Matrix solitons solutions of the modified Korteweg-de Vries equation

Sandra Carillo[1,2], Mauro Lo Schiavo[3], and Cornelia Schiebold[4,5]

[1] Università di Roma La Sapienza, Dip.S.B.A.I., 16 Via A. Scarpa, Rome, Italy,
Sandra.Carillo@uniroma1.it,
WWW home page: https://www.sbai.uniroma1.it/~sandra.carillo/index.html
[2] I.N.F.N. - Sezione Roma1, Gr. IV - M.M.N.L.P., Rome, Italy
[3] Università di Roma La Sapienza, Dip.D.I.S.G., 18 Via Eudossiana, Rome, Italy,
[4] Department of Mathematics and Science Education, Mid Sweden University, Sundsvall, Sweden
[5] Instytut Matematyki, Uniwersytet Jana Kochanowskiego w Kielcach, Poland

Abstract. Nonlinear non-Abelian Korteweg-de Vries (KdV) and modified Korteweg-de Vries (mKdV) equations and their links via Bäcklund transformations are considered. The focus is on the construction of *soliton solutions* admitted by matrix modified Korteweg-de Vries equation. Matrix equations can be viewed as a specialisation of operator equations in the finite dimensional case when operators admit a matrix representation. Bäcklund transformations allow to reveal structural properties [10] enjoyed by non-commutative KdV-type equations, such as the existence of a recursion operator. *Operator methods* combined with Bäcklund transformations, allow to construct explicit solution formulae [11]. The latter are adapted to obtain solutions admitted by the 2×2 and 3×3 matrix mKdV equation. Some of these matrix solutions are visualised to show the *solitonic* behaviour they exhibit. A further key tool used to obtain the presented results is an *ad hoc* construction of computer algebra routines to implement non-commutative computations.

Keywords: Nonlinear non-commutative[6] equations, Korteweg-de Vries and modified Korteweg-de Vries non-commutative equations, matrix soliton solutions, Bäcklund transformations, computer algebra manipulation.

1 Introduction

So called *soliton equations* are widely investigated since the late 1960's when the first exact solution of the Korteweg-de Vries equation was obtained, as reported, for instance, in the well known book by Calogero and Degasperis [2]. The name *soliton equations* is generally used to indicate nonlinear evolution equations which admit exact solutions in the Schwartz space of smooth *rapidly*

[6] termed also non-Abelian

decreasing functions[7]. The relevance of Bäcklund transformations in studying nonlinear evolution equations is well known both under the viewpoint of finding solutions to given initial boundary value problems, see [25] - [27] as well as in giving insight in the study of their structural properties, such as symmetry properties, admitted conserved quantities and Hamiltonian structure, see e.g. [6] and references therein. The important role played by Bäcklund transformations to investigate properties enjoyed by nonlinear evolution equations is based on the fact that most of the properties of interest are preserved under Bäcklund transformations [16], [17]. Thus, the construction of a net of Bäcklund transformation to connect nonlinear evolution equations allowed to prove new results both in the case of *scalar* equations [3]-[5], [18]-[19], as well as in the generalised case of operator equations [7]-[12] [8] A comparison between the scalar (Abelian) and the operator (non-Abelian) cases referring to third order KdV-type equations, Bäcklund transformations connecting them and related properties is comprised in [9]. Notably, Bäcklund transformations indicate a way to construct solutions to nonlinear evolution equations [20] and also to nonlinear ordinary differential equations, [13] and [14]. Then, the operator approach [1], [15], [24], [28], [29], allows to construct solutions admitted by the whole hierarchies of nonlinear operator equations which are connected via Bäcklund transformations, see [3]-[11] and references therein. The special case under investigation concerns solutions of the 2×2 and 3×3 mKdV matrix equation. Matrix equations are studied in [23] and in [21], where, respectively, solutions admitted by Burgers and KdV equations are obtained. A motivation for investigations on matrix equations is connected to quantum mechanics.

The material is organised as follows. Section 2 briefly reminds the needed notions on Bäcklund transformation, recalling also the links among KdV-type equations. In the subsequent Section 3, a short overview is provided on the method to obtain matrix solutions in the finite dimensional case. Then, some solutions of the matrix mKdV equation are graphically represented and, finally, relevant remarks and research perspectives are in the closing Section 4.

2 Noncommutative potential KdV, KdV and modified KdV hierarchies

The well known potential Korteweg deVries (pKdV), Korteweg deVries (KdV) and modified Korteweg deVries (mKdV) equations, in turn

$$w_t = w_{xxx} + 3w_x^2 \quad , \quad u_t = u_{xxx} + 6uu_x \quad \text{and} \quad v_t = v_{xxx} + 6v^2 v_x \tag{1}$$

are nonlinear evolution equations in the unknown real functions, respectively w, u and v. All the equations (1) admit *soliton solutions*, see, among a wide

[7] It is generally assumed that M is space of functions $u(x,t)$ which, $\forall$ fixed t, belong to the Schwartz space S of *rapidly decreasing functions* on $\mathbb{R}^n$, in the i.e. $S(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) : \|f\|_{\alpha,\beta} < \infty, \forall \alpha, \beta\}$, where $\|f\|_{\alpha,\beta} := sup_{x \in \mathbb{R}^n} \left| x^\alpha D^\beta f(x) \right|$, $D^\beta = \partial^\beta / \partial x^\beta$.

[8] An overview on non-commutative equations is given in [22].

literature, e.g. [2], namely solutions which represent a nonlinear waves which propagate preserving energy and shape. The aim of the present investigation is to consider a generalisation of equations (1), on introduction of operator valued equations according to the approach in [1], [15]. Thus, the non-commutative equations, counterpart, respectively, of (1), are given by:

$$W_t = W_{xxx} + 3W_x^2, \quad U_t = U_{xxx} + 3\{U, U_x\}, \quad \text{and} \quad V_t = V_{xxx} + 3\{V^2, V_x\}, \tag{2}$$

where the square and curly brackets denote, in turn, the commutator and the anti-commutator, that is $\forall T, S$, $[T, S] := TS - ST$ and $\{T, S\} := TS + ST$.

According to the definition, [16], [17], two different evolution equations, e.g. $(1)_1$ and $(1)_2$, are termed *connected via the Bäcklund transformation* $B(u, v) = 0$ whenever given two solutions they admit, say, $u(x, t)$ and $v(x, t)$, if

$$B(u(x,t), v(x,t))|_{t=0} = 0 \quad \text{implies} \quad B(u(x,t), v(x,t))|_{t=\tau} = 0, \forall \tau > 0. \tag{3}$$

Well known examples of Bäcklund transformations are the introduction of a *bona fide* potential and the Miura transformation which, respectively, relate the pKdV to the KdV and the latter to the mKdV equation. The non-commutative extensions are

$$B_1 : U - W_x = 0 \qquad M : \quad U - iV_x - V^2 = 0 \ . \tag{4}$$

Notably, [10], [31], the links via Bäcklund transformations, combined with the knowledge of the hereditary recursion operator of the KdV equation, allow to construct the recursion operator admitted by the pKdV as well as by mKdV equation and, hence, to extend the same links to the whole corresponding hierarchies (see [7] and references therein for details).

That is, given the KdV recursion operator $\Phi(U)$ then, the mKdV recursion operator and the pKdV recursion operator are obtained [10]. All the corresponding members in the three *hierarchies* of pKdV, KdV and mKdV equations follows to be linked via the transformations (4) Remarkably, these connections [10] allow, given a solution of the noncommutative pKdV equation, to construct the corresponding solutions of the noncommutative KdV and mKdV equations.

3 Matrix soliton solutions

This section aims to summarise the essential steps in the construction of solutions on application of the operator method devised in [1], [15], further developed in [29], [30], and extended to hierarchies in [11]. A very short outline on how this method can be adopted to construct solutions of the matrix mKdV equation is provided. Indeed, the study in [10], [7], [8], is devoted to study nonlinear evolution equations in which the unknown is an operator on a Banach space. The idea of the method can be sketched as follows:

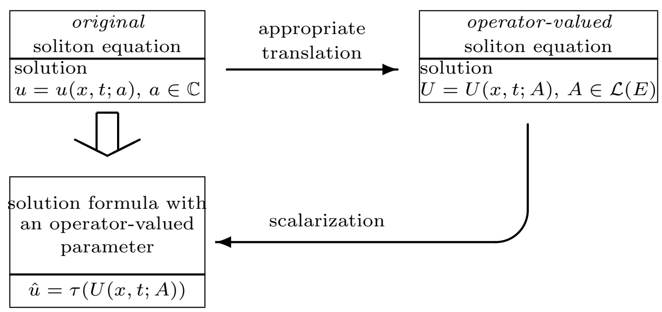
- Consider the operator method to obtain pKdV solutions.

- Use the links, via Bäcklund transformations, between the pKdV, KdV and mKdV equations to construct solutions of the KdV and mKdV equations.
- Implementation and visualisation of the explicit solutions via computer algebra routines.

The quite involved technical details can be found in [11]. The following first two subsections provide the schematic idea of the adopted method. In the third subsection some explicit soliton solutions of the 2×2 and 3×3-matrix mKdV equation are presented.

3.1 Operator method

The solution method, [1], [29], can be sketched via the following diagram



where a and A are parameters which, in turn, represent a generally complex number and an operator.

3.2 Soliton solutions of mKdV matrix equation

The following theorem (cf. Theorem 11 b) in [11] where the matrix mKdV hierarchy is treated) allows to construct noncommutative soliton solutions of the mKdV equation

$$V_t = V_{xxx} + 3\{V^2, V_x\}. \tag{5}$$

More precisely, (5) is to be read as an equation in which the dependent variable $V = V(x,t)$ takes values in the $\mathsf{d} \times \mathsf{d}$-matrices. For the proof and the notation related to Banach spaces we refer to [11].

Theorem 1. *Let E be a Banach space and $\mathcal{A}$ a quasi-Banach ideal equipped with a continuous determinant δ. Let $A \in \mathcal{L}(E)$ with $\mathrm{spec}(A) \subseteq \{\lambda \in \mathbb{C} | \mathrm{Re}(\lambda) > 0\}$. Moreover, assume that $B \in \mathcal{A}(E)$ satisfies the d-dimensionality condition*

$$AB + BA = \sum_{j=1}^{\mathsf{d}} d^{(j)} \otimes c^{(j)} \tag{6}$$

with linearly independent $d^{(j)} \in E'$ and $c^{(j)} \in E$, $j = 1, \ldots, \mathsf{d}$. Then the matrix-function

$$V = \frac{\mathrm{i}}{2} \left(\frac{\delta\left(I + \mathrm{i}(L + L^{(i,j)}) \right)}{\delta\left(I + \mathrm{i}L \right)} - \frac{\delta\left(I - \mathrm{i}(L + L^{(i,j)}) \right)}{\delta\left(I - \mathrm{i}L \right)} \right)_{i,j=1}^{d},$$

where

$$L^{(i,j)} = d^{(i)} \otimes \Big(\exp\left(Ax + A^3 t\right) c^{(j)} \Big), \quad L = \exp\left(Ax + A^3 t\right) B, \tag{7}$$

and $I = I_E$ *denotes the identity operator on* E*, solves the matrix mKdV* (5) *with values in the* $\mathsf{d} \times \mathsf{d}$*-matrices on every product domain* $(-\infty, c) \times G$ *(*$c \in \mathbb{R}$*,* G *a domain in* $\mathbb{R}^{n-1}$*) on which* $\delta(I \pm \mathrm{i}L) \neq 0$.

3.3 Some explicit solutions of the matrix modified KdV equation

The final step consists in the implementation, via computer algebra, of suitable routines amenable to visualise the matrix solutions of the mKdV equation. Indeed, multisoliton solutions admitted by the matrix KdV equation are given by Goncharenko in [21]: these solutions are obtained via a generalisation of the Inverse Scattering Method. As shown in [11], the solution class of the matrix KdV equation that corresponds to the class constructed in Theorem 1 comprises Goncharenko's multisoliton solutions. In the present subsection, soliton solutions of the matrix *modified* KdV are visualised using Mathematica. Some of such explicit 2-soliton solutions of the matrix mKdV, in the present subsection, are visualised using *Mathematica*.

Without going into details, we briefly explain the choices in Theorem 1 to realize 2-soliton solutions of the $\mathsf{d} \times \mathsf{d}$-matrix mKdV. To reflectionless spectral data with discrete eigenvalues k_1, k_2 and spectral matrices B_1, B_2 of size $\mathsf{d}\times\mathsf{d}$, we associate $A = \begin{pmatrix} k_1 I_d & 0 \\ 0 & k_2 I_d \end{pmatrix}$, $\quad B = \left(\frac{\mathrm{i}}{k_i + k_j} B_j \right)_{i,j=1}^{2}$, In particular, $E = \mathbb{C}^{2\mathsf{d}}$. In the plots below the case $B_1 = B_2$ is considered.

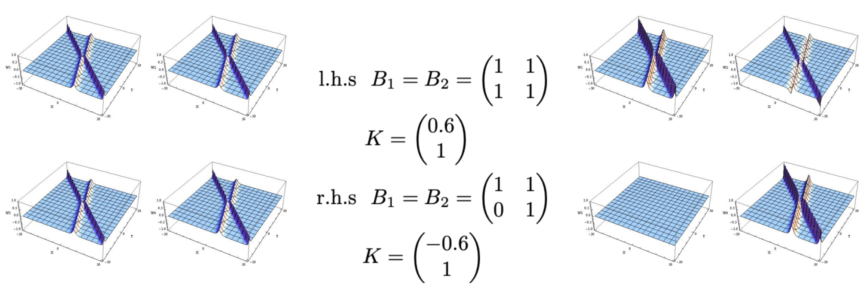
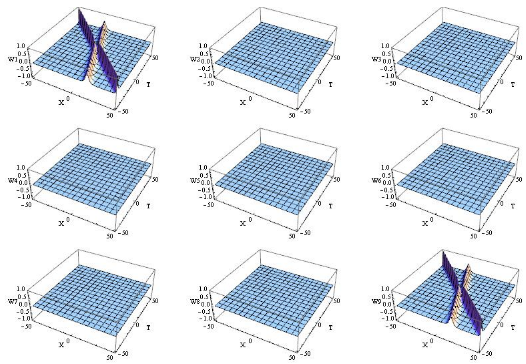


Fig. 1. Two examples of two-soliton solutions of 2×2 matrix mKdV equation; the elements of the spectral matrices are specified.

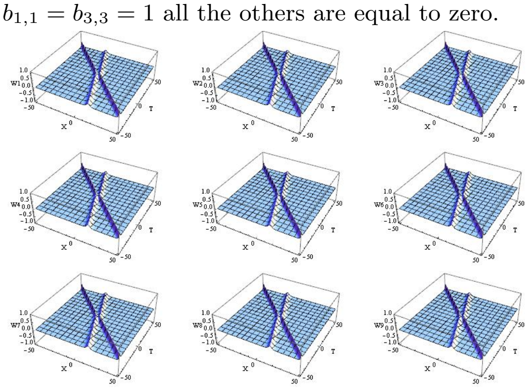
The examples of solutions of the mKdV equation, graphically represented in the Fig.s 2 - 4, represent different 3×3 matrix solutions: the elements of the spectral matrices are indicated in the caption. The pictures show the behaviour of *two-soliton solutions* admitted by the matrix mKdV equation.



$$B_1 = B_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$K = \begin{pmatrix} 0.6 \\ 1 \end{pmatrix}$$

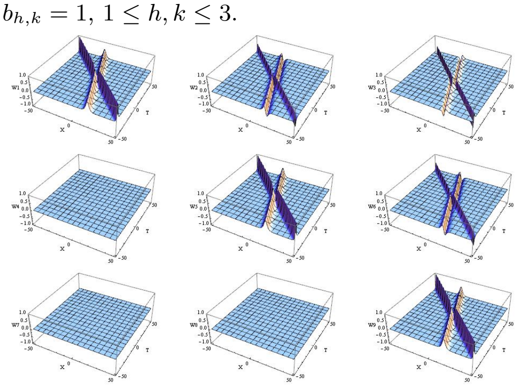
Fig. 2. Two-soliton solutions of 3×3 matrix mKdV equation; the two diagonal elements $b_{1,1} = b_{3,3} = 1$ all the others are equal to zero.



$$B_1 = B_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$K = \begin{pmatrix} 0.6 \\ 1 \end{pmatrix}$$

Fig. 3. Two-soliton solution of 3×3 matrix mKdV equation; all the matrix elements $b_{h,k} = 1$, $1 \leq h, k \leq 3$.



$$B_1 = B_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$K = \begin{pmatrix} 0.6 \\ 1 \end{pmatrix}$$

Fig. 4. Two-soliton solution of 3×3 matrix mKdV equation; all the matrix elements $b_{h,h} = 1$, $1 \leq h \leq 3$ and $b_{h,h+1} = 1$, $1 \leq h \leq 2$ all the others are equal to zero.

Note that the crucial features of interaction between two different solitons, well known on the scalar case seems to characterise also matrix solutions. These pictures represents only an example of a study, currently in progress, aiming to construct additional solutions as well as their interpretation. In particular, if the spectral matrices are diagonal, then also the solution enjoys the same property, see Fig. 2. Conversely, Fig. 4, shows that, when *off diagonal* elements in the spectral matrix are different from zero, the situation changes. For example, given the initial datum $b_{1,3} = 0$, in the spectral matrices, the corresponding element of the two-soliton solution, depicted in Fig. 4, is not zero. A detailed study on these solutions is currently under investigation. One of the main issues concerns the energy conservation and its partition among the matrix elements. The appropriate functional which represents the energy of the interacting solitons is expected to play a crucial role to understand the phenomenology under investigation.

4 Concluding remarks

The aim of the present study is to emphasise some of the properties of solutions admitted by matrix mKdV equation. Notably, depending on the spectral data, a variety of soliton solutions may be observed. The crucial feature seem to be the appearance, also in the matrix case, of localised solutions which can be termed *solitons* on the basis of their interaction properties. A much richer interaction phenomenology, with respect to the scalar case, can be observed when matrix solutions are investigated. Indeed, as soon as the spectral matrices have non-zero off-diagonal terms, the solution exhibits non-zero solutions in further matrix elements. In addition, a variety of different solutions can be observed. However, in all cases, a form of energy distribution seems to be observed: the most appropriate way to define a functional suitable to represent the *energy* related to a matrix soliton solution is, in the authors opinion, one of the interesting questions this work arises. The obtained results motivate further investigations to provide a better understanding of the interesting phenomenology already observed.

Acknowledgements

Under the financial support of G.N.F.M.-I.N.d.A.M., I.N.F.N. and Università di Roma La Sapienza, Rome, Italy. C.Schiebold acknowledges Dip. S.B.A.I., Università di Roma La Sapienza, for the kind hospitality.

References

1. H. Aden and B. Carl. On realizations of solutions of the KdV equation by determinants on operator ideals, *J. Math. Phys.* **37**, 1833-1857 (1996).
2. F. Calogero and A. Degasperis: Spectral Transform and Solitons I, Studies in Mathematics and its Application 13, North Holland, Amsterdam, 1980.
3. S Carillo, Nonlinear Evolution Equations: Bäcklund Transformations and Bäcklund charts, *Acta Applicandae Math.* **122**, 1 (2012), 93-106.
4. S. Carillo, KdV-type equations linked via Bäcklund transformations: remarks and perspectives, *Appl. Num. Math.*, **141** (2019), 81-90. doi:10.1016/j.apnum.2018.10.002
5. S. Carillo and B. Fuchssteiner, The abundant symmetry structure of hierarchies of nonlinear equations obtained by reciprocal links. *J. Math.Phys.* **30**, 1606-1613 (1989).

6. S. Carillo, M. Lo Schiavo, E. Porten, C. Schiebold, A novel noncommutative KdV-type equation, its recursion operator and solitons, *J. Math. Phys.*, **59**, (4)(2018).
7. S. Carillo, M. Lo Schiavo and C. Schiebold, Bäcklund transformations and non abelian nonlinear evolution eq.s: a novel Bäcklund chart, *SIGMA*, **12**, 087 (2016).
8. S. Carillo, M. Lo Schiavo and C. Schiebold, Recursion Operators admitted by non-Abelian Burgers equations: Some Remarks, *Mat.Comp.in Sim.***147C**, (2018) 40-51.
9. S. Carillo, M. Lo Schiavo and C. Schiebold, Abelian versus non-Abelian Bäcklund Charts: some remarks, *E.E.C.T.*, **8** (1), 43-55 (2019). doi:10.3934/eect.2019003
10. S. Carillo and C. Schiebold, Non-commutative KdV and mKdV hierarchies via recursion methods, *J. Math. Phys.* **50** (2009), 073510.
11. S. Carillo and C. Schiebold. Matrix Korteweg-de Vries and modified Korteweg-de Vries hierarchies:non-commutative soliton solutions.*J.Math.Phys.***52**, 053507 (2011).
12. S. Carillo and C. Schiebold, On the recursion operator for the noncommutative Burgers hierarchy, *J. Nonlinear Math. Phys.*, **19** (1), 27-37 (2012).
13. S. Carillo and F. Zullo, Ermakov-Pinney and Emden-Fowler eq.s: new solutions from novel Bäcklund transf.s, *Theor. Math. Phys.*, **196** (3), 1268-1281 (2018).
14. S. Carillo and F. Zullo, The Gross-Pitaevskii equation: Bäcklund transformations and admitted solutions, ArXiv **1803.09228**, *Ricerche di Matematica*, (2019).
15. B. Carl and C. Schiebold, *Nonlinear equations in soliton physics and operator ideals,* Nonlinearity **12**, 333-364 (1999).
16. A.S. Fokas and B. Fuchssteiner, Bäcklund transformation for hereditary symmetries, *Nonlin. Anal., Theory Methods Appl.* **5**, 4 (1981), 423-432.
17. B. Fuchssteiner, Application of hereditary symmetries to nonlinear evolution equations, *Nonlin. Anal., Th. Meth. Appl.* **3**, 6 (1979), 849-862.
18. B. Fuchssteiner and S. Carillo, Soliton structure versus singularity analysis: Third order completely integrable nonlin. eq.ns in 1+1-dim.s, *Phys. A* **154**, 467-510 (1989).
19. B. Fuchssteiner and S. Carillo, The Action-Angle Transformation for Soliton Equations, *Physica A*, **166**, (1990), 651-676,
20. B. Fuchssteiner, T. Schulze, S. Carillo, *Explicit solutions for the Harry Dym equation*, J. Phys. A: Math. Gen., **25**, 223-230, (1992).
21. V. M. Goncharenko, Multisoliton solutions of the matrix KdV equation, *Theoret. Math. Phys.*, **126**, 81-91 (2001).
22. B. A. Kupershmidt. "KP or mKP. Noncommutative Mathematics of Lagrangian, Hamiltonian, and Integrable Systems". Providence, American Math. Society 2000.
23. D.Levi, O.Ragnisco, M.Bruschi Continuous and discrete matrix Burgers' hierarchies, *Nuovo Cimento B*, **74**, no. 1, 33-51(1983).
24. V. A. Marchenko, *Nonlinear Equations and Operator Algebras,* Reidel, 1988.
25. C. Rogers and W. F. Ames *Nonlinear Boundary Value Problems in Science and Engineering* Academic Press, Boston , 1989
26. C. Rogers and W. K. Schief. *Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory.* Camb. Univ.Press, Cambridge (2002).
27. C. Rogers, W.F. Shadwick. Bäcklund Transformations and Their Applications, Math. in Science and Eng, Vol. 161, Academic Press, New York, 1982.
28. C. Schiebold. An operator-theoretic approach to the Toda lattice equation. *Phys. D* **122**, 37–61 (1998).
29. C. Schiebold. Explicit solution formulas for the matrix-KP. *Glasgow Math. J.* **51**, 147–155 (2009).
30. C. Schiebold. Noncommutative AKNS Systems and multisolitons solutions to the matrix sine-Gordon equation. *Disc. Cont. Dyn. Sys. Suppl.* **2009**, 678–690 (2009).
31. C. Schiebold. Structural properties of the noncommutative KdV recursion operator, *J. Math. Phys.* **52**, 113504 (2011).