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Leader-Following consensus for nonlinear agents with measurement feedback[†]

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Abstract

The Leader-Following consensus problem is investigated for large classes of nonlinear identical agents. Sufficient conditions are provided for achieving consensus via state and measurement feedback laws based on a local (i.e. among neighbors) information exchange. The leader's trajectories are assumed bounded without knowledge of the containing compact set and the agents' trajectories possibly unbounded under the action of a bounded input. Generalizations to heterogeneous agents and robustness are also discussed.

KEYWORDS:
Leader-Following consensus, state and measurement feedback laws, incremental homogeneity.

1 | INTRODUCTION

Multiagent systems and their collective behavior are topics of increasing interest due to the wide fields of applications, such as robotics, telecommunications and biology. The extensive literature on these topics focuses mainly on linear models (e.g.[4],[20],[22],[23],[27] and the references therein). The range of issues considered goes from the general consensus problem, where the agents have to reach a common objective, to the leader consensus problem, where the agents's objective consists in tracking a given dynamics called leader. In this sense the leader following consensus problem is a particular problem of consensus. In general the leader's trajectory is assumed to be generated by a suitable dynamic system and the output of each agent (the tracking output) must track the leader's output (or trajectory). Several results are available in the literature: in the linear deterministic case we recall[12],[17],[24] where delays are considered,[11],[16],[20],[21] where the topology of the network is switching.

Subsequently, agents described by nonlinear dynamics were considered for the first time in[18] and then in[15], where the nonlinearities in each agent dynamics is a convex combination of the nonlinearities of its neighbors. In[10], instead, the leader has a constant trajectory, in[9] the nonlinearity in the agents dynamics is assumed to be lower triangular and globally Lipschitz.

In[19] state feedback consensus laws (using the tracking outputs of each agent and its neighbors) were designed assuming each agent identical and passive with respect to its tracking output (controllers (6), (7) in[19]). Subsequently, the tracking outputs are estimated through invariant-manifold-based observers using the measured outputs to implement the state feedback consensus laws as output feedback laws. However, globally Lipschitz conditions are required on the zero dynamics of the agents (assumption (H2.1) of theorem 1) and global stability of suitable incremental systems associated to the agents' dynamics (systems (19) in[19]) is required having the effect of trivializing the design of the observers.

In[29] feedback consensus laws are designed for each heterogeneous agent using its tracking output and state together with the neighboring agents' tracking output (controllers (7) in[29]). The leader's dynamics is assumed to be linear and internally stable. Uncertainties modeled as constant parameters in a fixed compact set are included in the agents' dynamics. Practical design for robust consensus laws is illustrated on a benchmark class of lower triangular heterogeneous agents.

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In[5] and, subsequently, in[7] an output regulation approach is adopted for achieving consensus. Agents' consensus is first achieved on the leader's dynamics through a local internal model for each agent which is then used for designing local output regulators. Heterogeneous agents are considered while the leader's dynamics is a linear system in prime form (A, B, C) in state feedback interconnection with a nonlinearity (exosystem (7) in[7]) and it is assumed to have an attractive known compact set under the action of external inputs. The uncertainties are modeled as constant parameters in a fixed compact set. Consensus laws for each agent are designed using its tracking output together with the neighboring agents' measured outputs and local internal models' states (controllers (4) in[7]) with the assumptions that the measured outputs are equal to the tracking outputs. Practical design for robust consensus laws is illustrated on a benchmark class of heterogeneous agents in global normal form.

Finally, in[13],[6] heterogeneous agents are considered while the leader's dynamics is assumed to be linear and internally stable. Consensus laws for each agent are designed using its tracking output and (partial) state together with the neighboring agents' measured outputs (controllers (4) in[13]), again with the assumptions that the measured outputs are equal to the tracking outputs. Uncertainties modeled as constant parameters in a fixed compact set are also included in the agents' dynamics and practical design for robust consensus laws is illustrated on a benchmark class of lower triangular heterogeneous agents.

In the present paper the leader-following consensus problem is investigated for nonlinear systems under general assumptions and considering different frameworks (preliminary results have been given in[2]):

- (*leader-following consensus for identical agents*) The leader's dynamics is generically nonlinear with bounded trajectories for all times, neither knowledge of the compact set in which the trajectories are contained nor existence of globally attractive compact sets under the effect of external inputs (compare with[7]). Under this regard, our boundedness assumption may follow naturally for instance from any agents' passivity property. Relatedly, the agents' dynamics, identical to the leader's, have no finite escape time under the effect of external inputs ([29]). The consensus law for each follower is designed using only its measured output together with the neighboring agents' measured outputs (compare with[6],[7],[13],[26] and[29]). The consensus laws design is split up into two steps: in the first step consensus laws are designed with neighbors' state feedback, in the second step observers are introduced to recover the local state information and output consensus laws are obtained using a separation principle (i.e. state feedback laws are recovered when estimation errors are zero). The dynamics of each agent has a controllable and observable linearization and a general form with its nonlinearities assumed to be incrementally homogeneous in the upper bound[1], a sufficiently general assumption for achieving stability by feedback with a guaranteed region of attraction for wide classes of nonlinear systems including upper/lower triangular and intertwined structures. At this stage we do not consider the robustness issue as in[13] and[29], where uncertainty is modeled as a constant unknown parameter with values in some known compact set or a bounded time-varying disturbance.

- (*leader-following consensus for heterogeneous agents*) In Section 8.3 it is shown how to modify the assumptions in order to cope with heterogeneous agents. This extension brings our contribution to a closer comparison with[6],[7],[13],[26] and[29]. As a matter of fact we give additional assumptions in terms of the existence of a globally invariant manifold on which leader-following consensus is achieved, exactly as in the case of consensus in the aforementioned contributions. Therefore, the problem of achieving leader-following consensus boils down to make this invariant manifold globally attractive using the stabilization techniques in the case of identical agents.

- (*leader-following consensus for identical agents with uncertainties*) In Section 8.4 it is shown how to modify the initial assumptions when identical agents and time-varying bounded uncertainties are considered. In this case a disturbance-to-consensus IIS-type result is obtained, with a result comparable to[29] (although with the restrictions on the class of systems enlightened above) while in[29] consensus is achieved despite the unknown constant parameter.

The paper is organized as follows: in Section 2 the consensus problem is stated. In Section 3 some recalls on the notations and the properties of incrementally homogeneous systems are given. In Section 4 the error dynamics associated to the agents with respect to the leader is characterized using the assumption of incremental homogeneity and feedback laws are designed using local (i.e. from the neighbors) state information to make all the agents achieve consensus with the leader for a prescribed compact set of initial leader's state configurations. In Section 5 we assume that each agent exchanges with its neighbors the residual between the output consensus error and its dynamical estimate (implemented by some observer) and we design observers for reconstructing the local missing information on the consensus error. In Section 6 we put together the result of Sections 4 and 5 to design feedback laws using local output information to make all the agents achieve consensus with the leader for a prescribed compact set of initial leader's state configurations. These feedback laws are obtained from state feedback laws replacing the states of neighboring agents with dynamical estimates implemented by local observers. In Section 7 simulations are given for identical agents given by a Van Der Pol oscillator. In Section 8.3 the case of heterogeneous agents and in Section 8.4 the case of identical agents and time-varying uncertainties are discussed.

2 | PROBLEM STATEMENT

Consider the multiagent system

$$\begin{aligned} \dot{x}^{(0)}(t) &= Ax^{(0)}(t) + f(x^{(0)}(t)), \\ y^{(0)} &= Cx^{(0)}(t), \end{aligned} \tag{1}$$

$$\begin{aligned} \dot{x}^{(i)}(t) &= Ax^{(i)}(t) + f(x^{(i)}(t)) + Bu^{(i)}(t), \\ y^{(i)} &= Cx^{(i)}(t),\ i \in [1, N], \end{aligned} \tag{2}$$

consisting of a leader (eq. (1)) with $x^{(0)} \in R^n$, $y^{(0)} \in R$ and f a locally Lipschitz function, and N identical agents (eq. (2)), where $x^{(i)} \in R^n$, $y^{(i)} \in R$ and $u^{(i)} \in R$, $i \in [1, N]$, are the state, measured output and control input of the i-th agent. We will assume that (2) is controllable and observable in the first approximation around the origin and, without loss of generality, the pair (A, B) is in the form

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}. \tag{3}$$

Consequently $f(x^{(0)})$ in (1) can be written as

$$f(x^{(0)}) = BKx^{(0)} + f_0(x^{(0)}), \qquad \frac{\partial f_0(x^{(0)})}{\partial x^{(0)}}|_{x^{(0)}=0} = 0. \tag{4}$$

Furthermore we initially restrict the study to the case in which the matrix C is in the form

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \tag{5}$$

so that the triple (A, B, C) is in prime form. Such an assumption is taken only in order to simplify the notation. The general case with the pair (A,C) observable but with no special assumptions on C is discussed in Section 8.1, while how to handle nonlinearities in the output is discussed in section 8.2.

Moreover, on the leader's and, respectively, agents's trajectories we assume that

($\mathcal{A}0$) *(Leader) The trajectories of the system (1) are bounded for all $t \geq 0$ and initial conditions $x^{(0)}(0)$.*

($\mathcal{A}1$) *(Agents) There exist non-decreasing continuous functions $\alpha, \beta, \gamma : [0, +\infty) \to [0, +\infty)$ such that the trajectories of the system (2) satisfy*

$$\sup_{s\in[0,t]} \|x^{(i)}(s)\| \leq \alpha(t) + \beta(\|x^{(i)}(0)\|) + \gamma(\sup_{s\in[0,t]} |u^{(i)}(s)|),\ \forall i \in [1, N], \tag{6}$$

for all $t \geq 0$, initial conditions $x^{(i)}(0)$ and continuous input function $u^{(i)}(\cdot)$.

Assumption ($\mathcal{A}0$) requires that the leader's trajectories $x^{(0)}(t)$ are bounded for all times, without any knowledge of the containing compact set (depending on the initial condition $x^{(0)}(0)$). The boundedness of the leader's trajectory (or the consensus trajectory) is a common hypothesis in most papers ([7], [13], [6], [26] and [29]), specifically the compact set is assumed to be known or the existence of a known attracting compact set is assumed for the dynamical system which generates the leader's trajectories (exosystem). In this sense, our assumption is more general, since the knowledge of the compact set is not required. From a practical point of view, it is reasonable to assume boundedness of the leader's trajectories. Moreover, unbounded leader's trajectories would give place to unbounded controllers. Moreover, the dynamical system which generates the leader's trajectories ((1) in our paper) is assumed to be linear or globally Lipschitz in other similar contexts ([7], [13], [6], [26] and [29]).

Assumption ($\mathcal{A}1$) is a restriction on the agents' trajectories, which may not escape to infinity in finite time as long as the input is bounded. ISS properties of the agents' dynamics with respect to fixed compact sets have been required in papers as[7] (assumption 2),[13] (assumption 5) and[29] (formulas (44), (45)). In this sense and when the comparison is carried out in a leader following framework, our assumption is more general, since we simply require that the agents's trajectories do no not explode in finite time under the effect of bounded inputs. The technical reason for this weaker assumption is that it is sufficient to give the agents' controllers the time to recover the information on the (unknown) magnitude of the leader's state while maintaining their state bounded. It should be remarked that [13] and [29] are focused on the problem of achieving global agents' consensus over a network (i.e. no leaders) and, mainly, the ISS assumptions are motivated by the global achievement of the consensus. However, in a semiglobal framework as in the present paper, assumption (A1) is still weaker.

Remark 1. Van Der Pol oscillators

$$\dot{x}_1 = x_2 \tag{7}$$
$$\dot{x}_2 = (1 - x_1^2)x_2 - x_1 + u \tag{8}$$

satisfy $(\mathcal{A}0)$ *and* $(\mathcal{A}1)$ *with* $x^{(i)} = (x_1, x_2)^T$, $f(x^{(i)}) = (0, (1 - x_1^2)x_2 - x_1)^T$, *for* $i \in [0, N]$. *Indeed, if* $V(x) = \|x\|^2$ *we have* $\dot{V}(t) \leq 3V(t) + \|u(t)\|^2$ *which implies (6) with* $\alpha(s) = 2e^{6s}$, $\beta(s) = 4s^4$ *and* $\gamma(s) = 4s^4$.

Many other benchmark examples can be given that satisfy $(\mathcal{A}0)$ *and* $(\mathcal{A}1)$ *such as the following FitzHugh-Nagumo-type oscillator*

$$\dot{x}_1 = x_2 + a - bx_1^3 \tag{9}$$
$$\dot{x}_2 = x_2 - x_2^3 - x_1 + \cos t + u \tag{10}$$

with parameters $a, b > 0$ *(time-varying terms as* $\cos t$ *are not considered in (1), (2) for simplicity) or the tunnel-diode*

$$\dot{x}_1 = -\frac{1}{C}x_1 - \frac{1}{C}h(x_2) \tag{11}$$
$$\dot{x}_2 = -\frac{R}{L}x_1 - \frac{1}{L}x_2 + \sin t + \frac{1}{L}u \tag{12}$$

with nonlinearitiy h*, resistance* R *and capacitance* C*.*

The information exchange between the N agents and the leader will be represented by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2 \ldots, N\}$ is the set of vertices representing the N agents and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges of the graph. Edge (i, j), indicates that agents i and j can exchange information. The graph is undirected if the edges (i, j) and $(j, i) \in \mathcal{E}$ are considered to be the same. Two nodes i and j are neighbors to each other if $(i, j) \in \mathcal{E}$. The set of neighbors of node i is denoted by $\mathcal{N}^{(i)} := \{j \in \mathcal{V} : (j, i) \in \mathcal{E}, j \neq i\}$. A path is a sequence of connected edges in a graph. The graph is connected if there is a path (e.g. a sequence of connected edges) between every pair of vertices. The adjacency matrix $\mathcal{Q} = [q_j^i]$ of a graph $\mathcal{G}$ is an $N \times N$ matrix, whose (i, j)-th entry q_j^i is 1 if (i, j) is an edge of $\mathcal{G}$ and 0 if it is not or equivalently if the agent j is a neighbor of the agent i. The degree matrix $\mathcal{D}$ of $\mathcal{G}$ is a diagonal matrix whose i-th diagonal element is equal to the cardinality of $\mathcal{N}^{(i)}$. The Laplacian of $\mathcal{G}$ is a $N \times N$ matrix $\mathcal{L} = [l_j^i]$ such that $\mathcal{L} = -\mathcal{Q} + \mathcal{D}$. Moreover, $\mathcal{L}$ is symmetric if and only if the graph is undirected and $\mathcal{L}$ has all its eigenvalues in the closed right half plane and one eigenvalue at zero if and only if the graph is connected.
The leader is represented by vertex 0 and information is exchanged between the leader and the agents which are neighbors of the leader. Then, we have a graph $\widetilde{\mathcal{G}}$, which consists of graph $\mathcal{G}$, vertex 0 and edges between the leader 0 and its neighbors. Let $\mathcal{L}_0 = \text{diag}\{\ell_0^1, \cdots, \ell_0^N\}$ where ℓ_0^i is 1 if the leader is a neighbor of agent i and 0 else. The undirected graph $\widetilde{\mathcal{G}}$ is connected if and only if $\widehat{\mathcal{L}} = \mathcal{L} + \mathcal{L}_0$ is positive definite [20].

In this network, for each agent we consider control inputs using local (i.e. from the neighbors) information to make all the N agents achieve consensus with the leader for a prescribed compact set of initial leader's state configurations. We first consider the case of full state information (i.e. each agent exchanges locally the consensus error) and, then, using state reconstruction, we consider the case of partial information (i.e. each agent exchanges locally the output and the estimate of the consensus error).

In the case of full state information, the control input $u^{(i)}$ for the agent i, $i \in [1, N]$, is a function $v(\zeta^{(i)})$ of the *consensus law*

$$\zeta^{(i)} := \sum_{j=1}^{N} \ell_j^i (x^{(0)} - x^{(j)}) + \ell_0^i (x^{(0)} - x^{(i)}) \tag{13}$$

where ℓ_j^i are the entries of the Laplacian of $\mathcal{G}$. By the consensus law (13) each agent i exchanges its consensus error $x^{(0)} - x^{(i)}$ with its neighbors: indeed, $\sum_{j=1}^{N} \ell_j^i (x^{(0)} - x^{(j)}) = \sum_{j \in \mathcal{N}^{(i)}} ((x^{(j)} - x^{(0)}) - (x^{(i)} - x^{(0)}))$. Also notice that by the properties of $\mathcal{L}$

$$\zeta^{(i)} = \sum_{j=1}^{N} \ell_j^i (x^{(0)} - x^{(j)}) + \ell_0^i (x^{(0)} - x^{(i)}) = -\sum_{j=1}^{N} \ell_j^i e^{(j)} + \ell_0^i e^{(i)} \tag{14}$$

where $e^{(i)} = x^{(i)} - x^{(0)}$ is the consensus error for the agent i. For any given set, let $\Omega \subset \mathbb{R}^n$ with $\Omega^{\times N} = \overbrace{\Omega \times \cdots \times \Omega}^{N \text{times}}$.

Semiglobal Leader Following with Full State Information (Problem I).
Consider the multiagent network (1), (2). Given a fixed communication topology $\mathcal{G}$, described by an undirected and connected graph $\widetilde{\mathcal{G}}$, together with a compact set $\Omega \subset \mathbb{R}^n$, containing the origin, find a state feedback $u^{(i)} = v(\zeta^{(i)})$, $i \in [1, N]$, such that the dynamics of the consensus error $e = (e^{(1)^T}, \cdots, e^{(N)^T})^T$, $e^{(i)} = x^{(i)} - x^{(0)}$, $i \in [1, N]$, associated to (1), (2), is asymptotically stable with basin of attraction containing $\Omega^{\times N}$.

In other words, consensus with the leader must be achieved by all the agents for all the leader's state trajectories which at initial time may differ from each agent's state trajectories by a prescribed amount (quantified by the compact set Ω). Such a problem is addressed in Section 4.

In the case of partial information, the control input $u^{(i)}$ for the agent i, $i \in [1, N]$, is defined as follows:

(i) introduce for each agent i the consensus law

$$\chi^{(i)} = \sum_{j=1}^{N} \ell^i_j (y^{(j)} - y^{(0)} - C\xi^{(j)}) + \ell^i_0 (y^{(i)} - y^{(0)} - C\xi^{(i)}), \; i \in [1, N], \tag{15}$$

i.e. each agent exchanges locally its residual $y^{(i)} - y^{(0)} - C\xi^{(i)}$ where $\xi^{(i)}$ is an estimate of the consensus error $e^{(i)}$ (indeed, $\sum_{j=1}^{N} \ell^i_j (y^{(j)} - y^{(0)} - C\xi^{(j)}) = -\sum_{j \in \mathcal{N}^{(i)}} ((y^{(j)} - y^{(0)} - C\xi^{(j)}) - (y^{(i)} - y^{(0)} - C\xi^{(j)})))$,

(ii) the estimate $\xi^{(i)}$, $i \in [1, N]$, of the consensus error $e^{(i)}$, $i \in [1, N]$, is provided by an observer of the form

$$\dot{\xi}^{(i)} = A\xi^{(i)} + Bu^{(i)} + \Delta f(S(\xi^{(0)}), S(\xi^{(i)})) + w(\chi^{(i)}) \tag{16}$$

where S is a suitable saturation of $\xi^{(0)}$ and $\xi^{(i)}$ and $\Delta f(x^{(0)}, x^{(i)}) := f(x^{(0)}) - f(x^{(i)})$ and for some function $w(\cdot)$ of the *consensus law* (15). According if the leader's state is measured or not, $\xi^{(0)}$ is $x^{(0)}$ or a suitable estimate provided by an observer of the form

$$\dot{\xi}^{(0)} = A\xi^{(0)} + f(S(\xi^{(0)})) + w(y^{(0)} - C\xi^{(0)}), \tag{17}$$

(iii) the control input $u^{(i)}$ for the agent $i \in [1, N]$, is defined from the function $v(\cdot)$ (Problem I) as follows

$$u^{(i)} = v(\hat{\zeta}^{(i)}) \tag{18}$$

where the *estimated consensus law* $\hat{\zeta}^{(i)}$ is obtained from the consensus law $\zeta^{(i)}$ in (14) by replacing each consensus error $e^{(i)}$, $i \in [1, N]$, with its estimate $\xi^{(i)}$, $i \in [1, N]$.

Semiglobal Leader Following with Partial State Information (Problem II).
Consider the multiagent network (1), (2). Given a fixed communication topology $\mathcal{G}$, described by an undirected and connected graph $\widetilde{\mathcal{G}}$, together with a compact set $\Omega \subset \mathbb{R}^n$, containing the origin, find a dynamic measurement feedback (16), (17), (18) such that the dynamics of the consensus error $e = (e^{(1)^T}, \cdots, e^{(N)^T})^T$, $e^{(i)} = x^{(i)} - x^{(0)}$, $i \in [1, N]$, associated to (1), (2), is asymptotically stable with basin of attraction containing $\Omega^{\times N}$.

Problem II is solved in Section 6 by using the results of Section 4 and those in Section 5 where dynamical observers are designed to to reconstruct the consensus errors among the neighboring agents. For purpose of illustration, we split the presentation of the results into two parts: 1) the leader's state trajectories are contained in some known compact set Ω (sections 4-6) and 2) no containing (or attractive) set Ω is known.

3 | RECALLS AND PRELIMINARY RESULTS

Notations and definitions recalled hereafter are issued from[1]. More precisely,

• $\mathbb{R}^n$ is the set of n-dimensional real column vectors. $\mathbb{R}^n_>$ ($\mathbb{R}^{(N)}_\geq$) denotes the set of vectors in $\mathbb{R}^n$ with real positive (respectively non-negative) entries. A similar notation is used for matrices.

• For any $V \in \mathbb{R}^{p\times n}$ we denote by $V_{i,j}$ the (i,j)-th entry of V. For any $v \in \mathbb{R}^n$, $\mathrm{diag}\{v\}$ is the diagonal $n \times n$ matrix with diagonal elements $v_1, \dots, v_n$; v_i^k will denote the i-th component of the k-th block. We retain a similar notation for functions. $\mathbf{1}_m$ will denote the vector of $\mathbb{R}^m$ with all the m components equal to 1.

• $|a|$ denotes the absolute value of $a \in \mathbb{R}$, $\|a\|$ the Euclidean norm of $a \in \mathbb{R}^n$, $\|A\|$ the norm of $A \in \mathbb{R}^{n\times n}$ induced from the Euclidean norm $\|a\|$, while $\langle\langle A\rangle\rangle$ is the matrix obtained from A by substituting each element $a_{i,j}$ with its absolute value $|a_{i,j}|$. For any $v, w \in \mathbb{R}^n$, $v \preceq w$ means that $v_i \leq w_i$ for all $i \in [1, N]$.

• We denote by $\mathbb{C}^j(\mathcal{X}, \mathcal{Y})$, with $j \geq 0$, $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{Y} \subset \mathbb{R}^p$, the set of j-times continuously differentiable functions $f : \mathcal{X} \to \mathcal{Y}$. We also have the following notation: for any function $f : \mathbb{R}^n \to \mathbb{R}^p$ and for each pair of points $x', x'' \in \mathbb{R}^n$

$$\Delta f(x', x'') = f(x') - f(x'') \tag{19}$$

and when f is the identity function $\Delta(x', x'') = x' - x''$.

• Given a $m \times n$ matrix A and a $p \times q$ matrix B the Kronecker product $A \otimes B$ is a $(mp) \times (nq)$ matrix P, where the element a_{ij} in A is replaced by the block $a_{ij}B$. If A and B are square and invertible then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$. Given the matrices A, B, C and D, and assuming the products AC and BD defined, then, $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$. With such notation, for a given $\mathfrak{f} \in \mathbb{R}^n$, $\mathbf{1}_m \otimes \mathfrak{f}$ is the vector of $\mathbb{R}^{m\times n}$ equal to $(\mathfrak{f}^T, \dots, \mathfrak{f}^T)^T$. When no confusion is possible the subindex m will be omitted.

• A *saturation function* σ_h *with saturation levels* $h \in \mathbb{R}^n_>$ is a function $\sigma_h(x) := (\sigma_{h_1}(x_1), \dots, \sigma_{h_n}(x_n))^T$, $x \in \mathbb{R}^n$, such that for each $i \in [1, N]$ and $x_i \in \mathbb{R}$:

$$\sigma_{h_i}(x_i) = \begin{cases} x_i & |x_i| \leq h_i \\ \mathrm{sign}(x_i)h_i & \text{otherwise.} \end{cases} \tag{20}$$

• For any $\epsilon \in \mathbb{R}_>$ and for any vector $\mathfrak{r} \in \mathbb{R}^n_>$, we define $\epsilon^{\mathfrak{r}} := (\epsilon^{\mathfrak{r}_1}, \cdots, \epsilon^{\mathfrak{r}_n})^T$. The dilation of a vector $x \in \mathbb{R}^n$ with weights $\mathfrak{r}$ is denoted by $\epsilon^{\mathfrak{r}} \diamond x$ and is given by $\epsilon^{\mathfrak{r}} \diamond x := (\epsilon^{\mathfrak{r}_1} x_1, \cdots, \epsilon^{\mathfrak{r}_n} x_n)^T$. Note that for any $x, y \in \mathbb{R}^n$, $\mathfrak{r}_1, \mathfrak{r}_2 \in \mathbb{R}^n_>$ and $\epsilon \in \mathbb{R}_>$,

$$\epsilon^{\mathfrak{r}_1} \diamond \epsilon^{\mathfrak{r}_2} \diamond x = \epsilon^{\mathfrak{r}_2} \diamond \epsilon^{\mathfrak{r}_1} \diamond x = \epsilon^{\mathfrak{r}_1+\mathfrak{r}_2} \diamond x, \tag{21}$$

and

$$(\epsilon^{\mathfrak{r}_1} \diamond x)^T(\epsilon^{\mathfrak{r}_2} \diamond y) = (\epsilon^{\mathfrak{r}_2} \diamond x)^T(\epsilon^{\mathfrak{r}_1} \diamond y) = (\epsilon^{\mathfrak{r}_1+\mathfrak{r}_2} \diamond x)^T y = x^T(\epsilon^{\mathfrak{r}_1+\mathfrak{r}_2} \diamond y). \tag{22}$$

3.1 | Homogeneity and Incremental Homogeneity in the generalized sense

To cope with nonlinearities, we will use the notions of generalized homogeneity and incremental homogeneity in the upper bound introduced in [1] in the context of semi-global stabilization and observer design problems. Here we recall these notions in a slightly more general form. Let $\Delta f(x', x'') := f(x') - f(x'')$ and if f is the identity we simply write $\Delta(x', x'') := x' - x''$.

Definition 1. *A function* $\phi_\epsilon \in \mathbb{C}^0(\mathbb{R}^n, \mathbb{R}^l)$, $\epsilon \in \mathbb{R}_>$, *is said to be incrementally homogeneous (i.h.) with quadruple* $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \Phi)$ *if there exist* $\mathfrak{d} \in \mathbb{R}^l$, $\mathfrak{h} \in \mathbb{R}^n$, $\mathfrak{r} \in \mathbb{R}^n_>$ *and* $\Phi \in \mathbb{C}^0(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^{l\times n})$ *such that for all* $\epsilon > 0$ *and* $x', x'' \in \mathbb{R}^n$

$$\Delta\phi_\epsilon(\epsilon^{\mathfrak{r}} \diamond x', \epsilon^{\mathfrak{r}} \diamond x'') = \epsilon^{\mathfrak{d}} \diamond \left(\Phi(x', x'')\Delta(\epsilon^{\mathfrak{h}} \diamond x', \epsilon^{\mathfrak{h}} \diamond x'')\right)$$

The notion of incremental homogeneity incapsulates as a particular case the notion of homogeneity in the classical sense. When the variation Δ of f is computed in between the dilated point $x' \in \mathbb{R}^n$ and $x'' = 0$, with $f(0) = 0$, we say that ϕ_ϵ *is homogeneous with quadruple* $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \Phi')$ with $\Phi'(x) = \Phi(x, 0)$.

Example. *The function* $\phi_\epsilon(x) := x_1 + x_2^3$ *(in this case* ϕ_ϵ *does not depend on the dilating parameter) is i.h. with quadruple* $(\mathfrak{r}, 0, \mathfrak{h}, \phi)$*, where* $\mathfrak{r} := (1,2)^T$, $\mathfrak{h} := (1,6)^T$ *and* $\phi(x', x'') := (1, (x_2')^2 + (x_2'')^2 + x_2'' x_2')$. *The function* $\phi_\epsilon(x) := \epsilon(x_1 + x_2^3)$ *(here* ϕ_ϵ *does depend on the dilating parameter) is i.h. with quadruple* $(\mathfrak{r}, 1, \mathfrak{h}, \phi)$ *and the same* ϕ *above.*

There are functions, like $\sin x$, which are not i.h. but behave in the upper bound as an i.h. function. This motivates the following definition ($\langle\langle a\rangle\rangle$ denotes the column vector of the absolute values of the elements of $a \in \mathbb{R}^n$).

Definition 2. *A function* $\phi_\epsilon \in \mathbb{C}^0(\mathbb{R}^n, \mathbb{R}^l)$, $\epsilon \in \mathbb{R}_>$, *is said to be incrementally homogeneous in the upper bound (i.h.u.b.) with quadruple* $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \Phi)$ *if there exist* $\mathfrak{d} \in \mathbb{R}^l$, $\mathfrak{h} \in \mathbb{R}^n$, $\mathfrak{r} \in \mathbb{R}^n_>$, $\Phi \in \mathbb{C}^0(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^{l\times n}_\geq)$ *such that for all* $\epsilon \geq 1$ *and* $x', x'' \in \mathbb{R}^n$

$$\left\langle\left\langle \Delta\phi_\epsilon(\epsilon^{\mathfrak{r}} \diamond x', \epsilon^{\mathfrak{r}} \diamond x'')\right\rangle\right\rangle \preceq \epsilon^{\mathfrak{d}} \diamond \left(\Phi(x', x'')\left\langle\left\langle \Delta(\epsilon^{\mathfrak{h}} \diamond x', \epsilon^{\mathfrak{h}} \diamond x'')\right\rangle\right\rangle\right)$$

When the variation Δ of f is computed in between the dilated point $x' \in \mathbb{R}^n$ and $x'' = 0$, with $f(0) = 0$, we say that ϕ_ϵ *is homogeneous in the upper bound with quadruple* $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \Phi')$ with $\Phi'(x) = \Phi(x, 0)$.

Example. *The function* $\phi_\epsilon(x) := \epsilon\left(x_2 \;\; x_2^3\psi(x_1)\right)^T$, $\psi \in \mathbf{C}^0(\mathbb{R},\mathbb{R})$ *any bounded and globally Lipschitz function, is i.h.u.b. with triple* $(\mathfrak{r},\mathfrak{d},\mathfrak{h},\Phi)$*, where* $\mathfrak{r} := (1,2)^T$, $\mathfrak{d} := (3,7)^T$, $\mathfrak{h} := (1,0)^T$ *and the matrix* $\Phi(x',x'')$ *defined as*

$$[\Phi(x',x'')] = \begin{pmatrix} 0 & 1 \\ (x_2'')^3 \frac{|g(x_1')-g(x_1'')|}{|x_1'-x_1''|} & |(x_2')^2 + (x_2'')^2 + x_2'x_2''||g(x_1')| \end{pmatrix}.$$

Remark 2. The properties of incremental homogeneity are discussed in[1]*. Here, we only notice that if* ϕ_ϵ *is i.h.u.b. with quadruple* $(\mathfrak{r},\mathfrak{d},\mathfrak{h},\Phi)$ *then* $\psi_\epsilon(x,y) = \Delta\phi_\epsilon(y+x,x)$ *is h.u.b. with quadruple* $(\mathfrak{r},\mathfrak{d},\mathfrak{h},\Psi)$ *with* $\Psi(x,y) = (0,\Phi(y+x,x))$.

4 | CONSENSUS WITH FULL STATE INFORMATION

In this Section we see how to solve Problem I which is instrumental to solve Problem II. As a matter of fact, out design approach is based on a kind of separation principle for output feedback design where an output feedback controller is obtained from a state feedback controller by dynamically reconstructing the state (through observers) and replacing in the state feedback controller the state with its dynamical estimate.

4.1 | The error dynamics

Before going into the technical details, let us rewrite the given agents' and leader's dynamics into the coordinate framework which describes the error dynamics $e^{(i)} = x^{(i)} - x^{(0)}$, $i \in [1,N]$, between the leader's trajectory and each follower's trajectory. Setting $e = \left(e^{(1)T} \;\cdots\; e^{(N)T}\right)^T$ and $u = \left(u^{(1)T} \;\cdots\; u^{(N)T}\right)^T$, from (1), (2)

$$\dot{x}^{(0)} = Ax^{(0)} + f(x^{(0)}) \tag{23}$$

$$\dot{e} = (I_N \otimes A)e + (I_N \otimes B)u + F(x^{(0)},e) \tag{24}$$

where, using the notation (19),

$$F(x^{(0)},e) = \begin{pmatrix} \Delta f(e^{(1)} + x^{(0)}, x^{(0)}) \\ \vdots \\ \Delta f(e^{(N)} + x^{(0)}, x^{(0)}) \end{pmatrix}. \tag{25}$$

The idea is to change the error coordinates in such a way to decouple the linearization of the collective error dynamics (25).

4.2 | The action of a class of linear change of coordinates

Now we will study the properties of a certain class of coordinates transformations, which can be used to decouple the linearization of the collective error dynamics (25). To this aim, given an invertible $N \times N$ matrix T, consider the class of change of coordinates

$$\begin{pmatrix} x^{(0)} \\ e \end{pmatrix} \to \begin{pmatrix} x^{(0)} \\ \widetilde{e} \end{pmatrix}, \; \widetilde{e} := (T \otimes I_n)e. \tag{26}$$

Lemma 1. Let T be an invertible $N \times N$ matrix, and consider the transformation (26). In the new coordinates system (23), (24) reads

$$\begin{aligned} \dot{x}^{(0)} &= Ax^{(0)} + f(x^{(0)}) \\ \dot{\widetilde{e}} &= (I_N \otimes A)\widetilde{e} + (T \otimes B)u + \widetilde{F}(x^{(0)},\widetilde{e}), \end{aligned} \tag{27}$$

where

$$\widetilde{F}(x^{(0)},\widetilde{e}) = (T \otimes I_n)F\left(x^{(0)},(T^{-1} \otimes I_n)\widetilde{e}\right) \tag{28}$$

Proof. Starting from (24) under the transformation (26), the transformed dynamics is given by

$$\dot{\widetilde{e}} = (T \otimes I_n)\left((I_N \otimes A)e + (I_N \otimes B)u + \hat{F}(x^{(0)},e)\right)\Big|_{e=(T^{-1}\otimes I_n)\widetilde{e}} = (I_N \otimes A)\widetilde{e} + (T \otimes B)u + \widetilde{F}(x^{(0)},\widetilde{e})$$

where $\widetilde{F}(x^{(0)},\widetilde{e})$ is given by (28). □

The particular structure of the change of coordinates considered guarantees that any assumption of incremental homogeneity on f in (1) reflects into an analog assumption on $\widetilde{F}$ in (27). In particular, on account of Remark 2 of the previous section, if $f : R^n \to R^n$ in (2) is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}+\mathfrak{d}, \mathfrak{h}, \Phi)$, where $\Phi = \langle\langle BK\rangle\rangle + \Phi_0$ with $\Phi_0(0,0) = 0$ and for some weights $\mathfrak{r} \in \mathbb{R}^n_>$ and degrees $\mathfrak{d}, \mathfrak{h} \in \mathbb{R}^n$, then F, defined by (25), is h.u.b. with quadruple $\left(\mathbf{1}_{N+1} \otimes \mathfrak{r}, \mathbf{1}_N \otimes (\mathfrak{r}+\mathfrak{d}), \mathbf{1}_{N+1} \otimes \mathfrak{h}, \widehat{\Phi}\right)$ and

$$\widehat{\Phi}(x^{(0)}, e) = \begin{pmatrix} \mathbf{0} & \Phi(e^{(1)} + x^{(0)}, x^{(0)}) & \cdots & \mathbf{0} \\ \vdots & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \Phi(e^{(N)} + x^{(0)}, x^{(0)}) \end{pmatrix}, \quad \widehat{\Phi}(0,0) = \begin{pmatrix} \mathbf{0}_{N\times 1} & I_N \end{pmatrix} \otimes \langle\langle BK\rangle\rangle,$$

where the $\mathbf{0}$'s are matrices with consistent dimensions and since $\Phi(0,0) = \langle\langle BK\rangle\rangle$. As it will be shown in the next lemma, under the change of coordinates (26), the h.u.b. function F defined by (25) transforms into $\widetilde{F}$ which retains the h.u.b. properties and with the same weights and degrees as F. Recall that $A \preceq B$, $A, B \in \mathbb{R}^{m\times l}$, means $A_{ij} \le B_{ij}$ for all $i = 1, \dots m$, $j = 1, \dots, l$, while $\max_{\theta\in Q} \Phi(\theta)$, $\Phi(\theta) \in \mathbb{R}^{m\times l}$ for each $\theta \in \mathbb{R}^n$ and compact $Q \subset \mathbb{R}^n$, represents any matrix M such that $\Phi(\theta) \preceq M$ for all $\theta \in Q$. If $Q(\gamma)$ is a family of compact sets $Q(\gamma) \subset \mathbb{R}^n$ for each $\gamma \in \mathbb{R}_>$ and such that $Q(\gamma) \to \{0\}$ as $\gamma \to 0$ then $\max_{\theta\in Q(\gamma)} \Phi(\theta)$ is taken in such a way that $\max_{\theta\in Q(\gamma)} \Phi(\theta) \to \Phi(0)$ as $\gamma \to 0$.

Lemma 2. Assume that $f : R^n \to R^n$ in (2), is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}+\mathfrak{d}, \mathfrak{h}, \Phi)$, where $\Phi = \langle\langle BK\rangle\rangle + \Phi_0$ with $\Phi_0(0,0) = 0$. Then under the change of coordinates (26) the function $\widetilde{F} : R^{(N+1)n} \to R^{Nn}$ defined by (28) is h.u.b. with quadruple $\left(\mathbf{1}_{N+1} \otimes \mathfrak{r}, \mathbf{1}_N \otimes (\mathfrak{r}+\mathfrak{d}), \mathbf{1}_{N+1} \otimes \mathfrak{h}, \widetilde{\Phi}\right)$, where $\widetilde{\Phi}$ is given by

$$\widetilde{\Phi}(x^{(0)}, \widetilde{e}) = \left(\langle\langle T\rangle\rangle \otimes I_n\right) \left(\max_{\|e\| \le \|(\langle\langle T^{-1}\rangle\rangle \otimes I_n)\widetilde{e}\,\|} \widehat{\Phi}(x^{(0)}, e) \right) \mathrm{diag}\left\{ I_n \,, \langle\langle T^{-1}\rangle\rangle \otimes I_n \right\} \tag{29}$$

Moreover $\widetilde{\Phi}(0,0) = \begin{pmatrix} \mathbf{0}_{N\times 1} & \langle\langle T\rangle\rangle\langle\langle T^{-1}\rangle\rangle \end{pmatrix} \otimes \langle\langle BK\rangle\rangle$.

Proof. First note that given a $N \times N$ matrix Q, then $\varphi(\xi) := (Q \otimes I_n)\xi$, $\xi \in R^{Nn}$, is h.u.b. with quadruple

$$\left(\mathbf{1}_N \otimes \mathfrak{r}, \mathbf{1}_N \otimes (\mathfrak{r}+\mathfrak{d}), -\mathbf{1}_N \otimes \mathfrak{d}, \langle\langle Q\rangle\rangle \otimes I_n\right), \tag{30}$$

where $\mathfrak{d}$ can be chosen in an arbitrary way. This fact follows directly from the definition of homogeneity in the upper bound, the structure of the matrix $Q \otimes I_n$ and consequently of $\langle\langle Q\rangle\rangle \otimes I_n$, and equation (21). In fact, for $q', q'' \in R^{Nn}$

$$(Q \otimes I_n)\left(\epsilon^{\mathbf{1}_N\otimes\mathfrak{r}} \diamond (q' - q'')\right) = \epsilon^{\mathbf{1}_N\otimes\mathfrak{r}} \diamond \left((Q \otimes I_n)(q' - q'')\right)$$

so that

$$\left\langle\left\langle \varphi(\epsilon^{\mathbf{1}_N\otimes\mathfrak{r}} \diamond q') - \varphi(\epsilon^{\mathbf{1}_N\otimes\mathfrak{r}} \diamond q'')\right\rangle\right\rangle = \epsilon^{\mathbf{1}_N\otimes(\mathfrak{r}+\mathfrak{d})} \diamond \left(\left(\langle\langle Q\rangle\rangle \otimes I_n\right)\left\langle\left\langle \epsilon^{\mathbf{1}_N\otimes(-\mathfrak{d})} \diamond (q' - q'')\right\rangle\right\rangle\right)$$

Accordingly $v \to \varphi_1(v) := (T \otimes I_n)v$ is h.u.b. with quadruple $\left(\mathbf{1}_N \otimes \mathfrak{r}, \mathbf{1}_N \otimes (\mathfrak{r}+\mathfrak{d}), -\mathbf{1}_N \otimes \mathfrak{d}, \langle\langle T\rangle\rangle \otimes I_n\right)$, obtained by setting $Q = T$ and $\mathfrak{d} = \mathfrak{f}$ in (30). Similarly one can conclude that $w \to \varphi_2(w) := (T^{-1} \otimes I_n)w$ is h.u.b. with quadruple $\left(\mathbf{1}_N \otimes \mathfrak{r}, \mathbf{1}_N \otimes (\mathfrak{r}-\mathfrak{h}), \mathbf{1}_N \otimes \mathfrak{h}, \langle\langle T^{-1}\rangle\rangle \otimes I_n\right)$, obtained by setting $Q = T^{-1}$ and $\mathfrak{d} = -\mathfrak{h}$ in (30), as well as with quadruple $\left(\mathbf{1}_N \otimes \mathfrak{r}, \mathbf{1}_N \otimes \mathfrak{r}, \mathbf{1}_N \otimes 0, \langle\langle T^{-1}\rangle\rangle \otimes I_n\right)$, obtained by setting again $Q = T^{-1}$ and $\mathfrak{d} = 0$ in (30). By extension, $(w_1, w_2) \to \varphi_3(w_1, w_2) := (w_1, \varphi_2(w_2))$, $w_1 \in \mathbb{R}^n$ and $w_2 \in \mathbb{R}^{Nn}$, is h.u.b. with quadruple $\left(\mathbf{1}_{N+1} \otimes \mathfrak{r}, \mathbf{1}_{N+1} \otimes (\mathfrak{r}-\mathfrak{h}), \mathbf{1}_N \otimes \mathfrak{h}, \mathrm{diag}\{I_n \,, \langle\langle T^{-1}\rangle\rangle \otimes I_n\}\right)$.

We have already proved that, under the assumptions of lemma 2, F, defined by (25), is h.u.b. with quadruple $\left(\mathbf{1}_{N+1} \otimes \mathfrak{r}, \mathbf{1}_N \otimes (\mathfrak{r}+\mathfrak{d}), \mathbf{1}_{N+1} \otimes \mathfrak{h}, \widehat{\Phi}\right)$. Using the composition rule in[1], we conclude that $F(w_1, \varphi_2(w_2)) = (F\circ\varphi_3)(w_1, w_2)$ is h.u.b. with quadruple

$$\left(\mathbf{1}_{N+1} \otimes \mathfrak{r}, \mathbf{1}_N \otimes (\mathfrak{r}+\mathfrak{d}), \mathbf{1}_{N+1} \otimes \mathfrak{h}, \left(\max_{\|e\| \le \|(\langle\langle T^{-1}\rangle\rangle \otimes I_n)\widetilde{e}\,\|} \widehat{\Phi}(x^{(0)}, e) \right) \mathrm{diag}\left\{ I_n \,, \langle\langle T^{-1}\rangle\rangle \otimes I_n \right\} \right).$$

Using once more the composition rule in[1], $(w_1, w_2) \to \varphi_1(F(w_1, \varphi_2(w_2))) = (\varphi_1\circ F\circ\varphi_3)(w_1, w_2)$ is i.h.u.b. with quadruple $\left(\mathbf{1}_{N+1} \otimes \mathfrak{r}, \mathbf{1}_N \otimes (\mathfrak{r}+\mathfrak{d}), \mathbf{1}_{N+1} \otimes \mathfrak{h}, \widetilde{\Phi}\right)$, where $\widetilde{\Phi}$ is given in (29). Consequently, by (29)

$$\begin{aligned} \widetilde{\Phi}(0,0) &= \left(\langle\langle T\rangle\rangle \otimes I_n\right) \widehat{\Phi}(0,0)\mathrm{diag}\left\{ I_n \,, \langle\langle T^{-1}\rangle\rangle \otimes I_n \right\} = \left(\langle\langle T\rangle\rangle \otimes I_n\right) \left(\begin{pmatrix} \mathbf{0}_{N\times 1} & I_N \end{pmatrix} \otimes \langle\langle BK\rangle\rangle\right) \mathrm{diag}\left\{ I_n \,, \langle\langle T^{-1}\rangle\rangle \otimes I_n \right\} \\ &= \left(\langle\langle T\rangle\rangle \otimes I_n\right) \begin{pmatrix} \mathbf{0}_{Nn\times n} & \langle\langle T^{-1}\rangle\rangle \otimes \langle\langle BK\rangle\rangle \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{N\times 1} & \langle\langle T\rangle\rangle\langle\langle T^{-1}\rangle\rangle \end{pmatrix} \otimes \langle\langle BK\rangle\rangle. \end{aligned} \tag{31}$$

□

4.3 | The full state information feedback law

Next, we design for each agent the control input $u^{(i)}$ which solves Problem I. We are looking for a linear feedback law of the form $u^{(i)} = \Pi\zeta^{(i)}$, $i \in [1, N]$, where $\Pi \in (\mathbb{R}^n)^*$ is a row vector and $\zeta^{(i)}$ is the consensus law for the i-th agent, so that:

$$u^{(i)} = \Pi\zeta^{(i)} = \Pi\Big[\sum_{j=1}^{N} \ell^i_j(x^{(0)} - x^{(j)}) + \ell^i_0(x^{(0)} - x^{(i)})\Big] = -\Pi\Big[\sum_{j=1}^{N} \ell^i_j e^{(j)} + \ell^i_0 e^{(i)}\Big].$$

Therefore, since $\hat{\mathcal{L}} = \mathcal{L} + \mathcal{L}_0$, with $\mathcal{L}_0 = \mathrm{diag}(l^1_0, \cdots, l^N_0)$ and $\mathcal{L}$ the Laplacian, the overall control input is

$$u = \left(u^{(1)} \cdots u^{(N)}\right)^T = -(I_N \otimes \Pi)(\hat{\mathcal{L}} \otimes I_n)e \tag{32}$$

In the coordinates (26)

$$u = \left(u^{(1)} \cdots u^{(N)}\right)^T = -(I_N \otimes \Pi)\left[(\hat{\mathcal{L}} \otimes I_n)(T^{-1} \otimes I_n)\tilde{e}\right] = -(\hat{\mathcal{L}}T^{-1} \otimes \Pi)\tilde{e}.$$

Since $\hat{\mathcal{L}}$ is symmetric and definite positive ($\tilde{\mathcal{G}}$ is undirected and connected,[20]), it is diagonalizable by a transformation T i.e. $T\hat{\mathcal{L}}T^{-1} = \overline{\mathcal{L}} = diag(\bar{\lambda}_1, \cdots, \bar{\lambda}_N)$ and has positive eigenvalues $\bar{\lambda}_1, \cdots, \bar{\lambda}_N$. For the closed-loop system resulting from (27)

$$(T \otimes B)u = (T \otimes B)\left((\hat{\mathcal{L}}T^{-1}) \otimes \Pi\right)\tilde{e} = (T\hat{\mathcal{L}}T^{-1}) \otimes (B\Pi)\tilde{e} = \overline{\mathcal{L}} \otimes (B\Pi)\tilde{e}$$

and with $\tilde{e} = (T \otimes I_n)e$

$$\begin{aligned} \dot{x}^{(0)} &= Ax^{(0)} + f(x^{(0)}) \\ \dot{\tilde{e}} &= \left((I_N \otimes A) - \overline{\mathcal{L}} \otimes (B\Pi)\right)\tilde{e} + \tilde{F}(x^{(0)}, \tilde{e}) \end{aligned} \tag{33}$$

In[20], relying upon the fact that each agent is linear, it is shown that the change of coordinates $\tilde{e} = (T \otimes I_n)e$ is such that (33) consists of $N+1$ independent dynamics, including the leader dynamics. In a nonlinear context this is not true any more as shown in the next lemma. However even if we do not have perfectly decoupled nonlinear dynamics, we still preserve the incremental homogeneity property, which is sufficient for achieving consensus.

In order to get ready for the proof of the main result of this Section, we want to rewrite (33) in some more tractable form. To this aim, let us permute the coordinates $(x^{(0)}, \tilde{e}) \mapsto (x^{(0)}, \bar{e}) = P(x^{(0)}, \tilde{e})$ as follows (recall that $\tilde{e} = (\tilde{e}^{(1)^T}, \cdots, \tilde{e}^{(N)^T})^T$)

$$\bar{e}_j = (\tilde{e}^{(1)}_j, \cdots, \tilde{e}^{(N)}_j)^T, \ \bar{e} = (\bar{e}_1{}^T, \cdots, \bar{e}_n{}^T)^T$$

i.e. collect together all the j-th coordinates of each agent, $j \in [1, N]$. Accordingly, if $\tilde{F} = (\tilde{F}^{(1)^T}, \cdots, \tilde{F}^{(N)^T})^T$ set

$$\begin{aligned} &\overline{F}_j = (\tilde{F}^{(1)}_j \circ P^{-1}, \cdots, \tilde{F}^{(N)}_j \circ P^{-1})^T, \ \overline{F} = (\overline{F}_1{}^T, \cdots, \overline{F}_n{}^T)^T \\ &\overline{A} = A \otimes I_N, \ \overline{B} = B \otimes \overline{\mathcal{L}}, \ \overline{\Pi} = \Pi \otimes I_N. \end{aligned} \tag{34}$$

With these positions,

$$\begin{aligned} \dot{x}_0 &= Ax^{(0)} + f(x^{(0)}) \\ \dot{\bar{e}} &= (\overline{A} - \overline{B\Pi})\bar{e} + \overline{F}(x^{(0)}, \bar{e}). \end{aligned} \tag{35}$$

Clearly, if f is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r} + \mathfrak{d}, \mathfrak{h}, \Phi)$ and with the help of Lemma 2, then $\overline{F}$ is h.u.b. with quadruple $\left(\mathfrak{r} \otimes \mathbf{1}_{N+1}, (\mathfrak{r} + \mathfrak{d}) \otimes \mathbf{1}_N, \mathfrak{h} \otimes \mathbf{1}_{N+1}, \overline{\Phi}\right)$, where $\overline{\Phi}$ is defined starting from $\tilde{\Phi} = (\tilde{\Phi}^T_1, \cdots, \tilde{\Phi}^T_N)^T$ in the following way:

$$\overline{\Phi}_j = ((\tilde{\Phi}^{(1)}_j \circ P^{-1})^T, \cdots, (\tilde{\Phi}^{(N)}_j \circ P^{-1})^T)^T \circ P^{-1}, \ \overline{\Phi} = (\overline{\Phi}_1{}^T, \cdots, \overline{\Phi}_n{}^T)^T \tag{36}$$

with

$$\overline{\Phi}(0, 0) = \left(\mathbf{0}_{Nn \times n} \quad B \otimes \langle\langle K \rangle\rangle \otimes (\langle\langle T \rangle\rangle \langle\langle T^{-1} \rangle\rangle)\right). \tag{37}$$

Lemma 3. Assume that $f : R^n \to R^n$ in (2), is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r} + \mathfrak{d}, \mathfrak{h}, \Phi)$, where $\Phi = \langle\langle BK \rangle\rangle + \Phi_0$ with $\Phi_0(0, 0) = 0$. Then under the permutation (34) the function $\overline{F} : R^{(N+1)n} \to R^{Nn}$ defined by (34) is h.u.b. with quadruple $\left(\mathfrak{r} \otimes \mathbf{1}_{N+1}, (\mathfrak{r} + \mathfrak{d}) \otimes \mathbf{1}_N, \mathfrak{h} \otimes \mathbf{1}_{N+1}, \overline{\Phi}\right)$, where $\overline{\Phi}$ is given in (36) with $\overline{\Phi}(0, 0)$ satisfying (37).

4.4 | Achieving consensus with full state information

In this Section we will show how to choose the matrix $\overline{\Pi} := \Pi \otimes I_N$ in (35) (and therefore Π in (32)). To this aim, we assume that the leader's state trajectories are contained in some known compact set Ω and moreover the following basic assumptions on the nonlinearities affecting the dynamics of each agent in terms of incremental homogeneity:

($\mathcal{A}0'$) *The leader's state trajectories are contained in some known compact set* Ω.

($\mathcal{A}2$) *The nonlinear function* $f \in \mathbf{C}^0(R^n, R^n)$ *in (4) is incrementally homogeneous in the upper bound with quadruple* $\left(\mathfrak{r}, \mathfrak{r}+\mathfrak{f}, \hat{\mathfrak{f}}, \Phi\right)$, *with* $\Phi = \langle\langle BK\rangle\rangle + \Phi_0$, $\Phi_0(0,0) = 0$, *and for* $j \in [1, n-1]$,

$$\hat{\mathfrak{f}}_1 := \mathfrak{f}_1, \qquad \hat{\mathfrak{f}}_{j+1} := \mathfrak{r}_{j+1} - \mathfrak{r}_j - \mathfrak{f}_j,\ j = 2, \dots, n, \tag{38}$$

and

$$\mathfrak{f}_j \le \hat{\mathfrak{f}}_{j+1} \le \mathfrak{f}_{j+1},\ j = 1, \dots, n-1. \tag{39}$$

Remark 3. A consequence of (38) is that the sequence $\{\mathfrak{f}_j\}_{j\in[1,N]}$ *is non-decreasing. For instance, the dynamics (8) satisfy assumption* ($\mathcal{A}2$) *with* $\mathfrak{r}_1 = 1/8, \mathfrak{r}_2 = 3/8, \mathfrak{f}_1 = 1/8, \mathfrak{f}_2 = 1/8$.

The key Theorem to show achievement of consensus with the leader is the following.

Theorem 1. *Assume that* ($\mathcal{A}0'$) *and* ($\mathcal{A}2$) *hold true. There exist* $\epsilon^* > 1$ *and a positive definite diagonal* Γ *such that for all* $\epsilon \ge \epsilon^*$ *the feedback law* $u = (u^{(1)}, \dots, u^{(N)})^T$, $u^{(i)} = -\Pi\zeta^{(i)}$, $i \in [1, N]$, *with (13) and*

$$\Pi = B^T H(I_n - A^T H)^{-1},\ H = \Gamma \mathrm{diag}\{\epsilon^{2\mathfrak{f}_1}, \cdots \epsilon^{2\mathfrak{f}_n}\} \tag{40}$$

solves the Semiglobal Leader Following problem with full information for (1), (2).

Proof. Consider the error system (35) with leader dynamics. Next, consider a further change of coordinates

$$\begin{pmatrix} x^{(0)} \\ \overline{e} \end{pmatrix} \to \begin{pmatrix} x^{(0)} \\ \hat{e} \end{pmatrix},\ \hat{e} = \overline{Z}^{-1}\overline{e} \tag{41}$$

where

$$\overline{Z} = I_{Nn} - \overline{A}^T\overline{H},\ \overline{H} = H \otimes I_N. \tag{42}$$

Notice also that

$$I_{Nn} - \overline{A}^T\overline{H} = I_n \otimes I_N - (A\otimes I_N)^T(H\otimes I_N) = I_n \otimes I_N - (A^T\otimes I_N)(H\otimes I_N) = (I_n - A^T H)\otimes I_N.$$

Due to the particular choice of the change of coordinates it is easily verified that

$$\dot{\hat{e}} = (\hat{A} - \hat{B}\hat{\Pi})\hat{e} + \overline{Z}^{-1}\overline{F}(x^{(0)}, \overline{Z}\hat{e}) \tag{43}$$

where

$$\hat{A} = \overline{Z}^{-1}\overline{AZ},\ \hat{B} = \overline{Z}^{-1}\overline{B} = \overline{B},\ \hat{\Pi} = \overline{\Pi Z} = \overline{\mathcal{L}}^{-1}\overline{B}^T\overline{H}.$$

By noting that $(\overline{A}^T\overline{H})\overline{A}(\overline{A}^T\overline{H}) = \overline{A}^T\overline{H}^2$ then

$$\begin{aligned}\overline{Z}^{-1}\overline{AZ} &= \overline{Z}^{-1}(\overline{A} - \overline{AA}^T\overline{H}) = \overline{Z}^{-1}[\overline{A} - \overline{AA}^T\overline{H} + (\overline{A}^T\overline{H})\overline{A}(\overline{A}^T\overline{H}) - \overline{A}^T\overline{H}^2] \\ &= \overline{Z}^{-1}[-\overline{ZA}(\overline{A}^T\overline{H}) + \overline{A} - \overline{A}^T\overline{H}^2] = -\overline{AA}^T\overline{H} + \overline{Z}^{-1}(\overline{A} - \overline{A}^T\overline{H}^2)\end{aligned}$$

so that

$$\hat{A} - \hat{B}\hat{\Pi} = -\overline{AA}^T\overline{H} + \overline{Z}^{-1}(\overline{A} - \overline{A}^T\overline{H}^2) - \hat{B}\hat{\Pi} = -\hat{H} + \overline{Z}^{-1}(\overline{A} - \overline{A}^T\overline{H}^2)$$

where $\hat{H} := \mathrm{diag}\{I_N, \dots, I_N, \overline{\mathcal{L}}\}\overline{H}$. As a consequence,

$$\dot{\hat{e}} = -\hat{H}\hat{e} + \hat{\rho}(x^{(0)}, \hat{e})$$

where

$$\hat{\rho}(x^{(0)}, \hat{e}) = \overline{Z}^{-1}\left[(\overline{A} - \overline{A}^T\overline{H}^2)\hat{e} + \overline{F}(x^{(0)}, \overline{Z}\hat{e})\right].$$

By using (A2) and the conclusions in Appendix 9 we show that $\hat{\rho}$ is h.u.b. with quadruple $\left(\mathfrak{r}\otimes\mathbf{1}_{N+1},(\mathfrak{r}+\mathfrak{f})\otimes\mathbf{1}_N,\mathfrak{f}\otimes\mathbf{1}_{N+1},\hat{R}\right)$, where

$$\hat{R}(x^{(0)},\hat{e}) := (I_{Nn}-\overline{A}^T\overline{\Gamma})^{-1}\left[\begin{pmatrix}\mathbf{0}_{Nn\times n} & \overline{A}+\overline{A}^T\overline{\Gamma}^2\end{pmatrix}+\max_{\|e'\|\le\|(I_{Nn}+\overline{A}^T\overline{\Gamma})\hat{e}\|}\overline{\Phi}(x^{(0)},e')\right]\begin{pmatrix}I_n & \mathbf{0}_{n\times Nn}\\ \mathbf{0}_{Nn\times n} & I_{Nn}+\overline{A}^T\overline{\Gamma}\end{pmatrix} \tag{44}$$

with $\overline{\Gamma}=\Gamma\otimes I_N$ and $\overline{\Phi}$ defined in (36). Notice also that $\hat{R}$ can be also decomposed as

$$\hat{R}(x^{(0)},\hat{e}) := \begin{pmatrix}\mathbf{0}_{Nn\times n} & \hat{Q}(x^{(0)},\hat{e})\end{pmatrix},$$

for some $\hat{Q}$ such that

$$\begin{aligned}\hat{Q}(0,0) &= (I_{Nn}-\overline{A}^T\overline{\Gamma})^{-1}\left[\overline{A}+\overline{A}^T\overline{\Gamma}^2+\left(B\otimes K\otimes(\langle\langle T\rangle\rangle\langle\langle T^{-1}\rangle\rangle)\right)(I_{Nn}+\overline{A}^T\overline{\Gamma})\right]\\ &= (I_{Nn}-\overline{A}^T\overline{\Gamma})^{-1}\left[\overline{A}+\overline{A}^T\overline{\Gamma}^2+B\otimes\left(\left(K\otimes(\langle\langle T\rangle\rangle\langle\langle T^{-1}\rangle\rangle)\right)(I_{Nn}+\overline{A}^T\overline{\Gamma})\right)\right].\end{aligned}$$

Moreover, it is easy to see that $\hat{e}\to\hat{H}\hat{e}$ is i.h. with quadruple $\left(\mathfrak{r}\otimes\mathbf{1}_N,(\mathfrak{r}+\mathfrak{f})\otimes\mathbf{1}_N,\mathfrak{f}\otimes\mathbf{1}_N,\hat{\Gamma}\right)$ with $\hat{\Gamma} := \mathrm{diag}\{I_N,\dots,I_N,\overline{\mathcal{L}}\}\overline{\Gamma}$ and $\overline{\Gamma}=\Gamma\otimes I_N$. With $V(\hat{e})=\|\epsilon^{-\mathfrak{r}\otimes\mathbf{1}_N}\diamond\hat{e}\|^2$ it follows by straightforward computations that

$$\begin{aligned}\dot{V}(\hat{e}) &= \frac{\partial V}{\partial\hat{e}}(\hat{e})\left\{-\hat{H}\hat{e}+\hat{\rho}(x^{(0)},\hat{e})\right\}\\ &\le -\langle\langle\epsilon^{(\mathfrak{f}-\mathfrak{r})\otimes\mathbf{1}_N}\diamond\hat{e}\rangle\rangle^T\left[2\hat{\Gamma}-\hat{Q}\left(\epsilon^{-\mathfrak{r}\otimes\mathbf{1}_{N+1}}\diamond(x^{(0)},\hat{e})\right)-\hat{Q}^T\left(\epsilon^{-\mathfrak{r}\otimes\mathbf{1}_{N+1}}\diamond(x^{(0)},\hat{e})\right)\right]\langle\langle\epsilon^{(\mathfrak{f}-\mathfrak{r})\otimes\mathbf{1}_N}\diamond\hat{e}\rangle\rangle.\end{aligned}$$

After some lengthy manipulations, a positive definite diagonal $\hat{\Gamma}$ can be found such that

$$2\hat{\Gamma}-\hat{Q}(0,0)-\hat{Q}^T(0,0)\ge -2I.$$

Next, by compactness of Ω and the set $\{\hat{e}\in\mathbb{R}^{Nn}:V(\hat{e})\le\hat{c}^2\}$, it is possible to select $\hat{c}>0$ and $\epsilon_1>1$ such that

$$\hat{Q}(\epsilon^{-\mathfrak{r}\otimes\mathbf{1}_{N+1}}\diamond(x^{(0)},\hat{e}))-\hat{Q}(0,0)+\left(\hat{Q}(\epsilon^{-\mathfrak{r}\otimes\mathbf{1}_{N+1}}\diamond(x^{(0)},\hat{e}))-\hat{Q}(0,0)\right)^T\le I$$

for all $x^{(0)}\in\Omega$, $\hat{e}\in\mathbb{R}^{Nn}:V(\hat{e})\le\hat{c}^2$ and $\epsilon\ge\epsilon_1$. Therefore,

$$\dot{V}(\hat{e})\le-\|\epsilon^{(\mathfrak{f}-\mathfrak{r})\otimes\mathbf{1}_N}\diamond\hat{e}\|^2 \tag{45}$$

for all $x^{(0)}\in\Omega$, $\hat{e}\in\mathbb{R}^{Nn}:V(\hat{e})\le\hat{c}^2$ and $\epsilon\ge\epsilon_1$, i.e. the set $\{(x^{(0)},\hat{e})\in\Omega\times\mathbb{R}^{Nn}:V(\hat{e})\le\hat{c}^2\}$ is invariant for (43) together with the leader's dynamics (1). Since the leader has bounded trajectories in Ω (i.e. $x^{(0)}(t)\in\Omega$ for all $t\ge0$) by ($\mathcal{A}0'$), this proves that $\hat{e}(t)\to0$ as $t\to+\infty$ with initial consensus error $\hat{e}(0)\in\{\hat{e}\in\mathbb{R}^{Nn}:V(\hat{e})\le\hat{c}^2\}$ or equivalently that $e(t)\to0$ as $t\to+\infty$ with initial consensus error $e(0)\in\{e\in\mathbb{R}^{Nn}:V(\overline{Z}^{-1}P(T\otimes I_n)e)\le\hat{c}^2\}$. We will select $\epsilon\ge\epsilon_1$ in such a way that the set $\{e\in\mathbb{R}^{Nn}:V(\overline{Z}^{-1}P(T\otimes I_n)e)\le\hat{c}^2\}$ includes the prescribed compact set $\Omega^{\times N}$. Notice that $\hat{e}=\overline{Z}^{-1}P(T\otimes I_n)e$, where P is a permutation matrix. Therefore, $e(t)\to0$ as $t\to+\infty$ with initial consensus error $e(0)\in\{e\in\mathbb{R}^{Nn}:V(\overline{Z}^{-1}P(T\otimes I_n)e)\le\hat{c}^2\}$. Since $\mathfrak{f}_1\le\mathfrak{f}_2\le\dots\le\mathfrak{f}_n$ (remark 3), $\overline{e}\mapsto\overline{Z}^{-1}\overline{e}$ is i.h.u.b. with quadruple $\left(\mathfrak{r}\otimes\mathbf{1}_N,\mathfrak{r}\otimes\mathbf{1}_N,\mathbf{0}_n\otimes\mathbf{1}_N,(I_{Nn}-\overline{A}^T\overline{\Gamma})^{-1}\right)$. Therefore, there exists $\epsilon_2\ge\epsilon_1$ such that for all $\epsilon\ge\epsilon_2$

$$\begin{aligned}&\max_{e\in\Omega^{\times N}}V(\overline{Z}^{-1}P(T\otimes I_n)e)\le\max_{e\in\Omega^{\times N}}\left\|\epsilon^{-\mathfrak{r}\otimes\mathbf{1}_N}\diamond\left((I_{Nn}-\overline{A}^T\overline{H})^{-1}P(T\otimes I_n)e\right)\right\|^2\\ &\le\max_{e\in\Omega^{\times N}}\left\|(I_{Nn}-\overline{A}^T\overline{\Gamma})^{-1}\left\langle\left\langle\epsilon^{-\mathfrak{r}\otimes\mathbf{1}_N}\diamond P(T\otimes I_n)e\right\rangle\right\rangle\right\|^2\le\hat{c}^2.\end{aligned}$$

This concludes the proof with $\epsilon=\max\{\epsilon_1,\epsilon_2\}$. □

Remark 4. It is worth noting that the definition of H and, therefore, of Π in (40) is made according to a precise power rescaling of the parameter ϵ which depends on the homogeneity degrees of f.

5 | CONSENSUS ERRORS AND LEADER'S STATE ESTIMATION WITH PARTIAL INFORMATION

In the present Section we assume that for each agent the consensus error information of the neighbors are not available to the agent itself. More precisely, we assume that each agent exchanges with its neighbors the residual between the output consensus error and its dynamical estimate (implemented by some observer). We want to design observers for reconstructing the missing information on the consensus error. This result is of interest itself for distributed state estimation. For purpose of illustration, first we assume that the full leader's state is available to each agent. At the end of the Section, we show how to remove this restriction.

5.1 | Followers' outputs and leader's state information

Here, for each agent i we consider the consensus law $\chi^{(i)}$, defined in (15), and re-written in terms of the output consensus errors

$$\chi^{(i)} = \sum_{j=1}^{N} \ell^i_j (Ce^{(j)} - C\xi^{(j)}) + \ell^i_0 (Ce^{(i)} - C\xi^{(i)}),\ i \in [1, N], \tag{46}$$

where $\xi^{(j)}$, $j \in \{1, \dots, N\}$, are suitable estimates of $e^{(j)}$, $j \in \{1, \dots, N\}$. In other words, each agent j exchanges the residual $y^{(j)} - y^{(0)} - C\xi^{(j)} = Ce^{(j)} - C\xi^{(j)}$ (i.e. the difference between output consensus error and its estimate) with its neighbors.

Let us now consider the given system in the coordinates $(x^{(0)^T}, e^T)^T$, described by equations (23), (24) and restrict ourselves to trajectories $(x^{(0)}(t), e(t))$ such that $(x^{(0)}(t), e(t)) \in \Omega \times \mathcal{E}$ for all $t \geq 0$, $\mathcal{E} \subset \mathbb{R}^{Nn}$ a compact set containing $\Omega^{\times N}$. We want to design an observer for all such consensus error trajectories. To this aim, we propose for each agent i the observer

$$\dot{\xi}^{(i)} = A\xi^{(i)} + Bu^{(i)} + \Delta f \left(\mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(\xi^{(i)}) + \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(x^{(0)}), \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(x^{(0)}) \right) + \Pi_O \chi^{(i)},\ i \in [1, N].$$

for some design parameters $c > 0$, $\epsilon > 0$ and $\Pi_O \in \mathbb{R}^n$ (the **sat** function is defined in (20)). All these observers' equations together are recast into

$$\dot{\xi} = (I_N \otimes A)\xi + (I_N \otimes B)u + F\left(\mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(x^{(0)}), \mathbf{sat}_{\mathbf{1}_N \otimes c\epsilon^{\mathfrak{r}}}(\xi) \right) + (I_N \otimes \Pi_O)\chi \tag{47}$$

with $\xi = (\xi^{(1)^T}, \dots, \xi^{(N)^T})^T$ and $\chi = (\chi^{(1)}, \dots, \chi^{(N)})^T$. Notice the form of the saturated estimates $\mathbf{sat}_{\mathbf{1}_N \otimes c\epsilon^{\mathfrak{r}}}(\xi)$ where the saturation levels are taken to be power rescalings of the parameter ϵ depending on the homogeneity weights $\mathfrak{r}$ of f.

Assume that $c > 0$ and $\epsilon > 0$ have been selected so that for all $t \geq 0$

$$\begin{aligned} \mathbf{sat}_{\mathbf{1}_N \otimes c\epsilon^{\mathfrak{r}}}(e(t)) &= e(t), \\ \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(x^{(0)}(t)) &= x^{(0)}(t) \end{aligned} \tag{48}$$

(this is possible since $(x^{(0)}(t), e(t)) \in \Omega \times \mathcal{E}$ for all $t \geq 0$). On account of (48) and noticing that for all $v := (v^{(1)^T}, \cdots, v^{(N)^T})^T \in \mathbb{R}^{Nn}$

$$\mathbf{sat}_{\mathbf{1}_N \otimes c\epsilon^{\mathfrak{r}}}(v) = (\mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(v^{(1)}), \dots, \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(v^{(N)}))^T,$$

the system (23), (24), (47), (46), with $\eta = e - \xi$ and $X = \begin{pmatrix} x^{(0)^T} & e^T \end{pmatrix}^T$, reads as

$$\dot{X} = \begin{pmatrix} Ax^{(0)} + f(x^{(0)}) \\ (I_N \otimes A)e + (I_N \otimes B)u + F(X) \end{pmatrix} \tag{49}$$

$$\dot{\eta} = (I_N \otimes A)\eta - (I_N \otimes \Pi_O)\chi - \widehat{G}(X, \eta) \tag{50}$$

where

$$\widehat{G}(X, \eta) = \begin{pmatrix} \Delta f \left(\mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(e^{(1)} - \eta^{(1)}) + \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(x^{(0)}), \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(e^{(1)}) + \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(x^{(0)}) \right) \\ \vdots \\ \Delta f \left(\mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(e^{(N)} - \eta^{(N)}) + \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(x^{(0)}), \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(e^{(N)}) + \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(x^{(0)}) \right) \end{pmatrix}. \tag{51}$$

System (49), (50) is formally equivalent to (23), (24), under the following equivalences

$$X \leftrightarrow x^{(0)},\quad \eta \leftrightarrow e,\quad -(I_N \otimes \Pi_O)\chi \leftrightarrow (I_N \otimes B)u,\quad -\widehat{G} \leftrightarrow F.$$

With this in mind, we closely follow Sections 4.2, 4.3 and 4.4.

5.1.1 | The action of a class of linear change of coordinates

Given an invertible $N \times N$ matrix T, consider the class of change of coordinates $(X^T, \eta^T)^T \to (X^T, \widetilde{\eta}^T)$ with

$$\widetilde{\eta} = (T \otimes I_n)\eta. \tag{52}$$

The next lemma is proved as Lemma 1.

Lemma 4. Let T be an invertible $N \times N$ matrix, and consider the transformation (52). In the new coordinates system (49), (50) reads

$$\begin{aligned} \dot{X} &= \begin{pmatrix} Ax^{(0)} + f(x^{(0)}) \\ (I_N \otimes A)e + (I_N \otimes B)u + F(X) \end{pmatrix} \\ \dot{\widetilde{\eta}} &= (I_N \otimes A)\widetilde{\eta} - (T \otimes \Pi_O)\chi - \widetilde{G}(X, \widetilde{\eta}\,) \end{aligned} \tag{53}$$

where

$$\widetilde{G}(X, \widetilde{\eta}\,) = (T \otimes I_n)\widehat{G}\left(X, (T^{-1} \otimes I_n)\widetilde{\eta}\right). \tag{54}$$

The particular structure of the change of coordinates considered guarantees also in the framework that the assumption of i.h.u.b. on f in (1) reflects into an analog assumption on $\widetilde{G}$ in (54). For instance, if $f : R^n \to R^n$ in (2) is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r} + \mathfrak{d}, \mathfrak{h}, \Phi)$, where $\Phi = \langle\langle BK \rangle\rangle + \Phi_0$ with $\Phi_0(0, 0) = 0$, then $\widehat{G}$, defined by (51), is h.u.b. with quadruple $\left(\mathbf{1}_{2N+1} \otimes \mathfrak{r}, \mathbf{1}_N \otimes (\mathfrak{r} + \mathfrak{d}), \mathbf{1}_{2N+1} \otimes \mathfrak{h}, \widehat{\Phi}\right)$, where

$$\widehat{\Phi} = \begin{pmatrix} \mathbf{0}_{N\times(N+1)} & I_N \end{pmatrix} \otimes \left(\langle\langle BK \rangle\rangle + \max_{-2c\mathbf{1}_n \le p', p'' \le 2c\mathbf{1}_n} \Phi_0(p', p'') \right).$$

As a matter of fact, for all $v, w \in \mathbb{R}^n$

$$\begin{aligned} &-2c\mathbf{1}_n \le \epsilon^{-\mathfrak{r}} \diamond \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(v) + \epsilon^{-\mathfrak{r}} \diamond \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(w) \le 2c\mathbf{1}_n, \\ &\langle\langle \epsilon^{-\mathfrak{r}} \diamond \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(\epsilon^{\mathfrak{r}} \diamond v) - \epsilon^{-\mathfrak{r}} \diamond \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(\epsilon^{\mathfrak{r}} \diamond w) \rangle\rangle \le \langle\langle v - w \rangle\rangle. \end{aligned}$$

and as $c \to 0^+$

$$\widehat{\Phi} \to \begin{pmatrix} \mathbf{0}_{N\times(N+1)} & I_N \end{pmatrix} \otimes \langle\langle BK \rangle\rangle. \tag{55}$$

As it will be shown in the next lemma (which gives the analogue of lemma 2), under the change of coordinates (52), $\widehat{G}$ defined by (51), which is thus i.h.u.b., transforms into $\widetilde{G}$ which is still i.h.u.b. and with the same weights and degrees of $\widehat{G}$.

Lemma 5. Assume that $f : R^n \to R^n$ in (2), is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r} + \mathfrak{d}, \mathfrak{h}, \Phi)$, where $\Phi = \langle\langle BK \rangle\rangle + \Phi_0$ with $\Phi_0(0, 0) = 0$. Then under the change of coordinates (52) the function $\widetilde{G} : R^{(2N+1)n} \to R^{Nn}$ defined by (54) is h.u.b. with quadruple $\left(\mathbf{1}_{2N+1} \otimes \mathfrak{r}, \mathbf{1}_N \otimes (\mathfrak{r} + \mathfrak{d}), \mathbf{1}_{2N+1} \otimes \mathfrak{h}, \tilde{\Phi}\right)$, where $\widetilde{\Phi}$ is given by

$$\widetilde{\Phi} = (\langle\langle T \rangle\rangle \otimes I_n)\, \widehat{\Phi}\, \mathrm{diag}\left\{ I_{n(N+1)}\,, \langle\langle T^{-1} \rangle\rangle \otimes I_n \right\}. \tag{56}$$

Moreover as $c \to 0^+$

$$\widetilde{\Phi} \to \begin{pmatrix} \mathbf{0}_{N\times(N+1)} & \langle\langle T \rangle\rangle\langle\langle T^{-1} \rangle\rangle \end{pmatrix} \otimes \langle\langle BK \rangle\rangle. \tag{57}$$

Proof. The proof follows Lemma 2) for (56). For what concerns (57), indeed by (55) as $c \to 0^+$

$$\begin{aligned} \widetilde{\Phi} &\to \left(\langle\langle T \rangle\rangle \otimes I_n\right) \left(\begin{pmatrix} \mathbf{0}_{N\times(N+1)} & I_N \end{pmatrix} \otimes \langle\langle BK \rangle\rangle \right) \mathrm{diag}\left\{ I_{n(N+1)}\,, \langle\langle T^{-1} \rangle\rangle \otimes I_n \right\} \\ &= \left(\langle\langle T \rangle\rangle \otimes I_n\right) \begin{pmatrix} \mathbf{0}_{Nn\times n(N+1)} & I_N \otimes \langle\langle BK \rangle\rangle \end{pmatrix} \mathrm{diag}\left\{ I_{n(N+1)}\,, \langle\langle T^{-1} \rangle\rangle \otimes I_n \right\} \\ &= \left(\langle\langle T \rangle\rangle \otimes I_n\right) \begin{pmatrix} \mathbf{0}_{Nn\times n(N+1)} & \langle\langle T^{-1} \rangle\rangle \otimes \langle\langle BK \rangle\rangle \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{Nn\times n(N+1)} & \langle\langle T \rangle\rangle\langle\langle T^{-1} \rangle\rangle \otimes \langle\langle BK \rangle\rangle \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{0}_{N\times(N+1)} & \langle\langle T \rangle\rangle\langle\langle T^{-1} \rangle\rangle \end{pmatrix} \otimes \langle\langle BK \rangle\rangle. \end{aligned}$$

□

Next, we focus on the consensus term $-(T \otimes \Pi_O)\chi$ in (53). In the coordinates (52)

$$\chi = (I_N \otimes C)(\widehat{\mathcal{L}} \otimes I_n)(T^{-1} \otimes I_n)\widetilde{\eta} = (\widehat{\mathcal{L}}T^{-1} \otimes C)\widetilde{\eta}.$$

Since

$$(T \otimes \Pi_O)\left((\widehat{\mathcal{L}}T^{-1}) \otimes C \right) = (T\widehat{\mathcal{L}}T^{-1}) \otimes (\Pi_O C) = \overline{\mathcal{L}} \otimes (\Pi_O C)$$

the error system (53) becomes

$$\begin{aligned}\dot{X} &= \begin{pmatrix} Ax^{(0)} + f(x^{(0)}) \\ (I_N \otimes A)e + (I_N \otimes B)u + F(X) \end{pmatrix} \\ \dot{\widetilde{\eta}} &= \Big(I_N \otimes A - \overline{\mathcal{L}} \otimes (\Pi_O C)\Big)\widetilde{\eta} - \widetilde{G}(X, \widetilde{\eta}\,)\end{aligned} \tag{58}$$

Let us now permute the coordinates $(X, \widetilde{\eta}\,) \mapsto (X, \overline{\eta}) = P(X, \widetilde{\eta}\,)$ as follows (recall that $\overline{\eta} = (\widetilde{\eta}^{(1)^T}, \cdots, \widetilde{\eta}^{(N)^T})^T$)

$$\overline{\eta}_j = (\widetilde{\eta}_j^{(1)}, \cdots, \widetilde{\eta}_j^{(N)})^T, \; \eta = (\overline{\eta}_1{}^T, \cdots, \overline{\eta}_n{}^T)^T. \tag{59}$$

Accordingly set $\overline{\Lambda} = \mathrm{diag}\{\bar{\lambda}_1, \dots, \bar{\lambda}_N\}$ and

$$\begin{aligned}\overline{G}_j &= (\widetilde{G}_j^{(1)}, \cdots, \widetilde{G}_j^{(N)})^T, \; \overline{G} = (\overline{G}_1{}^T, \cdots, \overline{G}_n{}^T)^T \\ \overline{A} &= A \otimes I_N, \; \overline{C} = C \otimes \overline{\Lambda}, \; \overline{\Pi}_O = \Pi_O \otimes I_N.\end{aligned}$$

With these positions,

$$\begin{aligned}\dot{X} &= \begin{pmatrix} Ax^{(0)} + f(x^{(0)}) \\ (I_N \otimes A)e + (I_N \otimes B)u + F(X) \end{pmatrix} \\ \dot{\overline{\eta}} &= (\overline{A} - \overline{\Pi}_O\overline{C})\overline{\eta} + \overline{G}(X, \overline{\eta}).\end{aligned} \tag{60}$$

Clearly, if f is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}+\mathfrak{d}, \mathfrak{h}, \Phi)$ then $\overline{G}$ is h.u.b. with quadruple $\Big(\mathfrak{r} \otimes \mathbf{1}_{2N+1}, (\mathfrak{r}+\mathfrak{d}) \otimes \mathbf{1}_N, \mathfrak{h} \otimes \mathbf{1}_{2N+1}, \overline{\Phi}\Big)$, where $\overline{\Phi}$ is defined starting from $\widetilde{\Phi} = ((\widetilde{\Phi}^{(1)})^T, \cdots, (\widetilde{\Phi}^{(N)})^T)^T$ in the following way:

$$\overline{\Phi}_j = ((\widetilde{\Phi}_j^{(1)})^T, \cdots, (\widetilde{\Phi}_j^{(N)})^T)^T \circ P^{-1}, \; \overline{\Phi} = (\overline{\Phi}_1^T, \cdots, \overline{\Phi}_n^T)^T. \tag{61}$$

Moreover as $c \to 0^+$

$$\overline{\Phi} \to \left(\mathbf{0}_{Nn\times(N+1)n} \quad B \otimes \langle\langle K \rangle\rangle \otimes (\langle\langle T \rangle\rangle \langle\langle T^{-1} \rangle\rangle)\right). \tag{62}$$

Lemma 6. Assume that $f : R^n \to R^n$ in (2), is i.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{r}+\mathfrak{d}, \mathfrak{h}, \Phi)$, where $\Phi = \langle\langle BK \rangle\rangle + \Phi_0$ with $\Phi_0(0,0) = 0$. Then under the permutation (59) the function $\overline{G} : R^{(2N+1)n} \to R^{Nn}$ defined by (61) is h.u.b. with quadruple $\Big(\mathfrak{r} \otimes \mathbf{1}_{2N+1}, (\mathfrak{r}+\mathfrak{d}) \otimes \mathbf{1}_N, \mathfrak{h} \otimes \mathbf{1}_{2N+1}, \overline{\Phi}\Big)$, where $\overline{\Phi}$ is given in (61) and satisfies (62).

5.1.2 | Achieving consensus error estimation

In this Section we will show how to choose the matrix $\overline{\Pi}_O := \Pi_O \otimes I_N$ in (60) (and therefore Π_O in (47)). To this aim, we make the following basic assumptions on the nonlinearities affecting the dynamics of each agent in terms of incremental homogeneity, being a dual version of $(\mathcal{A}2)$.

$(\mathcal{A}3)$ *The nonlinear function* $f \in \mathbf{C}^0(R^n, R^n)$ *in (61) is incrementally homogeneous in the upper bound with quadruple* $\big(\mathfrak{r}, \mathfrak{r}+\widehat{\mathfrak{g}}, \mathfrak{g}, \Phi\big)$, *with* $\Phi = \langle\langle BK \rangle\rangle + \Phi_0$, $\Phi_0(0,0) = 0$, *and for* $j \in [1, n-1]$,

$$\begin{aligned}\widehat{\mathfrak{g}}_n &:= \mathfrak{g}_n, \\ \widehat{\mathfrak{g}}_j &:= \mathfrak{r}_{j+1} - \mathfrak{r}_j - \mathfrak{g}_{j+1},\end{aligned} \tag{63}$$

and

$$2\mathfrak{g}_{j+1} - \mathfrak{g}_j \le \widehat{\mathfrak{g}}_j \le \mathfrak{g}_j. \tag{64}$$

Remark 5. A consequence of (64) is that the sequence $\{\mathfrak{g}_j\}_{j\in[1,N]}$ *is non-increasing. As an example, the dynamics (8) satisfy assumption* $(\mathcal{A}3)$ *with* $\mathfrak{r}_1 = 1/8, \mathfrak{r}_2 = 3/8, \mathfrak{g}_1 = 1/2, \mathfrak{g}_2 = 1/4$.

The key Theorem to show achievement of consensus error estimation is the following.

Theorem 2. *Assume* $(\mathcal{A}0')$ *and* $(\mathcal{A}3)$ *and that the agents connected to the leader receive the leader's state information. Let* $\mathcal{E} \subset \mathbb{R}^{Nn}$ *be any compact set containing* $\Omega^{\times N}$. *There exist* $\epsilon_O > 1$, $c > 0$ *and a positive definite diagonal* Γ_O *such that for all* $\epsilon \ge \epsilon_O$ *(46), (47) with*

$$\Pi_O = (I_n - H_O A^T)^{-1} H_O C^T, \; H_O = \Gamma_O \mathrm{diag}\{\epsilon^{2\mathfrak{g}_1}, \cdots \epsilon^{2\mathfrak{g}_n}\}, \tag{65}$$

is an observer for all consensus error trajectories such that $e(t) \in \mathcal{E}$ *for all* $t \ge 0$.

Proof. Consider the change of coordinates

$$\begin{pmatrix} X \\ \overline{\eta} \end{pmatrix} \to \begin{pmatrix} X \\ \widehat{\eta} \end{pmatrix}, \ \widehat{\eta} = \overline{Z}_O \overline{\eta} \tag{66}$$

where setting $\overline{H}_O = H_O \otimes I_N$:

$$\overline{Z}_O = I_{Nn} - \overline{H}_O \overline{A}^T. \tag{67}$$

Also in this case notice that

$$I_{Nn} - \overline{H}_O \overline{A}^T = I_n \otimes I_N - (H_O \otimes I_N)(A \otimes I_N)^T = I_n \otimes I_N - (H_O \otimes I_N)(A^T \otimes I_N) = (I_n - H_O A^T) \otimes I_N.$$

Due to the particular choice of the change of coordinates it is easily verified that

$$\begin{aligned} \dot{X} &= \begin{pmatrix} Ax^{(0)} + f(x^{(0)}) \\ (I_N \otimes A)e + (I_N \otimes B)u + F(X) \end{pmatrix} \\ \dot{\widehat{\eta}} &= (\widehat{A} - \widehat{\Pi}_O \widehat{C})\widehat{\eta} + \overline{Z}_O \overline{G}(X, \overline{Z}_O^{-1} \widehat{\eta}) \end{aligned}$$

where

$$\widehat{A} = \overline{Z}_O \overline{A} \overline{Z}_O^{-1}, \ \widehat{C} = \overline{C} \overline{Z}_O^{-1} = \overline{C}, \ \widehat{\Pi}_O = \overline{Z}_O \overline{\Pi}_O. \tag{68}$$

By noting that $\overline{H}_O \overline{A}^T \overline{A} (\overline{H}_O \overline{A}^T) = \overline{H}_O^2 \overline{A}^T$ then

$$\begin{aligned} \overline{Z}_O \overline{A} \overline{Z}_O^{-1} &= (\overline{A} - \overline{H}_O \overline{A}^T \overline{A}) \overline{Z}_O^{-1} = [\overline{A} - \overline{H}_O \overline{A}^T \overline{A} + \overline{H}_O \overline{A}^T \overline{A} (\overline{H}_O \overline{A}^T) - \overline{H}_O^2 \overline{A}^T] \overline{Z}_O^{-1} \\ &= [-\overline{H}_O \overline{A}^T \overline{A} \overline{Z}_O + \overline{A} - \overline{A}^T \overline{H}_O^2] \overline{Z}_O^{-1} = -\overline{H}_O \overline{A}^T \overline{A} + (\overline{A} - \overline{H}_O^2 \overline{A}^T) \overline{Z}_O^{-1} \end{aligned}$$

so that

$$\widehat{A} - \widehat{B}\widehat{\Pi} = -\overline{H}_O \overline{A}^T \overline{A} + (\overline{A} - \overline{H}_O^2 \overline{A}^T) \overline{Z}_O^{-1} - \widehat{\Pi}_O \widehat{C} = -\widehat{H}_O + (\overline{A} - \overline{H}_O^2 \overline{A}^T) \overline{Z}_O^{-1}$$

where $\widehat{H}_O := \mathrm{diag}\{\overline{\mathcal{L}}, I_N, \dots, I_N\} \overline{H}_O$. As a consequence,

$$\dot{\widehat{\eta}} = -\widehat{H}_O \widehat{\eta} + \widehat{\rho}_O(X, \widehat{\eta})$$

where

$$\widehat{\rho}_O(X, \widehat{\eta}) = (\overline{A} - \overline{H}_O^2 \overline{A}^T) \overline{Z}_O^{-1} \widehat{\eta} + \overline{Z}_O \overline{G}(X, \overline{Z}_O^{-1} \widehat{\eta}).$$

Assumption ($\mathcal{A}3$) and similar conclusions to those of Appendix 9 (with $\mathfrak{d} := \hat{\mathfrak{g}}$ and $\mathfrak{h} := \mathfrak{g}$) show that $\widehat{\rho}_O$ is h.u.b. with quadruple $\left(\mathfrak{r} \otimes \mathbf{1}_{2N+1}, (\mathfrak{r} + \mathfrak{g}) \otimes \mathbf{1}_N, \mathfrak{g} \otimes \mathbf{1}_{2N+1}, \widehat{R}_O\right)$, where

$$\widehat{R}_O := \left[\begin{pmatrix} \mathbf{0}_{Nn \times (N+1)n} & \overline{A} + \overline{\Gamma}_O^2 \overline{A}^T \end{pmatrix} + (I_{Nn} + \overline{\Gamma}_O \overline{A}^T) \overline{\Phi}\right] \begin{pmatrix} I_{(N+1)n} & \mathbf{0}_{(N+1)n \times Nn} \\ \mathbf{0}_{Nn \times (N+1)n} & (I_{Nn} + \overline{\Gamma}_O \overline{A}^T)^{-1} \end{pmatrix} \tag{69}$$

$\overline{\Gamma}_O = \Gamma_O \otimes I_N$ and $\overline{\Phi}$ is defined in (36). The matrix $\widehat{R}_O$ can be also decomposed as

$$\widehat{R}_O = \begin{pmatrix} \mathbf{0}_{Nn \times (N+1)n} & \widehat{Q}_O \end{pmatrix}$$

and by (62) as $c \to 0^+$

$$\begin{aligned} \widehat{Q}_O \to \widehat{Q}_O(0) &:= \left(\overline{A} + \overline{\Gamma}^2 \overline{A}^T + B \otimes K \otimes (\langle\langle T \rangle\rangle \langle\langle T^{-1} \rangle\rangle)\right) (I_{Nn} - \overline{H}_O \overline{A}^T)^{-1} \\ &= (\overline{A} + \overline{\Gamma}^2 \overline{A}^T)(I_{Nn} - \overline{H}_O \overline{A}^T)^{-1} + B \otimes \left(\left(K \otimes (\langle\langle T \rangle\rangle \langle\langle T^{-1} \rangle\rangle)\right) (I_{Nn} - \overline{H}_O \overline{A}^T)^{-1}\right). \end{aligned} \tag{70}$$

Moreover, it is easy to see that $\widehat{\eta} \to \widehat{H}_O \widehat{\eta}$ is i.h. with quadruple $\left(\mathfrak{r} \otimes \mathbf{1}_N, (\mathfrak{r} + \mathfrak{g}) \otimes \mathbf{1}_N, \mathfrak{g} \otimes \mathbf{1}_N, \widehat{\Gamma}_O\right)$ with $\widehat{\Gamma}_O := \mathrm{diag}\{\overline{\mathcal{L}}, I_{N(n-1)}\} \overline{\Gamma}_O$ and $\overline{\Gamma}_O = \Gamma_O \otimes I_N$.

With $V_O(\widehat{\eta}) = \|\epsilon^{-\mathfrak{r} \otimes \mathbf{1}_N} \diamond \widehat{\eta}\|^2$ it follows by straightforward computations that

$$\dot{V}_O(\widehat{\eta}) = \frac{\partial V_O}{\partial \widehat{\eta}}(\widehat{\eta})\{-\widehat{H}_O \widehat{\eta} + \widehat{\rho}_O(X, \widehat{\eta})\} \le -\langle\langle \epsilon^{(\mathfrak{g}-\mathfrak{r}) \otimes \mathbf{1}_N} \diamond \widehat{\eta} \rangle\rangle^T \left(2\widehat{\Gamma}_O - \widehat{Q}_O - \widehat{Q}_O^T\right) \langle\langle \epsilon^{(\mathfrak{g}-\mathfrak{r}) \otimes \mathbf{1}_N} \diamond \widehat{\eta} \rangle\rangle$$

After some lengthy manipulations, it can be seen that there exists diagonal positive definite $\overline{\Gamma}_O$ (recall that $\overline{\Gamma}_O = \Gamma_O \otimes I_N$) such that $2\widehat{\Gamma}_O - \widehat{Q}_O(0) - \widehat{Q}_O^T(0) \geq -2I$. Next, using (70) it is possible to select $c > 0$ such that $\widehat{Q}_O - \widehat{Q}_O(0) + (\widehat{Q}_O - \widehat{Q}_O(0))^T \leq I$. Therefore, on account of (70), with such choice of c and for all $\epsilon \geq \epsilon_O$, where $\epsilon_O \geq 1$ is such that (48) is satisfied for all $\epsilon \geq \epsilon_O$, we obtain $\dot{V}_O(\hat{\eta}) \leq -\|\epsilon^{(\mathfrak{g}-\mathfrak{r})\otimes \mathbf{1}_N} \diamond \hat{\eta}\|^2$ which implies that $\hat{\eta}(t) \to 0$ (and therefore $\eta(t) \to 0$) as $t \to +\infty$ as long as $e(t) \in \mathcal{E}$ for all $t \geq 0$. This concludes the proof. □

5.2 | Followers' and leader's outputs information

In the present Section we remove the assumption that the leader's state is measured. Let us now consider the given system in the coordinates $(x^{(0)^T}, e^T)^T$, described by equations (23), (24) and restrict to trajectories $(x^{(0)}(t), e(t))$ such that $(x^{(0)}(t), e(t)) \in \Omega \times \mathcal{E}$ for all $t \geq 0$, $\mathcal{E} \subset \mathbb{R}^{Nn}$ any compact set containing $\Omega^{\times N}$. We want to design an observer for all such consensus error trajectories. We want to design an observer for both the leader's state and the consensus error e. To this aim, we propose the observer for the consensus error

$$\dot{\xi} = (I_N \otimes A)\xi + (I_N \otimes B)u + F(\mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(\xi^{(0)}), \mathbf{sat}_{\mathbf{1}_N \otimes c\epsilon^{\mathfrak{r}}}(\xi)) + (I_N \otimes \Pi_O)\chi \tag{71}$$

with $c > 0$, $\epsilon > 0$, Π_O and χ as in (47), and the observer for the leader's state

$$\dot{\xi}^{(0)} = A\xi^{(0)} + f\left(\mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(\xi^{(0)})\right) + \Pi_O(y^{(0)} - C\xi^{(0)}) \tag{72}$$

The observer (71) differs from the previous observer (47) in that the saturated estimate $\mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(\xi^{(0)})$ replaces the leader'state $x^{(0)}$. Our main Theorem is the following and follows the lines of Theorem 2.

Theorem 3. *Assume ($\mathcal{A}0'$) and ($\mathcal{A}3$) and that the agents connected to the leader receive the leader's state information. There exist $\epsilon_O > 1$, $c > 0$ and Γ_O such that for all $\epsilon \geq \epsilon_O$ (46), (65), (71), (72) is an observer for all consensus error trajectories such that $e(t) \in \mathcal{E}$ for all $t \geq 0$, $\mathcal{E} \subset \mathbb{R}^{Nn}$ any compact set containing $\Omega^{\times N}$.*

Proof. Assume that $c > 0$ and $\epsilon > 0$ have been selected as in (48). The system (23), (24), (46), (71), (72), reads with $\eta = e - \xi$, $\eta^{(0)} = x^{(0)} - \xi^{(0)}$, $\eta^E = (\eta^{(0)^T} \eta^T)^T$ and $X = \left(x^{(0)^T} \;\; e^T\right)^T$ as

$$\dot{X} = \begin{pmatrix} Ax^{(0)} + f(x^{(0)}) \\ (I_N \otimes A)e + (I_N \otimes B)u + F(X) \end{pmatrix} \tag{73}$$

$$\dot{\eta}^E = (I_{N+1} \otimes A)\eta^E - (I_{N+1} \otimes \Pi_O)\omega_O^E - \widehat{G}^E(X, \eta^E) \tag{74}$$

where $\omega^{(0)} = C\eta^{(0)}$, $\omega_O^E = \left(\omega^{(0)^T} \;\; \omega^T\right)^T$ and

$$\widehat{G}_0(X, \eta^E) = \Delta f\left(\mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(x^{(0)} - \eta^{(0)}), \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(x^{(0)})\right),$$

$$\widehat{G}_1(X, \eta^E) = \begin{pmatrix} \Delta f\left(\mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(e^{(1)} - \eta^{(1)}) + \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(x^{(0)} - \eta^{(0)}), \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(e^{(1)}) + \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(x^{(0)})\right) \\ \vdots \\ \Delta f\left(\mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(e^{(N)} - \eta^{(N)}) + \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(x^{(0)}), \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(e^{(N)}) + \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(x^{(0)})\right) \end{pmatrix}, \; \widehat{G}_2(X, \eta^E) = \mathbf{1}_N \otimes \widehat{G}_0(X, \eta^E),$$

$$\widehat{G}^E(X, \eta^E) = \begin{pmatrix} \widehat{G}_0(X, \eta^E) \\ \widehat{G}_1(X, \eta^E) - \widehat{G}_2(X, \eta^E) \end{pmatrix}.$$

System (73), (74) is formally equivalent to (23), (24), under the following equivalences

$$X \leftrightarrow x^{(0)}, \;\; \eta^E \leftrightarrow e, \;\; -(I_{N+1} \otimes \Pi_O)\omega^E \leftrightarrow (I_N \otimes B)u, \;\; -\widehat{G}^E \leftrightarrow F$$

and we can proceed as in Section 5.1 and give a sketch of the main steps. By assumption ($\mathcal{A}3$) $f : R^n \to R^n$ in (2) is i.h.u.b. with quadruple $\left(\mathfrak{r}, \mathfrak{r} + \hat{\mathfrak{g}}, \mathfrak{g}, \Phi\right)$, where $\Phi = \langle\langle BK \rangle\rangle + \Phi_0$ with $\Phi_0(0,0) = 0$, then $\widehat{G}^E$ is h.u.b. with quadruple

$\left(\mathbf{1}_{2(N+1)} \otimes \mathfrak{r}, \mathbf{1}_{N+1} \otimes (\mathfrak{r}+\widehat{\mathfrak{g}}), \mathbf{1}_{2(N+1)} \otimes \mathfrak{g}, \widehat{\Phi}\right)$, where

$$\begin{aligned}
\widehat{\Phi} &= \widehat{\Phi}_1 + \widehat{\Phi}_2, \\
\widehat{\Phi}_1 &= \begin{pmatrix} \mathbf{0}_{1\times(N+1)} & \mathbf{0} & \mathbf{0}_{1\times N} \\ \mathbf{0}_{N\times(N+1)} & \mathbf{1}_N & I_N \end{pmatrix} \otimes \left(\langle\langle BK \rangle\rangle + \max_{-2c\mathbf{1}_n \le p', p'' \le 2c\mathbf{1}_n} \Phi_0(p', p'') \right), \\
\widehat{\Phi}_2 &= \begin{pmatrix} \mathbf{0}_{1\times(N+1)} & 1 & \mathbf{0}_{1\times N} \\ \mathbf{0}_{N\times(N+1)} & \mathbf{1}_N & \mathbf{0}_{N\times N} \end{pmatrix} \otimes \left(\langle\langle BK \rangle\rangle + \max_{-c\mathbf{1}_n \le p', p'' \le c\mathbf{1}_n} \Phi_0(p', p'') \right).
\end{aligned}$$

After the change of coordinates

$$\widetilde{\eta}^{\,E} = \begin{pmatrix} \widetilde{\eta}^{\,(0)} \\ \widetilde{\eta} \end{pmatrix} := \left(\mathrm{diag}\{1,\ T\} \otimes I_n\right) \eta^E \tag{75}$$

we have from (74)

$$\dot{\widetilde{\eta}}^{\,E} = (I_{N+1} \otimes A)\widetilde{\eta}^{\,E} - \left(\mathrm{diag}\{1,\ T\} \otimes \Pi_O\right) \omega_O^E - \widetilde{G}^E(X, \widetilde{\eta}^{\,E}) \tag{76}$$

where

$$\widetilde{G}^E(X, \widetilde{\eta}^{\,E}) = \left(\mathrm{diag}\{1,\ T\} \otimes I_n\right) \widehat{G}^E \left(X, \left(\mathrm{diag}\{1,\ T^{-1}\} \otimes I_n\right) \widetilde{\eta}\ \right) \tag{77}$$

By assumption $(\mathcal{A}3)$ $f : R^n \to R^n$ in (2), is i.h.u.b. with quadruple $\left(\mathfrak{r}, \mathfrak{r}+\widehat{\mathfrak{g}}, \mathfrak{g}, \Phi\right)$, where $\Phi = \langle\langle BK \rangle\rangle + \Phi_0$ with $\Phi_0(0,0) = 0$. Then under the change of coordinates (75) the function $\widetilde{G}^E : R^{2(N+1)n} \to R^{(N+1)n}$ defined by (77) is h.u.b. with quadruple $\left(\mathbf{1}_{2(N+1)} \otimes \mathfrak{r}, \mathbf{1}_{N+1} \otimes (\mathfrak{r}+\widehat{\mathfrak{g}}), \mathbf{1}_{2(N+1)} \otimes \mathfrak{g}, \tilde{\Phi}\right)$, where $\widetilde{\Phi}$ is given by

$$\widetilde{\Phi} = \left(\mathrm{diag}\{1,\ \langle\langle T \rangle\rangle\} \otimes I_n\right) \widehat{\Phi} \left(\mathrm{diag}\{I_{N+2}, \langle\langle T^{-1} \rangle\rangle\} \otimes I_n\right). \tag{78}$$

Moreover as $c \to 0^+$

$$\widetilde{\Phi} \to \begin{pmatrix} \mathbf{0}_{1\times(N+1)} & 1 & \mathbf{0}_{1\times N} \\ \mathbf{0}_{N\times(N+1)} & 2\langle\langle T \rangle\rangle \mathbf{1}_N & \langle\langle T \rangle\rangle\langle\langle T^{-1} \rangle\rangle \end{pmatrix} \otimes \langle\langle BK \rangle\rangle. \tag{79}$$

Let us now permute the coordinates $(X, \widetilde{\eta}^{\,E}) \mapsto (X, \overline{\eta}^E) = P(X, \widetilde{\eta}^{\,E})$ as follows

$$\overline{\eta}_j = (\widetilde{\eta}_j^{(0)}, \cdots, \widetilde{\eta}_j^{(N)})^T,\ \overline{\eta} = (\overline{\eta}_1^{\ T}, \cdots, \overline{\eta}_n^{\ T})^T. \tag{80}$$

Accordingly set $\overline{\Lambda} = \mathrm{diag}\{1,\ \bar{\lambda}_1, \ldots, \bar{\lambda}_N\}$ and

$$\begin{aligned}
\overline{G}_j &= (\widetilde{G}_j^{(0)}, \cdots, \widetilde{G}_j^{(N)})^T,\ \overline{G} = (\overline{G}_1^{\ T}, \cdots, \overline{G}_n^{\ T})^T \\
\overline{A} &= A \otimes I_{N+1},\ \overline{C} = C \otimes \overline{\Lambda},\ \overline{\Pi}_O = \Pi_O \otimes I_{N+1}.
\end{aligned}$$

With these positions,

$$\dot{\overline{\eta}}^E = (\overline{A} - \overline{\Pi}_O \overline{C})\overline{\eta}^E + \overline{G}^E(X, \overline{\eta}^E). \tag{81}$$

Since $\widetilde{G}^E : R^{2(N+1)n} \to R^{(N+1)n}$ defined by (77) is h.u.b. with quadruple $\left(\mathbf{1}_{2(N+1)} \otimes \mathfrak{r}, \mathbf{1}_{N+1} \otimes (\mathfrak{r}+\widehat{\mathfrak{g}}), \mathbf{1}_{2(N+1)} \otimes \mathfrak{g}, \tilde{\Phi}\right)$, then $\overline{G}^E$ is h.u.b. with quadruple $\left(\mathfrak{r} \otimes \mathbf{1}_{2(N+1)}, (\mathfrak{r}+\widehat{\mathfrak{g}}) \otimes \mathbf{1}_{N+1}, \mathfrak{g} \otimes \mathbf{1}_{2(N+1)}, \overline{\Phi}\right)$, where $\overline{\Phi}$ is defined starting from $\widetilde{\Phi} = ((\widetilde{\Phi}^{(1)})^T, \cdots, (\widetilde{\Phi}^{(N)})^T)^T$ in the following way:

$$\overline{\Phi}_j = ((\widetilde{\Phi}_j^{(1)})^T, \cdots, (\widetilde{\Phi}_j^{(N)})^T)^T \circ P^{-1},\ \overline{\Phi} = (\overline{\Phi}_1^T, \cdots, \overline{\Phi}_n^T)^T \tag{82}$$

Moreover as $c \to 0^+$

$$\overline{\Phi} \to \left(\mathbf{0}_{(N+1)n\times(N+1)n} \quad B \otimes \langle\langle K \rangle\rangle \otimes \begin{pmatrix} 1 & \mathbf{0}_{1\times N} \\ 2\langle\langle T \rangle\rangle \mathbf{1}_N & \langle\langle T \rangle\rangle\langle\langle T^{-1} \rangle\rangle \end{pmatrix} \right). \tag{83}$$

Finally, with the change of coordinates

$$\begin{pmatrix} X \\ \overline{\eta}^E \end{pmatrix} \to \begin{pmatrix} X \\ \widehat{\eta}^E \end{pmatrix},\ \widehat{\eta}^E := \overline{Z}_O \overline{\eta}^E \tag{84}$$

where setting $\overline{H}_O = H_O \otimes I_{N+1}$:

$$\overline{Z}_O = I_{(N+1)n} - \overline{H}_O \overline{A}^{\,T} = (I_n - H_O A^T) \otimes I_{N+1} \tag{85}$$

(see similar derivation in the previous section), we get from (81)

$$\dot{\hat{\eta}}^E = -\hat{H}_O\hat{\eta}^E + \hat{\rho}_O^E(X,\hat{\eta}^E)$$

where

$$\hat{\rho}_O^E(X,\hat{\eta}^E) = (\overline{A} - \overline{H}_O^2\overline{A}^T)\overline{Z}_O^{-1}\hat{\eta}^E + \overline{Z}_O\overline{G}^E(X,\overline{Z}_O^{-1}\hat{\eta}^E).$$

With $V_O(\hat{\eta}^E) = \|\epsilon^{-\mathfrak{r}\otimes\mathbf{1}_{N+1}} \diamond \hat{\eta}^E\|^2$ it follows that $\dot{V}_O(\hat{\eta}^E) \le -\|\epsilon^{(\mathfrak{g}-\mathfrak{r})\otimes\mathbf{1}_{N+1}} \diamond \hat{\eta}^E\|^2$. □

6 | SEMIGLOBAL LEADER FOLLOWING WITH PARTIAL INFORMATION

In the present Section we address and solve the problem of achieving consensus assuming partial information (Problem II), using the state feedback consensus law and the observers for the consensus errors designed in Sections 4 and, respectively, 5. We assume that each agent exchanges its output with its neighbors and, in addition, measures the state of the leader (this is done for purpose of illustration and the leader's state availability is relaxed as in Section 5.2).

6.1 | Followers' outputs and leader's state information

In this case for each agent we will adopt a control $u^{(j)} = -\Pi\hat{\zeta}^{(j)}$, $j \in [1,N]$, where $\hat{\zeta}^{(j)}$ is the estimated consensus law for the j-th agent, obtained from $\zeta^{(j)}$ in (14) with the consensus error $e^{(j)}$ replaced by its estimate $\xi^{(j)}$:

$$u^{(j)} = \Pi\hat{\zeta}^{(j)} = -\Pi\Big[\sum_{j=1}^{N}\ell_j^i\mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(\xi^{(j)}) + \ell_0^i\mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(\xi^{(i)})\Big],\ j \in [1,N]. \tag{86}$$

Therefore, the control law is

$$u = -(I_N \otimes \Pi)(\hat{\mathcal{L}} \otimes I_n)\mathbf{sat}_{c\epsilon^{\mathbf{1}_N\otimes\mathfrak{r}}}(\xi), \tag{87}$$

together with the observer (47) for the followers' consensus error

$$\dot{\xi} = (I_N \otimes A)\xi + (I_N \otimes B)u + F\left(\mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(x^{(0)}), \mathbf{sat}_{\mathbf{1}_N\otimes c\epsilon^{\mathfrak{r}}}(\xi)\right) + (I_N \otimes \Pi_O)\chi. \tag{88}$$

The key Theorem to show achievement of consensus is the following.

Theorem 4. *Assume $(\mathcal{A}0')$, $(\mathcal{A}3)$ with $\mathfrak{f}_n < \mathfrak{g}_n$ and that the agents connected to the leader receive the leader's state information and, in addition, . There exist $\epsilon^* > 1$, $c > 0$ and positive definite diagonal matrices Γ, Γ_O such that for all $\epsilon \ge \epsilon^*$ (40), (65), (87), (88) solve the Semiglobal Leader Following problem with partial information for (1), (2).*

Proof. Consider the system (23), (24), (87) and (88) in coordinates $\hat{e} = Se$ and $\hat{\eta} = S_O\eta$ (follow Sections 4 and 5)

$$\begin{pmatrix}\dot{x}^{(0)}\\ \dot{\hat{e}}\end{pmatrix} = \begin{pmatrix}Ax^{(0)} + f(x^{(0)})\\ -\hat{H}\hat{e} + \hat{\rho}(x^{(0)},\hat{e}) - \overline{B\mathcal{L}}^{-1}\overline{B}^T\overline{H}S\left(-S^{-1}\hat{e} + \mathbf{sat}_{c\epsilon^{\mathbf{1}_N\otimes\mathfrak{r}}}(S^{-1}\hat{e} - S_O^{-1}\hat{\eta})\right)\end{pmatrix} \tag{89}$$
$$\dot{\hat{\eta}} = -\hat{H}_O\hat{\eta} + \hat{\rho}_O(x^{(0)}, S^{-1}\hat{e}, \hat{\eta}).$$

Notice that since $S = \overline{Z}^{-1}P(T \otimes I_n)$ and $\hat{e} \mapsto S^{-1}\hat{e}$ is h.u.b. with quadruple $\left(\mathfrak{r}\otimes\mathbf{1}_N, \mathbf{1}_N\otimes\mathfrak{r}, \mathbf{0}_n\otimes\mathbf{1}_N, (\langle\langle T^{-1}\rangle\rangle \otimes I_n)P^{-1}(I_{Nn} + \overline{A}^T\overline{\Gamma})\right)$ (remember that $\mathfrak{f}_1 \le \mathfrak{f}_2 \le \cdots \le \mathfrak{f}_n$ by remark 3)

$$\|\epsilon^{-\mathbf{1}_N\otimes\mathfrak{r}} \diamond S^{-1}\hat{e}\|^2 \le \|\left(\langle\langle T^{-1}\rangle\rangle \otimes I_n\right)P^{-1}(I_{Nn} + \overline{A}^T\overline{\Gamma})(\epsilon^{-\mathfrak{r}\otimes\mathbf{1}_N} \diamond \hat{e})\|^2 \le \|\left(\langle\langle T^{-1}\rangle\rangle \otimes I_n\right)P^{-1}(I_{Nn} + \overline{A}^T\overline{\Gamma})\|^2 V(\hat{e}) \tag{90}$$

where

$$V(\hat{e}) = \|\epsilon^{-\mathfrak{r}\otimes\mathbf{1}_N} \diamond \hat{e}\|^2.$$

Therefore, if $\mathcal{E} := \{\hat{e} \in \mathbb{R}^{Nn} : \|\epsilon^{-\mathfrak{r}\otimes\mathbf{1}_N} \diamond S^{-1}\hat{e}\| \le c\}$ and

$$\hat{c} := \frac{c}{\|\left(\langle\langle T^{-1}\rangle\rangle \otimes I_n\right)P^{-1}(I_{Nn} - \overline{A}^T\overline{\Gamma})^{-1}\|}$$

then

$$V(\hat{e}) \le \hat{c}^2 \Rightarrow \hat{e} \in \mathcal{E} \Rightarrow \mathbf{sat}_{c\epsilon^{\mathfrak{r}\otimes\mathbf{1}_N}}(S^{-1}\hat{e}) = S^{-1}\hat{e}. \tag{91}$$

We will select ϵ in such a way that the set $\{\hat{e} \in \mathbb{R}^{Nn} : V(\hat{e}) \leq \hat{c}^2\}$ is forward invariant and therefore $\hat{e}(t) \in \mathcal{E}$ for all times and this will guarantee $\hat{e}(t) \to 0$ (and therefore, $e(t) \to 0$) as $t \to +\infty$. To this aim, since $e \mapsto Se$ is h.u.b. with quadruple $\left(\mathbf{1}_N \otimes \mathfrak{r}, \mathfrak{r} \otimes \mathbf{1}_N, \mathbf{0}_n \otimes \mathbf{1}_N, (I_{Nn} - \overline{A}^T\overline{\Gamma})^{-1}P(\langle\langle T\rangle\rangle \otimes I_n)\right)$

$$
\begin{aligned}
&2(\epsilon^{-\mathfrak{r}\otimes\mathbf{1}_N} \diamond \hat{e})^T \epsilon^{-\mathfrak{r}\otimes\mathbf{1}_N} \diamond \overline{B\mathcal{L}}^{-1}\overline{B}^T\overline{H}S\left(-S^{-1}\hat{e} + \mathbf{sat}_{c\epsilon^{\mathbf{1}_N\otimes\mathfrak{r}}}(S^{-1}\hat{e} - S_O^{-1}\hat{\eta})\right) \\
&\leq \frac{1}{2}\|\epsilon^{(\mathfrak{f}-\mathfrak{r})\otimes\mathbf{1}_N} \diamond \hat{e}\|^2 + \left\|\epsilon^{(-\mathfrak{f}-\mathfrak{r})\otimes\mathbf{1}_N} \diamond \overline{B\mathcal{L}}^{-1}\overline{B}^T\overline{H}S\left(-S^{-1}\hat{e} + \mathbf{sat}_{c\epsilon^{\mathbf{1}_N\otimes\mathfrak{r}}}(S^{-1}\hat{e} - S_O^{-1}\hat{\eta})\right)\right\|^2 \\
&\leq \frac{1}{2}\|\epsilon^{(\mathfrak{f}-\mathfrak{r})\otimes\mathbf{1}_N} \diamond \hat{e}\|^2 + 2\epsilon^{2\mathfrak{f}_n}\left\|\overline{B\mathcal{L}}^{-1}\overline{B}^T\overline{H}\epsilon^{-\mathfrak{r}\otimes\mathbf{1}_N} \diamond S\left(-S^{-1}\hat{e} + \mathbf{sat}_{c\epsilon^{\mathbf{1}_N\otimes\mathfrak{r}}}(S^{-1}\hat{e} - S_O^{-1}\hat{\eta})\right)\right\|^2 \\
&\leq \frac{1}{2}\|\epsilon^{(\mathfrak{f}-\mathfrak{r})\otimes\mathbf{1}_N} \diamond \hat{e}\|^2 + 2\epsilon^{2\mathfrak{f}_n}Q\left\|\epsilon^{-\mathbf{1}_N\otimes\mathfrak{r}} \diamond \left(-S^{-1}\hat{e} + \mathbf{sat}_{c\epsilon^{\mathbf{1}_N\otimes\mathfrak{r}}}(S^{-1}\hat{e} - S_O^{-1}\hat{\eta})\right)\right\|^2
\end{aligned}
$$

where $Q := \|\overline{B\mathcal{L}}^{-1}\overline{B}^T\overline{\Gamma}(I_{Nn} - \overline{A}^T\overline{\Gamma})^{-1}P(\langle\langle T\rangle\rangle \otimes I_n)\|^2$.

With all this in mind and with $V(\hat{e}) = \|\epsilon^{-\mathfrak{r}\otimes\mathbf{1}_N} \diamond \hat{e}\|^2$ and $V_O(\hat{\eta}) = \|\epsilon^{-\mathfrak{r}\otimes\mathbf{1}_N} \diamond \hat{\eta}\|^2$, we obtain (follow the proof of Theorems 1 and 2)

$$
\begin{aligned}
\dot{V}(\hat{e}) &\leq -\frac{1}{2}\|\epsilon^{(\mathfrak{f}-\mathfrak{r})\otimes\mathbf{1}_N} \diamond \hat{e}\|^2 + 2N^2\epsilon^{2\mathfrak{f}_n}Q\left\|\epsilon^{-\mathbf{1}_N\otimes\mathfrak{r}} \diamond \left(-\mathbf{sat}_{c\epsilon^{\mathbf{1}_N\otimes\mathfrak{r}}}(S^{-1}\hat{e}) + \mathbf{sat}_{c\epsilon^{\mathbf{1}_N\otimes\mathfrak{r}}}(S^{-1}\hat{e} - S_O^{-1}\hat{\eta})\right)\right\|^2 \\
\dot{V}_O(\hat{\eta}) &\leq -\|\epsilon^{(\mathfrak{g}-\mathfrak{r})\otimes\mathbf{1}_N} \diamond \hat{\eta}\|^2
\end{aligned}
$$

for all $x^{(0)} \in \Omega$, $\hat{e} \in \mathbb{R}^{Nn} : V(\hat{e}) \leq \hat{c}^2$ and $\epsilon \geq \epsilon_1$ (here ϵ_1 is the same as in the proof of Theorem 1 while $c > 0$ is the same as in the proof of of Theorem 2). From the second inequality with Remark 5 we get $\dot{V}_O(\hat{\eta}) \leq -\|\epsilon^{(\mathfrak{g}-\mathfrak{r})\otimes\mathbf{1}_N} \diamond \hat{\eta}\|^2 \leq -\epsilon^{2\mathfrak{g}_n}V_O(\hat{\eta})$ which implies that for all $t \geq 0$

$$
V_O(\hat{\eta}(t)) \leq e^{-\epsilon^{2\mathfrak{g}_n}t}V_O(\hat{\eta}(0)) = e^{-\epsilon^{2\mathfrak{g}_n}t}\|\epsilon^{-\mathfrak{r}\otimes\mathbf{1}_N} \diamond \hat{\eta}(0)\|^2. \tag{92}
$$

But $\|\mathbf{sat}_{c\epsilon^{\mathbf{1}_N\otimes\mathfrak{r}}}(v_1) - \mathbf{sat}_{c\epsilon^{\mathbf{1}_N\otimes\mathfrak{r}}}(v_2)\| \leq \|\mathbf{sat}_{c\epsilon^{\mathbf{1}_N\otimes\mathfrak{r}}}(v_1 - v_2)\|$ for all $v_1, v_2 \in \mathbb{R}^{nN}$ so that

$$
\left\|\epsilon^{-\mathbf{1}_N\otimes\mathfrak{r}} \diamond \left(-\mathbf{sat}_{c\epsilon^{\mathbf{1}_N\otimes\mathfrak{r}}}(S^{-1}\hat{e}) + \mathbf{sat}_{c\epsilon^{\mathbf{1}_N\otimes\mathfrak{r}}}(S^{-1}\hat{e} - S_O^{-1}\hat{\eta})\right)\right\| \leq \min\left\{Nnc, \|\epsilon^{-\mathbf{1}_N\otimes\mathfrak{r}} \diamond S_O^{-1}\hat{\eta}\|\right\}
$$

and, since

$$
S_O = \overline{Z}_O P(T \otimes I_n)
$$

and $\hat{\eta} \mapsto S_O^{-1}\hat{\eta}$ is h.u.b. with quadruple $\left(\mathfrak{r} \otimes \mathbf{1}_N, \mathbf{1}_N \otimes (\mathfrak{r} - \mathfrak{g}), \mathbf{1}_N \otimes \mathfrak{g}, (\langle\langle T^{-1}\rangle\rangle \otimes I_n)P^{-1}(I_{Nn} - \overline{A}^T\overline{\Gamma}_O)^{-1}\right)$, with Remark 5

$$
\|\epsilon^{-\mathbf{1}_N\otimes\mathfrak{r}} \diamond S_O^{-1}\hat{\eta}\|^2 \leq \|\epsilon^{-\mathbf{1}_N\otimes\mathfrak{g}} \diamond \left(\langle\langle T^{-1}\rangle\rangle \otimes I_n\right)P^{-1}(I_{Nn} - \overline{A}^T\overline{\Gamma}_O)^{-1}(\epsilon^{(-\mathfrak{r}+\mathfrak{g})\otimes\mathbf{1}_N} \diamond \hat{\eta})\|^2 \leq \epsilon^{2(\mathfrak{g}_1-\mathfrak{g}_n)}Q_O V_O(\hat{\eta})
$$

for

$$
Q_O := \|\left(\langle\langle T^{-1}\rangle\rangle \otimes I_n\right)P^{-1}(I_{Nn} - \overline{A}^T\overline{\Gamma}_O)^{-1}\|^2.
$$

Therefore, from (92)

$$
\dot{V}(\hat{e}) \leq -\frac{1}{2}\|\epsilon^{(\mathfrak{f}-\mathfrak{r})\otimes\mathbf{1}_N} \diamond \hat{e}\|^2 + 2\epsilon^{2\mathfrak{f}_n}Q\min\{N^2n^2c^2, \epsilon^{2(\mathfrak{g}_1-\mathfrak{g}_n)}Q_O V_O(\hat{\eta})\} \tag{93}
$$

for all $x^{(0)} \in \Omega$, $\hat{e} \in \mathbb{R}^{Nn} : V(\hat{e}) \leq \hat{c}^2$ and $\epsilon \geq \epsilon_1$. On account of (92) and integrating (93) over $[0, t]$

$$
V(\hat{e}(t)) \leq V(\hat{e}(0)) + 2\epsilon^{2\mathfrak{f}_n}QN^2n^2c^2\int_0^t \min\left\{1, e^{-\epsilon^{2\mathfrak{g}_n s+\kappa(\epsilon)}}\right\}ds \leq V(Se(0)) + 4QN^2n^2c^2(\kappa(\epsilon)+1)\epsilon^{-2(\mathfrak{g}_n-\mathfrak{f}_n)} \tag{94}
$$

with $\kappa(\epsilon) = 2\left((\mathfrak{g}_1 - \mathfrak{g}_n)\ln\epsilon + \ln Q_O - \ln Nnc\right)$. Next, since by assumption of our Theorem $\mathfrak{f}_n < \mathfrak{g}_n$ and, then, $\lim_{\epsilon\to+\infty}(\kappa(\epsilon) + 1)\epsilon^{-2(\mathfrak{g}_n-\mathfrak{f}_n)} = 0$ we select $\epsilon^* \geq \epsilon_1$ in such a way that for all $\epsilon \geq \epsilon^*$ and $e(0) \in \Omega^{\times N}$ (see also the final Section of the proof of Theorem 1)

$$
V(Se(0)) + 2\epsilon^{2\mathfrak{f}_n}QN^2n^2c^2\int_0^t \min\left\{1, e^{-\epsilon^{2\mathfrak{g}_n s+\kappa(\epsilon)}}\right\}ds \leq \hat{c}^2 \tag{95}
$$

so that the set $\{\hat{e} \in \mathbb{R}^{Nn} : V(\hat{e}) \leq \hat{c}^2\}$ is forward invariant and by (91) $\hat{e}(t) \in \mathcal{E}$, $\forall t \geq 0$. Finally, as a consequence of (92), (93), it follows that $\hat{e}(t) \to 0$ (and therefore, $e(t) \to 0$) as $t \to +\infty$ for all $x^{(0)}(0) \in \Omega$ and $\hat{e}(0) \in \mathbb{R}^{Nn} : V(\hat{e}) \leq \hat{c}^2$ (and therefore, for all $x^{(0)}(0) \in \Omega$ and $e(0) \in \Omega^{\times N}$: see final part of the proof of Theorem 1). This proves our Theorem. ☐

6.2 | Followers' and leader's outputs information

In view of what we have discussed in the previous Section, we can state (without proof) the following intermediate important result of this paper. Here, remove the assumption that the leader's state is measured and we consider the additional observer dynamics (72) for estimating the leader's state.

Theorem 5. *Assume* $(\mathcal{A}0')$, $(\mathcal{A}2)$ *and* $(\mathcal{A}3)$ *with* $\mathfrak{f}_n < \mathfrak{g}_n$ *and that the agents connected to the leader receive the leader's state information. There exist* $\epsilon^* > 1$, $c > 0$ *and positive definite diagonal matrices* Γ, Γ_O *such that for all* $\epsilon \geq \epsilon^*$ *(40), (65), (72), (87), (88) solves the Semiglobal Leader Following problem with partial information for (35).*

Remark 6. A consequence of $(\mathcal{A}2)$ *and* $(\mathcal{A}3)$ *with* $\mathfrak{f}_n < \mathfrak{g}_n$ *is that the sequence non-decreasing* $\{\mathfrak{f}_j\}_{j\in[1,N]}$ *is strictly smaller than the non-increasing* $\{\mathfrak{g}_j\}_{j\in[1,N]}$. *The dynamics (8) satisfy assumption* $(\mathcal{A}2)$ *with* $\mathfrak{r}_1 = 1/8, \mathfrak{r}_2 = 3/8, \mathfrak{f}_1 = 1/8, \mathfrak{f}_2 = 1/8, \mathfrak{g}_1 = 1/2, \mathfrak{g}_2 = 1/4$.

In the remaining part of this Section we will give the details for relaxing $(\mathcal{A}0')$ to $(\mathcal{A}0)$ together with $(\mathcal{A}1)$, which is the final objective of our paper. In this case, since the compact set Ω is not known the observer (72) for the leader's state cannot be implemented. We replace (72) with a nonlinear observer, in particular we retain the same structure of (72) but the parameter ϵ is adapted on-line in such a way to estimate the magnitude of the leader'state trajectories. We propose the following observer

$$\dot{\xi}^{(0)} = A\xi^{(0)} + f(\mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(\xi^{(0)})) + \Pi_O(y^{(0)} - C\xi^{(0)}) \tag{96}$$

which is exactly (72) with, in addition, the adapting law

$$\dot{\epsilon} = \epsilon \left\{ |\epsilon^{-\mathfrak{r}_1+\mathfrak{g}_1}(y^{(0)} - C\xi^{(0)})|^2 + \|\epsilon^{-\mathfrak{r}+\mathfrak{g}} \diamond \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(\xi^{(0)} - \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(\xi^{(0)}))\|^2 \right\},\ \epsilon(0) = \epsilon^*. \tag{97}$$

The controller (87), (88) remains the same, with ϵ given by (97). From Theorem 4.2 of[3] we can prove the following.

Proposition 1. *Assume* $(\mathcal{A}0)$ *and* $(\mathcal{A}3)$. *There exist* $\epsilon^* > 1$, $c > 0$ *and positive definite diagonal matrices* Γ_O *such that for (1), (40), (65), (97) we have* $\lim_{t\to+\infty}(x^{(0)}(t) - \xi^{(0)}(t)) = 0$, $\lim_{t\to+\infty}\epsilon(t) = \epsilon^* < +\infty$ *and* $\lim_{t\to+\infty}(x^{(0)}(t) - \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(x^{(0)}(t))) = 0$ *for all initial states* $x^{(0)}(0)$.

Using Proposition 1 we can prove our final result.

Theorem 6. *Assume* $(\mathcal{A}0), (\mathcal{A}1), (\mathcal{A}2), (\mathcal{A}3)$ *with* $\mathfrak{f}_n < \mathfrak{g}_n$. *There exist* $\epsilon^* > 1$, $c > 0$ *and positive definite diagonal matrices* Γ, Γ_O *such that (40), (65), (87), (88), (96), (97) solve the Semiglobal Leader Following problem with partial information for (1), (2).*

The proof is based on the fact that, by $(\mathcal{A}1)$, the agents' state trajectories are bounded in time while estimating a bound on the magnitude of $x^{(0)}$ through (96), (97). After some time, by virtue of Proposition 1, $\|x^{(0)}(t) - \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(x^{(0)}(t))\|$ is small so that $x^{(0)}(t)$ is ultimately contained in some known compact set of the form $[-c\epsilon^{\mathfrak{r}}, c\epsilon^{\mathfrak{r}}]$.

7 | EXAMPLE AND SIMULATIONS

Consider, as the leader dynamics, the Van der Pol oscillator

$$\begin{aligned}\dot{x}_1^{(0)} &= x_2^{(0)}\\ \dot{x}_2^{(0)} &= (1 - x_1^{(0)^2})x_2^{(0)} - x_1^{(0)}\end{aligned}$$

which is in the form (1). Assume to have three agents and a leader connected as follows. Agent 2 is connected with the two other agents and the leader, while agents 1 and 3 are connected only with agent 2. Let $\mathcal{L}$ and $\mathcal{L}_0$ be respectively the Laplacians associated with the agents graph and the connection of the leader with the agents. Then they are given by

$$\mathcal{L} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix},\ \mathcal{L}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \hat{\mathcal{L}} = \mathcal{L}_0 + \mathcal{L} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

The given system satisfies assumptions $(\mathcal{A}0), (\mathcal{A}1), (\mathcal{A}2), (\mathcal{A}3)$, as pointed out in remarks 1, 3 and 5. We have taken Ω to be a 10×10 square centered at the origin. The parameters of the controller (87), (88), (96), (97) with (40), (65) are: $\Pi = (2\cdot10^5, 2\cdot10^3)$

and $\Pi_O = (10^3, 10)^T$ and $\epsilon(0) = 10$. The errors $e_1^{(i)}$ and $e_2^{(i)}$ for $i \in [1, 3]$ are reported in Figure 1 assuming that the agents are initialized as follows: Agent 1: $x^{(1)}(0) = (0, 0)^T$, Agent 2: $x^{(2)}(0) = (5, -5)^T$, Agent 3: $x^{(3)}(0) = (2, -2)^T$.

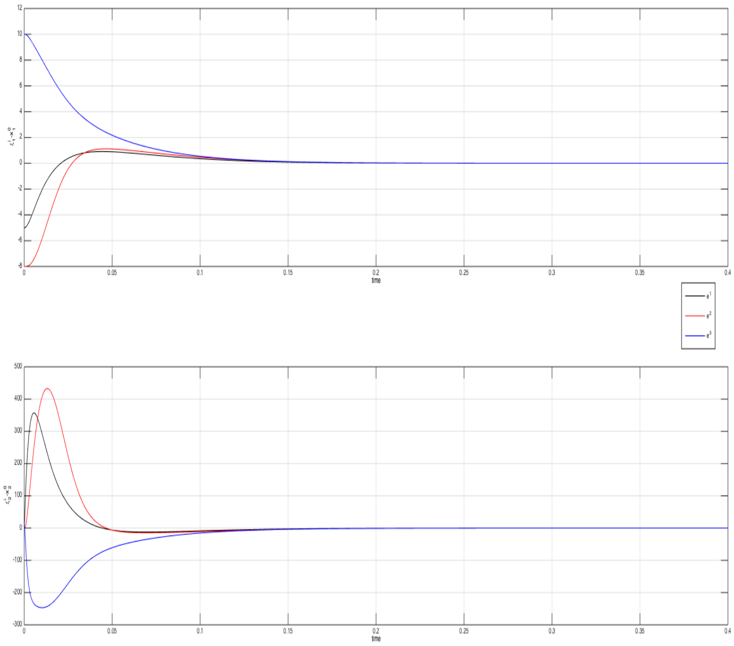

FIGURE 1 Consensus errors versus time

8 | EXTENSIONS

All the machinery introduced in the paper can be used in more general situations once appropriate additional assumptions are introduced in terms of incremental homogeneity.

8.1 | (A, B, C) not in prime form

The assumption of the triple (A,B,C) in prime form was considered in order to simplying the notation.Assume now to deal with the only case where the system is controllable and observable in the first approximation. Without loss of generality we can always assume that the matrices A and B are in the form (3) with $f(\cdot)$ given by (4). We can then apply the procedure proposed in section 4 to compute the static state feedback law $u = (u^{(1)}, \ldots, u^{(N)})^T$, $u^{(i)} = -\Pi\zeta^{(i)}$, $i \in [1, N]$, of theorem 1, assuming the incremental homogeneity assumption ($\mathcal{A}2$) on $f(\cdot)$ are satisfied. Consider now the change of coordinates $\tilde{x}^{(i)} = \tilde{T}x^{(i)}$, $i \in [0, N]$, with

$$\tilde{T} = \left(C^T \ (CA)^T \ \cdots \ (CA^{n-1})^T\right)^T \tag{98}$$

In the new coordinates $(\tilde{C}, \tilde{A})$ are in prime form while $\tilde{f}(\tilde{x}^{(i)}) = BK\tilde{x}^{(i)} + \tilde{f}_0(\tilde{x}^{(i)})$. We can then apply the procedure proposed in section 5.1 (and section 5.2) to compute the observer which estimates the consensus errors in the new coordinates with variables $\xi = (\xi^{(1)}, \ldots, , \xi^{(N)})^T$ assuming the incremental homogeneity assumption on $\tilde{f}(\cdot)$ are satisfied. Then the feedback law

$$u = -(I_N \otimes \Pi)(\hat{\mathcal{L}} \otimes I_n)\mathbf{sat}_{c\epsilon \mathbf{1}_N \otimes \mathbf{r}}((I_N \otimes \tilde{T}^{-1})\xi), \tag{99}$$

together with the observer (47) will achieve consensus.

8.2 | Output nonlinearities

In this case the leader's and agents's outputs are, more generally, assumed to be nonlinear:

$$\dot{x}^{(0)}(t) = Ax^{(0)}(t) + f(x^{(0)}(t)), \tag{100}$$
$$y^{(0)} = Cx^{(0)}(t) + h(x^{(0)}(t)), \tag{101}$$

$$\dot{x}^{(i)}(t) = Ax^{(i)}(t) + f(x^{(i)}(t)) + Bu^{(i)}(t), \tag{102}$$
$$y^{(i)} = Cx^{(i)}(t) + h(x^{(i)}(t)),\ i \in [1, N], \tag{103}$$

In this case, with (C, A) in prime form, we need the following additional assumptions on h in terms of incremental homogeneity:

($\mathcal{A}3'$) *The nonlinear function $C^T h \in \mathbf{C}^0(R^n, R^n)$ in (61) is incrementally homogeneous in the upper bound with quadruple $\left(\mathfrak{r}, \mathfrak{r} - \mathfrak{g}, \mathfrak{g}, C^T\Psi\right)$, with $\Psi = \alpha C$, $\alpha \in [0, 1)$ and $\mathfrak{g}$ satisfying (63), (64).*

This generalization is useful in many situations. For instance, saturated outputs can be written in the form (101) and (103) and satisfying ($\mathcal{A}3'$). The controller (87), (88) is modified by replacing $\chi^{(i)}$ in (46) with

$$\chi^{(i)} = \sum_{j=1}^{N} \ell_j^i \left(y^{(j)} - C\xi^{(j)} - H\left(\mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(x^{(0)}), \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(\xi^{(j)})\right)\right) + \ell_0^i \left(y^{(i)} - y^{(0)} - C\xi^{(i)} - H\left(\mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(x^{(0)}), \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(\xi^{(i)})\right)\right) \tag{104}$$

where $H(x^{(0)}, e^{(j)}) = h(e^{(j)} + x^{(0)}) - h(x^{(0)})$.

8.3 | Heterogeneous agents

In this case we assume heterogeneous agents, i.e.

$$\dot{x}^{(0)}(t) = Ax^{(0)}(t) + f^{(0)}(x^{(0)}(t)),\ y^{(0)} = Cx^{(0)}(t), \tag{105}$$
$$\dot{x}^{(i)}(t) = Ax^{(i)}(t) + f^{(i)}(x^{(i)}(t)) + Bu^{(i)}(t),\ y^{(i)} = Cx^{(i)}(t),\ i \in [1, N], \tag{106}$$

consisting of a leader (eq. (105)) with $x^{(0)} \in R^n$, $y^{(0)} \in R$ and $f^{(i)}$ locally Lipschitz function, and N heterogeneous agents (eq. (106)), where $x^{(i)} \in R^n$, $y^{(i)} \in R$ and $u^{(i)} \in R$, $i \in [1, N]$, are the state, measured output and control input of the i-th agent. Our main result remains true by the additional assumption on the agents

($\mathcal{A}4$) *There exist smooth maps $\Sigma^{(i)} : \mathbb{R}^n \to \mathbb{R}^n$ and $D^{(i)} : \mathbb{R}^n \to \mathbb{R}$ such that for all x*

$$\frac{\partial \Sigma^{(i)}}{\partial x}(x)[Ax + f^{(0)}(x)] = A\Sigma^{(i)}(x) + f^{(i)}(\Sigma^{(i)}(x)) + BD^{(i)}(x),\ i \in [1, N].$$

Indeed, under assumption ($\mathcal{A}4$) with $\mu^{(i)} := x^{(i)} - \Sigma^{(i)}(x^{(0)})$, $v^{(i)} := u^{(i)} - D^{(i)}(x^{(0)})$ and $w^{(i)} := y^{(i)} - C\Sigma^{(i)}(x^{(0)})$ (105), (106) becomes

$$\dot{x}^{(0)}(t) = Ax^{(0)}(t) + f^{(0)}(x^{(0)}(t)),\ y^{(0)} = Cx^{(0)}(t), \tag{107}$$
$$\dot{\mu}^{(i)}(t) = A\mu^{(i)}(t) + \Delta f^{(i)}(\mu^{(i)}(t) + \Pi^{(i)}(x^{(0)}), \Pi^{(i)}(x^{(0)})) + Bv^{(i)}(t),\ w^{(i)} = C\mu^{(i)}(t),\ i \in [1, N], \tag{108}$$

which can be assimilated to the system (23), (24) under the following identifications $e^{(i)} \leftrightarrow \mu^{(i)}$, $u^{(i)} \leftrightarrow v^{(i)}$ and $w^{(i)} \leftrightarrow y^{(i)}$. The consensus analysis follows as in Sections 4, 5 and 6.

8.4 | Robustness

In this case uncertain leader's and agents' dynamics are considered, with uncertainties modeled as a bounded time-varying disturbance (we consider identical agents for simplicity)

$$\begin{aligned} \dot{x}^{(0)}(t) &= Ax^{(0)}(t) + f(x^{(i)}(t), d(t)), \\ y^{(0)} &= Cx^{(0)}(t), \end{aligned} \tag{109}$$

$$\begin{aligned} \dot{x}^{(i)}(t) &= Ax^{(i)}(t) + f(x^{(i)}(t), d(t)) + Bu^{(i)}(t), \\ y^{(i)} &= Cx^{(i)}(t),\ i \in [1, N], \end{aligned} \tag{110}$$

consisting of a leader (eq. (105)) with $x^{(0)} \in R^n$, $y^{(0)} \in R$ and f a locally Lipschitz function, and N identical agents (eq. (110)), where $x^{(i)} \in R^n$, $y^{(i)} \in R$ and $u^{(i)} \in R$, $i \in [1, N]$, are the state, measured output and control input of the i-th agent with continuous bounded disturbance $d \in R$ (for simplicity we consider only one disturbance). Clearly, our incremental homogeneity assumptions must be modified to take into account the presence of disturbances d.

($\mathcal{A}2'$) *f is incrementally homogeneous in the upper bound with quadruple* $\left((\mathfrak{r}, \mathfrak{s}), \mathfrak{r} + \mathfrak{f}, (\hat{\mathfrak{f}}, \mathfrak{h}), \left(\Phi \ \Phi_d\right)\right)$, $\Phi = \langle\langle BK\rangle\rangle + \Phi_0$ *with* $\Phi_0(0, 0, 0) = 0$*,* $\mathfrak{r}, \mathfrak{f}$ *satisfying (38), (39) and* $\mathfrak{f}_1 > \mathfrak{h}$*,*

($\mathcal{A}3'$) *f is incrementally homogeneous in the upper bound with quadruple* $\left((\mathfrak{r}, \mathfrak{s}), \mathfrak{r} + \hat{\mathfrak{g}}, (\mathfrak{g}, \mathfrak{k}), \left(\Phi \ \Phi_d\right)\right)$*,* $\mathfrak{r}, \mathfrak{g}$ *satisfying (63), (64) and* $\mathfrak{g}_n > \mathfrak{k}$.

Notice that we had introduced additional weights $\mathfrak{s}$ and degrees $\mathfrak{h}$ in ($\mathcal{A}2'$) and $\mathfrak{k}$ in ($\mathcal{A}3'$) together with the matrix Φ_d in order to take into account the incremental contribution of d. Our result (the proof follows the same lines as with $d \equiv 0$) establishes an ISS property from the disturbance to the consensus error. The controller (86), (88) is modified as

$$u^{(j)} = \Pi\hat{\zeta}^{(j)} = -\Pi\left[\sum_{j=1}^{N} \ell^i_j \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(\xi^{(j)}) + \ell^i_0 \mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(\xi^{(i)})\right], \ j \in [1, N] \tag{111}$$

$$\dot{\xi} = (I_N \otimes A)\xi + (I_N \otimes B)u + F\left(\mathbf{sat}_{c\epsilon^{\mathfrak{r}}}(x^{(0)}), \mathbf{sat}_{\mathbf{1}_N \otimes c\epsilon^{\mathfrak{r}}}(\xi), 0\right) + (I_N \otimes \Pi_O)\chi. \tag{112}$$

where now

$$F(x^{(0)}, e, d) := \begin{pmatrix} f(x^{(0)}, d) - f(e^{(1)} + x^{(0)}, d) \\ \vdots \\ f(x^{(0)}, d) - f(e^{(N)} + x^{(0)}, d) \end{pmatrix}$$

We obtain the following result:

Theorem 7. *Assume* ($\mathcal{A}0'$), ($\mathcal{A}1$), ($\mathcal{A}2'$), ($\mathcal{A}3'$) *with* $\mathfrak{f}_n < \mathfrak{g}_n$*. There exist* $\epsilon^* > 1$, $c > 0$ *and positive definite diagonal matrices* Γ, Γ_O *such that (40), (65), (111), (112) solve the Semiglobal Leader Following problem with partial information for (1), (2) in the sense that the consensus error e, associated to (1), (2) and with initial values in* $\Omega^{\times N}$*, satisfies* $\limsup_{t\to+\infty} \|e(t)\| \le \alpha(\|d\|_\infty)$ *for some* $\alpha \in \mathcal{K}_\infty$.

9 | CONCLUSIONS

In the present paper sufficient conditions were given for the leader following consensus problem, assuming that the dynamics describing the agents are nonlinear and incrementally homogeneous. The study has been performed by assuming local full information and, secondly, local partial state information and finally recovering the leader's state information with a observer. The leader's trajectories are assumed bounded with possibly unknown containing compact set. An output feedback controller was proposed, using a state feedback controller coupled with high-gain observers which estimate the consensus errors and the leader's state. If the compact set in which the leader's trajectories are contained is unknown, the observer for the leader's state is an high-gain observer with adapted gains. Adaptation is used to estimate the magnitude of the leader's trajectories. Heterogeneous agents and robustness issues were also discussed. Further study will devoted to the leader-following consensus for a non-compact set of leader's trajectories.

□

APPENDIX

Proof of the h.u.b. of ρ.

Due to assumption (A2) and Lemma 4 with $\mathfrak{d} := \mathfrak{f}$ and $\mathfrak{h} := \hat{\mathfrak{f}}$, $\overline{F}$, defined in (36), is h.u.b. with quadruple $\left(\mathfrak{r} \otimes \mathbf{1}_{N+1}, \mathfrak{r} \otimes \mathbf{1}_N, \hat{\mathfrak{f}} \otimes \mathbf{1}_N, \overline{\Phi}\right)$, with $\overline{\Phi}$ is given in (36).

Using now the left inequality in (39), the composition and shifting rules in[1] with $\overline{Z}^{-1} = (I_{Nn} - \overline{A}^T\bar{H})^{-1} = \sum_{j=0}^{n-1}(\overline{A}^T\overline{H})^j$ and $(I_{Nn} - \overline{A}^T\overline{\Gamma})^{-1} = \sum_{j=0}^{n-1}(\overline{A}^T\overline{\Gamma})^j$, we find that the following functions are h.u.b.:

- with the assumption that $\mathfrak{d}_j \leq \mathfrak{d}_{j+1}$, $j \in [1, n-1]$ in (39), (42): $\xi \to \varphi_1(\xi) = \overline{Z}^{-1}\xi$ with quadruple $\left(\mathfrak{r}\otimes \mathbf{1}_N, (\mathfrak{r}+\mathfrak{f})\otimes \mathbf{1}_N, -\mathfrak{f}\otimes \mathbf{1}_N, (I_{Nn} - \overline{A}^T\overline{\Gamma})^{-1}\right)$;
- with the assumption that $\hat{\mathfrak{f}}_1 = \mathfrak{d}_1$ and $\hat{\mathfrak{f}}_j \leq \mathfrak{d}_j \leq \hat{\mathfrak{f}}_{j+1} \leq \mathfrak{d}_{j+1}$, $j \in [1, n-1]$ in (39), (42), $(x^{(0)}, \hat{e}) \to \varphi_2(x^{(0)}, \hat{e}) = (x^{(0)}, \overline{Z}\hat{e})$ with quadruple $\left(\mathfrak{r}\otimes \mathbf{1}_{N+1}, (\mathfrak{r}-\mathfrak{f})\otimes \mathbf{1}_{N+1}, \mathfrak{f}\otimes \mathbf{1}_{N+1}, \mathrm{diag}\{I_n, I_{Nn} + \overline{A}^T\overline{\Gamma}\}\right)$;
- with the assumption that $\mathfrak{d}_j \leq \mathfrak{d}_{j+1}$, $j \in [1, n-1]$ in (39), (42), $\hat{e} \to \varphi_3(\hat{e}) = (\overline{A} - \overline{A}^T\overline{H}^2)\hat{e}$ with quadruple $\left(\mathfrak{r}\otimes \mathbf{1}_N, (\mathfrak{r}+\mathfrak{f})\otimes \mathbf{1}_N, \mathfrak{f}\otimes \mathbf{1}_N, \overline{A} + \overline{A}^T\overline{\Gamma}^2\right)$.

Furthermore $\|\epsilon^{-\mathfrak{r}\otimes\mathbf{1}_N} \diamond (\overline{Z}(\epsilon^{\mathfrak{r}\otimes\mathbf{1}_N} \diamond \hat{e}))\| \leq \|(I_{Nn} + \overline{A}^T\overline{\Gamma})\hat{e}\|$ for all $\hat{e}$ and $\epsilon > 0$. By the composition rule in [1], we conclude that the following composite functions are i.h.u.b.:

- $\hat{e} \to (\varphi_1 \circ \varphi_3)(\hat{e})$ with quadruple $\left(\mathfrak{r}\otimes \mathbf{1}_N, (\mathfrak{r}+\mathfrak{f})\otimes \mathbf{1}_N, \mathfrak{f}\otimes \mathbf{1}_N, \Theta\right)$, where $\Theta = (I_{Nn} + \overline{A}^T\overline{\Gamma})^{-1}(\overline{A} + \overline{A}^T\overline{\Gamma}^2)$;
- $(x^{(0)}, \hat{e}) \to (\varphi_1 \circ \overline{F} \circ \varphi_2)(x^{(0)}, \hat{e})$ with quadruple $\left(\mathfrak{r}\otimes \mathbf{1}_{N+1}, (\mathfrak{r}+\mathfrak{f})\otimes \mathbf{1}_N, \mathfrak{f}\otimes \mathbf{1}_{N+1}, \hat{R}\right)$, where $\hat{R}$ are defined in (36).

This concludes the proof.

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