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Soliton Representations and Sobolev Diffeomorphism Symmetry in CFT

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Abstract

We show that any positive energy representation of $\mathrm{Diff}_+(S^1)$ can be extended to a strongly continuous unitary projective representation of the fractional Sobolev diffeomorphisms $\mathcal{D}^s(S^1)$, with $s > 3$. For some positive energy representations, i.e for the positive energy vacuum representations of $\mathrm{Diff}_+(S^1)$ with positive integer central charge, we can improve the implementation to the group $\mathcal{D}^s(S^1)$ with $s > 2$. We show that a conformal net of von Neumann algebras on the circle is always $\mathcal{D}^s(S^1)$-covariant, $s > 3$. Furthermore, we show that a given positive energy representation U of $\mathrm{Diff}_+(S^1)$ cannot be extended to some less-smooth diffeomorphisms, and from this fact we obtain an uncountable family of proper soliton representations. From these soliton representations we construct irreducible unitary projective positive energy representations of ΛG (resp. B_0) which do not extend to LG (resp. $\mathrm{Diff}_+(S^1)$).

Introduction

Conformal quantum field theory (CFT) in (1+1) dimension is a widely studied subject, with a plenty of physical applications [DMS97]. From the mathematical point of view, the interest in conformal filed theory is motivated by its connections to various areas of mathematics [EK98]. In $(1+1)$-dimensional conformal field theory, the symmetry group, i.e. the group of transformations which preserve space-time causality, is isomorphic to $\mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1)$, where each copy of $\mathrm{Diff}_+(S^1)$ acts on the respective chiral component.

The group of smooth diffeomorphisms of the circle $\mathrm{Diff}_+(S^1)$ is an object of particular interest. It is an infinite-dimensional Fréchet Lie group which is algebraically (and hence topologically) simple. Its representation theory is widely studied, as amongst other applications, it plays a pivotal role in conformal field theory. In the algebraic formulation, a chiral conformal field theory on S^1 is realized as a conformal net, namely an assignment $I \mapsto \mathcal{A}(I)$ where I is an open proper interval of the unit circle S^1 and $\mathcal{A}(I)$ a von Neumann algebra on a fixed Hilbert space, satisfying axioms dictated by natural physical requirements. From the irreducible positive energy representations of $\mathrm{Diff}_+(S^1)$ it is possible to construct models which constitute the building blocks of the theory, the Virasoro nets. In particular every conformal net is an extension of a Virasoro net. As is often claimed in the physical literature, the $\mathrm{Diff}_+(S^1)$ symmetry imposes a strong constraint on $(1+1)$-dimensional field theories as is evidenced by the fact that the conformal nets with central charge c in the discrete series are completely classified [KL04a, KL04b].

A natural question which arises when studying the representation theory of $\mathrm{Diff}_+(S^1)$ is the following. Given a positive energy representation U of $\mathrm{Diff}_+(S^1)$ how much can the regularity of the diffeomorphisms be weakened in order to obtain a representation of a larger group of non-smooth diffeomorphisms? In [CW05] Carpi and Weiner proved that the stress-energy tensor T associated to a given positive energy representation of $\mathrm{Diff}_+(S^1)$ can be evaluated on a certain class of non-smooth functions of S^1 retaining its self-adjointness. This fact, besides having remarkable applications such as uniqueness of conformal covariance [CW05] and positivity of energy of DHR sectors [Wei06], was an indication that a similar result could be transposed to the group level.

In Chapter 3 we show that it is possible to extend every positive energy projective unitary representation U of $\mathrm{Diff}_+(S^1)$ to the group of fractional Sobolev diffeomorphisms $\mathcal{D}^s(S^1)$ with $s > 3$ and in particular to the C^k diffeomorphisms of the circle with $k \geq 4$. It is not clear if the exponent $s > 3$ is optimal, uniformly on all projective representations of $\mathrm{Diff}_+(S^1)$, altough it seems that the methods used therein cannot be undertaken to proceed further. In Chapter 5 we show that for certain representations, namely the irreducible representations with integral central charge and with lowest weight zero, the latter result can be improved on, obtaining $\mathcal{D}^s(S^1)$ with $s > 2$.

The reverse problem, which is to understand whether a homeomorphism γ on the circle is not unitarily implementable in a compatible way with the representation U of $\mathrm{Diff}_+(S^1)$, is strictly related to the construction of soliton representations in conformal field theory presented in this thesis.

A soliton of a conformal net $\mathcal{A}$ is a family of (inclusion-preserving) normal representations indexed by open intervals of S^1 not containing the point -1. We say that a soliton is proper (or non-trivial) if does not extend to a representation of $\mathcal{A}$ on S^1.

The first rigorous approach in QFT to soliton representations is due to Roberts [Rob74] and Frohlich in [Frö76] gave concrete examples of solitons in many models. In our context, which is of chiral conformal field theory described by conformal nets, solitons were studied by Fredenhaghen in [Fre93] whilst Henriques had some results about the covariance of the soliton representations in specific models [Hen17a].

Since it is not possible to construct solitons for the Virasoro nets via α-induction because of their minimality [Car98], the existence of solitons for these models was unclear. Recently Henriques in [Hen17a] proved that the category of solitons $\mathrm{Sol}(\mathcal{A})$ of a finite index conformal net $\mathcal{A}$ is a bicommutant category whose Drinfel'd center corresponds to the category of DHR sectors of $\mathcal{A}$. This fact implies the existence of non-trivial soliton representations for all the conformal nets with central charge $c < 1$ and μ-index > 1.

In Chapter 4 we present an explicit construction of a family of proper irreducible (type I) soliton representations for any conformal nets. We consider a particular class of functions γ of the circle, namely orientation-preserving homeomorphisms which are C^∞ on $S^1 \setminus \{-1\}$ and fail to be differentiable in -1, from γ we construct a soliton representations σ_γ and we prove that is a proper soliton. The proof follows from showing that γ is not unitarily implementable, and this is done with the aid of the modular theory. This type of construction was already presented in [LX04, KLX05] but for a different class of functions and yielded non irreducible solitons of type III. In the case of the $U(1)$-current net and the virasoro net $\mathcal{A}_{\mathrm{Vir}_c}$ with $c \in \mathbb{Z}_+$ all the constructed solitons are covariant for B_0, the stabilizer subgroup of $\mathrm{Diff}_+(S^1)$ of the point -1, which contains translations and dilations. More generally we show that any soliton is translation covariant and has positive energy. The argument depends once again on the the already mentioned fact that the stress-energy tensor T can be evaluated on non-smooth functions and on quantum energy inequalities introduced in [FH05]. As an application, we construct irreducible unitary projective positive-energy representations of B_0 and of $\Lambda SU(N)$ (the subgroup of $LSU(N)$ consisting of loops with support not containing the point -1) which do not extend to $\mathrm{Diff}_+(S^1)$ and $LSU(N)$ respectively. These results can be seen as an application of the Tomita-Takesaki modular theory of von Neumann algebras to the representation theory of infinite-dimensional Lie groups.

The thesis is organized as follows: in Chapter 1 we introduce infinite-dimensional Lie Groups, with particular emphasis on the diffeomorphism group $\mathrm{Diff}_+(S^1)$ and on the loop groups. The last part of the Chapter is devoted to the groups of diffeomorphisms of Sobolev class. In Chapter 2 we recall the standard notions of conformal net and its representation theory together with examples coming from the unitary projective representations of the diffeomorphism group $\mathrm{Diff}_+(S^1)$ and loop groups. In Chapter 3 we extend every positive energy representation of $\mathrm{Diff}_+(S^1)$ to a strongly continuous projective unitary representation of $\mathcal{D}^s(S^1)$, $s > 3$ and we prove that any conformal net $\mathcal{A}$ is $\mathcal{D}^s(S^1)$-covariant, $s > 3$. In Chapter 4 we prove tha a conformal net $(\mathcal{A}, U, \Omega)$ is $\mathrm{Diff}_+^{1,\infty}(S^1)$-covariant, that every soliton is translation covariant with positive energy and we exihibit an explicit construction of proper solitons. Chapter 5 is dedicated to concrete examples: we use the results in Chapter 4 to prove that there exists irreducible positive energy representations $\Lambda SU(N)$ (resp. B_0) which do not extend to $LSU(N)$ (resp. $\mathrm{Diff}_+(S^1)$). Furthermore, we show that the $U(1)$-current net and the virasoro nets with positive integer central charge are $\mathcal{D}^s(S^1)$-covariant, $s > 2$.

The original research in Chapter 3 about positive energy representation of Sobolev diffeomorphism groups is due to a collaboration with Sebastiano Carpi, Simone Del Vecchio and Yoh Tanimoto, is contained in [CDIT18] and it has been submitted as a joint work. The results in Chapter 4 and 5 about soliton representations have been obtained in collaboration with Simone Del Vecchio and Yoh Tanimoto, is contained in [DIT18] and it has been submitted as a joint work.

Chapter 1

Groups of diffeomorphisms and Loop groups

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1.1 Infinite-dimensional Lie groups

We start this section introducing the fundamental notions that we need to talk about infinite-dimensional Lie groups.

Definition 1.1.1. A family $\{\rho_\alpha\}$ of seminorms on a complex vector space V is a family of maps $\rho_\alpha : V \to \mathbb{R}_+$ such that for every α, $\beta \in \mathbb{C}$, $v, v_1, v_2 \in V$ we have that $\rho_\alpha(v_1 + v_2) \leq \rho_\alpha(v_1) + \rho_\alpha(v_2)$ (subadditivity), $\rho_\alpha(\beta v) = |\beta|\rho_\alpha(v)$ (homogeneity). If in addition $\rho_\alpha(v) = 0$ for all α implies $v = 0$, we say that the family $\}\rho_\alpha\}$ separates point. A complex vector space V is a **locally convex space** if admits a family of seminorms separating points. The topology considered on V is the weakest topology such that all ρ_α are continuous, together with the addition operation in V.

From Definition 1.1.1, any locally convex space is an Hausdorff topological space. In addition, the topology is metrizable if and only if the collection of seminorms

$\{\rho_\alpha\}$ is countable. A sequence $\{v_i\} \subset V$ in a metrizable locally convex space V is **Cauchy** if $\rho_m(v_i - v_j) \to 0$ when $i, j \to \infty$, for all m. The space V is **complete** if every Cauchy sequence converges.

Definition 1.1.2. A **Fréchet space** is a complete metrizable locally convex space.

Definition 1.1.3. Let V, W be Fréchet spaces and $U \subset V$ an open set in V. A map $f : U \subset V \to W$ is said to be **differentiable** in $u \in U$ in the direction $v \in V$ if exists the limit

$$Df(u,v) := \lim_{t \to 0} \frac{f(u+tv) - f(u)}{t}, \tag{1.1.1}$$

and the function f is differentiable in U if the limit 1.1.1 exists for all $u \in U$ and $Df : U \times V \to W$ is continuous. Analogously, we can define the k-th derivative of f which is the function $D^k f : U \times \underbrace{V \times \cdots \times V}_{\text{k times}} \to W$ if it exists. The function f is said to be smooth (or C^∞) if $D^k f$ exists for all $k \in \mathbb{N}$ and is continuous.
A **Fréchet manifold** M is a topological Hausdorff space with an atlas $(U_\alpha, \varphi_\alpha)$ such that the coordinate charts φ_α take values in a Fréchet space and all the transition functions are C^∞.

Starting from Definition 1.1.3, given a Fréchet manifold M we can define tangent space, tangent bundle, vector fields, etc., as in the case of finite-dimensional manifolds.

Definition 1.1.4. A **Fréchet Lie Group** G is a Fréchet manifold together with a group structure such that the multiplication map which sends g_1, g_2 to $g_1 g_2$ and the inversion map which sends g to g^{-1} are C^∞.

Definition 1.1.5. Let G a Lie Group with identity element e. The **Lie algebra** $\mathfrak{g}$ of G is the tangent space at the identity e, with the usual bracket induced by the identification with the Lie algebra of left invariant vector fields of G.

1.2 The group $\mathrm{Diff}_+(S^1)$

Definition 1.2.1. We denote by $\mathrm{Diff}_+(S^1)$ the group of orientation preserving, smooth diffeomorphisms of the circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$.

Definition 1.2.2. We denote with $\mathrm{Vect}(S^1)$ the Lie algebra of smooth vector fields on S^1. We can identify $\mathrm{Vect}(S^1)$ with $C^\infty(S^1, \mathbb{R})$, since a vector field X on the circle can be written as $X(e^{i\theta}) = f(e^{i\theta}) \frac{d}{d\theta}$

The group $\mathrm{Diff}_+(S^1)$ is an infinite dimensional Lie group whose Lie algebra is $\mathrm{Vect}(S^1)$[Mil84]. Given $f \in \mathrm{Vect}(S^1)$ and $t \in \mathbb{R}$ we define $\mathrm{Exp} : \mathrm{Vect}(S^1) \to$

$\mathrm{Diff}_+(S^1)$ as the function which maps the field tf to the one-parameter group of diffeomorphism of S^1 $\mathrm{Exp}(tf) \in \mathrm{Diff}_+(S^1)$ satisfying the equation

$$\frac{dz(t)}{dt} = f(z(t))$$

where $z(t) = \mathrm{Exp}(tf)(z)$ and $\mathrm{Exp}(0)(z) = z$.

Proposition 1.2.3. *The exponential* $\mathrm{Exp} : \mathrm{Vect}(S^1) \to \mathrm{Diff}_+(S^1)$ *is not locally surjective.*

For an element $f \in C^\infty(S^1, \mathbb{R})$ we denote by f' the derivative of f with respect to the angle θ,

$$f'(z) = \frac{d}{d\theta} f(e^{i\theta})\Big|_{e^{i\theta}=z}.$$

We consider a diffeomorphism $\gamma \in \mathrm{Diff}_+(S^1)$ as a map from S^1 in $S^1 \subset \mathbb{C}$. With this convention, its action on $f \in \mathrm{Vect}(S^1)$ is

$$(\gamma_* f)(e^{i\theta}) = -ie^{-i\theta} \left(\frac{d}{d\theta} \gamma(e^{i\theta}) \right) \Big|_{\gamma^{-1}(e^{i\theta})} f(\gamma^{-1}(e^{i\theta})). \tag{1.2.1}$$

The following is an important fact about the diffeomorphism group $\mathrm{Diff}_+(S^1)$:

Theorem 1.2.4. *The group* $\mathrm{Diff}_+(S^1)$ *is algebraically simple.*

Corollary 1.2.5. *The group* $\mathrm{Diff}_+(S^1)$ *is generated by exponentials. Furthermore, every* $\gamma \in \mathrm{Diff}_+(S^1)$ *can be written as a finite product of exponential of localized fields, i.e. fields with support contained in a proper interval of* S^1.

Proof. Let $f \in \mathrm{Vect}(S^1)$ and $\gamma \in \mathrm{Diff}_+(S^1)$, then $\gamma \circ \mathrm{Exp}(f) \circ \gamma^{-1} = \mathrm{Exp}(\gamma_* f)$. □

Definition 1.2.6. We denote by $\mathrm{Diff}^k_+(S^1)$ the group of C^k-diffeomorphisms of S^1.

Note that this is not a Lie group, and indeed, the corresponding linear space $\mathrm{Vect}^k(S^1)$ of C^k-vector fields is not closed under the natural Lie bracket (see below).

The universal covering group of $\mathrm{Diff}_+(S^1)$ (resp. $\mathrm{Diff}^k_+(S^1)$), $\widetilde{\mathrm{Diff}_+(S^1)}$ (resp. $\widetilde{\mathrm{Diff}^k_+(S^1)}$), can be identified[1] with the group of C^∞-diffeomorphisms (resp. C^k-diffeomorphisms) γ of $\mathbb{R}$ which satisfy

$$\gamma(\theta + 2\pi) = \gamma(\theta) + 2\pi.$$

If $\gamma \in \widetilde{\mathrm{Diff}_+(S^1)}$, its image under the covering map is in the following denoted by $\dot\gamma \in \mathrm{Diff}_+(S^1)$, where $\dot\gamma(e^{i\theta}) = e^{i\gamma(\theta)}$. Conversely, if $\gamma \in \mathrm{Diff}_+(S^1)$, there is an

[1] The realization of $\widetilde{\mathrm{Diff}^k_+(S^1)}$ works in the same way as $\widetilde{\mathrm{Diff}_+(S^1)}$ as in [TL99, Section 6.1], see also [Ham82, Example 4.2.6].

element $\tilde{\gamma} \in \widetilde{\mathrm{Diff}_+(S^1)}$ whose image under the covering map is γ. Such a $\tilde{\gamma}$ is unique up to 2π and called a lift of γ.

The group $\mathrm{Diff}_+(S^1)$ admits the Bott-Virasoro cocycle $B : \mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1) \to \mathbb{R}$ (see e.g. [FH05]). The Bott-Virasoro group is then defined as the group with elements

$$(\gamma, t) \in \mathrm{Diff}_+(S^1) \times \mathbb{R}$$

and with multiplication

$$(\gamma_1, t_1) \circ (\gamma_2, t_2) = (\gamma_1 \circ \gamma_2, t_1 + t_2 + B(\gamma_1, \gamma_2)).$$

Note that, given a true (not projective) unitary irreducible representation V of the universal covering of the Bott-Virasoro group, one can obtain a unitary multiplier representation $\underline{V}(\gamma) := V(\gamma, 0)$ of $\widetilde{\mathrm{Diff}_+(S^1)}$ (with respect to the Bott-Virasoro cocycle B). Then the map $\underline{V} : \widetilde{\mathrm{Diff}_+(S^1)} \to U(\mathcal{H})$ satisfies

$$\underline{V}(\gamma_1)\underline{V}(\gamma_2) = e^{icB(\dot{\gamma}_1, \dot{\gamma}_2)}\underline{V}(\gamma_2)\underline{V}(\gamma_1),$$

where $c \in \mathbb{R}$ by irreducibility.

The Möbius group

The group $SL(2, \mathbb{R})$ of 2×2 real matrices with determinant one acts on the compactified real line $\mathbb{R} \cup \{\infty\}$ by fractional transformations:

$$g : x \to gx := \frac{ax + b}{cx + d} \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}).$$

The Kernel of this action is $\{\pm\mathbb{1}\}$.

By identifying the compactified real line $\mathbb{R} \cup \{\infty\}$ with the circle S^1 via Cayley transform

$$C : S^1 \setminus \{-1\} \to \mathbb{R}, \qquad z \mapsto i\frac{1 - z}{1 + z}, \tag{1.2.2}$$

with inverse

$$C^{-1} : \mathbb{R} \to S^1 \setminus \{-1\}, \qquad t \mapsto \frac{1 + it}{1 - it}, \tag{1.2.3}$$

the group $\mathrm{PSL}(2, \mathbb{R}) := SL(2, \mathbb{R}) / \{\pm\mathbb{1}\}$ can be identified with a subgroup of diffeomorphims of the circle S^1, the Möbius group. Using again the Cayley transform we can identify $SL(2, \mathbb{R})$ with $SU(1, 1) := \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 - |\beta|^2 = 1 \right\}$ which acts on $S^1 \subset \mathbb{C}$ by linear fractional transformation:

$$g : z \to gz := \frac{\alpha x + \beta}{\bar{\beta} x + \bar{\alpha}} \quad \text{for} \quad g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(1, 1).$$

It follows that $PSU(1,1) := SU(1,1)/\{\pm\mathbb{1}\} \simeq \mathrm{PSL}(2,\mathbb{R})$, and it will be clear from the context if we are dealing with elements of $PSU(1,1)$ acting on S^1 (circle picture) or with elements of $\mathrm{PSL}(2,\mathbb{R})$ acting on $\mathbb{R} \cup \{\infty\}$ (real line picture).

The following are important subgroups of $\mathrm{PSL}(2,\mathbb{R})$:

$$R(\theta) = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \quad \delta(s) = \begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{pmatrix}, \quad \tau(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

These are the rotation, dilation and traslation subgroup respectively and act in the following way (using the circle picture for rotations and the compactified real line for dilations and traslations)

$$\begin{aligned} R(\theta)z &= e^{i\theta}z \quad \text{on } S^1, \\ \delta(s)x &= e^{s}x \quad \text{on } \mathbb{R}, \\ \tau(t)x &= x + t \quad \text{on } \mathbb{R}. \end{aligned} \tag{1.2.4}$$

The generator of translations is by definition $T(x) := \frac{\partial}{\partial t}(\tau(t)x)\Big|_{t=0} = 1$. The corresponding field in angular coordinates, $z = e^{i\theta} \in S^1 \subset \mathbb{C}$, is

$$T(e^{i\theta}) = 1 + \cos(\theta). \tag{1.2.5}$$

The Stabilizer subgroup of one point in $\mathrm{Diff}_+(S^1)$

We denote with B_0 the subgroup of $\mathrm{Diff}_+(S^1)$ consisting of diffeomorphisms which fix the point $z = -1$. It is possible to consider B_0 as a Lie subgroup of $\mathrm{Diff}_+(S^1)$ with Lie algebra given by those vector fields $f \in \mathrm{Vect}(S^1)$ such that $f(-1) = 0$. It is very easy to see, passing to the circle picture, that the dilation and translation subgroups of $\mathrm{Diff}_+(S^1)$ are in B_0

The representation theory of B_0 it is not well understood. It possible to say something about the restriction of representation of $\mathrm{Diff}_+(S^1)$ to B_0: for example, the restriction to B_0 of an irreducible unitary projective positive-energy representation of $\mathrm{Diff}_+(S^1)$ is irreducible [Wei08, Corollary 3.6]. Two different inequivalent irreducible unitary projective positive-energy representations of $\mathrm{Diff}_+(S^1)$ may be equivalent when restricted to B_0 [Wei08, Corollary 6.4]. The question is wheter there exist some unitary representation of B_0 which don't extend to the whole $\mathrm{Diff}_+(S^1)$. If we think of B_0 as the group consisting of functions $\varphi : \mathbb{R} \to \mathbb{R}$ which are smooth and such that $\varphi(2\pi + x) = \varphi(x) + 2\pi$, $\varphi(-\pi) = -\pi$, we know the following result: [Tan10, Proposition 7.1]

Proposition 1.2.7. *If $\lambda \in \mathbb{R}$, the map $\pi : B_0 \to S^1$ such that*

$$\varphi \mapsto \pi(\varphi) := e^{i\lambda \log(\varphi'(0))}$$

is a unitary (not projective) one-dimensional representation of B_0 *and cannot be extended to* $\mathrm{Diff}_+(S^1)$.

1.2.1 The Virasoro algebra.

The space $\mathrm{Vect}(S^1)$ is endowed with the Lie algebra structure with the Lie bracket given by

$$[f, g] = f'g - fg'.$$

As a Lie algebra, $\mathrm{Vect}(S^1)$ admits the Gelfand–Fuchs two-cocycle

$$\omega(f, g) = \frac{1}{48\pi}\int_{S^1}(f(e^{i\theta})g'''(e^{i\theta}) - f'''(e^{i\theta})g(e^{i\theta}))d\theta.$$

The Virasoro algebra Vir is the central extension of the complexification of the algebra generated by the trigonometric polynomials in $\mathrm{Vect}(S^1)$ defined by the two-cocycle ω. It can be explicitly described as the complex Lie algebra generated by L_n, $n \in \mathbb{Z}$, and the central element $\mathbf{1}$, with brackets

$$[L_n, L_m] = (n - m)L_{n+m} + \delta_{n+m,0}\frac{n^3 - n}{12}\mathbf{1}.$$

Consider a representation $\pi : \mathrm{Vir} \to \mathrm{End}(V)$ of Vir on a complex vector space V endowed with a scalar product $\langle\cdot,\cdot\rangle$. We call π a **unitary positive energy representation** if the following hold

1. Unitarity: $\langle v, \pi(L_n)w\rangle = \langle\pi(L_{-n})v, w\rangle$ for every $v, w \in V$ and $n \in \mathbb{Z}$;
2. Positivity of the energy: $V = \bigoplus_{\lambda\in\mathbb{R}_+\cup\{0\}} V_\lambda$, where $V_\lambda := \ker(\pi(L_0) - \lambda\mathbb{1}_V)$. The lowest eigenvalue of $\pi(L_0)$ is called lowest weight;
3. Central charge: $\pi(\mathbf{1}) = c\mathbb{1}_V$;

There exists an irreducible unitary positive energy representation with central charge c and lowest weight h if and only if $c \geq 1$ and $h \geq 0$ (continuous series representation) or $(c, h) = (c(m), h_{p,q}(m))$, where $c(m) = 1 - \frac{6}{(m+2)(m+3)}$, $h_{p,q}(m) = \frac{(p(m+1)-qm)^2-1}{4m(m+1)}$, $m = 3, 4, \cdots$, $p = 1, 2, \cdots, m - 1$, $q = 1, 2, \cdots, p$, (discrete series representation) [KR87][DMS97]. In this case the representation space V is denoted by $\mathcal{H}^{\mathrm{fin}}(c, h)$. We denote by $\mathcal{H}(c, h)$ the Hilbert space completion of the vector space $\mathcal{H}^{\mathrm{fin}}(c, h)$ associated with the unique irreducible unitary positive energy representation of Vir with central charge c and lowest weight h.

In these representations, the conformal Hamiltonian $\pi(L_0)$ is diagonalized, and on the linear span of its eigenvectors $\mathcal{H}^{\mathrm{fin}}(c, h)$ (the space of finite energy vectors), the Virasoro algebra acts algebraically as unbounded operators.

1.2.2 The stress-energy tensor.

Let $\mathcal{H}(c,h)$ as above and, with abuse of notation, we denote by L_n the elements of Vir represented in $\mathcal{H}(c,h)$. For a smooth complex-valued function f on S^1 with finitely many non-zero Fourier coefficients, the (chiral) stress-energy tensor associated with f is the operator

$$T(f) = \sum_{n\in\mathbb{Z}} L_n \hat{f}_n$$

acting on $\mathcal{H}(c,h)$, where

$$\hat{f}_n = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-in\theta} f(e^{i\theta}).$$

by the linear energy bounds, yielding a self-adjoint unbounded operator $T(f)$. Moreover it can be extended to a particular class of non-smooth functions [CW05], retaining its self-adjointness. This fact will be used in this thesis and will be thus resumed in some detail in Section 1.2.3.

It is a crucial fact that the irreducible representations $\mathcal{H}(c,h)$ of Vir integrate to irreducible unitary strongly continuous representations of the universal covering of the Bott-Virasoro group [FH05]. In other words, denoting by q the quotient map $q : \mathcal{U}(\mathcal{H}(c,h)) \to \mathcal{U}(\mathcal{H}(c,h))/\mathbb{C}$ (we denote by $\mathcal{U}(\mathcal{K})$ the group of unitary operators on $\mathcal{K}$), there is an irreducible, unitary, strongly continuous multiplier representation U of $\widetilde{\mathrm{Diff}_+(S^1)}$, the universal covering of $\mathrm{Diff}_+(S^1)$, such that

$$q(U(\mathrm{Exp}(f))) = q(e^{iT(f)})$$

for all $f \in \mathrm{Vect}(S^1)$.

For the stress-energy tensor T, we have the following covariance [FH05, Proposition 5.1, Proposition 3.1].

Proposition 1.2.8. *The stress-energy tensor T on $\mathcal{H}(c,h)$ transforms according to*

$$U(\gamma)T(f)U(\gamma)^* = T(\dot{\gamma}_*(f)) + \frac{c}{24\pi}\int_0^{2\pi} \{\dot{\gamma}, z\}\Big|_{z=e^{i\theta}} f(e^{i\theta})e^{i2\theta}d\theta$$

on vectors in $\mathcal{H}^{\mathrm{fin}}(c,h)$, for $f \in \mathrm{Vect}(S^1)$ and $\gamma \in \widetilde{\mathrm{Diff}_+(S^1)}$. Furthermore the commutation relations

$$i[T(g),T(f)] = T(g'f - f'g) + c\omega(g,f),$$

hold for arbitrary $f,g \in C^\infty(S^1)$, on vectors $\psi \in \mathcal{H}^{\mathrm{fin}}(c,h)$.

Here

$$\{\dot{\gamma}, z\} = \frac{\frac{d^3}{dz^3}\dot{\gamma}(z)}{\frac{d}{dz}\dot{\gamma}(z)} - \frac{3}{2}\left(\frac{\frac{d^2}{dz^2}\dot{\gamma}(z)}{\frac{d}{dz}\dot{\gamma}(z)}\right)^2$$

is the Schwarzian derivative of $\dot\gamma$ and $\frac{d}{dz}\dot\gamma(z) = -i\bar z \frac{d}{d\theta}\dot\gamma(e^{i\theta})\Big|_{e^{i\theta}=z}$. Note that

$$\beta(\gamma, f) := \frac{c}{24\pi}\int_{S^1}\{\dot\gamma, z\} izf(z)dz$$

and $\omega(\cdot,\cdot)$ are related by

$$\frac{d}{dt}\beta(\mathrm{Exp}(tf), g)\Big|_{t=0} = -c\omega(f,g). \tag{1.2.1}$$

If we consider the Cayley transform (1.2.2)(1.2.3), a vector field $f \in \mathrm{Vect}(S^1)$ in real line coordinates is given by

$$C_*(f)(t) = \frac{2}{(1+t^2)} f(C^{-1}(t)).$$

With the Schwarz class functions $\mathscr{S}(\mathbb{R})$, the stress energy tensor satisfies the following quantum-energy inequalities [FH05, Theorem 4.1].

Theorem 1.2.9. *Let $f \in \mathrm{Vect}(S^1)$ with $C_*(f) \in \mathscr{S}(\mathbb{R})$ and $C_*(f)(t) \geq 0$ $\forall t \in \mathbb{R}$. For $\psi \in \mathscr{D}(L_0)$, it holds that*

$$(\psi, T(f)\psi) \geq -\frac{c}{12\pi}\int_{\mathbb{R}}\left(\frac{d}{dt}\sqrt{C_*(f)(t)}\right)^2 dt,$$

where the derivative is given by

$$\frac{d}{dt}\sqrt{C_*(f)(t)} = \begin{cases} (\frac{d}{dt}C_*(f)(t))/(2\sqrt{C_*(f)(t)}) & \text{if } C_*(f)(t) \neq 0 \\ 0 & \text{if } C_*(f)(t) = 0. \end{cases}$$

1.2.3 The stress-energy tensor on non-smooth vector fields

Let T be the stress-energy tensor on $\mathcal{H}(c,h)$. Given a not necessarily smooth real function f of S^1 it is possible to evaluate the stress-energy tensor on f [CW05, Proposition 4.5]. First of all we define for a real-valued function f of the circle

$$\|f\|_{\frac{3}{2}} := \sum_{n\in\mathbb{Z}} |\hat f_n|(1+|n|^{\frac{3}{2}}),$$

where $\hat f_n := \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(e^{i\theta})d\theta$ is the nth Fourier coefficient of f.

Definition 1.2.10. We denote with $\mathcal{S}_{\frac{3}{2}}(S^1,\mathbb{R})$ the class of functions $f \in L^1(S^1,\mathbb{R})$ such that $\|f\|_{\frac{3}{2}}$ is finite endowed with the topology induced by the norm $\|\cdot\|_{\frac{3}{2}}$.

The following is [CW05, Proposition 4.2, Theorem 4.4, Proposition 4.5].

Proposition 1.2.11. *If $f : S^1 \to \mathbb{C}$ is continuous and such that $\sum_{n\in\mathbb{Z}} |\hat{f}_n|(1 + |n|^{\frac{3}{2}}) < \infty$ then*

a) *the operator $T(f) = \sum_{n\in\mathbb{Z}} L_n \hat{f}_n$ on the domain $\mathcal{H}^{\mathrm{fin}}(c,h)$ is well defined, (i.e. the sum is strongly convergent on the domain);*

b) *$T(f)^*$ is an extension of the operator $T(f)^+ := \sum_{n\in\mathbb{Z}} L_n \bar{\hat{f}}_n$ (this is again understood as an operator on the domain $\mathcal{H}^{\mathrm{fin}}(c,h)$).*

c) *$T(f)$ is closable and $\overline{T(f)} = (T(f)^+)^*$, where $T(f)$ and $T(f)^+$ are considered as operators on the domain $\mathcal{H}^{\mathrm{fin}}(c,h)$. In particular, if $\hat{f}_n = \bar{\hat{f}}_{-n}$ for all $n \in \mathbb{Z}$ (i.e. if f is a real-valued function), then $T(f)$ is essentially self-adjoint on $\mathcal{H}^{\mathrm{fin}}(c,h)$.*

d) *If f is real, then for every $\xi \in \mathscr{D}(L_0)$ we have the following energy bounds*

$$\|T(f)\xi\| \leq r\|f\|_{\frac{3}{2}}\|(1 + L_0)\xi\|$$

where r is a positive constant. Consequently, $\mathscr{D}(L_0) \subset \mathscr{D}(T(f))$.

e) *If $\{f_n\}$ ($n \in \mathbb{N}$) is a sequence[2] of continuous real functions on S^1 of finite $\|\cdot\|_{\frac{3}{2}}$ norm and $\|f - f_n\|_{\frac{3}{2}}$ converges to 0 as n tends to ∞, then*

$$T(f_n) \to T(f)$$

in the strong resolvent sense.

It has been also shown that the class $\mathcal{S}_{\frac{3}{2}}(S^1, \mathbb{R})$ contains many non-smooth functions [Wei06, Lemma 2.2],[CW05, Lemma 5.3].

Proposition 1.2.12. *If a real-valued function f on the circle is piecewise smooth and once continuously differentiable on the whole S^1, then $f \in \mathcal{S}_{\frac{3}{2}}(S^1, \mathbb{R})$.*

1.3 Loop groups

Let G be a finite dimensional Lie group. The group of smooth maps from S^1 to G is denoted by LG. With ΛG we denote the group of smooth maps $\mathbb{R} \to G$ with compact support which is a subgroup of LG by embedding the real line in S^1 by Cayley transform.

The loop group LG is an infinite dimensional Lie group (see [Mil84]) with Lie algebra $L\mathfrak{g}$ consisting of smooth maps from S^1 to $\mathfrak{g}$. We want to study central

[2]This should be distinguished from the Fourier coefficients $\hat{f}_n$ of a single function f.

extensions of $L\mathfrak{g}$ or equivalently 2-cocycles. The important fact about 2-cocycles of $L\mathfrak{g}$ is that if $\mathfrak{g}$ is semisimple every continuous G-invariant 2-cocycle ω has the form

$$\omega(x,y) = \frac{1}{2\pi}\int_0^{2\pi} \langle x(\theta), y'(\theta)\rangle d\theta$$

where $\langle\cdot,\cdot\rangle$ is a symmetric invariant form on $\mathfrak{g}$. So the study of 2-cocycles for $L\mathfrak{g}$ reduces to the much simpler analysis of the symmetric invariant forms of $\mathfrak{g}$ which is a finite dimensional Lie algebra.

Theorem 1.3.1. *Let G be a compact, connected and simply connected Lie group. Then*

(i) a 2-cocycle ω on $L\mathfrak{g}$ gives rise to an extension of LG if and only if $[\omega/2\pi] \in H^2(LG, \mathbb{Z})$.

(ii) In this case the group extension $\widetilde{LG}$ is unique.

If G is a simple Lie group, i.e. has a simple Lie algebra $\mathfrak{g}$, then all the invariant inner products are proportional. The smallest one satisfying the integrality condition $\langle h_\alpha, h_\alpha\rangle \in 2\mathbb{Z}$ for every coroot h_α is called basic inner product and we denote it with the symbol $\langle\cdot,\cdot\rangle_{basic}$. It characterized by the following relation

$$\langle h_\alpha, h_\alpha\rangle_{basic} = 2$$

where α is the highest root and h_α is the associated coroot. The associated 2-cocycle of LG is denoted with ω_{basic}. Given an extension $\widetilde{LG}$, we define the **level** ℓ as the scalar in $\mathbb{Z}_+$ such that $\omega = \ell\omega_{basic}$.

Definition 1.3.2. A projective unitary representation of LG on a Hilbert space $\mathcal{H}$ is a map $U : LG \to \mathcal{U}(\mathcal{H})$ such that

$$U(g)U(h) = c(g,h)U(gh) \tag{1.3.1}$$

where $c(\cdot,\cdot)$ is a 2-cocycle of LG. A projective unitary representation of LG on $\mathcal{H}$ is said to satisfy the positive-energy condition if there exists a strongly continuous unitary representation R of $\mathbb{T}$ on the same Hilbert space with positive generator such that

$$R(\varphi)U(g)R(\varphi)^* = U(\tilde{R}(\varphi)g) \tag{1.3.2}$$

for all $g \in LG$ and $\varphi \in \mathbb{T}$, where $\tilde{R}(\theta)g(e^{i\theta}) := g(e^{i(\theta-\varphi)})$.

Correspondingly, a representation V of ΛG in $\mathcal{U}(\mathcal{H})$ has positive-energy if there exists a strongly continuous unitary representation T of the one parameter group of translations which intertwines V, i.e.

$$T(t)V(f)T(t)^* = V(\tilde{T}(t)f) \tag{1.3.3}$$

where $\tilde{T}(t)f(x) = f(x+t)$.

We have that [PS86][Proposition 9.2.6]

Proposition 1.3.3. *The restriction to ΛG of a positive energy representation of LG is a positive energy representation of ΛG.*

The interest in positive energy representations of loop group is partially motivated by the following facts [PS86][Theorem 9.3.1]:

Theorem 1.3.4. *A positive energy representation of LG is*

(i) completely reducible, i.e. is a direct sum of irreducible representations;

(ii) has an intertwining action of $\mathrm{Diff}_+(S^1)$.

In special cases we have the following classification result about the irreducible positive energy representations of LG [Was98][Corollary, section 9].

Theorem 1.3.5. *If G is a compact, simple and connected Lie Group, an irreducible positive energy representation of LG is uniquely determined by the level ℓ determined by the cocycle in 1.3.1 and by the lowest eigenspace $H(0)$ of L_0.*

1.4 Groups of diffeomorphisms of Sobolev class H^s

We introduce (see [EK14, Section 2] and [EK14, Definition 2.2], respectively)

$$H^s(S^1) := \{f \in L^2(S^1) : \|f\|_{H^s} < \infty\}, \text{ where } \|f\|_{H^s} := \left(\sum_{n\in\mathbb{Z}}(1+n^2)^s|\hat{f}_n|^2\right)^{\frac{1}{2}}$$

$$\mathcal{D}^s(S^1) := \{\gamma \in \mathrm{Diff}^1_+(S^1) : \tilde{\gamma} - \iota \in H^s\},$$

where $\tilde{\gamma}$ is a lift of γ to $\mathbb{R}$.

It is easy to see that $\mathrm{Diff}^k_+(S^1)$ is continuously embedded in $\mathcal{D}^k(S^1)$. and by the Sobolev-Morrey embedding [IKT13, Proposition 2.2], it follows that $\mathcal{D}^s \hookrightarrow \mathrm{Diff}^k_+(S^1)$ if $s > k + \frac{1}{2}$.

From [IKT13, Lemma 2.3] and [IKT13, Lemma B.4] we have that:

Lemma 1.4.1. *Let $s > \frac{1}{2}$. Then $H^s(S^1)$ is an algebra and $\|fg\|_{H^s} \leq C_s\|f\|_{H^s}\|g\|_{H^s}$. If $g \in H^s(S^1)$ and $\inf_\theta(1 + g(\theta)) > 0$, then $\frac{1}{1+g} \in H^s(S^1)$.*

The following is a special case of [IKT13, Theorem B.2] and an analogue of [IKT13, Proposition B.7]. According to [Kol13, P.12], Lemma 1.4.2(a) for integer s has been first established in [Ebi68].

Lemma 1.4.2. *Let $s > \frac{3}{2}$. Then*

a) $(\gamma, f) \mapsto f \circ \gamma$, $\mathcal{D}^s(S^1) \times H^s(S^1) \to H^s(S^1)$ *is continuous.*

b) $\gamma \mapsto \gamma^{-1}$, $\mathcal{D}^s(S^1) \to \mathcal{D}^s(S^1)$ *is continuous.*

c) $\mathcal{D}^s(S^1)$ *is a topological group.*

By applying these results, we obtain the following

Lemma 1.4.3. *We have the following.*

a) *Let* $s > 2$. *The embedding* $H^s(S^1) \hookrightarrow \mathcal{S}_{\frac{3}{2}}(S^1)$ *is continuous.*

b) *Let* $s > \frac{3}{2}$. *The map*

$$\mathcal{D}^{s+1}(S^1) \times H^s(S^1) \to H^s(S^1)$$
$$(\gamma, f) \mapsto \gamma_*(f),$$

where $\gamma_*(f)$ *is as in* (1.2.1), *is continuous.*

c) *Let* $s > 3$. $\beta(\gamma, f)$ *extends continuously to* $\gamma \in \mathcal{D}^s(S^1), f \in L^2(S^1)$.

Proof. (a) follows from

$$\sum_{k\neq 0} |\hat{f}_k||k|^{\frac{3}{2}} = \sum_{k\neq 0} |\hat{f}_k||k|^{2+\epsilon} \frac{1}{|k|^{\frac{1}{2}+\epsilon}} \leq \sqrt{\sum_{k\neq 0} \frac{1}{k^{1+2\epsilon}}} \sqrt{\sum_{k\neq 0} |\hat{f}_k|^2 |k|^{4+2\epsilon}}.$$

for any $\epsilon > 0$.

(b) follows from Lemmas 1.4.2 and 1.4.1 and (1.2.1).

(c) Note that, with $s > 3$, $\mathcal{D}^s(S^1) \ni \gamma \mapsto \{\dot{\gamma}, z\} \in L^2(S^1)$ is continuous. To see it, in the definition

$$\{\dot{\gamma}, z\} = \frac{\frac{d^3}{dz^3}\dot{\gamma}(z)}{\frac{d}{dz}\dot{\gamma}(z)} - \frac{3}{2}\left(\frac{\frac{d^2}{dz^2}\dot{\gamma}(z)}{\frac{d}{dz}\dot{\gamma}(z)}\right)^2,$$

the maps $\gamma \mapsto \frac{d^3}{dz^3}\dot{\gamma}(z) \in L^2(S^1)$ and $\gamma \mapsto \frac{1}{\frac{d}{dz}\dot{\gamma}(z)} \in H^{s-1}(S^1) \subset L^\infty(S^1)$ are continuous, hence their product is continuous in $L^2(S^1)$. The second derivative $\gamma \mapsto \frac{d^2}{dz^2}\dot{\gamma}(z) \in H^{s-2}(S^1)$ is continuous hence so is $\gamma \mapsto \left(\frac{\frac{d^2}{dz^2}\dot{\gamma}(z)}{\frac{d}{dz}\dot{\gamma}(z)}\right)^2 \in H^{s-2}(S^1)$ (by Lemma 1.4.1), hence we obtain the continuity of $\gamma \mapsto \{\dot{\gamma}, z\}$ by Lemma 1.4.1. Now the claim is immediate because $\beta(\gamma, f) = \frac{c}{24\pi}\int_{S^1}\{\dot{\gamma}, z\} i z f(z) dz$ □

Chapter 2

Conformal Nets

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2.1 Möbius covariant net

Let $\mathcal{I}$ be the set consisting of all open, non-empty, non-dense and connected subsets of the circle S^1. For a given $I \in \mathcal{I}$, we denote with I' the interior of the complement of I, namely $(S^1 \setminus I)^\circ$.

A **Möbius covariant net** on S^1 is a triple $(\mathcal{A}, U, \Omega)$ where $\mathcal{A} := \{\mathcal{A}(I)\}_{I\in\mathcal{I}}$ is a family of von Neumann algebras on a fixed complex Hilbert space $\mathcal{H}$ indexed by elements of $\mathcal{I}$, U is strongly continuous unitary representation of $\mathrm{PSL}(2, \mathbb{R})$ always on $\mathcal{H}$, and Ω is a vector of $\mathcal{H}$, which together satisfy the following properties:

A.1 Isotony: $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$, if $I_1 \subset I_2$, $I_1, I_2 \in \mathcal{I}$.

A.2 Locality: $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)'$, if $I_1 \cap I_2 = \emptyset$, $I_1, I_2 \in \mathcal{I}$.

A.3 Möbius covariance: for $g \in \mathrm{PSL}(2, \mathbb{R})$, $I \in \mathcal{I}$,

$$U(g)\mathcal{A}(I)U(g)^{-1} = \mathcal{A}(gI)$$

where $\mathrm{PSL}(2,\mathbb{R})$ acts on S^1 by Möbius transformations.

A.4 Positivity of the energy: the representation U has positive energy, i.e. the conformal Hamiltonian L_0, which is the generator of the one-parameter group of rotations and is defined by the relation $U(R_\theta) = e^{i\theta L_0}$, has non-negative spectrum.

A.5 Existence of the vacuum vector: up to a scalar, there exists a unique vector $\Omega \in \mathcal{H}$ which is invariant for the action of $\mathrm{PSL}(2,\mathbb{R})$, i.e. $U(g)\Omega = \Omega$ for all $g \in \mathrm{PSL}(2,\mathbb{R})$.

A.6 Ciclicity of the vacuum: Ω is cyclic for the algebra generated by all the local algebras, $\mathcal{A}(S^1) := \bigvee_{I\in\mathcal{I}} \mathcal{A}(I)$.

The uniqueness of the vacuum is equivalent to the irreducibility of the net in the following sense, see [GL96, Proposition 1.2]:

Proposition 2.1.1. *The following properties for a Möbius covariant net $(\mathcal{A}, U, \Omega)$ are equivalent:*

i) $\mathbb{C}\Omega$ *are the only U-invariant vectors.*

ii) The local algebras $\mathcal{A}(I)$, $I \in \mathcal{I}$ are type III_1 factors.

iii) If I_n is a family of intervals in $\mathcal{I}$ which intersects in one point, then $\wedge_n \mathcal{A}(I_n) = \mathbb{C}$.

iv) The net $(\mathcal{A}, U, \Omega)$ is irreducible, in the sense that the von Neumann algebra generated by all the local algebras $\vee_{I\in\mathcal{I}}\mathcal{A}(I)$ is equal to $\mathcal{B}(\mathcal{H})$.

Remark 2.1.2. Suppose to have a triple $(\mathcal{A}, U, \Omega)$ satisfying all the axioms except for axiom **A.6**. We can always obtain an irreducible Möbius covariant net taking the restriction to the space $\mathcal{H}_\mathcal{A} := \overline{\mathcal{A}(S^1)\Omega}$.

The following properties are a consequence of the axioms:

Theorem 2.1.3. *Let $(\mathcal{A}, U, \Omega)$ Möbius covariant net. The following properties are automatic [GF93][Theorem 2.19 ii), Corollary 2.8]*

- **Additivity:** *if $I \in \mathcal{I}$ is an interval and I_n is a collection of intervals in $\mathcal{I}$ such that $I = \cup_n I_n$, then $\mathcal{A}(I) \subset \bigvee_n \mathcal{A}(I_n)$.*
- **Haag duality:** *for every $I \in \mathcal{I}$, $\mathcal{A}(I') = \mathcal{A}(I)'$.*
- **Semicontinuity:** *if $I_n \in \mathcal{I}$ is a decreasing family of intervals and $I = (\bigcap_n I_n)^\circ$ then $\mathcal{A}(I) = \bigwedge_n \mathcal{A}(I_n)$.*

- **Reeh-Schlieder property:** *the vacuum vector* Ω *is cyclic and separating for each* $\mathcal{A}(I)$.

It must be stressed that if $(\mathcal{A}, U, \Omega)$ is a Möbius covariant net, using the Reeh-Schlieder property 2.1.3 we can associate to each local algebra $\mathcal{A}(I)$ the modular operator Δ_I using the Tomita-Takesaki modular theory. It is an important fact that the representation U is completely characterized by the local algebras $\{\mathcal{A}(I)\}$ and the vacuum vector Ω:

Theorem 2.1.4. Bisognano-Wichmann property: *the modular operator* Δ_I *associated to* $\mathcal{A}(I)$ *with respect to the vacuum vector* Ω *has a geometrical meaning in the following sense*

$$U(\delta_I(2\pi t)) = \Delta^{it}$$

where δ_I *is the one-parameter group of dilations associated to* I, *i.e. the elements in* $\mathrm{PSL}(2,\mathbb{R})$ *which preserve the interval* I.

2.1.1 Diffeomorphism covariant nets

By a conformal net (or diffeomorphism covariant net) we shall mean a Möbius covariant net which satisfies the following additional properties:

A.7 There exists a projective unitary representation U of $\mathrm{Diff}_+(S^1)$ on $\mathcal{H}$ extending the unitary representation of $\mathrm{PSL}(2,\mathbb{R})$ such that for all $I \in \mathcal{I}$ we have

$$U(\gamma)\mathcal{A}(I)U(\gamma)^* = \mathcal{A}(\gamma I), \quad \gamma \in \mathrm{Diff}_+(S^1),$$

and

$$U(\gamma)xU(\gamma)^* = x, \quad x \in \mathcal{A}(I), \gamma \in \mathrm{Diff}_+(I') \tag{2.1.1}$$

where $\mathrm{Diff}_+(I')$ denotes the subgroup of diffeomorphisms γ such that $\gamma(z) = z$ for all $z \in I$.

2.2 Representation theory

2.2.1 DHR representations

Definition 2.2.1. A **representation** (or **DHR representation**)π of a conformal net $\mathcal{A}$ is a family of maps

$$I \in \mathcal{I} \to \pi_I,$$

where π_I ia representation of the von Neumann algebra $\mathcal{A}(I)$ on a fixed Hilbert space $\mathcal{H}_\pi$, with the isotony property

$$\pi_J|_{\mathcal{A}(I)} = \pi_I, \quad I \subset J.$$

If the representations π_I are normal for every $I \in \mathcal{I}$ we say tha the representation π il **locally normal**. The representation π is automatically locally normal if the Hilbert space $\mathcal{H}_\pi$ is separable [Tak02, Theorem 5.1].

We say that two representations π and ρ are **equivalent** if there exists an intertwining unitary operator U from $\mathcal{H}_\pi$ and $\mathcal{H}_\rho$, i.e. $U\pi_I(x) = \rho_I(x)U$ for every $x \in \mathcal{A}(I)$ and $I \in \mathcal{I}$. The unitary equivalence class of a representation π of a net $\mathcal{A}$ is denoted with $[\pi]$ and the unitary equivalence classes of irreducible representations are called sectors, where a representation π is irreducible if and only if $\bigwedge \pi_I(\mathcal{A}(I))' = \mathbb{C}\mathbb{1}$.

The vacuum representation π_0 on $\mathcal{H}_{\pi_0} := \mathcal{H}$ is $\pi_0(x) := x$, for every $x \in \mathcal{A}(I)$, $I \in \mathcal{I}$. We say that a representation π on the vacuum Hilbert space $\mathcal{H}$ is localized in $I \in \mathcal{I}$ if $\pi|_{\mathcal{A}(I')} = \mathrm{id}|_{\mathcal{A}(I')}$. It follows from Haag duality that $\pi_J(\mathcal{A}(J)) \subset \mathcal{A}(J)$ for every $J \subset I$, in other words that π_J is and endomorphism of $\mathcal{A}(J)$. A representation π of a net $\mathcal{A}$ which is localized in some interval $I \in \mathcal{I}$ is said a localized endomorphism, and it turns out that if the representation space $\mathcal{H}_\pi$ is separable π is always unitary equivalent to a representation localized in an interval I, for every $I \in \mathcal{I}$.

The representation π is said to be **Möbius covariant** (resp.**diffeomorphism covariant**) if there exists a unitary strongly continuous projective representation U_π of the universal covering of the Möbius group (resp. of the universal covering of $\mathrm{Diff}_+(S^1)$) such that

$$U_\pi(g)\pi_I(x)U_\pi(g)^* = \pi_{\dot{g}I}(U(g)xU(g)^*),$$

for all $g \in \widetilde{\mathrm{PSL}(2,\mathbb{R})}$ (resp $g \in \widetilde{\mathrm{Diff}_+(S^1)}$), where $\dot{g}$ is the image of g in $\mathrm{PSL}(2,\mathbb{R})$ (resp. $\mathrm{Diff}_+(S^1)$) under the covering map.

2.2.2 Soliton representations

Let $\mathcal{I}_\mathbb{R}$ be the class of elements consisting of open, non-empty, connected subsets of the real line $\mathbb{R}$, identified with $S^1 \setminus \{-1\}$ via Cayley transform. Namely, $\mathcal{I}_\mathbb{R}$ is the family of bounded open intervals and of open half-lines of $\mathbb{R}$.

Definition 2.2.2. A soliton σ of a conformal net $\mathcal{A}$ is a map

$$I \in \mathcal{I}_\mathbb{R} \to \sigma_I$$

where σ_I is a normal representation of the von Neumann algebra $\mathcal{A}(I)$ on a fixed Hilbert space $\mathcal{H}_\sigma$ with the isotony property

$$\sigma_J|_{\mathcal{A}(I)} = \sigma_I, \quad I \subset J.$$

We say thet the soliton π is **proper** if there is no representation of the conformal net $\mathcal{A}$ which agrees with π when restricted to the family of intervals $\mathcal{I}_\mathbb{R}$.

Definition 2.2.3. A soliton σ of $\mathcal{A}$ on the Hilbert space $\mathcal{H}_\sigma$ is B_0-**covariant** if there is a unitary projective representation U_σ of B_0 (the universal cover of B_0) on $\mathcal{H}_\sigma$ such that

$$\mathrm{Ad}\, U_\sigma(\gamma)(\sigma_I(x)) = \sigma_{\gamma(I)}(\mathrm{Ad}\, U_0(\gamma)(x)) \tag{2.2.1}$$

with $x \in \mathcal{A}(I)$ and U_0 is the unitary projective representation of $\mathrm{Diff}_+(S^1)$ restricted to B_0. In addition, we say that the soliton σ has positive-energy if the unitary projective representation U_σ above can be choosen in such a way that the restriction to the one-parameter subgroup of translations of B_0 lifts to a true strongly continuous representation which has a positive self-adjoint generator.

2.3 Subnets

A conformal subnet of a conformal net $(\mathcal{A}, U, \Omega)$ on $\mathcal{H}_\mathcal{A}$ consists of a family $\mathcal{B} = \{\mathcal{B}(I)\}_{I \in \mathcal{I}}$ of von Neumann algebras always acting on $\mathcal{H}$ such that

1. $\mathcal{B}(I) \subset \mathcal{A}(I)$ for every $I \in \mathcal{I}$,
2. $\mathcal{B}(I) \subset \mathcal{B}(J)$ if $I \subset J$, $I, J \in \mathcal{I}$,
3. $U(g)\mathcal{B}(I)U(g)* = \mathcal{B}(gI)$ if $I \in \mathcal{I}$ and $g \in \mathrm{PSL}(2, \mathbb{R})$.

Note that $(\mathcal{B}, U, \Omega)$ it is not a Möbius covariant net because in general does not satisfy axiom **A.6**. We can always obtain a Möbius covariant net from $(\mathcal{B}, U, \Omega)$. Consider the Hilbert space $\mathcal{H}_\mathcal{B} := \overline{\bigvee_I \mathcal{B}(I)\Omega} \subset \mathcal{H}_\mathcal{A}$. We define the family $\hat{\mathcal{B}} := \{\hat{B}(I)\}$, where with $\hat{\mathcal{B}}(I)$ we mean the restriction of all the operators in $\mathcal{B}(I)$ to the subspace $\mathcal{H}_\mathcal{B}$. In a similar fashion, we define $\hat{U} := U|_{\mathcal{H}_\mathcal{B}}$ as the restriction of the representation U to the subspace $\mathcal{H}_\mathcal{B}$. The triple $(\hat{\mathcal{B}}, \hat{U}, \Omega)$ is a Möbius covariant net on $\mathcal{H}_\mathcal{B}$, see remark 2.1.2. By the Reeh-Schlieder property 2.1.3 the map $\mathcal{B}(I) \ni b \mapsto b|_{\mathcal{H}_\mathcal{B}} \in \hat{\mathcal{B}}(I)$ is an isomorphism of von Neumann algebras.

2.4 The Virasoro net

The Virasoro net with central charge c is the conformal net induced by the Vir representation $\mathcal{H}(c, h)$

$$\mathcal{A}_{(\mathrm{Vir},c)}(I) = \{e^{iT_{(c,0)}(f)} : f \in C^\infty(S^1), \text{real-valued}, \mathrm{supp} f \subset I\}''$$

It enjoys all the listed properties in the definition of a conformal net

It's representation theory is completely understood.

Proposition 2.4.1. *If U is a strongly continuous positive energy projective unitary irreducible representation of* $\mathrm{Diff}_+(S^1)$ *on a Hilbert space $\mathcal{H}$ (which is necessarily separable) then U is unitarily equivalent to $U_{(c,h)}$ which is the unique unitary projective representation obtained by the integration of the module $L(c,h)$.*

2.5 Loop group conformal net

From the class of irreducible positive energy representations of LG it is possible to choose a particular subclass, the irreducible vacuum representations, which have a unique lowest eigenvalue vector for L_0 which is invariant for the action of $\mathrm{PSL}(2,\mathbb{R})$. If we fix the level ℓ we have only one irreducible vacuum representation for LG [GF93][Section III.8].

Definition 2.5.1. If $U_{\ell,0}$ is the vacuum representation of level ℓ then the family of von Neumann algebras

$$\mathcal{A}_{G,\ell}(I) := \{U_{\ell,0}(f) : \mathrm{supp}\,(f) \subset I\}'' \tag{2.5.1}$$

is a conformal net, where the vacuum vector Ω is the lowest eigenvalue vector of L_0 and the diffeomorphism covariance follows from 1.3.4, see [GF93][Theorem 3.2].

In the case of $G = SU(N)$ we mention the following facts [Was98][Theorem B, Section 17], [Tan18]:

Theorem 2.5.2. *Let $G = SU(N)$ and U_{ℓ_1,h_1}, U_{ℓ_2,h_2} be two irreducible positive energy representations of $LSU(N)$ of level ℓ_1 and ℓ_2 and lowest weights h_1 and h_2 respectively. Then $\ell_1 = \ell_2$ if and only if U_{ℓ_1,h_1} and U_{ℓ_2,h_2} are locally equivalent, namely, for every interval I of S^1 there exist a unitary operator W_I such that*

$$W_I U_{\ell_1,h_1}(g) W_I^* = U_{\ell_2,h_2}(g)$$

when $supp(g) \subset I$.

Theorem 2.5.3. *Let $G = SU(N)$. There exists a one-to-one correspondence between the irreducible positive energy representations of level ℓ of $LSU(N)$ and irreducible representations of the conformal net $\mathcal{A}_{SU(N),\ell}$.*

The one-to-one correspondence is given by

$$U_{\ell,h} \mapsto \pi_{\ell,h}, \tag{2.5.2}$$

where $\pi_{\ell,h}(U_{\ell,0}(g)) := U_{\ell,h}(g)$ for $\mathrm{supp}\,(g) \subset I$ and $\pi_{\ell,h}$ is extended to $\mathcal{A}_{SU(N),\ell}(I)$ by local equivalence, Theorem 2.5.2.

Chapter 3

Extension of the $\mathrm{Diff}_+(S^1)$ representations to Sobolev diffeomorphisms

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3.1 Irreducible case

Our purpose is to extend the positive energy projective representation U on $\mathcal{H}(c,h)$ of $\mathrm{Diff}_+(S^1)$ to $\mathcal{D}^s(S^1)$ with $s > 3$. In the following $s > 3$ will be always assumed.

An element $\gamma \in \mathcal{D}^s(S^1)$ acts on $f \in \mathrm{Vect}(S^1)$ via (1.2.1). If T is the energy-momentum operator associated with a positive energy unitary representation of the Virasoro algebra Vir with central charge c and lowest weight h, we define a new class of operators

$$T^\gamma(f) \coloneqq T(\gamma_* f) - \beta(\gamma, f),$$

where $f \in \mathrm{Vect}(S^1)$ and $\beta(\gamma, f) = \frac{c}{24\pi}\int_{S^1}\{\gamma, z\} izf(z)dz$, which makes sense for $\gamma \in \mathcal{D}^s(S^1)$ by Lemma 1.4.3 and Proposition 1.2.11(*a*)). The fact that $\gamma_* f$ is in $\mathcal{S}_{\frac{3}{2}}(S^1, \mathbb{R})$ ensures that $T(\gamma_* f)$ is an essentially self-adjoint operator on $\mathcal{H}^{\mathrm{fin}}(c,h)$ and so is $T^\gamma(f)$ by Proposition 1.2.11(*c*)). We denote its closure by the same symbol $T^\gamma(f)$, so long as no confusion arises.

Note that, if $\gamma \in \mathrm{Diff}_+(S^1)$, then we have

$$T^\gamma(f) = \mathrm{Ad}\, U(\gamma)(T(f)). \tag{3.1.1}$$

Indeed, by definition $T^\gamma(f) = T(\gamma_* f) - \beta(\gamma, f)$ and by Proposition 1.2.8, (3.1.1) holds on $\mathscr{D}(L_0)$, and the both operators are essentially self-adjoint there, hence they must coincide. As they are unitarily implemented, the energy bound holds as well:

$$\|T^\gamma(f)\xi\| \le r\|f\|_{\frac{3}{2}} \cdot \|(1 + L_0^\gamma)\xi\|, \tag{3.1.2}$$

where $L_0^\gamma := T^\gamma(1)$.

We define for $\gamma_1, \gamma_2 \in \mathcal{D}^s(S^1)$

$$(T^{\gamma_1})^{\gamma_2}(f) := T^{\gamma_1}((\gamma_2)_* f) - \beta(\gamma_2, f).$$

Proposition 3.1.1. *Let $\gamma_1, \gamma_2 \in \mathcal{D}^s(S^1)$, $s > 3$, and $f \in \mathrm{Vect}(S^1)$. Then $(T^{\gamma_1})^{\gamma_2}(f) = T^{\gamma_1 \circ \gamma_2}(f)$.*

Proof. Using the properties of the Schwarzian derivative [OT05]

$$\{\gamma_1 \circ \gamma_2, z\} = \{\gamma_1, \gamma_2(z)\}\left(\frac{d}{dz}\gamma_2(z)\right)^2 + \{\gamma_2, z\}$$

where $y = \gamma_2(z)$, we infer that

$$\begin{aligned}
\beta(\gamma_1 \circ \gamma_2, f) &= -\frac{c}{24\pi}\int_0^{2\pi} \{\gamma_1 \circ \gamma_2, z\}\Big|_{z=e^{i\theta}} f(e^{i\theta})e^{i2\theta}d\theta \\
&= -\frac{c}{24\pi}\int_0^{2\pi} \{\gamma_1, y\}\Big|_{y=\gamma_2(e^{i\theta})}\left(\frac{d}{dz}\gamma_2(z)\right)^2\Big|_{z=e^{i\theta}} f(e^{i\theta})e^{i2\theta}d\theta \\
&\quad - \frac{c}{24\pi}\int_0^{2\pi} \{\gamma_2, z\}\Big|_{z=e^{i\theta}} f(e^{i\theta})e^{i2\theta}d\theta \\
&= -\frac{c}{24\pi}\int_0^{2\pi} \{\gamma_1, y\}\Big|_{y=e^{i\varphi}} \cdot (-i)\frac{d}{d\theta}\left(\gamma_2(e^{i\theta})\right)\Big|_{e^{i\theta}=\gamma_2^{-1}(e^{i\varphi})} f(\gamma_2^{-1}(e^{i\varphi}))e^{i\varphi}d\varphi \\
&\quad - \frac{c}{24\pi}\int_0^{2\pi} \{\gamma_2, z\}\Big|_{z=e^{i\theta}} f(e^{i\theta})e^{i2\theta}d\theta \\
&= -\frac{c}{24\pi}\int_0^{2\pi} \{\gamma_1, y\}\Big|_{y=e^{i\varphi}} \cdot (-i)e^{-i\varphi}\frac{d}{d\theta}\left(\gamma_2(e^{i\theta})\right)\Big|_{e^{i\theta}=\gamma_2^{-1}(e^{i\varphi})} f(\gamma_2^{-1}(e^{i\varphi}))e^{i2\varphi}d\varphi \\
&\quad - \frac{c}{24\pi}\int_0^{2\pi} \{\gamma_2, z\}\Big|_{z=e^{i\theta}} f(e^{i\theta})e^{i2\theta}d\theta \\
&= \beta(\gamma_1, \gamma_{2_*}(f)) + \beta(\gamma_2, f),
\end{aligned}$$

where we used the change of variables $e^{i\varphi} = \gamma_2(e^{i\theta})$, hence $e^{i\theta}\frac{d\theta}{d\varphi}\frac{d\gamma_2}{dz}(e^{i\theta})|_{\gamma_2(e^{i\theta})=e^{i\varphi}} = e^{i\varphi}$, $\frac{d\gamma_2}{dz}(e^{i\theta}) = -ie^{-i\theta}\frac{d}{d\theta}\gamma_2(e^{i\theta})$ and (1.2.1).

So $(T^{\gamma_1})^{\gamma_2}(f) = T((\gamma_1)_*((\gamma_2)_* f)) - \beta(\gamma_1, \gamma_{2*}f) - \beta(\gamma_2, f) = T((\gamma_1 \circ \gamma_2)_* f) - \beta(\gamma_1 \circ \gamma_2, f) = T^{\gamma_1 \circ \gamma_2}(f)$. □

Lemma 3.1.2. *$\mathscr{D}(L_0) = \mathscr{D}(L_0^\gamma)$ for every $\gamma \in \mathcal{D}^s(S^1)$, where $L_0^\gamma := T^\gamma(1)$ and here we denote by 1 the constant function with the value 1.*

Proof. By Lemma 3.2.2 we can take a sequence $\{\gamma_n\}$ in $\mathrm{Diff}_+(S^1)$ convergent to γ in the topology of $\mathcal{D}^s(S^1)$. We observe that $1 = \lim_n \gamma_{n*}(\gamma_*^{-1}(1))$ in the topology of $\mathcal{S}_{\frac{3}{2}}(S^1)$. For $\xi \in \mathscr{D}(L_0)$ we know from Proposition 1.2.11(*e*)) and (3.1.2) that

$$\begin{aligned}\|L_0\xi\| &= \lim_{n\to\infty} \| \left(T^{\gamma_n}((\gamma_*^{-1})(1)) + \beta(\gamma_n, \gamma_*^{-1}(1))\right)\xi\| \\ &\leq \left(\lim_{n\to\infty} r\|\gamma_*^{-1}(1)\|_{\frac{3}{2}} \cdot \|(1+L_0^{\gamma_n})\xi\| + |\beta(\gamma_n,\gamma_*^{-1}(1))|\|\xi\|\right) \\ &= r\|\gamma_*^{-1}(1)\|_{\frac{3}{2}} \cdot \|(1+L_0^{\gamma})\xi\| + |\beta(\gamma,\gamma_*^{-1}(1))|\|\xi\|,\end{aligned}$$

Recall that we know that $\mathscr{D}(L_0) \subset \mathscr{D}(L_0^\gamma)$ from Proposition 1.2.11(*d*)) and L_0^γ is essentially self-adjoint on $\mathscr{D}(L_0)$. From the above inequality, we infer that any sequence $\xi_n \in \mathscr{D}(L_0)$ converging to $\xi \in \mathscr{D}(L_0^\gamma)$ in the graph norm of L_0^γ is also convergent in the graph norm of L_0, and therefore, we have $\mathscr{D}(L_0^\gamma) = \mathscr{D}(L_0)$. □

Proposition 3.1.3 (energy bounds for T^γ)**.** *Let $\gamma \in \mathcal{D}^s(S^1)$. Then*

$$\|T^\gamma(f)\xi\| \leq r\|f\|_{\frac{3}{2}}\|(1+L_0^\gamma)\xi\|$$

for all $\xi \in \mathscr{D}(L_0)$.

Proof. Let $\{\gamma_n\}$ a sequence of elements in $\mathrm{Diff}_+(S^1)$ converging to $\gamma \in \mathcal{D}^s(S^1)$ as in Lemma 3.2.2. By Proposition 1.2.11(*e*)) and (3.1.2),

$$\begin{aligned}\|T^\gamma(f)\xi\| = \lim_{n\to\infty}\|T^{\gamma_n}(f)\xi\| \leq \lim_{n\to\infty} r\|f\|_{\frac{3}{2}}\|(1+L_0^{\gamma_n})\xi\| = \\ = r\|f\|_{\frac{3}{2}}\|(1+L_0^{\gamma})\xi\|,\end{aligned}$$

which is the desired inequality. □

Theorem 3.1.4. *T^γ yields an irreducible unitary positive energy representation of* Vir *with central charge c and lowest weight h on $\mathcal{H}(c,h)$.*

Proof. We are going to prove the Virasoro relations on $C^\infty(L_0^\gamma)$. For this purpose, we have to take under control the action of various exponentiated operators.

Computations on $\mathscr{D}(L_0)$. We start by noting that $e^{iT^\gamma(g)}\mathscr{D}(L_0) \subset \mathscr{D}(L_0)$. Indeed, using [FH05, Proposition 3.1] we have, for $\xi \in \mathscr{D}(L_0)$ and $\gamma_n \in \mathrm{Diff}_+(S^1)$ as in Lemma 3.2.2,

$$L_0 e^{iT^{\gamma_n}(g)}\xi = e^{iT^{\gamma_n}(g)}(T((\gamma_n \mathrm{Exp}(-g)\gamma_n^{-1})_*(1)) - \beta(\gamma_n\mathrm{Exp}(-g)\gamma_n^{-1}, 1))\xi,$$

and the right-hand side converges as $n \to \infty$ by Proposition 1.2.11(*e*)). Therefore, since both $e^{iT^{\gamma_n}(g)}\xi$ and $L_0 e^{iT^{\gamma_n}(g)}\xi$ are convergent, it follows that $e^{iT^{\gamma}(g)}\xi \in \mathscr{D}(L_0)$ and

$$L_0 e^{iT^{\gamma}(g)}\xi = e^{iT^{\gamma}(g)}(T((\gamma\mathrm{Exp}(-g)\gamma^{-1})_*(1)) - \beta(\gamma\mathrm{Exp}(-g)\gamma^{-1}, 1))\xi.$$

For vectors $\xi \in \mathscr{D}(L_0)$ and $\gamma_n \in \mathrm{Diff}_+(S^1)$, by Proposition 1.2.8 we have the operator equality

$$e^{iT^{\gamma_n}(g)}T^{\gamma_n}(f)e^{-iT^{\gamma_n}(g)} = T^{\gamma_n}(\mathrm{Exp}(g)_*(f)) - \left(\frac{c}{24\pi}\int_{S^1}\{\mathrm{Exp}(g), z\}izf(z)dz\right),$$

and we saw above that for $\xi \in \mathscr{D}(L_0)$ and $\gamma_n \in \mathrm{Diff}_+(S^1)$, it holds that $e^{-iT^{\gamma_n}(g)}\xi \in \mathscr{D}(L_0) \subset \mathscr{D}(T^{\gamma_n}(f))$, therefore, we have

$$e^{iT^{\gamma_n}(g)}T^{\gamma_n}(f)e^{-iT^{\gamma_n}(g)}\xi = T^{\gamma_n}(\mathrm{Exp}(g)_*(f))\xi - \left(\frac{c}{24\pi}\int_{S^1}\{\mathrm{Exp}(g), z\}izf(z)dz\right)\xi.$$

We apply to the operator equality the function

$$h_k : s \in \mathbb{R} \to s\chi_{(-k,k)}$$

where χ is the characteristic function of the interval $(-k, k) \subset \mathbb{R}$. By bounded functional calculus, we obtain for any $\xi \in \mathscr{D}(L_0)$

$$h_k(e^{iT^{\gamma_n}(g)}T^{\gamma_n}(f)e^{-iT^{\gamma_n}(g)})\xi = e^{iT^{\gamma_n}(g)}h_k(T^{\gamma_n}(f))e^{-iT^{\gamma_n}(g)}\xi, \tag{3.1.3}$$

and the right-hand side tends to $e^{iT^{\gamma}(g)}h_k(T^{\gamma}(f))e^{-iT^{\gamma}(g)}\xi$ as $n \to \infty$, because we have convergence of $T^{\gamma_n}(f)$ to $T^{\gamma}(f)$ and $T^{\gamma_n}(g)$ to $T^{\gamma}(g)$ in the strong resolvent sense, and their bounded functional calculus $e^{iT^{\gamma_n}(g)}, h_k(T^{\gamma_n}(f))$ converge to $e^{iT^{\gamma}(g)}, h_k(T^{\gamma_n}(f))$, respectively. On the other hand, the left-hand side of (3.1.3) can be rewritten as

$$h_k\left(T^{\gamma_n}(\mathrm{Exp}(g)_*(f)) - \frac{c}{24\pi}\int_{S^1}\{\mathrm{Exp}(g), z\}izf(z)dz\right)\xi$$

and this converges to

$$h_k\left(T^{\gamma}(\mathrm{Exp}(g)_*(f)) - \frac{c}{24\pi}\int_{S^1}\{\mathrm{Exp}(g), z\}izf(z)dz\right)\xi$$

as $n \to \infty$, again by the convergence of $\{T^{\gamma_n}(\mathrm{Exp}(g)_*(f))\}$ in the strong resolvent sense and bounded functional calculus with h_k. Altogether, we know that the following equality holds:

$$e^{iT^{\gamma}(g)}h_k(T^{\gamma}(f))e^{-iT^{\gamma}(g)}\xi = h_k\left(T^{\gamma}(\mathrm{Exp}(g)_*(f)) - \frac{c}{24\pi}\int_{S^1}\{\mathrm{Exp}(g), z\}izf(z)dz\right)\xi.$$

By taking the limit for $k \to \infty$, we get for every $\xi \in \mathscr{D}(L_0)$

$$e^{iT^\gamma(g)}T^\gamma(f)e^{-iT^\gamma(g)}\xi = T^\gamma(\mathrm{Exp}(g)_*(f))\xi - \left(\frac{c}{24\pi}\int_{S^1}\{\mathrm{Exp}(g), z\}izf(z)dz\right)\xi. \tag{3.1.4}$$

Recall that $\mathscr{D}(L_0) = \mathscr{D}(L_0^\gamma)$. We get in particular

$$e^{itL_0^\gamma}T^\gamma(f)e^{-itL_0^\gamma}\xi = T^\gamma(f_t)\xi, \tag{3.1.5}$$

where $f_t(e^{i\theta}) = f(e^{i(\theta-t)})$.

Computations on $C^\infty(L_0^\gamma)$. The right-hand side of (3.1.5) is differentiable with respect to t when $\xi \in \mathscr{D}(L_0)$ since for the right hand side we get

$$\lim_{t\to 0}\frac{1}{t}(T^\gamma(f_t) - T^\gamma(f))\xi = \lim_{t\to 0}T^\gamma(\tfrac{1}{t}(f_t - f))\xi = T^\gamma(-f')\xi = -T^\gamma(f')\xi,$$

by the continuity of T^γ in the topology of $\mathcal{S}_{\frac{3}{2}}(S^1)$ (Proposition 3.1.3). Let us specialize it to $\xi \in C^\infty(L_0^\gamma) := \bigcap_n \mathscr{D}((L_0^\gamma)^n)$. For the left-hand side of (3.1.5), we have

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=0} e^{itL_0^\gamma}T^\gamma(f)e^{-itL_0^\gamma}\xi \\
&= \lim_{t\to\infty}\left(\frac{1}{t}\left(e^{itL_0^\gamma}T^\gamma(f)e^{-itL_0^\gamma} - e^{itL_0^\gamma}T^\gamma(f)\right)\xi + \frac{1}{t}\left(e^{itL_0^\gamma}T^\gamma(f) - T^\gamma(f)\right)\xi\right).
\end{aligned} \tag{3.1.6}$$

The first term converges to $-iT^\gamma(f)L_0\xi$. Indeed, by Proposition 3.1.3,

$$\begin{aligned}
&\left\|\frac{1}{t}\left(e^{itL_0^\gamma}T^\gamma(f)e^{-itL_0^\gamma} - e^{itL_0^\gamma}T^\gamma(f)\right)\xi + ie^{itL_0^\gamma}T^\gamma(f)L_0^\gamma\xi\right\| \\
&= \left\|\frac{1}{t}\left(T^\gamma(f)e^{-itL_0^\gamma} - T^\gamma(f)\right)\xi + iT^\gamma(f)L_0^\gamma\xi\right\| \\
&\le r\|f\|_{\frac{3}{2}}\left\|(1+L_0^\gamma)\left(\frac{e^{-itL_0^\gamma}-1}{t} + iL_0^\gamma\right)\xi\right\| \\
&= r\|f\|_{\frac{3}{2}}\left\|\left(\frac{e^{-itL_0^\gamma}-1}{t} + iL_0^\gamma\right)(1+L_0^\gamma)\xi\right\|.
\end{aligned}$$

Since $\xi \in C^\infty(L_0^\gamma)$, by Stone's theorem [RS80, Theorem VIII.7(c)] the above converges to 0 as $t \to 0$. Thus the limit exists also for the second term of (3.1.6), and by applying Stone's theorem [RS80, Theorem VIII.7(d)], we get $T^\gamma(f)\xi \in \mathscr{D}(L_0^\gamma)$, and the second term converges to $iL_0^\gamma T^\gamma(f)\xi$. or in other words, $T^\gamma(f)C^\infty(L_0) \subset \mathscr{D}(L_0^\gamma)$ (actually, we proved $T^\gamma(f)\mathscr{D}((L_0^\gamma)^2) \subset \mathscr{D}(L_0^\gamma)$). Thus we have established the following commutation relation on $C^\infty(L_0^\gamma)$:

$$[L_0^\gamma, T^\gamma(f)]\xi = iT^\gamma(f')\xi. \tag{3.1.7}$$

It follows that $C^\infty(L_0^\gamma)$ is an invariant domain for every $T^\gamma(f)$ with $f \in C^\infty(S^1, \mathbb{R})$. Indeed, for $T^\gamma(f)\xi$, with $\xi \in C^\infty(L_0^\gamma)$ and $f \in C^\infty(S^1, \mathbb{R})$, (3.1.7) is equivalent to

$$L_0^\gamma T^\gamma(f)\xi = [L_0^\gamma, T^\gamma(f)]\xi + T^\gamma(f)L_0^\gamma\xi = iT^\gamma(f')\xi + T^\gamma(f)L_0^\gamma\xi. \tag{3.1.8}$$

Now we go by induction in k. Assume that $T^\gamma(f)\xi \in \mathscr{D}((L_0^\gamma)^k)$ and all $f \in C^\infty(S^1, \mathbb{R})$. It then follows from (3.1.8) that $L_0^\gamma T^\gamma(f)\xi \in \mathscr{D}((L_0^\gamma)^k)$, i.e. $T^\gamma(f)\xi \in \mathscr{D}((L_0^\gamma)^{k+1})$. We thus get the desired claim $T^\gamma(f)C^\infty(L_0^\gamma) \subset C^\infty(L_0^\gamma)$.

The Virasoro relations. Finally we show that the stress-energy tensor T^γ indeed yields a representation of $\mathrm{Vect}(S^1)$. For $\xi \in C^\infty(L_0^\gamma)$,

$$\begin{aligned}&\left.\frac{d}{dt}\right|_{t=0} e^{itT^\gamma(g)}T^\gamma(f)e^{-itT^\gamma(g)}\xi \\ &= \lim_{t\to 0}\left(\frac{1}{t}\left(e^{itT^\gamma(g)}T^\gamma(f)e^{-itT^\gamma(g)} - e^{itT^\gamma(g)}T^\gamma(f)\right) + \frac{1}{t}\left(e^{itT^\gamma(g)}T^\gamma(f) - T^\gamma(f)\right)\right)\xi.\end{aligned} \tag{3.1.9}$$

As for the left-hand side, from (3.1.4), we obtain $(T^\gamma(g'f - gf') + c\omega(g,f))\xi$ by (1.2.1).

Let us see the right-hand side of (3.1.9) term by term. As for the first term, we have

$$\begin{aligned}&\left\|\frac{1}{t}\left(e^{itT^\gamma(g)}T^\gamma(f)e^{-itT^\gamma(g)} - e^{itT^\gamma(g)}T^\gamma(f)\right)\xi + e^{itT^\gamma(g)}\cdot iT^\gamma(f)T^\gamma(g)\xi\right\| \\ &= \left\|\frac{1}{t}\left(T^\gamma(f)e^{-itT^\gamma(g)} - T^\gamma(f)\right)\xi + iT^\gamma(f)T^\gamma(g)\xi\right\| \\ &\le r\|f\|_{\frac{3}{2}}\left\|(1+L_0^\gamma)\frac{1}{t}\left(e^{-itT^\gamma(g)} - 1\right)\xi + (1+L_0^\gamma)\cdot iT^\gamma(g)\xi\right\| \\ &\le r\|f\|_{\frac{3}{2}}\left(\left\|\left(\frac{1}{t}\left(e^{-itT^\gamma(g)} - 1\right) + iT^\gamma(g)\right)\xi\right\| + \left\|\left(\frac{1}{t}L_0^\gamma\left(e^{-itT^\gamma(g)} - 1\right) + iL_0^\gamma T^\gamma(g)\right)\xi\right\|\right).\end{aligned} \tag{3.1.10}$$

The first term of (3.1.10) goes to 0 by Stone's theorem [RS80, Theorem VIII.7(c)]. The second term can be treated by (3.1.4) and (3.1.7) as follows:

$$\begin{aligned}&\left\|\frac{1}{t}L_0^\gamma(e^{-itT^\gamma(g)} - 1)\xi + iL_0^\gamma T^\gamma(g)\xi\right\| \\ &= \left\|\frac{1}{t}\left(e^{-itT^\gamma(g)}(T^\gamma(\mathrm{Exp}(tg)_*(1)) - \beta(\mathrm{Exp}(tg),1)) - L_0^\gamma\right)\xi + i(iT^\gamma(g') + T^\gamma(g)L_0^\gamma)\xi\right\| \\ &\le \left\|\frac{1}{t}(e^{-itT^\gamma(g)}T^\gamma(\mathrm{Exp}(tg)_*(1)) - e^{-itT^\gamma(g)}L_0^\gamma)\xi - T^\gamma(g')\xi\right\| \\ &\qquad + \left\|\frac{1}{t}(e^{-itT^\gamma(g)}L_0^\gamma - L_0^\gamma)\xi + iT^\gamma(g)L_0^\gamma\xi\right\| + \left|\frac{1}{t}\beta(\mathrm{Exp}(tg),1)\right|\|\xi\|.\end{aligned}$$

each term can be seen to converge to 0: the first term is done by noting that $L_0^\gamma = T^\gamma(1)$, continuity of T^γ (Proposition 3.1.3), $[g,1] = g'$ and unitarity of $e^{-itT^\gamma(g)}$. The second term vanishes by using Stone's theorem. The last term also converges to zero by (1.2.1) and using the fact that $\omega(g,1) = 0$. To summarize, the first term of the right-hand side of (3.1.9) tends to $-iT^\gamma(f)T^\gamma(g)$.

The second term of (3.1.9) is equal to $iT^\gamma(g)T^\gamma(f)$. Indeed, since $C^\infty(L_0^\gamma)$ is invariant under the action of $T^\gamma(f)$, this follows by Stone's theorem.

Altogether, we obtained the equality $i[T^\gamma(g), T^\gamma(f)] = T^\gamma(g'f - gf') + c\omega(g,f)$ on $C^\infty(L_0^\gamma)$, which is the Virasoro commutation relation.

Note that until here we have only used that T is a positive energy representation of the Virasoro algebra with the central charge c with diagonalizable L_0, but not irreducibility. Therefore, one can iterate our construction for another element in $\mathcal{D}^s(S^1)$. In particular, by taking γ^{-1}, we obtain by Proposition 3.1.1

$$(T^\gamma)^{\gamma^{-1}}(f) = T(f). \tag{3.1.11}$$

We claim that the new representation T^γ is irreducible and has the same lowest weight h. Indeed, by (3.1.11), one can approximate $T(f)$ by $T^\gamma(\gamma_{n*}^{-1}f) + \beta(\gamma, (\gamma_n^{-1})_*(f))$ in the strong resolvent sense, where $\{\gamma_n\} \subset \mathrm{Diff}_+(S^1)$ and $\gamma_n \to \gamma$ in the topology of $\mathcal{D}^s(S^1)$. As $\{e^{iT(f)} : f \in \mathrm{Vect}(S^1)\}$ generates $\mathcal{B}(\mathcal{H}(c,h))$, so does $\{e^{iT^\gamma(f)} : f \in \mathrm{Vect}(S^1)\}$, and this shows that T^γ is a irreducible representation of the Virasoro algebra. Furthermore, the new conformal Hamiltonian $L_0^\gamma = T^\gamma(1)$ has spectrum which is a subset of the spectrum of the old conformal Hamiltonian L_0 since it is obtained as a limit in the strong resolvent sense of $\{\mathrm{Ad}\, U(\gamma_n)(L_0)\}$ with the same spectrum [RS80, Theorem VIII.24(a)]. Again by iteration, we have

$$\mathrm{sp}\, L_0 = \mathrm{sp}\, (T^\gamma)^{\gamma^{-1}}(1) \subset \mathrm{sp}\, L_0^\gamma = \mathrm{sp}\, T^\gamma(1) \subset \mathrm{sp}\, L_0,$$

therefore, all these sets must coincide. In particular, h is the lowest eigenvalue of L_0^γ. □

As T and T^γ are equivalent as irreducible representations of $\mathrm{Vect}(S^1)$ and thus of the Virasoro algebra, there is an intertwiner $U(\gamma)$, defined up to a scalar: $U(\gamma)T(f) = T^\gamma(f)U(\gamma)$.

Corollary 3.1.5. *The map $\gamma \mapsto U(\gamma)$ where $\gamma \in \mathcal{D}^s(S^1)$, $s > 3$, is a unitary projective representation of $\mathcal{D}^s(S^1)$, i.e. $U(\gamma_1 \circ \gamma_2) = U(\gamma_1)U(\gamma_2)$ up to a phase factor.*

Proof. We know that for $\gamma_1, \gamma_2 \in \mathcal{D}^s(S^1)$

$$U(\gamma_1)T(f) = T^{\gamma_1}(f)U(\gamma_1),$$
$$U(\gamma_2)T(f) = T^{\gamma_2}(f)U(\gamma_2)$$

hold for every $f \in \mathrm{Vect}(S^1)$. So

$$\begin{aligned} U(\gamma_1)U(\gamma_2)T(f) &= U(\gamma_1)T^{\gamma_2}(f)U(\gamma_2) = U(\gamma_1)(T(\gamma_{2*}f) - \beta(\gamma_2, f))U(\gamma_2) = \\ &= (T^{\gamma_1}(\gamma_{2*}f) - \beta(\gamma_2, f))U(\gamma_1)U(\gamma_2) = \\ &= (T((\gamma_1 \circ \gamma_2)_* f) - \beta(\gamma_1, \gamma_{2*}f) - \beta(\gamma_2, f))U(\gamma_1)U(\gamma_2). \end{aligned}$$

Consequently by the computations of Proposition 3.1.1

$$U(\gamma_1)U(\gamma_2)T(f) = T^{\gamma_1 \circ \gamma_2}(f)U(\gamma_1)U(\gamma_2),$$

therefore, $U(\gamma_1 \circ \gamma_2) = U(\gamma_1)U(\gamma_2)$ up to a phase because we are dealing with irreducible representations of the Virasoro algebra. □

Corollary 3.1.6. *Let $U_{(c,h)}$ be the irreducible unitary projective representation of $\mathrm{Diff}_+(S^1)$ with central charge c and lowest weight h. $U_{(c,h)}$ extends to a strongly continuous irreducible unitary projective representation of $\mathcal{D}^s(S^1)$.*

Proof. We only need to be prove, i.e. that the action $\alpha : \mathcal{D}^s(S^1) \to \mathrm{Aut}(\mathcal{B}(\mathcal{H}(c,h)))$, $\gamma \mapsto \mathrm{Ad}\, U(\gamma)$ is pointwise continuous in the strong operator topology of $\mathcal{B}(\mathcal{H}(c,h))$.

Let $\{\gamma_n\} \subset \mathrm{Diff}_+(S^1)$, $\gamma \in \mathcal{D}^s(S^1)$ with $\gamma_n \to \gamma$ in the topology of $\mathcal{D}^s(S^1)$. Then

$$\lim_{n\to\infty} U(\gamma_n)e^{itT(f)}U(\gamma_n)^* = \lim_{n\to\infty} e^{itT^{\gamma_n}(f)} = e^{itT^{\gamma}(f)}$$

where the limit is meant in the strong topology. By taking $f = 1$, we obtain the convergence of $L_0^{\gamma_n}$ to L_0^{γ} in the strong resolvent sense. As they are in the (c,h)-representation of the Virasoro algebra, the lowest eigenprojections E_0, E_0^{γ} are one-dimensional, and it holds that $\lim_{n\to\infty} \mathrm{Ad}\, U(\gamma_n)(E_0) = E_0^{\gamma}$. Let Ω, Ω^{γ} be the lowest eigenvectors. By fixing the scalars, we may assume that $\Omega^{\gamma_n} := U(\gamma_n)\Omega \to \Omega^{\gamma}$.

With this $U(\gamma_n)$ with fixed phase, the sequence

$$U(\gamma_n)e^{iT(f_1)} \cdots e^{iT(f_k)}\Omega = e^{iT^{\gamma_n}(f_1)} \cdots e^{iT^{\gamma_n}(f_k)}\Omega^{\gamma_n}$$

is convergent to $e^{iT^{\gamma}(f_1)} \cdots e^{iT^{\gamma}(f_k)}\Omega^{\gamma}$, because all the operators $e^{iT^{\gamma_n}(f_1)}, \cdots, e^{iT^{\gamma_n}(f_k)}$ are uniformly bounded and convergent in the strong operator topology. Since vectors of the form $e^{iT(f_1)} \cdots e^{iT(f_k)}\Omega$ span a dense subspace of the whole Hilbert space $\mathcal{H}(c,h)$, together with the uniform boundedness of $U(\gamma_n)$, we obtain the convergence of $U(\gamma_n)$ to $U(\gamma)$ in the strong operator topology.

The continuity follows, since for any $x \in \mathcal{B}(\mathcal{H})$, $\mathrm{Ad}\, U(\gamma_n)(x)$ is convergent in the strong operator topology because $U(\gamma_n)$ is uniformly bounded.

□

Corollary 3.1.7. *Let $U_{(c,h)}$ be the irreducible unitary projective representation of $\mathrm{Diff}_+(S^1)$ with central charge c and lowest weight h. $U_{(c,h)}$ extends to a strongly continuous irreducible unitary projective representation of $\mathrm{Diff}^k_+(S^1)$ with $k \geq 4$.*

Proof. This is an immediate corollary of the continuous embedding $\mathrm{Diff}^k_+(S^1) \hookrightarrow \mathcal{D}^s(S^1)$, $s \leq k$. □

3.2 Direct sum of irreducible representations

Here we prove that every positive energy projective unitary representation of $\mathrm{Diff}_+(S^1)$ extends to a unitary projective representation of $\mathcal{D}^s(S^1)$ for $s > 3$. A similar result holds for the universal covering groups provided that the representation is assumed to be a direct sum of irreducibles. This is not an immediate consequence of Corollary 3.1.6, because, in general, the direct sum of projective representations does not make sense: $\mathcal{U}(\mathcal{H}_j)/\mathbb{C}$ is not a linear space. On the other hand, if we have **multiplier representations** of a group G with the same cocycle, $U_j(g_1)U_j(g_2) = \omega(g_1, g_2)U_j(g_1g_2)$ where $\omega(g_1, g_2)$ is a 2-cocycle $H^2(G, \mathbb{C})$ of G, then the direct sum $\bigoplus_j U_j(g)$ is again a multiplier representation with the same cocycle ω. If we are interested in a projective representation of a certain quotient G/H by a normal subgroup H we have to make sure that the direct sum $\bigoplus U_j(h)$ reduces to a scalar when $h \in H$.

First of all, we need that elements in $\mathcal{D}^s(S^1)$ with compact support can be approximated by elements $\mathcal{D}^s(S^1)$ with slightly larger support.

Lemma 3.2.1. *For a fixed $f \in H^s(S^1)$, the rotation $\mathbb{R} \ni t \mapsto f_t = f(e^{i(\cdot - t)}) \in H^s(S^1)$ is continuous.*

Proof. We have $\hat{f}_{t,k} = e^{ikt}\hat{f}_k$, and hence $|\hat{f}_{t,k}| = |\hat{f}_k|$ and $\hat{f}_{t,k} \to \hat{f}_k$ as $t \to 0$. By Lebesgue's dominated convergence theorem (applied to the measure space $\mathbb{Z}$ with the counting measure, with the dominating function $k \mapsto 4|(1+k^2)^s\hat{f}_k|^2$)

$$\sum_k (1+k^2)^s |\hat{f}_{t,k} - \hat{f}_k|^2 \to 0.$$

This means $\|f - f_t\|_{H^s} \to 0$. □

Lemma 3.2.2. *For every $\gamma \in \mathcal{D}^s(S^1)$, there exists a sequence $\{\gamma_n\}$ converging to γ in the topology of $\mathcal{D}^s(S^1)$. Furthermore, if γ is supported in I, we can take γ_n such that $\operatorname{supp}\gamma_n \supset \gamma_{n+1}$ and $\bigcap_n \operatorname{supp}\gamma_n = I$.*

Proof. Let $\gamma \in \mathcal{D}^s(S^1)$ and $\varphi \in \widetilde{\mathcal{D}^s(S^1)}$ such that $\varphi(\theta + 2\pi) = \varphi(\theta) + 2\pi$ and $\gamma(e^{i\theta}) = e^{i\varphi(\theta)}$. If γ is supported in a proper interval we may assume without loss of generality that $\varphi(\theta) = \theta$ if $\theta \in [-\pi, a) \cup (b, \pi]$. The function $\psi := \varphi' - 1$ is 2π-periodic and has compact support $[a, b]$ as a function on $[-\pi, \pi]$.

We now choose a set of C^∞-functions $\{g_n\}$ with compact support strictly contained in $[-\pi, \pi]$ such that for all $n \in \mathbb{N}$ $g_n \geq 0$, $\int g_n = 1$, $\operatorname{supp}(g_n) \supset \operatorname{supp}(g_{n+1})$, $\operatorname{supp}(g_n) \to \{0\}$. In addition, if γ is supported in $[a, b]$, we may assume that $[a, b] + \operatorname{supp}(g_n) \supset \operatorname{supp}(\psi * g_n)$, where the convolution is defined on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ as an abelian group.

To obtain the claim, it is enough to show that $\|\psi - \psi * g_n\|_{H^s} \to 0$ as $n \to 0$. This follows from

$$\|\psi - \psi * g_n\|_{H^s} \leq \int_{S^1} g_n(t)\|\psi - \psi_t\|_{H^s} dt$$

and Lemma 3.2.1.

□

Lemma 3.2.3. *Let $U_{(c,h_1)}, U_{(c,h_2)}$ be irreducible, projective representations of $\widetilde{\mathcal{D}^s(S^1)}$ with central charge c and lowest weight h_1, h_2 respectively, constructed as in Section 3. Let I be a proper interval of S^1. Then the projective representations $U_{(c,h_1)}$ and $U_{(c,h_2)}$ restricted to $\mathcal{D}^s(I)$ are unitarily equivalent. Furthermore, a unitary U intertwines $U_{(c,h_1)}$ and $U_{(c,h_2)}$ restricted to $\mathcal{D}^s(I)$ if and only if it intertwines $T_{(c,h_1)}(f)$ and $T_{(c,h_2)}(f)$ for every $f \in \mathrm{Vect}(S^1)$ with support in I.*

Proof. Let $\tilde{I}$ an open proper interval of S^1 such that $\tilde{I} \supset \overline{I}$. By [Wei17, Theorem 5.6] there exists a unitary W which intertwines the representations $U_{(c,h_1)}, U_{(c,h_2)}$ when restricted to $\mathrm{Diff}_+(\tilde{I})$. Let $\gamma \in \mathcal{D}^s(I)$, then by Lemma 3.2.2 there exists a sequence of C^∞-diffeomorphisms $\{\gamma_n\} \subset \mathrm{Diff}_+(\tilde{I})$ converging to γ. By Corollary 3.1.6,

$$\mathrm{Ad}\, W U_{(c,h_1)}(\gamma) W^* = \mathrm{Ad} \lim_{n\to\infty} W U_{(c,h_1)}(\gamma_n) W^* = \mathrm{Ad} \lim_{n\to\infty} U_{(c,h_2)}(\gamma_n) = \mathrm{Ad}\, U_{(c,h_2)}(\gamma).$$

The last assertion follows from [Wei17, Lemma 2.1]. □

We are going to show that we can take the direct sum of irreducible projective representations of $\mathcal{D}^s(S^1)$, $\{U_{(c,h_j)}\}$, with the same central charge c but possibly different lowest weights $\{h_j\}$ where differences $h_j - h_{j'}$ are integers. We split the proof into two steps. First, we make $U_{(c,h_j)}$ into continuous multiplier representations with the same cocycle in some neighborhood $\mathcal{V}$ of the identity diffeomorphism $\iota \in \widetilde{\mathcal{D}^s(S^1)}$. Then it is straightforward to take the direct sum. Next, we show that the direct sum representation reduced to a projective representation of $\mathcal{D}^s(S^1)$ if the differences $h_j - h_{j'}$ are integers.

Let G and G' be two topological groups. Given a neighborhood $\mathcal{V}$ of the identity in G, a continuous map $\mu : \mathcal{V} \to G'$ is a local homomorphism if $\mu(g_1)\mu(g_2) = \mu(g_1 g_2)$ for all $g_1, g_2 \in \mathcal{V}$ and $g_1 g_2 \in \mathcal{V}$.

We say that a map U is a local unitary multiplier representation of a topological group G on a neighborhood $\mathcal{V}$ of the identity if U is a map from $\mathcal{V}$ to the unitary group $\mathcal{U}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ which satisfies the equality $U(g_1)U(g_2) = \omega(g_1, g_2)U(g_1 g_2)$, where $\omega : \mathcal{V} \times \mathcal{V} \to \mathbb{T}$ and $\omega(g_1, g_2)\omega(g_1 g_2, g_3) = \omega(g_1, g_2 g_3)\omega(g_2, g_3)$ whenever g_1, g_2, g_3, $g_1 g_2$ and $g_2 g_3$ are in $\mathcal{V}$. The following is obtained by reversing the idea of [Tan18].

Proposition 3.2.4. *For a family $\{(c, h_j)\}$ of pairs with the same central charge c, there is a neighborhood $\mathcal{V}$ of $\widetilde{\mathcal{D}^s(S^1)}$ such that the irreducible unitary projective representations $U_{(c,h_j)}$ lift to local multiplier representations of $\mathcal{V}$ with the same cocycle $c(\cdot,\cdot)$.*

Proof. Let us take h_1. By [Bar54][Mor17, Proposition 12.44], in a neighborhood $\hat{\mathcal{V}}$ of the identity $\iota \in \widetilde{\mathrm{Diff}^4_+(S^1)}$, $U_{(c,h_1)}$ lifts to a continuous multiplier representation, with some continuous cocycle $c(\cdot,\cdot)$, which we will denote by U_1.

Because $\widetilde{\mathcal{D}^s(S^1)}$ is a topological group, and by Lemmas C.0.3, C.0.4, for each neighborhood $\mathcal{W}$, there is a smaller neighborhood $p(\mathcal{W})$ such that $p(\mathcal{W})^2 \subset \mathcal{W}$ and $\chi_k(\gamma), \chi_k^{(k)}(\gamma), \chi_{k+1}^{(k)}(\gamma) \subset \mathcal{W}$ for $\gamma \in p(\mathcal{W})$. We take $\mathcal{V} = p^{11}(\hat{\mathcal{V}}) = \underbrace{p(p(p(\cdots \hat{\mathcal{V}} \cdots)))}_{11\text{-times}}$.

Construction of multiplier representations U_j. We show that we can take U_j with the same cocycle $c(\cdot,\cdot)$.

We fix a covering $\{I_k\}$ of S^1 as in Lemma C.0.3. For $\gamma \in p(\hat{\mathcal{V}})$, we define U_j as follows: By Lemma 3.2.3, there are unitary intertwiners $\{V_{j,k}\}$ between $U_{(c,h_1)}$ and $U_{(c,h_j)}$ restricted to $\mathcal{D}^s(I_k)$. We set

$$U_j(\chi_k(\gamma)) = \mathrm{Ad}\, V_{j,k}(U_1(\gamma_k)),$$

which makes sense because $p(\hat{\mathcal{V}}) \subset \hat{\mathcal{V}}$. Note that $U_j(\chi_k(\gamma))$ does not depend on the choice of unitary intertwiner $V_{j,k}$, since, if $V_{j,k}$ and $\hat{V}_{j,k}$ are both unitary intertwiners, then by Lemma 3.2.3

$$\mathrm{Ad}\, V_{j,k}^* \hat{V}_{j,k}(U_j(\chi_k(\gamma))) = U_j(\chi_k(\gamma))$$

for γ smooth, and by continuity of U_1 for $\chi_k(\gamma) \in \mathcal{D}^s(I_k) \cap \hat{\mathcal{V}}$.

Let us denote $\gamma_k = \chi_k(\gamma)$ for simplicity. Now, since $\gamma = \gamma_1\gamma_2\gamma_3$ with $\gamma_k \in \mathcal{D}^s(I_k) \cap \hat{\mathcal{V}}$, we can define $U_j(\gamma)$ by

$$U_j(\gamma) = U_j(\gamma_1)U_j(\gamma_2)U_j(\gamma_3)c(\gamma_1,\gamma_2)^{-1}c(\gamma_1\gamma_2,\gamma_3)^{-1}, \tag{3.2.1}$$

and note that the corresponding equation holds for U_1.

Well-definedness. We used a particular set of maps χ_k to define U_j, but actually they do not depend on the choice of such map χ_k if γ satisfies certain properties and is sufficiently close to ι. Namely, we take two decompositions $\gamma = \gamma_1\gamma_2\gamma_3 = \gamma_1'\gamma_2'\gamma_3'$ where $\gamma_k, \gamma_k' \in \mathcal{D}^s(I_k) \cap p^5(\hat{\mathcal{V}})$.

It holds that $\gamma_3^{-1}\gamma_2^{-1}\gamma_1^{-1}\gamma_1'\gamma_2'\gamma_3' = \iota$ in $\widetilde{\mathcal{D}^s(S^1)}$ and $U_1(\gamma_1)^* = c(\gamma_1,\gamma_1^{-1})U_1(\gamma_1^{-1})$, hence we have

$$c(\gamma_1,\gamma_2,\gamma_3,\gamma_1',\gamma_2',\gamma_3') := U_1(\gamma_3)^*U_1(\gamma_2)^*U_1(\gamma_1^{-1}\gamma_1')U_1(\gamma_2')U_1(\gamma_3') \in \mathbb{C}.$$

Furthermore, as U_1 is a multiplier representation in $\hat{\mathcal{V}}$, we have

$$\begin{aligned}U_1(\gamma) &= U_1(\gamma_1)U_1(\gamma_2)U_1(\gamma_3)c(\gamma_1,\gamma_2)^{-1}c(\gamma_1\gamma_2,\gamma_3)^{-1}\\ &= U_1(\gamma_1')U_1(\gamma_2',)U_1(\gamma_3')c(\gamma_1',\gamma_2')^{-1}c(\gamma_1'\gamma_2',\gamma_3')^{-1}.\end{aligned}$$

By putting all factors in one side, we obtain

$$c(\gamma_1,\gamma_2,\gamma_3,\gamma_1',\gamma_2',\gamma_3')c(\gamma_1^{-1},\gamma_1')c(\gamma_1,\gamma_1^{-1})c(\gamma_1,\gamma_2)c(\gamma_1\gamma_2,\gamma_3)c(\gamma_1',\gamma_2')^{-1}c(\gamma_1'\gamma_2',\gamma_3')^{-1} = 1. \tag{3.2.2}$$

Note that U_j is unitarily equivalent to U_1 on any proper interval, therefore, $U_j(\gamma_1)^*U_j(\gamma_1') = c(\gamma_1^{-1},\gamma_1')c(\gamma_1,\gamma_1^{-1})U_j(\gamma_1^{-1}\gamma_1')$, and $\gamma_1^{-1}\gamma_1' = \gamma_2\gamma_3\gamma_3'^{-1}\gamma_2'^{-1}$ has support in $I_2\cup I_3$. Then we can again use the unitary equivalence between U_j and U_1 on $I_2\cup I_3$ to obtain

$$U_j(\gamma_3)^*U_j(\gamma_2)^*U_j(\gamma_1^{-1}\gamma_1')U_j(\gamma_2')U_j(\gamma_3') = c(\gamma_1,\gamma_2,\gamma_3,\gamma_1',\gamma_2',\gamma_3'),$$

which is, by (3.2.2), equivalent to the equality

$$\begin{aligned}&U_j(\gamma_1)U_j(\gamma_2)U_j(\gamma_3)c(\gamma_1,\gamma_2)^{-1}c(\gamma_1\gamma_2,\gamma_3)^{-1}\\ =\,&U_j(\gamma_1')U_j(\gamma_2')U_j(\gamma_3')c(\gamma_1',\gamma_2')^{-1}c(\gamma_1'\gamma_2',\gamma_3')^{-1}.\end{aligned}$$

In other words, U_j is well-defined on $p^6(\hat{\mathcal{V}})$.

Cocycle relations. Next we show that U_j is a local multiplier representation on $\mathcal{V}$. Let $\gamma,\gamma'\in\mathcal{V} = p^{11}(\hat{\mathcal{V}})$ and we take decompositions $\gamma = \gamma_1\gamma_2\gamma_3, \gamma' = \gamma_1'\gamma_2'\gamma_3'$. We first look at the product $\gamma_3\gamma_1'$. This is supported in $I_1\cup I_3$, and we can find another decomposition $\gamma_3\gamma_1' = \gamma_1''\gamma_3''$ using Lemma C.0.4, where $\gamma_j''\in\mathcal{D}^s(I_j)\cap p^8(\hat{\mathcal{V}})$. By repeating such operations and taking new decompositions in proper intervals, we find

$$\begin{aligned}\gamma\gamma' &= \gamma_1\gamma_2\gamma_3\gamma_1'\gamma_2'\gamma_3'\\ &= \gamma_1\gamma_2\gamma_1''\gamma_3''\gamma_2'\gamma_3'\\ &= \gamma_1\gamma_1'''\gamma_2'''\gamma_2''''\gamma_3''''\gamma_3',\end{aligned}$$

where $\gamma_j^{(k)}\in\mathcal{D}^s(I_j)\cap p^6(\hat{\mathcal{V}})$.

Again, by considering the multiplier representation U_1, we can prove the following relations

$$\begin{aligned}U_1(\gamma_3)U_1(\gamma_1') &= U_1(\gamma_1'')U_1(\gamma_3'')c(\gamma_3,\gamma_1',\gamma_1'',\gamma_3''),\\ U_1(\gamma_2)U_1(\gamma_1'') &= U_1(\gamma_1''')U_1(\gamma_2''')c(\gamma_2,\gamma_1'',\gamma_1''',\gamma_2'''),\\ U_1(\gamma_3'')U_1(\gamma_2') &= U_1(\gamma_2'''')U_1(\gamma_3'''')c(\gamma_3'',\gamma_2',\gamma_2'''',\gamma_3''''),\end{aligned} \tag{3.2.3}$$

where $c(\gamma_3, \gamma_1', \gamma_1'', \gamma_3''), c(\gamma_2, \gamma_1'', \gamma_1''', \gamma_2'''), c(\gamma_3'', \gamma_2', \gamma_2'''', \gamma_3'''') \in \mathbb{C}$ are defined through these equalities. Therefore, as U_1 has the cocycle c,

$$\begin{aligned}
&c(\gamma, \gamma')U_1(\gamma\gamma') \\
&= U_1(\gamma)U_1(\gamma') \\
&= c(\gamma_1, \gamma_2)^{-1}c(\gamma_1\gamma_2, \gamma_3)^{-1}c(\gamma_1', \gamma_2')^{-1}c(\gamma_1'\gamma_2', \gamma_3')^{-1} \\
&\quad \times U_1(\gamma_1)U_1(\gamma_2)U_1(\gamma_3)U_1(\gamma_1')U_1(\gamma_2')U_1(\gamma_3') && \text{by (3.2.1)} \\
&= c(\gamma_1, \gamma_2)^{-1}c(\gamma_1\gamma_2, \gamma_3)^{-1}c(\gamma_1', \gamma_2')^{-1}c(\gamma_1'\gamma_2', \gamma_3')^{-1} \\
&\quad \times U_1(\gamma_1)U_1(\gamma_1''')U_1(\gamma_2''')U_1(\gamma_2'''')U_1(\gamma_3'''')U_1(\gamma_3') && \text{by (3.2.3)} \\
&\quad \times c(\gamma_3, \gamma_1', \gamma_1'', \gamma_3'')c(\gamma_2, \gamma_1'', \gamma_1''', \gamma_2''')c(\gamma_3'', \gamma_2', \gamma_2'''', \gamma_3'''') \\
&= c(\gamma_1, \gamma_2)^{-1}c(\gamma_1\gamma_2, \gamma_3)^{-1}c(\gamma_1', \gamma_2')^{-1}c(\gamma_1'\gamma_2', \gamma_3')^{-1} \\
&\quad \times c(\gamma_3, \gamma_1', \gamma_1'', \gamma_3'')c(\gamma_2, \gamma_1'', \gamma_1''', \gamma_2''')c(\gamma_3'', \gamma_2', \gamma_2'''', \gamma_3'''') \\
&\quad \times c(\gamma_1, \gamma_1''')c(\gamma_2'''\gamma_2'''')c(\gamma_3''''\gamma_3') \cdot U_1(\gamma_1\gamma_1''')U_1(\gamma_2'''\gamma_2'''')U_1(\gamma_3''''\gamma_3') \\
&= c(\gamma_1, \gamma_2)^{-1}c(\gamma_1\gamma_2, \gamma_3)^{-1}c(\gamma_1', \gamma_2')^{-1}c(\gamma_1'\gamma_2', \gamma_3')^{-1} \\
&\quad \times c(\gamma_3, \gamma_1', \gamma_1'', \gamma_3'')c(\gamma_2, \gamma_1'', \gamma_1''', \gamma_2''')c(\gamma_3'', \gamma_2', \gamma_2'''', \gamma_3'''') \\
&\quad \times c(\gamma_1, \gamma_1''')c(\gamma_2'''\gamma_2'''')c(\gamma_3''''\gamma_3') \cdot c(\gamma_1\gamma_1''', \gamma_2'''\gamma_2'''')c(\gamma_1\gamma_1'''\gamma_2'''\gamma_2'''', \gamma_3''''\gamma_3')U_1(\gamma\gamma')
\end{aligned}$$

or equivalently, the following relation between scalars:

$$\begin{aligned}
c(\gamma, \gamma') = {} & c(\gamma_1, \gamma_2)^{-1}c(\gamma_1\gamma_2, \gamma_3)^{-1}c(\gamma_1', \gamma_2')^{-1}c(\gamma_1'\gamma_2', \gamma_3')^{-1} \\
& \times c(\gamma_3, \gamma_1', \gamma_1'', \gamma_3'')c(\gamma_2, \gamma_1'', \gamma_1''', \gamma_2''')c(\gamma_3'', \gamma_2', \gamma_2'''', \gamma_3'''') \\
& \times c(\gamma_1, \gamma_1''')c(\gamma_2'''\gamma_2'''')c(\gamma_3''''\gamma_3') \cdot c(\gamma_1\gamma_1''', \gamma_2'''\gamma_2'''')c(\gamma_1\gamma_1'''\gamma_2'''\gamma_2'''', \gamma_3''''\gamma_3').
\end{aligned} \tag{3.2.4}$$

Since U_j is locally equivalent to U_1, the following also follows from (3.2.3):

$$\begin{aligned}
U_j(\gamma_3)U_j(\gamma_1') &= U_j(\gamma_1'')U_j(\gamma_3'')c(\gamma_3, \gamma_1', \gamma_1'', \gamma_3''), \\
U_j(\gamma_2)U_j(\gamma_1'') &= U_j(\gamma_1''')U_j(\gamma_2''')c(\gamma_2, \gamma_1'', \gamma_1''', \gamma_2'''), \\
U_j(\gamma_3'')U_j(\gamma_2') &= U_j(\gamma_2'''')U_j(\gamma_3'''')c(\gamma_3'', \gamma_2', \gamma_2'''', \gamma_3''''),
\end{aligned} \tag{3.2.5}$$

Now, in order to show that U_j is a local multipler representation with the cocycle

c, we only have to compute

$$\begin{aligned}
&U_j(\gamma)U_j(\gamma') \\
&= c(\gamma_1,\gamma_2)^{-1}c(\gamma_1\gamma_2,\gamma_3)^{-1}c(\gamma_1',\gamma_2')^{-1}c(\gamma_1'\gamma_2',\gamma_3')^{-1} \\
&\quad\times U_j(\gamma_1)U_j(\gamma_2)U_j(\gamma_3)U_j(\gamma_1')U_j(\gamma_2')U_j(\gamma_3') && \text{by (3.2.1)} \\
&= c(\gamma_1,\gamma_2)^{-1}c(\gamma_1\gamma_2,\gamma_3)^{-1}c(\gamma_1',\gamma_2')^{-1}c(\gamma_1'\gamma_2',\gamma_3')^{-1} \\
&\quad\times U_j(\gamma_1)U_j(\gamma_1''')U_j(\gamma_2''')U_j(\gamma_2'''')U_j(\gamma_3'''')U_j(\gamma_3') && \text{by (3.2.5)} \\
&\quad\times c(\gamma_3,\gamma_1',\gamma_1'',\gamma_3'')c(\gamma_2,\gamma_1'',\gamma_1''',\gamma_2''')c(\gamma_3'',\gamma_2',\gamma_2'''',\gamma_3'''') \\
&= c(\gamma,\gamma')\left(c(\gamma_1,\gamma_1''')c(\gamma_2'''\gamma_2'''')c(\gamma_3''''\gamma_3')\cdot c(\gamma_1\gamma_1''',\gamma_2'''\gamma_2'''')c(\gamma_1\gamma_1'''\gamma_2'''\gamma_2'''',\gamma_3''''\gamma_3')\right)^{-1} && \text{by (3.2.4)} \\
&\quad\times U_j(\gamma_1)U_j(\gamma_1''')U_j(\gamma_2''')U_j(\gamma_2'''')U_j(\gamma_3'''')U_j(\gamma_3') \\
&= c(\gamma,\gamma')\left(c(\gamma_1\gamma_1''',\gamma_2'''\gamma_2'''')c(\gamma_1\gamma_1'''\gamma_2'''\gamma_2'''',\gamma_3''''\gamma_3')\right)^{-1} \\
&\quad\times U_j(\gamma_1\gamma_1''')U_j(\gamma_2'''\gamma_2'''')U_j(\gamma_3''''\gamma_3') \\
&= c(\gamma,\gamma')U_j(\gamma\gamma'),
\end{aligned}$$

where we used local equivalence between U_j and U_1 in and 4th equalities, and the well-definedness (independence of the partition of a group element into $\mathcal{D}^s(I_k)\cap p^5(\hat{\mathcal{V}})$) in the 5th equality. Namely, U_j has the cocycle c on $\mathcal{V} = p^{11}(\hat{\mathcal{V}})$. □

Direct sum of multiplier representations. Since all the projective representations U_j can be made into the local multiplier representations with the same cocycle c, the direct sum $U := \bigoplus_j U_j$ is again a local multiplier representation of $\widetilde{\mathcal{D}^s(S^1)}$ on $\mathcal{V}$. By forgetting the phase, we can interpret that U is a local projective representation of $\mathcal{V} \subset \widetilde{\mathcal{D}^s(S^1)}$, or in other words, a continuous local group homomorphism from $\mathcal{V}$ into $\mathcal{U}(\mathcal{H})/\mathbb{T}$ (see Section A), where $\mathcal{H} = \bigoplus_j \mathcal{H}(c,h_j)$. As $\widetilde{\mathcal{D}^s(S^1)}$ is simply connected and locally connected, U extends to a continuous projective representation of $\widetilde{\mathcal{D}^s(S^1)}$ [Pon46, Theorem 63].

Theorem 3.2.5. *For a family $\{(c,h_j)\}$ of pairs with the same central charge c such that $h_j - h_{j'} \in \mathbb{N}$, the direct sum projective representation U of $\widetilde{\mathcal{D}^s(S^1)}$ as above satisfies $U(\rho(2\pi)) \in \mathbb{C}$, where $\rho(\cdot)$ is the lift of rotations to $\widetilde{\mathcal{D}^s(S^1)}$, or in other words, U is a projective representation of $\mathcal{D}^s(S^1)$.*

Proof. Let $\tilde{U}_{(c,h_j)}$ the irreducible global multiplier representation of $\widetilde{\mathrm{Diff}_+(S^1)}$ with central charge c and lowest weight h_j associated to the Bott-Virasoro cocycle. As a projective representation, we have $U\big|_{\widetilde{\mathrm{Diff}_+(S^1)}} = \bigoplus_j \tilde{U}_{(c,h_j)}$: this is because, by definition of U, they agree on a neighborhood of the identity of $\widetilde{\mathrm{Diff}_+(S^1)}$, and since $\widetilde{\mathrm{Diff}_+(S^1)}$ is simply connected they agree globally. Since $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ is a simple Lie group, $U\big|_{\widetilde{\mathrm{PSL}(2,\mathbb{R})}}$ extends to a true representation of $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ by changing $U(\gamma)$ only by a scalar [Bar54][Mor17, Theorem 12.72] (see also [Mor17, Example 12.77]).

The lift to a true representation of $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ is unique, since if V_1 and V_2 are true representations which give rise to the same projective representation, we have that $V_1(g) = \chi(g)V_2(g)$ for all $g \in \widetilde{\mathrm{PSL}(2,\mathbb{R})}$, where χ is a character. Since $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$ is a perfect group, $\chi(g) = 1$ for all g. By the uniqueness of the lift of $U|_{\widetilde{\mathrm{PSL}(2,\mathbb{R})}}$ to a true representation V, we have that $V = \bigoplus_j V_{(c,h_j)}$, where $V_{(c,h_j)}$ is the lift of $\tilde{U}_{(c,h_j)}|_{\widetilde{\mathrm{PSL}(2,\mathbb{R})}}$ to a true representation. As we assume that $h_j - h_{j'}$ are integers, $V(\rho(2\pi)) \in \mathbb{C}$.

□

From the previous theorem, it follows that every positive energy projective unitary representation of $\mathrm{Diff}_+(S^1)$ extends to a unitary projective representation of $\mathcal{D}^s(S^1)$ using the following well-known fact that we here prove for completeness.

Proposition 3.2.6. *Let U be a positive energy unitary projective representation of* $\mathrm{Diff}_+(S^1)$ *on the Hilbert space $\mathcal{H}$. Then U is unitarily equivalent to a direct sum of irreducible positive energy unitary projective representation of* $\mathrm{Diff}_+(S^1)$ *and extends to $\mathcal{D}^s(S^1)$, $s > 3$.*

Proof. As in the proof of Theorem 3.2.5, we have that $U|_{\mathrm{PSL}(2,\mathbb{R})}$ can be lifted to a true representation of $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$. Thus we can take the generator of rotations L_0 and, since $e^{i2\pi L_0} \in \mathbb{C}\mathbb{1}$ from the fact that U is a projective representation of $\mathrm{Diff}_+(S^1)$, it follows that L_0 is diagonalizable with spectrum $\mathrm{Sp}(L_0) \subset \{h_1 + \mathbb{N}\}$ with $h_1 \in \mathbb{R}$, $h_1 \geq 0$. Let $\mathcal{H}^{\mathrm{fin}}$ be the dense subspace of $\mathcal{H}$ generated by the eigenvectors of L_0. We can apply [CKLW18, Theorem 3.4] to conclude that there exists a positive energy unitary representation π_U of Vir on $\mathcal{H}^{\mathrm{fin}}$.

The representation of Vir on $\mathcal{H}^{\mathrm{fin}}$ is equivalent to an algebraic orthogonal direct sum of multiples of irreducible positive energy representations of Vir in the following sense. Let V_1 be the smallest π_U-invariant subspace of $\mathcal{H}^{\mathrm{fin}}$ which contains $\ker(L_0 - h_1\mathbb{1}_{\mathcal{H}^{\mathrm{fin}}})$ where h_1 is the smallest eigenvalue of L_0. By induction let V_n be the smallest π_U-invariant subspace of $(V_1 \oplus V_2 \oplus \cdots \oplus V_{n-1})^\perp \cap \mathcal{H}^{\mathrm{fin}}$ which contains $(V_1 \oplus V_2 \oplus \cdots \oplus V_{n-1})^\perp \cap \ker(L_0 - h_n\mathbb{1}_{\mathcal{H}^{\mathrm{fin}}})$ where h_n is the smallest eigenvalue of L_0 restricted to $(V_1 \oplus V_2 \oplus \cdots \oplus V_{n-1})^\perp \cap \mathcal{H}^{\mathrm{fin}}$. It is straightforward to see that $\mathcal{H}^{\mathrm{fin}} = \bigoplus_n V_n$ in the algebraic sense. Now choose an orthonormal basis $\{e_j^n\}$ of $W_n := V_n \cap \ker(L_0 - h_n\mathbb{1}_{\mathcal{H}^{\mathrm{fin}}})$. We define H_j^n to be the smallest π_U-invariant subspace of W_n which contains the vector e_j^n. By construction H_j^n has no proper π_U-invariant subspaces, H_j^n and H_k^n are orthogonal subspaces for $j \neq k$ and $\overline{V_n} = \bigoplus_j \overline{H_j^n}$. Let T be the stress-energy tensor associated to the representation π_U of Vir. By construction $T(f)|_{H_j^n}$ is essentially self-adjoint on H_j^n.

To conclude the decomposition of U, we have to show that $e^{iT(f)}\overline{H_j^n} \subset \overline{H_j^n}$ for all $f \in \mathrm{Vect}(S^1)$. We note that $\mathscr{D}\left(\left(\overline{(T(f)|_{H_j^n})}\right)^\ell\right) \subset \mathscr{D}(T(f)^\ell)$ and if $\xi \in$

$\mathscr{D}\left(\left(\overline{(T(f)|_{H_j^n})}\right)^\ell\right)$ then $\left(\overline{T(f)|_{H_j^n}}\right)^\ell \xi = (T(f))^\ell \xi$. Thus the analytic vectors for $\overline{(T(f)|_{H_j^n})}$ are also analytic for $T(f)$ and $e^{i\overline{(T(f)|_{H_j^n})}}\xi = e^{iT(f)}\xi$. Using the density of the analytic vectors in $\overline{H_j^n}$, we obtain that $e^{i\overline{(T(f)|_{H_j^n})}} = e^{iT(f)}\big|_{H_j^n}$. Irreducibility of $U|_{\overline{H_j^n}}$ follows because $T|_{H_j^n}$ is irreducible.

The extension to $\mathcal{D}^s(S^1)$ is now a mere corollary of Theorem 3.2.5. □

Corollary 3.2.7. *Let U be a positive energy unitary projective representation of $\mathrm{Diff}_+(S^1)$ on the Hilbert space $\mathcal{H}$. Then U is unitarily equivalent to a direct sum of irreducible positive energy unitary projective representation of $\mathrm{Diff}_+(S^1)$ and extends to $\mathrm{Diff}_+^k(S^1)$ with $k \geq 4$.*

Proof. This again follows from Proposition 3.2.6 and the continuous embedding $\mathrm{Diff}_+^k(S^1) \hookrightarrow \mathcal{D}^s(S^1)$, $s \leq k$. □

We do not know whether our local multiplier representations can be extended to a global multiplier representation of $\widetilde{\mathcal{D}^s(S^1)}$. It is also open whether the global multiplier representation of $\mathrm{Diff}_+(S^1)$ with the Bott-Virasoro cocycle [FH05, Proposition 5.1] extends to $\widetilde{\mathcal{D}^s(S^1)}$ by continuity.

3.3 Conformal nets and diffeomorphism covariance

Consider a conformal net $(\mathcal{A}, U, \Omega)$, see 3.3. By definition, U is a positive energy representation of $\mathrm{Diff}_+(S^1)$ and is equivalent to a direct sum of irreducible representations, see Proposition 3.2.6. Every irreducible component U_j in decomposition has the same value of the central charge c and if h_j is the lowest weight of U_j, $h_j - h_k \in \mathbb{Z}$ for every j, k. This fact is crucial for our purpose, which is to extend the conformal symmetry of the net to the larger group $\mathcal{D}^s(S^1)$, $s > 3$, in the sense that we want to show that the conditions in (2.1.1) are satisfied for arbitrary γ in $\mathcal{D}^s(S^1)$ and $\mathcal{D}^s(I')$ respectively.

Proposition 3.3.1. *A conformal net $(\mathcal{A}, U, \Omega)$ is $\mathcal{D}^s(S^1)$-covariant, $s > 3$.*

Proof. Let $\{\gamma_n\}$ be a sequence of diffeomorphisms in $\mathrm{Diff}_+(S^1)$ converging to $\gamma \in \mathcal{D}^s(S^1)$ in the topology of $\mathcal{D}^s(S^1)$ as in Lemma 3.2.2. For all $n \in \mathbb{N}$ it holds that

$$U(\gamma_n)\mathcal{A}(I)U(\gamma_n)^* = \mathcal{A}(\gamma_n I) \subset \mathcal{A}(\textstyle\bigcup_{k=m}^n \gamma_k I),$$

where we used isotony of the net $\mathcal{A}$. For $x \in \mathcal{A}(I)$, it follows for $m \leq n$ that

$$U(\gamma_n)xU(\gamma_n)^* \in \mathcal{A}(\textstyle\bigcup_{k=m}^n \gamma_k I) = \bigvee_{k=m}^\infty \mathcal{A}(\gamma_k I),$$

by additivity. By Proposition 3.1.6 it follows that $U(\gamma)xU(\gamma)^* = \lim_{n\to\infty} U(\gamma_n)xU(\gamma_n)^*$ (convergence in the strong operator topology) is in $\bigcup_{k=m}^{\infty} \mathcal{A}(\gamma_k \cdot I)$ for any m, hence we have by upper semicontinuity that

$$U(\gamma)\mathcal{A}(I)U(\gamma)^* \subset \bigcap_m \mathcal{A}(\textstyle\bigcup_{k=m}^{\infty} \gamma_k I) = \mathcal{A}(\gamma I).$$

The other inclusion follows by applying $\mathrm{Ad}\, U(\gamma^{-1})$.

Now consider $\gamma \in \mathcal{D}^s(I')$ and $x \in \mathcal{A}(I)$. We know from lemma 3.2.2 that exists a sequence $\{\gamma_n\} \subset \mathrm{Diff}_+(I'_n)$ converging to γ in the topology of $\mathcal{D}^s(S^1)$ and a decreasing sequence of intervals $I'_n \supset \mathrm{supp}\,(\gamma_n) \supset I'$ such that $\bigcap_n I'_n = I'$. For $x \in \mathcal{A}(I_n)$, $U(\gamma_m)xU(\gamma_m)^* = x$ if $m \geq n$, hence by Proposition 3.1.6 we obtain $U(\gamma)xU(\gamma)^* = x$. As n is arbitrary, this holds for any $x \in \mathcal{A}(\bigcup_n I_n) = \mathcal{A}(I)$ by additivity. □

Chapter 4

General results about soliton representations

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In this Chapter we prove that any soliton representation is translation covariant and has always positive energy (in [Hen17b, Section 3.3.1], Henriques already observed that the translation covariance holds for any soliton representations). Furthermore we construct a class of inequivalent irreducible proper soliton representations. In the first part of the chapter we prove that any C^1 piecewise smooth diffeomorphism is imlementable by an unitary operator in any conformal net $(\mathcal{A}, U, \Omega)$ and the proof is based on an idea of André Henriques. This technical statement is used in Section 4.2 and 4.3.

4.1 C^1 piecewise smooth diffeomorphisms

Definition 4.1.1. With $\mathrm{Diff}_+^{1,\infty}(S^1)$ we denote the group of C^1 diffeomorphisms of the circle which are piecewise smooth. In the sequel, we denote with $\mathrm{Diff}_{+,0}^{1,\infty}$ the subgroup of $\mathrm{Diff}_+^{1,\infty}(S^1)$ consisting of elements γ such that $\gamma(-1) = -1$ and with $\mathrm{Diff}_{+,1}^{1,\infty}$ the subgroup of $\mathrm{Diff}_{+,0}^{1,\infty}$ consisting of elements γ such that $\gamma'(-1) = 1$.

Let $\gamma \in \mathrm{Diff}_+^{1,\infty}(S^1)$ and let $\tilde{\gamma}$ be a lift of γ to the universal covering $\widetilde{\mathrm{Diff}_+^1(S^1)}$. Recall that as a consequence of Borel's lemma [Hör90][Theorem 1.2.6], there exists an open interval I of S^1 which contains $p = -1$ and $\gamma_{I_-}, \gamma_{I_+} \in \mathrm{Diff}_+(S^1)$ such that γ agrees with γ_{I_-} in I_- and with γ_{I_+} in I_+, where I_- and I_+ are the connected components of $I \setminus \{-1\}$.

For $f \in C^\infty(S^1, \mathbb{R})$ and $\gamma \in \mathrm{Diff}_+(S^1)$ we define

$$f^{(k)}(e^{i\theta}) := \frac{d^k}{d\theta^k} f(e^{i\theta}) \tag{4.1.1}$$

and

$$\gamma^{(k)}(e^{i\theta}) := \frac{d^k}{d\theta^k} \tilde{\gamma}(\theta) \tag{4.1.2}$$

where $\tilde{\gamma}$ is the lift of γ in $\widetilde{\mathrm{Diff}_+(S^1)}$.

Lemma 4.1.2. *Let $\{\lambda_n\}_{n\geq 2}$ be a sequence of real numbers. There exists $g \in C^\infty(S^1, \mathbb{R})$ such that $\mathrm{Exp}(g)^{(n)}(-1) = \lambda_n$ for all $n \geq 2$.*

Proof. Consider the following Lie subalgebras of $C^\infty(S^1, \mathbb{R})$

$$\mathfrak{b}_n = \left\{ f \in C^\infty(S^1, \mathbb{R}) : f^{(k)}(-1) = 0, \text{for } 0 \leq k \leq n \right\}, \tag{4.1.3}$$

and $\mathfrak{b}_\infty = \left\{ f \in C^\infty(S^1, \mathbb{R}) : f^{(k)}(-1) = 0, \text{for all } k \in \mathbb{N} \right\}$.

To each algebra corresponds a Lie subgroup of $\mathrm{Diff}_+(S^1)$,

$$B_n := \left\{ \gamma \in \mathrm{Diff}_+(S^1) : \gamma(-1) = -1, \quad \gamma^{(1)}(-1) = 1, \gamma^{(k)}(-1) = 0, \ \text{for } 2 \leq k \leq n \right\}, \tag{4.1.4}$$

and $B_\infty := \left\{ \gamma \in \mathrm{Diff}_+(S^1) : \gamma(-1) = -1, \gamma^{(1)}(-1) = 1, \gamma^{(k)}(-1) = 0, \ \text{for all } k \geq 2 \right\}$.

By explicit calculations, B_n is a normal subgroup of B_1 for every $n \geq 1$, as in [Tan10]. The quotient B_1/B_n is a finite-dimensional Lie group with Lie algebra $\mathfrak{b}_1/\mathfrak{b}_n$. An element $[\gamma] \in B_1/B_n$ is completely determined by the numbers $\{\gamma^{(k)}(-1)\}_{k=2}^n$ and the product is

$$\left(\{\gamma_1^{(k)}(-1)\}_{k=2}^n\right) \cdot \left(\{\gamma_2^{(k)}(-1)\}_{k2}^n\right) = \left\{ (\gamma_1 \circ \gamma_2)^{(k)} (-1) \right\}_{k=2}^n.$$

Analogously, every element $[f]$ of the Lie algebra $\mathfrak{b}_1/\mathfrak{b}_n$ is completely determined by the numbers $\{f^{(k)}(-1)\}_{k=2}^n$.

The colimit of the sequence of Lie algebras

$$\mathfrak{b}_1/\mathfrak{b}_2 \longleftarrow \cdots \longleftarrow \mathfrak{b}_1/\mathfrak{b}_n \longleftarrow \mathfrak{b}_1/\mathfrak{b}_{n+1} \longleftarrow \cdots \tag{4.1.5}$$

is the Lie algebra $x^2\mathbb{C}[[x]]$, where sum, product and derivation are the usual ones for formal power series and the Lie bracket is $[f, g] := f'g - g'f$, $f, g \in x^2\mathbb{C}[[x]]$. Note that $x^2\mathbb{C}[[x]] \simeq \mathfrak{b}_1/\mathfrak{b}_\infty$ by Borel's Lemma.

The colimit of the sequence of groups

$$B_1/B_2 \longleftarrow \cdots \longleftarrow B_1/B_n \longleftarrow B_1/B_{n+1} \longleftarrow \cdots \tag{4.1.6}$$

is the group $x + x^2\mathbb{C}[[x]]$ with product given by the composition of formal power series.

Since $\mathfrak{b}_1/\mathfrak{b}_n$ is a nilpotent Lie algebra, the exponential map $\mathrm{Exp}_n : \mathfrak{b}_1/\mathfrak{b}_n \longrightarrow B_1/B_n$ is surjective [CG90][Theorem 1.2.1]. We prove that Exp_n agrees with the projection of the exponential Exp of $\mathrm{Diff}_+(S^1)$ on $\mathfrak{b}_1/\mathfrak{b}_n$. Let $f, g \in \mathfrak{b}_1$ such that $[f] = [g]$ in $\mathfrak{b}_1/\mathfrak{b}_n$, i.e. $f = g + h$ with $h \in \mathfrak{b}_n$. We need to show that $\mathrm{Exp}(f) \circ \mathrm{Exp}(-g) \in B_n$ or $\mathrm{Exp}(f)^{(k)}(-1) = \mathrm{Exp}(g)^{(k)}(-1)$ for $0 \le k \le n-1$. We have that

$$\frac{d}{dt}\left(\frac{d^k}{d\theta^k}\mathrm{Exp}(tf)(e^{i\theta})\Big|_{e^{i\theta}=-1}\right) = \frac{d^k}{d\theta^k}\left(\frac{d}{dt}\mathrm{Exp}(tf)(e^{i\theta})\right)\Big|_{e^{i\theta}=-1} = \frac{d^k}{d\theta^k} f(\mathrm{Exp}(tf)(e^{i\theta}))\Big|_{e^{i\theta}=-1} \tag{4.1.7}$$

and observe that making explicit calculations, in last term of the equation $\frac{d^k}{d\theta^k}\mathrm{Exp}(tf)(e^{i\theta})\Big|_{e^{i\theta}=-1}$ does not appear because $f^{(1)}(-1) = 0$. Reasoning by induction on k and since $\frac{d^k}{d\theta^k}\mathrm{Exp}(tf)(e^{i\theta})\Big|_{e^{i\theta}=-1}$ and $\frac{d^k}{d\theta^k}\mathrm{Exp}(tg)(e^{i\theta})\Big|_{e^{i\theta}=-1}$ satisfy the same differential equation with the same initial data, we can conclude that $\mathrm{Exp}(f)^{(k)}(-1) = \mathrm{Exp}(g)^{(k)}(-1)$ for $0 \le k \le n-1$.

The colimit of the Exp_n maps is in particular surjective. Furthermore it agrees with Exp projected on $\mathfrak{b}_1/\mathfrak{b}_\infty$.

□

Proposition 4.1.3. *Let $\{\lambda_n^+\}_{n\ge2}, \{\lambda_m^-\}_{m\ge2}$ be two sequences of real numbers. There exists $g \in C^1(S^1,\mathbb{R})$, g smooth on $S^1 \setminus \{-1\}$, such that $\mathrm{Exp}(g) \in \mathrm{Diff}_+^{1,\infty}(S^1)$, $\mathrm{Exp}(g)$ is smooth on $S^1 \setminus \{-1\}$ and $\partial_+^n\mathrm{Exp}(g)(-1) = \lambda_n^+$, $\partial_-^m\mathrm{Exp}(g)(-1) = \lambda_m^-$ for all $n, m \ge 2$.*

Proof. From Lemma 4.1.2 applied to $\{\lambda_n^+\}_{n\ge2}, \{\lambda_m^-\}_{m\ge2}$, there exist $g_+, g_- \in C^\infty(S^1,\mathbb{R})$ such that $\mathrm{Exp}(g_+)^{(n)}(-1) = \{\lambda_n^+\}$ and $\mathrm{Exp}(g_-)^{(n)}(-1) = \{\lambda_n^-\}$, $m, n \ge 2$. From g_+ and g_- we can construct a g which is smooth on $S^1 \setminus \{-1\}$, g is in $C^1(S^1,\mathbb{R})$ and $g|_{I_+} = g_+|_{I_+}$, $g|_{I_-} = g_-|_{I_-}$. □

Proposition 4.1.4. *Let $\gamma \in \mathrm{Diff}_+^{1,\infty}(S^1)$, smooth on $S^1 \setminus \{-1\}$. There exist $g \in C^{1,\infty}(S^1,\mathbb{R}), \varphi \in \mathrm{Diff}_+(S^1)$ such that $\gamma = \mathrm{Exp}(g) \circ \varphi$.*

Proof. Up to composing γ with a dilation and a rotation, we can assume that $\gamma(-1) = -1$ and $\gamma^{(1)}(-1) = 1$. Let $\{\lambda_n^+\}_{n\ge2} := \{\partial_+^n\gamma(-1)\}$ and $\{\lambda_m^-\}_{m\ge2} := \{\partial_-^m\gamma(-1)\}$. By Proposition 4.1.3 there exists $g \in C^1(S^1,\mathbb{R})$, g smooth in $S^1 \setminus \{-1\}$ such that $\mathrm{Exp}(g) \in \mathrm{Diff}_+^{1,\infty}(S^1)$ and $\partial_+^n\mathrm{Exp}(g)(-1) = \lambda_n^+$, $\partial_-^m\mathrm{Exp}(g)(-1) = \lambda_m^-$ for all $n, m \geq 2$. It follows that $\varphi := \gamma \circ \operatorname{Exp}(-g)$ is an element of $\operatorname{Diff}^{1,\infty}_+(S^1)$ such that $\partial^k_+\varphi(-1) = \partial^k_-\varphi(-1) = 0$ for all $k \geq 2$ and in particular is an element of $B_\infty \subset \operatorname{Diff}_+(S^1)$. □

Corollary 4.1.5. *Let $(\mathcal{A}, U, \Omega)$ be a conformal net. The representation U extends to $\operatorname{Diff}^{1,\infty}_+(S^1)$ and the net is covariant with respect to $\operatorname{Diff}^{1,\infty}_+(S^1)$.*

Proof. If γ fixes the point -1 and $\gamma^{(1)}(-1) = 1$, we define $U(\gamma) := U(\operatorname{Exp}(g))U(\varphi)$. It is enough to show the covariance for exponentials. Let $\{f_n\} \in C^\infty(S^1, \mathbb{R})$ converging to $f \in C^{1,\infty}(S^1, \mathbb{R}) \subset \mathcal{S}_{\frac{3}{2}}(S^1)$, see [CW05, Lemma 4.6]. Let $\gamma_n := \operatorname{Exp}(f_n)$. By Proposition 1.2.11 it follows that $e^{iT(f_n)}$ converges strongly to $e^{iT(f)}$. The rest of the proof is the same as in Proposition 3.3.1.
We want to show that the map U is a well-defined. This is clear if γ fixes the point -1 and $\gamma^{(1)}(-1) = 1$, since the action of $U(\gamma) := U(\operatorname{Exp}(g))U(\varphi)$ on the local algebras is defined by U and $\bigvee_{I \in \mathcal{I}_\mathbb{R}} \mathcal{A}(I) = \mathcal{B}(H)$. If γ has only one non-smooth point we can write $\gamma = \gamma_1\hat{\gamma}\gamma_2$ with γ_1, γ_2 smooth and $\hat{\gamma}$ which fixes -1 and $\hat{\gamma}^{(1)}(-1) = 1$ and define $U(\gamma) := U(\gamma_1)U(\hat{\gamma})U(\gamma_2)$. If γ has a finite number of non-smooth points, we can write $\gamma = \hat{\gamma}\bar{\varphi}$ with $\hat{\gamma}$ which fixes the non-smooth points and $\operatorname{supp}(\hat{\gamma})$ is a disjoint union of intervals. We define $U(\gamma)$ as the product of each non-smooth component as defined above. We want to show now that U is a unitary projective representation of $\operatorname{Diff}^{1,\infty}_+(S^1)$. If If γ_1, γ_2 fix the point -1 and $\gamma_1^{(1)}(-1) = \gamma_2^{(1)}(-1) = 1$, $\operatorname{Ad} U(\gamma_1\gamma_2)$ and $\operatorname{Ad} U(\gamma_1)U(\gamma_2)$ implement the same action on the local algebras, so $U(\gamma_1\gamma_2)$ and $U(\gamma_1)U(\gamma_2)$ differ by a scalar. This is true also when γ_1 and γ_2 have only one and the same non-smooth point. If γ_1 and γ_2 have a finite number of non-smooth points, let $\gamma_i = \hat{\gamma}_i\bar{\varphi}_i$ as above. We have that the components of $\bar{\varphi}_1\hat{\gamma}_2\bar{\varphi}_1^{-1}$ and $\hat{\gamma}_1$ are either disjoint or have a common non-smooth point. In the first case the representation U commute. In the second case, we have the homomorphism property as above. So we the decomposition $\gamma_1\gamma_2 = (\hat{\gamma}_1\bar{\varphi}_1\hat{\gamma}_2\bar{\varphi}_1^{-1})\bar{\varphi}_1\bar{\varphi}_2$, with $\hat{\gamma}_1\bar{\varphi}_1\hat{\gamma}_2\bar{\varphi}_1^{-1}$ supported around the non-smooth points and $\bar{\varphi}_1\bar{\varphi}_2 \in \operatorname{Diff}_+(S^1)$ and we have that $U(\gamma_1)U(\gamma_2) = U(\hat{\gamma}_1)U(\bar{\varphi}_1)U(\hat{\gamma}_2)U(\bar{\varphi}_2) = U(\hat{\gamma}_1\bar{\varphi}_1\hat{\gamma}_2\bar{\varphi}_1^{-1})U(\bar{\varphi}_1\bar{\varphi}_2) = U(\gamma_1\gamma_2)$. □

4.2 Positivity of energy

Let us first observe that $\operatorname{Exp}(tg)$ makes sense if g is C^1, because then the existence and uniqueness of the ODE are assured. We need some preparatory results on representations of these elements.

Lemma 4.2.1. *Let $g \in C^\infty(S^1)$ and f be a real piecewise smooth and C^1-function on S^1. Then it holds that*

$$\operatorname{Ad} e^{iT(g)}(e^{iT(f)}) = e^{i(T(\operatorname{Exp}(g)_*(f)) + \beta(\operatorname{Exp}(g), f))}.$$

Proof. Let $f_n \in C^\infty(S^1)$. We use Proposition 1.2.8, so

$$\operatorname{Ad} e^{iT(g)}(e^{iT(f_n)}) = e^{i(T(\operatorname{Exp}(g)_*(f_n))+\beta(\operatorname{Exp}(g),f_n))}. \tag{4.2.1}$$

We choose an s such that $2 < s < \frac{5}{2}$. We want to show that $f \in H^s(S^1)$. Indeed, f'' is everywhere defined except a finite number of points and is of bounded variation, so using [Wei06, Lemma 2.2], we have $|k^2 \hat{f}_k| \leq \left|\frac{\operatorname{Var}(f'')}{k}\right|$, where $\operatorname{Var}(f'')$ is the variation of f''. From this follows that $|k|^{2s}|\hat{f}_k|^2 \leq \left|\frac{\operatorname{Var}(f'')^2}{k^{6-2s}}\right|$ and the right-hand side is summable in k as $6-2s > 1$, hence $f \in H^s(S^1)$.

Next, let us observe that $H^s(S^1) \subset \mathcal{S}_{\frac{3}{2}}(S^1)$. Indeed,

$$\sum_k (1+|k|)^{\frac{3}{2}}|\hat{f}_k| \leq \sum_k (1+|k|)^s |\hat{f}_k| \cdot (1+|k|)^{\frac{3}{2}-s} \leq 2\sum_k (1+|k|^2)^{\frac{s}{2}}|\hat{f}_k| \cdot (1+|k|)^{\frac{3}{2}-s}$$

and the right-hand side can be seen as a scalar product of two $\ell^2(\mathbb{Z})$ sequences (because $s > 2$), hence it holds that $\|f\|_{\frac{3}{2}} \leq \text{Const.}\|f\|_{H^s}$, where the constant depends on s but not on f.

We can choose a sequence $\{f_n\} \subset C^\infty(S^1), \|f - f_n\|_{H^s} \to 0$, so in particular $f_n \to f$ in $\mathcal{S}_{\frac{3}{2}}(S^1)$. By [IKT13, Lemma B.2], $f \mapsto \operatorname{Exp}(g)_*(f)$ is continuous in $H^s(S^1)$, hence $\operatorname{Exp}(g)_*(f_n) \to \operatorname{Exp}(g)_*(f)$ in $\mathcal{S}_{\frac{3}{2}}(S^1)$. By [CW05, Proposition 4.5], $T(\operatorname{Exp}(g)_*(f_n)) \to T(\operatorname{Exp}(g)_*(f))$ in the strong resolvent sense, and $\beta(\operatorname{Exp}(g), f_n) \to \beta(\operatorname{Exp}(g), f)$. Taking the limit of (4.2.1), we obtain the claim. □

Remark 4.2.2. If $f \in C^1$ and not C^2, then $f \notin H^s(S^1), s > \frac{5}{2}$ since with such s it holds that $H^s(S^1) \subset C^2(S^1)$ by the Sobolev-Morrey embedding.

Lemma 4.2.3. *Let $f, g \in C^\infty(S^1)$ and $g(-1) = g'(-1) = f(-1) = f'(-1) = 0$ and compactly supported. Let $I_\pm$ be disjoint intervals in S^1 one of whose boundary points is -1. Let $f = f_- + f_+, f_\pm \in \mathcal{S}_{\frac{3}{2}}(S^1)$ be the decomposition of f into two pieces cut at the point -1 (which is possible by [Wei06, Lemma 2.2]), and similarly introduce $g = g_- + g_+, g_\pm \in \mathcal{S}_{\frac{3}{2}}(S^1)$, and assume that $\operatorname{supp} f_\pm, \operatorname{supp} g_\pm \subset I_\pm$.*

Then it holds that

$$\operatorname{Ad} e^{iT(g_-)}(T(f_-)) = T(\operatorname{Exp}(g_-)_*(f_-)) + \beta(\operatorname{Exp}(g_-), f_-),$$

where $\beta(\operatorname{Exp}(g_-), f_-)$ is defined by a similar formula as before:

$$\beta(\operatorname{Exp}(g_-), f) := \frac{c}{24\pi}\int_{\operatorname{supp} g_-} \{\operatorname{Exp}(g_-), z\}\Big|_{z=e^{i\theta}} f(e^{i\theta})e^{i2\theta}d\theta, \tag{4.2.2}$$

where the integral is restricted to $\operatorname{supp} g_-$ in which the Schwarzian derivative is defined.

Proof. Let $t \in \mathbb{R}$. Since f_- is piecewise smooth and C^1 and g is smooth, by Lemma 4.2.1 we have

$$\mathrm{Ad}\, e^{iT(g)}(e^{iT(tf_-)}) = e^{iT(\mathrm{Exp}(g)_*(tf_-))} e^{i\beta(\mathrm{Exp}(g),tf_-)}$$

and

$$\begin{aligned}\beta(\mathrm{Exp}(g), tf_-) &= \frac{c}{24\pi}\int_0^{2\pi} \{\mathrm{Exp}(g), z\}\Big|_{z=e^{i\theta}} tf_-(e^{i\theta})e^{i2\theta}d\theta \\ &= \frac{c}{24\pi}\int_{\mathrm{supp}\, g_-} \{\mathrm{Exp}(g_-), z\}\Big|_{z=e^{i\theta}} tf_-(e^{i\theta})e^{i2\theta}d\theta \\ &= \beta(\mathrm{Exp}(g_-), tf_-),\end{aligned}$$

because $\mathrm{Exp}(g_-)$ and f_- has support contained in a common interval where $\mathrm{Exp}(g_-)$ is smooth.

By [Wei06, Proposition 2.3], $e^{iT(tf_\pm)}$ and $e^{iT(g_\pm)}$ are affiliated to $\mathcal{A}(I_\pm)$. Note that $g_\pm \in \mathcal{S}_{\frac{3}{2}}(S^1)$, hence it follows that $e^{iT(g)} = e^{iT(g_-)}e^{iT(g_+)}$. By the assumed support property, we have

$$\mathrm{Ad}\, e^{iT(g)}(e^{iT(tf_-)}) = \mathrm{Ad}\,(e^{iT(g_-)}\cdot e^{iT(g_+)})(e^{iT(tf_-)}) = e^{iT(\mathrm{Exp}(g_-)_*(tf_-))}\cdot e^{i\beta(\mathrm{Exp}(g_-),tf_-)}.$$

By taking the derivative with respect to t, we obtain

$$\mathrm{Ad}\, e^{iT(g)}(T(f_-)) = \mathrm{Ad}\, e^{iT(g_-)}(T(f_-)) = T(\mathrm{Exp}(g_-)_*(f_-)) + \beta(\mathrm{Exp}(g_-), f_-),$$

on the full domain.

□

Theorem 4.2.4. *A soliton σ of a conformal net $(\mathcal{A}, U, \Omega)$ is $\mathit{Diff}^{1,\infty}_{+,1}$-covariant and has positive energy.*

Proof. The strategy is to write the translation as a product of three elements: two of them are localized in half-lines and the other on an interval. First of all, we define $I_{(\theta_1,\theta_2)} := \{e^{i\theta} : \theta_1 < \theta < \theta_2\} \subset S^1$. Then we take a C^∞ function $h_+ : S^1 \setminus \{-1\} \to \mathbb{R}$ which is equal to 0 on $I_{(-\pi,0)}$ and equal to 1 on $I_{(\frac{\pi}{2},0)}$. Similarly, let $h_- : S^1\setminus\{-1\} \to \mathbb{R}$ be a C^∞ function which is equal to 1 on $I_{(-\pi,-\frac{\pi}{2})}$ and equal to 0 on $I_{(0,\pi)}$. The two functions have disjoint supports.

Let us first prove the following relation:

$$\mathrm{Ad}\, e^{itT(h_-\tau)}(T(\tau)) = T(\mathrm{Exp}(h_-\tau)_*(\tau)) + \beta(\mathrm{Exp}(h_-\tau), \tau), \qquad (4.2.3)$$

with τ the generator of translations. Note that $h_-\tau$ is supported in a certain interval I_-, one of whose boundary is -1, hence so is $\mathrm{Exp}(th_-\tau)$. We decompose τ into two pieces $\tau_+, \tau_- \in \mathcal{S}_{\frac{3}{2}}(S^1)$ such that $\tau_-(e^{i\theta}) = \tau(e^{i\theta})$ on I_- and $\tau_+ = \tau - \tau_-$. Note that $\beta(\mathrm{Exp}(th_-\tau), \tau_-) = \beta(\mathrm{Exp}(th_-\tau), \tau)$, since the supports of $\mathrm{Exp}(th_-\tau)$ and of τ_+ are disjoint (4.2.2). As $h_-\tau$ coincides with τ on a neighborhood of -1, we can apply Lemma 4.2.3 to obtain

$$\begin{aligned}\mathrm{Ad}\, e^{itT(h_-\tau)}(T(\tau_-)) &= T(\mathrm{Exp}(th_-\tau)_*(\tau_-)) + \beta(\mathrm{Exp}(th_-\tau), \tau_-)\\ &= T(\mathrm{Exp}(th_-\tau)_*(\tau_-)) + \beta(\mathrm{Exp}(th_-\tau), \tau).\end{aligned}$$

One the other hand, since $h_-\tau$ and τ_+ have disjoint support, we have

$$\mathrm{Ad}\, e^{itT(h_-\tau)}(T(\tau_+)) = T(\tau_+).$$

Note that $\mathrm{Exp}(th_-\tau)_*\tau = \mathrm{Exp}(th_-\tau)_*\tau_+ + \mathrm{Exp}(th_-\tau)_*\tau_- = \tau_+ + \mathrm{Exp}(th_-\tau)_*\tau_-$. By adding these operator equations, we obtain on the intersection of the domains

$$\mathrm{Ad}\, e^{itT(h_-\tau)}(T(\tau)) = T(\mathrm{Exp}(th_-\tau)_*(\tau)) + \beta(\mathrm{Exp}(th_-\tau), \tau).$$

The intersection contains $C^\infty(L_0)$, hence the right-hand side is essentially self-adjoint. Hence the left-hand side is a self-adjoint extension of the right-hand side, and therefore, they must coincide on the full domain.

Next, we write $e^{itT(\tau)}$ as

$$e^{itT(\tau)} = e^{itT(h_-\tau)} \cdot e^{-itT(h_-\tau)} e^{itT(\tau)} e^{-itT(h_+\tau)} \cdot e^{itT(h_+\tau)}.$$

We claim that $e^{-itT(h_-\tau)} e^{itT(\tau)} e^{-itT(h_+\tau)}$ is localized on a bounded interval (the interval depends on t). This claim follows from (4.2.3). Indeed, $\mathrm{Exp}(th_-\tau)_*(\tau)$ agrees with τ in a neighborhood of the point at infinity (depending on t),

$$\begin{aligned}e^{-itT(h_-\tau)} e^{itT(\tau)} e^{-itT(h_+\tau)} &= e^{itT(\mathrm{Exp}(th_-\tau)_*(\tau))} e^{i\beta(\mathrm{Exp}(th_-\tau)_*(\tau),\tau)} e^{-itT(h_-\tau)} e^{-itT(h_+\tau)}\\ &= e^{itT(\mathrm{Exp}(th_-\tau)_*(\tau))} e^{i\beta(\mathrm{Exp}(th_-\tau)_*(\tau),\tau)} e^{-itT(h_-\tau+h_+\tau)},\end{aligned}$$

where we used the linearity of T on functions of class $\mathcal{S}_{\frac{3}{2}}(S^1)$, and the last expression is localized in a bounded interval: as $h_-\tau + h_+\tau$ equals τ in a neighborhood of $-1 \in S^1$, $\mathrm{Ad}\, e^{-itT(h_-\tau+h_+\tau)}$ implements the same action on $\mathcal{A}(I_{t,\epsilon})$ for some neighborhood $I_{t,\epsilon}$ for small t as the action of $\mathrm{Ad}\, e^{itT(\mathrm{Exp}(h_+\tau)_*(\tau))}$. In other words, $\mathrm{Ad}\, e^{-itT(h_-\tau+h_+\tau)} e^{itT(\mathrm{Exp}(h_+\tau)_*(\tau))}$ is trivial on $\mathcal{A}(I_{t,\epsilon})$, which implies that $e^{-itT(h_-\tau+h_+\tau)} e^{itT(\mathrm{Exp}(h_+\tau)_*(\tau))}$ is localized in $I'_{t,\epsilon}$.

We introduce a representation of the translation group by

$$U_\sigma(t) := \sigma(e^{itT(h_-\tau)}) \sigma(e^{-itT(h_-\tau)} e^{itT(\tau)} e^{-itT(h_+\tau)}) \sigma(e^{itT(h_+\tau)})$$

By noting that h_- and h_+ have disjoint supports, this yields a one parameter group in t:

$$\begin{aligned}
U_\sigma(t_1)U_\sigma(t_2) &= \sigma(e^{it_1T(h_-\tau)})\sigma(e^{-it_1T(h_-\tau)}e^{it_1T(\tau)}e^{-it_1T(h_+\tau)})\sigma(e^{it_1T(h_+\tau)}) \\
&\quad\cdot \sigma(e^{it_2T(h_-\tau)})\sigma(e^{-it_2T(h_-\tau)}e^{-it_2T(\tau)}e^{it_2T(h_+\tau)})\sigma(e^{it_2T(h_+\tau)}) \\
&= \sigma(e^{it_1T(h_-\tau)})\sigma(e^{it_2T(h_-\tau)})\sigma(e^{-it_1T(h_-\tau)}e^{it_1T(\mathrm{Exp}(t_2h_-\tau)_*(\tau))}e^{i\beta(\mathrm{Exp}(t_2h_-\tau),\tau)}e^{-it_1T(h_+\tau)}) \\
&\quad\cdot \sigma(e^{it_1T(h_+\tau)})\sigma(e^{-it_2T(h_-\tau)}e^{it_2T(\tau)}e^{-it_2T(h_+\tau)})\sigma(e^{it_2T(h_+\tau)}) \\
&= \sigma(e^{it_1T(h_-\tau)})\sigma(e^{it_2T(h_-\tau)})\sigma(e^{-it_1T(h_-\tau)}e^{it_1T(\mathrm{Exp}(t_2h_-\tau)_*(\tau))}e^{i\beta(\mathrm{Exp}(t_2h_-\tau),\tau)}e^{-it_1T(h_+\tau)}) \\
&\quad\cdot \sigma(e^{-it_2T(h_-\tau)}e^{it_2T(\mathrm{Exp}(t_1h_+)_*\tau)}e^{i\beta(\mathrm{Exp}(t_1h_+\tau),\tau)}e^{-it_2T(h_+\tau)})\sigma(e^{it_1T(h_+\tau)})\sigma(e^{it_2T(h_+\tau)}) \\
&= \sigma(e^{it_1T(h_-\tau)})\sigma(e^{it_2T(h_-\tau)}) \\
&\quad\cdot \sigma(e^{-it_1T(h_-\tau)}e^{-it_2T(h_-\tau)}e^{it_1T(\tau)}e^{i\beta(\mathrm{Exp}(-t_2h_-\tau),\mathrm{Exp}(t_2h_-\tau)_*(\tau))}e^{i\beta(\mathrm{Exp}(t_2h_-\tau),\tau)}e^{-it_1T(h_+\tau)}) \\
&\quad\cdot \sigma(e^{it_2T(\mathrm{Exp}(t_1h_+)_*\tau)}e^{i\beta(\mathrm{Exp}(t_1h_+\tau),\tau)}e^{-it_2T(h_+\tau)})\sigma(e^{it_1T(h_+\tau)})\sigma(e^{it_2T(h_+\tau)}) \\
&= \sigma(e^{it_1T(h_-\tau)}e^{it_2T(h_-\tau)})\sigma(e^{-it_1T(h_-\tau)}e^{-it_2T(h_-\tau)}e^{it_1T(\tau)}e^{it_2T(\tau)}e^{-it_1T(h_+\tau)}e^{-it_2T(h_+\tau)}) \\
&\quad\cdot \sigma(e^{it_1T(h_+\tau)}e^{it_2T(h_+\tau)})e^{i\beta(\mathrm{Exp}(t_2h_-\tau),\tau)}e^{i\beta(\mathrm{Exp}(-t_2h_-\tau),\mathrm{Exp}(t_2h_-\tau)_*(\tau))} \\
&\quad\cdot e^{i\beta(\mathrm{Exp}(t_1h_+\tau),\tau)}e^{i\beta(\mathrm{Exp}(-t_1h_+\tau),\mathrm{Exp}(t_1h_+\tau)_*(\tau))} \\
&= U_\sigma(t_1+t_2),
\end{aligned}$$

where we used the equality

$$\beta(\gamma_1\circ\gamma_2, f) = \beta(\gamma_1, \gamma_{2*}(f)) + \beta(\gamma_2, (f)),$$

which implies

$$\begin{aligned}
0 = \beta(\mathrm{id},\tau) = \beta(\mathrm{Exp}(-t_2h_-\tau),\mathrm{Exp}(t_2h_-\tau)_*(\tau)) + \beta(\mathrm{Exp}(t_2h_-\tau),\tau), \\
0 = \beta(\mathrm{id},\tau) = \beta(\mathrm{Exp}(-t_1h_+\tau),\mathrm{Exp}(t_1h_+\tau)_*(\tau)) + \beta(\mathrm{Exp}(t_1h_+\tau),\tau).
\end{aligned}$$

It remains to prove the positivity of energy. We do this by showing that $U_\sigma(t)$ can be obtained as a limit in the strong resolvent sense of a sequence of one-parameter unitary groups with positive generator.

Let $\tau_1 : \mathbb{R} \to \mathbb{R}$ be a C^∞ vector field equal to 1 on $(-\infty, 1)$ and equal to 0 on $(2, +\infty)$. From τ_1 we construct a sequence of vector fields

$$\tau_n(x) := \tau_1\left(\frac{x}{n}\right), \qquad x \in \mathbb{R},\ n \in \mathbb{N}.$$

We fix $2 < s < \frac{5}{2}$ (cf. the proof of Lemma 4.2.1). Let us show that $\tau_n \to \tau$ in the $H^s(S^1)$ topology as vector fields on S^1. The expression of τ_n in angular coordinates is

$$\tau_n(x(\theta)) = (1+\cos(\theta))\,\tau_1\left(\frac{x(\theta)}{n}\right).$$

For this it is sufficient to show that $\left\{\frac{d^3}{d\theta^3}\tau_n\right\}_{n\in\mathbb{N}}$ is a sequence of functions in $L^1(S^1)$ uniformly bounded in n and that $\tau_n \to \tau$ in $L^1(S^1)$: this implies that

$|k^3\hat{\tau}_n(k)| <$ Const., where $\hat{\tau}_n(k)$ is the k-th Fourier coefficient of τ_n, or equivalently, $|k^{2s}\hat{\tau}_n(k)| < \frac{\text{Const.}}{k^{6-2s}}$, and the right-hand side is summable in k since $6-2s > 1$. From the convergence $\tau_n \to \tau$ in L^1 we obtain the convergence of each $\hat{\tau}_n(k)$, Theorefore, by the Lebesgue dominiated convergence theorem (applied to the measurable set $\mathbb{Z}$ with the counting measure), we obtain the convergence $\tau_n \to \tau$ in $H^s(S^1)$.

The third derivative of τ_n is

$$\begin{aligned}\frac{d^3}{d\theta^3}\tau_n(x(\theta)) = \sin(\theta)\tau_1\left(\frac{x(\theta)}{n}\right) - \frac{1}{n}\frac{2\cos(\theta)}{(1+\cos(\theta))}\frac{d}{dx}\tau_1\left(\frac{x(\theta)}{n}\right) \\ - \frac{1}{n}\frac{\sin^2(\theta)}{(1+\cos(\theta))^2}\frac{d}{dx}\tau_1\left(\frac{x(\theta)}{n}\right) + \frac{1}{n^3}\frac{1}{(1+\cos(\theta))^2}\frac{d^3}{dx^3}\tau_1\left(\frac{x(\theta)}{n}\right).\end{aligned} \tag{4.2.4}$$

The first term of the right-hand side of (4.2.4) is clearly uniformly bounded in n on S^1. For the second term of the right-hand side of (4.2.4) we have:

$$\begin{aligned}&\int_0^{2\pi}\left|\frac{1}{n}\frac{2\cos(\theta)}{(1+\cos(\theta))}\frac{d}{dx}\tau_1\left(\frac{x(\theta)}{n}\right)\right|d\theta = \int_n^{2n}\left|\frac{2\cos(\theta(x))}{n}\frac{d}{dx}\tau_1\left(\frac{x}{n}\right)\right|dx \\ &= \int_1^2\left|2\cos(\theta(y))\left(\frac{d}{dx}\tau_1\left(\frac{x}{n}\right)\Big|_{\frac{x}{n}=y}\right)\right|dy\end{aligned}$$

which does not depend on n.

The third term is

$$\begin{aligned}&\int_0^{2\pi}\left|\frac{1}{n}\frac{\sin^2(\theta)}{(1+\cos(\theta))^2}\frac{d}{dx}\tau_1\left(\frac{x(\theta)}{n}\right)\right|d\theta = \int_n^{2n}\left|\frac{1}{n}\frac{\sin^2(\theta)}{(1+\cos(\theta))}\frac{d}{dx}\tau_1\left(\frac{x}{n}\right)\right|dx \\ &\le \int_n^{2n}\left|\frac{1}{n}\frac{\sin^2(\theta(n))}{(1+\cos(\theta(2n))}\frac{d}{dx}\tau_1\left(\frac{x}{n}\right)\right|dx = \int_n^{2n}\left|\frac{2n^2(1+4n^2)}{n(1+n^2)^2}\frac{d}{dx}\tau_1\left(\frac{x}{n}\right)\right|dx \\ &= \int_1^2\left|\frac{2n^2(1+4n^2)}{(1+n^2)^2}\left(\frac{d}{dx}\tau_1\left(\frac{x}{n}\right)\Big|_{\frac{x}{n}=y}\right)\right|dy\end{aligned}$$

which is uniformly bounded in n.

The fourth term is uniformly bounded in n since

$$\begin{aligned}&\int_0^{2\pi}\left|\frac{1}{n^3}\frac{1}{(1+\cos(\theta))^2}\frac{d^3}{dx^3}\tau_1\left(\frac{x(\theta)}{n}\right)\right|d\theta = \int_n^{2n}\left|\frac{1}{n^3}\left(\frac{1}{1+\cos(\theta(x))}\right)\frac{d^3}{dx^3}\tau_1\left(\frac{x}{n}\right)\right|dx \\ &\le \int_n^{2n}\left|\frac{1}{n^3}\left(\frac{1}{1+\cos(\theta(2n))}\right)\frac{d^3}{dx^3}\tau_1\left(\frac{x}{n}\right)\right|dx = \int_n^{2n}\left|\frac{1+4n^2}{2n^3}\frac{d^3}{dx^3}\tau_1\left(\frac{x}{n}\right)\right|dx \\ &= \int_1^2\left|\frac{1+4n^2}{2n^3}\left(\frac{d^3}{dx^3}\tau_1\left(\frac{x}{n}\right)\Big|_{\frac{x}{n}=y}\right)\right|dy.\end{aligned}$$

We need only to show that $\tau_n \to \tau$ in $L^1(S^1)$:

$$\int_0^{2\pi} \big|\tau(\theta) - \tau_n(\theta)\big| d\theta = \int_0^{2\pi} \left| \left(\frac{1+\cos(\theta)}{2}\right)\left(1 - \tau_1\left(\frac{x(\theta)}{n}\right)\right)\right| d\theta$$
$$= \int_n^{+\infty} \left| \left(\frac{(1+\cos(\theta(x)))^2}{2}\right)\left(1-\tau_1\left(\frac{x}{n}\right)\right)\right| dx = \int_1^{+\infty} \left|\frac{n}{2}(1+\cos(\theta(ny)))^2\,(1-\tau_1(y))\right| dy$$
$$= \int_1^{+\infty} \left| \left(\frac{\sqrt{2n}}{1+n^2y^2}\right)^2 (1-\tau_1(y))\right| dy \le \frac{2}{n^3}\int_1^{+\infty} \frac{1}{y^4} dy \longrightarrow 0 \qquad (\text{as } n \to \infty).$$

The representation $U_\sigma(t)$ can be obtained as the limit of $\sigma(e^{itT(\tau_n)})$ in the strong topology. Indeed,

$$\sigma(e^{itT(\tau_n)}) = \sigma(e^{itT(h_-\tau_n)})\sigma(e^{-itT(h_-\tau_n)}e^{itT(\tau_n)}e^{-itT(h_+\tau_n)})\sigma(e^{itT(h_+\tau_n)})$$

Note that h_-, h_+, τ_n belong to $H^s(S^1)$, and the product is (jointly) continuous [IKT13, Lemma B.4], hence both $h_-\tau_n$ and $h_+\tau_n$ are convergent in $H^s(S^1)$, and by the argument of Lemma 4.2.1, they are convergent in $\mathcal{S}_{\frac{3}{2}}(S^1)$, hence the corresponding operators are convergent in the strong resolvent sense. Furthremore, each of these sequences are localized in a fixed interval or a half line, by the normality of σ on half lines, the convergence follows.

We have by Theorem 1.2.9 that $T(\tau_1) \ge \alpha$. By the fact that the Schwarz derivative of a Möbius transformation is 0, it follows that the quantum energy inequalities are invariant under Möbius transformations and thus we have that

$$T(\delta_*^n(\tau_1)) = T(n\tau_n) \ge \alpha,$$

which implies

$$T(\tau_n) \ge \frac{\alpha}{n}.$$

Since $T(\tau_n)$ is localized on a half-line, by local normality of σ, the generator of the one-parameter group $\sigma(e^{itT(\tau_n)})$ is bounded from below. By [RS75, Theorem VIII.23], the generator of $U_\sigma(t)$, T^σ, is positive as well.

By Section 4.1, the net $(\mathcal{A}, U, \Omega)$ is $\mathrm{Diff}_+^{1;\infty}(S^1)$-covariant. Any element $\gamma \in \mathrm{Diff}_{+,1}^{1,\infty}(S^1)$ can be decomposed into a product $\gamma = \gamma_- \circ (\gamma_-^{-1} \circ \gamma \circ \gamma_+^{-1}) \circ \gamma_+$, where $\gamma_\pm \in \mathrm{Diff}_{+,1}^{1;\infty}(S^1)$ as in the proof for $U^\sigma(t)$. The definition

$$U^\sigma(\gamma) := \sigma(U(\gamma_-))\sigma(U(\gamma_-^{-1} \circ \gamma \circ \gamma_+^{-1}))\sigma(U(\gamma_+))$$

does not depend on the decomposition of γ. If I is a left half-line, we can choose γ_- such that $I \cap \mathrm{supp}\,\gamma_+ = \emptyset$ and $\mathrm{supp}\,(\gamma_-) \supset I$. Now for $x \in \mathcal{A}(I)$ the covariance $\sigma(\mathrm{Ad}\,U(\gamma)(x)) = \mathrm{Ad}\,U^\sigma(\gamma)(\sigma(x))$ follows because the both sides are localized in I_-, and by the definition $U^\sigma(\gamma \circ \gamma_+^{-1}) = \sigma(U(\gamma \circ \gamma_+^{-1}))$. □

4.3 Solitons from nonsmooth diffeomorphisms

Here we construct a continuous family of proper solitons for any conformal net $\mathcal{A}$, using the diffeomorphism covariance.

Let $\mathcal{F} \subset \mathrm{Homeo}_+(S^1)$ be the class of orientation preserving homeomorphism γ of S^1, which have the following properties

1. $\gamma(-1) = -1$,
2. γ is a smooth function in $S^1 \setminus \{-1\}$, the half-sided derivates exist even at the point -1 at all orders with the first derivatives different from zero.

As a consequence of Borel's lemma [Hör90][Theorem 1.2.6], there exists an open interval I of S^1 which contains $p = -1$ and $\gamma_{I_-}, \gamma_{I_+} \in \mathrm{Diff}_+(S^1)$ such that γ agrees with γ_{I_-} in I_- and with γ_{I_+} in I_+, where I_- and I_+ are the connected components of $I \setminus \{-1\}$.

Let $\mathcal{A}$ be a conformal net on S^1 on the Hilbert space $\mathcal{H}$ and U its associated projective representation of $\mathrm{Diff}_+(S^1)$. For $\gamma \in \mathcal{F}$ and for every $I \in \mathcal{I}_\mathbb{R}$ we choose $\gamma_I \in \mathrm{Diff}_+(S^1)$ which agrees with γ on I (there is such γ_I even if I is a half-line by the remark above). We denote by σ_γ the family of maps $\sigma_\gamma \coloneqq \{\sigma_\gamma^I\}$ where

$$\begin{aligned} \sigma_\gamma^I : & \mathcal{A}(I) \to \mathcal{B}(\mathcal{H}) \\ & x \mapsto \sigma_\gamma^I(x) = \mathrm{Ad}\, U(\gamma_I)(x) \end{aligned}$$

and $I \in \mathcal{I}_\mathbb{R}$, $\gamma \in \mathcal{F}$.

Proposition 4.3.1. *Let $\gamma \in \mathcal{F}$. The family of maps σ_γ is a soliton of the conformal net $\mathcal{A}$.*

Proof. Local normality follows because each map σ_γ^I is given by the adjoint action $\mathrm{Ad}\, U(\gamma_I)$. We show that the family of maps σ_γ is compatible, namely that, if $I \subset J$ for $I, J \in \mathcal{I}_\mathbb{R}$, then $\sigma_\gamma^J \restriction_{\mathcal{A}(I)} = \sigma_\gamma^I$. By definition, $\gamma_I, \gamma_J \in \mathrm{Diff}_+(S^1)$ agree with γ on I and J, respectively, hence they agree on I. Then on $\mathcal{A}(I)$ we have

$$\mathrm{Ad}\, U(\gamma_I) = \mathrm{Ad}\, U(\gamma_J) \circ \mathrm{Ad}\, U(\gamma_J^{-1} \circ \gamma_I) = \mathrm{Ad}\, U(\gamma_J),$$

because $\gamma_J^{-1} \circ \gamma_I$ is a diffeomorphism of the circle localized in I' and by conformal covariance (2.1.1). □

Now we show that if γ has different left and right derivatives, then σ_γ is a proper soliton. Modular theory is used as a tool to show non-triviality of the constructed soliton. Let us introduce the notation for left and right derivatives:

$$\partial_\pm \gamma(-1) = \lim_{\theta \to 0^\pm} \frac{\tilde{\gamma}(-e^{i\theta}) - \tilde{\gamma}(-1)}{\theta}. \tag{4.3.1}$$

with $\tilde{\gamma}$ the lift of γ in $\widetilde{\mathrm{Homeo}}_+(S^1)$. Furthermore, denote their ratio by

$$R_\gamma := \frac{\partial_+\gamma(-1)}{\partial_-\gamma(-1)} \tag{4.3.2}$$

which is an element of $\mathbb{R}_+$ by definition.

4.3.1 Type I solitons

Now we show that the construction in 4.3.1 using functions in a subclass of $\mathcal{F}$ indeed yields proper solitons, i.e. solitons which cannot be obtained as restrictions of representations of the conformal net on S^1. Modular theory is used as a tool to show non-triviality of the constructed soliton.

Let $\mathcal{F}_\delta$ be the class of functions in $\mathcal{F}$ of the form

$$\gamma(e^{i\theta}) := \begin{cases} e^{i\theta} & \text{if } \theta \in [-\pi, \alpha) \\ k(e^{i\theta}) & \text{if } \theta \in [\alpha, \beta) \\ \delta(s)(e^{i\theta}) & \text{if } \theta \in [\beta, \pi) \end{cases} \tag{4.3.1}$$

where $\delta(s)$ is the dilation as in equation 1.2.4, $0 < \alpha < \beta < \pi$ and k is a smooth function on $[\alpha, \beta)$ such that $\gamma \in \mathcal{F}$. Note that $\partial_-\gamma(-1) = 1$ and $\partial_+\gamma(-1) = e^s$, so the value $s = 0$ must be excluded.

Theorem 4.3.2. *Let $\gamma \in \mathcal{F}_\delta$. Then σ_γ is a proper, irreducible soliton of $\mathcal{A}$.*

Proof. From $\gamma \in \mathcal{F}_\delta$ it is possible to construct a new function σ on the circle which is always continuous but fails to be differentiable in two points, the points -1 and 1:

$$\sigma := \gamma \circ R_\pi \circ \gamma^{-1} \circ R_\pi \tag{4.3.2}$$

with R_π the rotation of π. The function

$$\psi(e^{i\theta}) := \begin{cases} e^{i\theta} & \text{if } \theta \in [-\pi, 0) \\ \delta(s)(e^{i\theta}) & \text{if } \theta \in [0, \pi) \end{cases} \tag{4.3.3}$$

is continuous and like σ fails to be differentiable in $-1, 1$. In fact there is a $\phi \in \mathrm{Diff}_+(S^1)$ such that $\phi \circ \sigma = \psi$.

If we consider the map

$$\tilde{\sigma} := \mathrm{Ad}\, U(\phi) \circ \sigma_\gamma \circ \mathrm{Ad}\, U(R_\pi) \circ \sigma_{\gamma^{-1}} \circ \mathrm{Ad}\, U(R_\pi)$$

defined on $\mathcal{A}((-\pi, 0)) \cup \mathcal{A}((0, \pi))$, then we have that $\tilde{\sigma}(x) = x$ for $x \in \mathcal{A}((-\pi, 0))$ and $\tilde{\sigma}(x) = \mathrm{Ad}\, U(\delta(s))$ for $x \in \mathcal{A}((0, \pi))$.

Suppose that σ_γ is not a proper soliton of the Virasoro net, i.e. it is the restriction of a DHR representation. In particular σ_γ is rotation covariant, [DFK04, Theorem 6], namely there is a unitary representation of the universal covering of S^1, $\theta \to U^\gamma(R_\theta)$, such that

$$\operatorname{Ad} U^\gamma(R_\theta) \circ \sigma_\gamma = \sigma_\gamma \circ \operatorname{Ad} U(R_\theta). \tag{4.3.4}$$

Then it follows that $\tilde{\sigma}$ is implemented by a unitary since

$$\begin{aligned}\tilde{\sigma} :=& \operatorname{Ad} U(\phi) \circ \sigma_\gamma \circ \operatorname{Ad} U(R_\pi) \circ \sigma_{\gamma^{-1}} \circ \operatorname{Ad} U(R_\pi)\\ =& \operatorname{Ad} U(\phi) \circ \operatorname{Ad} U^\gamma(R_\pi) \circ \sigma_\gamma \circ \sigma_{\gamma^{-1}} \circ \operatorname{Ad} U(R_\pi) =\\ =& \operatorname{Ad} U(\phi) \circ \operatorname{Ad} U^\gamma(R_\pi) \circ \operatorname{Ad} U(R_\pi).\end{aligned}$$

This unitary must belong to $\mathcal{A}((0,\pi))$ by Haag duality since $\tilde{\sigma}(x) = x$ for $x \in \mathcal{A}(-\pi, 0)$. At the same time, by the Bisognano-Wichmann theorem 2.1.4, it must implement a modular automorphism of $\mathcal{A}(0,\pi)$ with respect to the vacuum vector since $\tilde{\sigma}(x) = \operatorname{Ad} U(\delta(s))$ for $x \in \mathcal{A}((0,\pi))$. We have a contradiction because the modular automorphisms cannot be inner for a type III factor.

□

Remark 4.3.3. The functions γ are taken to be in $\mathcal{F}_\delta$ so that $\phi := \psi \circ (\gamma \circ R_\pi \circ \gamma^{-1} \circ R_\pi)^{-1}$ is a smooth diffeomorphism, where ψ is in particular a function which we know is not unitarily implementable with the aid of modular theory. If we take any $\gamma \in \mathcal{F}$, then the resulting ϕ will not necessarily be smooth but at best piecewise smooth. Thus, in order to show that any $\gamma \in \mathcal{F}$ determines a proper soliton, we need the results in Section 4.1, i.e. that a piecewise smooth diffeomorphisms ϕ is unitarily implementable in a local net $\mathcal{A}$.

Proposition 4.3.4. *Let $\mathcal{A}$ be a conformal net. If $\gamma \in \mathcal{F}$ then σ_γ is a proper soliton representation.*

Proof. Let $\varphi \in \mathcal{F}_\delta$ with $R_\gamma = R_\varphi$ and note that $\sigma_\gamma = \operatorname{Ad} U(\gamma \circ \varphi^{-1}) \circ \sigma_\varphi$ is a proper soliton for $\mathcal{A}$ since $\gamma \circ \varphi^{-1} \in \operatorname{Diff}^{1,\infty}_+(S^1)$. The equation does not depend on the choice of φ. □

Proposition 4.3.5. *Let $\mathcal{A}$ be a conformal net. Let $\gamma_1, \gamma_2 \in \mathcal{F}$, then $\sigma_{\gamma_1} \simeq \sigma_{\gamma_2}$ if and only if $R_{\gamma_1} = R_{\gamma_2}$.*

Proof. $\sigma_{\gamma_1} \simeq \sigma_{\gamma_2} \Leftrightarrow \sigma_{\gamma_1 \circ \gamma_2^{-1}} = \operatorname{Ad} W$, with W a unitary in $\mathcal{B}(\mathcal{H})$. By Theorem 4.3.2 this is true if and only if $\gamma_1 \circ \gamma_2^{-1}$ is at least in $\operatorname{Diff}^{1,\infty}_+(S^1)$, or equivalently if and only if $R_{\gamma_1} = R_{\gamma_2}$. □

Remark 4.3.6. It follows easily that alpha-induction is not a surjective map in the case of a finite-index conformal extension $\mathcal{A} \subset \mathcal{B}$, i.e. $\alpha^\pm$:DHR$\{\mathcal{A}\} \to \operatorname{Sol}^\pm(\mathcal{B})$.

4.3.2 Type III solitons

Instead of considering functions $\gamma \in \mathcal{F}$, we can repeat the same construction in Definition 4.3.1 using functions in a different class. Let $\mathcal{G}$ be the set maps from S^1 to S^1 φ with the following properties

1. φ is smooth on $S^1 \setminus \{-1\}$ and the half-sided derivatives at all orders exist even at the point -1 (the left and right first derivatives are non-zero),

2. φ is injective and orientation preserving,

3. $\varphi(S^1 \setminus \{-1\})$ is a proper interval of S^1.

If we take $\varphi \in \mathcal{G}$, σ_φ still yields a soliton of the conformal net $\mathcal{A}$, since the conclusions of Proposition 4.3.1 are still true.

This type of construction was already presented in [LX04] and [KLX05]. In this case one obtains solitons σ_φ which are of type III (namely $\sigma_\varphi(\mathcal{A}(\mathbb{R}))'$ is a type III factor). For completeness in the following proposition we show that this type of construction also yields a proper soliton.

Proposition 4.3.7. *Given $\varphi \in \mathcal{G}$, then σ_φ is a proper soliton of type III.*

Proof. We must show that the representation σ_φ does not extend to a representation of the net $\mathcal{A}$ of the circle. Consider the set $E := I_1 \cup I_2$, where I_1 and I_2 are two disjoint intervals of the circle which have the point $p = -1$ as a common end-point. Suppose now that σ_φ extends to a representation of the net $\mathcal{A}$ on S^1. In this case we have an action of σ_φ^E on $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ such that

$$\sigma_\varphi^E(\mathcal{A}(I_1) \vee \mathcal{A}(I_2)) = \sigma_\varphi^{I_1}(\mathcal{A}(I_1)) \vee \sigma_\varphi^{I_2}(\mathcal{A}(I_2)) \simeq \sigma_\varphi^{I_1}(\mathcal{A}(I_1)) \otimes \sigma_\varphi^{I_2}(\mathcal{A}(I_2)) \tag{4.3.1}$$

where we used the fact that the net satisfies the split property. But $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ is not isomorphic to $\mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$, so the contradiction [Buc74][page 292, Example b)]. □

4.3.3 Covariance for soliton representations

In section 4.1 we proved that every conformal net $\mathcal{A}$ is $\mathrm{Diff}_+^{1,\infty}(S^1)$-covariant. We now use this fact to see that all the constructed soliton representations σ_γ, $\gamma \in \mathcal{F}$, are B_0-covariant.

Proposition 4.3.8. *Let $(\mathcal{A}, U, \Omega)$ be a $\mathrm{Diff}_+^{1,\infty}(S^1)$-covariant net and let $\gamma \in \mathcal{F}$. Then the soliton σ_γ is B_0-covariant.*

Proof. Let $\gamma \in \mathcal{F}$ and σ_γ the associated soliton. Given $g \in B_0$, $\gamma \circ g \circ \gamma^{-1}$ is a C^1-diffeomorphism which is locally C^∞, so it follows that there exists $U^\sigma(g) := U(\gamma \circ g \circ \gamma^{-1})$ which is a map from B_0 to $\mathcal{U}(\mathcal{H})/\mathbb{T}$. The covariance on σ_γ holds, indeed

$$U^\sigma(g)\sigma_\gamma^I(x)U^\sigma(g)^* = \sigma_\gamma^{gI}(U(g)xU(g)^*).$$

□

Remark 4.3.9. Let $\mathcal{A}$ be a $\mathrm{Diff}_+^{1,\infty}(S^1)$-covariant net, U its covariance representation and $\gamma \in \mathcal{F}$. Note that the representation U_γ defined by the equation 5.2.2 is irreducible when $\mathcal{A}$ is the Virasoro net $\mathcal{A}_{\mathrm{Vir}_c}$ with $c \in \mathbb{Z}_+$.
For $\varphi \in \mathrm{Diff}_+(I)$,

$$\sigma_\gamma(U_{c,0}(\varphi)) = \mathrm{Ad}\, U(\gamma_I)(U_{c,0}(\varphi)) = U_{c,0}(\gamma_I \circ \varphi \circ \gamma_I^{-1}) = U_\gamma(\varphi).$$

This relation is similar to the correspondence between irreducible, unitary, positive energy representations of $\mathrm{Diff}_+(S^1)$ and DHR sectors of the Vir_c nets. It would be suggestive to think that solitons of the type σ_γ exhaust all unitary equivalence classes of irreducible solitons for Virasoro nets.

Chapter 5

Further results on concrete examples

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5.1 The U(1)-current net

Let $\mathcal{K}$ be a real Hilbert space with a nondegenerate symplectic bilinear form σ and J a complex structure on $\mathcal{K}$. The C*-algebra generated by the non-zero operator $W(f)$, $f \in \mathcal{K}$, satisfying the relations $W(f)W(g) = e^{-i\sigma(f,g)_1/2}W(f+g)$ and $W(0) = \mathbb{1}$ is called the CCR algebra. If $f \in \mathcal{K}$ and A is an invertible operator on $\mathcal{K}$ which preserves the symplectic bilinear form, then the map $W(f) \mapsto W(Af)$ is a *-automprhism of the CCR algebra. Such a *-automorphism is unitary implemented if and only if $A \in Sp_2(\mathcal{K})$, i.e. $\frac{1}{2}J[A,J]$ is an Hilbert-Schmidt operator. Such a unitary is unique up to a phase factor, see [Ott95][Theorem 16].

Let $C^\infty(S^1,\mathbb{R}) \subset L^2(S^1)$ be the space of real-valued smooth function on S^1. We define a seminorm on it

$$\|f\| := \sum_{k\in\mathbb{N}} k|\hat{f}_k|^2$$

which is induced by the semi-inner product

$$(f,g)_{1/2} := \frac{1}{2}\sum_{k\in\mathbb{N}} k(\overline{\hat{f}_k}\hat{g}_k + \hat{f}_k\overline{\hat{g}_k}).$$

It is also possible to induce a complex structure on $C^\infty(S^1,\mathbb{R})$ by means of the operator J

$$J\left(\sum_{k\in\mathbb{Z}\setminus\{0\}} f_k e_k\right) := \sum_{k\in\mathbb{N}}(if_k)e_k + \sum_{k\in\mathbb{N}}(-if_{-k})e_{-k},$$

where $e_k(e^{i\theta}) := e^{ik\theta}$.

The space $C^\infty(S^1,\mathbb{R})$ equipped with J modulo the null space $\{f\in C^\infty(S^1,\mathbb{R}) : \|f\|=0\}$ is denoted with $\mathcal{H}_{1/2}$ and is a realization of the complex Hilbert space $\mathcal{H}_1$, namely the representation space of the irreducible unitary representation U_1 of $\mathrm{PSL}(2,\mathbb{R})$ with lowest weight 1. The action of $\mathrm{PSL}(2,\mathbb{R})$ on $C^\infty(S^1,\mathbb{R})$

$$U_1(\gamma)(f) := f\circ\gamma^{-1} \tag{5.1.1}$$

extends to $\mathcal{H}_{1/2}$.

Let $\Gamma(\mathcal{H}_{1/2})$ be the second quantization space constructed from $\mathcal{H}_{1/2}$ and $\Gamma_+(\mathcal{H}_{1/2})$ the associated symmetric Fock space. For any $f\in\mathcal{H}_{1/2}$ the Weyl operators $W(f)$ on $\Gamma_+(\mathcal{H}_{1/2})$ are unitary operators which satisfy

- **commutation relations:** $W(f)W(g) = e^{-i\frac{\mathrm{Im}\,(f,g)_{1/2}}{2}}W(f+g)$
- **strong continuity:** if $f_n\to f$ in $\mathcal{H}_1$ then $\|(W(f_n)-W(f))v\|\to 0$ for every $v\in\Gamma_+(\mathcal{H}_1)$.

The bilinear form $\sigma(f,g) := \mathrm{Im}(f,g)_{1/2}$ is clearly invariant under the action of $U_1(\gamma)$ for all $f,g\in C^\infty(S^1,\mathbb{R})$. Furthermore, the unitary operators $U_1(\gamma)\in\mathcal{U}(\mathcal{H}_{1/2})$, $\gamma\in\mathrm{PSL}(2,\mathbb{R})$ act on $\Gamma_+(\mathcal{H}_{1/2})$ via the second quantization functor, and we define $U(\gamma) := \Gamma(U_1(\gamma))$. The adjoint action of $U(\gamma)$ on the Weyl operators is particularly simple, since

$$\mathrm{Ad}\,U(\gamma)W(f) = W(U_{1/2}(\gamma)f). \tag{5.1.2}$$

Definition 5.1.1. The family of von Neumann algebras

$$\mathcal{A}_{U(1)}(I) := \{W(f) : f\in C^\infty(S^1,\mathbb{R})\subset\mathcal{H}_{1/2},\ \mathrm{supp}\,(f)\subset I\}''$$

is a Möbius covariant net on S^1, where the vacuum vector is $1\in\mathbb{C}\subset\Gamma_+(\mathcal{H}_{1/2})$ and the Möbius covariance follows immediately by 5.1.2.

The representation U_1 of $\mathrm{PSL}(2,\mathbb{R})$ can be extended to a projective representation U of $\mathrm{Diff}_+(S^1)$ in such a way that $\mathcal{A}_{U(1)}$ is actually a diffeomorphism covariant net, see [PS86][Theorem 9.3.1].

Lemma 5.1.2. *Let $\gamma\in\mathcal{D}^s(S^1)$, $s>3/2$, the image of $\tilde\gamma\in\widetilde{\mathcal{D}^s(S^1)}$ through the covering map and $\lambda_{m,n} := \frac{1}{2\pi}\int_0^{2\pi} e^{\pm i|m|\tilde\gamma(\theta)}e^{\pm i|n|\theta}d\theta$. Then there exists $C_s\geq 0$ such that*

$$|\lambda_{m,n}| \leq \frac{C_s\|\tilde\gamma^{-1}\|_{s-1}}{(|m|+|n|)^{s-1}}.$$

Proof. As in the proof of [Seg81, Proposition 5.3], consider the path $\tilde{\gamma}_t$ in $\widetilde{\mathcal{D}^s(S^1)}$:

$$[0,1] \ni t \mapsto \tilde{\gamma}_t := t\tilde{\gamma} + (1-t)\mathrm{id} \in \widetilde{\mathcal{D}^s}(S^1).$$

This is indeed a path in $\widetilde{\mathcal{D}^s(S^1)}$, because $\tilde{\gamma}_t(\theta) > 0$. By setting $t = \frac{|m|}{|m|+|n|}$, we have

$$\lambda_{m,n} = \frac{1}{2\pi}\int_0^{2\pi} e^{\pm i(|n|+|m|)\tilde{\gamma}_t(\theta)} d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{\pm i(|n|+|m|)\varphi} \left(\tilde{\gamma}_t^{-1}\right)'(\varphi)d\varphi.$$

Since $\left(\tilde{\gamma}_t^{-1}\right)' \in H^{s-1}$, $|\widehat{\left(\tilde{\gamma}_t^{-1}\right)'}_{(|m|+|n|)}| \leq \frac{\|\left(\tilde{\gamma}_t^{-1}\right)'\|_{s-1}}{(|m|+|n|)^{s-1}}$. The map $t \mapsto \|\left(\tilde{\gamma}_t^{-1}\right)'\|_{s-1}$ is continuous, which proves the statement. □

Proposition 5.1.3. *Let $\gamma \in \mathcal{D}^s(S^1)$, $s > 2$, and $f \in C^\infty(S^1,\mathbb{R})$. The map $V(\gamma)[f] := [f\circ\gamma^{-1}] = f\circ\gamma^{-1} - (\widehat{f\circ\gamma^{-1}})_0$ induces an action to the CCR algebra which is implemented by an unitary operator $U(\gamma)$.*

Proof. Let $f, g \in C^\infty(S^1,\mathbb{R})$. The (real) symplectic bilinear form $\sigma([f],[g]) := \mathrm{Im}\,\langle f, g\rangle$ can be written as follows:

$$\sigma([f],[g]) = \frac{1}{4\pi}\int_0^{2\pi} f(e^{i\theta})g'(e^{i\theta})d\theta.$$

As a consequence, for $\gamma \in \mathcal{D}^s(S^1)$, $s > 2$, the map $V(\gamma)$ preserves the symplectic form because γ is in particular in $\mathrm{Diff}^1_+(S^1)$. Following [Vro13, Theorem 24] we only need to show that the Hilbert-Schmidt norm of the operator $A_{V(\gamma)} := \frac{1}{2}J[V(\gamma), J]$

$$\|A_{V(\gamma)}\|^2_{HS} = \sum_{m>0,n<0} \frac{|m|}{|n|}|\lambda_{m,n}|^2 \leq \sum_{m>0,n<0}\frac{|m|}{|n|}C_s^2\frac{1}{(|m|+|n|)^{2(s-1)}}$$

is finite. Let $p := |m|+|n|$, then

$$\sum_{m>0,n>0}\frac{m}{n\,(m+n)^{2(s-1)}} = \sum_{p>0}\frac{1}{p^{2(s-1)}}\sum_{n=1}^{p-1}\frac{p-n}{n} \leq \sum_{p>0}\frac{p-1}{p^{2(s-1)}}\sum_{n=1}^{p-1}\frac{1}{n} \leq \sum_{p>0}\frac{(p-1)\,(2+\log(p))}{p^{2(s-1)}}$$

which converges if $s > 2$. □

Theorem 5.1.4. *The map $\alpha : \mathcal{D}^s(S^1) \to Aut(\mathcal{B}(\mathcal{H}))$ such that $\gamma \mapsto \alpha_\gamma := \mathrm{Ad}\,U(\gamma)$ is pointwise strongly continuous if $s > 2$.*

Proof. Let $f \in C^\infty(S^1,\mathbb{R})$ and $\{\gamma_n\} \subset \mathcal{D}^s(S^1)$ a sequence converging to γ in $\mathcal{D}^s(S^1)$. Recall that $C^\infty(S^1,\mathbb{R}) \subset H^s(S^1)$ for every s and that if $f \in H^s(S^1)$, $s \geq 1/2$, then $\|f\|_{1/2} \leq \|f\|_s$. By Lemma 1.4.2, the map $(f,\gamma) \mapsto f\circ\gamma^{-1}$ is continuous for $s > 3/2$. Using Proposition 5.1.3 and the strong continuity of the Weyl operators, it follows that for $s > 2$, the map $\alpha_{\gamma_n}(W([f])) \to \alpha_\gamma(W([f]))$, $f \in C^\infty(S^1,\mathbb{R})$

Thus we have that there is a strongly dense set R of $\mathcal{B}(\mathcal{H})$ for which $\lim_{n\to\infty} U(\gamma_n)xU(\gamma_n)^* = U(\gamma)xU(\gamma)^*$ in the strong topology for every $x \in R$.
Now let $\{\xi_n\} \subset \mathcal{H}$ be a dense sequence. Let $A \in B(\mathcal{H})$. By Kaplanski's theorem we can choose a sequence $\{A_m\} \subset R$ for which $A_m \to A$ strongly.
Thus we have for every ξ_n

$$\lim_{m\to\infty} U(\gamma)A_mU(\gamma)^*\xi_n = U(\gamma)AU(\gamma)^*\xi_n,$$

i.e. $f_n(\gamma) := U(\gamma)AU(\gamma)^*\xi_n$ is the pointwise limit of $f_n^m(\gamma) := U(\gamma)A_mU(\gamma)^*\xi_n$. Note that $\mathcal{D}^s(S^1)$ is a Baire set, since it is an open set of a complete metric space [IKT13, page 37]. By Baire-Osgood's theorem [Car00, Theorem 11.20] we get that the set

$$S(f_n) := \{\gamma \in \mathcal{D}^s(S^1) : f_n \text{ is not continuous in } \gamma\}$$

is meager. Thus also $\cup_n S(f_n)$ is meager. It follows that $\mathcal{D}^s(S^1)\backslash\cup_n S(f_n)$ is nonempty and thus $\exists\ \gamma_0 \in \mathcal{D}^s(S^1)$ for which all f_n are continuous. It easily follows that

$$\gamma \mapsto U(\gamma)AU(\gamma)^*\xi =: f_\xi^A(\gamma)$$

is continuous for γ_0 for every $\xi \in \mathcal{H}$. Define $h := \gamma_0^{-1}\gamma$, then

$$g_\xi^A(h) := U(h)AU(h)^*\xi = U(\gamma_0^{-1})^*U(\gamma)AU(\gamma)^*U(\gamma_0^{-1})^*\xi = U(\gamma_0^{-1})f^A_{U(\gamma_0)\xi}(\gamma)$$

is continuous in the identity $e \in \mathcal{D}^s(S^1)$ for every $A \in \mathcal{B}(\mathcal{H})$ and for every $\xi \in \mathcal{H}$.

Since the map

$$\gamma \mapsto \mathrm{Ad}\,(U(\gamma)) \in \mathrm{Aut}(\mathcal{B}(\mathcal{H}))$$

is a group homomorphism and is continuous in e (where $\mathrm{Aut}(\mathcal{B}(\mathcal{H}))$ is equipped with the topology of pointwise strong convergence) it is continuous for every $\gamma \in \mathcal{D}^s(S^1)$. □

Definition 5.1.5. We denote with $\mathrm{Diff}_+^{1,\infty}(S^1)$ the group of orientation preserving, C^1 diffeomorphisms of the circle which are piecewise C^∞.

Remark 5.1.6. The group of piecewise Möbius transformations defined in [Wei05] is contained in $\mathrm{Diff}_+^{1,\infty}(S^1)$.

Lemma 5.1.7. *The group* $\mathrm{Diff}_+^{1,\infty}(S^1) \subset \mathcal{D}^s(S^1)$ *if* $s < 5/2$.

Proof. Follows immediately from $|k^2\hat{\gamma}_k| \leq \left|\frac{\mathrm{Var}(\gamma'')}{k}\right|$, where $\mathrm{Var}(\gamma'')$ is the total variation of γ'', see [Kat04][Theorem 4.5]. □

Corollary 5.1.8. *The U(1)-current net* $\mathcal{A}_{U(1)}$ *is* $\mathcal{D}^s(S^1)$*-covariant,* $s > 2$*, and in particular is* $\mathrm{Diff}_+^{1,\infty}(S^1)$*-covariant.*

Proof. The proof is the same as in Proposition 3.3.1.

□

Corollary 5.1.9. *The Virasoro net $\mathcal{A}_{\mathrm{Vir}_1}$ is $\mathcal{D}^s(S^1)$-covariant, $s > 2$, and in particular is $\mathrm{Diff}_+^{1;\infty}(S^1)$-covariant.*

Proof. For the theorem 5.1.4 map $\alpha : \gamma \mapsto \alpha_\gamma := \mathrm{Ad}U(\gamma)$, $\gamma \in \mathcal{D}^s(S^1)$, is continuous. Let $\mathcal{A} := \mathcal{A}_{\mathrm{Vir}_1}$ the Virasoro net of central charge $c = 1$ and $\mathcal{B}$ che $U(1)$-current net on the Hilbert space $\mathcal{H}$. The projection E on $\mathcal{H}_{\mathcal{A}} := \overline{\bigvee_I \mathcal{A}(I)\Omega}$ is clearly invariant for the action of α_γ due to continuity of α and so we have the desired claim. □

Remark 5.1.10. The action of $\mathrm{Diff}_+^{1;\infty}(S^1)$ in Corollary 5.1.8 and Corollary 5.1.9 is continuous. On the contrary, the action in Corollary 4.1.5 is in general not continuous.

Remark 5.1.11. Let $U_{(1,0)}$ the irreducible positive energy projective unitary representation of $\mathrm{Diff}_+(S^1)$ with central charge 1 and lowest weight 0. Define $U_n := \bigotimes_n U_{(1,0)}$, which is a positive energy projective representation of $\mathrm{Diff}_+(S^1)$ which contains $U_{(n,0)}$ as a subrepresentation. Using Corollary 5.1.9 we can deduce that all the Virasoro nets with positive integral central charge are $\mathcal{D}^s(S^1)$-covariant, $s > 2$ and that all the representations $U_{(n,0)}$ extend to $\mathcal{D}^s(S^1)$, $s > 2$.

Remark 5.1.12. It should be stressed that all the extended representations are continuous.

Lemma 5.1.13. *Let $\mathring{g} \in B_0$, $\mathring{\gamma} \in \mathcal{F}$ and $2 < s < 5/2$. The homomorphism $\alpha_{\mathring{\gamma}} : B_0 \longrightarrow \mathcal{D}^s(S^1)$, $\mathring{g} \mapsto \alpha_{\mathring{\gamma}}(\mathring{g}) := \mathring{\gamma} \circ \mathring{g} \circ \mathring{\gamma}^{-1}$, where B_0 is equipped with the C^∞-topology, is continuous.*

Proof. Let $\{\mathring{g}_n\} \subset B_0$ be a sequence converging to $\mathring{g} \in B_0$ with respect to the C^∞-topology. We denote with γ the lift to $\widetilde{\mathrm{Diff}}_+^0(S^1)$ of $\mathring{\gamma}$ and with g_n and g the lift to $\widetilde{B_0}$ of $\mathring{\gamma}_n$ and $\mathring{g}$, respectively. We use the same strategy of Lemma 4.2.3. Namely, the convergence $\gamma \circ g_n \circ \gamma^{-1} \to \gamma \circ g \circ \gamma^{-1}$ in the $L^1(S^1)$-topology is clear. Then, by

$$\left| \left(\gamma \circ \widehat{g_n \circ} \gamma^{-1}\right)_k \right| \leq \frac{\mathrm{Var}\left((\gamma \circ g_n \circ \gamma^{-1})''\right)}{k^3}$$

it is sufficient to show that the right-hand side is uniformly bounded in n. The second derivative of $\gamma \circ g_n \circ nu^{-1}$ is

$$\begin{aligned} \frac{d^2}{d\theta^2}\left(\gamma \circ g_n \circ \gamma^{-1}\right)(\theta) = {} & \gamma''(g_n(\gamma^{-1}(\theta)))g_n'(\gamma^{-1}(\theta))^2 \frac{1}{\gamma'(\gamma^{-1}(\theta))^2} \\ & + \gamma'(g_n(\gamma^{-1}(\theta)))g_n''(\gamma^{-1}(\theta)) \frac{1}{\gamma'(\gamma^{-1}(\theta))^2} \\ & - \gamma'(g_n(\gamma^{-1}(\theta)))g_n'(\gamma^{-1}(\theta)) \frac{\gamma''(\gamma^{-1}(\theta))}{\gamma'(\gamma^{-1}(\theta))^3}. \end{aligned} \tag{5.1.3}$$

To evaluate its total variation, we use the following facts: for every pair of functions f_1, f_2 with bounded variation, it holds [Pau15, Theorem 3.7] that

$$\begin{aligned} \mathrm{Var}(f_1 \cdot f_2) &\leq \|f_1\|_\infty \mathrm{Var}(f_2) + \|f_2\|_\infty \mathrm{Var}(f_1) + 3\mathrm{Var}(f_1)\mathrm{Var}(f_2) \\ \mathrm{Var}(f_1 \circ f_2) &\leq L_{f_1} \mathrm{Var}(f_2), \end{aligned}$$

where f_1 is Lipschitz and L_{f_1} is the Lipschitz constant of f_1. Now, the total variations of the second and the third terms are uniformly bounded in n since $L_{g_n^{(k)}}$ are uniformly bounded in n. As for the first term, we have $\mathrm{Var}(\gamma'' \circ g_n \circ \gamma^{-1}) \leq 2\pi \left\|(\gamma'' \circ g_n \circ \gamma^{-1})'\right\|_{L^\infty(0,2\pi)} + |\gamma''(2\pi) - \gamma''(0)|$, and this is again uniformly bounded since γ'' has a bounded derivative on the open interval $(0, 2\pi)$ and $L_{g_n^{(k)}}$ are uniformly bounded in n. □

5.2 Non-extendable representations of $\Lambda SU(N)$ and B_0

5.2.1 Representations of $\Lambda SU(N)$

Proposition 5.2.1. *There exist irreducible positive energy representations of $\Lambda SU(N)$ which do not extend to positive energy representations of $LSU(N)$.*

Proof. Fix a level ℓ and consider the conformal net $\mathcal{A}_{SU(N),\ell}$ induced by the vacuum representation of level ℓ and lowest weight 0, $U_{\ell,0}$, of $LSU(N)$. Then we can construct a representation of $\Lambda SU(N)$, $U^\gamma_{\ell,0}$, by defining

$$U^\gamma_{\ell,0} := \sigma_\gamma \circ U_{\ell,0}, \tag{5.2.1}$$

where σ_γ is a proper soliton of the conformal net $\mathcal{A}_{SU(N),\ell}$ with $\gamma \in \mathcal{F}$.
Clearly by Proposition 4.2.4, $U^\gamma_{\ell,0}$ it has positive energy. To check that $U^\gamma_{\ell,0}$ does not extend to a positive energy representation of $LSU(N)$ we proceed by contradiction. Suppose that $U^\gamma_{\ell,0}$ does indeed extend to a positive energy representation of $LSU(N)$. Then $U^\gamma_{\ell,0}$ is also irreducible as a representation of $LSU(N)$. $U^\gamma_{\ell,0}$ must have level ℓ by Theorem 2.5.2 since it is locally equivalent to $U_{\ell,0}$ by its defining equation 5.2.1. Then by the correspondence 2.5.2 applied to $U^\gamma_{\ell,0}$, the corresponding representation of the conformal net $\mathcal{A}_{SU(N),\ell}$ is an extension of σ_γ, which does not exist by Theorem 4.3.2. □

5.2.2 Representations of the one point stabilizer subgroup of $\mathrm{Diff}_+(S^1)$

We know want to use the results in Section 4.3 to construct unitary projective representations of B_0. Let $\gamma \in \mathcal{F}$, set

$$\begin{aligned} \alpha_\gamma : B_0 &\to \mathrm{Diff}_+^{1,\infty}(S^1) \\ g &\mapsto \gamma \circ g \circ \gamma^{-1}. \end{aligned} \tag{5.2.1}$$

Clearly α_γ is an homomorphism of the stabilizer group of the point at infinity B_0 into the group $\mathrm{Diff}_+^{1,\infty}(S^1)$. Note that the function $\gamma \circ g \circ \gamma^{-1}$ is indeed a C^1 function as the discontinuity of the first derivative of γ at the point of infinity is eliminated. We construct a projective unitary representation U_γ of B_0 induced from γ in the following way:

$$\begin{aligned} U_\gamma : B_0 &\to \mathcal{U}(\mathcal{H}) \\ g &\mapsto U_\gamma(g) := (U \circ \alpha_\gamma)(g) \end{aligned} \tag{5.2.2}$$

where U is a projective unitary representation of $\mathrm{Diff}_+^{1,\infty}(S^1)$ on the Hilbert space $\mathcal{H}$.

Proposition 5.2.2. *Let $\gamma \in \mathcal{F}$ and $U = U_{(c,0)}$ the irreducible positive-energy unitary projective representation of $\mathrm{Diff}_+(S^1)$ with central charge c. The maps U_γ defined by the equation 5.2.2 are unitary projective representations of B_0 which do not extend to $\mathrm{Diff}_+(S^1)$. In addition, $U_{\gamma_1} \simeq U_{\gamma_2}$ if and only if $R_{\gamma_1} = R_{\gamma_2}$.*

Proof. It follows easily from the definition of U_γ, see for instance the proof in Proposition 5.2.1. □

Proposition 5.2.3. *Let $2 < s < 5/2$ and $\gamma \in \mathcal{F}$. The map $U_\gamma := U \circ \alpha_\gamma$ is a strongly continuous unitary projective representation of B_0 when $U = U_{n,0}$, $n \in \mathbb{Z}_+$.*

Furthermore, let $\mathcal{A}$ be the $\mathrm{U}(1)$-current net or the Virasoro net $\mathcal{A}_{\mathrm{Vir}_c}$ with $c \in \mathbb{Z}_+$ and $\gamma \in \mathcal{F}$. Every soliton σ_γ of $\mathcal{A}$ as in Section 4.3 is continuously B_0-covariant with respect to the representation U_γ.

Proof. This is clear from Theorem 5.1.4 and Lemma 5.1.13. □

Appendix A

Projective unitary representations

In this section we collect the basic definitions on projective unitary representations of (topological) groups.

Definition A.0.1. A strongly continuous unitary projective representation of a topological group G is a pair $(U, \mathcal{H})$ where $\mathcal{H}$ is a Hilbert space and U is a continuous group homomorphism from G to $\mathcal{U}(\mathcal{H})/\mathbb{T}$, where $\mathcal{U}(\mathcal{H})$ is equipped with the strong operator topology and $\mathcal{U}(\mathcal{H})/\mathbb{T}$ with the quotient topology by the quotient map q.

The subbasis elements which contain $q(u)$ are $\{\mathcal{U}_{q(u),\xi,\varepsilon}\}_{\xi\in\mathcal{H},\varepsilon>0}$, where

$$\mathcal{U}_{q(u),\xi,\varepsilon} = \{q(v) : \text{ there are } u', v' \in \mathcal{U}(\mathcal{H}), q(u) = q(u'), q(v) = q(v'), \text{ and } \|(v'-u')\xi\| < \varepsilon\}.$$

Therefore, it is clear that a net $\{q(u_\lambda)\}$ has limit $q(u)$ if and only if for each $\xi \in \mathcal{H}$ there is $z_{\xi,\lambda}, \hat{z}_{\xi,\lambda} \in \mathbb{T}$ such that $\|z_{\xi,\lambda}u_\lambda\xi - \hat{z}_{\xi,\lambda}u\xi\| \to 0$ if and only if there is $z_{\xi,\lambda} \in \mathbb{T}$ such that[1] $z_{\xi,\lambda}u_\lambda\xi \to u\xi$. Actually, $z_{\xi,\lambda}$ does not depend on ξ (because, if $z_{\xi,\lambda}u_\lambda\eta$ were not convergent for $\eta \perp \xi$, $z_{\xi,\lambda}u_\lambda(\xi+\eta)$ would not be convergent in $\mathcal{H}/\mathbb{T}$, hence convergence holds for any η), hence $q(u_\lambda)$ is convergent if and only if there is a net $z_\lambda \in \mathbb{T}$ such that $z_\lambda u_\lambda$ is convergent in the strong operator topology.

We have the following result, see [Bar54]

Theorem A.0.2. $U(g_\lambda) \to U(g)$ *in* $\mathcal{U}(\mathcal{H})/\mathbb{T}$ *if and only if* $\mathrm{Ad}\, U(g_\lambda)(x) \to \mathrm{Ad}\, U(g)(x)$.

We can consider $U(g)$ as an operator acting on $\mathcal{H}$ determined up to a phase factor. Two projective unitary representations $(U_1, \mathcal{H}_1)$ and $(U_2, \mathcal{H}_2)$ are said to be equivalent if exists an unitary $W : \mathcal{H}_1 \to \mathcal{H}_2$ such that $WU_1(g) = U_2(g)W$ for every $g \in G$ up to a phase factor.

Definition A.0.3. A **unitary multiplier representation** of G is a pair $(U, \mathcal{H})$ were $U : G \to \mathcal{U}(\mathcal{H})$ is a map such that $U(g_1)U(g_2) = \omega(g_1, g_2)U(g_1g_2)$ and $\omega :$ $G \times G \to \mathbb{T}$ is a map which satisfies the equality

$$\omega(g_1, g_2)\omega(g_1g_2, g_3) = \omega(g_1, g_2g_3)\omega(g_2, g_3).$$

[1] One can concretely make the following choice: $z_{\xi,\lambda} = \frac{\overline{\langle u\xi, u_\lambda\xi\rangle}}{|\langle u\xi, u_\lambda\xi\rangle|}$, then $z_{\xi,\lambda}u_\lambda\xi$ converges to $u\xi$.

A unitary multiplier representation U of G is strongly continuous if $U(g)v$ tends to $U(g_0)v$ for all $v \in \mathcal{H}$ if g tends to g_0.

Appendix B

Central extensions

In this section we introduce central extensions of groups and of Lie algebras. Most of the definitions and facts are taken from [KW09, Sch08].

B.1 Central extensions of groups

Let G and H be two arbitrary groups.

Definition B.1.1. An extension of G by the group H is an exact sequence of homomorphisms

$$1 \longrightarrow H \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1.$$

The extension is **central** if H is abelian and is in the center of $\hat{G}$.

We say that two central extensions of G by H are **equivalent** if the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \longrightarrow & \hat{G}_1 & \longrightarrow & G & \longrightarrow & 1 \\ & & \big\downarrow {\scriptstyle \mathrm{id}} & & \big\downarrow {\scriptstyle \Phi} & & \big\downarrow {\scriptstyle \mathrm{id}} & & \\ 1 & \longrightarrow & H & \longrightarrow & \hat{G}_2 & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

is commutative, with $\Phi : \hat{G}_1 \longrightarrow \hat{G}_2$ a group homomorphism.

A map $c : G \times G \to H$ is said to be a **2-cocycle** of G with values in H if

$$c(g_1, g_2)c(g_1g_3, g_3) = c(g_1, g_2g_3)c(g_2, g_3)$$

and $c(1, 1) = 1$, for every $g_1, g_2, g_3 \in G$. We say that two 2-cocycles c_1 and c_2 are equivalent, $c_1 \backsim c_2$, iff there exists a map $\alpha : G \to H$ such that $\alpha(g_1g_2) = c_1(g_1, g_2)c_2(g_1, g_2)^{-1}\alpha(g_1)\alpha(g_2)$.

We define the second cohomology group $H^2(G, H)$ of G with coefficients in H as the set of 2-cocycles modulo the equivalence relation

$$H^2(G, H) \coloneqq \{c \text{ is a 2-cocycle}\}/ \backsim$$

and with the group operation given by the pointwise product.

We now construct a central extension of G by H using a 2-cocycle ω in this way: the exact sequence which determines the central extension is

$$1 \longrightarrow H \xrightarrow{\iota} H \times_\omega G \xrightarrow{\pi_2} G \longrightarrow 1$$

where $H \times_\omega G$ is equal to $H \times G$ as a set and is endowed with the multiplication

$$(h_1, g_1) \cdot (h_2, g_2) \coloneqq (\omega(g_1, g_2)h_1 h_2, g_1 g_2),$$

and π_2 is te projection map from $H \times G$ onto G. With this in mind, the following holds:

Proposition B.1.2. *There exists a correspondence between the 2-cocycles of G with values in H and central extensions of G by H. The second cohomology group $H^2(G, H)$ is in one-to-one correspondence with the equivalence classes of central extensions.*

B.2 Central extensions of Lie algebras

Definition B.2.1. A **central extension** of a Lie algebra $\mathfrak{g}$ by a vector space $\mathfrak{h}$ is a Lie algebra $\tilde{\mathfrak{g}}$ which is equal to $\mathfrak{g} \oplus \mathfrak{h}$ as a vector space, and with bracket

$$[(g_1, h_1), (g_2, h_2)] \coloneqq ([g_1, g_2], \omega(g_1, g_2)), \tag{B.2.1}$$

where $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}$ is a continuous bilinear map.

Since the bracket associated to $\tilde{\mathfrak{g}}$ does not depend on elements in $\mathfrak{h}$, it is clear that $\mathfrak{h}$ is in the center of $\tilde{\mathfrak{g}}$. From equation (B.2.1) ω has to be bilinear, antisymmetric and has to satisfy the equation

$$\omega([g_1, g_2], g_3) + \omega([g_2, g_3], g_1) + \omega([g_3, g_1], g_2) = 0 \tag{B.2.2}$$

for every $g_1, g_2, g_3 \in \mathfrak{g}$ (cocycle relation). A continuous function $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}$ which is bilinear, antisymmetric and satisfies the cocycle relation is called a **2-cocycle**. We denote with the symbol $Z^2(\mathfrak{g}, \mathfrak{h})$ the space of 2-cocycles of $\mathfrak{g}$ with values in $\mathfrak{h}$. An element $\omega \in Z^2(\mathfrak{g}, \mathfrak{h})$ is a **2-coboundary** if $\omega(g_1, g_2) = \alpha(g_1, g_2)$ for every $g_1, g_2 \in \mathfrak{g}$, where $\alpha : \mathfrak{g} \to \mathfrak{h}$ is a linear map. The space of 2-coboundaries of $\mathfrak{g}$ with values in $\mathfrak{h}$ is denoted with $B^2(\mathfrak{g}, \mathfrak{h})$.

Definition B.2.2. The **second cohomolgy group** of $\mathfrak{g}$ awith values in $\mathfrak{h}$ is $H^2(\mathfrak{g}, \mathfrak{h}) \coloneqq Z^2(\mathfrak{g}, \mathfrak{h})/B^2(\mathfrak{g}, \mathfrak{h})$.

In a different way, we can define a central extension of a Lie algebra $\mathfrak{g}$ by the Lie algebra $\mathfrak{h}$ as an exact sequence

$$0 \longrightarrow \mathfrak{h} \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

with $\mathfrak{h}$ in the center of $\tilde{\mathfrak{g}}$. A morphism of central extensions of $\mathfrak{g}$ is a pair (μ, ν) of Lie algebra homomorphisms such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{h}_1 & \longrightarrow & \tilde{\mathfrak{g}}_1 & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathfrak{h}_2 & \longrightarrow & \tilde{\mathfrak{g}}_2 & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \end{array}$$

is commutative. The extensions of $\mathfrak{g}$ by $\mathfrak{h}$ are equivalent if ν is an isomorphism of Lie algebras and μ is the identity map.

The following is a well-known fact, see [Sch08][Remark 4.7]:

Proposition B.2.3. *There exists a bijection between $H^2(\mathfrak{g}, \mathfrak{h})$ and the set of equivalence classes of central extensions of $\mathfrak{g}$ by $\mathfrak{h}$.*

Let $\mathfrak{g}$ a Lie algebra and $\tilde{\mathfrak{g}}$ a central extension of $\mathfrak{g}$. If for any other central extension $\tilde{\mathfrak{g}}_*$ there exists an unique morphism of central extension between $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}_*$, then $\tilde{\mathfrak{g}}$ is called **universal**.

The following Theorem is a classical result of Bargmann [Bar54]:

Theorem B.2.4. *Let G be a connected and simply connected finite-dimensional Lie group with Lie algebra $\mathfrak{g}$. If $H^2(\mathfrak{g}, \mathbb{R}) = 0$, then every unitary projective representation U of G lifts to a true unitary representation.*

Appendix C

Continuous fragmentation of $\widetilde{\mathcal{D}^s(S^1)}$.

If I is a proper open interval of S^1, we denote with $I' = (S^1 \setminus I)^\circ$ the interior of its complement. Let $\overline{I}$ be the closure of I. With $\mathrm{Diff}_+(I)$ we denote the group

$$\mathrm{Diff}_+(I) \coloneqq \left\{\gamma \in \mathrm{Diff}_+(S^1) : \gamma(x) = x \text{ if } x \in I'\right\} \tag{C.0.1}$$

and with $\mathcal{D}^s(I)$ the group

$$\mathcal{D}^s(I) \coloneqq \left\{\gamma \in \mathcal{D}^s(S^1) : \gamma(x) = x \text{ if } x \in I'\right\}. \tag{C.0.2}$$

In different words, we say that an element $\gamma \in \mathrm{Diff}_+(I)$ (or an element $\gamma \in \mathcal{D}^s(I)$) is supported in I, where the **support** of a (non necessarily smooth) diffeomorphism γ is the closure of the set $\{x \in S^1 : \gamma(x) \neq x\}$.

Let $\{I_j\}_{j=1,2,3}$ be a cover of the unit circle as Fig. C.1: $I_k \coloneqq (a_k, b_k)$ where a_k, b_k are the endpoints. We take a smaller interval $\hat{I}_k = (\hat{a}_k, \hat{b}_k) \subset I_k$ which still consist a cover of S^1 points $\breve{a}_1, \breve{b}_1$, c.f. [DFK04]. Furthermore, we choose $\hat{b}_2, \breve{b}_2$ such that $\hat{a}_1 < \hat{b}_2 < \breve{b}_2 < b_2$.

Definition C.0.1. We say that a group $G \subset \mathrm{Homeo}_+(S^1)$ has the **fragmentation property** if for any finite open cover $\mathcal{U} = \{I_i\}_{i=1}^n$ of S^1 and for any element $\gamma \in G$ there exist $\{\gamma_i\}_{i=1}^m \subset G$ such that $\gamma = \gamma_1 \circ \dots \gamma_m$ and $\mathrm{supp}\,(\gamma_j)$ is contained in some $I_i \in \mathcal{U}$.

We denote with $\mathrm{Homeo}_0(S^1)$ the connected component of $\mathrm{Homeo}(S^1)$ containing the identity. Since $\mathrm{Homeo}_0(S^1)$ is algebraically simple [Man15][Corollary 1.10], and $\mathrm{Homeo}_+(S^1)$ is connected and normal $\mathrm{Homeo}_0(S^1)$, they coincide. Here we mention an important fact about the group of orientation preserving homeomorphisms (for a sketch of the proof and for references see [Man15]):

Theorem C.0.2. *The group $\mathrm{Homeo}_+(S^1)$ has the fragmentation property.*

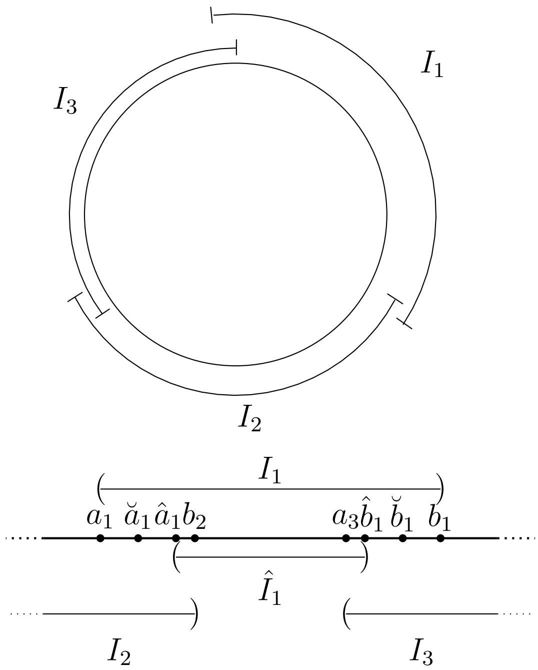

Figure C.1: The covering of the unit circle.

By the above theorem any given diffeomorphism γ can be written as a product of elements supported in I_k. For our purpose we need a slightly refined version of it, namely, if $\gamma \in \mathcal{V}$, where $\mathcal{V}$ is in a small neighborhood of the unit element ι, then the fragments γ_k can be taken in a small, but larger neighborhood $\hat{\mathcal{V}}$:

Lemma C.0.3. *There is a neighborhood $\mathcal{V}$ of the unit element ι of $\widetilde{\mathcal{D}^s(S^1)}$ and continuous localizing maps $\chi_k : \mathcal{V} \to \widetilde{\mathcal{D}^s(I_k)}$ with*

$$\gamma = \chi_1(\gamma)\chi_2(\gamma)\chi_3(\gamma)$$

and $\chi_k(\iota) = \iota$, $\operatorname{supp}\chi_k(\gamma) \subset I_k$. If $\operatorname{supp}\gamma \subset \breve{I}_k \cup \breve{I}_{k+1}$, then $\chi_{k+2}(\gamma) = \iota$, where $k = 1, 2, 3 \mod 3$.

Proof. We may assume without loss of generality that $0 < a_1 < \breve{a}_1 < \hat{a}_1 < b_2 < a_3 < \hat{b}_1 < \breve{b}_1 < b_1 < 2\pi$, (see Figure C.1).

We choose 2π-periodic function $D_{c,1}$ with $D_{c,1}(t) = 1$ for $t \in \hat{I}_1 = [\hat{a}_1, \hat{b}_1]$ and $D_{c,1}(t) = 0$ for $t \in [0, \breve{a}_1] \cup [\breve{b}_1, 2\pi]$ and $0 \leq D_{c,1}(t) \leq 1$ everywhere. Let $0 \leq D_{l,1}(t) \leq 1$ be another smooth 2π-periodic function with support in $(a_1, \breve{a}_1)$ and with $\int_0^{2\pi} D_{l,1}(t)dt = \int_{a_1}^{\breve{a}_1} D_{l,1}(t)dt = \frac{1}{2}(\breve{a}_1 - a_1)$ (which is possible because the interval $(a_1, \breve{a}_1)$ is longer than $\frac{1}{2}(a_1, \breve{a}_1)$). Similarly, let $0 \leq D_{r,1}(t) \leq 1$ be a smooth 2π-periodic function with support in $(\breve{b}_1, b_1)$ and with $\int_0^{2\pi} D_{r,1}(t)dt = \frac{1}{2}(b_1 - \breve{b}_1)$.

We consider the following open neighborhood of the unit element of $\widetilde{\mathcal{D}^s(S^1)}$

$$\mathcal{V}_\varepsilon := \left\{\gamma \in \widetilde{\mathcal{D}^s(S^1)} : |\gamma(\theta) - \iota(\theta)| < \varepsilon, |\gamma'(\theta) - 1| < \varepsilon \text{ for } \theta \in [0, 2\pi]\right\}.$$

Suppose $\gamma \in \mathcal{V}_\varepsilon$. We set

$$M := \max\{D_{\mathrm{c},1}(t), t \in [0, 2\pi]\}$$

and define the constant $\alpha(\gamma)$ by

$$\alpha_1(\gamma) = \frac{2}{\breve{a}_1 - a_1}\left(\gamma(\hat{a}_1) - \hat{a}_1 - \int_0^{\hat{a}_1} (\gamma'(t) - 1)D_{\mathrm{c},1}(t)dt\right). \tag{C.0.3}$$

It follows that

$$|\alpha_1(\gamma)| \leq \frac{2}{|\breve{a}_1 - a_1|}\varepsilon(1 + \hat{a}_1 M) \tag{C.0.4}$$

by the definition of $\mathcal{V}_\varepsilon$ and

$$\gamma(\hat{a}_1) = \int_0^{\hat{a}_1} ((\gamma'(t) - 1)D_{\mathrm{c},1}(t) + 1 + \alpha_1(\gamma)D_{\mathrm{l},1}(t))dt.$$

Similarly, set the constant $\beta_1(\gamma)$ by

$$\begin{aligned}\beta_1(\gamma) &= \frac{-2}{b_1 - \breve{b}_1}\left(\int_0^{2\pi} ((\gamma'(t) - 1)D_{\mathrm{c},1}(t) + \alpha_1(\gamma)D_{\mathrm{l},1}(t))dt\right) \\ &\left(= \frac{2}{b_1 - \breve{b}_1}\left(\hat{b}_1 - \gamma(\hat{b}_1) - \int_{\hat{b}_1}^{b_1} (\gamma'(t) - 1)D_{\mathrm{c},1}(t)\right)\right),\end{aligned} \tag{C.0.5}$$

then it follows that

$$|\beta_1(\gamma)| \leq \frac{2}{|b_1 - \breve{b}_1|}\varepsilon(|\hat{b}_1 - b_1|M + 1) \tag{C.0.6}$$

and

$$b_1 = \int_0^{b_1} ((\gamma'(t) - 1)D_{\mathrm{c},1}(t) + 1 + \alpha_1(\gamma)D_{\mathrm{l},1}(t) + \beta_1(\gamma)D_{\mathrm{r},1}(t))dt.$$

The function

$$\gamma_1(\theta) = \int_0^{\theta} ((\gamma'(t) - 1)D_{\mathrm{c},1}(t) + 1 + \alpha_1(\gamma)D_{\mathrm{l},1}(t) + \beta_1(\gamma)D_{\mathrm{r},1}(t))dt \tag{C.0.7}$$

is 2π-periodic and the first derivative

$$\gamma_1'(\theta) = (\gamma'(\theta) - 1)D_{\mathrm{c},1}(\theta) + 1 + \alpha_1(\gamma)D_{\mathrm{l},1}(\theta) + \beta_1(\gamma)D_{\mathrm{r},1}(\theta)$$

is positive if we take ε sufficiently small because we can control $|\alpha_1(\gamma)|$ and $|\beta_1(\gamma)|$ by (C.0.4), (C.0.6). Furthermore $\gamma_1' - 1 \in H^{s-1}(S^1)$ because $H^{s-1}(S^1)$ is an algebra by Lemma 1.4.1 and $\gamma - \iota \in H^s$. In conclusion, γ_1 can be regarded as an element in $\widetilde{D^s(S^1)}$. It also has the desired properties, namely $\gamma_1(\theta) = \theta$ for $\theta \in I_1'$ and $\gamma_1(\theta) = \gamma(\theta)$ for $\theta \in \hat{I}_1$. From (C.0.7)(C.0.3)(C.0.5) follows that the map $\mathcal{V}_\varepsilon \to \widetilde{\mathcal{D}^s(S^1)}$, $\gamma \mapsto \gamma_1$ is continuous.

We choose ε such that γ_1' is positive for $\gamma \in \mathcal{V}_\varepsilon$. Furthermore the assignment $\mathcal{V}_\varepsilon \to \widetilde{\mathcal{D}^s(S^1)}$, $\gamma \mapsto \gamma\gamma_1^{-1}$ is continuous by Lemma 1.4.2. We take $\mathcal{V} \subset \mathcal{V}_\varepsilon$ to be the neighborhood of the identity of $\widetilde{D^s(S^1)}$ such that for $\gamma \in \mathcal{V}$ we have $\gamma\gamma_1^{-1} \in \mathcal{V}_{\varepsilon_1}$ where ε_1 is small enough that we obtain $\gamma_2 \in \widetilde{D^s(S^1)}$ (in particular γ_2' is positive) if we do an analogous construction on I_2 for $\gamma\gamma_1^{-1}$.

For $\gamma \in \mathcal{V}$ we set $\chi_1(\gamma) = \gamma_1$. The continuity of the map χ_1 in the topology of $\widetilde{\mathcal{D}^s(S^1)}$ is clear.

Next we construct $\chi_2(\gamma)$. By construction $(\gamma\gamma_1^{-1})(\theta) = \theta$ for $\theta \in \hat{I}_1$, therefore , $\operatorname{supp} \gamma\gamma_1^{-1} \subset I_2 \cup I_3$. We can apply an analogous construction to I_2 and $\gamma\gamma_1^{-1}$ to obtain γ_2 such that $\operatorname{supp} \gamma_2 \subset \hat{I}_2$, $\gamma_2(\theta) = (\gamma\gamma_1^{-1})(\theta)$ for $\theta \in \hat{I}_2$. In this way we obtain the continuous map $\chi_2(\gamma) := \gamma_2$. Furthermore, by our choice $\hat{a}_1 < \hat{b}_2 < \check{b}_2 < b_2$, $\gamma_2(\theta) = (\gamma\gamma_1^{-1})(\theta)$ for $\theta \in \hat{I}_1$ where both are equal to θ, hence for $\hat{I}_1 \cup \hat{I}_2$.

Now we have $(\gamma\gamma_1^{-1}\gamma_2^{-1})(\theta) = \theta$ for $\theta \in \hat{I}_1 \cup \hat{I}_2$, and as $\{\hat{I}_k\}$ is a cover of S^1, $(\hat{I}_1 \cup \hat{I}_2)' \subset \hat{I}_3$. Therefore, if we set $\chi_3(\gamma) = \gamma\gamma_1^{-1}\gamma_2^{-1}$, it is supported in $\hat{I}_3 \subset I_3$ and the map χ_3 is continuous because it is a composition of continuous maps (Lemma 1.4.2). □

If γ is not supported on all S^1 but is localized in some proper interval, we can improve the previous statement.

Lemma C.0.4. *Let $k \in \{1, 2, 3\} \mod 3$ and $\tilde{I}_k = I_k \cup I_{k+1}$. There is a neighborhood $\mathcal{V}$ of the unit element ι of $\widetilde{\mathcal{D}^s(S^1)}$ and continuous localizing maps*

$$\chi_k^{(k)} : \mathcal{V} \cap \widetilde{\mathcal{D}^s(\tilde{I}_k)} \to \widetilde{\mathcal{D}^s(I_k)},$$
$$\chi_{k+1}^{(k)} : \mathcal{V} \cap \widetilde{\mathcal{D}^s(\tilde{I}_k)} \to \widetilde{\mathcal{D}^s(I_{k+1})}$$

with $\gamma = \chi_k^{(k)}(\gamma)\chi_{k+1}^{(k)}(\gamma)$ and $\chi_k^{(k)}(\iota) = \chi_{k+1}^{(k)}(\iota) = \iota$.

Proof. Without loss of generality, we may assume $k = 2$. This is done by applying the steps of constructing χ_2 and χ_3 in the proof of Lemma C.0.3 to slightly enlarged I_2 and $\hat{I}_2$, so that $\chi_2^{(2)}(\gamma)(\theta) = \gamma(\theta)$ for $\theta \in I_3'$. □

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