let us take at first $\phi_\epsilon(G_{k_{1,n}}(w_n))=\left[(\epsilon+|G_{k_{1,n}}(w_n)|)^{2\gamma-1}-\epsilon^{2\gamma-1}\right]\text{sign}(w_n)$ as a test function in (3.3). Following the same arguments of Step 1 we get

$$\alpha(2\gamma-1)\int_\Omega\frac{|\nabla G_{k_{1,n}}(w_n)|^2}{(\epsilon+|G_{k_{1,n}}(w_n)|)^{2-2\gamma}}$$

$$\leq\left[\left(\int_{A_{0,k_{1,n}}}|E|^N\right)^{\frac{1}{N}}\|\nabla G_{k_{1,n}}(w_n)\|_{L^{m^*}(\Omega)}+\|f\|_{L^m(\Omega)}\right]\|\phi_\epsilon(G_{k_{1,n}}(w_n))\|_{L^{m'}(\Omega)},$$

and that

$$\int_\Omega |\nabla G_{k_{1,n}}(w_n)|^{m^*} \le \left(\frac{C_{\alpha,\gamma}}{\mathcal{S}_{m^*}}\right)^{\frac{m^*}{2}} \left[\left(\int_{A_{0,k_{1,n}}} |E|^N\right)^{\frac{1}{N}} \|\nabla G_{k_{1,n}}(w_n)\|_{L^{m^*}(\Omega)} + \|f\|_{L^m(\Omega)}\right]^{\frac{m^*}{2}} \times \left(\int_\Omega |\nabla G_{k_{1,n}}(w_n)|^{m^*}\right)^{\frac{1}{2}},$$

that in turn implies (thanks to (3.7))

$$\|\nabla G_{k_{1,n}}(w_n)\|_{L^{m^*}(\Omega)} \le C_1 \|f\|_{L^m(\Omega)}, \tag{3.8}$$

where again $C_1 = \frac{2^{\frac{2}{m^*}}}{\alpha(2\gamma-1)\mathcal{S}_{m^*}}$ (the same of (3.6)!).
Let us chose now $\phi_\epsilon(T_{k_{1,n}}(w_n)) = \left[(\epsilon + |T_{k_{1,n}}(w_n)|)^{2\gamma-1} - \epsilon^{2\gamma-1}\right]\operatorname{sign}(w_n)$ as a test function in (3.3). We get

$$\begin{aligned}
\alpha(2\gamma-1)\int_\Omega \frac{|\nabla T_{k_{1,n}}(w_n)|)|^2}{(\epsilon + |T_{k_{1,n}}(w_n)|)|)^{2-2\gamma}}
&\le \int_\Omega |E||\nabla w_n||\phi_\epsilon(T_{k_{1,n}}(w_n))| + \|f\|_{L^m(\Omega)}\|\phi_\epsilon(T_{k_{1,n}}(w_n))\|_{L^{m'}(\Omega)} \\
&= \int_{A_{0,k_{1,n}}} |E||\nabla w_n||\phi_\epsilon(T_{k_{1,n}}(w_n))| + \int_{A_{k_{1,n},\infty}} |E||\nabla w_n||\phi_\epsilon(T_{k_{1,n}}(w_n))| \\
&\quad + \|f\|_{L^m(\Omega)}\|\phi_\epsilon(T_{k_{1,n}}(w_n))\|_{L^{m'}(\Omega)} \\
&\le \left(\int_{A_{0,k_{1,n}}} |E|^N\right)^{\frac{1}{N}} \|\nabla T_{k_{1,n}}(w_n)\|_{L^{m^*}(\Omega)}\|\phi_\epsilon(T_{k_{1,n}}(w_n))\|_{L^{m'}(\Omega)} \\
&\quad + \left[\|E\|_{L^N(\Omega)}\|\nabla G_{k_{1,n}}(w_n)\|_{L^{m^*}(\Omega)} + \|f\|_{L^m(\Omega)}\right]\|\phi_\epsilon(T_{k_{1,n}}(w_n))\|_{L^{m'}(\Omega)}.
\end{aligned} \tag{3.9}$$

Taking advantage of the estimate above and (3.8) and taking the limit as $\epsilon \to 0$, we also deduce that

$$\int_\Omega |\nabla T_{k_{1,n}}(w_n)|^{m^*} \le \left[\frac{C_{\alpha,\gamma}}{\mathcal{S}_{m^*}}\left(\int_{A_{0,k_{1,n}}} |E|^N\right)^{\frac{1}{N}}\right]^{\frac{m^*}{2}} \int_\Omega |\nabla T_{k_{1,n}}(w_n)|^{m^*} + \left(\frac{C_{\alpha,\gamma}}{\mathcal{S}_{m^*}}\right)^{\frac{m^*}{2}} \left[C_1\|E\|_{L^N(\Omega)} + 1\right]^{\frac{m^*}{2}} \|f\|^{\frac{m^*}{2}}_{L^m(\Omega)} \left(\int_\Omega |\nabla T_{k_{1,n}}(w_n)|^{m^*}\right)^{\frac{1}{2}}.$$

We hence take advantage once more of (3.7) to infer that

$$\|\nabla T_{k_{1,n}}(w_n)\|_{L^{m^*}(\Omega)} \le C_1\big(C_1\|E\|_{L^N(\Omega)} + 1\big)\|f\|_{L^m(\Omega)},$$

with $C_1 = \frac{2^{\frac{2}{m^*}}}{\alpha(2\gamma-1)\mathcal{S}_{m^*}}$. Thus

$$\|\nabla w_n\|_{L^{m^*}(\Omega)} \le C_1\left[1 + \big(C_1\|E\|_{L^N(\Omega)} + 1\big)\right]\|f\|_{L^m(\Omega)}. \tag{3.10}$$

**Step 3.** Let us use once more Lemma 3.2 to deduce the existence of $+\infty = k_{0,n} > k_{1,n} > \cdots > k_{I_n,n} > k_{I_n+1,n} = 0$ such that

$$\int_{A_{k_{j+1,n},k_{j,n}}(w_n)} |E|^N = \left(\alpha\mathcal{S}_{m^*}\frac{2\gamma-1}{2^{\frac{2}{m^*}}}\right)^N \quad \text{for } j = 0, \cdots, I_n - 1$$

$$\text{and} \quad \int_{A_{k_{I_n+1,n},k_{I_n,n}}(w_n)} |E|^N \leq \left(\alpha \mathcal{S}_{m^*} \frac{2\gamma - 1}{2^{\frac{2}{m^*}}}\right)^N .$$

Exactly as in Step 2, taking $\phi_\epsilon(G_{k_{1,n},k_{0,n}}(w_n)) = \left[(\epsilon + |G_{k_{1,n},k_{0,n}}(w_n)|)^{2\gamma-1} - \epsilon^{2\gamma-1}\right]\text{sign}(w_n)$ as a test function in (3.3), we recover that

$$\|\nabla G_{k_{1,n},k_{0,n}}(w_n)\|_{L^{m^*}(\Omega)} \leq C_1 \|f\|_{L^m(\Omega)}. \tag{3.11}$$

For $j \in \{1, \cdots, I_n\}$ let us chose $\phi_\epsilon(G_{k_{j+1,n},k_{j,n}}(w_n)) = \left[(\epsilon + |G_{k_{j+1,n},k_{j,n}}(w_n)|)^{2\gamma-1} - \epsilon^{2\gamma-1}\right]\text{sign}(w_n)$ as a test function in (3.3). Splitting the contribution of the lower order term in slices we get

$$\begin{aligned}
&\int_\Omega \frac{|\nabla G_{k_{j+1,n},k_{j,n}}(w_n)|}{(\epsilon + |G_{k_{j+1,n},k_{j,n}}(w_n)|)^{2-2\gamma}} \\
&\qquad \leq \left(\int_{A_{k_{j+1,n},k_{j,n}}} |E|^N\right)^{\frac{1}{N}} \|\nabla T_{k_{1,n}}(w_n)\|_{L^{m^*}(\Omega)} \|\phi_\epsilon(T_{k_{1,n}}(w_n))\|_{L^{m'}(\Omega)} \\
&\qquad\quad + \left[\|E\|_{L^N(\Omega)} \sum_{i=0}^{j-1} \|\nabla G_{k_{i+1,n},k_{i,n}}(w_n)\|_{L^{m^*}(\Omega)} + \|f\|_{L^m(\Omega)}\right] \|\phi_\epsilon(T_{k_{1,n}}(w_n))\|_{L^{m'}(\Omega)}.
\end{aligned}$$

As before from the previous inequality we infer that

$$\begin{aligned}
\int_\Omega |\nabla G_{k_{j+1,n},k_{j,n}}(w_n)|^{m^*} &\leq \left[\frac{C_{\alpha,\gamma}}{\mathcal{S}_{m^*}} \left(\int_{A_{k_{j+1,n},k_{j,n}}} |E|^N\right)^{\frac{1}{N}}\right]^{\frac{m^*}{2}} \int_\Omega |\nabla G_{k_{j+1,n},k_{j,n}}(w_n)|^{m^*} \\
&\quad + \left(\frac{C_{\alpha,\gamma}}{\mathcal{S}_{m^*}}\right)^{\frac{m^*}{2}} \left[\|E\|_{L^N(\Omega)} \sum_{i=0}^{j-1} \|\nabla G_{k_{i+1,n},k_{i,n}}(w_n)\|_{L^{m^*}(\Omega)} + \|f\|_{L^m(\Omega)}\right]^{\frac{m^*}{2}} \\
&\qquad \times \left(\int_\Omega |\nabla G_{k_{j+1,n},k_{j,n}}(w_n)|^{m^*}\right)^{\frac{1}{2}},
\end{aligned}$$

that, thanks to the definition of $A_{k_{j+1,n},k_{j,n}}$, can be rewritten as

$$\begin{aligned}
&\left(\int_\Omega |\nabla G_{k_{j+1,n},k_{j,n}}(w_n)|^{m^*}\right)^{\frac{1}{m^*}} \\
&\qquad \leq C_1 \left[\|E\|_{L^N(\Omega)} \sum_{i=0}^{j-1} \|\nabla G_{k_{i+1,n},k_{i,n}}(w_n)\|_{L^{m^*}(\Omega)} + \|f\|_{L^m(\Omega)}\right],
\end{aligned}$$

with $C_1 = \frac{2^{\frac{2}{m^*}}}{\alpha(2\gamma-1)\mathcal{S}_{m^*}}$. Notice at this point that the sum in the square bracket in the second line above involves a finite number of contributions, hence it can be iteratively estimated starting from (3.11). We claim that the previous inequality can be rewritten as

$$\left(\int_\Omega |\nabla G_{k_{j+1,n},k_{j,n}}(w_n)|^{m^*}\right)^{\frac{1}{m^*}} \leq C_1(1 + \|E\|_{L^N(\Omega)} C_1)^j \|f\|_{L^m(\Omega)}.$$

Let us prove the claim by induction. For the case $j = 1$ look at (3.10). Moreover we have that

$$\left(\int_\Omega |\nabla G_{k_{j+1,n},k_{j,n}}(w_n)|^{m^*}\right)^{\frac{1}{m^*}} \le C_1\left[\|E\|_{L^N(\Omega)}\sum_{i=0}^{j-1}C_1(1+\|E\|_{L^N(\Omega)}C_1)^i+1\right]\|f\|_{L^m(\Omega)}$$

$$= C_1(1+\|E\|_{L^N(\Omega)}C_1)\left[\|E\|_{L^N(\Omega)}\sum_{i=1}^{j-1}C_1(1+\|E\|_{L^N(\Omega)}C_1)^{i-1}+1\right]\|f\|_{L^m(\Omega)}$$

$$= C_1(1+\|E\|_{L^N(\Omega)}C_1)^2\left[\|E\|_{L^N(\Omega)}\sum_{i=2}^{j-1}C_1(1+\|E\|_{L^N(\Omega)}C_1)^{i-2}+1\right]\|f\|_{L^m(\Omega)}$$

$$= C_1(1+\|E\|_{L^N(\Omega)}C_1)^{j-1}\left[\|E\|_{L^N(\Omega)}C_1(1+\|E\|_{L^N(\Omega)}C_1+1\right]\|f\|_{L^m(\Omega)}$$

and the claim is proved. Thus we have that

$$\left(\int_\Omega |\nabla w_n|^{m^*}\right)^{\frac{1}{m^*}} \le C_1\sum_{j=0}^{I_n}(1+\|E\|_{L^N(\Omega)}C_1)^j\|f\|_{L^m(\Omega)}$$

$$\le C_1\sum_{j=0}^{I}(1+\|E\|_{L^N(\Omega)}C_1)^j\|f\|_{L^m(\Omega)},$$

since by construction $I_n \le I$. □

**Lemma 3.4.** *Let us assume* (2.2) *that* $E \in L^N(\Omega)$ *and* $f \in L^m(\Omega)$ *with* $\frac{2N}{N+2} < m < \frac{N}{2}$ *and consider the solutions of the family of approximating problems* (3.3). *Hence there exists a constant* $C = C(\alpha, E, m, N)$ *such that*

$$\|w_n\|_{L^m(\Omega)^{**}} \le C.$$

*Proof.* Take $\gamma := \frac{m^{**}}{2^*}$ (in this case $\frac{1}{2} < \gamma \le 1$) and define

$$J_n = \frac{\int_{\{|\nabla w_n|\neq 0\}}|E|^N}{\left(\alpha\mathcal{S}_2\frac{2\gamma-1}{2\gamma}\right)^N} \quad\text{and}\quad I_n = \begin{cases} J_n & \text{if } J_n\in\mathbb{N}\\ [J_n] & \text{if } J_n\notin\mathbb{N}\end{cases} \tag{3.12}$$

$$J = \frac{\int_\Omega|E|^N}{\left(\alpha\mathcal{S}_2\frac{2\gamma-1}{2\gamma}\right)^N} \quad\text{and}\quad I = \begin{cases} J & \text{if } J\in\mathbb{N}\\ [J] & \text{if } J\notin\mathbb{N}\end{cases}$$

where $\mathcal{S}_2$ is the Sobolev constant relative to $W_0^{1,m^*}(\Omega)$. Note by definition that $I_n \le I$. Lemma 3.2 assures the existence of $+\infty = k_{0,n} > k_{1,n} > \cdots > k_{I_n,n} > k_{I_n+1,n} = 0$ such that

$$\int_{A_{k_{j+1,n},k_{j,n}}(w_n)}|E|^N = \left(\alpha\mathcal{S}_2\frac{2\gamma-1}{2\gamma}\right)^N \quad\text{for } j = 0,\cdots,I_n-1$$

$$\text{and}\quad \int_{A_{k_{I_n+1,n},k_{I_n,n}}(w_n)}|E|^N \le \left(\alpha\mathcal{S}_2\frac{2\gamma-1}{2\gamma}\right)^N.$$

Let us take $\phi = |G_{k_{1,n}}(w_n)|^{2\gamma-1}\text{sign}(w_n)$ as test function in (3.3). We get

$$\alpha(2\gamma-1)\int_\Omega|\nabla G_{k_{1,n}}(w_n)|^2|G_{k_{1,n}}(w_n)|^{2\gamma-2}$$

$$\le \int_\Omega |E||\nabla w_n||G_{k_{1,n}}(w_n)|^{2\gamma-1} + \int_\Omega |f||G_{k_{1,n}}(w_n)|^{2\gamma-1}$$

$$\le \frac{\gamma}{\mathcal{S}_2}\left(\int_{A_{k_{1,n},\infty}(w_n)} |E|^N\right)^{\frac{1}{N}} \int_\Omega |\nabla G_{k_{1,n}}(w_n)|^2 |G_{k_{1,n}}(w_n)|^{2\gamma-2} + \int_\Omega |f||G_{k_{1,n}}(w_n)|^{2\gamma-1}.$$

where we have use Hölder Inequality with exponents $\frac{1}{N}+\frac{1}{2}+\frac{1}{2^*}=1$ and Sobolev Inequality. Thanks to the choice of $k_1$, using Sobolev Inequality and recalling the definition of $\gamma$, the inequality above becomes

$$\int_\Omega |\nabla G_{k_{1,n}}(w_n)|^2 |G_k(w_n)|^{2\gamma-2} \le C\|f\|_{L^m(\Omega)} \left(\int_\Omega |\nabla G_{k_{1,n}}(w_n)|^2 |G_k(w_n)|^{2\gamma-2}\right)^{\frac{2^*}{2m'}}.$$

Thus we have the following estimates

$$\frac{\mathcal{S}_2}{\gamma}\left(\int_\Omega |G_{k_{1,n}}(w_n)|^{2^*\gamma}\right)^{\frac{2}{2^*}} \le \int_\Omega |\nabla G_{k_{1,n}}(w_n)|^2 |G_{k_{1,n}}(w_n)|^{2\gamma-2} \le C\|f\|^{\frac{2m^{**}}{2^*}}_{L^m(\Omega)}. \tag{3.13}$$

If $I_n = 0$ we have finished. Otherwise let us take $\phi = |G_{k_{j+1,n},k_{j,n}}(w_n)|^{2\gamma-1}\text{sign}(w_n)$ for $j = 1,\cdots,I_n$ in order to obtain

$$(2\gamma-1)\int_\Omega |\nabla G_{k_{j+1,n},k_{j,n}}(w_n)|)|^2 |G_{k_{j+1,n},k_{j,n}}(w_n)^{2\gamma-2}$$

$$= \int_{A_{k_{j+1,n},k_{j,n}}} |E||\nabla w_n||G_{k_{j+1,n},k_{j,n}}(w_n)|^{2\gamma-1}$$

$$+\sum_{i=0}^{j-1}\int_{A_{k_{i+1,n},k_{i,n}}} |E||\nabla w_n||G_{k_{j+1,n},k_{j,n}}(w_n)|^{2\gamma-1} + \int_\Omega |f||G_{k_{i,n},k_{i-1,n}}(w_n)^{2\gamma-1}|.$$

Thanks to the choice of $k_{j+1,n}, k_{j,n}$ we can absorb the first integral in the right hand side above into the principal part. Moreover by means of Hölder and Sobolev Inequalities we get

$$\int_\Omega |\nabla G_{k_{j+1,n},k_{j,n}}(w_n)^\gamma|^2 \le C\left[\|f\|_{L^m(\Omega)}\left(\int_\Omega |\nabla G_{k_{j+1,n},k_{j,n}}(w_n)^\gamma|^2\right)^{\frac{2^*}{2m'}} + \|E\|_{L^N(\Omega)}\sum_{i=0}^{j-1}\|\nabla G_{k_{i+1,n},k_{i,n}}(w_n)^\gamma\|_{L^2(\Omega)}\left(\int_\Omega |\nabla G_{k_{j+1,n},k_{j,n}}(w_n)^\gamma|^2\right)^{\frac{1}{2}}\right]. \tag{3.14}$$

Let us use now Hölder Inequality to get

$$\left(\int_\Omega |\nabla G_{k_{j+1,n},k_{j,n}}(w_n)^\gamma|^2\right)^{\frac{1}{2}} \le \mathcal{C}\left[\|f\|^{\frac{m^{**}}{2^*}}_{L^m(\Omega)} + \|E\|_{L^N(\Omega)}\sum_{i=0}^{j-1}\|\nabla G_{k_{i+1,n},k_{i,n}}(w_n)^\gamma\|_{L^2(\Omega)}\right],$$

where $\mathcal{C} = \mathcal{C}(\alpha, N, E, m)$. Arguing as in Lemma 3.2 we recover

$$\left(\int_\Omega |\nabla G_{k_{j+1,n},k_{j,n}}(w_n)^\gamma|^2\right)^{\frac{1}{2}} \le \mathcal{C}(1+\|E\|_{L^N(\Omega)}\mathcal{C})^j \|f\|^{\frac{m^{**}}{2^*}}_{L^m(\Omega)}.$$

Summing over $j$ from 1 to $I$ and using Sobolev Inequality, we conclude that there exists $C = C(\alpha, E, N)$ such that

$$\|w_n\|_{L^m(\Omega)^{**}} \le C\|f\|_{L^m(\Omega)}.$$

□

We state and prove now the existence and regularity result of this subsection.

**Theorem 3.5.** *Assume* (2.2), (2.5) *and* $f \in L^m(\Omega)$ *with* $1 < m < \frac{N}{2}$. *Hence there exists* $u$ *solution of* (3.2) *such that*

- *if* $1 < m \le (2^*)'$, *then* $u \in W_0^{1,m^*}(\Omega)$,
- *if* $(2^*)' < m < \frac{N}{2}$, *then* $u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$.

**Comment 3.6.** *Theorem 3.5 is obtained in [30] by means of an alternative approach partially based on a duality argument.*

*Proof.* **Case** $1 < m \le (2^*)'$**.** Thanks to Lemma 3.3 we infer that there exist a function $w \in W_0^{1,m^*}(\Omega)$ such that up to a subsequence $w_n \rightharpoonup w$ in $W_0^{1,m^*}(\Omega)$. Since $m^* > \frac{N}{N-1}$ we take advantage of Lemma 1.11 to deduce that

$$\nabla w_n \to \nabla w \quad a.e. \quad \text{in} \quad \Omega.$$

Thus we can pass to the limit as $n$ diverges in (3.3): the first and the last term are trivial; for the second one notice that

$$\frac{E_n}{1 + \frac{1}{n}|\nabla w_n|} \to E \quad \text{in} \quad L^{(m^*)'}(\Omega)$$

because $(m^*)' < N$ and the almost everywhere convergence of $\{|\nabla w_n|\}$.

**Case** $(2^*)' < m < \frac{N}{2}$**.** Thanks to the previous step we already know that up to a subsequence $w_n \rightharpoonup w$ in $W_\Omega^{1,2}$, where $w$ solves problem (3.2). Moreover taking advantage of Lemma 3.4 we also know that $\{w_n\}$ is bounded in $L^m(\Omega)^{**}$ and so $w$ itself belongs to such space. □

### 3.1.2 Data in Marcinkiewicz spaces

Let us now deal with the case

$$|E| \in L^N(\Omega) \quad \text{and} \quad f \in M^m(\Omega) \quad \text{with} \quad 1 < m < \frac{N}{2}.$$

The general strategy is close to the one of Subsection 2.1.2, namely comparison estimate for $\overline{w}_n$ through the rearrangements of the solutions of a suitable symmetrized problem, and the estimate for $\overline{|\nabla w_n|}$, similar to the one given by Lemma 2.6. Anyway, despite this similarity, the different structure of the lower order term gives rise to a different simmetrizing procedure.

Now we give the two Lemmas concerning the estimate for $\overline{w}_n$ and $\overline{|\nabla w_n|}$.

**Lemma 3.7.** *For any* $n \in \mathbb{N}$, *let* $w_n$ *be the solution of* (3.3) *and denote with* $\overline{w}_n$ *its decreasing rearrangement. It follows that*

$$\overline{w}_n(\tau) \le \overline{z}(\tau) := C \int_\tau^{|\Omega|} t^{\frac{2}{N}-2+\gamma} \int_0^t \overline{f}(s) s^{-\gamma} ds dt. \tag{3.15}$$

*for* $\gamma = \frac{1}{2m'}$ *and* $C = C(\alpha, m, E, N, \gamma)$.

**Remark 3.8.** *The function* $z(x) = z(\omega_N |x|^N)$ *solves the symmetrized problem*

$$\begin{cases} -\Delta z = \mathcal{C}_1 \nabla z \dfrac{x}{|x|^2} + \mathcal{C}_2 \tilde{f}(\omega_N |x|^N) & \text{in } B_\Omega, \\ z = 0 & \text{on } \partial B_\Omega, \end{cases}$$

*where* $B_\Omega$ *is the ball centered at the origin sucht that* $|B_\Omega| = |\Omega|$ *and* $\mathcal{C}_i = \mathcal{C}_i(N, \alpha, E, m, \gamma)$ *for* $i = 1, 2$.

*Proof.* Taking $\frac{T_h(G_k(w_n))}{h}$, with $h>0$ and $k\geq 0$, as test function in (3.3) we get

$$\frac{\alpha}{h}\int_{\{k<|w_n|<k+h\}}|\nabla w_n|^2\leq \int_{\{|w_n|>k\}}|f|+\int_{\{|w_n|>k\}}|\nabla w_n||E_n(x)| \tag{3.16}$$

and, passing to the limit with respect to $h\to 0$,

$$-\frac{d}{dk}\int_{\{|w_n|>k\}}|\nabla w_n|^2\leq \frac{1}{\alpha}\int_{\{|w_n|>k\}}|f|+\frac{1}{\alpha}\int_{\{|w_n|>k\}}|\nabla w_n||E_n(x)|.$$

Notice that the last integral above can be estimate as follows

$$\int_{\{|w_n|>k\}}|\nabla w_n||E_n(x)|=\int_k^{+\infty}\left(\frac{d}{ds}\int_{\{|w_n|>s\}}|\nabla w_n||E_n(x)|\right)ds$$

$$\leq \int_k^{+\infty}\left(-\frac{d}{ds}\int_{\{|w_n|>s\}}|D(x)|^2\right)^{\frac12}\left(-\frac{d}{ds}\int_{\{|w_n|>s\}}|\nabla w_n|^2\right)^{\frac12}ds.$$

Let us set for any $n\in\mathbb{N}$ and $k>0$

$$A_n(k)=|\{|w_n|>k\}|,$$

namely $A_n(k)$ is the distribution function of $w_n$. Consider moreover $Q_n(s)$, with $s\in(0,|\Omega|)$, the pseudo rearrangement of $|E_n|^2$ with respect to $w_n$ (see (1.5) for the definition). Thanks to Lemma 1.5 we have that for all $k>0$

$$Q_n(A_n(k))(-A_n'(k))=-\frac{d}{dk}\int_{\{|w_n|>k\}}|E_n(x)|^2 \quad\text{and that}\quad \|Q_n\|_{L^{N/2}(0,|\Omega|)}\leq \|E\|_{L^N(\Omega)}. \tag{3.17}$$

Thus we have

$$-\frac{d}{dk}\int_{\{|w_n|>k\}}|\nabla w_n|^2\leq \frac{1}{\alpha}\int_{\{|w_n|>k\}}|f|$$

$$+\frac{1}{\alpha}\int_k^{+\infty}Q_n(A_n(s))^{\frac12}(-A_n'(s))^{\frac12}\left(-\frac{d}{ds}\int_{\{|w_n|>s\}}|\nabla w_n|^2\right)^{\frac12}ds.$$

Using (1.10) we obtain

$$\left(-\frac{d}{dk}\int_{\{|w_n|>k\}}|\nabla w_n|^2\right)^{\frac12}\leq \frac{1}{\alpha\sigma_N}A_n(k)^{\frac1N-1}(-A_n'(k))^{\frac12}\int_{\{|w_n|>k\}}|f|$$

$$+\frac{1}{\alpha\sigma_N}A_n(k)^{\frac1N-1}(-A_n'(k))^{\frac12}\int_k^{+\infty}Q_n(A_n(s))^{\frac12}(-A_n'(s))^{\frac12}\left(-\frac{d}{ds}\int_{\{|w_n|>s\}}|\nabla w_n|^2\right)^{\frac12}ds.$$

Let us use Lemma 1.12 and make a change of variable to obtain that

$$\left(-\frac{d}{dk}\int_{\{|w_n|>k\}}|\nabla w_n|^2\right)^{\frac12}\leq \frac{1}{\sigma_N}A_n(k)^{\frac1N-1}(-A_n'(k))^{\frac12}\int_0^{A_n(k)}\overline{f}$$

$$+\frac{1}{\alpha^2\sigma_N^2}A_n(k)^{\frac1N-1}(-A_n'(k))^{\frac12}\int_0^{A_n(k)}Q_n(s)^{\frac12}s^{\frac1N}\tilde{f}(s)e^{\frac{1}{\alpha\sigma_N}\int_s^{A_n(k)}Q_n(\tau)^{\frac12}\tau^{\frac1N-1}d\tau}ds.$$

Recalling that $\tilde{f}(s) = \frac{1}{s}\int_0^s \overline{f}(\tau)d\tau$, we note that the integral in the second line above can be written as

$$-\alpha\sigma_N \int_0^{A_n(k)} \left(\int_0^s \overline{f}(\tau)d\tau\right) \frac{d}{ds}\left(e^{\frac{1}{\alpha\sigma_N}\int_s^{A_n(k)} Q_n(\tau)^{\frac{1}{2}}\tau^{\frac{1}{N}-1}d\tau}\right) ds.$$

Thus integrating by parts[1] we get

$$\left(-\frac{d}{dk}\int_{\{|w_n|>k\}} |\nabla w_n|^2\right)^{\frac{1}{2}} \\ \le \frac{1}{\alpha\sigma_N} A_n(k)^{\frac{1}{N}-1}(-A_n'(k))^{\frac{1}{2}} \int_0^{A_n(k)} \overline{f}(s) e^{\frac{1}{\alpha\sigma_N}\int_s^{A_n(k)} Q_n(\tau)^{\frac{1}{2}}\tau^{\frac{1}{N}-1}d\tau} ds. \quad (3.18)$$

Using once more (1.10) and writing the differential inequality for $\frac{d}{ds}\overline{w}_n(s)$, we get

$$-\frac{d}{dt}\overline{w}_n(t) \le \frac{1}{\alpha\sigma_N^2} t^{\frac{2}{N}-2} \int_0^t \overline{f}(s) e^{\frac{1}{\sigma_N}\int_s^t Q_n(\tau)^{\frac{1}{2}}\tau^{\frac{1}{N}-1}d\tau} ds.$$

To estimate the exponential in the right hand side above we take advantage of (3.17), to infer that for $\gamma = \frac{1}{2m'}$

$$e^{\frac{1}{\sigma_N}\int_s^t Q_n(\tau)^{\frac{1}{2}}\tau^{\frac{1}{N}-1}d\tau} \le e^{C_\gamma \|E\|^2_{L^N(\Omega)}} \left(\frac{t}{s}\right)^\gamma.$$

Integrating between $\tau$ and $|\Omega|$ we obtain

$$\overline{w}_n(\tau) \le C \int_\tau^{|\Omega|} t^{\frac{2}{N}-2+\gamma} \int_0^t \overline{f}(s) s^{-\gamma} ds dt.$$

□

**Lemma 3.9.** *Let $\overline{|\nabla w_n|}$ be the decreasing rearrangement of $|\nabla w_n|$. For any $n \in \mathbb{N}$, it result that*

$$\frac{1}{\tau}\int_0^\tau \overline{|\nabla w_n|} \le C\left[\frac{1}{\tau}\int_0^\tau s^{\frac{1}{N}-1+\gamma}\left(\int_0^s t^{-\gamma}\overline{f}(t)dt\right) ds \right. \\ \left. + \left(\frac{1}{\tau}\int_\tau^{|\Omega|} s^{2\left(\frac{1}{N}-1+\gamma\right)}\left(\int_0^s t^{-\gamma}\overline{f}(t)dt\right)^2 ds\right)^{\frac{1}{2}}\right] \quad (3.19)$$

*where $C = C(N, \alpha, m, \|B\|_{L^N(\Omega)})$.*

*Proof.* Recalling Lemma 1.3 and Remark 1.4, we obtain that

$$\int_0^s \overline{|\nabla w_n|} d\tau = \int_{\widetilde{\Omega}_n(s)} |\nabla w_n| dx$$

$$= \int_{\widetilde{\Omega}_n(s)\cap\{|w_n|>\overline{w}_n(s)\}} |\nabla w_n| dx + \int_{\widetilde{\Omega}_n(s)\cap\{|w_n|\le\overline{w}_n(s)\}} |\nabla w_n| dx$$

[1] We can perform integration by parts because

$$\lim_{s\to 0}\int_0^s \overline{f}(\tau)d\tau e^{\frac{1}{\alpha\sigma_N}\int_s^{A_n(k)} Q_n(\tau)^{\frac{1}{2}}\tau^{\frac{1}{N}-1}d\tau} \le C \lim_{s\to 0}\frac{1}{s}\int_0^s \overline{f}(\tau)d\tau A_n(k) = 0.$$

$$\leq \int_{\{|w_n|>\overline{w}_n(s)\}} |\nabla w_n| dx + \left( \int_{\{|w_n|\leq \overline{w}_n(s)\}} |\nabla w_n|^2 dx \right)^{\frac{1}{2}} |\widetilde{\Omega}_n(s)|^{\frac{1}{2}} \leq I_1(s) + I_2^{\frac{1}{2}}(s) s^{\frac{1}{2}}.$$

**Estimate of $I_2$.** From (3.18) we also have

$$-\frac{d}{dk}\int_{\{|w_n|>k\}} |\nabla w_n|^2 \leq C A_n(k)^{2\left(\frac{1}{N}-1+\gamma\right)} \left( \int_0^{A_n(k)} t^{-\gamma} \overline{f}(t) dt \right)^2 (-A_n'(k)),$$

from which we infer that (see (2.23))

$$\frac{d}{ds}\int_{\{|w_n|>\overline{w}_n(s)\}} |\nabla w_n|^2 \leq C s^{2\left(\frac{1}{N}-1+\gamma\right)} \left( \int_0^s t^{-\gamma} \overline{f}(t) dt \right)^2.$$

Integrating between $\tau$ and $\Omega$ we get

$$I_2 \leq C \int_\tau^{|\Omega|} s^{2\left(\frac{1}{N}-1+\gamma\right)} \left( \int_0^s t^{-\gamma} \overline{f}(t) dt \right)^2 ds.$$

**Estimate of $I_1(s)$.** To deal with $I_1$ recall (2.24) so that

$$\frac{d}{ds}\int_{\{|w_n|>\overline{u}_n(s)\}} |\nabla w_n| \leq \left( \frac{d}{ds}\int_{\{|w_n|>\overline{u}_n(s)\}} |\nabla w_n|^2 \right)^{\frac{1}{2}} \leq C s^{\frac{1}{N}-1+\gamma} \int_0^s t^{-\gamma} \overline{f}(t) dt.$$

Integrating between 0 and $\tau$ we get

$$I_1 = \int_{\{|w_n|>\overline{w}_n(\tau)\}} |\nabla w_n| \leq C \int_0^\tau s^{\frac{1}{N}-1+\gamma} \int_0^s t^{-\gamma} \overline{f}(t) dt ds.$$

□

Let us now combine the previous results to state and prove the existence and regularity Theorem of this subsection.

**Theorem 3.10.** *Let us assume (2.2), that $E \in \left(L^N(\Omega)\right)^N$ and that $f \in M^m(\Omega)$ with $1 < m < \frac{N}{2}$. Hence there exists $w$ solution of (3.2). Moreover*

- *if $1 < m < (2^*)'$, then $u \in M^{m^{**}}(\Omega)$ and $|\nabla u| \in M^{m^*}(\Omega)$,*
- *if $(2^*)' < m < \frac{N}{2}$, then $u \in W_0^{1,2}(\Omega) \cap M^{m^{**}}(\Omega)$.*

*Proof of Theorem 3.10.* From (3.15) it follows that for every $1 < m < \frac{N}{2}$

$$\overline{w}_n(\tau) \leq C\|f\|_{M^m(\Omega)} \int_\tau^{|\Omega|} t^{\frac{2}{N}-2+\gamma} \int_0^t s^{-\frac{1}{m}-\epsilon} ds dt \tag{3.20}$$

$$\leq C\|f\|_{M^m(\Omega)} \int_\tau^{|\Omega|} t^{\frac{2}{N}-\frac{1}{m}-1} \leq C\|f\|_{M^m(\Omega)} \tau^{-\frac{1}{m^{**}}}.$$

where we have used at first that $\epsilon < \frac{1}{m'}$ and that $\frac{2}{N} - \frac{1}{m} < 0$. Let us now split the proof in two cases.
**Case $1 < m < (2^*)'$.** As far as the gradient is concerned, thanks to (3.19) we have that

$$\frac{1}{\tau}\int_0^\tau \overline{|\nabla w_n|} \leq C\left[ \frac{1}{\tau}\int_0^\tau s^{\frac{1}{N}-1+\gamma} \left( \int_0^s t^{-\gamma} \overline{f}(t) dt \right) ds \right.$$

$$+\left(\frac{1}{\tau}\int_{\tau}^{|\Omega|} s^{2\left(\frac{1}{N}-1+\gamma\right)}\left(\int_0^s t^{-\gamma}\overline{f}(t)dt\right)^2 ds\right)^{\frac{1}{2}}\Bigg]$$

$$\leq C\|f\|_{M^m(\Omega)}\left[\frac{1}{\tau}\int_0^{\tau} s^{\frac{1}{N}-\frac{1}{m}}ds+\left(\frac{1}{\tau}\int_{\tau}^{|\Omega|} s^{\frac{2}{N}-\frac{2}{m}}\right)^{\frac{1}{2}}\right]\leq C\|f\|_{M^m(\Omega)}\tau^{-\frac{1}{m^*}},$$

where we have used that $\epsilon < \frac{1}{m'}$ and that $m^* < 2$, namely $m < (2^*)'$. Thus we infer that $\{|\nabla w_n|\}$ is bounded in $W_0^{1,r}(\Omega)$ with $\frac{N}{N-1} < r < m^*$. Thus there exist a function $v \in W_0^{1,r}(\Omega)$ such that up to a subsequence

$$w_n \rightharpoonup v \quad \text{in} \quad W_0^{1,r}(\Omega).$$

Thanks to this weak convergence we can pass to the limit with respect to $n \to \infty$ both in the principal part of (3.3) and in the first order term since

$$\frac{E_n}{1+\frac{1}{n}|\nabla w_n|} \to E \quad \text{in} \quad L^{(r^*)'}(\Omega).$$

Hence we proved that there exist a solution of (3.2). Thanks to Proposition 1.1 we also have that $|\nabla w| \in M^{m^*}(\Omega)$ and $w \in M^{m^{**}}(\Omega)$.

**Case** $(2^*)' < m < \frac{N}{2}$**.** Notice at first that in this range of the parameter $m$ estimate (3.20) implies that $\{w_n\}$ is bounded in $L^{(2^*)'}(\Omega)$. Taking hence $w_n$ as a test function in (3.3) we easily obtain that

$$\|\nabla w_n\|_{L^2(\Omega)} \leq C\|f\|_{L^{(2^*)'}(\Omega)}.$$

Thus we have a weak $W_0^{1,2}(\Omega)$ limit $w$ solution of (3.2). Moreover $w \in M^{m^{**}}(\Omega)$. □

## 3.2 Drift term in $M^N(\Omega)$

We now focus on problem (3.1) with a drift term with Marcinkiewicz coefficient. As in Chapter 2, we have to impone a size restriction on $\overline{E}(s)$.

### 3.2.1 Data in Lebesgue spaces

Let us state the following result.

**Theorem 3.11.** *Assume (2.2), $f \in M^m(\Omega)$ with $1 < m < \frac{2}{N}$ and*

$$E \in M^N(\Omega) \quad \textit{with} \quad \|E\|_{M^N(\Omega)} < \alpha\omega_N^{\frac{1}{N}} N\frac{m-1}{m}. \tag{3.21}$$

*Then there exist $u$ distributional solution of (2.3) and moreover*

- *if $1 < m < (2^*)'$, then $u \in M^{m^{**}}(\Omega)$ and $|\nabla u| \in M^{m^*}(\Omega)$,*
- *if $(2^*)' < m < \frac{N}{2}$, then $u \in W_0^{1,2}(\Omega) \cap M^{m^{**}}(\Omega)$.*

**Comment 3.12.** *A special case of Theorem 3.11 is $0 \in \Omega$ and*

$$E(x) = B\frac{x}{|x|^2} \quad \textit{with} \quad B < \alpha\frac{N-2m}{m}.$$

*Such case has been treated in [69].*

*Proof.* It is really close to the one of Theorem 2.8 and we omit it here for brevity. □

### 3.2.2 Data in Marcinkiewicz spaces

Let us generalize now Theorem 3.10 in the case $E \in \left(M^N(\Omega)\right)^N$

**Theorem 3.13.** *Assume* (2.2), $f \in M^m(\Omega)$ *with* $1 < m < \frac{2}{N}$ *and* $E \in M^N(\Omega)$ *such that*

$$E = \mathcal{F} + \mathcal{E} \quad with \quad \mathcal{F} \in (L^\infty(\Omega))^N \quad and \quad \overline{\mathcal{E}}(s) \leq \frac{B}{s^{\frac{1}{N}}} \quad with \quad B < \alpha\omega_N^{\frac{1}{N}} N \frac{m-1}{m}. \tag{3.22}$$

*Then there exist* $u$ *distributional solution of* (2.3) *and moreover*

- *if* $1 < m < (2^*)'$, *then* $u \in M^{m^{**}}(\Omega)$ *and* $|\nabla u| \in M^{m^*}(\Omega)$,
- *if* $(2^*)' < m < \frac{N}{2}$, *then* $u \in W_0^{1,2}(\Omega) \cap M^{m^{**}}(\Omega)$.

**Comment 3.14.** *See Comment 2.11 for the reasons of assumption* (3.22).

*Proof.* As in Chapter 2, to take $|E| \in M^N(\Omega)$ is a variation on the theme of the Lebesgue case, with the non negligible difference that some control on the size of $E$ is required. Here we follow strategy of Subsection 3.1.2, stressing the main differences. We split the proof in the following steps.
**Step 1.** Pointwise estimate for $\overline{u}_n$ and $\overline{|\nabla u_n|}$.
**Step 2.** Marcinkiewicz a priori estimate for $\overline{u}_n$ and $\overline{|\nabla u_n|}$.
**Step 3.** Existence and regularity for $1 < m < (2^*)'$.
**Step 4.** Existence and regularity for $(2^*)' < m < \frac{N}{2}$.

**Step 1.** As in the proof of Theorem 2.10, in order to handle assumption (3.22), it is convenient to consider the following sequence $\{w_n\} \subset W_0^{1,2}(\Omega)$ of solutions of

$$\int_\Omega A(x)\nabla w_n \nabla\varphi = \int_\Omega (\mathcal{F}_n(x) + \mathcal{E}_n(x)) \frac{\nabla w_n}{1 + \frac{1}{n}|\nabla w_n|}\varphi + \int_\Omega f\varphi \quad \forall\, \varphi \in W_0^{1,2}(\Omega),$$

whose existence is assured by Schauder's fixed point theorem. As usual for $k > 0$ we set

$$A_n(k) = |\{|w_n| > k\}|.$$

Taking $\frac{T_h(G_k(u_n))}{h}$, with $h > 0$ and $k \geq 0$, as test function and following Lemma 3.7 we obtain

$$\left(-\frac{d}{dk}\int_{\{|w_n|>k\}} |\nabla w_n|^2\right)^{\frac{1}{2}} \tag{3.23}$$

$$\leq \frac{1}{\sigma_N} A_n(k)^{\frac{1}{N}-1}(-A_n'(k))^{\frac{1}{2}} \int_0^{A_n(k)} \overline{f}(s) e^{\frac{1}{\alpha\sigma_N}\int_s^{A_n(k)}\left(Q_{1,n}(\tau)^{\frac{1}{2}} + Q_{2,n}(\tau)^{\frac{1}{2}}\right)\tau^{\frac{1}{N}-1}d\tau} ds$$

and

$$-\frac{d}{dt}\overline{w}_n(t) \leq \frac{1}{\alpha\sigma_N^2} t^{\frac{2}{N}-2} \int_0^t \overline{f}(s) e^{\frac{1}{\sigma_N}\int_s^t\left(Q_{1,n}(\tau)^{\frac{1}{2}} + Q_{2,n}(\tau)^{\frac{1}{2}}\right)\tau^{\frac{1}{N}-1}d\tau} ds,$$

where

$$Q_{1,n}(s) := \frac{d}{ds}\int_{\Omega_n(s)} |\mathcal{F}(x)|^2 dx \quad \text{and} \quad Q_{2,n}(s) := \frac{d}{ds}\int_{\Omega_n(s)} |\mathcal{E}(x)|^2 dx.$$

By construction we have that

$$\frac{1}{\alpha\sigma_N}\int_t^s Q_{1,n}(\tau)^{\frac{1}{2}}\tau^{\frac{1}{N}-1}d\tau \leq \|\mathcal{F}\|_{L^\infty(\Omega)} \frac{N}{\alpha\sigma_N}|\Omega|^N$$

and, recalling (2.33), also that

$$\frac{1}{\alpha\sigma_N}\int_t^s Q_{2,n}(\tau)^{\frac{1}{2}}\tau^{\frac{1}{N}-1}d\tau \le \frac{NB}{2\alpha\sigma_N(N-2)} + \frac{B}{\alpha\sigma_N}\log\left(\frac{s}{t}\right).$$

Thus

$$\overline{w}_n(\tau) \le C\int_\tau^{|\Omega|} t^{\frac{2}{N}-2+\frac{B}{\alpha\sigma_N}}\int_0^t \overline{f}(s)s^{-\frac{B}{\alpha\sigma_N}}dsdt. \tag{3.24}$$

Once we have (3.23) and (3.24) we infer (as in Lemma 3.9) that

$$\frac{1}{\tau}\int_0^\tau \overline{|\nabla w_n|} \le C\left[\frac{1}{\tau}\int_0^\tau s^{\frac{1}{N}-1+\frac{B}{\alpha\sigma_N}}\int_0^s t^{-\frac{B}{\alpha\sigma_N}}\overline{f}(t)dtds\right. \tag{3.25}$$

$$\left. + \left(\frac{1}{\tau}\int_\tau^{|\Omega|} s^{2\left(\frac{1}{N}-1+\frac{B}{\alpha\sigma_N}\right)}\left(\int_0^s t^{-\frac{B}{\alpha\sigma_N}}\overline{f}(t)dt\right)^2 ds\right)^{\frac{1}{2}}\right].$$

**Step 2.** Here the assumption $B < \alpha\omega^{\frac{1}{N}}N\frac{m-1}{m}$ plays its central role in the achievement of the Marcinkiewicz estimate for $\{w_n\}$ and $\{|\nabla w_n|\}$. From (3.24) we deduce that

$$\overline{w}_n(\tau) \le C\int_\tau^{|\Omega|} t^{\frac{2}{N}-2+\frac{B}{\alpha\sigma_N}}\int_0^t \overline{f}(s)s^{-\frac{B}{\alpha\sigma_N}}dsdt \tag{3.26}$$

$$\le C\|f\|_{M^m(\Omega)}\int_\tau^{|\Omega|} t^{-\frac{1}{m^{**}}-1} \le C\|f\|_{M^m(\Omega)}\tau^{-\frac{1}{m^{**}}}$$

where we used assumption (2.30). Coupling (3.25) with (3.26) it also results (see the second part of the proof of Theorem 3.10) that

$$\frac{1}{\tau}\int_0^\tau \overline{|\nabla w_n|} \le C\|f\|_{M^m(\Omega)}\left[\frac{1}{\tau}\int_0^\tau s^{-\frac{1}{m^*}} + \left(\frac{1}{\tau}\int_\tau^{|\Omega|} s^{-\frac{2}{m^*}}\right)^{\frac{1}{2}}\right] \le C\|f\|_{M^m(\Omega)}\tau^{-\frac{1}{m^*}}.$$

**Step 3.** From the previous Step we infer the existence of a function $w \in W_0^{1,r}(\Omega)$ with $1 < r < \frac{Nm}{N-m}$ such that, up to a subsequence

$$w_n \rightharpoonup w \quad \text{in} \quad W_0^{1,r}(\Omega).$$

To show that such a $w$ is indeed a solution of (3.2) we have to pass to the limit as $n$ diverge in the family of equations (3.3). The principal part pass to the limit thanks to the weak convergence of $\{w_n\}$. In order to deal with the lower order term notice that for any subset $A \subset \Omega$ it results (recall that $m > 1$)

$$\int_A |\nabla w_n||E_n(x)| \le \int_0^{|A|}\overline{|\nabla w_n|}(s)\overline{B}(s)ds \le C\int_0^{|A|} t^{-\frac{1}{m^*}-\frac{1}{N}} \le C|A|^{\frac{1}{m'}},$$

that is the equi-integrability of the sequence

$$\left\{\frac{\nabla w_n \cdot E(x)}{1+\frac{1}{n}|\nabla w_n|}\right\}.$$

This and the almost everywhere convergence of the gradients assured by Lemma 1.11 allows us to take advantage of Vitali theorem and conclude that the function $w$ satisfies (3.2). Moreover thanks to Proposition 1.1 it follows that

$$\|w\|_{M^{m^{**}}(\Omega)} + \|\nabla w\|_{M^{m^*}(\Omega)} \le C\|f\|_{M^m(\Omega)}.$$

**Step 4.** Form estimate (3.26) we know that $\{w_n\}$ is bounded in $L^q(\Omega)$ for $2^* < q < m^{**}$. Thus

$$\int_0^{|\Omega|} t^{-\frac{N}{2}} \overline{w}_n^2(t)dt \leq \left(\int_0^{|\Omega|} \overline{w}_n^q(t)dt\right)^{\frac{2}{q}} \left(\int_0^{|\Omega|} t^{-\frac{2q}{N(q-2)}}\right)^{\frac{q-2}{q}} \leq C$$

since $1 - \frac{2q}{N(q-2)} > 0$. Let us take now $w_n$ as a test function in (3.3). Using Hölder's inequality we get

$$\alpha \left(\int_\Omega |\nabla w_n|^2\right)^{\frac{1}{2}} \leq \left(\int_\Omega |E|^2|w|^2\right)^{\frac{1}{2}} + \frac{1}{\mathcal{S}_2}\|f\|_{L^{(2^*)'}(\Omega)}$$

$$\leq \|E\|_{M^N(\Omega)} \left(\int_\Omega t^{-\frac{N}{2}} \overline{w}_n^2\right)^{\frac{1}{2}} + \frac{1}{\mathcal{S}_2}\|f\|_{L^{(2^*)'}(\Omega)} \leq C.$$

Hence up to a subsequence $\{|\nabla w_n|\}$ weakly converge in $W_0^{1,2}(\Omega)$ to a function $w \in W_0^{1,2}(\Omega)$. To show that $w$ solves (3.2) and belongs to $M^{m^{**}}(\Omega)$ we follow exactly the same reasoning of the previous Step. □

# Chapter 4

# Nonlinear operator

Looking at Chapters 2 and 3 one naturally wonders if is possible to extend the results contained therein to a non linear setting. Moreover the structure of the rearrangement-type estimates suggest to consider data in a general Lorentz space rather then confine the analysis to Lebesgue or Marcinkiewicz spaces. Here we provide this type of generalizations.

## 4.1 Convection lower order term

Given $p > 1$, the problem we consider in this section is

$$\begin{cases} -\text{div}\,(a(x,\nabla u)) = -\text{div}\big(u|u|^{p-2}E(x)\big) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where the Carathéodory function $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^{\mathbb{N}}$ satisfies (1.14), the vector field $E : \Omega \to \mathbb{R}^{\mathbb{N}}$ is such that (see Remark 2.11)

$$E = \mathcal{F} + \mathcal{E} \quad \text{with} \quad \mathcal{F} \in (L^\infty(\Omega))^N \quad \text{and} \quad \overline{\mathcal{E}}(s) \le \frac{B}{s^{\frac{p-1}{N}}} \quad \text{with} \quad B < \alpha^{\frac{1}{p-1}} \omega_N^{\frac{1}{N}} \frac{N - pm}{(p-1)m}, \tag{4.2}$$

and the datum $f$ belongs to $L^1(\Omega)$ or to a Lorentz space $L^{m,q}(\Omega)$ to be specified later. Problem (4.1) has to be intended in the following weak formulation

$$u \in W_0^{1,1}(\Omega) \quad : \quad \begin{array}{c} |\nabla u|^{p-1} \in L^1(\Omega), \quad |u|^{p-1}|E(x)| \in L^1(\Omega) \quad \text{and} \\ \displaystyle\int_\Omega a(x,\nabla u)\nabla\phi = \int_\Omega |u|^{p-2}u\, E(x)\nabla\phi + \int_\Omega f(x)\phi \quad \forall\, \phi \in C_0^1(\Omega). \end{array} \tag{4.3}$$

Let us present the first result of this section.

**Theorem 4.1.** *Let us assume $f \in L^{m,q}(\Omega)$ and that conditions (1.14) and (4.2) hold true. If $\max\{1, \frac{N}{N(p-1)+1}\} < m < (p^*)'$ and $0 < q \le \infty$, then there exists $u$ solution of (4.3) such that*

$$|u| \in L^{\frac{(p-1)Nm}{N-mp},(p-1)q}(\Omega) \quad \textit{and} \quad |\nabla u| \in L^{\frac{(p-1)Nm}{N-m},(p-1)q}(\Omega).$$

Theorem 4.1 provides a generalization of Table 1 for non-linear setting and Lorentz data. Anyway it is well known in the literature that, in the case of both $p$ and $m$ close to 1, some subtleties arise. Indeed, if $1 < p < 2-\frac{1}{N}$ and $1 < m < \frac{N}{N(p-1)+1}$, the notion of distributional solution is not any more adequate and entropy solutions have to be introduced, see as an example [28]. We do not treat this case and instead focus on the bordeline values $m = \max\left\{1, \frac{N}{N(p-1)+1}\right\}$.

**Theorem 4.2.** *Let us assume* $m = \max\{1, \frac{N}{N(p-1)+1}\}$ *and that conditions* (1.14) *and* (4.2) *hold true. (i) If* $p > 2 - \frac{1}{N}$ *and* $f \in L^1(\Omega)$*, then there exists* $u$ *solution of* (4.3) *such that*

$$|u| \in L^{\frac{(p-1)N}{N-p},\infty}(\Omega) \quad and \quad |\nabla u| \in L^{\frac{(p-1)N}{N-1},\infty}(\Omega).$$

*(ii) If* $p > 2 - \frac{1}{N}$ *and* $f \in \mathbb{L}^{1,q}(\Omega)$ *with* $0 < q \leq \infty$*, then there exists* $u$ *solution of* (4.3) *such that*

$$|u| \in L^{\frac{(p-1)N}{N-p},(p-1)q}(\Omega) \quad and \quad |\nabla u| \in L^{\frac{(p-1)N}{N-1},(p-1)q}(\Omega).$$

*(iii) If* $p = 2 - \frac{1}{N}$ *and* $f \in \mathbb{L}^{1,q}(\Omega)$ *with* $0 < q \leq \frac{1}{p-1} = \frac{N}{N-1}$*, then there exists* $u$ *solution of* (4.3) *such that*

$$|u| \in L^{\frac{N}{N-1},\frac{N-1}{N}q}(\Omega) \quad and \quad |\nabla u| \in L^{1,(p-1)q}(\Omega).$$

*(iv) If* $p < 2 - \frac{1}{N}$ *and* $f \in L^{m,q}(\Omega)$ *with* $m = \frac{N}{N(p-1)+1}$ $0 < q \leq \frac{1}{p-1}$*, then there exists* $u$ *solution of* (4.3) *such that*

$$|u| \in L^{\frac{N}{N-1},(p-1)q}(\Omega) \quad and \quad |\nabla u| \in L^{1,(p-1)q}(\Omega).$$

The main observation on Theorems 4.1 and 4.2 is that, also in this nonlinear Lorentz setting, we recover the same results of the case $E \equiv 0$ (see [4], [35], [76] and reference therein). Let us briefly comment Theorem 4.2. In points $(i)$ and $(ii)$ the summability of the data assures that $|\nabla u|$ belongs to a Lebesgue space smaller (more regular) than $L^1(\Omega)$. We refer to the Appendix the definition of $\mathbb{L}^{1,q}(\Omega)$ and the fact that $\mathbb{L}^{1,1}(\Omega) = L\log L(\Omega) \subset L^1(\Omega)$ (see Lemma 1.8 in Chapter 1, [18]).
On the contrary, in points *(iii)* and *(iv)*, the gradient belongs to Lorentz spaces with first exponent equal to 1. Such spaces are contained at best in $L^1(\Omega)$ and this make more difficult the proof of the result because $L^1(\Omega)$ is not reflexive. We refer to [35] for related results restricted to the Lebesgue framework.

### 4.1.1 Proof of the results

Before proving our main results, we need three preliminary Lemmas. The first one is devoted to the achievement of a point-wise estimate for $\overline{u}_n$, the solution of (1.15). The second Lemma gives the estimate relative to the decreasing rearrangement of $\nabla u_n$. The third one provides the required Lorentz bounds for the sequences $\{u_n\}$ and $\{|\nabla u_n|\}$.

**Lemma 4.3.** *Let us assume* (1.14) *and* (4.2)*. For any* $n \in \mathbb{N}$*, let* $u_n$ *be the solution of* (1.15) *and denote with* $\overline{u}_n$ *its decreasing rearrangement. It follows that*

$$\overline{u}_n(t) \leq \overline{v}(t) := \frac{C}{t^\gamma} \int_t^{|\Omega|} s^{\frac{p'}{N}+\gamma-1} \tilde{f}(s)^{\frac{1}{p-1}} ds, \tag{4.4}$$

*where* $C = C(N, \alpha, p, E, m)$ *and* $\gamma < \frac{N-pm}{(p-1)Nm}$.

**Remark 4.4.** *See Remark 2.5 for the meaning of* $\overline{v}(t)$.

*Proof.* Let us take $\frac{T_h(G_k(u_n))}{h}$ with $h > 0$ and $k \geq 0$ as test function in (1.15). Thanks to assumption (1.14) we get

$$\frac{\alpha}{h} \int_{\{k<|u_n|<k+h\}} |\nabla u_n|^p \leq \int_{\{|u_n|>k\}} |f| + \frac{(k+h)^{p-1}}{h} \int_{\{k<|u_n|<k+h\}} |E||\nabla u_n|. \tag{4.5}$$

Let us set for any $n \in \mathbb{N}$ and $k > 0$

$$A_n(k) = |\{|u_n| > k\}|,$$

namely $A_n(k)$ is the distribution function of $u_n$. Consider moreover $D_{1,n}(s)$ and $D_{2,n}(s)$, with $s \in (0, |\Omega|)$, the pseudo rearrangements of $|\mathcal{F}|^2$ and $|\mathcal{E}|^2$ with respect to $u_n$ (see (1.5) for the definition). Thanks to Lemma 1.5 we have that for $k > 0$

$$D_{1,n}(A_n(k))(-A_n'(k)) = -\frac{d}{dk} \int_{\{|u_n|>k\}} |\mathcal{F}|^{p'} \text{ and } D_{2,n}(A_n(k))(-A_n'(k)) = -\frac{d}{dk} \int_{\{|u_n|>k\}} |\mathcal{E}|^{p'}.$$

Setting $\mathcal{D}_n(s) = D_{1,n}(s) + D_{2,n}(s)$, to have a more compact notation, and following the same argument of Lemma 2.4 and Theorem 2.10, we obtain

$$\left(-\frac{d}{dk}\int_{\{|u_n|>k\}}|\nabla u_n|^p\right)^{\frac{1}{p'}} \leq \frac{A_n(k)^{\left(\frac{1}{N}-1\right)}}{\alpha\sigma_N}\int_{\{|u_n|>k\}}|f|\big(-A_n'(k)\big)^{\frac{1}{p'}} + \frac{k^{p-1}}{\alpha}\mathcal{D}_n(A_n(k))^{\frac{1}{p'}}\big(-A_n'(k)\big)^{\frac{1}{p'}}, \quad (4.6)$$

that can be rewritten, using (1.10), as

$$1 \leq \left[\frac{A_n(k)^{p\left(\frac{1}{N}-1\right)}}{\alpha\sigma_N^p}\int_{\{|u_n|>k\}}|f| + \frac{k^{p-1}}{\alpha\sigma_N^{p-1}}\mathcal{D}_n(A_n(k))^{\frac{1}{p'}}A_n(k)^{\left(\frac{1}{N}-1\right)(p-1)}\right](-A_n'(k))^{p-1}.$$

Hence

$$\begin{aligned}-\frac{d}{ds}\overline{u}_n(s) &\leq \left[\frac{s^{p\left(\frac{1}{N}-1\right)}}{\alpha\sigma_N^p}\int_0^s\bar{f} + \frac{1}{\alpha\sigma_N^{p-1}}\mathcal{D}_n(s)^{\frac{1}{p'}}s^{\left(\frac{1}{N}-1\right)(p-1)}\overline{u}_n^{p-1}(s)\right]^{\frac{1}{p-1}}\\ &\leq C_\delta s^{p'\left(\frac{1}{N}-1\right)}\left(\int_0^s\bar{f}(\tau)d\tau\right)^{\frac{1}{p-1}} + \frac{\delta}{\alpha^{\frac{1}{p-1}}\sigma_N}\mathcal{D}_n(s)^{\frac{1}{p}}s^{\frac{1}{N}-1}\overline{u}_n(s),\end{aligned}$$

where $\delta > 1$ is such that

$$\gamma = \frac{\delta B}{\alpha^{\frac{1}{p-1}}\sigma_N} < \frac{N-pm}{(p-1)Nm}.$$

This is possible thanks to assumption (4.2). Defining the auxiliary function

$$R_n(s) = e^{\frac{\gamma}{B}\int_t^s\mathcal{D}_n(\tau)^{\frac{1}{p}}\tau^{\frac{1}{N}-1}d\tau},$$

we finally deduce that

$$-\frac{d}{ds}\big(R(s)\overline{u}_n(s)\big) \leq Cs^{p'\left(\frac{1}{N}-1\right)}R_n(s)\left(\int_0^s\bar{f}(\tau)d\tau\right)^{\frac{1}{p-1}}.$$

In order to estimate $R_n(s)$ we recall the definition of $\mathcal{D}_n$ and Lemma 1.6. It results that

$$\int_t^s D_{1,n}(\tau)^{\frac{1}{p}}\tau^{\frac{1}{N}-1}d\tau \leq \|\mathcal{F}\|_{L^\infty(\Omega)}\frac{N}{\alpha^{\frac{1}{p-1}}\sigma_N}|\Omega|^{\frac{1}{N}}$$

and that

$$\begin{aligned}\int_t^s D_{2,n}(\tau)^{\frac{1}{p}}\tau^{\frac{1}{N}-1}d\tau &\leq \frac{1}{pB^{p-1}}\int_t^s D_{2,n}(\tau)\tau^{\frac{p}{N}-1}d\tau + \frac{B}{p'}\int_t^s\frac{1}{\tau}d\tau\\ \leq \frac{1}{pB^{p-1}}&\left[s^{\frac{p}{N}-1}\int_0^s\overline{\mathcal{E}}^{p'} - t^{\frac{p}{N}-1}\int_0^t\overline{\mathcal{E}}^{p'} - \frac{p-N}{N}\int_t^s\tau^{\frac{p}{N}-2}\int_0^\tau\overline{\mathcal{E}}^{p'}d\tau\right] + \frac{B}{p'}\log\left(\frac{s}{t}\right)\\ &\leq \frac{NB}{p(N-p)} + B\log\left(\frac{s}{t}\right),\end{aligned}$$

where we have used Young Inequality, integration by parts and assumption (4.2). Thus we have that

$$R_n(s) = e^{\frac{\gamma}{B}\int_t^s\mathcal{D}_n(\tau)^{\frac{1}{p}}\tau^{\frac{1}{N}-1}d\tau} \leq C\left(\frac{s}{t}\right)^\gamma.$$

Integrating between $t$ and $|\Omega|$ and recalling that by definition of both $\overline{u_n}(|\Omega|) = 0$ and $R(t) = 1$, we get

$$\overline{u}_n(t) = -R(|\Omega|)\overline{u}_n(|\Omega|) + R(t)\overline{u}_n(t) \leq \frac{C_1}{t^\gamma}\int_t^{|\Omega|}s^{p'\left(\frac{1}{N}-1\right)+\gamma}\left(\int_0^s\bar{f}(\tau)d\tau\right)^{\frac{1}{p-1}}ds.$$

□

The next Lemma provide the estimate relative to the decreasing rearrangement of $\nabla u_n$.

**Lemma 4.5.** *Let us assume* (1.14) *and* (4.2). *Let* $\overline{|\nabla u_n|}$ *be the decreasing rearrangement of* $|\nabla u_n|$. *There exists* $C = C(N, \alpha, p, E, m)$ *such that*

$$\frac{1}{s}\int_0^s \overline{|\nabla u_n|}^{p-1} \leq C\left[\frac{1}{s}\int_0^s \left(v(t)^{p-1}\mathcal{D}_n(t)^{\frac{1}{p'}} + \tilde{f}(t)t^{\frac{1}{N}}\right)dt + \left(\frac{1}{s}\int_s^{|\Omega|}\left(v(t)^p\mathcal{D}_n(t) + \tilde{f}(t)^{p'}t^{\frac{p'}{N}}\right)dt\right)^{\frac{1}{p'}}\right], \quad (4.7)$$

*where* $v(t)$ *is defined in* (4.4).

*Proof.* Taking advantage of Lemma 1.3 (see Remark 1.4), it follows that

$$\int_0^s \overline{|\nabla u_n|}^{p-1}d\tau = \int_{\widetilde{\Omega}_s}|\nabla u_n|^{p-1}dx$$
$$= \int_{\widetilde{\Omega}_s\cap\{|u_n|>\overline{u}_n(s)\}}|\nabla u_n|^{p-1}dx + \int_{\widetilde{\Omega}_s\cap\{|u_n|\leq\overline{u}_n(s)\}}|\nabla u_n|^{p-1}dx$$
$$\leq \int_{\{|u_n|>\overline{u}_n(s)\}}|\nabla u_n|^{p-1}dx + \left(\int_{\{|u_n|\leq\overline{u}_n(s)\}}|\nabla u_n|^p dx\right)^{\frac{1}{p'}}|\widetilde{\Omega}_s|^{\frac{1}{p}} \leq I_1(s) + I_2^{\frac{1}{p'}}(s)s^{\frac{1}{p}}.$$

As far as $I_2$ is concerned we infer from (4.6) that

$$\frac{d}{ds}\int_{\{|u_n|>\overline{u}_n(s)\}}|\nabla u_n|^p = \frac{d}{dk}\int_{\{|u_n|>k\}}|\nabla u_n|^p\Big|_{k=\overline{u}_n(s)}\frac{d}{ds}\overline{u}_n(s)$$
$$\leq C\left[\overline{u}_n(s)^p\mathcal{D}_n(s) + s^{\frac{p'}{N}}\tilde{f}(s)^{p'}\right].$$

Integrating between $s$ and $|\Omega|$, we get

$$I_2 = \int_{\{|u_n|\leq\overline{u}_n(s)\}}|\nabla u_n|^p = -\int_{\{|u_n|>\overline{u}_n(s)\}}|\nabla u_n|^p + \int_\Omega|\nabla u_n|^p$$
$$\leq C\left[\int_s^{|\Omega|}\overline{u}_n(t)^p\mathcal{D}_n(t) + t^{\frac{p'}{N}}\tilde{f}(t)^{p'}dt\right].$$

In order to estimate $I_1$ notice that

$$\int_{\{\overline{u}_n(s)\leq|u_n|<\overline{u}_n(s+h)\}}|\nabla u_n|^{p-1}$$
$$\leq \left(\int_{\{\overline{u}_n(s)\leq|u_n|<\overline{u}_n(s+h)\}}|\nabla u_n|^p\right)^{\frac{1}{p'}}|\{\overline{u}_n(s)\leq|u_n|<\overline{u}_n(s+h)\}|^{\frac{1}{p}},$$

that, passing to the limit as $h \to 0$ and recalling that $|\{|u_n| > \overline{u}_n(s)\}|' \leq 1$, gives

$$\frac{d}{ds}\int_{\{|u_n|>\overline{u}_n(s)\}}|\nabla u_n|^{p-1} \leq \left(\frac{d}{ds}\int_{\{|u_n|>\overline{u}_n(s)\}}|\nabla u_n|^p\right)^{\frac{1}{p'}} \leq C\left(\overline{u}_n(s)^{p-1}\mathcal{D}_n^{\frac{1}{p'}}(s) + \tilde{f}(s)s^{\frac{1}{N}}\right)$$

Hence we have the following estimate for $I_1$

$$I_1 \leq C\int_0^s\left(\overline{u}_n(t)^{p-1}\mathcal{D}_n^{\frac{1}{p'}}(t) + \tilde{f}(t)t^{\frac{1}{N}}\right)dt.$$

Putting together the obtained information for $I_1$ and $I_2$ we prove (4.7). □

The previous estimates on the decreasing rearrangements of $u_n$ and $\nabla u_n$ allow us to obtain the following Lorentz estimates in function of the Lorentz summability of the datum $f$.

**Lemma 4.6.** *(i) If $f \in L^{m,q}(\Omega)$ with $1 < m < (p^*)'$ and $0 < q \le \infty$, then*

$$\|u_n\|_{L^{\frac{(p-1)Nm}{N-pm},(p-1)q}(\Omega)} \le C\|f\|_{L^{m,q}(\Omega)} \quad \text{and} \quad \|\nabla u_n\|_{L^{\frac{(p-1)Nm}{N-m},(p-1)q}(\Omega)} \le C\|f\|_{L^{m,q}(\Omega)}.$$

*(ii) If $f \in \mathbb{L}^{1,q}(\Omega)$ with $0 < q \le \infty$*

$$\|u_n\|_{L^{\frac{(p-1)N}{N-p},(p-1)q}(\Omega)} \le C\|f\|_{\mathbb{L}^{1,q}(\Omega)} \quad \text{and} \quad \|\nabla u_n\|_{L^{\frac{(p-1)N}{N-1},(p-1)q}(\Omega)} \le C\|f\|_{f\in\mathbb{L}^{1,q}(\Omega)}.$$

*(iii) If $f \in L^1(\Omega)$, then*

$$\|u_n\|_{L^{\frac{(p-1)N}{N-p},\infty}(\Omega)} \le C\|f\|_{L^1(\Omega)} \quad \text{and} \quad \|\nabla u_n\|_{L^{\frac{(p-1)N}{N-1},\infty}(\Omega)} \le C\|f\|_{L^1(\Omega)}.$$

*Proof.* **Point *(i)*.** Le us start with the $f \in L^{m,q}(\Omega)$ with $1 \le m < (p^*)'$ and $0 < q < \infty$. *Estimate for $\{w_n\}$.* Using (4.4) we get

$$\||u|^{p-1}\|^q_{L^{\frac{Nm}{N-pm},q}(\Omega)} = \int_0^{+\infty} t^{\frac{q(N-pm)}{Nm}} \overline{u}(t)^{(p-1)q}\frac{dt}{t}$$

$$\le C\int_0^{+\infty} t^{\frac{q(N-pm)}{Nm}-\gamma(p-1)q}\left(\int_t^{|\Omega|} s^{\frac{p'}{N}+\gamma-1}\tilde{f}(s)^{\frac{1}{p-1}}ds\right)^{(p-1)q}\frac{dt}{t}$$

$$= C\int_0^{\infty} t^{\frac{q(N-pm)}{Nm}-\gamma(p-1)q+(p-1)q}\left(\frac{\int_t^{|\Omega|} s^{\frac{p'}{N}+\gamma-1}\tilde{f}^{\frac{1}{p-1}}}{t}\right)^{(p-1)q}\frac{dt}{t} \le C\int_0^{\infty} t^{\frac{q}{m}}\overline{f}^q\frac{dt}{t},$$

where the last inequality comes from Lemma 1.9 with $\delta = \frac{N-pm}{Nm(p-1)} - \gamma + 1$, that is strictly bigger the one thanks to the choice of $\gamma$. In the case $q = +\infty$, we obtain directly from (4.4) that

$$\overline{u}(s) \le \frac{C}{s^{\frac{N-pm}{Nm(p-1)}}}\|f\|_{L^{m,\infty}(\Omega)}.$$

*Estimate for $\{\nabla w_n\}$.* Thank to Lemma 1.6 estimate (4.7) can be rewritten as

$$\frac{1}{s}\int_0^s \overline{|\nabla u_n|}^{p-1} \le C\left[\frac{1}{s}\int_0^s \left(v(t)^{p-1}t^{-\frac{p-1}{N}} + \tilde{f}t^{\frac{1}{N}}\right)dt + \left(\frac{1}{s}\int_s^{|\Omega|}\left(v(t)^p t^{-\frac{p}{N}} + \tilde{f}^{p'}t^{\frac{p'}{N}}\right)dt\right)^{\frac{1}{p'}}\right]. \tag{4.8}$$

In order to prove the membership of the four terms above to $L^{m^*,q}(\Omega)$ we use Lemma 1.9

$$\int_0^{\infty} s^{\frac{q}{m^*}}\left(\frac{1}{s}\int_0^s v(t)^{p-1}t^{-\frac{p-1}{N}}dt\right)^q\frac{ds}{s} \le C\int_0^{\infty} s^{\frac{q(N-pm)}{Nm}}v(s)^{(p-1)q}\frac{ds}{s} < \infty,$$

$$\int_0^{\infty} s^{\frac{q}{m^*}}\left(\frac{1}{s}\int_0^s \tilde{f}(t)t^{\frac{1}{N}}dt\right)^q\frac{ds}{s} \le C\int_0^{\infty} s^{\frac{q}{m}}\tilde{f}(s)^q\frac{ds}{s} < \infty$$

and

$$\int_0^{+\infty} s^{\frac{q}{m^*}}\left(\frac{1}{s}\int_s^{|\Omega|} \tilde{f}(t)^{p'}t^{\frac{p'}{N}}dt\right)^{\frac{q}{p'}}\frac{ds}{s} \le \int_0^{+\infty} s^{\frac{q}{m}}\tilde{f}(s)^q\frac{ds}{s} < \infty,$$

$$+\int_0^{+\infty} s^{\frac{q}{m^*}}\left(\frac{1}{s}\int_s^{|\Omega|} \frac{v(t)^p}{t^{\frac{p}{N}}}dt\right)^{\frac{q}{p'}}\frac{ds}{s} \le \int_0^{+\infty} s^{\frac{q(N-pm)}{Nm}}v(s)^{(p-1)q}\frac{ds}{s} < \infty.$$

where we take $\delta = \frac{1}{m^*} < 1$ in the first two cases and $\delta = \frac{p'}{m^*} < 1$ (recall that $m < (p^*)'$) in the last ones. Hence we have that

$$\|\nabla u_n\|^q_{L^{\frac{(p-1)Nm}{N-m},(p-1)q}(\Omega)} \leq \int_0^\infty \tau^{\frac{q}{m^*}} \left( \frac{1}{s} \int_0^\tau \overline{|\nabla w_n|}^{p-1}(t)dt \right)^q \frac{d\tau}{\tau} \leq C\|f\|_{L^{m,q}(\Omega)}.$$

In the case $q = \infty$ we obtain by direct calculation from (4.8) that

$$\|\nabla u\|_{L^{\frac{(p-1)Nm}{N-m},\infty}(\Omega)} \leq C\|f\|_{L^{m,\infty}(\Omega)}.$$

**Point *(ii)*.** It follows exactly the same argument of Point *(i)*.
**Point *(iii)*.** Inequality (4.4) becomes

$$\overline{u}_n(t) \leq v(t) \leq C\|f\|_{L^1(\Omega)} \frac{1}{t^\gamma} \int_t^{|\Omega|} s^{p'\left(\frac{1}{N}-1\right)+\gamma} ds \leq C\|f\|_{L^1(\Omega)} t^{-\frac{N-p}{(p-1)N}},$$

where we have used the fact that $p'\left(\frac{1}{N} - 1\right) + \gamma + 1 < 0$. On the other hand we have that

$$\frac{1}{t}\int_0^t \overline{|\nabla u_n|}^{p-1} \leq C\|f\|_{L^1(\Omega)} \left[ \frac{1}{t}\int_0^t s^{\frac{1}{N}-1} ds + \left( \frac{1}{t}\int_0^t s^{p'\left(\frac{1}{N}-1\right)} ds \right)^{\frac{1}{p'}} \right] \leq C\|f\|_{L^1(\Omega)} t^{-\frac{N-1}{N}},$$

and thus the proof is concluded. □

Now we are in the position of proving Theorems 4.1 and 4.2. We start from the latter.

*Proof of Theorem 4.2.* **Case *(i)*.** Let us start with the case $p > 2 - \frac{1}{N}$ and $f \in L^1(\Omega)$. From Lemma 4.6 we deduce that the $\{|\nabla u_n|\}$ is bounded in $L^{\frac{(p-1)N}{N-1},\infty}(\Omega)$ and, in turn, in $L^r(\Omega)$ for $1 < r < \frac{(p-1)N}{N-1}$. Hence there exists $u \in W_0^{1,r}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,r}(\Omega)$. Thanks to the almost everywhere convergence of the gradients proved in Lemma 1.10, we infer that

$$\nabla u_n \to \nabla u \quad \text{in} \quad L^{\frac{r}{p-1}}(\Omega).$$

Recalling that it is possible to choose $r$ so that $\frac{r}{p-1} > 1$, we also have that

$$|\nabla u_n|^{p-2}\nabla u_n \to |\nabla u|^{p-2}\nabla u \quad \text{in} \quad L^1(\Omega).$$

Thus we can pass to the limit in the right hand side of (1.15) for every $\phi \in C_0^1(\Omega)$. In order to handle the lower order term, notice that for every measurable $\omega \subset \Omega$ it follows that

$$\int_\omega |u_n|^{p-1}|E_n| \leq \int_0^{|\omega|} v^{p-1}(t) t^{-\frac{p-1}{N}} dt \leq C\|f\|_{L^1(\Omega)} \int_0^{|\omega|} t^{-\frac{N-p}{N}-\frac{p-1}{N}} \leq C|\omega|^{\frac{1}{N}}, \tag{4.9}$$

where we used Lemma 4.6. This together with the $a.e.$ convergence of $u_n$ allows us to take advantage of Vitali Theorem and prove that

$$\int_\Omega a(x,\nabla u)\nabla\phi = \int_\Omega u|u|^{p-2}\, E(x)\nabla\phi + \int_\Omega f(x)\phi \quad \forall\, \phi \in C_0^1(\Omega).$$

From Proposition 1.1 we easily infer that

$$|u|^{p-1} \in L^{\frac{N}{N-p},\infty}(\Omega) \quad \text{and} \quad |\nabla u|^{p-1} \in L^{\frac{N}{N-1},\infty}(\Omega).$$

**Case *(ii)*.** If $p > 2 - \frac{1}{N}$ and $f \in \mathbb{L}^{1,q}(\Omega)$ with $0 < q \leq \infty$, since $\frac{N(p-1)}{N-1} > 1$ and following the same arguments of the previous step, we infer that there exists $u$ distributional solution of (4.1) such that

$$|u| \in L^{\frac{(p-1)N}{N-p},(p-1)q}(\Omega) \quad \text{and} \quad |\nabla u| \in L^{\frac{(p-1)N}{N-1},(p-1)q}(\Omega).$$

**Case *(iii)*.** On the other hand, if $p = 2 - \frac{1}{N}$ and $f \in \mathbb{L}^{1,q}(\Omega)$ with $0 < q \leq \frac{1}{p-1} = \frac{N}{N-1}$, Lemma 4.6 implies that $\{|\nabla u_n|\}$ is bounded in $L^1(\Omega)$. Since $L^1(\Omega)$ is not reflexive, this is not enough to assure the existence of a weakly converging subsequence. In order to recover a compactness property for $\{|\nabla u_n|\}$, we need to prove its equi-integrability (see [35]). For it, let $\omega$ be a measurable subset of $\Omega$ and notice that

$$\int_\omega |\nabla u_n(x)|dx \leq \int_0^{|\omega|} \overline{|\nabla u_n|}(t)dt \leq \int_0^{|\omega|} t \left( \frac{1}{t} \int_0^s \overline{|\nabla u_n|}^{p-1} \right)^{\frac{N}{N-1}} \frac{dt}{t} \tag{4.10}$$

$$\leq \int_0^{|\omega|} t \left( \frac{1}{t} \int_0^t \left(v(s)^{p-1} s^{-\frac{p-1}{N}} + \tilde{f} s^{\frac{1}{N}}\right)ds + \left( \frac{1}{t} \int_t^{|\Omega|} \left(v(s)^p s^{-\frac{p}{N}} + \tilde{f}^{p'} s^{\frac{p'}{N}}\right)ds \right)^{\frac{1}{p'}} \right)^{\frac{N}{N-1}} \frac{dt}{t}$$

where the last inequality comes from (4.8). Lemma 4.6 with $f \in \mathbb{L}^{1,\frac{N}{N-1}}(\Omega)$ implies that

$$\int_0^{|\Omega|} t \left( \frac{1}{t} \int_0^t \left(v(s)^{p-1} s^{-\frac{p-1}{N}} + \tilde{f} s^{\frac{1}{N}}\right)ds + \left( \frac{1}{t} \int_t^{|\Omega|} \left(v(s)^p s^{-\frac{p}{N}} + \tilde{f}^{p'} s^{\frac{p'}{N}}\right)ds \right)^{\frac{1}{p'}} \right)^{\frac{N}{N-1}} \frac{dt}{t}$$

$$\leq C \int_0^{|\Omega|} \left(t\tilde{f}(t)\right)^{\frac{N}{N-1}} \frac{dt}{t} = C\|f\|_{\mathbb{L}^{1,\frac{N}{N-1}}(\Omega)}.$$

This means that the function

$$\left( \frac{1}{t} \int_0^t \left(v(s)^{p-1} s^{-\frac{p-1}{N}} + \tilde{f} s^{\frac{1}{N}}\right)ds + \left( \frac{1}{t} \int_t^{|\Omega|} \left(v(s)^p s^{-\frac{p}{N}} + \tilde{f}^{p'} s^{\frac{p'}{N}}\right)ds \right)^{\frac{1}{p'}} \right)^{\frac{N}{N-1}}$$

belongs to $L^1(0, |\Omega|)$. This consideration and inequality (4.10) imply that for every $\epsilon$ there exists $\delta > 0$ such that

$$\int_\omega |\nabla u_n(x)|dx \leq \epsilon \quad \forall\, \omega \subset \Omega \quad \text{with} \quad |\omega| < \delta.$$

Hence we take advantage of Dunford-Pettis Theorem to infer the existence of a vector field $L \in \left(L^1(\Omega)\right)^N$ such that

$$\nabla u_n \rightharpoonup L \quad \text{in} \quad \left(L^1(\Omega)\right)^N.$$

By the very definition of weak gradient of a Sobolev function it results that

$$\int_\Omega \nabla u_n F = -\int_\Omega u_n \text{div}(F) \quad \forall \quad F \in (C_0^\infty(\Omega))^N. \tag{4.11}$$

Thanks to the weak convergence of $\nabla u_n$ in $\left(L^1(\Omega)\right)^N$ and the strong convergence of $u_n$ in $L^1(\Omega)$ (Lemma 4.6 says that indeed $u_n$ strongly converge to $u$ in $L^r(\Omega)$ with $1 < r < \frac{N}{N-1}$), we can pass to the limit in the equation above and deduce that $F \equiv \nabla u$.

At this point, thanks to the almost convergence of $\nabla u_n$ to $\nabla u$ (see Lemma 1.10), we can infer that indeed

$$\nabla u_n \to \nabla u \quad \text{in} \quad \left(L^1(\Omega)\right)^N.$$

Since $p - 1 = 1 - \frac{1}{N} < 1$, we also have that $|\nabla u_n|^{p-2}\nabla u_n \to |\nabla u|^{p-2}\nabla u$ in $L^1(\Omega)$. We follow the arguments of the previous step to conclude that $u$ is a solution of (4.1). Moreover, thanks to the almost everywhere of both $\{u_n\}$ and $\{|\nabla u_n|\}$, it results

$$|u| \in L^{\frac{N-1}{N-p},(p-1)q}(\Omega) \quad \text{and} \quad |\nabla u| \in L^{1,(p-1)q}(\Omega).$$

**Case (iv).** The case $p < 2 - \frac{1}{N}$ and $f \in L^{m,q}(\Omega)$ with $m = \frac{N}{N(p-1)+1}$ and $0 < q \le \frac{1}{p-1}$ is handled similarly to the Case *(ii)*. Indeed for the considered values of $m$ it results $\frac{(p-1)Nm}{N-pm} = 1$, thus Lemma 4.6 implies that $\{|\nabla u_n|\}$ is bounded in $L^1(\Omega)$. Reasoning as in (4.10), (4.11) and using the almost everywhere convergence of the gradient, we conclude that

$$\nabla u_n \to \nabla u \quad \text{in} \quad \left(L^1(\Omega)\right)^N.$$

From now on the proof is close to the one of the previous case. □

*Proof of Theorem 4.1.* **Case *(i)*.** Following the same argument of the first step of the proof of Theorem 4.1. We infer that there exists $u \in W_0^{1,r}(\Omega)$ with $1 < r < \frac{Nm(p-1)}{N-m}$ such that up to a subsequence

$$\nabla u_n \to \nabla u \quad \text{in} \quad L^{\frac{r}{p-1}}(\Omega).$$

Since it is possible to chose $r$ so that $\frac{r}{p-1} > 1$, we deduce that

$$|\nabla u_n|^{p-2}\nabla u_n \to |\nabla u|^{p-2}\nabla u \quad \text{in} \quad L^1(\Omega).$$

To to pass to the limit in (1.15) it is enough to notice that (4.9) is still valid. We also have that

$$|u|^{p-1} \in L^{\frac{N}{N-p},\infty}(\Omega) \quad \text{and} \quad |\nabla u|^{p-1} \in L^{\frac{N}{N-1},\infty}(\Omega).$$

□

## 4.2 Drift lower order term

Finally let us focus on nonlinear drift term. Let us consider, for $p > 1$,

$$\begin{cases} -\text{div}\,(a(x,\nabla w)) = |\nabla w|^{p-2}\nabla w E(x) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.12}$$

where $\Omega$ is a bounded open set of $\mathbb{R}^N$ with $N > 2$, the Carathéodory function $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^{\mathbb{N}}$ satisfies (1.14), the datum $f$ belongs to some Lebesgue or Lorentz space to be specified later and the vector field $E : \Omega \to \mathbb{R}^{\mathbb{N}}$ is such that

$$E = \mathcal{F} + \mathcal{E} \quad \text{with} \quad \mathcal{F} \in (L^\infty(\Omega))^N \quad \text{and} \quad \overline{\mathcal{E}}(s) \le \frac{B}{s^{\frac{1}{N}}} \quad \text{with} \quad B < \alpha\omega_N^{\frac{1}{N}} N \frac{m-1}{m}. \tag{4.13}$$

We consider the following weak formulation of problem (4.12).

$$u \in W_0^{1,1}(\Omega) \quad : \quad \begin{array}{l} |\nabla u|^{p-1} \in L^1(\Omega), \quad |\nabla u|^{p-1}|E(x)| \in L^1(\Omega) \quad \text{and} \\ \displaystyle\int_\Omega a(x,\nabla u)\nabla\phi = \int_\Omega |\nabla u|^{p-2}\nabla u E(x)\phi + \int_\Omega f(x)\phi \quad \forall\, \phi \in C_0^1(\Omega). \end{array} \tag{4.14}$$

The main result of this section is the following.

**Theorem 4.7.** *Let us assume $f \in L^{m,q}(\Omega)$ and that conditions (1.14) and (4.13) hold true.*
*(i) If $\max\{1, \frac{N}{N(p-1)+1}\} < m < (p^*)'$ and $0 < q \le \infty$, then there exist $u$ solution of (4.14) such that*

$$|u| \in L^{\frac{(p-1)Nm}{N-pm},(p-1)q}(\Omega) \quad \textit{and} \quad |\nabla u| \in L^{\frac{(p-1)Nm}{N-m},(p-1)q}(\Omega).$$

Notice that assumption (4.13) becomes more restrictive as $m \to 1$. Here we not consider the limit case $m = 1$ and refer the interested reader to [50] and [21].

### 4.2.1 Proof of the results

In the next Lemma we recover the pointwise estimate for the the rearrangement of $w_n$, the solution of (1.16), and $\nabla w_n$.

**Lemma 4.8.** *The sequence* $\{w_n\}$ *of solution of* (1.16) *satisfies the following estimates:*

$$\overline{w}_n(\tau) \leq z(t) := C\int_\tau^{|\Omega|} t^{p'\left(\frac{1}{N}-1\right)+\frac{B}{\alpha\sigma_N(p-1)}}\left(\int_0^t \overline{f}(s)s^{-\frac{B}{\alpha\sigma_N}}\,ds\right)^{\frac{1}{p-1}} dt \tag{4.15}$$

*and*

$$\frac{1}{s}\int_0^s \overline{|\nabla w_n|}^{p-1} \leq C\left[\frac{1}{s}\int_0^s t^{\frac{1}{N}-1+\frac{B}{\alpha\sigma_N}}\left(\int_0^t \overline{f}(\tau)\tau^{-\frac{B}{\alpha\sigma_N}}\,d\tau\right)dt + \left(\frac{1}{s}\int_s^{|\Omega|} t^{p'\left(\frac{1}{N}-1+\frac{B}{\alpha\sigma_N}\right)}\left(\int_0^t \overline{f}(\tau)\tau^{-\frac{B}{\alpha\sigma_N}}\,d\tau\right)^{p'} dt\right)^{\frac{1}{p'}}\right]. \tag{4.16}$$

*Proof.* Let us set for any $n\in\mathbb{N}$ and $k>0$

$$A_n(k) = |\{|w_n|>k\}|,$$

namely $A_n(k)$ is the distribution function of $w_n$. Following the same arguments of Lemma 3.7 we obtain that

$$\left(-\frac{d}{dk}\int_{\{|w_n|>k\}}|\nabla w_n|^p\right)^{\frac{1}{p'}} \leq \frac{1}{\alpha\sigma_N}A_n(k)^{\frac{1}{N}-1}(-A_n'(k))^{\frac{1}{p'}}\int_0^{A_n(k)}\overline{f}(s)e^{\frac{1}{\alpha\sigma_N}\int_s^{A_n(k)}\left(Q_{1,n}(\tau)^{\frac{1}{p}}+Q_{2,n}(\tau)^{\frac{1}{p}}\right)\tau^{\frac{1}{N}-1}d\tau}ds.$$

where

$$Q_{1,n}(s) := \frac{d}{ds}\int_{\Omega_n(s)}|\mathcal{F}(x)|^p dx \quad\text{and}\quad Q_{2,n}(s) := \frac{d}{ds}\int_{\Omega_n(s)}|\mathcal{E}(x)|^p dx.$$

By construction and Lemma 1.6 we deduce that $\|Q_{1,n}\|_{L^\infty(\Omega)} \leq C\|\mathcal{F}\|^p_{L^\infty(\Omega)}$ and moreover, by means of Young Inequality and integration by parts, we have that

$$\int_s^t Q_n(\tau)^{\frac{1}{p}}\tau^{\frac{1}{N}-1}d\tau \leq \frac{NB}{p(N-p)} + B\log\left(\frac{t}{s}\right).$$

Thus we recover the following estimate for $\overline{w}_n$

$$\overline{w}_n(\tau) \leq C\int_\tau^{|\Omega|} t^{p'\left(\frac{1}{N}-1\right)+\frac{B}{\alpha\sigma_N(p-1)}}\left(\int_0^t \overline{f}(s)s^{-\frac{B}{\alpha\sigma_N}}\,ds\right)^{\frac{1}{p-1}} dt.$$

In order to obtain the estimate for the gradient, we follow Lemma 3.9 to get

$$\frac{1}{s}\int_0^s \overline{|\nabla w_n|}^{p-1} \leq C\left[\frac{1}{s}\int_0^s t^{\frac{1}{N}-1+\frac{B}{\alpha\sigma_N}}\left(\int_0^t \overline{f}(\tau)\tau^{-\frac{B}{\alpha\sigma_N}}\,d\tau\right)dt + \left(\frac{1}{s}\int_s^{|\Omega|} t^{p'\left(\frac{1}{N}-1+\frac{B}{\alpha\sigma_N}\right)}\left(\int_0^t \overline{f}(\tau)\tau^{-\frac{B}{\alpha\sigma_N}}\,d\tau\right)^{p'} dt\right)^{\frac{1}{p'}}\right]$$

□

**Lemma 4.9.** *There exist two constant $C = C(\alpha, p, E, N)$ and $\tilde{C} = \tilde{C}(\alpha, p, E, N)$ such that*

$$\|w_n\|_{L^{[(p-1)m^*]^*,(p-1)q}(\Omega)} \leq C\|f\|_{L^{m,q}(\Omega)} \quad \textit{and} \quad \|\nabla w_n\|_{L^{(p-1)m^*,(p-1)q}(\Omega)} \leq \tilde{C}\|f\|_{L^{m,q}(\Omega)}.$$

*Proof. Estimate for* $\{w_n\}$. Assume tha $q > \infty$. From (4.15) it follows that

$$\|w_n\|^q_{L^{\frac{(p-1)Nm}{N-pm},(p-1)q}(\Omega)} = \int_0^{+\infty} t^{\frac{q(N-pm)}{Nm}} \overline{w}_n(t)^q \frac{dt}{t}$$

$$\leq C\int_0^{+\infty} \tau^{\frac{q(N-pm)}{Nm}+(p-1)q} \left(\frac{1}{\tau}\int_\tau^{|\Omega|} t^{p'\left(\frac{1}{N}-1\right)+\frac{B}{\alpha\sigma_N(p-1)}} \left(\int_0^t \overline{f}(s)s^{-\frac{B}{\alpha\sigma_N}}ds\right)^{\frac{1}{p-1}} dt\right)^{(p-1)q} \frac{d\tau}{\tau}$$

$$\leq C\left[\int_0^{+\infty} \tau^{\frac{q}{m}+\frac{qB}{\alpha\sigma_N}} \left(\tau^{-1}\int_0^\tau t^{-\frac{B}{\alpha\sigma_N}}\overline{f}(t)dt\right)^q \frac{d\tau}{\tau}\right] \leq C\int_0^\infty t^{\frac{q}{m}}\overline{f}^q\frac{dt}{t},$$

where we used Lemma 1.9 twice, once with $\delta = \frac{N-pm}{(p-1)Nm} + 1 > 1$ and once with $\delta = \frac{1}{m} + \frac{B}{\alpha\sigma_N} < 1$. If $q = \infty$ directly from (4.15) we obtain that

$$\overline{H_0^1(\Omega)}_n \leq C\|f\|_{L^{m,\infty}(\Omega)}t^{-\frac{N-pm}{(p-1)Nm}}.$$

*Estimate for* $\{|\nabla w_n|\}$. Let us start with

$$\int_0^\infty \tau^{\frac{q}{m^*}} \left(\frac{1}{\tau}\int_0^\tau s^{\frac{1}{N}-1+\frac{B}{\alpha\sigma_N}} \int_0^s t^{-\frac{B}{\alpha\sigma_N}}\overline{f}(t)dtds\right)^q \frac{d\tau}{\tau}$$

$$\leq C\int_0^\infty \tau^{\frac{q}{m}+q\frac{B}{\alpha\sigma_N}} \left(\frac{1}{\tau}\int_0^\tau t^{-\frac{B}{\alpha\sigma_N}}\overline{f}(t)dt\right)^q \frac{d\tau}{\tau}$$

$$\leq C\int_0^\infty \tau^{\frac{q}{m}}\overline{f}(\tau)^q\frac{d\tau}{\tau}.$$

Moreover

$$\int_0^\infty \tau^{\frac{q}{m^*}} \left(\frac{1}{\tau}\int_\tau^{|\Omega|} t^{p'\left(\frac{1}{N}-1+\frac{B}{\alpha\sigma_N}\right)} \left(\int_0^t \overline{f}(\tau)\tau^{-\frac{B}{\alpha\sigma_N}}d\tau\right)^{p'} dt\right)^{\frac{q}{p'}} \frac{d\tau}{\tau}$$

$$\leq C\int_0^\infty \tau^{\frac{q}{m}+\frac{qB}{\alpha\sigma_N}} \left(\frac{1}{s}\int_0^\tau t^{-\frac{B}{\alpha\sigma_N}}\overline{f}(t)dt\right)^q \frac{d\tau}{\tau} \leq C\int_0^\infty \tau^{\frac{q}{m}}\overline{f}^q\frac{d\tau}{\tau},$$

where we used Lemma 1.9 twice, once with $\delta = \frac{p'}{m^*} > 1$ and once with $\delta = \frac{1}{m} + \frac{B}{\alpha\sigma_N} < 1$. Hence we have that

$$\|\nabla w_n\|^q_{L^{\frac{(p-1)Nm}{N-m},(p-1)q}(\Omega)} \leq \int_0^\infty \tau^{\frac{q}{m^*}} \left(\frac{1}{s}\int_0^\tau \overline{|\nabla w_n|}^{p-1}(t)dt\right)^q \frac{d\tau}{\tau} \leq \int_0^\infty \tau^{\frac{q}{m}}\overline{f}^q\frac{d\tau}{\tau}.$$

If $q = \infty$ directly from (4.16) we obtain that

$$\overline{|\nabla w_n|} \leq C\|f\|_{L^{m,\infty}(\Omega)}t^{-\frac{N-m}{(p-1)Nm}}.$$

□

*Proof of Theorem 4.7.* Is really close to the one of Theorem 3.13 and we omit it for brevity. □

# Chapter 5

# Elliptic problems with $L^1(\Omega)$ coefficients

The main topic of this chapter is the problem

$$\begin{cases} -\mathrm{div}(A(x)|\nabla u|^{p-2}\nabla u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $\Omega$ is an open bounded subset of $\mathbb{R}^N$, with $N > 2$, and $p$ is a real number such that $1 < p < N$ and the function $A(x)$ satisfies, for $\alpha > 0$,

$$A \in L^1(\Omega), \quad A(x) \geq \alpha. \tag{5.2}$$

We show that, despite the presence of the singular coefficient $A(x)$, for any $f \in L^{(p^*)'}(\Omega)$ there exists $u \in W_0^{1,p}(\Omega)$ distributional solution of (5.1); if $f$ belongs merely to $L^1(\Omega)$, we still prove that there exists a solution of (5.1) in the entropy sense (see next section for the precise meaning of solutions).
Thereafter we use the techniques developed to solve problem (5.1) to obtain existence results for problems with semilinear and quasilinear lower order terms. Finally we consider a more general nonlinear differential operator in divergence form.

The linear version of problem (5.1), with general $L^1(\Omega)$ coefficients, has been addressed for the first time by Trudinger in [88], using duality method between weighted Sobolev Spaces. More recently, always in the linear case, problem (5.1) is considered in [32] without the weighted framework but with stronger assumptions on the summability of $A(x)$, that has to be at least an $L^2(\Omega)$ function. We stress that the previously quoted papers deal also with possibly degenerate coefficients, while here we always assume (5.2).
As far as semilinear lower order terms are concerned, we refer to the classical paper [43] and to the more recent ones [38] and [46]. For quasilinear lower order terms, we follow the monograph [33] (see the bibliography therein for the original references). Finally we consider some generalizations of Leray-Lions operators in the spirit of [68] and [80].

With respect to the existent literature, our contribution is to generalize some of the results of [88] to a wide class of nonlinear problems. The difficulties, arising from both the nonlinearity of the operator and the singularity of the coefficient, will be solved by coupling energy estimates of type

$$\int A(x)|\nabla u|^p < \infty,$$

with a modified version of the classical Minty Lemma (see [40]). We stress that in our treatment we avoid the use of weighted Sobolev spaces.

Before stating our results we give some notation. In the sequel we are going to use the following useful cutoff functions (see [85]): the truncation function $T_j : \mathbb{R} \to \mathbb{R}$ with $j > 0$, defined as

$$T_j(s) = \max\{\min\{j, s\}, -j\},$$

and the complementary of the truncation function $G_j : \mathbb{R} \to \mathbb{R}$ with $j > 0$, defined as

$$G_j(s) = s - T_j(s).$$

Note that the previous functions are Lipschitz and their value at 0 is 0. Hence if $v \in W_0^{1,p}(\Omega)$ it follows that $T_j(v), G_j(v) \in W_0^{1,p}(\Omega)$.
We moreover define the space of functions

$$X_0^p(\Omega) := \left\{ \varphi \in W_0^{1,p}(\Omega) \quad \text{such that} \quad \int_\Omega A(x)|\nabla\varphi|^p < \infty \right\}. \tag{5.3}$$

We also adopt the following notation

$$A_n = T_n(A), \quad \text{and} \quad f_n = T_n(f).$$

With $C_i$, $i \in \mathbb{N}$, we indicate generic positive constants that may depend on the dimension $N$, on the real number $p$, on the domain $\Omega$, on the Sobolev constant and on the other data of the problem, while with $\epsilon_n$ we indicate a generic sequence that goes to zero as $n$ diverges.

## 5.1 Statement of the main results

We say that a measurable function $u : \Omega \to \mathbb{R}$ is a distributional solution of (5.1) if

$$u \in X_0^p(\Omega) \quad \text{and} \quad \int_\Omega A(x)|\nabla u|^{p-2}\nabla u \nabla\phi = \int_\Omega f(x)\phi \quad \forall\, \phi \in X_0^p(\Omega). \tag{5.4}$$

The second meaning of solution that we consider is the one introduced for the first time in [17]. Let us define

$$\mathcal{T}_0^{1,p}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that } T_k(u) \in W_0^{1,p}(\Omega) \ \ \forall\, k > 0 \right\},$$

i.e. the set of measurable functions whose truncates belong to $W_0^{1,p}(\Omega)$ (we refer to [17] for more details). We say that $u \in \mathcal{T}_0^{1,p}(\Omega)$ is an entropy solution of (5.1) if

$$\begin{gathered} \forall\, k > 0, \quad \int_\Omega A(x)|\nabla T_k(u)|^p < \infty \quad \text{and} \\ \int_\Omega A(x)|\nabla u|^{p-2}\nabla u \nabla T_k(u-\phi) = \int_\Omega f(x)T_k(u-\phi) \quad \forall\, \phi \in X_0^p(\Omega) \cap L^\infty(\Omega). \end{gathered} \tag{5.5}$$

We state now our first result.

**Theorem 5.1.** *Let us assume that the function $A(x)$ satisfies (5.2) and that $f \in L^{(p^*)'}(\Omega)$. Hence there exists $u \in X_0^p(\Omega)$ distributional solution of (5.1). If $f \in L^1(\Omega)$ there exists $u \in \mathcal{T}_0^{1,p}(\Omega)$ entropy solution of (5.1).*

Let us recall that, for $p$ large enough ($p > 2 - \frac{1}{N}$), $f \in L^1(\Omega)$ and $A \in L^\infty(\Omega)$, problem (5.1) admits a distributional solution in $W_0^{1,q}(\Omega)$ with $q < \frac{(p-1)N}{N-1}$ (see [17]). Unfortunately, in this unbounded setting, we are not able to recover the same type of result and thus we have to use the weaker notion (5.5) of entropy solutions (see Remark 5.9). The main obstacle is that there exist no weighted versions of Sobolev Embeddings for a general $L^1(\Omega)$ weight.

The second type of problems we deal with presents a semilinear lower order term with sign condition. It is known in the literature that such a lower order term can give a regularizing effect to the summability of the solution and of its gradient (see [38] and [46]). Let us hence consider

$$\begin{cases} -\mathrm{div}(A(x)|\nabla u|^{p-2}\nabla u) + B(x)|u|^{q-2}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.6}$$

where $q > 1$ and the the function $B(x)$ satisfies, for $\delta > 0$,

$$B \in L^1(\Omega), \quad \delta A(x) \leq B(x). \tag{5.7}$$

We state then our second existence result.

**Theorem 5.2.** *Let us assume* (5.2)*,* $q > \max\{p, \frac{p}{p-1}\}$ *and* $f \in L^1(\Omega)$*. Then there exists* $u \in W_0^{1,r}(\Omega)$ $r < \frac{p(q-1)}{q}$ *solution of* (5.6) *in the following weak sense*

$$\begin{gathered} \int_\Omega A(x)|\nabla u|^r + \int_\Omega B(x)|u|^{q-1} < \infty \quad \textit{and} \\ \int_\Omega A(x)|\nabla u|^{p-2}\nabla u \nabla\phi + \int_\Omega B(x)|u|^{q-2}u\phi = \int_\Omega f(x)\phi \qquad \forall \phi \in C_0^1(\Omega). \end{gathered} \tag{5.8}$$

*Moreover, if* $f \in L^{q'}(\Omega)$ *with* $q > p^*$*, it holds that* $u \in W_0^{1,p}(\Omega)$ *and*

$$\int_\Omega A(x)|\nabla u|^p + \int_\Omega B(x)|u|^q < \infty. \tag{5.9}$$

**Remark 5.3.** *Note that* $q > \frac{p}{p-1}$ *implies* $\frac{p(q-1)}{q} > 1$ *and hence there exist values of* $r$ *such that* $1 < r < \frac{p(q-1)}{q}$*. On the other hand, the fact that* $q > p$ *implies that* $\frac{r}{p-1} > 1$*; thus we can use Hölder inequality (with coefficients* $\frac{r}{p-1}$ *and* $\frac{r}{r-p+1}$*) to check that the first integral in the left hand side of* (5.8) *is well defined. Finally note that if* $q > p^*$*, then* $q' < (p^*)'$ *and thus the enhanced regularity* (5.9) *is due to a further regularizing effect of the semilinear lower order term. Other regularizing effects given by the interaction of the coefficients of semilinear elliptic problems are considered in [9].*

The third type of problems that we study is

$$\begin{cases} -\mathrm{div}(A(x)|\nabla u|^{p-2}\nabla u) + D(x)g(u)|\nabla u|^p = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.10}$$

where $A(x)$ is as in (5.2), $D(x) \geq 0$ belongs to $L^1(\Omega)$ and the function $g : \mathbb{R} \to \mathbb{R}$ satisfies

$$g \in C(\mathbb{R}) \quad \text{and} \quad g(s)s \geq 0 \text{ for } s \in \mathbb{R}. \tag{5.11}$$

The literature about problems with lower order terms with natural growth with respect to the gradient is huge (see the monograph [33] and reference therein) and the main novelty here is that the functions $A(x)$ and $D(x)$ are not bounded. It is worthy to stress that, if $D \in L^1(\Omega)$, there are no results available for problems like

$$\begin{cases} -\Delta_p u + D(x)g(u)|\nabla u|^p = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.12}$$

because, roughly speaking, it is not possible to control the term $D(x)g(u)|\nabla u|^p$ with the principal operator. On the contrary in (5.10), the interplay between the coefficients $A(x)$ and $D(x)$ allows to overcome such an obstacle, as it is shown in the next Theorem.

**Theorem 5.4.** *Let us assume that the functions $A$ and $g \in C(\mathbb{R})$ satisfy respectively (5.2) and (5.11), that $D(x)$ belongs to $L^1(\Omega)$, with $D(x) \geq 0$ and that there exists a constant $\sigma > 0$ such that*

$$D(x) \leq \sigma A(x) \quad a.e. \quad in\, \Omega. \tag{5.13}$$

*Then if $f \in L^{(p^*)'}(\Omega)$ there exists $u \in W_0^{1,p}(\Omega)$ solution of (5.10) in the following weak sense*

$$\int_\Omega A(x)|\nabla u|^p + \int_\Omega D(x)g(u)|\nabla u|^p < \infty \quad and$$

$$\int_\Omega A(x)|\nabla u|^{p-2}\nabla u\nabla\phi + \int_\Omega D(x)g(u)|\nabla u|^p\phi = \int_\Omega f(x)\phi \qquad \forall\phi \in C_0^1(\Omega).$$

*If moreover there exist $\gamma, \tilde{s} > 0$ such that*

$$g(s)sign(s) \geq \gamma \quad for \quad |s| > \tilde{s} \tag{5.14}$$

*and there exists $0 < \tau < \sigma$ such that*

$$\tau A(x) \leq D(x) \quad a.e. \quad in\, \Omega, \tag{5.15}$$

*then, for any $f \in L^1(\Omega)$, problem (5.10) admits a solution $u \in W_0^{1,p}(\Omega)$ in the previous weak sense.*

**Remark 5.5.** *Assumption (5.13) says, roughly speaking, that the lower order term is controlled by the principal part of the operator. On the other hand (5.15) implies that the two terms have the same weight in the estimates and this gives rise to a regularizing effect as in [37].*

Let us compare the hypothesis on the coefficient $B$ of the semilinear problem (5.6) with the ones of the coefficient $D$ of the quasilinear problem (5.10). Indeed while there is correspondence between (5.7) and (5.15), there is no need of a semilinear counterpart of (5.13). This is because problem (5.6) is solved for bounded $A(x)$ and general positive $B(x)$ in $L^1(\Omega)$ (see [43]), while, as we already said, problem (5.12) is still unsolved.

Lastly we generalize Theorem 5.1 for a wide class of elliptic operators (see [68] and [80]). Let us hence consider

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u)) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.16}$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^{\mathbb{N}}$ is a Carathéodory function that satisfies

$$\begin{gathered} a(x,s,\xi)\xi \geq A(x)h(|s|)|\xi|^p, \\ |a(x,s,\xi)| \leq \gamma A(x)(1 + h(|s|))\,|\xi|^{p-1}], \\ [a(x,s,\xi) - a(x,s,\xi^*)][\xi - \xi^*] > 0, \end{gathered} \tag{5.17}$$

for almost every $x \in \Omega$, every $s \in \mathbb{R}$ and every $\xi, \xi^* \in \mathbb{R}^N$ with $\xi \neq \xi^*$, with $A(x)$ as in (5.2), $\gamma > 0$ and $h \in C(\mathbb{R}^+)$ such that, for $\beta > 0$,

$$h(|s|) \geq \beta > 0. \tag{5.18}$$

**Theorem 5.6.** *Assume that (5.17)-(5.18) hold true. Then for every $f \in L^1(\Omega)$ there exist $u \in \mathcal{T}_0^{1,p}(\Omega)$ entropy solution of (5.16) in the following sense*

$$\forall\, k > 0 \quad \int_\Omega A(x)h(|u|)^{p'}|\nabla T_k(u)|^p < \infty \quad and$$

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\phi) = \int_\Omega f(x)T_k(u-\phi) \quad \forall\,\phi \in C_0^1(\Omega).$$

*If moreover $f$ belongs to the space $L^{(p^*)'}(\Omega)$, then $u \in W_0^{1,p}(\Omega)$*

$$\int_\Omega A(x)h(|u|)^{p'}|\nabla u|^p < \infty \quad and$$

$$\int_\Omega a(x,u,\nabla u)\nabla\phi = \int_\Omega f(x)\phi \qquad \forall\,\phi \in C_0^1(\Omega).$$

## 5.2 Proof of the results

Before the proofs, we set the following notation

$$A_n(x) := T_n(A(x)), \quad B_n(x) := T_n(B(x)), \quad D_n(x) := T_n(D(x)), \quad n \in \mathbb{N},$$

where $T_n$ is the truncate at level $n$. We moreover state and prove a preliminary Lemma (see [41] Lemma 3.3) that will be often used in the sequel.

**Lemma 5.7.** *Let $\sigma_n$ be a sequence of nonnegative bounded functions, almost everywhere convergent to some function $\sigma$, and let $\rho_n : \mathbb{R}^N \to \mathbb{R}^N$ be a sequence of functions which is weakly convergent in $L^q(\Omega)^N$ $(q > 1)$ to some function $\rho$. If the sequence $\sigma_n|\rho_n|^q$ is bounded in $L^1(\Omega)$, then $\sigma|\rho|^q$ belongs to $L^1(\Omega)$ and*

$$\int_\Omega \sigma|\rho|^q \le \liminf_{n\to+\infty} \int_\Omega \sigma_n|\rho_n|^q . \tag{5.19}$$

*If moreover $\sigma_n \to \sigma$ strongly in $L^1(\Omega)$, then it holds true also that*

$$\sigma_n^{\frac{1}{q}}\rho_n \rightharpoonup \sigma^{\frac{1}{q}}\rho \quad \text{weakly in} \quad L^q(\Omega)^N. \tag{5.20}$$

*Proof.* Note that for every $k > 0$ and every $\psi \in (L^{q'}(\Omega))^N$ we have that

$$T_k(\sigma_n)^{\frac{1}{q}}\rho_n \rightharpoonup T_k(\sigma)^{\frac{1}{q}}\rho \quad \text{weakly in } L^q(\Omega)^N.$$

Hence by the lower semi-continuity of the $L^q$-norm it results

$$\int_\Omega T_k(\sigma)|\rho|^q \le \liminf_{n\to+\infty}\int_\Omega T_k(\sigma_n)|\rho_n|^q \le \liminf_{n\to+\infty}\int_\Omega \sigma_n|\rho_n|^q \le C_1.$$

Then letting $k$ tend to infinity and using the monotone convergence theorem we obtain (5.19).

Let now $\Psi$ be an arbitrary element of $L^{q'}(\Omega)^N$, then

$$\int_\Omega \sigma_n^{\frac{1}{q}}\rho_n\cdot\Psi = \int_{\{\sigma_n\le k\}}\sigma_n^{\frac{1}{q}}\rho_n\cdot\Psi + \int_{\{\sigma_n>k\}}\sigma_n^{\frac{1}{q}}\rho_n\cdot\Psi = \int_{\{\sigma\le k\}}\sigma^{\frac{1}{q}}\rho\cdot\Psi + \epsilon_n + \int_{\{\sigma_n>k\}}\sigma_n^{\frac{1}{q}}\rho_n\cdot\Psi,$$

Moreover

$$\int_\Omega \sigma^{\frac{1}{q}}\rho\cdot\Psi = \int_{\{\sigma_n\le k\}}\sigma^{\frac{1}{q}}\rho\cdot\Psi + \int_{\{\sigma_n>k\}}\sigma^{\frac{1}{q}}\rho\cdot\Psi = \int_{\{\sigma\le k\}}\sigma^{\frac{1}{q}}\rho\cdot\Psi + \epsilon_n + \int_{\{\sigma_n>k\}}\sigma^{\frac{1}{q}}\rho\cdot\Psi.$$

It follows that

$$\left|\int_\Omega [\sigma_n^{\frac{1}{q}}\rho_n - \sigma^{\frac{1}{q}}\rho]\cdot\Psi\right| \le \epsilon_n + \int_{\{\sigma_n>k\}}\sigma_n^{\frac{1}{q}}|\rho_n||\Psi| + \int_{\{\sigma_n>k\}}\sigma^{\frac{1}{q}}|\rho||\Psi|.$$

Using Hölder inequality we have that

$$\int_{\{\sigma_n>k\}}\sigma_n^{\frac{1}{q}}|\rho_n||\Psi| \le \left(\int_\Omega \sigma_n|\rho_n|^q\right)^{\frac{1}{q}}\left(\int_{\{\sigma_n>k\}}|\Psi|^q\right)^{\frac{1}{q'}}$$

and

$$\int_{\{\sigma_n>k\}}\sigma^{\frac{1}{q}}|\rho||\Psi| \le \left(\int_\Omega \sigma|\rho|^q\right)^{\frac{1}{q}}\left(\int_{\{\sigma_n>k\}}|\Psi|^{q'}\right)^{\frac{1}{q'}}.$$

Thanks to the assumption $\sigma_n \to \sigma$ in $L^1(\Omega)$ we deduce that the limit with respect to $k \to \infty$ of the quantities above is zero uniformly with respect to $n$. So we can conclude that $\sigma_n^{\frac{1}{q}}\rho_n \rightharpoonup \sigma^{\frac{1}{q}}\rho$ weakly in $L^q(\Omega)^N$. □

Now we are ready to prove our first existence result.

*Proof of Theorem 5.1. Case* $f \in L^{(p^*)'}(\Omega)$.
**Step 1.** It is standard to prove the existence of functions $u_n \in W_0^{1,p}(\Omega)$ that, for any $n \in \mathbb{N}$, satisfy

$$\int_\Omega A_n(x)|\nabla u_n|^{p-2}\nabla u_n \nabla \phi = \int_\Omega f_n \phi \quad \forall \phi \in W_0^{1,p}(\Omega). \tag{5.21}$$

Taking $u_n$ as test function in the equation above, we get the following estimate

$$\alpha \int_\Omega |\nabla u_n|^p \leq \int_\Omega A_n(x)|\nabla u_n|^p \leq \frac{\mathcal{S}_p^{\frac{p}{p-1}}}{\alpha^{\frac{1}{(p-1)}}}\|f\|_{L^{(p^*)'}(\Omega)}^{\frac{p}{p-1}},$$

that implies that there exists $u \in W_0^{1,p}(\Omega)$ such that, up to a not relabeled subsequence,

$$\begin{aligned} &u_n \rightharpoonup u && \text{weakly in } W_0^{1,p}(\Omega),\\ &u_n \to u && \text{strongly in } L^q(\Omega) \quad \text{for any } 1 \leq q < p^*,\\ &u_n \to u && a.e. \text{ in } \Omega. \end{aligned}$$

Moreover by means of Lemma 5.7, with $\sigma_n = A_n$ and $\rho_n = \nabla u_n$, we deduce that

$$A(x)|\nabla u|^p \in L^1(\Omega) \tag{5.22}$$

and

$$A_n(x)^{\frac{1}{p}}\nabla u_n \rightharpoonup A(x)^{\frac{1}{p}}\nabla u \text{ weakly in } L^p(\Omega)^N. \tag{5.23}$$

**Step 2.** Let $\varphi$ belong to $X_0^p(\Omega)$, the space defined in (5.3), and take $\phi = (u_n - \varphi)$ as a test function in (5.21). We get

$$\int_\Omega A_n(x)|\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - \varphi) = \int_\Omega f_n(x)(u_n - \varphi).$$

Adding and subtracting in the equation above the term

$$\int_\Omega A_n(x)|\nabla \varphi|^{p-2}\nabla\varphi\nabla(u_n - \varphi),$$

we can take advantage of the monotonicity of the $p-$laplace operator to obtain

$$\int_\Omega A_n(x)|\nabla \varphi|^{p-2}\nabla\varphi\nabla(u_n - \varphi) \leq \int_\Omega f_n(x)(u_n - \varphi).$$

Thanks to (5.23), we pass to the limit in the inequality above obtaining

$$\int_\Omega A(x)|\nabla \varphi|^{p-2}\nabla\varphi\nabla(u - \varphi) \leq \int_\Omega f(x)(u - \varphi) \quad \forall\, \varphi \in X_0^p(\Omega). \tag{5.24}$$

To recover (5.4) we use a weighted version of the classical Minty Lemma. Recalling (5.22), we can chose $\varphi = u - tw$ as test function in (5.24), where $t \in \mathbb{R}$ and $w \in X_0^p(\Omega)$. It follows

$$t\int_\Omega A(x)|\nabla(u - tw)|^{p-2}\nabla(u - tw)\nabla w \leq t\int_\Omega f(x)w.$$

If $t \to 0^+$ we get

$$\int_\Omega A(x)|\nabla u|^{p-2}\nabla u \nabla w \leq \int_\Omega f(x)w.$$

On the other hand, if $t \to 0^-$, we get the opposite inequality. Then it results that

$$\int_\Omega A(x)|\nabla u|^{p-2}\nabla u\nabla w = \int_\Omega f(x)w \quad \forall\, w \in X_0^p(\Omega).$$

*Case* $f \in L^1(\Omega)$.

**Step 1.** Due to the poor summability of the datum, we take $T_k(u_n)$ with $k > 0$ as a test function in (5.21) and we get the following energy estimate for the truncates

$$\alpha \int_\Omega |\nabla T_k(u_n)|^p \le \int_\Omega A_n(x)|\nabla T_k(u_n)|^p \le k\int_\Omega |f|,$$

from which we can infer, arguing as in Theorem 6.1 of [17], that there exists a function $u \in \mathcal{T}_0^{1,p}(\Omega)$ such that, up to a not relabeled subsequence, for every $k > 0$

$$\begin{aligned} T_k(u_n) &\rightharpoonup T_k(u) && \text{weakly in } W_0^{1,p}(\Omega),\\ T_k(u_n) &\to T_k(u) && \text{strongly in } L^q(\Omega)\ \forall\, 1 \le q < \infty,\\ u_n &\to u && a.e. \text{ in } \Omega. \end{aligned}$$

In this case from Lemma 5.7, with $\sigma = A_n$ and $\rho_n = \nabla T_k(u_n)$, we deduce that for every $k > 0$

$$A(x)|\nabla T_k(u)|^p \in L^1(\Omega) \quad\text{and}\quad A_n(x)^{\frac{1}{p}}\nabla T_k(u_n) \rightharpoonup A(x)^{\frac{1}{p}}\nabla T_k(u) \quad \text{weakly in } L^p(\Omega)^N. \tag{5.25}$$

**Step 2.** To pass to the limit in equation (5.21), we follow the approach developed in [40]. Let $\varphi$ belong to $X_0^p(\Omega) \cap L^\infty(\Omega)$, where $X_0^p(\Omega)$ is the space defined in (5.3), and take $\phi = T_k(u_n - \varphi)$ as a test function in (5.21). We get

$$\int_\Omega A_n(x)|\nabla u_n|^{p-2}\nabla u_n \nabla T_k(u_n - \varphi) = \int_\Omega f_n(x)T_k(u_n - \varphi).$$

Adding and subtracting in the equation above the term

$$\int_\Omega A_n(x)|\nabla\varphi|^{p-2}\nabla\varphi\nabla T_k(u_n - \varphi),$$

we can take advantage of the monotonicity of the $p-$Laplace operator to obtain

$$\int_\Omega A_n(x)|\nabla\varphi|^{p-2}\nabla\varphi\nabla T_k(u_n - \varphi) \le \int_\Omega f_n(x)T_k(u_n - \varphi). \tag{5.26}$$

In order to pass to the limit in (5.26) with respect to $n$, let us notice that the right hand side above can be written as

$$\begin{aligned} &\int_\Omega A_n(x)|\nabla\varphi|^{p-2}\nabla\varphi\nabla T_k(u_n - \varphi)\\ &\qquad = \int_{\{|u_n-\varphi|\le k\}} A_n(x)^{\frac{1}{p'}}|\nabla\varphi|^{p-2}\nabla\varphi\nabla u_n A_n(x)^{\frac{1}{p}} - \int_{\{|u_n-\varphi|\le k\}} A_n(x)|\nabla\varphi|^p. \end{aligned}$$

Since $\{|u_n - \varphi| \le k\} \subset \{|u_n| \le k + \|\varphi\|_{L^\infty(\Omega)}\}$, recalling the properties of $\varphi$ and using (5.25), we can pass to the limit with respect to $n$ in (5.26) in order to obtain

$$\int_\Omega A(x)|\nabla\varphi|^{p-2}\nabla\varphi\nabla T_k(u - \varphi) \le \int_\Omega f(x)T_k(u - \varphi) \quad \forall\, \varphi \in X_0^p(\Omega) \cap L^\infty(\Omega). \tag{5.27}$$

To prove that (5.27) implies (5.5) we take advantage of the $L^1$ version of the Minty Lemma proved in Lemma 7 of [40], that can be adapted to our case with minor modifications. We give just a sketch of the proof. Choosing

$\varphi = T_h(u) + tT_k(u-\psi)$, with $h > 0$, $|t| < 1$ and $\psi \in X_0^p(\Omega) \cap L^\infty(\Omega)$, as a test function in (5.27), we have

$$\begin{aligned}&\int_{\{|G_h(u)-tT_k(u-\psi))|\le k\}} A(x)|\nabla(T_h(u)+tT_k(u-\psi))|^{p-2}\nabla(T_h(u)+tT_k(u-\psi))\nabla G_h(u)\\ &\quad - t\int_{\{|G_h(u)-tT_k(u-\psi))|\le k\}} A(x)|\nabla(T_h(u)+tT_k(u-\psi))|^{p-2}\nabla(T_h(u)+tT_k(u-\psi))\nabla T_k(u-\psi)\\ &\qquad\le \int_\Omega f(x)T_k(G_h(u)-tT_k(u-\psi)) \quad \forall\, \phi \in X_0^p(\Omega)\cap L^\infty(\Omega).\end{aligned}$$

Taking the limit with respect to $h \to \infty$ on both side of the inequality above, it follows that

$$-t\int_\Omega A(x)|\nabla(u+tT_k(u-\psi))|^{p-2}\nabla(u+tT_k(u-\psi))\nabla T_k(u-\psi) \le -t\int_\Omega f(x)T_k(u-\psi).$$

As $t \to 0^-$ we get

$$\int_\Omega A(x)|\nabla u|^{p-2}\nabla u\nabla T_k(u-\psi) \le \int_\Omega f(x)T_k(u-\psi).$$

As $t \to 0^+$ we get the opposite inequality and hence the thesis. □

**Remark 5.8.** *Note that indeed we have proved that $u$ and $T_k(u)$ can be taken respectively as test functions in* (5.4) *and* (5.5).

**Remark 5.9.** *In Theorem 5.1 we have considered only the extreme cases $f \in L^1(\Omega)$ or $f \in L^{(p^*)'}(\Omega)$, but that there is no gain in considering an intermediate situation. Indeed as soon as $f$ belongs to $L^m(\Omega)$ with $1 \le m < (p^*)'$, we are not able to recover a reasonable estimate of the type*

$$\int A_n(x)|\nabla u_n|^q \le C_2 \quad \textit{for some} \quad q \le \frac{(p-1)Nm}{N-m},$$

*and thus there is no hope to pass to limit in* (5.21) *without the notion of entropy solutions.*

As already said in the Introduction, adding a lower order term it is possible to observe some regularizing effect and this is what we are going to show proving Theorem 5.2.

*Proof of Theorem 5.2.* Let us divide the proof in four steps.
*Step 1.* Approximation and a priori estimates.
*Step 2.* Almost everywhere converge of the gradients.
*Step 3.* Passage to the limit and conclusion.
*Step 4.* The case $f \in L^{q'}(\Omega)$ with $q > p^*$.
**Step 1.** Thanks to [46] for any $n \in \mathbb{N}$ there exists $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ solution of the following approximating problem

$$\int_\Omega A_n|\nabla u_n|^{p-2}\nabla u_n\nabla\phi + \int_\Omega B_n(x)|u_n|^{q-2}u_n\phi = \int_\Omega f_n\phi \qquad \forall\phi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega). \tag{5.28}$$

Let us start with some estimates of the lower order term. Taking $\phi = \frac{T_j(G_k(u_n))}{j}$, with $j > 0$ and $k \ge 0$, we have

$$\begin{aligned}&\int_{\{|u_n|>k+j\}} B_n(x)|u_n|^{q-1}\\ &\qquad \le \frac{1}{j}\int_\Omega A_n(x)|\nabla T_j(G_k(u_n))|^p + \int_\Omega B_n(x)|u_n|^{q-2}u_n\frac{T_j(G_k(u_n))}{j} \le \int_{\{|u_n|>k\}} |f(x)|,\end{aligned}$$

that, using Fatou Lemma as $j \to 0$, becomes

$$\int_{\{|u_n|>k\}} B_n(x)|u_n|^{q-1} \le \int_{\{|u_n|>k\}} |f|. \tag{5.29}$$

In particular, for $k = 0$, we get

$$\int_\Omega B_n(x)|u_n|^{q-1} \le ||f||_{L^1(\Omega)}. \tag{5.30}$$

Following [38], let us take now $\phi = [1 - (1 + |u_n|)^{1-p(1-\lambda)}]\text{sgn}(u_n)$ with $0 < \lambda < 1 - \frac{1}{p}$ as a test function in (5.28), noticing that, thanks to the choice of $\lambda$, it follows that $|\phi| \le 1$. Taking advantage of the sign condition on the lower order term, we drop a positive term and obtain

$$\int_\Omega \frac{A_n|\nabla u_n|^p}{(1 + |u_n|)^{p(1-\lambda)}} \le \frac{||f||_{L^1(\Omega)}}{1 - p(1 - \lambda)}. \tag{5.31}$$

Being our aim to obtain a uniform estimate of $A_n|\nabla u_n|^r$ in $L^1(\Omega)$ for some $r > 1$, let us consider

$$\begin{aligned}\int_\Omega A_n|\nabla u_n|^r &= \int_\Omega \frac{A_n^{\frac{r}{p}}|\nabla u_n|^r}{(1 + |u_n|)^{r(1-\lambda)}} A_n^{1-\frac{r}{p}}(1 + |u_n|)^{r(1-\lambda)} \\ &\le \left(\int_\Omega \frac{A_n|\nabla u_n|^p}{(1 + |u_n|)^{p(1-\lambda)}}\right)^{\frac{r}{p}} \left(\int_\Omega A_n(1 + |u_n|)^{\frac{pr(1-\lambda)}{p-r}}\right)^{1-\frac{r}{p}} \\ \le C_3 + C_4 \left(\int_\Omega A_n|u_n|^{\frac{pr(1-\lambda)}{p-r}}\right)^{1-\frac{r}{p}} &\le C_3 + C_4\delta^{\frac{r}{p}-1} \left(\int_\Omega B_n(x)|u_n|^{\frac{pr(1-\lambda)}{p-r}}\right)^{1-\frac{r}{p}},\end{aligned}$$

where we used Hölder inequality with exponents $\frac{p}{r}$ and $\frac{p}{p-r}$, estimate (5.31) and assumption (5.7). In order to control the last integral above with estimate (5.30), we have to impose that

$$\frac{pr(1 - \lambda)}{p - r} = q - 1 \quad \text{namely} \quad r = \frac{p(q - 1)}{p(1 - \lambda) + q - 1} < \frac{p(q - 1)}{q},$$

where the last inequality is due to the fact that $\lambda < 1 - \frac{1}{p}$. Recalling that $q > p'$ implies $\frac{p(q-1)}{q} > 1$, we have that

$$\alpha \int_\Omega |\nabla u_n|^r \le \int_\Omega A_n|\nabla u_n|^r \le C_5 \quad \text{with} \quad 1 < r < \frac{p(q - 1)}{q}.$$

From the previous estimate we deduce that there exists $u \in W_0^{1,r}(\Omega)$ such that, up to a not relabeled subsequence, $u_n \rightharpoonup u$ weak in $W_0^{1,r}(\Omega)$ and $u_n \to u$ $a.e.$ in $\Omega$. Using Lemma 5.7 we also have that

$$A(x)|\nabla u|^r \in L^1(\Omega), \tag{5.32}$$

and that

$$A_n(x)^{\frac{1}{r}}\nabla u_n \rightharpoonup A(x)^{\frac{1}{r}}\nabla u \quad \text{weakly in} \quad L^r(\Omega)^N.$$

Notice moreover that, thanks to the sign condition on the lower order term, taking $T_k(u_n)$ as test function in (5.28), we deduce, as in Theorem 5.1, the estimate

$$\alpha \int_\Omega |\nabla T_k(u_n)|^p \le \int_\Omega A_n(x)|\nabla T_k(u_n)|^p \le k \int_\Omega |f|.$$

From which if follows that, for any $k > 0$,

$$A_n(x)^{\frac{1}{p}}\nabla T_k(u_n) \rightharpoonup A(x)^{\frac{1}{p}}\nabla T_k(u) \quad \text{weakly in} \quad L^p(\Omega)^N$$

and (see [17]) that

$$u_n \rightharpoonup u \quad \text{in} \quad W_0^{1,\rho}(\Omega) \quad \text{for any} \quad 1 \le \rho < \frac{N(p-1)}{N-1}.$$

**Step 2.** Following [24], let us take $\phi = T_j(u_n - T_k(u))$, with $0 < j < k$, as test function in (5.28) in order to obtain

$$\int_\Omega A_n \left[ |\nabla u_n|^{p-2}\nabla u_n - |\nabla T_k(u)|^{p-2}\nabla T_k(u) \right] \nabla T_j(u_n - T_k(u))$$
$$\le 2j\|f\|_{L^1(\Omega)} - \int_\Omega A_n |\nabla T_k(u)|^{p-2}\nabla T_k(u) \nabla T_j(u_n - T_k(u)), \quad (5.33)$$

where we have used that, thanks to (5.30),

$$\int_\Omega B_n(x)|u_n|^{q-1}|T_j(u_n - T_k(u))| \le j\|f\|_{L^1(\Omega)}.$$

The weak convergence of $A_n(x)^{\frac{1}{p}}\nabla T_k(u_n)$, proved at the end of Step 1, and the inclusion $\{|u_n - T_k(u)| < j\} \subset \{|u_n| < 2k\}$ allow us to infer that the last term in (5.33) goes to zero as $n$ diverges. Hence we get

$$\limsup_{n\to\infty} \int_\Omega \left[ |\nabla u_n|^{p-2}\nabla u_n - |\nabla T_k(u)|^{p-2}\nabla T_k(u) \right] \nabla T_j(u_n - T_k(u)) \le \frac{2j\|f\|_{L^1(\Omega)}}{\alpha}.$$

At this point, recalling the uniform bound of $u_n$ in $W_0^{1,\rho}(\Omega)$ with $\rho < \frac{N(p-1)}{N-1}$, we apply the argument of Step 3 in the proof of Theorem 2.1 in [24] to conclude that

$$\nabla u_n \to \nabla u \;\; a.e. \;\text{ in }\; \Omega.$$

**Step 3.** Thanks to the assumption $q > p$, it is possible to choose $r < \frac{p(q-1)}{q}$ such that $r > p - 1$. With this fact in mind, let us rewrite the first term of (5.28) as

$$\int_\Omega A_n^{\frac{p-1}{r}} |\nabla u_n|^{p-2}\nabla u_n \nabla\phi A_n^{\frac{r-p+1}{r}}.$$

Notice that with the previous choice of $r$ it follows that

$$\int_\Omega \left| A_n^{\frac{p-1}{r}} |\nabla u_n|^{p-2}\nabla u_n \right|^{\frac{r}{p-1}} = \int_\Omega A_n |\nabla u_n|^r \le C_6,$$

that, together with the almost everywhere convergence of the gradients proved in Step 2, implies

$$A_n^{\frac{p-1}{r}} |\nabla u_n|^{p-2}\nabla u_n \rightharpoonup A(x)^{\frac{p-1}{r}} |\nabla u|^{p-2}\nabla u \quad \text{weakly in} \quad L^{\frac{r}{p-1}}(\Omega)^N.$$

On the other hand for every $\phi \in C_c^1(\Omega)$

$$A_n^{\frac{r-p+1}{r}} \nabla\phi \to A(x)^{\frac{r-p+1}{r}} \nabla\phi \quad \text{strongly in} \quad L^{\frac{r}{r-p+1}}(\Omega)^N.$$

This two pieces of information allow us to pass to the limit in the second order term of (5.28). It only remains to pass to the limit in the lower order term. We show that $B_n(x)|u_n|^{q-2}u_n \to B(x)|u|^{q-2}u$ in $L^1(\Omega)$ using Vitali Theorem. For every measurable set $E \subset \Omega$ we have

$$\int_E B_n(x)|u_n|^{q-1} \le k^{t-1} \int_{E\cap\{|u_n|\le k\}} B_n(x) + \int_{E\cap\{|u_n|>k\}} B_n(x)|u_n|^{q-1}$$
$$\le k^{q-1} \int_E B(x) + \int_{\{|u_n|>k\}} |f(x)|,$$

where we used estimate (5.29). From the estimate above easily follows that the sequence $B_n(x)|u_n|^{q-2}u_n$ is equi-integrable. Moreover the $a.e.$ convergence of the $u_n$ allow us to apply the Vitali Convergence theorem to conclude that $B_n(x)|u_n|^{q-2}u_n \to B(x)|u|^{q-2}u$ in $L^1(\Omega)$. Hence we have proved (5.8).

**Step 4.** Let us assume now that $q > p^*$ and that $f$ belongs to $L^{q'}(\Omega)$ (note that $q > p^*$ implies that $f \notin W^{-1,p'}(\Omega)$). Choosing $u_n$ as test function we get

$$\int_\Omega A_n(x)|\nabla u_n|^p + \int_\Omega B_n(x)|u_n|^q \le \int_\Omega |f||u_n|. \tag{5.34}$$

Dropping the first integral in the left hand side and using assumption (5.7) and (5.2), we obtain

$$\alpha\delta \int_\Omega |u_n|^q \le \int_\Omega B_n(x)|u_n|^q \le \|f\|_{L^{q'}(\Omega)} \left(\int_\Omega |u_n|^q\right)^{\frac{1}{q}}. \tag{5.35}$$

that is

$$\left(\int_\Omega |u_n|^q\right)^{\frac{1}{q'}} \le \frac{\|f\|_{L^{q'}(\Omega)}}{\alpha\delta}.$$

Hence we can go back to (5.34) and drop the second term in the left hand side in order to get

$$\int_\Omega A_n(x)|\nabla u_n|^p \le \|f\|_{L^{q'}(\Omega)} \left(\int_\Omega |u_n|^q\right)^{\frac{1}{q}} \le \frac{\|f\|^{q'}_{L^{q'}(\Omega)}}{(\alpha\delta)^{\frac{1}{q-1}}}. \tag{5.36}$$

Thus using the estimate (5.35) and (5.36), the a.e. convergence of the sequence $u_n$ and Lemma 5.7, we prove (5.9) in the case $f \in L^{q'}(\Omega)$ with $q > p^*$.

□

**Remark 5.10.** *With minor modifications of the proof it is possible to consider more general lower order terms as* $d(x, u)$*, where* $d : \mathbb{R} \times \Omega \to \mathbb{R}$ *is a Carathéodory function satisfying*

$$d(x, s)\mathit{sign}(s) \ge A(x)|s|^{q-1} \quad \mathit{for} \quad |s| \gg 1,$$

*and* $A(x)$ *as in* (5.2)*. See for example [43] and [46].*

Now we give the proof of Theorem 5.4. We shall see that the main concern is to absorb the lower order term into the second order operator, in order to obtain the strong convergence of the truncates of the approximating solutions in the energy space and the almost everywhere convergence of the gradients. This is done combining Lemma 5.7, assumption (5.13) and standard techniques developed for problems with first order terms with natural growth. We stress that the stronger assumptions (5.14) and (5.15) give rise to a regularizing effect as in [37].

*Proof of Theorem 5.4.* Let us at first analyze the case $f \in L^{(p^*)'}(\Omega)$ following three steps.
*Step 1.* Approximation and a priori estimates.
*Step 2.* Strong convergence of truncations and a.e. convergence of the gradients.
*Step 3.* Passage to the limit and conclusion.
**Step 1.** For any $n \in \mathbb{N}$, let us consider the function $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, solution of the following approximating problem

$$\int_\Omega A_n|\nabla u_n|^{p-2}\nabla u_n \nabla\phi + \int_\Omega D_n(x)g(u_n)|\nabla u_n|^p\phi = \int_\Omega f_n\phi, \tag{5.37}$$

with $\phi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, whose existence is proved for example in [33]. Choosing $u_n$ as test function in (5.37) and dropping a positive term (recall the sign assumption on $g$), we get

$$\int_\Omega A_n(x)|\nabla u_n|^p \le \|f\|_{L^{(p^*)'}(\Omega)}\|u_n\|_{L^{p^*}(\Omega)} \le \mathcal{S}_p \frac{\|f\|_{(p^*)'}}{\alpha^{\frac{1}{p}}}\|A_n^{\frac{1}{p}}\nabla u_n\|_{L^p(\Omega)},$$

that, thanks to assumption (5.2), becomes

$$\alpha \int_\Omega |\nabla u_n|^p \le \int_\Omega A_n(x)|\nabla u_n|^p \le \frac{\mathcal{S}_p^{\frac{p}{p-1}}}{\alpha^{\frac{1}{(p-1)}}}\|f\|^{\frac{p}{p-1}}_{L^{(p^*)'}(\Omega)}, \tag{5.38}$$

This estimate assures that, up to a not relabeled subsequence, $u_n$ converges to some $u \in W_0^{1,p}(\Omega)$ weakly in $W_0^{1,p}(\Omega)$, strongly in $L^q(\Omega)$ with $q < p^*$, and $a.e.$ in $\Omega$. Moreover, thanks to Lemma 5.7, it follows that

$$A|\nabla u|^p \in L^1(\Omega) \quad \text{and} \quad A_n^{\frac{1}{p}}\nabla u_n \rightharpoonup A(x)^{\frac{1}{p}}\nabla u \quad \text{weakly in} \quad L^p(\Omega)^N. \tag{5.39}$$

As far as the lower order term is concerned, take $\frac{T_j(G_k(u_n))}{j}$, with $j > 0$ and $k \ge 0$, as test function in (5.37). Dropping a positive term we get

$$\int_{\{|u_n|>k+j\}} D_n(x)|g(u_n)||\nabla u_n|^p \le \int D_n(x)|g(u_n)||\nabla u_n|^p \frac{|T_j(G_k(u_n))|}{j} \le \int_{\{|u_n|>k\}} |f|.$$

Using Fatou Lemma with respect to $j \to 0$, we obtain

$$\int_{\{|u_n|>k\}} D_n(x)|g(u_n)||\nabla u_n|^p \le \int_{\{|u_n|>k\}} |f| \quad \text{for } k \ge 0. \tag{5.40}$$

Moreover, for $k = 0$, we can apply Lemma 5.7 to infer that

$$\int_\Omega D(x)|g(u)||\nabla u|^p \le \liminf_{n\to\infty} \int_\Omega D_n(x)|g(u_n)||\nabla u_n|^p \le \int_\Omega |f|,$$

i.e. $D(x)|g(u)||\nabla u|^p$ belongs to $L^1(\Omega)$.

**Step 2.** In this step we follow the approach of [37]. Let us use $\phi = \varphi_\lambda(z_n)$, with $z_n = T_k(u_n) - T_k(u)$, $\varphi_\lambda(t) = (e^{\lambda|t|} - 1)\text{sgn}(t)$ and $\lambda > 0$, as test function in (5.37). It is worthy to note that $|\varphi_\lambda(z_n)| + \varphi'_\lambda(z_n) \le (1+\lambda)e^{2\lambda k}$ and that, thanks to the a.e. convergence of $u_n$ proved in Step 1, $\varphi_\lambda(z_n) \to 0$ and $\varphi'_\lambda(z_n) \to 1$ $a.e.$ in $\Omega$. We get

$$\underbrace{\int_\Omega A_n|\nabla u_n|^{p-2}\nabla u_n \nabla(T_k(u_n) - T_k(u))\varphi'_\lambda(z_n)}_{I_1} + \underbrace{\int_\Omega D_n(x)g(u_n)|\nabla u_n|^p \varphi_\lambda(z_n)}_{I_2}$$

$$= \underbrace{\int_\Omega f(x)\varphi_\lambda(z_n)}_{I_3}.$$

In order to rewrite $I_1$ in a more convenient way note at first that the term

$$\int_\Omega A_n|\nabla G_k(u_n)|^{p-2}\nabla G_k(u_n)\nabla T_k(u)\varphi'_\lambda(z_n)$$

converges to zero as $n$ diverges. Indeed, on one hand, estimate (5.38) tells us that

$$A^{\frac{1}{p'}}|\nabla G_k(u_n)|^{p-2}\nabla G_k(u_n)$$

is weakly convergent in $W_0^{1,p'}(\Omega)$; on the other, Lebesgue Theorem, together with the almost everywhere convergence of $u_n$ to $u$, (5.39) and the properties of $\varphi'_\lambda$, assures that

$$A_n(x)^{\frac{1}{p}}\nabla T_k(u)\chi_{\{|u_n|>k\}}\varphi'_\lambda(z_n) \to 0 \quad \text{strongly in} \quad L^p(\Omega)^N.$$

Secondly, using (5.39), it results that

$$A_n(x)^{\frac{1}{p}}\nabla(T_k(u_n)-T_k(u)) \rightharpoonup 0 \quad \text{weakly in} \quad L^p(\Omega)^N.$$

Thus we have that

$$\begin{aligned}I_1 = &\int_\Omega A_n \left[|\nabla T_k(u_n)|^{p-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\right]\nabla(T_k(u_n)-T_k(u))\varphi_\lambda'(z_n)\\ &+\int_\Omega A_n|\nabla T_k(u)|^{p-2}\nabla T_k(u)\nabla(T_k(u_n)-T_k(u))\varphi_\lambda'(z_n)\\ &+\int_\Omega A_n|\nabla G_k(u_n)|^{p-2}\nabla G_k(u_n)\nabla T_k(u)\varphi_\lambda'(z_n)\\ =&\int_\Omega A_n \left[|\nabla T_k(u_n)|^{p-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\right]\nabla(T_k(u_n)-T_k(u))\varphi_\lambda'(z_n)+\epsilon_n,\end{aligned}$$

where $\lim_{n\to\infty}\epsilon_n = 0$. With regard to $I_2$, using the sign assumption (5.11), we get

$$\begin{aligned}I_2 = &\int_{\{|u_n|>k\}} D_n(x)g(u_n)|\nabla u_n|^p\varphi_\lambda(z_n) + \int_{\{|u_n|\le k\}} D_n(x)g(u_n)|\nabla u_n|^p\varphi_\lambda(z_n)\\ &\ge \int_{\{|u_n|\le k\}} D_n(x)g(u_n)|\nabla u_n|^p\varphi_\lambda(z_n) \ge -\sigma\bar g_k\int_\Omega A_n(x)|\nabla T_k(u_n)|^p|\varphi_\lambda(z_n)|\\ =& -\sigma\bar g_k\int_\Omega A_n\left[|\nabla T_k(u_n)|^{p-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\right]\nabla(T_k(u_n)-T_k(u))|\varphi_\lambda(z_n)|\\ &-\sigma\bar g_k\int_\Omega A_n|\nabla T_k(u)|^{p-2}\nabla T_k(u)\nabla(T_k(u_n)-T_k(u))|\varphi_\lambda(z_n)|\\ &+\sigma\bar g_k\int_\Omega A_n|\nabla T_k(u_n)|^{p-2}\nabla T_k(u_n)\nabla T_k(u)|\varphi_\lambda(z_n)|,\end{aligned}$$

where $\bar g_k = \max_{|s|\le k} g(s)$. Using (5.39) and the definition of $\varphi_\lambda(z_n)$, we deduce that the last two integrals above converge to zero as $n$ diverges. Moreover it easily follows that $I_3 = \epsilon_n$ as $n\to\infty$. Lastly notice that if $\lambda = 1+\sigma\bar g_k$, then

$$\varphi_\lambda' - \sigma\bar g_k|\varphi_\lambda(z_n)| > 1.$$

Thus summing up the information on $I_1$, $I_2$, $I_3$ and with the previous choice of $\lambda$ we get

$$\lim_{n\to\infty}\int_\Omega A_n(x)\left[|\nabla T_k(u_n)|^{p-2}\nabla T_k(u_n) - |\nabla T_k(u)|^{p-2}\nabla T_k(u)\right]\nabla(T_k(u_n)-T_k(u) = 0.$$

Recalling assumption (5.2), we can infer that (see for example [66]) for every $k>0$

$$\nabla u_n \to \nabla u \quad a.e. \quad \text{and that} \quad A_n(x)^{\frac{1}{p}}\nabla T_k(u_n) \to A(x)^{\frac{1}{p}}\nabla T_k(u) \quad \text{strongly in} \quad L^p(\Omega)^N.$$

**Step 3.** Thanks to Step 1 and Step 2 we can pass to the limit in the first integral of (5.37). Now we claim that

$$D_n(x)g(u_n)|\nabla u_n|^p \to D(x)g(u)|\nabla u|^p \quad \text{strongly in } L^1(\Omega).$$

In order to prove the claim, let $E$ be any measurable subset of $\Omega$ and recall assumption (5.13) and estimate (5.40). We get

$$\begin{aligned}&\int_E D_n(x)|g(u_n)||\nabla u_n|^p\\ &\quad\le \bar g_k\int_{E\cap\{|u_n|\le k\}} D_n(x)|\nabla T_k(u_n)|^p + \int_{E\cap\{|u_n|>k\}} D_n(x)|g(u_n)||\nabla G_k(u_n)|^p\\ &\quad\le \sigma\bar g_k\int_{E\cap\{|u_n|\le k\}} A_n|\nabla T_k(u_n)|^p + \int_{\{|u_n|>k\}}|f|.\end{aligned}$$

From the strong convergence of the sequences $u_n$ and $A_n(x)|\nabla T_k(u_n)|^p$ in $L^1(\Omega)$ (proved respectively in Step 1 and Step 2), it follows that the sequence $D_n(x)g(u_n)|\nabla u_n|^p$ is equi-integrable. Taking advantage of the a.e. convergence of the gradients proved in Step 2, we can apply Vitali Theorem and the claim is proved.
At this point we can easily pass to the limit in (5.37) and conclude the proof of Theorem 5.4 in the case $f \in L^{(p^*)'}(\Omega)$.

In order to deal with the case $f \in L^1(\Omega)$, with the additional assumptions (5.14) and (5.15), let us take $T_{\tilde{s}}(u_n)$, with $\tilde{s}$ as in (5.14), as test function in (5.37). We get

$$\int_\Omega A_n|\nabla T_{\tilde{s}}(u_n)|^p + \tau\gamma\tilde{s}\int_\Omega A_n|\nabla G_{\tilde{s}}(u_n)|^p \le \tilde{s}\|f\|_{L^1(\Omega)},$$

that is

$$\int_\Omega A_n|\nabla u_n|^p \le \frac{\tilde{s}}{\min\{1,\tau\gamma\tilde{s}\}}\|f\|_{L^1(\Omega)}.$$

Hence using once more Lemma 5.7 we obtain that

$$\int_\Omega A(x)|\nabla u|^p \le \frac{\tilde{s}}{\min\{1,\tau\gamma\tilde{s}\}}\|f\|_{L^1(\Omega)}$$

and that

$$A_n^{\frac{1}{p}}\nabla u_n \rightharpoonup A(x)^{\frac{1}{p}}\nabla u \quad \text{weakly in} \quad L^p(\Omega)^N.$$

Note now that Step 2 and Step 3 hold true also for $L^1(\Omega)$ data. Hence exactly as in the previous case we obtain the thesis. □

In this last part we prove Theorem 5.6.

*Proof of Theorem 5.6.* Let us set for $n \in \mathbb{N}$

$$a_n(x,s,\xi) := \frac{a(x,s,\xi)}{A(x)h(|s|)}A_n(x)h_n(|s|), \tag{5.41}$$

where $h_n(s) := T_n(h(s))$. Note that by definition $a_n(x,s,\xi)$ satisfies

$$a_n(x,s,\xi)\xi \ge A_n(x)h_n(|s|)|\xi|^p,$$

$$|a_n(x,s,\xi)| \le \gamma A_n(x)(1+h_n(|s|))\,|\xi|^{p-1}],$$

$$[a_n(x,s,\xi) - a_n(x,s,\xi^*)][\xi-\xi^*] > 0,$$

for almost every $x \in \Omega$, every $s \in \mathbb{R}$ and every $\xi, \xi^* \in \mathbb{R}^N$ with $\xi \neq \xi^*$. Thus we can use for example [33] (see also [66]) to infer the existence of $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ solution of

$$\int_\Omega a_n(x,u_n,\nabla u_n)\nabla\phi = \int_\Omega f_n\phi \quad \forall\phi \in W_0^{1,p}(\Omega). \tag{5.42}$$

We divide the rest of the proof into several steps; in the first three we assume only $f \in L^1(\Omega)$ while in the last one we consider $L^{(p^*)'}(\Omega)$ data.
*Step 1.* A priori estimates.
*Step 2.* Entropy solutions.
*Step 3.* $\nabla u_n \to \nabla u$ $a.e.$ in $\Omega$.
*Step 4.* Distributional solutions.
**Step 1.** In order to recover the basic energy estimate for the truncates of $u_n$, we need to define the following auxiliary functions (see [80])

$$H_n(s) := \int_0^s h_n(|\tau|)^{\frac{1}{p-1}}d\tau \quad \text{and} \quad H(s) := \int_0^s h(|\tau|)^{\frac{1}{p-1}}d\tau.$$

By construction, $H_n$ is a locally Lipschitz function and $H_n(0) = 0$, thus we can take $H(T_k(u_n))$, with $k > 0$, as test function in (5.42) obtaining

$$\alpha\beta^{p'}\int_\Omega |\nabla T_k(u_n)|^p \le \beta^{p'}\int_\Omega A_n(x)|\nabla T_k(u_n)|^p \le \int_\Omega A_n(x)h_n(T_k(u_n))^{p'}|\nabla T_k(u_n)|^p$$
$$\le \int_\Omega a_n(x,u_n,\nabla u_n)\nabla T_k(u_n)h_n(|T_k(u_n)|)^{\frac{1}{p-1}} \le H_n(k)\|f\|_{L^1(\Omega)} \le H(k)\|f\|_{L^1(\Omega)}.$$

This estimate implies, on one hand, that there exists a function $u \in \mathcal{T}_0^{1,p}(\Omega)$ (see [17]) such that, up to a not relabeled sub sequence,

$$\begin{aligned} &T_k(u_n) \rightharpoonup T_k(u) \quad \text{in} \quad W_0^{1,p}(\Omega), \\ &u_n \rightharpoonup u \quad \text{in} \quad W_0^{1,\rho}(\Omega) \quad \text{for any} \quad 1 \le \rho < \frac{N(p-1)}{N-1}, \\ &u_n \to u \quad a.e. \quad \text{in} \quad \Omega, \end{aligned}$$

and on the other, by Lemma 5.7, that for every $k > 0$

$$A(x)|\nabla T_k(u)|^p \in L^1(\Omega) \quad \text{and} \quad A_n(x)^{\frac{1}{p}}\nabla T_k(u_n) \rightharpoonup A(x)^{\frac{1}{p}}\nabla T_k(u) \quad \text{weakly in} \quad L^p(\Omega)^N. \tag{5.43}$$

**Step 2.** Let $\varphi$ belong to $X_0^p(\Omega) \cap L^\infty(\Omega)$, where $X_0^p(\Omega)$ is the space defined in (5.3), and take $\phi = T_k(u_n - \varphi)$ as a test function in (5.42). Then it follows that

$$\int_\Omega a_n(x,u_n,\nabla u_n)\nabla T_k(u_n-\varphi) = \int_\Omega f_n(x)T_k(u_n-\varphi).$$

Adding and subtracting in the equation above the term

$$\int_\Omega a_n(x,u_n,\nabla\varphi)\nabla T_k(u_n-\varphi)$$

and taking advantage of the monotonicity condition of (5.17) we get

$$\int_\Omega a_n(x,u_n,\nabla\varphi)\nabla T_k(u_n-\varphi) \le \int_\Omega f_n(x)T_k(u_n-\varphi). \tag{5.44}$$

Noticing that $\{|u_n-\varphi| \le k\} \subset \{|u_n| \le k + \|\varphi\|_{L^\infty(\Omega)}\}$ and that

$$\int_\Omega a_n(x,u_n,\nabla\varphi)\nabla T_k(u_n-\varphi) = \int_{\{|u_n-\varphi|\le k\}} \frac{a(x,u_n,\nabla\varphi)}{A(x)h(u_n)}A_n(x)h_n(u_n)\nabla(u_n-\varphi),$$

we can pass to the limit in (5.44) using assumption (5.17), the $a.e.$ convergence of $u_n$ and the weak convergence of $A_n(x)^{\frac{1}{p}}\nabla T_k(u_n)$ proved in Step 1. Hence we obtain, for every $k > 0$,

$$\int_\Omega a(x,u,\nabla\varphi)\nabla T_k(u-\varphi) \le \int_\Omega f(x)T_k(u-\varphi), \quad \forall\,\varphi \in X_0^p(\Omega)\cap L^\infty(\Omega).$$

As in the last part of Theorem 5.1 we take advantage of Lemma 7 of [40] to infer that, for every $k > 0$,

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\varphi) = \int_\Omega f(x)T_k(u-\varphi) \quad \forall\,\varphi \in X_0^p(\Omega)\cap L^\infty(\Omega).$$

**Step 3.** In order to prove the almost everywhere convergence of the gradients we adapt the methods used in [24]. Using $T_j(u_n - T_k(u))$ with $0 < j < k$ as test function in (5.42) we get

$$\int_\Omega \left[a_n(x,u_n,\nabla u_n) - a_n(x,u_n,\nabla T_k(u))\right]\nabla T_j(u_n - T_k(u))$$
$$\le j\|f\|_{L^1(\Omega)} - \int_\Omega a_n(x,u_n,\nabla T_k(u))\nabla T_j(u_n - T_k(u)).$$

The last term above can be written as

$$\int_{|u_n-T_k(u)|<j} \frac{a(x,u_n,\nabla T_k(u))}{A(x)h(|u_n|)} A_n(x)h_n(|u_n|)\nabla T_j(u_n-T_k(u)),$$

that converges to zero as $n$ diverges thanks to (5.43), the $a.e.$ convergence of $u_n$ and the fact that $\{|u_n-T_k(u)|<j\}\subset\{|u_n|<2k\}$. Thus, recalling the definition of $a_n(x,s,\xi)$ and assumptions (5.2) and (5.18), it follows

$$\limsup_{n\to\infty} \alpha\beta \int_\Omega \left[\frac{a(x,u_n,\nabla T_k(u_n))}{A(x)h(|u_n|)} - \frac{a(x,u_n,\nabla T_k(u))}{A(x)h(|u_n|)}\right]\nabla(u_n-T_k(u)) \le \frac{j\|f\|_{L^1(\Omega)}}{\alpha\beta}.$$

Noticing that the Carathéodory function

$$\frac{a(x,s,\xi)}{A(x)h(|s|)}$$

satisfies the assumptions of a Leray-Lions operator and thanks to the weak convergence of $u_n$ in $W_0^{1,\rho}(\Omega)$ for any $\rho<\frac{N(p-1)}{N-1}$, we can use the argument of Step 3 in the proof of Theorem 2.1 in [24] to conclude that $\nabla u_n\to\nabla u$ $a.e.$ in $\Omega$.

**Step 4.** In this last part of the proof we consider the stronger assumption $f\in L^{(p^*)'}(\Omega)$. Recalling that $u_n\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, we can choose $H_n(u_n)$ as test function in (5.42), where as before $H_n(s)=\int_0^s h_n(|t|)^{\frac{1}{p-1}}dt$. We obtain

$$\int_\Omega A_n(x)|\nabla H(u_n)|^p = \int_\Omega A_n(x)h_n(|u_n|)^{p'}|\nabla u_n|^p \le \|f\|_{L^{(p^*)'}(\Omega)}\|H(u_n)\|_{L^{p^*}(\Omega)}.$$

Using Sobolev inequality we get

$$\int_\Omega A_n(x)h_n(|u_n|)^{p'}|\nabla u_n|^p \le \frac{\mathcal{S}_p^{\frac{p}{p-1}}}{\alpha^{\frac{1}{p-1}}}\|f\|^{\frac{p}{p-1}}_{L^{(p^*)'}(\Omega)}$$

and, thanks to the $a.e.$ convergence of $\nabla u_n$ to $\nabla u_n$ and Fatou Lemma,

$$\int_\Omega A(x)h(|u|)^{p'}|\nabla u|^p \le \frac{\mathcal{S}_p^{\frac{p}{p-1}}}{\alpha^{\frac{1}{p-1}}}\|f\|^{\frac{p}{p-1}}_{L^{(p^*)'}(\Omega)}.$$

Thus we have that the sequence

$$\frac{a(x,u_n,\nabla u_n)}{A(x)h(|u_n|)}A_n(x)^{\frac{1}{p'}}h_n(|u_n|) \quad \text{is bounded in} \quad L^{p'}(\Omega)^N$$

and converges almost everywhere to $a(x,u,\nabla u)A^{\frac{1}{p'}-1}$ thanks to Step 3. Hence it also converges weakly in $L^{p'}(\Omega)^N$ to its $a.e.$-limit. Noticing that for every $\phi\in X_0^p(\Omega)$

$$A_n(x)^{\frac{1}{p}}\nabla\phi \to A(x)^{\frac{1}{p}}\nabla\phi \quad \text{strongly in} \quad L^p(\Omega)^N,$$

we can pass to the limit in (5.42) and conclude the proof. □

# Chapter 6

# Gradient behaviour for large solutions to semilinear elliptic problems

As already said in the introduction given $p > 1$ and $f$ Lipschitz, under appropriate assumptions on the smoothness of the bounded domain $\Omega \subset \mathbb{R}^N,\ N \geq 1$, we give a precise description of the asymptotic behaviour of the gradient of the unique solution of

$$\begin{cases} -\Delta u + |u|^{p-1}u = f & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega. \end{cases}$$

In particular we show that there exists a corrector function $S$, finite sum of singular terms, such that

$$z := u - S \in W^{1,\infty}(\Omega).$$

Moreover we prove that

$$\forall\, \bar{x} \in \partial\Omega \qquad z(\bar{x}) = 0 \ \text{ and } \ \frac{\partial z}{\partial \nu}(\bar{x}) = 0,$$

where $\nu$ is the outward unit normal to $\partial\Omega$.

## 6.1 Statement of the main results

Before stating precisely our main results, we need to give some notation.

### 6.1.1 Notation

We shall often work in tubular neighborhoods of $\partial\Omega$ of the type

$$\Omega_\delta = \{y \in \Omega \ : \ \text{dist}(u, \partial\Omega) < \delta\},\ \delta > 0.$$

We recall that $\Omega$ is always at least of class $C^2$. Hence the function $\text{dist}(\cdot, \partial\Omega)$ distance from the boundary is well defined and twice differentiable near $\partial\Omega$. More precisely the following Theorem, proved in [64], gives the relationship between the regularity of the boundary and the regularity of the distance function.

**Theorem 6.1** (Theorem 3 in [64]). *Let $\Omega$ be a domain of class $C^\gamma$ with $\gamma \geq 2$. Then*

$$\exists \bar{\bar{\delta}} > 0 \quad \textit{such that} \qquad dist(\cdot, \partial\Omega) \in C^\gamma(\overline{\Omega_{\bar{\bar{\delta}}}}). \tag{6.1}$$

Thanks to the previous Theorem we can define the following smooth versions of the distance function.

**Definition 6.2.** *Let $\Omega$ be a domain of class $C^\gamma$ with $\gamma \geq 2$ and let $\bar\delta > 0$ be given by (6.1). Then we define the regularized distance as a function $d \in C^\gamma(\Omega)$ such that $d(x) = dist(x, \partial\Omega)$ for every $x$ that belongs to $\Omega_{\bar\delta}$. We moreover denote $d_n(x) := d(x) + \frac{1}{n}$.*

It is worthy to stress that $d_n(\cdot)$ inherits from $\text{dist}(\cdot, \partial\Omega)$ the following important properties

$$|\nabla d_n(x)|^2 = 1, \quad \nabla d_n(x) = -\nu(x) \quad \text{and} \quad \Delta d_n(x) = -(N-1)H(x), \qquad x \in \Omega_{\bar\delta}$$

where the vector filed $\nu$ and the function $H$ are such that, for any $\bar x \in \partial\Omega$, $\nu(\bar x)$ is the outward normal to $\partial\Omega$ at $\bar x$ and $H(\bar x)$ is the mean curvature of $\partial\Omega$ at $\bar x$.

Finally, unless otherwise specified, we indicate with $C$ a constant that depends only on the data of the problem and that can vary line to line also in the proof on the same theorem.

## 6.2 Main results

The ansatz that guides our approach is that it is possible to give an explicit description of the explosive behaviour of the large solution $u$ and of its gradient $\nabla u$ by means of a finite sum of singular terms. Inspired by (34), (36) and (37) we conjecture that

$$u(x) \sim \sigma_0 d^{-\alpha} + \sigma_1 d^{-\alpha+1} + \sigma_2 d^{-\alpha+2} + \cdots,$$

where $\sigma_k$ with $k = 0, 1, \cdots$ are smooth functions, and define the following *regularized* function

$$z := u - \left(\sigma_0 d^{-\alpha} + \sigma_1 d^{-\alpha+1} + \sigma_2 d^{-\alpha+2} + \cdots\right). \tag{6.2}$$

Hence the first question we want to answer is:

Can we find $\sigma_k$ with $k = 0, 1, \cdots$ such that $z$ and $|\nabla z|$ belong to $L^\infty(\Omega)$?

Of course the functions $\sigma_1, \ldots, \sigma_k$ shall take into account several characteristics of the problem, among others the geometry of the domain. Notice moreover that the definition (6.2) suggests that we need $[\alpha] + 2$ terms for having $z \in W^{1,\infty}(\Omega)$. Indeed we have the following result.

**Theorem 6.3.** *Let us assume $p > 1$ and fix $\alpha := \frac{2}{p-1}$. Let $\Omega$ be a bounded domain of class $C^{[\alpha]+5}$ with $[\alpha]$ the integer part of $\alpha$, let $f$ belong to $W^{1,\infty}(\Omega)$, and let $u$ be the unique large solution of (38). Let us define the following functions*

$$\begin{aligned}
\sigma_0 :=& [\alpha(\alpha+1)]^{\frac{1}{p-1}} \\
\sigma_1(x) :=& -\frac{1}{2}\frac{\alpha\sigma_0}{1+2\alpha}\Delta d(x) = \sigma_0 \frac{\alpha(N-1)H(x)}{2(1+2\alpha)} \\
\sigma_k(x) :=& \frac{(\alpha+1-k)[\sigma_{k-1}(x)\Delta d(x) + 2\nabla\sigma_{k-1}(x)\nabla d(x)] + \Delta\sigma_{k-2}(x)}{(k-\alpha)(k-\alpha-1) - (2+\alpha)(\alpha+1)} \\
&+ \frac{\sigma_0^p}{(k-\alpha)(k-\alpha-1) - (2+\alpha)(\alpha+1)} \sum_{j=2}^{k} \left[\binom{p}{j}\sigma_0^{-j} \sum_{i_1+\cdots+i_j=k} \sigma_{i_1}(x)\cdots\sigma_{i_j}(x)\right]
\end{aligned} \tag{6.3}$$

*for $k = 2 \cdots [\alpha]+1$ and $i_1, \cdots, i_j$ positive integers.*

*Then $\sigma_k \in C(\overline{\Omega})^{[\alpha]+5-k}$ with $k = 0, \cdots, [\alpha]+1$, and the function $S \in C^4(\Omega)$, defined as*

$$S(x) = \sum_{k=0}^{[\alpha]+1} \sigma_k(x)\, d^{k-\alpha}(x), \tag{6.4}$$

*is such that*

$$z(x) := u(x) - S(x) \in W^{1,\infty}(\Omega).$$

*Moreover it holds true that*

$$z(\bar{x}) \quad \text{and} \quad \frac{\partial z}{\partial \nu}(\bar{x}) = 0 \quad \forall \bar{x} \in \Omega. \tag{6.5}$$

**Remark 6.4.** *Let us stress that the higher the value of $\alpha$ (i.e. the closer $p$ is to 1), the higher the number of singular terms is and the higher the regularity of $\Omega$ has to be.*
*Moreover if we split the above estimate along normal and tangential directions we get a very precise estimate of all the singular terms in the expansion of the gradient. More specifically we have that*

$$\lim_{x\to\partial\Omega} \frac{\partial u}{\partial \nu} - \sum_{k=0}^{[\alpha]+1} (\alpha - k)\sigma_k(x)\, d^{k-\alpha-1}(x) + \frac{\partial \sigma_k(x)}{\partial \nu}\, d^{k-\alpha}(x) = 0 \tag{6.6}$$

*while*

$$\lim_{x\to\partial\Omega} \left| \frac{\partial u}{\partial \tau} - \sum_{k=0}^{[\alpha]+1} \frac{\partial \sigma_k(x)}{\partial \tau}\, d^{k-\alpha}(x) \right| = 0 \tag{6.7}$$

$\forall \tau \in \mathbb{S}^{N-1}$ *such that $\tau \cdot \nu = 0$. Let us stress that with the notation $x \to \partial\Omega$ we mean that the limits above are uniform with respect $\partial\Omega$ (see Theorems 6.12 and 6.13).*

From (6.6) and (6.7) we easily obtain the second order asymptotic of the gradient (40) mentioned in the introduction.

The core of Theorem 6.3 is a Bernstein type estimate for $|\nabla z|$. This type of technique, already used in the framework of large solutions for quasilinear problem in [70] (see also [71]), has been originally developed in [72] and [73] and it allows to obtain $L^\infty(\Omega)$-estimates for solutions of a vast class of boundary value problems. Of course we do not know a priori the boundary condition (if any) satisfied by $u - S$; thus it is not possible to obtain Bernstein estimates directly for $z$ and $|\nabla z|$. We overcome this obstacle arguing by approximation and considering the following regularized corrector function

$$S_n(x) = \sum_{k=0}^{[\alpha]+1} \sigma_k(x) d_n^{k-\alpha}(x), \qquad d_n = d(x) + \frac{1}{n}, \tag{6.8}$$

where $\sigma_0, \cdots, \sigma_{[\alpha]+1}$ are the functions defined in (6.3), and the following approximated problem

$$\begin{cases} -\Delta u_n + |u_n|^{p-1} u_n = f, & \text{in } \Omega \\ \frac{\partial u_n}{\partial \nu} = \frac{\partial S_n}{\partial \nu} & \text{on } \partial\Omega. \end{cases} \tag{6.9}$$

Moreover we define $z_n(x) := u_n(x) - S_n(x)$, that solves

$$\begin{cases} -\Delta z_n + |z_n + S_n|^{p-1}(z_n + S_n) - |S_n|^{p-1} S_n = \widetilde{f}_n & \text{in } \Omega \\ \dfrac{\partial z_n}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.10}$$

where

$$\widetilde{f}_n = f + \Delta S_n - |S_n|^{p-1} S_n. \tag{6.11}$$

Let us stress that the choice of the Neumann boundary condition in (6.9) and in turn in (6.10) is not the only possible, but it is the most convenient for our scope; indeed Neumann problems are particularly suited for the implementation of the previously mentioned Bernstein estimates. Observe at this point that $u_n$ converges, at least in $C^2_{\text{loc}}(\Omega)$ (see Proposition 6.11) to the unique large solution to (38), and this in turn implies that $z_n \to z := u - S$

in $C^2_{loc}(\Omega)$ where $S = \lim_{n\to\infty} S_n$. Hence, once a uniform estimate (with respect to $n$) in $W^{1,\infty}(\Omega)$ is obtained for the solution $z_n$ of (6.10), it can be inherited by $z$ as $n$ diverges.

The proof of Theorem 6.3 is divided into the following main steps:

- we prove that there exists a constant $\bar{C} = \bar{C}(\sigma_0, \cdots, \sigma_{[\alpha]+1}, f)$ such that $d_n|\widetilde{f}_n| + d_n^2|\nabla\widetilde{f}_n| \le \bar{C}d^{1+[\alpha]-\alpha}$ for every $n \in \mathbb{N}$;
- we show that for every $n \in \mathbb{N}$ problem (6.9) admits a solution $u_n$ and we describe the first order behaviour of $u_n$ near the boundary;
- through a Bernstein type estimate, we show that there exists a positive constant $B = B(\sigma_0, \cdots, \sigma_{[\alpha]+1}, f)$ such that $\|z_n\|_{W^{1,\infty}(\Omega)} \le B$ for every $n \in \mathbb{N}$. This implies that $\|z\|_{W^{1,\infty}(\Omega)} \le B$.

Hence Theorem 6.3 tells us that $z \in C^2(\Omega)$ satisfies

$$\begin{cases} -\Delta z + |z+S|^{p-1}(z+S) - |S|^{p-1}S = \widetilde{f} & \text{in } \Omega, \\ z \in W^{1,\infty}(\Omega). \end{cases}$$

where

$$\widetilde{f}(x) = f(x) + \Delta S(x) - |S(x)|^{p-1}S(x) \tag{6.12}$$

and $\widetilde{f} = \lim_{n\to\infty} \widetilde{f}_n$. Note that so far we do not have any information on the boundary behaviour of $z$, apart from the fact that is globally Lipschitz continuous. Thus, it is natural to wonder if $z$ satisfies some boundary condition; and indeed, coupling the previously mentioned Bernstein technique with sub and super solutions method, we prove we prove (6.5).

Let us now consider a class of nonlinearities for which Theorem 6.3 can be generalized with minor modifications. Let us thus focus on the following problem

$$\begin{cases} -\Delta u + h(x)|u|^{p-1}u = r(x,u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{6.13}$$

where $p > 1$, $h \in C^4(\bar{\Omega})$ is such that for $0 < A < B$

$$A \le h(x) \le B \quad \forall\, x \in \bar{\Omega}, \tag{6.14}$$

and $r \in C^1(\bar{\Omega} \times \mathbb{R})$ satisfies

$$r(x,s)s \ge 0 \quad \text{and} \quad \frac{\partial}{\partial s}\left(h(x)|s|^{p-1}s - r(x,s)\right) \ge 0 \quad \forall\,(x,s) \in \Omega \times \mathbb{R}. \tag{6.15}$$

In Theorem 2.7 of [12] it is proved that for any bounded domain $\Omega$ of class $C^2$, under the assumptions (6.14) and (6.15), problem (6.13) admits a positive large solution; moreover every large solution $u$ of (6.13) has the following asymptotic behaviour near $\partial\Omega$

$$\lim_{d(x)\to 0} \frac{u(x)}{\sigma_0\left(\sqrt{h(x)}d(x)\right)^{-\alpha}} = 1. \tag{6.16}$$

Now we make additional growth assumptions on the function $r(x,s)$ in order to be able to implement the Bernstein technique as in Theorem 6.3. We require that

$$\begin{cases} \sup_{0<s<1} |r(x,s^{-\alpha})|s \in L^\infty(\Omega), \\ \sup_{0<s<1} |r_x(x,s^{-\alpha})|s^2 \in L^\infty(\Omega), \\ \sup_{0<s<1} |r_s(x,s^{-\alpha})|s^2 = o(1), & \text{as } d(x) \to 0, \\ \sup_{0<s<1} |r_s(x,s^{-\alpha})|s^{-\alpha+1} \in L^\infty(\Omega), \end{cases} \tag{6.17}$$

where $r_x := \nabla_x r$ and $r_s := \frac{\partial r}{\partial s}$. As a first consequence of (6.17) we deduce that for $1 < q < p$ the function $g(x,s) := h(x)|s|^{p-1}s - r(x,s)$ satisfies

$$\frac{g(x,s)}{s^q} \quad \text{is increasing for large values of } s. \tag{6.18}$$

Indeed

$$\frac{d}{ds}\frac{g(x,s)}{s^q} = \frac{(p-q)s^{p-1} - r_s(x,s) + r(x,s)s^{-1}}{s^q} > 0 \quad \text{for large value of } s, \forall x \in \Omega.$$

Thus using (6.16) and (6.18) we can take advantage of (the proof of) Theorem 2 of [58] to infer that problems (6.13) admits a unique large solution.

We stress here that we obtain the asymptotic expansion of large solutions and their gradient via an approximation procedure; thus, in the absence of a uniqueness result, our method gives a description only of the large solution obtained by the approximating scheme, i.e. the minimal large solution.

We can state our last result.

**Theorem 6.5.** *Let us assume $p > 1$, fix $\alpha := \frac{2}{p-1}$ and let $\Omega$ be a bounded domain of class $C^{[\alpha]+5}$. Assume moreover that (6.14), (6.15) and (6.17) hold true. Then there exist functions $\sigma_{h,k} = \sigma_{h,k}(p,\Omega,h)$ (see (6.39) for the precise definition) with $\sigma_{h,k} \in C(\overline{\Omega})^{[\alpha]+5-k}$ $k = 0,\cdots,[\alpha]+1$, such that, defining $S_h$ as*

$$S_h(x) = \sum_{k=0}^{[\alpha]+1} \sigma_{h,k}(x) \left(\sqrt{h(x)}d(x)\right)^{k-\alpha}, \tag{6.19}$$

*it results $u - S_h \in W^{1,\infty}(\Omega)$ and $z(\bar{x}) := u(\bar{x}) - S_h(\bar{x}) = 0$ for every $\bar{x} \in \partial\Omega$. If moreover we assume*

$$\sup_{0<s<1} |r(x, s^{-\alpha})| = o(1) \quad \text{as } d(x) \to 0, \tag{6.20}$$

*it holds true that*

$$z(\bar{x}) \quad \text{and} \quad \frac{\partial z}{\partial \nu}(\bar{x}) = 0 \quad \forall \bar{x} \in \Omega.$$

Let us stress that the functions $\sigma_{h,k}$ do not depend on the function $r$, due to assumptions (6.17). Indeed these growth conditions imply that the contribution of the perturbation $r(x,s)$ does not affect the asymptotic behaviour prescribed by $h(x)|s|^{p-1}s$.
On the other hand the presence of the weight $h$ requires some modifications in the definition of the corrector function $S_h$. This in turn yield to even more involved formulas for $\sigma_{h,0},\cdots,\sigma_{h,[\alpha]+1}$ than (6.3). Notice that

$$\begin{aligned}\sigma_{h,0} =& [\alpha(\alpha+1)]^{\frac{1}{p-1}} \equiv \sigma_0,\\ \sigma_{h,1}(x) =& \alpha\sigma_0 h^{-1}(x)\frac{\alpha h^{-\frac{1}{2}}(x)\nabla h(x)\nabla d_n(x) + h^{\frac{1}{2}}(x)(N-1)H(x)}{2(1+2\alpha)},\end{aligned}$$

namely the first order behaviour does not see the influence of the weight, that comes into play from the second one onward. As a last comment to Theorem 6.5, notice that, in order to recover the Neumann boundary conditions for $z$, the additional growth assumption (6.20) is needed (see Remark 6.7).

Unfortunately we are not able to treat problem (29) with $g$ that satisfies just (30) and (31). The main obstacles in considering a general $g(s)$ (that satisfies anyway (30) and (31)) are, on one side, that the simple structure of the corrector function $S$ is lost and, on the other, that it becomes much harder to manipulate terms as $g(z+S)-g(S)$.

## 6.3 Gradient bound

### 6.3.1 The choice of $S_n$.

In this first section we determine the regularity of the functions $\sigma_k$, $k = 0, \cdots, [\alpha]+1$, defined in (6.3) and we show that $\widetilde{f}_n$, defined in (6.11), is such that

$$\exists\, \bar{C} = \bar{C}(\sigma_0, \cdots, \sigma_{[\alpha]+1}, f) \quad : \quad d_n|\widetilde{f}_n| + d_n^2|\nabla \widetilde{f}_n| \leq \bar{C} d_n^{1+[\alpha]-\alpha}.$$

We prove a slightly more general result that emphasizes the relationship between the number of elements of $S_n$ and the required regularity of $\Omega$.

**Proposition 6.6.** *Let us take a natural number $M \in [0, [\alpha]+1]$, $\Omega$ a bounded open domain of class $C^{M+4}$, $\sigma_k$ as in (6.3) with $k = 0, \cdots, M$ and $S_n$ as in (6.8). Then we have that $\sigma_k \in C(\bar{\Omega})^{M+3-k}$ with $k = 0, \cdots, M$ and that there exists $n_0 = n_0(\sigma_0, \cdots, \sigma_M)$ such that for every $n > n_0$*

$$|(\Delta S_n - |S_n|^{p-1} S_n) d_n| + |\nabla(\Delta S_n - |S_n|^{p-1} S_n) d_n^2| \leq C d_n^{M-\alpha} \quad \text{in } \Omega, \tag{6.21}$$

*where $C = C(N, \alpha, \partial\Omega)$.*

*Proof.* Note at first that the positive root of $(k-\alpha)(k-\alpha-1) - (2+\alpha)(\alpha+1) = 0$ (seen as an equation in the variable $k$) is bigger then $[\alpha]+1$: indeed denoting by $k_i$ $i = 1, 2$ the two roots, we have that

$$k_1 < 0 < k_2 = \frac{2\alpha + 1 + \sqrt{(2\alpha+1)^2 + 2(\alpha+1)}}{2} \qquad \text{and} \qquad k_2 > 2\alpha + 1 > [\alpha] + 1 \quad \alpha > 0.$$

Thus the denominator in (6.3) is always different from zero. As far as the regularity of the terms involved in (6.21) is concerned, Theorem 6.1 assures us that $d_n \in C^{M+4}(\Omega)$ (see also [59] and [64]); moreover, as it is clear from the formulas in (6.3), the evaluation of $\sigma_k$ involves derivatives of $d_n$ of order $k+1$. Hence the regularity of $\sigma_k$ is $M + 4 - (k+1) = M + 3 - k$, i.e. $\sigma_k \in C^{M+3-k}(\bar{\Omega})$ with $k = 1, \cdots, M$.
Let us show now that such a choice of $\sigma_k$ implies that (6.21) holds true. Thanks to the proved regularity property, we are allowed to compute both the gradient and the lApllacian of $S_n(x)$. Recalling that by definition $\nabla d_n(x) = \nabla d(x)$ and $\Delta d_n(x) = \Delta d(x)$, we have that

$$\Delta S_n(x) = \sum_{k=0}^{M} [(k-\alpha)(k-\alpha-1)\sigma_k d_n^{k-\alpha-2}(x)|\nabla d(x)|^2 + (k-\alpha)\sigma_k(x) d_n^{k-\alpha-1}(x)\Delta d(x)$$
$$+ 2(k-\alpha) d_n^{k-\alpha-1}(x)\nabla\sigma_k(x)\nabla d(x) + \Delta\sigma_k(x) d_n^{k-\alpha}(x)].$$

Ordering the previous expression according to the power of the distance function and working in $\Omega_{\bar{\delta}}$, in order to use that $|\nabla d|^2 = 1$, we obtain

$$\Delta S_n(x) = \alpha(\alpha+1)\sigma_0 d_n^{-\alpha-2}(x) + \big[\alpha(\alpha-1)\sigma_1(x) - \alpha\sigma_0\Delta d(x)\big] d_n^{-\alpha-1}(x)$$
$$+ \sum_{k=2}^{M} \Big\{ (k-\alpha)(k-\alpha-1)\sigma_k(x) + (k-\alpha-1)\big[\sigma_{k-1}(x)\Delta d(x) + 2\nabla\sigma_{k-1}(x)\nabla d(x)\big] + \Delta\sigma_{k-2}(x) \Big\} d_n^{k-\alpha-2}(x)$$
$$+ r(x) d_n^{M-\alpha-1}(x) \quad \text{in } \Omega_{\bar{\delta}},$$

where $r = r(\sigma_{M-1}, \sigma_M) \in C^1(\bar{\Omega})$.

Now let us focus on the non linear term $|S_n|^{p-1}S_n$. Since any $\sigma_k$ is bounded, there exists $\delta_0 = \delta_0(\sigma_0, \cdots, \sigma_M)$, with $\delta_0 < \bar{\delta}$, $n_0 = n_0(\delta_0)$ and a function $R = R(\sigma_0, \cdots, \sigma_M) \in C^1(\bar{\Omega})$ such that

$$|S_n|^{p-1}S_n = \sigma_0^p d_n^{-\alpha-2} \left( \sum_{k=0}^{M} \frac{\sigma_k}{\sigma_0} d_n^k \right)^p = \sigma_0^p d_n^{-\alpha-2} + p\sigma_0^{p-1}\sigma_1 d_n^{-\alpha-1}$$
$$+ \sum_{k=2}^{M} d_n^{k-\alpha-2} \left\{ p\sigma_0^{p-1}\sigma_k + \sigma_0^p \sum_{j=2}^{k} \left[ \binom{p}{j} \sigma_0^{-j} \sum_{i_1+\cdots+i_j=k} \sigma_{i_1}\cdots\sigma_{i_j} \right] \right\} + R(x) d_n^{M-\alpha-1}$$
$$\text{in } \Omega_{\delta_0} \text{ and } n \geq n_0.$$

Now it becomes clear that the choice of $\sigma_0, \cdots, \sigma_M$ in (6.3) is the unique for which

$$\big|\Delta S_n(x) - |S_n(x)|^{p-1}S_n(x)\big| d_n(x) = \big|r(x) - R(x)\big| d_n^{M-\alpha}(x) \leq C d_n^{M-\alpha}(x) \quad \text{in } \Omega_{\delta_0} \text{ and } n \geq n_0\,,$$

and moreover

$$\big|\nabla(\Delta S_n - |S_n|^{p-1}S_n) d_n^2(x)\big| \leq (\alpha + 1 - M)\big|\nabla(r(x) - R(x))\big| d_n^{M-\alpha}(x) \leq C d_n^{M-\alpha}(x)$$
$$\text{in } \Omega_{\delta_0} \text{ and } n \geq n_0,$$

with $C = C(\sigma_0, \cdots, \sigma_M)$. The estimate in $\Omega \setminus \Omega_{\delta_0}$ it is straightforward thanks to the regularity of $\sigma_k$. □

**Remark 6.7.** *For the proof of Theorem 6.3 we take $M = [\alpha] + 1$, so that (6.21) becomes*

$$|(\Delta S_n - |S_n|^{p-1}S_n) d_n| + |\nabla(\Delta S_n - |S_n|^{p-1}S_n) d_n^2| \leq C d_n^{1+[\alpha]-\alpha} \quad \textit{in } \Omega$$

*Since $f \in W^{1,\infty}(\Omega)$, recalling the definition (6.11) of $\widetilde{f}_n$, it follows*

$$\exists \bar{C} = \bar{C}(\sigma_0, \ldots, \sigma_M, f) \quad \textit{such that} \qquad d_n|\widetilde{f}_n| + d_n^2|\nabla \widetilde{f}_n| \leq \bar{C} d_n^{1+[\alpha]-\alpha} \leq \widetilde{C}\,. \tag{6.22}$$

**Remark 6.8.** *For the sake of completenesses we explicitly compute the expression for $\sigma_2$:*

$$\sigma_2(x) = \frac{(\alpha+2)\sigma_0^{p-2}\sigma_1^2(x)d(x) + (\alpha-1)[\sigma_1(x)\, d(x)\Delta d(x) + \nabla\sigma_1(x)\nabla d(x)]}{(2-\alpha)(1-\alpha) - (2+\alpha)(\alpha+1)}.$$

*Of course $\sigma_0$ and $\sigma_1(x)$ coincide with the ones already known in literature.*

### 6.3.2 Existence and preliminary estimates for $u_n$

In this section we find suitable sub- and super-solutions for problem (6.9) in order to prove both existence and some preliminary estimates on the solutions $u_n$ of (6.9).
We first observe that

$$\frac{\partial S_n}{\partial \nu}\bigg|_{\partial\Omega} = \alpha\sigma_0 n^{\alpha+1} + n^{\alpha} \sum_{k=1}^{M} \left[(\alpha - k)\sigma_k n^{1-k} + \nabla\sigma_k \cdot \nu n^{-k}\right] = \alpha\sigma_0 n^{\alpha+1} + n^{\alpha}\psi_n \quad \text{if } \alpha \neq 1$$

while

$$\frac{\partial S_n}{\partial \nu}\bigg|_{\partial\Omega} = \sigma_0 n^2 + \nabla\sigma_1 \cdot \nu + \nabla\sigma_2 \cdot \nu \frac{1}{n} - \sigma_2 = \sigma_0 n^2 + \psi_n \quad \text{if } \alpha = 1,$$

where $\psi_n \in C(\partial\Omega)$ is uniformly bounded (with respect to $n$). More precisely

$$\exists\, T = T(N, \alpha, \partial\Omega)\ : \qquad \|\psi_n\|_{L^\infty(\partial\Omega)} \leq T \qquad \forall n \in \mathbb{N}. \tag{6.23}$$

Such a bound is crucial in order to prove that problem (6.9) admits a pair of sub- and super-solutions.

**Proposition 6.9.** *Let $p > 1$, $f \in W^{1,\infty}(\Omega)$, $S_n$ as (6.8) and $T$ as in (6.23). Hence problem (6.9) admits a pair of (classical) sub- and super-solutions.*

*Proof. Case $\alpha > 1$, sub-solution.* We prove that it is possible to chose $M_1$ and $M_2$ positive constants such that $w_n := \sigma_0 d_n^{-\alpha} - M_1 d_n^{1-\alpha} - M_2$ is a sub-solution for (6.9). Fix at first

$$M_1 \geq \max\left\{\frac{(p-1)\alpha\sigma_0||\Delta d||_{L^\infty(\Omega)}}{(p+3)}, \frac{T}{\alpha-1}\right\}$$

and observe that this choice of $M_1$ implies that

$$\frac{\partial w_n}{\partial \nu} - \alpha\sigma_0 n^{\alpha+1} - \psi_n n^\alpha = [-(\alpha-1)M_1 - \psi_n]n^\alpha < 0 \quad \text{on } \partial\Omega. \tag{6.24}$$

Moreover, using the monotonicity of the function $s \to |s|^{p-1}s$, let us fix $\delta_0 = \delta_0(M_1) < \bar{\delta}$ and $n_0 = n_0(\delta_0)$ so that

$$\begin{aligned}|w_n|^{p-1}w_n &\leq |\sigma_0 d_n^{-\alpha} - M_1 d_n^{1-\alpha}|^{p-1}(\sigma_0 d_n^{-\alpha} - M_1 d_n^{1-\alpha}) = (\sigma_0 d_n^{-\alpha} - M_1 d_n^{1-\alpha})^p \\ &= \sigma_0^p d_n^{-\alpha-2}\left(1 - \frac{M_1}{\sigma_0}d_n\right)^p = \sigma_0^p d_n^{-\alpha-2}\left[1 - p\frac{M_1}{\sigma_0}d_n + O(d_n^2)\right] \quad \text{in } \Omega_{\delta_0},\ n > n_0.\end{aligned}$$

On the other hand an easy computation shows that

$$\Delta w_n = \alpha(\alpha+1)\sigma_0 d_n^{-\alpha-2} - \alpha d_n^{-\alpha-1}\Big[\sigma_0\Delta d + (\alpha-1)M_1\Big] + (\alpha-1)M_1 d_n^{-\alpha}\Delta d \quad \text{in } \Omega_{\delta_0},\ n > n_0.$$

Recalling that

$$p\sigma_0^{p-1} - \alpha(\alpha-1) = p\alpha^2 + p\alpha - \alpha^2 + \alpha = 2\frac{p+3}{p-1},$$

we deduce that

$$\begin{aligned}-\Delta w_n + |w_n|^{p-1}w_n - f &\leq -\Delta w_n + (\sigma_0 d_n^{-\alpha} - M_1 d_n^{1-\alpha})^p - f \\ &\leq \left(-2\frac{p+3}{p-1}M_1 + \alpha\sigma_0\Delta d\right)d_n^{-\alpha-1} + O(d_n^{-\alpha}) \leq 0 \quad \text{in } \Omega_{\delta_0},\ n > n_0,\end{aligned}$$

where the last inequality holds true thanks to the choice of $M_1$. Now taking

$$M_2 \geq \sigma_0\delta_0^{-\alpha} + \left(\|\Delta w_n\|_{L^\infty(\Omega\setminus\Omega_{\delta_0})} + \|f\|_{L^\infty(\Omega)}\right)^{\frac{1}{p}}$$

and using once more the monotonicity of $s \to |s|^{p-1}s$, it follows that

$$-\Delta w_n + |w_n|^{p-1}w_n - f \leq -\Delta w_n - \|\Delta w_n\|_{L^\infty(\Omega\setminus\Omega_{\delta_0})} - \|f\|_{L^\infty(\Omega)} - f \leq 0 \quad \text{in } \Omega\setminus\Omega_{\delta_0}. \tag{6.25}$$

and we conclude that $w_n$ is a sub-solution of problem (6.9).

*Case $\alpha > 1$, super-solution.* Let us show now that it is possible to take $N_1 \geq M_1$ such that $v_n := \sigma_0 d_n^{-\alpha} + N_1 d_n^{1-\alpha}$ turns out to be a super-solution for (6.9). As far as the boundary condition is concerned we have

$$\left.\frac{\partial v_n}{\partial \nu}\right|_{\partial\Omega} - \sigma_0 n^{\alpha+1} - \psi_n n^\alpha = [(\alpha-1)N_1 - \psi_n]n^\alpha > 0 \quad \text{on } \partial\Omega,$$

where the last inequality follows from the previous choice of $N_1$.
Since $v_n$ is positive, thanks to the convexity of the function $(1+s)^p$ with $p > 1$, we have that

$$|v_n|^{p-1}v_n = \sigma_0^p d_n^{-\alpha-2}\left(1 + \frac{N_1}{\sigma_0}d_n\right)^p \geq \sigma_0^p d_n^{-\alpha-2}\left(1 + p\frac{N_1}{\sigma_0}d_n\right).$$

As in the previous case it follows that

$$-\Delta v_n + v_n^p - f \geq \left(2\frac{p+3}{p-1}N_1 + \alpha\sigma_0\Delta d_n\right) d_n^{-\alpha-1} + O(d_n^{-\alpha}) \quad \text{in } \Omega_{\bar\delta},\ n > n_0,$$

where we have used that $|\nabla d|^2 = 1$ in $\Omega_{\bar\delta}$. Thanks, once again, to the choice of $N_1$ we can conclude that $-\Delta v_n + v_n^p \geq f$ in $\Omega_{\bar\delta}$.
Finally we have

$$\begin{aligned} -\Delta v_n + v_n^p - f &= -\Delta v_n + (\sigma_0 d_n^{-\alpha} + N_1 d_n^{1-\alpha})^p - f \\ &\geq -\Delta v_n + N_1^p d_n^{p(1-\alpha)} - f \geq C_1 N_1^p - C_2 N_2 - C_3 \quad \text{in } \Omega \setminus \Omega_{\bar\delta}, \end{aligned}$$

where the last inequality comes from the fact that in $\Omega \setminus \Omega_{\bar\delta}$ $d_n \geq \bar\delta$ and that in $-\Delta v_n$ only linear powers of $N_1$ appears. So by increasing if necessary the value of $N_1$, we have $-\Delta v_n + v_n^p \geq f$ in $\Omega \setminus \Omega_{\bar\delta}$. It is then possible to conclude that $v_n$ is a super-solution of (6.9) in $\Omega$ and that $v_n \geq w_n$.

For the range $0 < \alpha \leq 1$ the proof is similar and we just stress the main differences.

*Case* $\alpha = 1$. Note that, with this choice of $\alpha$, we have $p = 3$, $\sigma_0 = \sqrt{2}$. We claim that $w_n := \sqrt{2}d_n^{-1} + M_3 \log d_n - M_4$ and $v_n := \sqrt{2}d_n^{-1} - N_3 \log d_n + N_4$, with $M_3, M_4, N_3, N_4 > 0$, are a sub- and a super-solution for (6.9). Let us choose $M_3 \geq T$ in order to obtain

$$\left.\frac{\partial w_n}{\partial \nu}\right|_{\partial\Omega} - \sqrt{2}n^2 - \psi_n n = [-M_3 - \psi_n]n < 0.$$

Then we fix $\delta_0 = \delta_0(M_3) < \min\{\bar\delta, 1\}$ (so that $\log(\delta_0) < 0$) and $n_0 = n_0(\delta_0)$ such that

$$w_n^3 \leq 2^{\frac{3}{2}} d_n^{-3}\left(1 + \frac{M_3}{\sqrt{2}} d_n \log(d_n)\right)^3 = 2^{\frac{3}{2}} d_n^{-3} + 6M_3 d_n^{-2}\log(d_n) + o(d_n^2 \log(d_n)) \quad \text{in } \Omega_{\delta_0},\ n > n_0,$$

Hence it follows that

$$-\Delta w_n + |w_n|^{p-1} w_n - f \leq +6M_3 d_n^{-2} \log d_n + o(d_n^2 \log(d_n)) \leq 0 \quad \text{in } \Omega_{\delta_0},\ n > n_0.$$

Now we fix

$$M_4 \geq \sqrt{2}\delta_0^{-1} + \left(\|\Delta w_n\|_{L^\infty(\Omega\setminus\Omega_{\delta_0})} + \|f\|_{L^\infty(\Omega)}\right)^{\frac{1}{p}},$$

that implies $-\Delta w_n + |w_n|^{p-1} w_n - f \leq 0$ in $\Omega \setminus \Omega_{\delta_0}$ and we in turn that $w_n$ is a sub-solution of problem (6.9).

For the super-solution $v_n$, we consider $N_3 \geq T$. Thus, exactly as in the previous case, we get

$$\left.\frac{\partial v_n}{\partial \nu}\right|_{\partial\Omega} - \sigma_0 n^2 - \psi_n n = [N_3 - \psi_n]n > 0.$$

Noticing that $v_n$ is positive and using the convexity of the function $(1+s)^3$, we obtain

$$v_n^3 = 2^{\frac{3}{2}} d_n^{-3}\left(1 - \frac{N_3}{\sqrt{2}} d_n \log(d_n) + N_4\right)^3 \geq 2^{\frac{3}{2}} d_n^{-3} - 6N_3 d_n^{-2} \log(d_n).$$

Moreover we fix $\delta_0 \leq \bar\delta$ and $n_0 = n_0(\delta_0)$ so that

$$-\Delta v_n + v_n^p - f \geq -6N_3 d_n^{-2} \log d_n + o(d_n^2 \log(d_n)) \geq 0 \quad \text{in } \Omega_{\delta_0},\ n > n_0.$$

At this point, choosing

$$N_4 \geq \sqrt{2}\delta_0^{-1} + \left(\|\Delta v_n\|_{L^\infty(\Omega\setminus\Omega_{\delta_0})} + \|f\|_{L^\infty(\Omega)}\right)^{\frac{1}{p}}$$

it follows that $-\Delta v_n + v_n^p - f \geq 0$ in $\Omega \setminus \Omega_{\delta_0}$ and we conclude.

*Case* $\alpha < 1$. Finally we consider $w_n = \sigma_0 d_n^{-\alpha} - M_5 + M_6 d_n^{-\alpha+1}$ and $v_n = \sigma_0 d_n^{-\alpha} + N_5 - N_6 d_n^{-\alpha+1}$ with $M_5, M_6, N_5, N_6 > 0$. Let us fix $M_6 \geq \frac{T}{1-\alpha}$, in order to have

$$\left.\frac{\partial w_n}{\partial \nu}\right|_{\partial\Omega} - \alpha\sigma_0 n^{\alpha+1} - \psi_n n^\alpha < 0.$$

Moreover, taking $M_5 > 0$ it is possible to select $\delta_0 \leq \bar{\delta}$ and $n_0 = n_0(\delta_0)$ such that

$$|w_n|^{p-1}w_n = \sigma_0^p d_n^{-\alpha-2} - p\sigma_0^{p-1}M_5 d_n^2 + o(d_n^2)$$

and that

$$-\Delta w_n + |w_n|^{p-1}w_n - f \leq -p\sigma_0^{p-1}M_5 d_n^{-2} + o(d_n^{-2}) \leq 0 \quad \text{in } \Omega_{\delta_0},\ n > n_0.$$

Finally increasing the value of $M_5$ so that

$$M_5 \geq \sigma_0\delta_0 + M_6\delta_0^{-\alpha+1} + \left(\|\Delta w_n\|_{L^\infty(\Omega\setminus\Omega_{\delta_0})} + \|f\|_{L^\infty(\Omega)}\right)^{\frac{1}{p}},$$

it follows

$$-\Delta w_n + |w_n|^{p-1}w_n - f \leq 0 \quad \text{in } \Omega \setminus \Omega_{\delta_0}$$

As far as $v_n$ is concerned, let us fix as before $N_6 \geq \frac{\|\psi_n\|_\infty}{1-\alpha}$ in order to get

$$\left.\frac{\partial v_n}{\partial \nu}\right|_{\partial\Omega} - \alpha\sigma_0 n^{\alpha+1} - \psi_n n^\alpha > 0.$$

Take now $N_5 > 0$ and fix $\delta_0 < \bar{\delta}$ and $n_0 = n_0(\delta_0)$ such that

$$|v_n|^{p-1}v_n = \sigma_0^p d_n^{-\alpha-2}\left(1 + \frac{N_5}{\sigma_0}d_n^\alpha - \frac{N_6}{\sigma_0}d_n\right)^p \geq \sigma_0^p d_n^{-\alpha-2} + p\sigma_0^{p-1}N_5 d_n^{-2} + o(d_n^{-2}) \quad \text{in } \Omega_{\delta_0},\ n > n_0.$$

and that

$$-\Delta v_n + |v_n|^{p-1}v_n - f \geq p\sigma_0^{p-1}N_5 d_n^{-2} + o(d_n^{-2}) \geq 0 \quad \text{in } \Omega_{\delta_0},\ n > n_0.$$

As in the previous cases, by increasing if necessary the value of $N_5$, we have $-\Delta v_n + |v_n|^{p-1}v_n - f \geq 0$ in $\Omega \setminus \Omega_{\delta_0}$ and that $v_n$ is a super-solution of (6.9). □

The ordered sub- and super-solutions obtained in the previous proposition allow us to prove existence of a solution for problem (6.9) and, on the other hand, give us a control on the behaviour of $u_n$ (and in turn of $z_n$) near $\partial\Omega$, which is crucial in order to prove the results of the next section.

**Theorem 6.10.** *Let $p > 1$, $f \in W^{1,\infty}(\Omega)$, $S_n$ as in (6.8). Then problem (6.9) has a unique classical solution $u_n$ for every $n \in \mathbb{N}$. Moreover*

$$\exists C = C(\alpha, N, \partial\Omega, f) \quad : \quad \left|\frac{z_n(x)}{S_n(x)}\right| = \left|\frac{u_n(x)}{S_n(x)} - 1\right| \leq C\varepsilon(d_n(x)) \tag{6.26}$$

*where*

$$\varepsilon(s) = \begin{cases} s & \text{if } \alpha > 1 \\ s(1 + |\log s|) & \text{if } \alpha = 1 \\ s^\alpha & \text{if } \alpha < 1\,. \end{cases} \tag{6.27}$$

*Proof.* The proof of the existence and uniqueness is standard and we sketch it here for the convenience of the reeder. In Proposition 6.9, for every $\alpha > 0$ we have constructed a pair of ordered sub- and super-solutions for problem (6.9)

$$w_n \le v_n \quad \text{in } \Omega.$$

Let us set $v_n^0 = v_n$, $C := \max\{||v_n||_{L^\infty(\Omega)}, ||w_n||_{L^\infty(\Omega)}\}$, $m > pC^{p-1}$ and let us define $v_n^i$ for $i = 1, 2, ...$ as the solutions of

$$\begin{cases} -\Delta v_n^i + m v_n^i = m v_n^{i-1} - |v_n^{i-1}|^{p-1} v_n^{i-1} + f, & \text{in } \Omega, \\ \frac{\partial v_n^i}{\partial \nu} = \frac{\partial S_n}{\partial \nu}, & \text{on } \partial\Omega. \end{cases}$$

The choice of $m$ allows us to say that the function $s \to ms - |s|^{p-1}s$ is increasing in $[-C, C]$ so that we can apply the standard procedure of the sub- and super-solution method for existence of solutions (see for instance [56]). We claim that $v_n^{i-1} \ge v_n^i$ for every $i = 1, 2, ....$ Indeed for $i = 1$ we have that the function $w := v_n^1 - v_n^0$ satisfies

$$\begin{cases} -\Delta w + mw \le 0, & \text{in } \Omega, \\ \frac{\partial w}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases}$$

Hopf's Lemma and the Strong Maximum Principle assure us that $w \le 0$, which implies $v_0 \ge v_1$ and we can conclude the proof of the claim by induction. Similarly we can prove that $w_n \le v_n^i$ for every $i = 1, 2, \cdots$. Then we have that $v_n^i \searrow u_n$ a.e in $\Omega$ as $i \to \infty$ and that

$$w_n \le u_n \le v_n;$$

moreover by compactness and regularity arguments (see respectively [2] and [1]) it is possible to prove that $u_n \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solves of (6.9). The uniqueness follows by Theorem 3.6 of [59].
In order to prove (6.27) we first consider the case $\alpha > 1$. We have that

$$\sigma_0 d_n^{-\alpha} - M_1 d_n^{1-\alpha} - M_2 \le u_n \le \sigma_0 d_n^{-\alpha} + N_1 d_n^{1-\alpha},$$

where $M_1$, $M_2$ and $N_1$ are the constant given by Proposition 6.9. Subtracting $S_n$ we get

$$-(M_1 - \sigma_1) d_n^{1-\alpha} + O(d_n^{2-\alpha}) \le u_n - S_n \le (N_1 - \sigma_1) d_n^{1-\alpha} + o(d_n^{1-\alpha}) + O(d_n^{2-\alpha})$$

with $b$ and $B$ bounded functions. Thanks to the choice of $M_1$ and $N_1$ it follows that there exists a positive constant constant $C = C(\alpha, N, \partial\Omega, f)$ such that

$$\left| \frac{u_n(x)}{S_n(x)} - 1 \right| \le C d_n(x) \quad \text{in } \Omega.$$

The case $\alpha \le 1$ follows similarly using the respective sub- and super-solutions and for brevity we omit the proof. □

We close this section proving the following Proposition.

**Proposition 6.11.** *The sequence $u_n$ of solutions of problem (6.9) converges in $C^2_{loc}(\Omega)$ to the solution of problem (38).*

*Proof.* Let us define $\psi_n := u_n - u_{n+1}$, which, for $n$ large enough, satisfies

$$\begin{cases} -\Delta \psi_n + |u_n|^{p-1} u_n - |u_{n+1}|^{p-1} u_{n+1} = 0, & \text{in } \Omega, \\ \frac{\partial \psi_n}{\partial \nu} < 0, & \text{on } \partial\Omega. \end{cases}$$

The Neumann boundary condition assures us that the maximum of $\psi_n$ cannot be reached on $\partial\Omega$. So let it be $x_0 \in \Omega$ the maximum point for $\psi_n$. This implies that $-\Delta \psi_n(x_0) \ge 0$ and then we obtain from the equation the following information

$$|u_n(x_0)|^{p-1} u_n(x_0) - |u_{n+1}(x_0)|^{p-1} u_{n+1}(x_0) \le 0.$$

But, since $s \to |s|^{p-1}s$ is increasing, it has to be $\psi_n(x_0) = u_n(x_0) - u_{n+1}(x_0) \le 0$. Being $x_0$ the maximum point, it follows that $u_n \le u_{n+1}$ in $\Omega$. So we know that the sequence $u_n$ is increasing and converges pointwise to some function $u$. Moreover we know, thanks to Theorem 6.10, that each $u_n$ is between the relative sub- and super-solutions $w_n \le u_n \le v_n$. Thus we have that for any $\omega$ compactly contained in $\Omega$ there exists $c = c(\omega, \alpha, N)$ such that

$$u_n(x) \le u_{n+1}(x) \le \cdots \le u(x) \le v(x) \le c \quad \forall\, x \in \omega,$$

where $v$ is the limit as $n$ diverges of the super-solutions $v_n$. On the other hand we also have that

$$w(x) \le u(x) \quad \forall\, x \in \Omega,$$

where $w$ is the limit of the sub-solutions. Thus using standard compactness and interior elliptic regularity arguments, we have that for every $\omega \subset\subset \Omega$ $u_n \to u$ in $C^2(\omega)$. Thus we can pass to the limit with respect to $n$ in (6.9) and moreover we also obtain

$$\lim_{x \to \partial\Omega} u(x) = \infty.$$

□

### 6.3.3 Estimates of $z_n$ and $|\nabla z_n|$ in $L^\infty(\Omega)$.

Now we are ready to prove the uniform estimate in $W^{1,\infty}(\Omega)$ for $z_n := u_n - S_n$, where $u_n$ are the solutions of problem (6.9) and $S_n$ has been defined in (6.8). Note that thanks to Proposition 6.11 we already know that for every $\omega$ compactly contained in $\Omega$ we have that

$$\forall \omega \subset\subset \Omega \quad \exists c_\omega : \qquad \|z_n\|_{W^{1,\infty}(\omega)} \le \|u_n\|_{W^{1,\infty}(\omega)} + \|S_n\|_{W^{1,\infty}(\omega)} \le c_\omega.$$

Thus the main concern here is to obtain a Lipschitz control in $\Omega_{\delta_0}$ for some $\delta_0 > 0$ small enough.

Let us start with the bound in $L^\infty(\Omega_{\delta_0})$.

**Theorem 6.12.** *Let $z_n$ be as above. Then there exist $1 < \beta < 2 + [\alpha] - \alpha \le 2$ and $A_1, A_2 > 0$ such that*

$$-A_1 d_n^\beta(x) \le z_n(x) \le A_2 d_n^\beta(x) \quad \forall\, x \in \Omega. \tag{6.28}$$

*Proof.* We build barriers in a neighborhood of $\partial\Omega$ through sub- and super-solutions method. In order to do it, let us fix $1 < \beta < 2 + [\alpha] - \alpha \le 2$ and $\epsilon > 0$ such that

$$\gamma_1 := -\beta^2 + \beta + (1+\epsilon)\frac{\sqrt{2}}{2} p\sigma_0^{p-1} > 0 \quad \text{and} \quad \gamma_2 := -\beta^2 + \beta + (1-\epsilon)\frac{\sqrt{2}}{2} p\sigma_0^{p-1} > 0.$$

This is always possible since $p\sigma_0^{p-1} = \frac{2p(p+1)}{(p-1)^2} > 2$ and the function $s \to -s^2 + s + \sqrt{2}$ admits $\frac{1+\sqrt{1+4\sqrt{2}}}{2} > 1$ as positive root. Moreover let us introduce the function $\tilde{d}_n(x) = \left(d(x)^{\frac{1}{2}} + \frac{1}{n^{\frac{1}{2}}}\right)^{\frac{1}{2}}$ and notice that

$$\frac{\sqrt{2}}{2} d_n \le \tilde{d}_n \le d_n. \tag{6.29}$$

The reason of considering this further regularization of the distance function is that $\frac{\partial \tilde{d}_n}{\partial \nu} = 0$ on $\partial\Omega$.

*Bound from above.* We claim that there exists $A_1 > 0$ such that $v_n(x) = A_1 \tilde{d}_n^\beta$ is a supersolution of (6.10). Simple computations show that

$$\nabla v_n = A_1 \beta d \left(d^2 + \frac{1}{n^2}\right)^{\frac{\beta}{2}-1} d\nabla d$$

and

$$\Delta v_n = A_1\beta\left(d^2+\frac{1}{n^2}\right)^{\frac{\beta}{2}-1}\left[d^2\left(\beta-2\right)\left(d^2+\frac{1}{n^2}\right)^{-1}+1+d\Delta d\right].$$

It is immediate to check that $v_n$ satisfies the Neumann boundary conditions. Moreover, by definition of $S_n$, there exist $\delta_0=\delta_0(\sigma_0,\cdots,\sigma_{[\alpha]+1})$ and $n_0=n_0(\delta_0)$ such that $S_n\geq 0$ and $pS_n^{p-1}\geq(1-\epsilon)p\sigma_0^{p-1}d_n^{-2}$ in $\Omega_{\delta_0}$ and $n>n_0$. Using (6.29) and the convexity of the function $(1+s)^p-1$ near zero, we obtain that

$$-\Delta v_n+S_n^p\left[\left(1+\frac{v_n}{S_n}\right)^p-1\right]-|\tilde f|\geq -A_1\beta\tilde d_n^{\beta-2}\left[\beta-1+d\|\Delta d\|_{L^\infty(\Omega)}\right]$$

$$+\frac{\sqrt{2}}{2}(1-\epsilon)pS_n^{p-1}\tilde d_n^{\beta-2}-|\tilde f|\geq A_1\tilde d_n^{\beta-2}[\gamma_1-\beta\delta_0\|\Delta d\|_{L^\infty(\Omega)}]-|\tilde f|\geq 0,$$

where the last inequality follows by a further decreasing of $\delta_0$ and (6.22). Since $z_n$ and $S_n$ are uniformly bounded in $\Omega\setminus\Omega_{\delta_0}$, we choose $A_1$ large enough to conclude that $v_n$ is a super solution of (6.10).

*Bound from below.* We want to prove that there exists $A_2>0$ such that $w_n=-A_2\tilde d_n^\beta$ is a sub solution of (6.10), where $\beta$ is the same of above. In this case we cannot take advantage of the convexity of the function $(1+s)^p-1$, but is always possible to conclude that there exist $\delta_0=\delta(\sigma_0,\cdots,\sigma_{[\alpha]+1},B)$, $n_0=n_0(\delta_0)$ such that

$$|S_n|^{p-1}S_n\left[\left|1+\frac{w_n}{S_n}\right|^{p-1}\left(1-\frac{B}{S_n}\right)-1\right]=S_n^p(x)\left[\left(1+\frac{w_n}{S_n(x)}\right)^p-1\right]$$

$$=S_n^p(x)\left[-A_1\frac{p\tilde d_n^\beta}{S_n(x)}+O(d_n^{2\alpha}w_n^2)\right]\leq -A_2(1+\epsilon)p\sigma_0^{p-1}d_n^{-2}\quad\text{in }\Omega_{\delta_0}\text{ and }\ n>n_0.$$

Thus, arguing as in the previous part, it is possible to choose $A_2$ large enough to conclude that

$$-\Delta w_n+|S_n|^{p-1}S_n\left[\left|1+\frac{w_n}{S_n}\right|^{p-1}\left(1+\frac{w_n}{S_n}\right)-1\right]-|\tilde f_n|\leq 0\quad\text{in }\Omega\text{ for }\ n>n_0.$$

Since the Neumann boundary conditions are satisfied we conclude the proof of the Theorem. □

We can now state and prove our main result.

**Theorem 6.13.** *Let $z_n$ be the functions defined in* (6.10)*. Then there exists $\delta_0=\delta_0(\sigma_0,\cdots,\sigma_{[\alpha]+1})$, $C=C(\alpha,N,\partial\Omega,f,\delta_0)$ and $0<\eta<1$ such that*

$$|\nabla z_n|\leq Cd_n^\eta\quad\text{in }\Omega_{\delta_0}. \tag{6.30}$$

*Proof.* We divide the proof in two steps.
*Step 1.* Inequality satisfied by $|\nabla z_n|^2$.
*Step 2.* Application of maximum principle to $w_n:=|\nabla z_n|^2e^{\lambda d_n}$ in $\Omega_{\delta_0}$.

**Step 1.** Thanks to (6.26) there exist $\delta_0<\bar\delta$ and $n_0=n_0(\delta_0)$ such that

$$\begin{aligned}0<C_1\leq\left(1+\frac{z_n}{S_n}\right)^{p-1}&\leq C_2\\ \left|\left(1+\frac{z_n(x)}{S_n(x)}\right)^p-1\right|&\leq C_3\frac{|z_n(x)|}{S_n(x)}\end{aligned}\qquad\forall n>n_0\ \ \forall x\in\Omega_{\delta_0}, \tag{6.31}$$

where the positive constants $C_1,\ C_2$ and $C_3$ depend only on $\alpha,N,\partial\Omega$ and $f$. Moreover from the definition of $S_n$ and recalling that $|\nabla d_n|=1$ in $\Omega_{\delta_0}$, it follows that

$$|\nabla S_n\nabla z_n|\leq C_4\frac{|\nabla z_n|}{d_n^{\alpha+1}}. \tag{6.32}$$

with $C_4 = C_4(\alpha, N, \partial\Omega, f)$. All the computations performed from now on are meant on $\Omega_{\delta_0}$ and with $n > n_0$. At first let us recover the equation satisfied by $|\nabla z_n|^2$ (see [72] and reference therein). In order to do it, it is useful to recall that

$$\nabla(|\nabla z_n|^2) = 2D^2 z_n \nabla z_n \quad \text{and that} \quad \Delta(|\nabla z_n|^2) = 2\nabla(\Delta z_n)\nabla z_n + 2|D^2 z_n|^2.$$

Hence, through Schwarz inequality, we get

$$\Delta(|\nabla z_n|^2) \geq 2\nabla\left[(z_n + S_n)^p - S_n^p\right]\nabla z_n - 2\nabla\widetilde{f}_n \nabla z_n + \frac{2}{N}(\Delta z_n)^2 .$$

Now we consider separately each one of the terms on the right hand side above.
*First term.* We rewrite it as

$$\nabla\left[(z_n + S_n)^p - S_n^p\right]\nabla z_n = \nabla\left[S_n^p\left[\left(1+\frac{z_n}{S_n}\right)^p - 1\right]\right]\nabla z_n$$
$$= pS_n^{p-1}\left(1+\frac{z_n}{S_n}\right)^{p-1}|\nabla z_n|^2 + p\left[S_n\left[\left(1+\frac{z_n}{S_n}\right)^p - 1\right] - z_n\left(1+\frac{z_n}{S_n}\right)^{p-1}\right]S_n^{p-2}\nabla S_n \nabla z_n . \tag{6.33}$$

Note that in the right hand side above the first term is the coercive one, while the other has to be absorbed. Thanks to (6.31) the coercive term of (6.33) becomes

$$\exists\gamma > 0 \ : \qquad pS_n^{p-1}\left(1+\frac{z_n}{S_n}\right)^{p-1}|\nabla z_n|^2 \geq 3\gamma\frac{|\nabla z_n|^2}{d_n^2} .$$

Recalling (6.27), Theorem 6.12 and (6.31), the last term of (6.33) can be controlled as follows

$$p\left|S_n\left[\left(1+\frac{z_n}{S_n}\right)^p - 1\right] - z_n\left(1+\frac{z_n}{S_n}\right)^{p-1}\right||S_n|^{p-2}|\nabla S_n \nabla z_n|$$
$$\leq C|z_n||S_n|^{p-2}\frac{|\nabla z_n|}{d_n^{\alpha+1}} \leq C\frac{|\nabla z_n|}{d_n^{3-\beta}} \leq \frac{C_\gamma}{d_n^{2(2-\beta)}} + \gamma\frac{|\nabla z_n|^2}{d_n^2},$$

where we have used (6.31), (6.32), (6.28) and Young's inequality. Then we get

$$\nabla\left[(z_n + S_n)^p - S_n^p\right]\nabla z_n \geq 2\gamma\frac{|\nabla z_n|^2}{d_n^2} - \frac{C_\gamma}{d_n^{2(2-\beta)}}.$$

*Second term.* We apply Young's inequality and use (6.22) to obtain

$$-\nabla\widetilde{f}_n\nabla z_n \geq -\gamma\frac{|\nabla z_n|^2}{d_n^2} - C_\gamma|\nabla\widetilde{f}_n|^2 d_n^2 \geq -\gamma\frac{|\nabla z_n|^2}{d_n^2} - \frac{C}{d_n^{2(\alpha-[\alpha])}}.$$

*Third term.*It is positive and we can drop it.

Hence, gathering together the inequalities above, we have that

$$\Delta(|\nabla z_n|^2) \geq \gamma\frac{|\nabla z_n|^2}{d_n^2} - C_1\left[\frac{1}{d_n^{2(2-\beta)}} + \frac{1}{d_n^{2(\alpha-[\alpha])}}\right] \tag{6.34}$$
$$\geq \gamma\frac{|\nabla z_n|^2}{d_n^2} - \frac{C_1}{d_n^\theta} \quad \forall n > n_0 \ \ \forall x \in \Omega_{\delta_0},$$

with $\theta = \max\{2(2-\beta), 2(\alpha-[\alpha])\} < 2$ and for some constant $C_1 = C_1(\alpha, N, \partial\Omega, f)$.

**Step 2.** As in [70] let us consider now $w_n := |\nabla z_n|^2 e^{\lambda d_n}$ with $\lambda > 2\|\Delta d\|_{L^\infty(\Omega)}$. Its boundary behaviour is described in Lemma 2.4 of [70] (using in turn an idea of [73]). For the convenience of the reader we report here the computations. Notice at first that the boundary condition $\frac{\partial z_n}{\partial \nu} = \nabla z_n \cdot \nu = 0$ implies that there exists a function $\mu \in L^\infty(\partial\Omega)$ such that

$$\nabla(\nabla z_n \nabla d_n)|_{\partial\Omega} = \mu\nu.$$

To get convinced of this fact just observe that the regular function $\nabla z_n \cdot d_n$ takes the value 0 on $\Omega$ (recall that $\nu = -\nabla d_n$). Then its gradient evaluated on the boundary cannot have any tangential component, otherwise the condition for $\nabla v_n \cdot d_n$ would be violated. Hence we have

$$\mu\nu \cdot \nabla z_n = \nabla(\nabla z_n \nabla d_n)\nabla z_n = D^2 z_n \nabla z_n \nabla d_n + D^2 d_n \nabla z_n \nabla z_n \quad \text{on } \partial\Omega.$$

But the left hand side above is zero, so that

$$\frac{\partial |\nabla z_n|^2}{\partial \nu} = 2D^2 d_n \nabla z_n \cdot \nu \leq 2||D^2 d||_\infty |\nabla z_n|^2$$

and as a consequence

$$\begin{aligned} \frac{\partial w_n}{\partial \nu} = \nabla(|\nabla z_n|^2 e^{\lambda d_n}) \cdot \nu =& \lambda w_n \nabla d_n \cdot \nu + e^{\lambda d_n} \nabla(|\nabla z_n|^2) \cdot \nu \\ \leq& [-\lambda + 2\|\Delta d\|_{L^\infty(\Omega)}] w_n \quad \text{on } \partial\Omega. \end{aligned} \tag{6.35}$$

Hence we can take $\lambda$ large enough to have

$$\frac{\partial w_n}{\partial \nu} \leq 0 \quad \text{on } \partial\Omega. \tag{6.36}$$

Taking in to account (6.34), it follows that $w_n$ satisfies

$$\Delta w_n \geq (\lambda^2 + \lambda \Delta d_n) w_n + 2\lambda \nabla w_n \nabla d_n - 2\lambda^2 w_n + \gamma \frac{w_n}{d_n^2} - \frac{C_1}{d_n^\theta},$$

that is

$$-\Delta w_n + [\gamma - (\lambda^2 + \lambda ||\Delta d_n||_{L^\infty(\Omega)}) d_n^2] \frac{w_n}{d_n^2} + 2\lambda \nabla w_n \nabla d_n \leq \frac{C_1}{d_n^\theta}$$

Hence, up to a decrease of $\delta_0$ and an increase of $n_0$, we get

$$-\Delta w_n + 2\lambda \nabla w_n \nabla d_n + \frac{\gamma}{2} \frac{w_n}{d_n^2} \leq \frac{C_2}{d_n^\theta} \quad \text{in} \quad \Omega_{\delta_0} \quad \text{and} \quad n > n_0. \tag{6.37}$$

Let us consider now $v_n = A\tilde{d}_n^\eta = A\left(d^2 + \frac{1}{n^2}\right)^{\frac{\eta}{2}}$ with $0 < \eta < 1$ and $A > 0$. Easy computations show that

$$-\Delta v_n + 2\lambda \nabla v_n \nabla d_n + \frac{\gamma}{2} \frac{v_n}{d_n^2} - \frac{C_2}{d_n^\theta} \geq A\tilde{d}_n^{\eta - 2} \left[ \frac{\sqrt{2}\gamma}{4} - \eta - \eta d \Delta d - C_2 d_n^{2-\eta-\theta} \right].$$

Thus recalling that $\theta < 2$, up to a further decrease of $\delta_0$ and increase of $n_0$, there exist $0 < \eta < 1$ and $A > 0$ such that $v_n$ is a super solution of

$$\begin{cases} -\Delta\psi + 2\lambda \nabla\psi \nabla d_n + \frac{\gamma}{2} \frac{\psi}{d_n^2} \leq \frac{C_2}{d_n^\theta} & \text{in } \Omega_{\delta_0}, \\ \frac{\partial \psi}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \psi = \max_{\partial\Omega_{\delta_0} \setminus \partial\Omega} |w_n| & \text{on } \partial\Omega_{\delta_0} \setminus \partial\Omega. \end{cases} \tag{6.38}$$

Since (6.36) and (6.37) assure us that $w_n$ is a sub solution of (6.38), we deduce that

$$w_n \leq v_n,$$

that conclude the proof of the Theorem. □

Let us now give the proof of Theorem 6.3.

*Proof of Theorem 6.3.* Thanks to Theorems 6.12 and 6.13 we have a uniform Lipschitz bound for the sequence $z_n = u_n - S_n$ in $\Omega_{\delta_0}$, while Proposition 6.11 assures the interior regularity. Thus we can deduce that there exists a constant $C = C(\alpha, N, \partial\Omega, f)$ such that

$$\|z_n\|_{W^{1,\infty}(\Omega)} \le \|z_n\|_{W^{1,\infty}(\Omega_{\delta_0})} + \|z_n\|_{W^{1,\infty}(\Omega\setminus\Omega_{\delta_0})} \le C$$

and passing to the limit with respect to $n$

$$\|u - S\|_{W^{1,\infty}(\Omega)} = \|z\|_{W^{1,\infty}(\Omega)} \le C.$$

Moreover, passing to the limit in (6.28) and (6.30) with respect to $n$, we infer (6.5). □

### 6.3.4 Generalizations

It this last section we give for brevity the sketch of the proof of Theorem 6.5, just stressing the main differences with respect to Theorem 6.3.

*Sketch of the proof of Theorem 6.5.* Let us give at first the compete expression of $\sigma_{h,0}, \cdots, \sigma_{h,[\alpha]+1}$

$$\begin{aligned}
\sigma_{h,0} &:= [\alpha(\alpha+1)]^{\frac{1}{p-1}},\\
\sigma_{h,1}(x) &:= \alpha\sigma_0 h^{-1}\frac{\alpha h^{-\frac12}\nabla h\nabla d_n + h^{\frac12}(N-1)H(x)}{2(1+2\alpha)},\\
\sigma_{h,k}(x) &:= \frac{L_k(\sigma_{h,k-1},\sigma_{h,k-2}) + P_k(\sigma_{h,k-1},\sigma_{h,k-2}) + Q_k(\sigma_{h,k-2})}{(k-\alpha)(k-\alpha-1)-(2+\alpha)(\alpha+1)}\\
&\quad + \frac{\sigma_{h,0}^p}{(k-\alpha)(k-\alpha-1)-(2+\alpha)(\alpha+1)}\sum_{j=2}^{k}\left[\binom{p}{j}\sigma_{h,0}^{-j}\sum_{i_1+\cdots+i_j=k}\sigma_{h,i_1}(x)\cdots\sigma_{h,i_j}(x)\right]\\
&\quad \text{for } k = 2\cdots[\alpha]+1 \text{ and } i_1,\cdots,i_j \text{ positive integers,}
\end{aligned} \tag{6.39}$$

where

$$\begin{aligned}
L_k(\sigma_{h,k-1},\sigma_{h,k-2}) :=& (\alpha+1-k)\left[\sigma_{k-1}\left(h^{-\frac12}\nabla h\nabla d_n + h^{\frac12}\Delta d_n + \Delta\sigma_{k-2}\right) + 2h^{\frac12}\nabla\sigma_{k-1}\nabla d_n\right]h^{-1}\\
P_k(\sigma_{h,k-1},\sigma_{h,k-2}) :=& (\alpha+2-k)\left[(k-\alpha-1)\sigma_{k-1}\nabla h\nabla d_n + \sigma_{k-2}\left(-\frac14 h^{\frac32}|\nabla h|^2 + \frac12 h^{-\frac12}\Delta h\right)\right]h^{-\frac32}\\
Q_k(\sigma_{h,k-2}) :=& (\alpha+2-k)\left[\nabla\sigma_{k-2}\nabla h + \frac{(k-\alpha-3)}{4}\sigma_{k-2}h^{-\frac94}|\nabla h|^2\right]
\end{aligned}$$

A tedious computation shows that with such a choice, there exist a positive constant $\widetilde{C}_h = \widetilde{C}_h(\alpha, N, \partial\Omega, h, r)$ such that

$$|(\Delta S_{h,n} - |S_{h,n}|^{p-1}S_{h,n})d_n| + |\nabla(\Delta S_{h,n} - |S_{h,n}|^{p-1}S_{h,n})d_n^2| \le \bar{C}_h d_n^{1+[\alpha]-\alpha} \le \widetilde{C}_h \quad \text{in } \Omega,$$

where $S_{h,n}(x) = \sum_{k=0}^{[\alpha]+1}\sigma_{h,k}(x)\left(\sqrt{h(x)}d_n(x)\right)^{k-\alpha}$. Hence we can define the approximated problems

$$\begin{cases} -\Delta u_{h,n} + h(x)|u_{h,n}|^{p-1}u_{h,n} = r(x, u_{h,n}), & \text{in } \Omega\\ \frac{\partial u_{h,n}}{\partial\nu} = \frac{\partial S_{h,n}}{\partial\nu} & \text{on } \partial\Omega. \end{cases} \tag{6.40}$$

For the sake of clarity we give some details of the construction of the sub-solution in the case $\alpha > 1$. Let us consider the function

$$w_{h,n} := \sigma_0 h^{-\frac{\alpha}{2}}(x)d_n^{-\alpha}(x) - M_{h,1}h^{\frac{1-\alpha}{2}}(x)d_n^{1-\alpha}(x) - M_{h,2},$$

with

$$M_{h,1} \geq \alpha\sigma_0(p-1)\frac{\alpha A^{-\frac{3}{2}}\|\nabla h\|_{L^\infty(\Omega)} + A^{-\frac{1}{2}}\|\Delta d_n\|}{p+3}.$$

Notice that thanks to (6.17)

$$|r(x, w_{h,n})d_n^{\alpha+1}| = |r(x, d_n^{-\alpha} + o(d_n^{-\alpha}))|d_n d_n^\alpha \leq \sup_{0<s<1}\left\{|r(x, s^{-\alpha})|s\right\} d_n^{-\alpha} = o(1) \quad \text{as } d \to 0,\ n \to \infty.$$

Then there exist $\delta_0 = \delta_0(M_{h,1}, r)$ and $n_0 = n_0(\delta_0)$ such that

$$\begin{aligned} &-\Delta w_{h,n} + |w_{h,n}|^{p-1}w_{h,n} - r(x, w_{h,n}) \leq \\ &\quad \left(-2\frac{p+3}{p-1}h^{\frac{-\alpha+1}{2}}M_{h,1} - \alpha^2\sigma_0 h^{-\frac{\alpha}{2}-1}\nabla h\nabla d_n + \alpha\sigma_0 h^{-\frac{\alpha}{2}}\Delta d_n - r(x, w_{h,n})d_n^{\alpha+1}\right) d_n^{-\alpha-1} \\ &\qquad + O(d_n^{-\alpha}) \leq 0 \quad \forall n > n_0 \ \ \forall x \in \Omega_{\delta_0}. \end{aligned}$$

Up to an increase of the value of $M_{h,1}$ and taking the value of $M_{h,2}$ large enough, we deduce (following the same arguments that have led to (6.24) and (6.25)) that $w_{h,n}$ is a sub-solution of (6.40).
Once that sub- and super-solutions are obtained, we proceed as in Proposition 6.9, Theorem 6.10 and Proposition 6.11 in order to deduce that the solution $u_{h,n}$ of (6.40) converges (as $n \to \infty$) in $C^2_{loc}(\Omega)$ to $u_h$, unique solution of (6.13). Moreover the following estimate is satisfied

$$\exists C = C(\alpha, N, \partial\Omega, h, r) \quad : \quad \left|\frac{u_{h,n}(x)}{S_{h,n}(x)} - 1\right| \leq C\varepsilon(d_n(x)) \tag{6.41}$$

where

$$\varepsilon(s) = \begin{cases} s & \text{if } \alpha > 1 \\ s(1 + |\log s|) & \text{if } \alpha = 1 \\ s^\alpha & \text{if } \alpha < 1\,. \end{cases}$$

Let us now define $z_{h,n} := u_{h,n} - S_{h,n}$, that solves

$$\begin{cases} -\Delta z_{h,n} + |z_{h,n} + S_h|^{p-1}(z_{h,n} + S_{h,n}) - |S_{h,n}|^{p-1}S_{h,n} = r(x, z_{h,n} - S_{h,n}) + \widetilde{f}_{h,n}, & \text{in } \Omega, \\ \frac{\partial z_{h,n}}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\widetilde{f}_n := \Delta S_{h,n} - |S_{h,n}|^{p-1}S_{h,n}$. Concerning the $L^\infty(\Omega)$ estimate for $z_{h,n}$, we adapt the proof of Theorem 6.12 and prove that there exists $1 < \beta < 2$ and a positive constant $C$ such that

$$|z_{h,n}| \leq C d_n^\beta.$$

Let us now have a closer glance to the perturbed version of Theorem 6.13, for which the growth conditions (6.17) are especially designed. Exactly as in the previous section, we obtain that there exist $\delta_0$ and $n_0$ such that

$$\begin{aligned} \Delta(|\nabla z_{h,n}|^2) \geq 2\nabla &\left[h(z_{h,n} + S_{h,n})^p - hS_{h,n}^p\right]\nabla z_{h,n} + \frac{2}{N}(\Delta z_{h,n})^2 \\ &+ 2\nabla r(x, z_{h,n} + S_{h,n})\nabla z_{h,n} - 2\nabla(\widetilde{f}_{h,n})\nabla z_{h,n} \quad \text{in } \Omega_{\delta_0},\ \forall\, n > n_0. \end{aligned}$$

The main concern of course is the third term on the right hand side above; we have

$$\begin{aligned} 2\nabla r(x, z_{h,n} + S_{h,n})\nabla z_{h,n} &\leq 2\nabla_x r\nabla z_{h,n} + 2\frac{\partial r}{\partial s}|\nabla z_{h,n}|^2 - C\frac{\partial r}{\partial s}d_n^{-\alpha-1}\nabla d_n\nabla z_{h,n} \\ &\leq \gamma\frac{|\nabla z_{h,n}|^2}{d_n^2} + C_\gamma|r_x|^2 d_n^2 + 2\,|r_s|\,|\nabla z_{h,n}|^2 + C\,|r_s|\,d_n^{-\alpha-1}. \end{aligned}$$

Let us focus on the three last terms on the right hand side above. Using assumption (6.17) and estimate (6.41) we get that for $d(x) \to 0$ and $n \to \infty$

$$
\begin{aligned}
|r_x(x, z_{h,n} + S_{h,n})| = \left| r_x\left(x, S_{h,n}\left(1 + \frac{z_{h,n}}{S_{h,n}}\right)\right)\right| \frac{d_n^2}{d_n^2} &= |\nabla_x r(x, \sigma_0 d_n^{-\alpha} + o(d_n^{-\alpha}))| \frac{d_n^2}{d_n^2} \\
&\leq \frac{\sup_{0<s<1}\left\{|\nabla_x r(x, s^{-\alpha})| s^2\right\}}{d_n^2} \leq \frac{C}{d_n^2},
\end{aligned}
$$

$$
|r_s(x, z_{h,n} + S_{h,n})||\nabla z_{h,n}|^2 \leq \sup_{0<s<1}\{|r_s(x, s^{-\alpha})| s^2\} \frac{|\nabla z_{h,n}|^2}{d_n^2} = o(1) \frac{|\nabla z_{h,n}|^2}{d_n^2},
$$

$$
|r_s(x, z_{h,n} + S_{h,n})| d_n^{-\alpha-1} \leq \frac{\sup_{0<s<1}\{|r_s(x, s^{-\alpha})| s^{-\alpha+1}\}}{d_n^2} \leq \frac{C}{d_n^2}.
$$

Thus up to a decrease on the value of $\delta_0$ and an increase on the one of $n_0$, we obtain

$$
2\nabla r(x, z_{h,n} + S_{h,n}) \nabla z_n \leq (\gamma + o(1)) \frac{|\nabla z_n|^2}{d_n^2} + \frac{C}{d_n^2} \quad \text{in } \Omega_{\delta_0}.
$$

At this point it is easy to deduce the counter of (6.34), i.e. that there exist some $\delta_0$ and $n_0 = n_0(\delta_0)$ such that

$$
\Delta(|\nabla z_{h,n}|^2) \geq \gamma \frac{|\nabla z_{h,n}|^2}{d_n^2} - \frac{C_1}{d_n^2} \quad \forall n > n_0 \;\; \forall x \in \Omega_0.
$$

From now on the proof follows closely the one of Theorem 6.13.

Hence we infer that there exists $z_h \in C^2(\Omega)$, such that $z_{h,n} \to z_h$ in $C^2_{loc}(\Omega)$, that solves

$$
\begin{cases}
-\Delta z_h + |z_h + S_h|^{p-1}(z_h + S_h) - |S_h|^{p-1} S_h = r(x, z_h - S_h) + \widetilde{f}_h, & \text{in } \Omega, \\
z_h \in W^{1,\infty}(\Omega),
\end{cases}
$$

and that moreover

$$
\left|\frac{z_h}{S_h}\right| \leq o(1) \quad \text{as} \quad d(x) \to 0 \quad \text{and} \quad |\widetilde{f}_h| d + |\nabla \widetilde{f}_h| d^2 \leq C,
$$

The rest of the proof closely follows the one of Theorem 6.13. □

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