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Quantum finite W-algebras for $\mathfrak{gl}_N$

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Introduction

Quantum finite W-algebras were first studied by B. Kostant at the end of the $'70$s. They are associative algebras built from a reductive or semisimple complex Lie algebra $\mathfrak{g}$ and a nilpotent element $f \in \mathfrak{g}$ through a procedure of Hamiltonian reduction.

Kostant ([Ko78]) only worked on W-algebras for principal nilpotent elements, but his work was generalized shortly after by D. Lynch ([Ly79]) to nilpotent elements with even Dynkin gradings. More recent and general results are due for instance to A. Premet ([Pr02], [Pr07]) or W. Gan and V. Ginzburg ([GG02]).

The same procedure of Hamiltonian reduction can be applied in the different context of Poisson algebras and Poisson vertex algebra to give rise to other two family of W-algebras, classical finite and classical affine respectively. A fourth family, the quantum affine W-algebra can on the other hand only be defined through a cohomological approach.

These four families, introduced and studied separately, and with different purposes, are connected by procedures that we call affinization, quantization, classical limit and Zhu functor. For instance, the classical cases can be obtained as classical limits of the corresponding quantum W-algebras. Keeping this complete picture in mind drives us to compare differences and similarities and to develop, as much as possible, a symmetric theory in each one of these cases. This goal also motivated the work of my thesis.

My first approach with W-algebras goes back to my master's thesis. Back then, with [Wa11] as a guiding reference, I learned about (quantum finite) W-algebras, their properties and their representation theory. It is remarkable for instance that various equivalent ([DCSHK]) definitions for the same W-algebra $W(\mathfrak{g}, f)$ can be given, but the quantum finite W-algebras ultimately depend only on the nilpotent orbits in $\mathfrak{g}$. This is obtained through the results of J. Brundan and S. Goodwin, showing that W-algebras associated with different gradings but the same nilpotent f are isomorphic ([BG05]), and through results of Gan and Ginzburg showing the independece of $W(\mathfrak{g}, f)$ from the choice of the isotropic subspace $\mathfrak{l}$ ([GG02]).

On the representation side, two beautiful theorems relate the representation of quantum finite W-algebras and other known objects: the first one with a subcategory of $U(\mathfrak{g})$-modules called Whittaker modules ([Sk02]), and the second with the affine degenerate Hecke algebra, in a generalizaton of the well-known Schur-Weyl duality ([BK08b]). The last one only holds for $\mathfrak{g} = \mathfrak{gl}_N$.

From the very beginning though, the issue of describing this associative algebra structure in full detail was clear. We can easily describe two particular cases, the W-algebra associated with the zero nilpotent element and the W-algebra associated with the principal nilpotent element. In fact, it follows from the definition that $W(\mathfrak{g}, 0) \cong U(\mathfrak{g})$, while, by a result of Kostant-Kazhdan, $W(\mathfrak{g}, f)$ is isomorphic to the center of $U(\mathfrak{g})$ when f is principal nilpotent.

As we know, the nilpotent elements of $\mathfrak{gl}_N$ are uniquely determined by partition of N, namely by the sizes of the Jordan blocks of the nilpotent: the element 0 corresponds to the partition (1^N), and the principal nilpotent f corresponds to the partition (N). In-between these extreme nilpotent cases lie all other W-algebras, whose structure is oftentimes obscure. With some restrictions on the partitions, similar results also hold for the other classical algebras, $\mathfrak{sp}_{2N}$, $\mathfrak{so}_{2N}$ and $\mathfrak{so}_{2N+1}$.

It is starting from these premises that at the beginning of my PhD program, the common goal that my advisor Alberto De Sole and I had in mind was to shed some light on the structure of quantum finite W-algebras, especially trying to get a better picture of generators and relations.

This was also motivated by the results of A. De Sole, V. Kac and D. Valeri in the context of classical affine W-algebras; we made various attempts to obtain explicit formulas for the generators mirroring the construction in [DSKV14], where the authors exploited the theory of Poisson vertex algebras to explicitly compute generators and their commutation relations for the classical affine W-algebras attached to minimal and short nilpotent elements.

When this approach showed its limits, we focused our attention on the relationship between quantum finite W-algebras and Yangians. This relationship in the case of $\mathfrak{g} = \mathfrak{gl}_N$ and a rectangular nilpotent element had been highlighted by many authors, as for instance Drinfeld [Dr88] or Ragoucy and Sorba [RS99]. However, the most significant contribution comes from the work of J. Brundan and A. Kleshchev, who in a series of papers ([BK05]-[BK08a]) deeply study quantum finite W-algebras for $\mathfrak{gl}_N$ and establish an isomorphism between a certain subquotient of a Yangian Y_n and the W-algebra $W(\mathfrak{gl}_N, f)$. This subquotient is called truncated shifted Yangian, and it depends on combinatorial parameters attached to the W-algebra, namely a good $\frac{1}{2}\mathbb{Z}$- grading and the partition corresponding to the nilpotent element f.

We then noticed a major resemblance between the defining relation for a Yangian and the defining relation of a pseudodifferential operator of Adler type, which in the most recent works of A. De Sole, V. Kac and D. Valeri aided the development of a new method for constructing integrable Hamiltonian hierarchies of Lax type equations. As an application, by showing that all classical affine W-algebras carry such a hierarchy, they were able to explicitly construct generators and to compute their λ-brackets.

Therefore, in analogy with the results obtained in the classical affine case, we construct an $r_1 \times r_1$ matrix of Yangian type, where r_1 is the number of maximal parts of the partition corresponding to f, encoding the generators and relations of the W-algebra.

In the case when Γ is a Dynkin grading and $\mathfrak{l} = 0$, such matrix $L(z)$ was defined in [DSKV16c], and it was there proved that it encodes the whole structure of the W-algebra. It is here defined in all generality in Definition 2.2.1.

Instead, we here describe it in an example. In the case of $\mathfrak{gl}_3$ with partition $(2, 1)$, and the even $\frac{1}{2}\mathbb{Z}$-grading whose associated pyramid is right-aligned we have for instance

$$L(z) := \begin{pmatrix} z + e_{11} & \boxed{e_{21}} & e_{31} \\ 1 & z + e_{22} - 2 & 0 \\ e_{13} & e_{23} & z + e_{33} \end{pmatrix} =$$

$$= e_{21} - \begin{pmatrix} z + e_{11} & e_{31} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e_{13} & z + e_{33} \end{pmatrix}^{-1} \begin{pmatrix} z + e_{22} - 2 \\ e_{23} \end{pmatrix},$$

where e_{ij} is the elementary matrix having 1 in position (i, j) and 0 elsewhere. In this special case of a right-aligned pyramid, we can prove that the entries of the matrix $L(z)$ satisfy a certain recursive relation (Theorem 2.2.2).

By a result of Premet ([Pr02]), there exists a choice of generators for $W(\mathfrak{g}, f)$ which depend on the centralizer $\mathfrak{g}^f$ of f such that they are compatible with the Kazhdan filtration on $W(\mathfrak{g}, f)$.

In [DSKV16c] the authors construct a matrix $W(z)$, whose entries are polynomials with Premet's generators, and conjecture that the matrix $L(z)$ can be obtained as a generalized quasideterminant of $W(z)$. Note that the entries of $L(z)$ are formal Laurent series in z^{-1}, and are not suitable to give a finite set of generators for $W(\mathfrak{g}, f)$, while the entries of $W(z)$ are polynomials in z. The description of the generators given by Brundan and Kleshchev is also not the most suitable for the conjecture. The analogue of this conjecture has a positive answer for any nilpotent in the classical affine case ([DSKV16b]). For the quantum finite case, the conjecture is tested in [DSKV16c] for the special cases of rectangular and minimal nilpotent elements.

The main result of this thesis is the proof this conjecture, working under the broader assumption of a generic good $\frac{1}{2}\mathbb{Z}$-grading for f and an arbitrary isotropic subspace $\mathfrak{l} \subset \mathfrak{g}_{\frac{1}{2}}$.

When the $\frac{1}{2}\mathbb{Z}$-grading is even, we define the $r \times r$ matrix $W(z)$, where r is the number of parts of the partition, through a recursive formula (Definition 3.2.1). For instance, in the case of $N = 3$ with partition $(2, 1)$ and grading as above, $W(z)$ is the 2×2 matrix with entries

$$W_{11}(z) = -z^2 - z(e_{11} + e_{22} - 2) + e_{21} - e_{11}(e_{22} - 2) + e_{31}e_{13}$$

$$W_{12}(z) = e_{31}$$

$$W_{21}(z) = e_{23} - e_{13}(e_{22} - 2) + e_{33}e_{13}$$

$$W_{22}(z) = z + e_{33}\,.$$

It is by studying both the recursion for $L(z)$ and the recursion for $W(z)$ that we are able to prove the conjecture in the case of an aligned pyramid. In fact, the recursive formula defining $W(z)$ partially agrees, but doesn't completely coincide, with the recursive relation that we have proved for $L(z)$. However, for a particular quasideterminant of the matrix $W(z)$ the same recursion of the matrix $L(z)$ holds, proving the conjecture. In the case of $N = 3$ and $(2, 1)$ as above, a simple computation shows that

$$L(z) = \begin{pmatrix} \boxed{W_{11}(z)} & W_{12}(z) \\ W_{21}(z) & W_{22}(z) \end{pmatrix} = -z^2 - z(e_{11} + e_{22} - 2) + e_{21} - e_{11}(e_{22} - 2) \\ + e_{31}e_{13} - e_{31}(z + e_{33})^{-1}(e_{23} - e_{13}(e_{22} - 2) + e_{33}e_{13}) \,.$$

The generalization of these results to arbitrary good $\frac{1}{2}\mathbb{Z}$-gradings is not straightforward, because these recursive constructions are not always possible. The same definition of $W(z)$ loses meaning when the grading is not even.

However, we use general results about the structure of a W-algebra to overcome this obstacle. Given an arbitrary good grading Γ, by repeatedly changing the isotropic subspace $\mathfrak{l}$, we build a chain of adjacent good gradings for f connecting Γ to the grading whose associated pyramid is right-aligned. We then use results of [BG05] and [GG02] to describe associative algebra isomorphisms between the W-algebras associated to the different gradings of the chain, and finally prove the conjecture.

Similarly, by an analysis of the recursion we prove that the generators on the right-aligned case are in Premet's form, and through a similar machinery we can extend this result in all generality.

The outline of the thesis is as follows:

In Chapter 1 we begin with a reminder of some basic notions that we will use throughout the work, and we fix the notation. Besides the concept of good $\frac{1}{2}\mathbb{Z}$-gradings and pyramids, for which we analyze the procedure of column removal that will be crucial further on, we define our main object, the quantum finite W-algebra $W(\mathfrak{g}, f)$, and recall the useful Kazhdan filtration and Rees algebra. Finally, after a review of the theory of quasideterminants we introduce the concept of an operator of Yangian type, which already appeared in [DSKV16c] and [DSKV17b].

In Chapter 2 we put the bases of our construction. First, we define a matrix $T(z)$ which will be of help in the proof of some properties later on, and most importantly sets a bridge between our construction and the one of Brundan and Kleshchev ([BK05]-[BK08a]). This relationship will be analyzed at the end of the chapter. Then, starting from the same combinatorial data associated with a W-algebra, we will construct a matrix $L(z)$ which we will then prove to be of Yangian type for the W-algebra (Theorems 3.3.1 and 3.3.2).

In Chapter 3 we introduce De Sole, Kac and Valeri's conjecture (Conjecture 3.1.1) for a finite set of generators for a W-algebra in the case of a Dynkin $\frac{1}{2}\mathbb{Z}$-grading and $\mathfrak{l} = 0$. We then recursively define a matrix with polynomial entries that is a suitable candidate to prove the conjecture. Under some restrictions, we are able to prove that the coefficients of the matrix $W(z)$ lie in the W-algebra.

In Chapter 4 we begin by proving the core of Conjecture 3.1.1, namely the relationship between a W-algebra and the operator of Yangian type $L(z)$ associated with the same data, in the particular case when the pyramid attached to the $\frac{1}{2}\mathbb{Z}$-is aligned (Theorem 4.1.1). The rest of the chapter is dedicated to extend these results to the general case; this is aided by an analysis of the properties of adjacent gradings and isotropic subspaces. A first attempt allows us, still under some restrictions, to prove an analogue recursive definition of the matrix $W(z)$, that however doesn't completely solve our problem. In the end, we will build a machinery that will allow us, starting from the results in Proposition 3.3.1 and Theorem 4.1.1 to finally give a positive answer to this conjecture for a general pair of good $\frac{1}{2}\mathbb{Z}$-grading for $\mathfrak{g}$ and an isotropic $\mathfrak{l}$ non-necessarily zero.

Chapter 1

Basic notions and definitions

1.1 Setup and notation

Throughout the thesis, we let $\mathfrak{g} = \mathfrak{gl}_N$ and $(x|y) = \mathrm{tr}(xy)$, where xy denotes the usual matrix multiplication. Most results can be stated and proved for arbitrary reductive $\mathfrak{g}$, but we will be interested in the case $\mathfrak{g} = \mathfrak{gl}_N$.

1.1.1 Good $\frac{1}{2}\mathbb{Z}$-gradings and pyramids

A $\frac{1}{2}\mathbb{Z}$-grading Γ for $\mathfrak{g}$ is a decomposition $\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for every i, $j \in \frac{1}{2}\mathbb{Z}$. We shall denote, for $j \in \frac{1}{2}\mathbb{Z}$, $\mathfrak{g}_{\geq j} = \bigoplus_{k \geq j} \mathfrak{g}_k$, $\mathfrak{g}_{>j} = \bigoplus_{k>j} \mathfrak{g}_k$, $\mathfrak{g}_{\leq j} = \bigoplus_{k \leq j} \mathfrak{g}_k$ and $\mathfrak{g}_{<j} = \bigoplus_{k<j} \mathfrak{g}_k$.

Moreover, for each $j \in \frac{1}{2}\mathbb{Z}$, let us denote by

$$\pi_j : \mathfrak{g} \twoheadrightarrow \mathfrak{g}_j \tag{1.1.1}$$

the projection with respect to the direct sum decomposition given by Γ. We will use the shorthands

$$\pi_{>j} : \mathfrak{g} \twoheadrightarrow \mathfrak{g}_{>j}, \quad \pi_{<j} : \mathfrak{g} \twoheadrightarrow \mathfrak{g}_{<j}, \quad \pi_{\geq j} : \mathfrak{g} \twoheadrightarrow \mathfrak{g}_{\geq j}, \quad \pi_{\leq j} : \mathfrak{g} \twoheadrightarrow \mathfrak{g}_{\leq j}\,.$$

Definition 1.1.1. We say that a $\frac{1}{2}\mathbb{Z}$-grading $\Gamma : \mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ is *even* if $\mathfrak{g}_j = 0$ for all $j \in \frac{1}{2} + \mathbb{Z}$. We say that a $\frac{1}{2}\mathbb{Z}$-grading Γ is *odd* if it is not even.

Example 1.1.1. Let $0 \neq f \in \mathfrak{g}$ be a nilpotent element. By the Jacobson-Morozov Theorem we know that there exists an $\mathfrak{sl}_2$-triple (e, h, f) associated to it such that f is the nilnegative element, e is the nilpositive element and h is a semisimple element. By the representation theory of $\mathfrak{sl}_2$, the eigenspace decomposition of $\mathfrak{g}$ relative to the adjoint action of $x = \frac{1}{2}h$ gives a $\frac{1}{2}\mathbb{Z}$-grading for $\mathfrak{g}$. We call this a *Dynkin* grading.

Definition 1.1.2. Let $f \in \mathfrak{g}$ be a nilpotent element. A *good* $\frac{1}{2}\mathbb{Z}$*-grading* for f is a $\frac{1}{2}\mathbb{Z}$-grading of $\mathfrak{g}$, $\Gamma : \mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ such that the following holds:

(i) $f \in \mathfrak{g}_{-1}$;

(ii) $\mathrm{ad}\, f : \mathfrak{g}_j \longrightarrow \mathfrak{g}_{j-1}$ is injective for $j \geq \frac{1}{2}$;

(iii) $\mathrm{ad}\, f : \mathfrak{g}_j \longrightarrow \mathfrak{g}_{j-1}$ is surjective for $j \leq \frac{1}{2}$.

We moreover ask that the center of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$.

In particular, if Γ is a good $\frac{1}{2}\mathbb{Z}$-grading for f, then $\mathrm{ad}f : \mathfrak{g}_{\frac{1}{2}} \longrightarrow \mathfrak{g}_{-\frac{1}{2}}$ is a bijection. It can be proved [Wa11, Lemma 8] that properties (ii) and (iii) are equivalent for every $\frac{1}{2}\mathbb{Z}$-grading of $\mathfrak{g}$.

Example 1.1.2. A Dynkin $\frac{1}{2}\mathbb{Z}$-grading is always a good $\frac{1}{2}\mathbb{Z}$-grading, since properties $(i) - (iii)$ derive from the definition of $\mathfrak{sl}_2$-triple and from the theory of maximal weight of $\mathfrak{sl}_2$ applied to $\mathfrak{g}$, seen as a $\mathfrak{sl}_2$-module under the adjoint action.

Proposition 1.1.1. *[Wa11, Lemma 4, Proposition 5] The following properties hold for every good $\frac{1}{2}\mathbb{Z}$-grading $\Gamma : \mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$:*

(a) There exists a semisimple element $h_\Gamma \in \mathfrak{g}$ such that Γ coincides with the eigenspace decomposition relative to the adjoint action of h_Γ, namely $\mathfrak{g}_j = \{x \in \mathfrak{g} \mid [h_\Gamma, x] = jx\}$;

(b) $\mathfrak{g}^f \subset \bigoplus_{j\leq 0} \mathfrak{g}_j$, where g^f is the centralizer of f in $\mathfrak{g}$;

(c) $(\mathfrak{g}_i|\mathfrak{g}_j) = 0$ unless $i + j = 0$. Namely, $\mathfrak{g}_{-i}$ is the dual of $\mathfrak{g}_i$ with respect to the bilinear form $(\cdot|\cdot)$;

(d) dim $\mathfrak{g}^f$ = dim $\mathfrak{g}_0$ + dim $\mathfrak{g}_{\frac{1}{2}}$.

Let $\lambda = (p_1 \geq p_2 \geq \ldots \geq p_r)$ be a partition of N.

Definition 1.1.3. A *pyramid* p of size N associated with the partition $\lambda = (p_1 \geq p_2 \geq \ldots \geq p_r)$ (namely, of shape λ) is a collection of 1×1 boxes arranged in r rows with centers $(i, j) \in \frac{1}{2}\mathbb{Z} \times \mathbb{Z}_{>0}$ such that

(i) For the bottom row, $j = 1$. The rows are then ordered increasingly from bottom to top: $j = 1, \ldots, r$;

(ii) Row j consists of p_j boxes;

(iii) For each row, the x-coordinates of the centers of the boxes form an arithmetic progression with common difference 1: $f_j, f_j + 1, \ldots, l_j$, where f_j (resp. l_j) is the x-coordinate of the center of the leftmost (resp. rightmost) box of row j;

(iv) The bottom row is symmetric with respect to the y-axis: $\frac{p_1-1}{2} \equiv l_1 = -f_1$;

(v) It graphically satisfies the condition of pyramid, namely $f_{j+1} \geq f_j$, $l_{j+1} \leq l_j$ for each $j = 1, \ldots, r - 1$.

To visualize this definition, here's an example for a pyramid associated with the partition $(6, 4, 2)$ of 12:

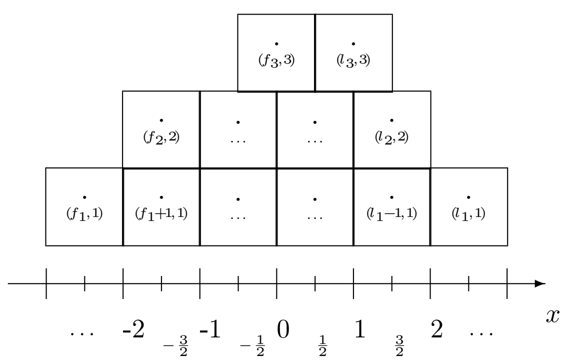


Figure 1.1: A pyramid of size 12 and shape $\lambda = (6, 4, 2)$

We shall denote by $t_1 \leq t_2 \leq \ldots \leq r \geq \ldots \geq s_2 \geq s_1$ the unimodal sequence of column lengths in p, from left to right, the total number of columns being p_1. We shall also denote by r_1 be the multiplicity of p_1 in the partition; as a consequence, the bottom of the pyramid p will therefore be a rectangular block of size $p_1 \times r_1$ that will also be invariant under the change of good $\frac{1}{2}\mathbb{Z}$-grading associated with the same partition of N. It is also clear that $r_1 = \min(t_1, s_1)$, namely r_1 is the height of the shortest column of p.

Definition 1.1.4. We say that a pyramid p is *even* if every box in the pyramid lies exactly on a box of the underlying row, and that it is *odd* if there is at least a box lying in between two boxes of the underlying row.

We say that a pyramid p is *symmetric* if it is symmetric with respect to the y-axis.

Given a pyramid p, we shall introduce, for each $1 \leq i, j \leq r$, the following (semi)-integers:

$$s_{ij} = \begin{cases} \text{difference of the number of boxes between row } i \text{ and row } j \text{ on the RHS of } p, & \text{if } i \leq j\,, \\ \text{difference of the number of boxes between row } i \text{ and row } j \text{ on the LHS of } p, & \text{if } i \geq j\,, \end{cases} \tag{1.1.2}$$

where the RHS (resp. LHS) of p corresponds to the boxes whose center has a non-negative (resp. non-positive) x-coordinate, and half boxes are counted. For the pyramid in Figure 1.1 above, the s_{ij}'s are displayed in the following matrix:

$$\begin{bmatrix} 0 & 1 & \frac{3}{2} \\ 1 & 0 & \frac{1}{2} \\ \frac{5}{2} & \frac{3}{2} & 0 \end{bmatrix}.$$

We have in particular,

$$s_{1i} = \text{difference of the number of boxes between row 1 and row } i \text{ on the RHS of } p\,,$$
$$s_{i1} = \text{difference of the number of boxes between row 1 and row } i \text{ on the LHS of } p\,.$$

Definition 1.1.5. Let $1 \leq i \leq r$. We say that row i is *integer* if $s_{i1} \in \mathbb{N}$, and that row i is *semi-integer* if $s_{i1} \in \frac{1}{2} + \mathbb{N}$. Note that this is equivalent to checking the (semi-)integrity of s_{1i} instead.

Let $\mathcal{T}$ be the following index set (of cardinality N)

$$\mathcal{T} = \left\{(i,h) \in \mathbb{Z}^2_{>0} \mid 1 \leq i \leq r,\, 1 \leq h \leq p_i\right\}. \tag{1.1.3}$$

Let p be a pyramid of size N associated with the partition $(p_1 \geq p_2 \geq \ldots \geq p_r)$. We label the boxes with elements (i,h) of $\mathcal{T}$ from bottom to top and from right to left (but any other bijective numbering with a set of cardinality N will do). For $N = 6$ and partition $(3,2,1)$, for instance, some of the possible (numbered) pyramids are presented in Figure 1.2,

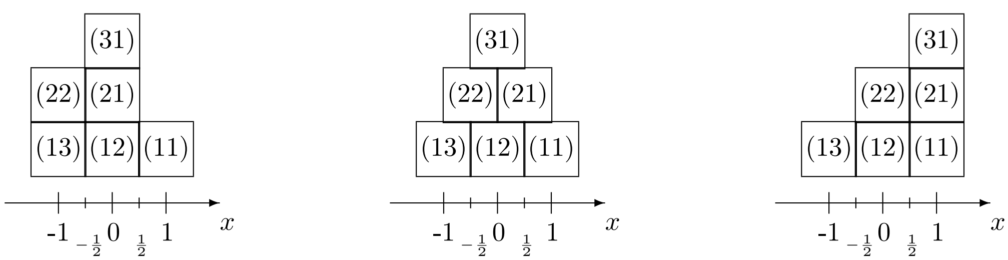


Figure 1.2: Pyramids for $N = 6$ and partition $(3,2,1)$

Elashvili and Kac in [EK05] provide a classification of good $\frac{1}{2}\mathbb{Z}$-gradings[1] for the classical Lie algebras $\mathfrak{gl}_N$, $\mathfrak{sp}_{2N}$, $\mathfrak{so}_N$. In particular, for the case of $\mathfrak{g} = \mathfrak{gl}_N$ which is the one we are eventually interested in, the following theorem holds.

Theorem 1.1.1. *[EK05, Section 4] Let $\mathfrak{g} = \mathfrak{gl}_N$ and let $\lambda = (p_1 \geq p_2 \geq \ldots \geq p_r)$ be a partition of N. Then, there exists a bijection between the set of pyramids of size N and shape λ, and the set of good $\frac{1}{2}\mathbb{Z}$-gradings for a nilpotent element $f \in \mathcal{O}_{J(\lambda)}$, up to GL_N-conjugation.*

Here, $J(\lambda)$ is the nilpotent $N \times N$ matrix in Jordan blocks form associated with the partition λ. The GL_N action on a grading $\Gamma : \mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ is given by: $A \cdot \Gamma = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} A\mathfrak{g}_j A^{-1}$, for any $A \in \mathrm{GL}_N$.

To understand this correspondence, let us fix some notation that we will use throughout the thesis.

Let V be the N-th dimensional vector space over $\mathbb{C}$ with basis

$$\{e_{(i,h)}\}_{(i,h) \in \mathcal{T}}\,.$$

[1]Actually, Elashvili and Kac work with good $\mathbb{Z}$-gradings; however, the two objects are in bijection by use of a dilation/contraction of factor $\frac{1}{2}$.

Thus, the Lie algebra $\mathfrak{gl}_N \cong \mathfrak{gl}(V)$ has a basis consisting of the elementary matrices

$$\{e_{(i,h)(j,k)}\}_{(i,h),(j,k)\in\mathcal{T}} \,. \tag{1.1.4}$$

Here and further, in order to avoid confusion, we denote by $e_{(i,h)(j,k)}$ the elementary matrices viewed as elements of the Lie algebra $\mathfrak{gl}_N$ and by $E_{(i,h)(j,k)}$ the same elementary matrices (having 1 in position $\big((i,h),(j,k)\big)$ and 0 elsewhere) viewed as elements of $\mathrm{Mat}_{N\times N}\mathbb{C}$. Moreover, we can think of each matrix $e_{(i,h)(j,k)}$ as an arrow on the pyramid pointing from the box labeled (j,k) to the box labeled (i,h).

Given a pyramid p of size N, we can associate to it the nilpotent element

$$f = J(\lambda) = \sum_{\substack{(i,h)\in\mathcal{T}\\ h<p_i}} e_{(i,h+1)(i,h)} \tag{1.1.5}$$

and the semisimple endomorphism $h_p = \sum_{(i,h)\in\mathcal{T}} x(ih)e_{(i,h)(i,h)}$, where $x(ih)$ denotes the x-coordinate of the center of the box labeled (i,h). Thus, from the pyramid p we we can define a $\frac{1}{2}\mathbb{Z}$-grading for $\mathfrak{gl}_N$ which is given by the eigenspace decomposition of $\mathfrak{gl}_N$ relative to the adjoint action of h_p:

$$\Gamma_p : \mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_j \,, \qquad \mathfrak{g}_j = \{x\in\mathfrak{g} \mid [h_p,x] = jx\} \,. \tag{1.1.6}$$

According to this definition,

$$\deg e_{(i,h)(j,k)} = x(ih) - x(jk) \,. \tag{1.1.7}$$

We can prove that the $\frac{1}{2}\mathbb{Z}$-grading from (1.1.6) actually is a good $\frac{1}{2}$-grading for f; from Equation (1.1.5) and Equation 1.1.7 we have for instance $f \in \mathfrak{g}_{-1}$. It is in fact clear that $x(i,h+1) = x(ih) - 1$ for any $(i,h)\in\mathcal{T}$.

The one-to-one correspondence from Theorem 1.1.1 therefore sends $p \mapsto \Gamma_p$. See [EK05, Section 4] or [Wa11, Theorem 40] on how to construct the inverse map, sending a given good $\frac{1}{2}\mathbb{Z}$-grading Γ for a nilpotent $f = J(\lambda)$ (where $\lambda = (p_1 \geq \ldots \geq p_r)$ is a partition of N) to its associated pyramid p.

Some remarks about the grading defined in (1.1.6):

- By (1.1.2), $x(ih) = -(s_{1i}+h) + \frac{p_1+1}{2}$, and Equation (1.1.7) becomes
$$\deg e_{(i,h)(j,k)} = s_{1j} - s_{1i} + k - h \,. \tag{1.1.8}$$
- Note that $d = p_1 - 1$ is the maximal degree of the grading, and it only depends on the given partition of N;
- Note that $p_1 = s_{1i} + p_i + s_{i1}$ for every $1 \leq i \leq r$. More generally, $|p_i - p_j| = s_{ij} + s_{ji}$. Moreover, for any $1 \leq i \leq r$, $s_{i1}, s_{1i} \in \mathbb{N}$ if p is even; in general though $s_{i1}, s_{1i} \in \frac{1}{2}\mathbb{N}$;
- Let $x := h_p \in \mathfrak{g}$ be the semisimple element which determines the grading and let X be the corresponding element in $\mathrm{Mat}_{N\times N}\mathbb{C} \cong \mathrm{End}\, V$. Then, the X-eigenspace decomposition of V is $V = \bigoplus_{k\in\frac{1}{2}\mathbb{Z}} V[k]$. Note that $\frac{d}{2}$ is the largest x-coordinate for the center of the boxes of the pyramid, hence it is the largest X-eigenvalue for V. The corresponding $\mathrm{ad}X$-eigenspace decomposition of $\mathrm{End}\, V$ is
$$\mathrm{End}\, V = \bigoplus_{k\in\frac{1}{2}\mathbb{Z}} (\mathrm{End}\, V)[k] \,.$$
Here, d is the largest eigenvalue for the $\mathrm{ad}X$-action.

Note that under the bijection of Theorem 1.1.1, even (resp. odd) pyramids correspond to even (resp. odd) $\frac{1}{2}\mathbb{Z}$-gradings. Moreover, symmetric pyramids correspond to Dynkin $\frac{1}{2}\mathbb{Z}$-gradings.

Another useful definition is the following:

Definition 1.1.6. A pyramid p is *right-aligned* if $x(i,1) = \frac{d}{2}$, for every $1 \leq i \leq r$. Similarly, a pyramid p is *left-aligned* if $x(j,p_j) = -\frac{d}{2}$, for every $1 \leq j \leq r$. Note that both cases force p to be even. Moreover, $s_{1i} = 0$ (resp. $s_{i1} = 0$) for every $1 \leq i \leq r$ when p is right-(resp. left-)aligned.

Remark 1.1.1. In [DSKV16c] the authors consider the specific case of a symmetric pyramid for $\mathfrak{g} = \mathfrak{gl}_N$. Thus, the x-coordinate of the center of the box labeled (i, h) is

$$\widetilde{x}(ih) = \frac{1}{2}(p_i + 1 - 2h) \,. \tag{1.1.9}$$

Equation (1.1.9) can be obtained by (1.1.8) when the pyramid p is symmetric. Namely, when $s_{i1} = s_{1i}$ for every $1 \leq i \leq r$. In fact, for p symmetric

$$x(ih) = -(s_{1i} + h) + \frac{p_1 + 1}{2} = -(s_{1i} + h) + \frac{p_i + s_{1i} + s_{i1} + 1}{2} = \frac{p_i + 2s_{1i} + 1 - 2s_{1i} - 2h}{2} = \widetilde{x}(ih) \,.$$

1.1.2 Isotropic subspaces, nilpotent subalgebras and other notable elements

Given the nilpotent element $f \in \mathfrak{g}$, let $(f|\cdot) \in \mathfrak{g}^*$ be the dual of f with respect to the trace form $(\cdot|\cdot)$. Let us now define a bilinear form ω on $\mathfrak{g}_{\frac{1}{2}}$:

$$\begin{aligned} \omega : \mathfrak{g}_{\frac{1}{2}} \times \mathfrak{g}_{\frac{1}{2}} &\longrightarrow \mathbb{C} \\ \omega(x, y) &:= (f|[x, y]) \,. \end{aligned} \tag{1.1.10}$$

It is well known that the following properties hold for ω:

Lemma 1.1.1. *The bilinear form ω on $\mathfrak{g}_{\frac{1}{2}}$ is antisymmetric and non-degenerate.*

Proof. Antisymmetry is clear by the definition. The non-degeneracy of the trace form on $\mathfrak{g}$ and the bijection $\mathrm{ad}_f : \mathfrak{g}_{\frac{1}{2}} \to \mathfrak{g}_{-\frac{1}{2}}$ together guarantee that ω is non-degenerate. ☐

Choose an isotropic subspace $\mathfrak{l} \subset \mathfrak{g}_{\frac{1}{2}}$ with respect to the bilinear form ω. Let $\mathfrak{l}^c$ be a complementary subspace to $\mathfrak{l}$ in $\mathfrak{g}_{\frac{1}{2}}$, namely $\mathfrak{g}_{\frac{1}{2}} = \mathfrak{l} \oplus \mathfrak{l}^c$, and let $\mathfrak{l}^\perp$ denote its orthogonal complement with respect to the bilinear form ω, i.e. $\omega(\mathfrak{l}, \mathfrak{l}^\perp) = 0$ and $\mathfrak{l}^\perp$ is the maximal subspace of $\mathfrak{g}_{\frac{1}{2}}$ with such property. Note that $\mathfrak{l} \subseteq \mathfrak{l}^\perp$ by definition of isotropic subspace, and moreover $\mathfrak{l} = \mathfrak{l}^\perp$ in the case that $\mathfrak{l}$ is Lagrangian.

Then $\mathfrak{m} = \mathfrak{l} \oplus \mathfrak{g}_{\geq 1}$ and $\mathfrak{n} = \mathfrak{l}^\perp \oplus \mathfrak{g}_{\geq 1}$ are nilpotent subalgebras of $\mathfrak{g}$ such that $\mathfrak{m} \subseteq \mathfrak{n} \subseteq \mathfrak{g}_{\geq \frac{1}{2}}$. Let

$$\pi : \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{m} \cong \mathfrak{l}^c \oplus \mathfrak{g}_{\leq 0} \tag{1.1.11}$$

be the induced quotient map. We shall denote by $\mathfrak{p} := \mathfrak{l}^c \oplus \mathfrak{g}_{\leq 0}$ this complementary subspace to $\mathfrak{m}$ in $\mathfrak{g}$, and by $\pi_{\mathfrak{p}} : \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{m} \cong \mathfrak{p}$ the corresponding projection.

Let

$$I_{\mathfrak{l}} := U(\mathfrak{g})\langle b - (f|b)\rangle_{b \in \mathfrak{m}} \tag{1.1.12}$$

be a left ideal of $U(\mathfrak{g})$. Let $M_{\mathfrak{l}} := U(\mathfrak{g})/I_{\mathfrak{l}}$ be the corresponding quotient. $M_{\mathfrak{l}}$ is a cyclic $U(\mathfrak{g})$-module generated by $\bar{1}_{\mathfrak{l}}$, the image of $1 \in U(\mathfrak{g})$ in the quotient $U(\mathfrak{g})/I_{\mathfrak{l}}$. Clearly, $u\bar{1}_{\mathfrak{l}} = 0$ in $M_{\mathfrak{l}}$ if and only if $u \in I_{\mathfrak{l}}$.

After choosing a PBW basis for $U(\mathfrak{g})$ corresponding to the choice of an ordered basis $\{\mathfrak{g}_{\leq 0}, \mathfrak{l}^c, \mathfrak{l}, \mathfrak{g}_{\geq 1}\}$ for $\mathfrak{g}$, we can identify

$$U(\mathfrak{g})/I_{\mathfrak{l}} \cong U(\mathfrak{g}_{\leq 0}) \otimes F(\mathfrak{l}^c) \,, \tag{1.1.13}$$

where $F(\mathfrak{l}^c)$ denotes the generalized Weyl algebra of $\mathfrak{l}^c$ (see Definition 1.1.7 below). Under this identification, we define

$$\begin{aligned} \rho_{\mathfrak{l}} : U(\mathfrak{g}) &\longrightarrow U(\mathfrak{g}_{\leq 0}) \otimes F(\mathfrak{l}^c) \cong U(\mathfrak{g})/I_{\mathfrak{l}} \\ \mathfrak{g} \ni a &\mapsto \rho_{\mathfrak{l}}(a) = \pi_{\mathfrak{p}}(a) + (f|a) \,. \end{aligned} \tag{1.1.14}$$

the (canonical) quotient map with $\operatorname{Ker} \rho_{\mathfrak{l}} = I_{\mathfrak{l}}$. Note that for $\mathfrak{l} = 0$ we have that ρ_0 coincides with the linear map defined in [DSKV16c, Section 3.2]. In this case, we shall simply denote $\rho := \rho_0$, $I := I_0$, $M := M_0$.

Definition 1.1.7. Let V a finite-dimensional vector space over $\mathbb{C}$, endowed with a (not necessarily non-degenerate) anti-symmetric bilinear form

$$\omega : V \times V \longrightarrow \mathbb{C} \,.$$

We define the *generalized Weyl algebra* of V with respect to the form ω to be the quotient of the tensor algebra of V, $T(V)$, by the two-sided ideal generated by the elements $\langle v\otimes w - w\otimes v - \omega(v,w)\rangle$, $v,\, w\in V$:

$$F(V) := T(V)/\langle v\otimes w - w\otimes v - \omega(v,w)\rangle_{v,w\in V}\,.$$

The following matrices will play an important role in our work:

$$E = \sum_{(i,h),(j,k)\in\mathcal{T}} e_{(j,k)(i,h)} E_{(i,h)(j,k)} \in \mathrm{Mat}_{N\times N} U(\mathfrak{g})\,, \tag{1.1.15}$$

the $N\times N$ matrix with entry $e_{(j,k)(i,h)}\in U(\mathfrak{g})$ in position $\big((i,h),(j,k)\big)$. And

$$D_{\mathfrak{l}} = \sum_{(i,h)\in\mathcal{T}} d_{(ih)} E_{(i,h)(i,h)} \in \mathrm{Mat}_{N\times N}\mathbb{C}\,, \tag{1.1.16}$$

the $N\times N$ scalar matrix with entry $d_{(ih)} = -\dim\,(\mathfrak{m}\, e_{(i,h)})$ in position $\big((i,h),(i,h)\big)$. Note that for $\mathfrak{l}=0$, then $-d_{(ih)}\in\mathbb{Z}$ just counts the number of blocks of the pyramid entirely to the right of the box labeled by (i,h).

Under the map $\rho_{\mathfrak{l}}$ we have $\rho_{\mathfrak{l}}(E) = F + \pi_{\mathfrak{p}}E$, where $F = \sum_{\substack{(i,h)\in\mathcal{T}\\ h<p_i}} E_{(i,h+1)(i,h)}$ is the matrix f viewed as an element of $\mathrm{Mat}_{N\times N}\mathbb{C}$.

We will often use the shorthand $\tilde{e}_{(j,k)(i,h)}$ for the sum $e_{(j,k)(i,h)} + \delta_{(i,h)(j,k)} d_{(ih)}$, namely the $\big((i,h),(j,k)\big)$-th coefficient of $E + D_{\mathfrak{l}}$.

1.1.3 Playing with pyramids: how to remove a column

By Theorem 1.1.1, a good $\frac{1}{2}\mathbb{Z}$-grading Γ for a nilpotent element $f\in\mathfrak{gl}_N$ associated with a partition $\lambda = (p_1 \geq \ldots \geq p_r)$ is in one-to-one correspondence with a pyramid p of size N and shape λ. Removing the leftmost (resp. rightmost) column of the pyramid p, when possible, we are left with a pyramid of size $N - t_1$ (resp. $N - s_1$), that we denote by ${}'p$ (resp. p'). Note that we are allowed to remove the leftmost (resp. rightmost) column of a pyramid p, therefore obtaining still a regular pyramid, only in the case when no block lies not even partially on the column we are about to remove. This is always the case for even pyramids. We are also clearly requiring $p_1 > 1$, namely that the pyramid p doesn't consist of a single column.

Example 1.1.3. Let p_1, p_2 and p_3 be as in Figures 1.3 -1.5:

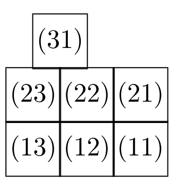


Figure 1.3: p_1

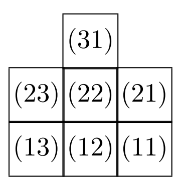


Figure 1.4: p_2

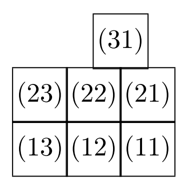


Figure 1.5: p_3

Then, we cannot remove the leftmost column in p_1, while we are allowed to do so in p_2 (which is even). Moreover, we can remove the leftmost column in p_3, even though it is an odd pyramid but, after doing so, we cannot also remove the central column.

We shall say that the pyramid ${}'p$ (resp. p') exists, if it is possible to remove the leftmost (resp. rightmost) column of p. The pyramids ${}'p$ and p', when they exist, are numbered by the index sets

$$\mathcal{T}^{'p} = \big\{(i,h)\in\mathbb{Z}^2_{>0} \mid 1\leq h\leq p_i - 1 \text{ for } 1\leq i\leq t_1, \text{ and } 1\leq h\leq p_i \text{ for } t_1+1\leq i\leq r\big\}$$

and

$$\mathcal{T}^{p'} = \big\{(i,h)\in\mathbb{Z}^2_{>0} \mid 1\leq h\leq p_i - 1 \text{ for } 1\leq i\leq s_1, \text{ and } 1\leq h\leq p_i \text{ for } s_1+1\leq i\leq r\big\}$$

respectively. Moreover, the partition $(p_1, \dots, p_r)$ of N reduces to the partition

$$(p_1 - 1, \dots, p_{t_1} - 1, p_{t_1+1}, \dots, p_r) =: (p_1^{'p}, \dots, p_r^{'p}) \tag{1.1.17}$$

of $N - t_1$ and to the partition

$$(p_1 - 1, \dots, p_{s_1} - 1, p_{s_1+1}, \dots, p_r) =: (p_1^{p'}, \dots, p_r^{p'}) \tag{1.1.18}$$

of $N - s_1$. We shall denote by $f^{'p}$ and $f^{p'}$ respectively the associated nilpotent elements.

The $\frac{1}{2}\mathbb{Z}$-gradings $\Gamma^{'p}$ and $\Gamma^{p'}$ on the pyramids $'p$ and p' are given explicitly as in (1.1.8):

$$\deg_{'p} e_{(i,h)(j,k)} = s_{1j}^{'p} - s_{1i}^{'p} + k - h\,, \qquad \deg_{p'} e_{(i,h)(j,k)} = s_{1j}^{p'} - s_{1i}^{p'} + k - h\,,$$

where $s_{1i}^{'p} = s_{1i}$ and $s_{1i}^{p'} = s_{1i} - \delta_{i>s_1}$ for every $1 \leq i \leq r$.

In Proposition 1.1.2, we describe the embeddings $\mathfrak{gl}_{N-t_1} \hookrightarrow \mathfrak{gl}_N \hookleftarrow \mathfrak{gl}_{N-s_1}$ and prove some of their properties. For instance, the grading defined in (1.1.8) will coincide with the induced $\frac{1}{2}\mathbb{Z}$-grading for the subalgebra $\mathfrak{gl}_{N-t_1}$ (resp. $\mathfrak{gl}_{N-s_1}$) of $\mathfrak{gl}_N$. We will often use the shorthand $\mathfrak{g}^{'p} := \mathfrak{g}_{N-t_1}$ and $\mathfrak{g}^{p'} := \mathfrak{gl}_{N-s_1}$.

Remark 1.1.2. We will add apexes or subscripts to characterize Lie algebras, $\frac{1}{2}\mathbb{Z}$-gradings, maps, subspaces, matrices, etc ... associated to different pyramids or gradings. When nothing is specified, the Lie algebra, grading, map, subspace, matrix, etc ... is associated to the whole pyramid p. In case we need to remove more than one column from the pyramid, we will use $\overbrace{'\cdots'}^{k}p$ or ${}^{k}p$ as a shorthand for the pyramid with its first leftmost k columns removed, and $p\overbrace{'\cdots'}^{k}$ or p^k as a shorthand for the pyramid with its first rightmost k columns removed.

Similarly, we use the notation $\mathfrak{g}^{(1)}$, $\mathfrak{g}^{(2)}$ whenever we will need to distinguish between two different $\frac{1}{2}\mathbb{Z}$-gradings Γ_1, Γ_2 for $\mathfrak{gl}_N$ associated with the same partition $(p_1 \geq p_2 \geq \ldots \geq p_r)$.

Proposition 1.1.2. *Suppose that it is possible to remove the leftmost (resp. rightmost) column from the pyramid p. Define Lie algebra maps*

$$\begin{aligned} &\sigma_l : \mathfrak{gl}_{N-t_1} \longrightarrow \mathfrak{gl}_N\,, && \sigma_r : \mathfrak{gl}_{N-s_1} \longrightarrow \mathfrak{gl}_N\,, \\ &e_{(i,h)(j,k)} \mapsto e_{(i,h)(j,k)}\,, && e_{(i,h)(j,k)} \mapsto e_{(i,h+\delta_{i\leq s_1})(j,k+\delta_{j\leq s_1})}\,, \end{aligned} \tag{1.1.19}$$

and extend them to associative algebra homomorphisms[2]

$$\begin{aligned} &\sigma_l : U(\mathfrak{gl}_{N-t_1}) \longrightarrow U(\mathfrak{gl}_N)\,, && \sigma_r : U(\mathfrak{gl}_{N-s_1}) \longrightarrow U(\mathfrak{gl}_N)\,, \\ &e_{(i,h)(j,k)} \mapsto e_{(i,h)(j,k)}\,, && e_{(i,h)(j,k)} \mapsto e_{(i,h+\delta_{i\leq s_1})(j,k+\delta_{j\leq s_1})} - \delta_{(i,h)(j,k)}s_1\,, \end{aligned} \tag{1.1.20}$$

Then, σ_l and σ_r are injective associative algebra homomorphisms and they induce well-defined quotient maps $\overline{\sigma_l}$ and $\overline{\sigma_r}$ that are injective. Namely, we have the following commutative diagrams

$$\begin{array}{ccc} U(\mathfrak{gl}_{N-t_1}) & \xrightarrow[\sigma_l]{} & U(\mathfrak{g}) \\ {\scriptstyle \rho_{\mathfrak{l}}^{'p}}\downarrow & & \downarrow{\scriptstyle \rho_{\mathfrak{l}}} \\ U(\mathfrak{gl}_{N-t_1})/I_{\mathfrak{l}^{'p}} & \xrightarrow[\overline{\sigma_l}]{} & U(\mathfrak{g})/I_{\mathfrak{l}} \end{array} \tag{1.1.21}$$

and

$$\begin{array}{ccc} U(\mathfrak{gl}_{N-s_1}) & \xrightarrow[\sigma_r]{} & U(\mathfrak{g}) \\ {\scriptstyle \rho_{\mathfrak{l}}^{p'}}\downarrow & & \downarrow{\scriptstyle \rho_{\mathfrak{l}}} \\ U(\mathfrak{gl}_{N-s_1})/I_{\mathfrak{l}^{p'}} & \xrightarrow[\overline{\sigma_r}]{} & U(\mathfrak{g})/I_{\mathfrak{l}} \end{array} \tag{1.1.22}$$

[2] We are actually twisting the extension $\sigma_r : U(\mathfrak{gl}_{N-s_1}) \longrightarrow U(\mathfrak{gl}_N)$, $e_{(i,h)(j,k)} \mapsto e_{(i,h+\delta_{i\leq s_1})(j,k+\delta_{j\leq s_1})}$ with the associative algebra homomorphism $\eta : U(\mathfrak{gl}_N) \longrightarrow U(\mathfrak{gl}_N)$, $e_{(i,h)(j,k)} \mapsto e_{(i,h)(j,k)} - \delta_{(i,h)(j,k)}s_1$.

where $\mathfrak{m}'^p = \mathfrak{l}'^p \oplus (\mathfrak{gl}_{N-t_1})_{\geq 1}$ *and* $\mathfrak{m}^{p'} = \mathfrak{l}^{p'} \oplus (\mathfrak{gl}_{N-s_1})_{\geq 1}$, *for isotropic subspaces* $\mathfrak{l}'^p$ *and* $\mathfrak{l}^{p'}$, *and*

$$I_{\mathfrak{l}'^p} = U(\mathfrak{g}_{N-t_1})\langle b - (f'^p|b)\rangle_{b\in\mathfrak{m}'^p}\,, \qquad I_{\mathfrak{l}^{p'}} = U(\mathfrak{gl}_{N-s_1})\langle b - (f^{p'}|b)\rangle_{b\in\mathfrak{m}^{p'}}\,.$$

Note that if $'p$ *(resp.* p'*) exists, then* $\mathfrak{g}_{\frac{1}{2}}$ *coincides with the image of* $\mathfrak{g}'^p_{\frac{1}{2}}$ *(resp.* $\mathfrak{g}^{p'}_{\frac{1}{2}}$*) under* σ_l *(resp.* σ_r*). We denote by* $\mathfrak{l}'^p$ *(resp.* $\mathfrak{l}^{p'}$*) the preimage of* $\mathfrak{l}$ *in* $g'^p_{\frac{1}{2}}$ *(resp.* $\mathfrak{g}^{p'}_{\frac{1}{2}}$*).*

The commutativity of the diagrams can be restated as $\rho_{\mathfrak{l}} \circ \sigma_l = \overline{\sigma_l} \circ \rho'^p_{\mathfrak{l}}$ *and* $\rho_{\mathfrak{l}} \circ \sigma_r = \overline{\sigma_r} \circ \rho^{p'}_{\mathfrak{l}}$*. Moreover, whenever the composition makes sense*[3] *we have*

$$\sigma_l \circ \sigma_r = \sigma_r \circ \sigma_l : U(\mathfrak{gl}_{N-t_1-s_1}) \longrightarrow U(\mathfrak{gl}_N)\,. \tag{1.1.23}$$

Finally, the algebra homomorphisms σ_l *and* σ_r *commute with the projections:*

$$\sigma_l \circ \pi'^p_k = \pi_k \circ \sigma_l : \mathfrak{g}'^p \longrightarrow \mathfrak{g}_k\,, \qquad \sigma_r \circ \pi'^p_k = \pi_k \circ \sigma_r : \mathfrak{g}^{p'} \longrightarrow \mathfrak{g}_k\,, \tag{1.1.24}$$

for each $k \in \frac{1}{2}\mathbb{Z}$*. Same holds for the projections* $\pi_{>k}$, $\pi_{<k}$, $\pi_{\geq k}$ *and* $\pi_{\leq k}$*.*

Note that the non-triviality of the map $\sigma_r : \mathfrak{gl}_{N-s_1} \longrightarrow \mathfrak{gl}_N$ is due to the particular numbering that we have chosen for the pyramids.

Proof. First observe that the map $\sigma_r : U(\mathfrak{gl}_{N-s_1}) \longrightarrow U(\mathfrak{gl}_N)$ obtained by the composition in (1.1.20) is an associative algebra homomorphism, since

$$\begin{aligned}
&\sigma_r([e_{(i_1,h_1)(j_1,k_1)}, e_{(i_2,h_2)(j_2,k_2)}]) \\
&= \delta_{(j_1,k_1)(i_2,h_2)} e_{(i_1,h_1+\delta_{i_1\leq s_1})(j_2,k_2+\delta_{j_2\leq s_1})} - \delta_{(j_2,k_2)(i_1,h_1)} e_{(i_2,h_2+\delta_{i_2\leq s_1})(j_1,k_1+\delta_{j_1\leq s_1})} \\
&= [\sigma_r(e_{(i_1,h_1)(j_1,k_1)}), \sigma_r(e_{(i_2,h_2)(j_2,k_2)})]\,.
\end{aligned}$$

Injectivity for both maps holds by the definition. Let us now consider the induced map $\overline{\sigma_l}$. First of all, σ_l sends $I_{\mathfrak{l}'^p} = U(\mathfrak{g}'^p)\langle b - (f'^p|b)\rangle_{b\in\mathfrak{m}'^p}$ to $I_{\mathfrak{l}} = U(\mathfrak{g})\langle b - (f|b)\rangle_{b\in\mathfrak{m}}$ since for $y \in U(\mathfrak{g}'^p)$ and $b \in \mathfrak{m}'^p$ the following holds

$$\sigma_l(y(b - (f'^p|b))) = \sigma_l(y)(\sigma_l(b) - (f|\sigma_l(b))) \in U(\mathfrak{g})\langle b - (f|b)\rangle_{b\in\mathfrak{m}}\,.$$

In fact, it is clear by 1.1.8 the $\frac{1}{2}\mathbb{Z}$-grading on $\mathfrak{g}'^p$ coincides with the one induced by the $\frac{1}{2}\mathbb{Z}$-grading on $\mathfrak{g}$, i.e. $\sigma_l(\mathfrak{g}'^p_k) = \mathfrak{g}_k \cap \sigma_l(\mathfrak{g}'^p)$:

$$\deg_p(\sigma_l(e_{(i,h)(j,k)})) = s_{1j} + k - s_{1i} - h = \deg_{'p}(e_{(i,h)(j,k)})\,.$$

A straightforward computation also shows that $(f'^p|b) = (f|\sigma_l(b))$ for every $b \in g'^p$. By the universal property of the quotient, there exists a well-defined $\overline{\sigma_l}$ such that the Diagram 1.1.21 commutes. Since we can decompose $\mathfrak{g}'^p = \mathfrak{g}'^p_{\leq 0} \oplus (\mathfrak{l}'^p)^c \oplus \mathfrak{l}'^p \oplus \mathfrak{g}'^p_{\geq 1}$ and $\mathfrak{g} = \mathfrak{g}_{\leq 0} \oplus \mathfrak{l}^c \oplus \mathfrak{l} \oplus \mathfrak{g}_{\geq 1}$, and since the induced map $\overline{\sigma}_l$ sends $\mathfrak{g}'^p_{\leq 0} \mapsto \mathfrak{g}_{\leq 0}$, $(\mathfrak{l}'^p)^c \mapsto \mathfrak{l}^c$, $\mathfrak{l}'^p \mapsto \mathfrak{l}$, $\mathfrak{g}'^p_{\geq 1} \mapsto \mathfrak{g}_{\geq 1}$, the injectivity of $\overline{\sigma}_l$ follows from the PBW Theorem, together with the fact that $(f'^p|\cdot) = (f|\cdot)_{|\sigma_l(\mathfrak{g}'^p)}$.

The proof in the case of $\overline{\sigma_r}$ runs analogously, keeping in mind that

(i) $s_{1i} = s^{p'}_{1i} + \delta_{i\geq s_1+1}$;

(ii) If $e_{(i,h)(j,k)} \in \mathfrak{m}^{p'}$, then $\sigma_r(e_{(i,h)(j,k)}) = e_{(i,h+\delta_{i\leq s_1})(j,k+\delta_{j\leq s_1})}$;

[3]Whenever we write a composition of the maps σ_l and σ_r, we are supposing that composition to make sense. Therefore, for instance

$$\sigma_l \circ \sigma_r : U(\mathfrak{gl}_{N-t_1-s_1}) \xrightarrow{\sigma_r} U(\mathfrak{gl}_{N-t_1}) \xrightarrow{\sigma_l} U(\mathfrak{gl}_N)$$

and

$$\sigma_l \circ \sigma_l : U(\mathfrak{gl}_{N-t_1-t_2}) \xrightarrow{\sigma_l} U(\mathfrak{gl}_{N-t_1}) \xrightarrow{\sigma_l} U(\mathfrak{gl}_N)\,.$$

(iii) Thanks to (i) we have

$$\begin{aligned}\deg_p(\sigma_r(e_{(i,h)(j,k)})) &= \deg_p e_{(i,h+\delta_{i\leq s_1})(j,k+\delta_{j\leq s_1})} = s_{1j} - s_{1i} + k + \delta_{j\leq s_1} - h + \delta_{i\leq s_1} \\ &= s^{p'}_{1j} + \delta_{j\geq s_1+1} - s^{p'}_{1i} - \delta_{i\geq s_1+1} + k + \delta_{j\leq s_1} - h + \delta_{i\leq s_1} \\ &= s^{p'}_{1j} + 1 - s^{p'}_{1i} - 1 + k - h = s^{p'}_{1j} - s^{p'}_{1i} + k - h \\ &= \deg_{p'} e_{(i,h)(j,k)}\,.\end{aligned}$$

Hence, $\sigma_r(\mathfrak{g}^{p'}_k) = \mathfrak{g}_k \cap \sigma_r(\mathfrak{g}^{p'})$;

(iv) If $e_{(i,h)(j,k)} \in \mathfrak{m}^{p'}$, then

$$(f^{p'}|e_{(i,h)(j,k)}) = \delta_{ij}\delta_{k,h+1} = \delta_{ij}\delta_{k+\delta_{j\leq s_1},h+\delta_{i\leq s_1}+1} = (f|\sigma_r(e_{(i,h)(j,k)}))\,.$$

Now, Equation 1.1.23 is an immediate consequence of the definition of the maps, since for $e_{(i,h)(j,k)} \in \mathfrak{gl}_{N-t_1-s_1}$ we have

$$\sigma_l \circ \sigma_r(e_{(i,h)(j,k)}) = e_{(i,h+\delta_{i\leq s_1})(j,k+\delta_{j\leq s_1})} = \sigma_r \circ \sigma_l(e_{(i,h)(j,k)})\,.$$

Finally, Equation 1.1.24 is a consequence of the fact that the grading on $\mathfrak{g}'^p$ and $\mathfrak{g}^{p'}$ is the one induced by the grading on $\mathfrak{g}$. □

Corollary 1.1.1. *Suppose that it is possible to remove the leftmost (resp. rightmost) column from the pyramid p. We can extend σ_l and σ_r to associative algebra homomorphisms to polynomials*

$$\sigma_l : U(\mathfrak{gl}_{N-t_1})[z] \longrightarrow U(\mathfrak{gl}_N)[z]\,, \qquad \sigma_r : U(\mathfrak{gl}_{N-s_1})[z] \longrightarrow U(\mathfrak{gl}_N)[z]\,,$$

and formal Laurent series

$$\sigma_l : U(\mathfrak{gl}_{N-t_1})((z^{-1})) \longrightarrow U(\mathfrak{gl}_N)((z^{-1}))\,, \qquad \sigma_r : U(\mathfrak{gl}_{N-s_1})((z^{-1})) \longrightarrow U(\mathfrak{gl}_N)((z^{-1}))\,,$$

by letting σ_l and σ_r act as the identity on $z^{\pm 1}$.

1.2 Definition of W-algebra

Consider the left ideal $I_\mathfrak{l} = U(\mathfrak{g})\langle b-(f|b)\rangle_{b\in\mathfrak{m}}$, and the corresponding left $U(\mathfrak{g})$-module $M_\mathfrak{l} = U(\mathfrak{g})/I_\mathfrak{l}$ defined in (1.1.12).

Lemma 1.2.1. *For the left ideal $I_\mathfrak{l}$ the following holds:*

$$U(\mathfrak{g})I_\mathfrak{l}U(\mathfrak{n}) \subset I_\mathfrak{l}\,.$$

Proof. Clearly, $U(\mathfrak{g})I_\mathfrak{l} \subset I_\mathfrak{l}$. We therefore need to prove that $I_\mathfrak{l}U(\mathfrak{n}) \subset I_\mathfrak{l}$. Let $h = y(b-(f|b)) \in I_\mathfrak{l}$, where $y \in U(\mathfrak{g})$ and $b \in \mathfrak{m}$. Given $a \in \mathfrak{n}$ we have

$$ha = y(b-(f|b))a = ya(b-(f|b)) + y[b,a]\,.$$

Obviously, the first summand in the RHS lies in $I_\mathfrak{l}$. The second summand also lies in $I_\mathfrak{l}$ because $[b,a] \subset [\mathfrak{l},\mathfrak{l}^\perp] \oplus \mathfrak{g}_{\geq\frac{3}{2}}$ and, by definition of $\mathfrak{l}^\perp$, $(f|[\mathfrak{l},\mathfrak{l}^\perp]) = 0$. □

Lemma 1.2.1 can be restated saying that $\mathfrak{g}$ acts on the module $M_\mathfrak{l}$ by left multiplication, while $\mathfrak{n}$ acts on it by both left and right multiplication. As a consequence, $\mathrm{ad}\mathfrak{n}(I_\mathfrak{l}) \subset I_\mathfrak{l}$.

Definition 1.2.1. The *quantum finite W-algebra* associated with $\mathfrak{g}$, f, $\Gamma, \mathfrak{l}$ is

$$W(\mathfrak{g},f,\Gamma,\mathfrak{l}) := \Big(U(\mathfrak{g})/I_\mathfrak{l}\Big)^{\mathrm{ad}\mathfrak{n}}. \tag{1.2.1}$$

Proposition 1.2.1. *$W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$ has a natural structure of a unital associative algebra, induced by that of $U(\mathfrak{g})$. Namely, for $\overline{y}_1, \overline{y}_2 \in W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$ we have $\overline{y}_1 \cdot \overline{y}_2 = \overline{y_1\, y_2}$.*

An alternative construction works as follows. Consider the subspace

$$\widetilde{W} := \{w \in U(\mathfrak{g}) \mid [a, w] \in I_{\mathfrak{l}}, \text{ for all } a \in \mathfrak{n}\} \subset U(\mathfrak{g})\,. \tag{1.2.2}$$

Lemma 1.2.2. *(i) $I_{\mathfrak{l}} \subset \widetilde{W}$;*

(ii) $I_{\mathfrak{l}}\widetilde{W} \subset I_{\mathfrak{l}}$;

(iii) $\widetilde{W}$ is a subalgebra of $U(\mathfrak{g})$;

(iv) $I_{\mathfrak{l}}$ is a proper two-sided ideal of $\widetilde{W}$.

Proof. By Lemma 1.2.1, $\mathrm{ad}\mathfrak{n}(I_{\mathfrak{l}}) \subset I_{\mathfrak{l}}$, proving (i). For (ii), let $h = y(b-(f|b)) \in I_{\mathfrak{l}}$ as above, and let $w \in \widetilde{W}$. Then,

$$hw = y(b-(f|b))w = yw(b-(f|b)) + y[b, w]\,.$$

The first summand in the RHS clearly lies in $\in I_{\mathfrak{l}}$, and the second summand also lies in $I_{\mathfrak{l}}$ by definition of $\widetilde{W}$, since $b \in \mathfrak{m} \subset \mathfrak{n}$. This proves part (ii). About part (iii), for $w_1, w_2 \in \widetilde{W}$ and $a \in \mathfrak{n}$ we have

$$[a, w_1 w_2]\bar{1}_{\mathfrak{l}} = [a, w_1]w_2\bar{1}_{\mathfrak{l}} + w_1[a, w_2]\bar{1}_{\mathfrak{l}}\,.$$

By assumption, we have $[a, w_2] \in I_{\mathfrak{l}}$. On the other hand, we also have $[a, w_1] \in I_{\mathfrak{l}}$ and then by part (ii), $[a, w_1]w_2 \in I_{\mathfrak{l}}$. Finally, for part (iv) recall that $I_{\mathfrak{l}}$ is a left ideal of $U(\mathfrak{g})$ and hence of $\widetilde{W}$; by part (ii), $I_{\mathfrak{l}}$ also is a right ideal of $\widetilde{W}$, proving (iv). □

Part (iv) of Lemma 1.2.2 in particular implies that the quotient $\widetilde{W}/I_{\mathfrak{l}}$ has a well-defined associative algebra structure. We moreover have

$$\widetilde{W}/I_{\mathfrak{l}} = \Big(U(\mathfrak{g})/I_{\mathfrak{l}}\Big)^{\mathrm{ad}\ \mathfrak{n}} = W(\mathfrak{g}, f, \Gamma, \mathfrak{l})\,.$$

Multiple formulations are actually possible for the definition of the quantum finite W algebra $W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$; Definition 1.2.1 corresponds to the so-called Whittaker model definition for a W-algebra.

Remark 1.2.1. When the $\frac{1}{2}\mathbb{Z}$-grading is even, necessarily $\mathfrak{l} = \mathfrak{l}^{\perp} = 0$ because $\mathfrak{g}_{\frac{1}{2}} = 0$. In this case, $\mathfrak{m} = \mathfrak{g}_{\geq 1}$, whereas $\mathfrak{p} = \mathfrak{g}_{\leq 0}$ is a subalgebra of $\mathfrak{g}$. By the PBW Theorem we can decompose $U(\mathfrak{g}) = U(\mathfrak{g}_{\leq 0}) \oplus I$ and the projection along this direct sum decomposition gives an isomorphism

$$U(\mathfrak{g})/I \xrightarrow{\sim} U(\mathfrak{g}_{\leq 0})\,.$$

As a consequence, $W(\mathfrak{g}, f, \Gamma, 0)$ can be regarded as a subalgebra of $U(\mathfrak{g}_{\leq 0})$.

Regardless of the numerous choices that we had to make, the resulting W-algebras will be isomorphic. By a result of Kostant all $\mathfrak{sl}_2$-triples containing the nilpotent element f are conjugate in $\mathfrak{g}$. It follows that the W-algebra $W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$ only depends on the adjoint nilpotent orbit of f. Moreover, by a result of Gan and Ginzburg [GG02], W-algebras associated with different isotropic subspaces are isomorphic, and by a result of Brundan and Goodwin [BG05], the same holds for W-algebras associated to different good $\frac{1}{2}\mathbb{Z}$-gradings for f. Both the results in [GG02] and [BG05] are useful for our purposes, and will be treated in more details in Section 4.2.

Let us consider two particularly special cases:

Example 1.2.1. Let $f = 0$. Then, $(f|\cdot) = 0$ and it is easily seen that $\mathfrak{g} = \mathfrak{g}_0$, $\mathfrak{m} = 0 = \mathfrak{n}$; therefore $I_{\mathfrak{l}} = 0$ and $W(\mathfrak{g}, f, \Gamma, \mathfrak{l}) = U(\mathfrak{g})$.

Example 1.2.2. Let f be principal nilpotent. By a result of Kostant, the associated W-algebra is in this case isomorphic to the center of the enveloping algebra $U(\mathfrak{g})$; $W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$ is therefore isomorphic to a polynomial algebra in N indeterminates $\mathbb{C}[x_1, \ldots, x_N]$.

1.3 Kazhdan filtration

Let $\Gamma : \mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ be a good $\frac{1}{2}\mathbb{Z}$-grading for $f \in \mathfrak{g}$ as before. Consider the Kazhdan filtration on $U(\mathfrak{g})$, given by

$$F_\Delta U(\mathfrak{g}) = \sum_{\Delta_1+\ldots+\Delta_s\leq\Delta} \mathfrak{g}_{1-\Delta_1}\cdot\ldots\cdot\mathfrak{g}_{1-\Delta_s}. \tag{1.3.1}$$

It is an increasing associative algebra filtration, depending on the conformal weight Δ, given by $\Delta(x) = 1 - j$ for $x \in \mathfrak{g}_j$:

$$F_{\Delta_1}U(\mathfrak{g})\cdot F_{\Delta_2}U(\mathfrak{g}) \subset F_{\Delta_1+\Delta_2}U(\mathfrak{g}),$$
$$[F_{\Delta_1}U(\mathfrak{g}), F_{\Delta_2}U(\mathfrak{g})] \subset F_{\Delta_1+\Delta_2-1}U(\mathfrak{g}).$$

Also let

$$\mathrm{gr}U(\mathfrak{g}) := \bigoplus_\Delta \mathrm{gr}_\Delta U(\mathfrak{g}), \quad \mathrm{gr}_\Delta U(\mathfrak{g}) := F_\Delta U(\mathfrak{g})/F_{\Delta-\frac{1}{2}}U(\mathfrak{g}), \tag{1.3.2}$$

be the associated graded algebra. It follows that $\mathrm{gr}U(\mathfrak{g})$ is a graded Poisson algebra, isomorphic to the symmetric algebra $S(\mathfrak{g})$, endowed with the Kirillov-Kostant Poisson bracket, and graded by the conformal weight.

Computing $F_0U(\mathfrak{g})$ we get

$$\begin{aligned} F_0U(\mathfrak{g}) &= \mathbb{C} + \sum_{\Delta_1\leq 0}\mathfrak{g}_{1-\Delta_1} + \sum_{\Delta_1+\Delta_2\leq 0}\mathfrak{g}_{1-\Delta_1}\mathfrak{g}_{1-\Delta_2} + \sum_{\Delta_1+\Delta_2+\Delta_3\leq 0}\mathfrak{g}_{1-\Delta_1}\mathfrak{g}_{1-\Delta_2}\mathfrak{g}_{1-\Delta_3} + \ldots \\ &= \mathbb{C} + \sum_{j_1\geq 1}\mathfrak{g}_{j_1} + \sum_{j_1+j_2\geq 2}\mathfrak{g}_{j_1}\mathfrak{g}_{j_2} + \sum_{j_1+j_2+j_3\geq 3}\mathfrak{g}_{j_1}\mathfrak{g}_{j_2}\mathfrak{g}_{j_3} + \ldots \end{aligned}$$

Since $\mathfrak{g}_{\geq 1} \subset I_\mathfrak{l}$, it follows that $F_0U(\mathfrak{g}) \subseteq \mathbb{C} + I_\mathfrak{l}$. Moreover, since the element $b - (f|b)$, $b \in \mathfrak{l} \oplus \mathfrak{g}_{\geq 1}$, is homogeneous with respect to this filtration, we get an induced Kazhdan filtration on the ideal $I_\mathfrak{l}$:

$$F_\Delta I_\mathfrak{l} = F_\Delta U(\mathfrak{g}) \cap I_\mathfrak{l} = (F_{\Delta-\frac{1}{2}}U(\mathfrak{g}))\mathfrak{l} + (F_\Delta U(\mathfrak{g}))\langle b - (f|b)\rangle_{b\in\mathfrak{g}_1} + \sum_{j>1}(F_{\Delta+j-1}U(\mathfrak{g}))\mathfrak{g}_j. \tag{1.3.3}$$

We can therefore deduce the following properties:

$(a.i)$ $F_\Delta U(\mathfrak{g}) = F_\Delta I_\mathfrak{l}, \quad \Delta < 0$;

$(a.ii)$ $F_0U(\mathfrak{g}) = \mathbb{C} \oplus F_0 I_\mathfrak{l}$.

By the property $(a.ii)$, $F_0I_\mathfrak{l} \subset F_0U(\mathfrak{g})$ is a two-sided ideal of codimension 1, and the corresponding quotient map is the algebra homomorphism $\epsilon_0 : F_0U(\mathfrak{g}) \longrightarrow \mathbb{C}$ given by the formula

$$\epsilon_0(\sum a_1\cdots a_k) = \sum (f|a_1)\cdots(f|a_k). \tag{1.3.4}$$

Therefore, we have an induced Kazhdan filtration at the quotient $M_\mathfrak{l} = U(\mathfrak{g})/I_\mathfrak{l}$: $F_\Delta M_\mathfrak{l} = F_\Delta U(\mathfrak{g})/F_\Delta I_\mathfrak{l}$, and $F_\Delta M_\mathfrak{l} = \delta_{\Delta,0}\mathbb{C}\bar{1}_\mathfrak{l}$ for every $\Delta \leq 0$.

The action of $F_0U(\mathfrak{g})$ on $F_0M_\mathfrak{l} = \mathbb{C}\bar{1}_\mathfrak{l}$ is induced by the map (1.3.4), i.e. $u\bar{1}_\mathfrak{l} = \epsilon_0(u)\bar{1}_\mathfrak{l}$ for every $u \in F_0U(\mathfrak{g})$.

Finally, we have induced Kazhdan filtrations on the subspace $\widetilde{W} \subset U(\mathfrak{g})$ and on the quotient $\widetilde{W}/I_\mathfrak{l} = W(\mathfrak{g}, f, \Gamma, \mathfrak{l}) \subset M_\mathfrak{l}$ such that

$$\begin{aligned} &F_\Delta\widetilde{W} = \widetilde{W} \cap F_\Delta U(\mathfrak{g}), \\ &F_\Delta W(\mathfrak{g}, f, \Gamma, \mathfrak{l}) = F_\Delta\widetilde{W}/F_\Delta I_\mathfrak{l} = (\widetilde{W} \cap F_\Delta U(\mathfrak{g}))/F_\Delta I_\mathfrak{l}. \end{aligned} \tag{1.3.5}$$

Note that, thanks to properties $(a.i) - (a.ii)$, $F_\Delta W(\mathfrak{g}, f, \Gamma, \mathfrak{l}) = 0$ for $\Delta < 0$. Moreover, for the associated graded the following inclusion of commutative algebras holds

$$\mathrm{gr}\, W(\mathfrak{g}, f, \Gamma, \mathfrak{l}) \subset \mathrm{gr}\,\big(U(\mathfrak{g})/I_\mathfrak{l}\big) = \mathrm{gr}\, U(\mathfrak{g})/\,\mathrm{gr}\, I_\mathfrak{l} \cong S(\mathfrak{g})/\,\mathrm{gr}\, I_\mathfrak{l}, \tag{1.3.6}$$

where $\operatorname{gr} I_{\mathfrak{l}}$ is the associated graded ideal of $I_{\mathfrak{l}}$, which we also see as the two-sided ideal $S(\mathfrak{g})\langle b-(f|b)\rangle_{b\in\mathfrak{m}}$ of $S(\mathfrak{g})$.

Let us now define a refinement of this filtration that also takes into account the number of factors appearing:

$$F_{\Delta,n}U(\mathfrak{g}) = F_{\Delta}U(\mathfrak{g}) \cap \{\text{conformal weight} = \Delta \Rightarrow \text{at least } n \text{ factors}\}. \tag{1.3.7}$$

Then, $F_{\Delta,n+1}U(\mathfrak{g}) \subseteq F_{\Delta,n}U(\mathfrak{g})$. Moreover, $F_{\Delta,0}U(\mathfrak{g}) = F_{\Delta}U(\mathfrak{g})$ and we shall set $F_{\Delta,\infty}(\mathfrak{g}) := F_{\Delta-1}U(\mathfrak{g})$.

This is still an associative algebras filtration, since

$$F_{\Delta_1,n_1}U(\mathfrak{g}) \cdot F_{\Delta_2,n_2}U(\mathfrak{g}) \subset F_{\Delta_1+\Delta_2,n_1+n_2}U(\mathfrak{g})$$

and

$$[F_{\Delta_1,n_1}U(\mathfrak{g}), F_{\Delta_2,n_2}U(\mathfrak{g})] \subset F_{\Delta_1+\Delta_2-1,n_1+n_2-1}U(\mathfrak{g})\,.$$

As for the classical Kazhdan filtration, we can extend this refined version to $I_{\mathfrak{l}}, M_{\mathfrak{l}}$ and $\widetilde{W}$ as

$$\begin{aligned} F_{\Delta,n}I_{\mathfrak{l}} &:= F_{\Delta,n}U(\mathfrak{g}) \cap I_{\mathfrak{l}} \\ F_{\Delta,n}M_{\mathfrak{l}} &:= F_{\Delta,n}U(\mathfrak{g})/F_{\Delta,n}I_{\mathfrak{l}} \\ F_{\Delta,n}\widetilde{W} &:= F_{\Delta,n}U(\mathfrak{g}) \cap \widetilde{W}\,. \end{aligned}$$

We obtain for $W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$:

$$F_{\Delta,n}W(\mathfrak{g}, f, \Gamma, \mathfrak{l}) = F_{\Delta,n}\widetilde{W}/F_{\Delta,n}I_{\mathfrak{l}} = (\widetilde{W} \cap F_{\Delta,n}U(\mathfrak{g}))/F_{\Delta,n}I_{\mathfrak{l}}.$$

Finally, let us consider the extension of the Kazhdan filtration of $U(\mathfrak{g})$ and $W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$ obtained by letting z have Kazhdan degree equal to 1. In other words, we let

$$\begin{aligned} F_{\Delta,n}(U(\mathfrak{g})[z]) &:= \sum_{k\geq 0}(F_{\Delta-k,n}U(\mathfrak{g}))(-z)^k\,, \\ F_{\Delta,n}(W(\mathfrak{g}, f, \Gamma, \mathfrak{l})[z]) &:= \sum_{k\geq 0}(F_{\Delta-k,n}W(\mathfrak{g}, f, \Gamma, \mathfrak{l}))(-z)^k\,. \end{aligned} \tag{1.3.8}$$

This filtration is obviously an associative algebra filtration:

$$F_{\Delta_1,n_1}(U(\mathfrak{g})[z]) \cdot F_{\Delta_2,n_2}(U(\mathfrak{g})[z]) \subset F_{\Delta_1+\Delta_2,n_1+n_2}(U(\mathfrak{g})[z]) \tag{1.3.9}$$

and moreover

$$[F_{\Delta_1,n_1}(U(\mathfrak{g})[z]), F_{\Delta_2,n_2}(U(\mathfrak{g})[z])] \subset F_{\Delta_1+\Delta_2-1,n_1+n_2-1}(U(\mathfrak{g})[z])\,. \tag{1.3.10}$$

The same holds for $W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$.

1.4 Rees algebras and modules

Definition 1.4.1. The *(completed) Rees algebra* $\mathcal{R}U(\mathfrak{g})$ associated with the Kazhdan filtration of Section 1.3 is the subalgebra of $U(\mathfrak{g})((z^{-\frac{1}{2}}))$ consisting of the Laurent series in $z^{-\frac{1}{2}}$ with the property that the coefficient of z^n lies in $F_{-n}U(\mathfrak{g})$, for all $n \in \frac{1}{2}\mathbb{Z}$:

$$\mathcal{R}U(\mathfrak{g}) = \widehat{\sum}_{n\geq -N\in\frac{1}{2}\mathbb{Z}} z^{-n}F_nU(\mathfrak{g}) \subset U(\mathfrak{g})((z^{-\frac{1}{2}}))\,, \tag{1.4.1}$$

where the hat denote the completion of the usual Rees algebra in which only finite sums appear. This completion is obtained by allowing series with infinitely many negative half-integer powers of z.

Note that

(i) $F_0U(\mathfrak{g}) \subset \mathcal{R}U(\mathfrak{g})$ is a subalgebra of the Rees algebra, but $F_nU(\mathfrak{g}) \not\subset \mathcal{R}U(\mathfrak{g})$ for $n > 0$;

(ii) Since $1 \in F_0U(\mathfrak{g}) \subset F_{\frac{1}{2}}U(\mathfrak{g})$, we have that $z^{-\frac{1}{2}}$ is an element of the Rees algebra $\mathcal{R}U(\mathfrak{g})$, which is moreover central. Therefore, it defines an injective endomorphism of $\mathcal{R}U(\mathfrak{g})$ (given by multiplication by $z^{-\frac{1}{2}}$) which commutes with multiplication by elements of $\mathcal{R}U(\mathfrak{g})$. However, for any $n > 0$ we have that z^n does not lie in $\mathcal{R}U(\mathfrak{g})$.

Given the left ideal $I_{\mathfrak{l}}$ and the $U(\mathfrak{g})$-module $M_{\mathfrak{l}}$ we can define the following:

$$\mathcal{R}I_{\mathfrak{l}} = \widehat{\sum}_{n\in\frac{1}{2}\mathbb{Z}} z^{-n}F_nI_{\mathfrak{l}} \subset I_{\mathfrak{l}}((z^{-1}))\,, \qquad \mathcal{R}M_{\mathfrak{l}} = \mathcal{R}U(\mathfrak{g})/\mathcal{R}I_{\mathfrak{l}}\,,$$

left ideal and Rees module of $\mathcal{R}U(\mathfrak{g})$ respectively. Note that $\mathcal{R}U(\mathfrak{g})/\mathcal{R}I_{\mathfrak{l}} = \mathbb{C}\bar{1}_{\mathfrak{l}} \oplus \mathcal{R}_-M_{\mathfrak{l}}$, where $\mathcal{R}_-M_{\mathfrak{l}} = \widehat{\sum}_{n\geq\frac{1}{2}} z^{-n}F_nM_{\mathfrak{l}} \subset z^{-\frac{1}{2}}M_{\mathfrak{l}}[[z^{-\frac{1}{2}}]]$ is a submodule of $\mathcal{R}M_{\mathfrak{l}}$ of codimension 1.

The algebra homomorphism (1.3.4) can be extended to a surjective linear map (see [DSKV16c, Lemma 5.5] for a proof)

$$\begin{gathered}\epsilon : \mathcal{R}U(\mathfrak{g}) \longrightarrow \mathbb{C}\,,\\ a(z) = \sum_{N\geq n\in\frac{1}{2}\mathbb{Z}} a_nz^n \mapsto \epsilon(a(z)) := \epsilon_0(a_0) = (f|a_0)\,.\end{gathered} \tag{1.4.2}$$

The following proposition summarizes remarkable properties of the Rees module $\mathcal{R}M_{\mathfrak{l}}$, see [DSKV16c, Lemma 5.6, Proposition 5.7] for a proof.

Proposition 1.4.1. *(i)* $\mathcal{R}I_{\mathfrak{l}} \cdot \mathcal{R}M_{\mathfrak{l}} \subset z^{-1}\mathcal{R}M_{\mathfrak{l}}$ *and* $z^{-\frac{1}{2}}\mathcal{R}M_{\mathfrak{l}} \subset \mathcal{R}_-M_{\mathfrak{l}}$*;*

(ii) The action of $\mathcal{R}U(\mathfrak{g})$ *on the quotient module* $\mathcal{R}M_{\mathfrak{l}}/\mathcal{R}_-M_{\mathfrak{l}} = \mathbb{C}\bar{1}_{\mathfrak{l}}$ *is induced by the map (1.4.2). Namely,* $a(z)\bar{1}_{\mathfrak{l}} \equiv \epsilon(a(z))\bar{1}_{\mathfrak{l}}$ *mod* $\mathcal{R}_-M_{\mathfrak{l}}$ *for every* $a(z) \in \mathcal{R}U(\mathfrak{g})$*;*

(iii) An element $a(z) \in \mathcal{R}U(\mathfrak{g})$ *acts as an invertible morphism of* $\mathcal{R}M_{\mathfrak{l}}$ *if and only if* $\epsilon(a(z)) \neq 0$*.*

Proposition 1.4.1 gives us a characterization elements of $\mathcal{R}U(\mathfrak{g})$ acting as invertible morphisms of $\mathcal{R}M_{\mathfrak{l}}$. However, it may happen that the inverse of such an element $a(z) \in RM_{\mathfrak{l}}$ does not exist in the Rees algebra $\mathcal{R}U(\mathfrak{g})$ because it might involve an infinite number of positive powers of z. A solution to this issue consists of the definition of an extension $\mathcal{R}_\infty U(\mathfrak{g})$ of $\mathcal{R}U(\mathfrak{g})$ with the property that all elements $a(z) \in \mathcal{R}U(\mathfrak{g})$ such that $\epsilon(a(z)) \neq 0$ are invertible in $\mathcal{R}_\infty U(\mathfrak{g})$. The construction of $\mathcal{R}_\infty U(\mathfrak{g})$ goes through a limiting procedure for which we address to [DSKV16c, Section 5.4]; for our purposes we will however list some of its main properties.

Proposition 1.4.2 ([DSKV16c, Section 5.4])**.** *There exists an algebra extension* $\mathcal{R}_\infty U(\mathfrak{g})$ *of the Rees algebra* $\mathcal{R}U(\mathfrak{g})$*, satisfying the following properties:*

(a) The map (1.4.2) extends to an algebra homomorphism $\epsilon : \mathcal{R}_\infty U(\mathfrak{g}) \to \mathbb{C}$*.*

(b) The left action of $\mathcal{R}U(\mathfrak{g})$ *on the Rees module* $\mathcal{R}M_{\mathfrak{l}}$ *extends to a left action of* $\mathcal{R}_\infty U(\mathfrak{g})$ *on* $\mathcal{R}M$*, and* $\mathcal{R}_-M$ *is preserved by this action.*

(c) The action of $\mathcal{R}_\infty U(\mathfrak{g})$ *on the quotient module* $\mathcal{R}M_{\mathfrak{l}}/\mathcal{R}_-M_{\mathfrak{l}} = \mathbb{C}\bar{1}_{\mathfrak{l}}$ *is induced by the map* ϵ *in (a), i.e.* $\alpha(z)\bar{1}_{\mathfrak{l}} \equiv \epsilon(\alpha(z))\bar{1}_{\mathfrak{l}} \mod \mathcal{R}_-M_{\mathfrak{l}}$ *for every* $\alpha(z) \in \mathcal{R}_\infty U(\mathfrak{g})$*.*

(d) For every $\alpha(z) \in \mathcal{R}_\infty U(\mathfrak{g})$ *and every integer* $N \geq 0$*, there exist* $\alpha_N(z) \in \mathcal{R}U(\mathfrak{g})$ *such that* $(\alpha(z)-\alpha_N(z))\cdot \mathcal{R}M_{\mathfrak{l}} \subset z^{-N-1}\mathcal{R}M$*;*

(e) For an element $\alpha(z) \in \mathcal{R}_\infty U(\mathfrak{g})$*, the following conditions are equivalent:*

  (i) $\alpha(z)$ *is invertible in* $\mathcal{R}_\infty U(\mathfrak{g})$*;*

  (ii) $\alpha(z)$ *acts as an invertible endomorphism of* $\mathcal{R}M_{\mathfrak{l}}$*;*

  (iii) $\epsilon(\alpha(z)) \neq 0$*.*

(f) An operator $A(z) \in \mathcal{R}_\infty U(\mathfrak{g}) \otimes \mathrm{Hom}(V_1, V_2)$*, where* V_1, V_2 *are vector spaces, is invertible if and only if* $\epsilon(A(z)) \in \mathrm{Hom}(V_1, V_2)$ *is invertible.*

Given the W-algebra $W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$ we may also consider its Rees algebra, which is induced from $M_{\mathfrak{l}}$:

$$\mathcal{R}W(\mathfrak{g}, f, \Gamma, \mathfrak{l}) = \widehat{\sum}_{n\geq 0} z^{-n} F_n W(\mathfrak{g}, f, \Gamma, \mathfrak{l}) \subset \mathcal{R}M_{\mathfrak{l}}\,.$$

Note that $\mathcal{R}W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$ is a subalgebra of the algebra $W(\mathfrak{g}, f, \Gamma, \mathfrak{l})[[z^{-\frac{1}{2}}]]$, and the following holds

Proposition 1.4.3 ([DSKV16c, Proposition 5.14])**.** *Let $a(z) \in \mathcal{R}_\infty U(\mathfrak{g})$, $g(z) \in \mathcal{R}U(\mathfrak{g})$ and $w(z) \in \mathcal{R}M_{\mathfrak{l}}$ be such that $\alpha(z)\bar{1}_{\mathfrak{l}} = g(z)\bar{1}_{\mathfrak{l}} = w(z)$. Then, the following conditions are equivalent:*

(i) $[a, \alpha(z)]\bar{1}_{\mathfrak{l}} = 0$*, for all* $a \in \mathfrak{g}_{\geq \frac{1}{2}}$*;*

(ii) $[a, g(z)]\bar{1}_{\mathfrak{l}} = 0$*, for all* $a \in \mathfrak{g}_{\geq \frac{1}{2}}$*;*

(iii) $w(z) \in \mathcal{R}W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$*.*

1.5 (Generalized) quasideterminants

Quasideterminants arise as an attempt to define a determinant for matrices with noncommutative entries, attempt that also counts for instance quantum determinants, Capelli determinants, and the probably most famous and widely used example of Dieudonné determinant.

They were introduced by Gelfand and Retakh in the early ′90s ([GR91],[GR92],[GR93]) for matrices over noncommutative division rings, and they have proved to be extremely useful and versatile, finding applications is many areas including noncommutative symmetric functions, noncommutative integrable systems, quantum algebras and Yangians.

Since they are also a key tool in our construction, we review here the definition and some properties of quasideterminants, addressing for instance to [GGRW05] for an extended treatment of the subject.

Definition 1.5.1. Let $A = (a_{ij})_{i,j=1}^N$ be an $N \times N$ matrix over a ring R with 1. Denote by A^{ij} the matrix obtained from A by deleting the i-th row and j-th column. Suppose that the matrix A^{ij} is invertible. Then the (ij)*-th quasideterminant* of A is defined by

$$|A|_{ij} = a_{ij} - r_i^j (A^{ij})^{-1} c_j^i\,, \tag{1.5.1}$$

where r_i^j denotes the row matrix obtained by the i-th row of A by deleting the j-th entry and c_j^i denotes the column matrix obtained by the j-th column of A by deleting the i-th entry.

Example 1.5.1. Let A be the 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then, provided that they exist, we have 4 possible quasideterminants:

$$|A|_{11} = a_{11} - a_{12}a_{22}^{-1}a_{21}\,, \qquad |A|_{12} = a_{12} - a_{11}a_{21}^{-1}a_{22}\,,$$
$$|A|_{21} = a_{21} - a_{22}a_{12}^{-1}a_{11}\,, \qquad |A|_{22} = a_{22} - a_{21}a_{11}^{-1}a_{12}\,.$$

Equivalently, the quasideterminant of A can be defined as follows:

Proposition 1.5.1. *Suppose that the inverse matrix A exists, and that its (ji)-th entry $(A^{-1})_{ji}$ is an invertible element of the ring R. Then the (ij)-th quasideterminant of A is defined as*

$$|A|_{ij} = ((A^{-1})_{ji})^{-1}\,. \tag{1.5.2}$$

Proof. See [Mo07, Proposition 1.10.4] □

Remark 1.5.1. If the entries of the matrix A commute, from (1.5.2) we get $|A|_{ij} = (-1)^{i+j} \frac{\det A}{\det A^{ij}}$, where det denotes the usual determinant for commutative algebras.

Let $\mathcal{I}$, $\mathcal{J} \subset \{1,\dots,N\}$ subsets of the same cardinality $M \le N$, and denote by $\mathcal{I}^c$, $\mathcal{J}^c$ the complements of $\mathcal{I}$ and $\mathcal{J}$ in $\{1,\dots,N\}$. Definition 1.5.1 can be generalized as follows

Definition 1.5.2. The $(\mathcal{I},\mathcal{J})$-*th quasideterminant* of A is defined as

$$|A|_{\mathcal{I}\mathcal{J}} = A_{\mathcal{I}\mathcal{J}} - A_{\mathcal{I}\mathcal{J}^c}(A_{\mathcal{I}^c\mathcal{J}^c})^{-1}A_{\mathcal{I}^c\mathcal{J}} \in \mathrm{Mat}_{M\times M}R\,, \tag{1.5.3}$$

provided that $A_{\mathcal{I}^c\mathcal{J}^c}$ is invertible in $\mathrm{Mat}_{(N-M)\times(N-M)}R$. We denote by $A_{\mathcal{I}\mathcal{J}}$ the $M\times M$ submatrix of A obtained by taking rows from the set I and columns from the set J (same holds for $A_{\mathcal{I}\mathcal{J}^c}$, $A_{\mathcal{I}^c\mathcal{J}^c}$, $A_{\mathcal{I}^c\mathcal{J}}$). Equivalently, ([DSKV16a, Proposition 4.2])

$$|A|_{\mathcal{I}\mathcal{J}} = ((A^{-1})_{\mathcal{J}\mathcal{I}})^{-1}\,, \tag{1.5.4}$$

provided that both A and $(A^{-1})_{\mathcal{J}\mathcal{I}}$ are invertible (in $\mathrm{Mat}_{N\times N}R$ and $\mathrm{Mat}_{M\times M}R$ respectively).

One of the main properties of quasideterminants is that they satisfy the so-called hereditary property, which can be stated as follows:

Proposition 1.5.2 (Hereditary Property)**.** *Let $\mathcal{I}_1 \subset \mathcal{I} \subset \{1,\dots,N\}$ and $\mathcal{J}_1 \subset \mathcal{J} \subset \{1,\dots,N\}$ be subsets of the same cardinality, $|\mathcal{I}_1| = |\mathcal{J}_1|$ and $|\mathcal{I}| = |\mathcal{J}|$. Assume that both the $(\mathcal{I},\mathcal{J})$-th quasideterminant and the $(\mathcal{I}_1,\mathcal{J}_1)$-th quasideterminant of A exist. Then*

$$|A|_{\mathcal{I}_1\mathcal{J}_1} = ||A|_{\mathcal{I}\mathcal{J}}|_{\mathcal{I}_1\mathcal{J}_1}\,. \tag{1.5.5}$$

Proof. By the Definition 1.5.1 of quasideterminant we have

$$||A|_{\mathcal{I}\mathcal{J}}|_{\mathcal{I}_1\mathcal{J}_1} = (((((A^{-1})_{\mathcal{J}\mathcal{I}})^{-1})^{-1})_{\mathcal{J}_1\mathcal{I}_1})^{-1} = (((A^{-1})_{\mathcal{J}\mathcal{I}})_{\mathcal{J}_1\mathcal{I}_1})^{-1} = ((A^{-1})_{\mathcal{J}_1\mathcal{I}_1})^{-1} = |A|_{\mathcal{I}_1\mathcal{J}_1}\,.$$

□

The notion of quasideterminant can be further generalized as follows: let A be an $N\times N$ matrix over a ring R with 1, as before, and let $I \in \mathrm{Mat}_{N\times M}R$, $J \in \mathrm{Mat}_{M\times N}R$ for some $M \le N$.

Definition 1.5.3. The (I,J)-*th generalized quasideterminant* of A is

$$|A|_{IJ} = (JA^{-1}I)^{-1} \in \mathrm{Mat}_{M\times M}R\,, \tag{1.5.6}$$

provided both A and $JA^{-1}I$ are invertible (in $\mathrm{Mat}_{N\times N}R$ and in $\mathrm{Mat}_{M\times M}R$ respectively).

Definition 1.5.3 is a generalization of Definition 1.5.2 in the sense that given the subsets $\mathcal{I} = \{i_1,\dots,i_m\}$ and $\mathcal{J} = \{j_1,\dots,j_m\}$ of $\{1,\dots,N\}$, then the quasideterminant $|A|_{\mathcal{I}\mathcal{J}}$ defined in (1.5.4) coincides with the quasideterminant $|A|_{I,J}$ defined in (1.5.6), where I is the $N\times M$ matrix with entries 1 in position (i_k,k) for $1\le k\le M$ and zero otherwise, and J is the $M\times N$ matrix with entries 1 in position (k,j_k) for $1\le k\le M$ and zero otherwise. In fact we have

$$A_{\mathcal{J}\mathcal{I}} = JAI\,,$$

which implies

$$|A|_{IJ} = (JA^{-1}I)^{-1} = ((A^{-1})_{\mathcal{J}\mathcal{I}})^{-1} = |A|_{\mathcal{I}\mathcal{J}}\,.$$

For the results in Section 2.2.1 it also useful to introduce the following definition of quasideterminant, that clearly generalizes Definition 1.5.3. Let R be a unital associative algebra over $\mathbb{C}$ and let

$$\begin{aligned}\chi_1 &: 0 \longrightarrow U_1 \xrightarrow{\Psi_1} V_1 \xrightarrow{\Pi_1} W_1 \longrightarrow 0\\ \chi_2 &: 0 \longrightarrow U_2 \xrightarrow{\Psi_2} V_2 \xrightarrow{\Pi_2} W_2 \longrightarrow 0\end{aligned} \tag{1.5.7}$$

be two short exact sequences of R-modules. Let moreover $A : V_1 \longrightarrow V_2$ be an R-module homomorphism.

Definition 1.5.4. [DSKV17b, Section 2.2] The *generalized quasideterminant* of the R-module homomorphism A with respect to the maps Ψ_2 and Π_1 in (1.5.7) is the R-module homomorphism

$$|A|_{\Psi_2,\Pi_1} := (\Pi_1 A^{-1}\Psi_2)^{-1} : W_1 \longrightarrow U_2\,, \tag{1.5.8}$$

provided that it exists, i.e. provided that $A : V_1 \longrightarrow V_2$ is invertible and that $\Pi_1 A^{-1}\Psi_2 : U_2 \longrightarrow W_1$ is invertible.

By [DSKV17b, Proposition 2.4], we obtain a very useful formula for the generalized quasideterminant (1.5.8). It in fact states that the generalized quasideterminant $|A|_{\Psi_2,\Pi_1}$, if it exists, coincides with the *Dirac reduction* for the R-module homomorphism $A : V_1 \longrightarrow V_2$ (see [DSKV17b, Lemma 2.1] for the general definition):

$$|A|_{\Psi_2,\Pi_1} = \Psi_2^{-1}(A - A\Psi_1(\Pi_2 A\Psi_1)^{-1}\Pi_2 A)\Pi_1^{-1} : W_1 \longrightarrow U_2\,. \tag{1.5.9}$$

1.5.1 Operators of Yangian type

Definition 1.5.5. Let U be an associative algebra. We say that a matrix $A(z) \in \mathrm{Mat}_{M\times N}U((z^{-1}))$ is an *operator of Yangian type* for the associative algebra U if, for every $i,h \in \{1,\dots,M\}$, $j,k \in \{1,\dots,N\}$, it satisfies the following *Yangian identity*:

$$(z-w)[A_{ij}(z), A_{hk}(w)] = A_{hj}(w)A_{ik}(z) - A_{hj}(z)A_{ik}(w)\,. \tag{1.5.10}$$

The name Yangian identity is due to the fact that in the case $M = N$ Equation 1.5.10 coincides up to an overall sign to the defining relation of the Yangian for $\mathfrak{gl}_N$ (see [Dr86], or [Mo07] for a review on the topic). Equation 1.5.10 also has a classical version, which holds in the context of Poisson algebras; the identity is the same but the associative algebra commutator on the left hand side is replaced by the Poisson bracket. In the context of Poisson vertex algebras a "chiralization" of the Posson algebras identity was introduced in [DSKV15, DSKV16a, DSKV16b] and it goes under the name of *Adler identity*. The Adler identity serves as the defining relation of the so-called operators of Adler type, which constitute the classical affine analogue of the operators of Yangian type and whose properties in relation with the classical affine W-algebras have inspired this work.

Example 1.5.2. Let $U = U(\mathfrak{gl}_N)$ and let E be as in Equation (1.1.15). Then it is easily checked that the $N \times N$ matrix $A(z) = z\mathbb{1}_N + E$ is an operator of Yangian type for $U(\mathfrak{gl}_N)$.

The following Proposition aims to prove that every generalized quasideterminant of an operator of Yangian type is again of Yangian type.

Proposition 1.5.3. *Let U be a unital associative algebra and suppose that $A(z) \in Mat_{M\times N}U((z^{-1}))$ is an operator of Yangian type.*

(i) Let $Z \subset U$ be the center of U, and let $J \in Mat_{P\times M}Z$, $I \in Mat_{N\times Q}Z$. Then $JA(z)I \in Mat_{P\times Q}U((z^{-1}))$ is an operator of Yangian type.

(ii) Assume that $M = N$ and that $A(z)$ is invertible in $Mat_{N\times N}U((z^{-1}))$. Then the inverse matrix $A(z)^{-1}$ is an operator of Yangian type with respect to the opposite product of the algebra U.

Proof. See [DSKV16c, Proposition 2.9]. □

As a clear consequence of Proposition 1.5.3 we have

Theorem 1.5.1. *Let U be a unital associative algebra and let $Z \subset U$ be its center. Let $A(z) \in Mat_{N\times N}U((z^{-1}))$ be an operator of Yangian type. Then, for every $J \in Mat_{M\times N}Z$, $I \in Mat_{N\times M}Z$ with $M \le N$, the generalized quasideterminant $|A(z)|_{IJ}$, provided that it exists, is an operator of Yangian type.*

Chapter 2

The building blocks: matrices $T(z)$ and $L(z)$

As in Chapter 1, let $f \in \mathfrak{gl}_N$ be a nilpotent element corresponding to the partition $\lambda = (p_1 \geq \ldots \geq p_r)$ of N. Moreover, let $\Gamma : \mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ be a good $\frac{1}{2}\mathbb{Z}$-grading for f and let p be the associated pyramid of size N and shape λ. We also suppose that $\mathfrak{l} \subset \mathfrak{g}_{\frac{1}{2}}$ is an isotropic subspace with respect to the bilinear form ω as in (1.1.10).

2.1 Definition of the matrix $T(z)$

The matrix $T(z)$ is the first step of our construction, and it will moreover constitute a bridge between our construction and the one by Brundan and Kleshchev, that in [BK06] describe the structure of the W-algebra for $\mathfrak{gl}_N$. We will analyze this connection in Section 2.3.

Let us introduce the following index subsets of $\mathcal{T}$ (cf. (1.1.3))

$$\begin{aligned} &\mathcal{I} = \{(i,1) \mid 1 \leq i \leq r\}, && \mathcal{J} = \{(j,p_j) \mid 1 \leq j \leq r\}, \\ &\mathcal{I}^c = \{(i,h) \mid 1 \leq i \leq r, 1 < h \leq p_i\}, && \mathcal{J}^c = \{(j,k) \mid 1 \leq j \leq r, 1 \leq k < p_j\}, \end{aligned} \tag{2.1.1}$$

We also fix a bijection between the sets $\mathcal{I}^c$ and $\mathcal{J}^c$:

$$\begin{aligned} \mathcal{I}^c &\longrightarrow \mathcal{J}^c, \\ (i,h) &\mapsto (i,h-1). \end{aligned} \tag{2.1.2}$$

Definition 2.1.1. Define the matrix $T(z) \in \mathrm{Mat}_{r \times r} U(\mathfrak{g})((z^{-1}))$ as the following quasideterminant:

$$T(z) = |z\mathbb{1}_N + F + \pi_{\mathfrak{p}} E + D_{\mathfrak{l}}|_{\mathcal{I}\mathcal{J}}\,. \tag{2.1.3}$$

Example 2.1.1. For $p_1 = 1$, Definition 2.1.1 simply becomes

$$T(z) = z\mathbb{1}_N + E\,. \tag{2.1.4}$$

Proposition 2.1.1. *The following identity holds for $T_{ij}(z) \in U(\mathfrak{g})((z^{-1}))$, $1 \leq i,j \leq r$:*

$$\begin{aligned} T_{ij}(z) = \delta_{(i,1)(j,p_j)} z + \tilde{e}_{(j,p_j)(i,1)} - \sum_{l \geq 0} (-1)^l \sum_{(i_0,h_0),\ldots,(i_l,h_l) \in \mathcal{J}^c} &\big(\delta_{(i,1)(i_0,h_0)} z + \pi_{\mathfrak{p}} \tilde{e}_{(i_0,h_0)(i,1)}\big) \\ \times \big(\delta_{(i_0,h_0+1)(i_1,h_1)} z + \pi_{\mathfrak{p}} \tilde{e}_{(i_1,h_1)(i_0,h_0+1)}\big) \cdots \big(\delta_{(i_{l-1},h_{l-1}+1)(i_l,h_l)} z + \pi_{\mathfrak{p}} \tilde{e}_{(i_l,h_l)(i_{l-1},h_{l-1}+1)}\big) \\ \times \big(\delta_{(i_l,h_l+1)(j,p_j)} z + \pi_{\mathfrak{p}} \tilde{e}_{(j,p_j)(i_l,h_l+1)}\big)\,. \end{aligned} \tag{2.1.5}$$

Proof. Let $A := z\mathbb{1}_N + F + \pi_{\mathfrak{p}} E + D_{\mathfrak{l}}$. By Definition 1.5.2, we have

$$T(z) = |A|_{\mathcal{I}\mathcal{J}} = A_{\mathcal{I}\mathcal{J}} - A_{\mathcal{I}\mathcal{J}^c}(A_{\mathcal{I}^c\mathcal{J}^c})^{-1}A_{\mathcal{I}^c\mathcal{J}} \,. \tag{2.1.6}$$

Note that for every (i,h), $(j,k) \in \mathcal{T}$, the $((i,h),(j,k))$-th entry of A is

$$A_{(i,h)(j,k)} = \delta_{(i,h)(j,k)} z + \pi_{\mathfrak{p}} \tilde{e}_{(j,k)(i,h)} + \delta_{(i,h)(j,k+1)} \,.$$

In view of the bijection (2.1.2) we can write $A_{\mathcal{I}^c\mathcal{J}^c}$ as

$$A_{\mathcal{I}^c\mathcal{J}^c} = \sum_{(i,h)(j,k)\in\mathcal{J}^c} (\delta_{(i,h+1)(j,k)} z + \pi_{\mathfrak{p}} \tilde{e}_{(j,k)(i,h+1)} + \delta_{(i,h)(j,k)}) E_{(i,h)(j,k)} \,.$$

Choose the following ordering for the elements in $\mathcal{T}$:

$$(i,h) < (j,k) \text{ if } x(ih) > x(jk), \text{ or if } x(ih) = x(jk) \text{ and } i < j \,. \tag{2.1.7}$$

By definition of the map $\pi_{\mathfrak{p}}$ and Equation (1.1.8), $\pi_{\mathfrak{p}} \tilde{e}_{(j,k)(i,h+1)} = 0$ if $x(jk) - x(ih) \geq 0$. Note that this is not a necessary condition in the case $\mathfrak{l} \neq 0$. It follows that $(A_{\mathcal{I}^c\mathcal{J}^c})_{(i,h)(j,k)} = \delta_{(i,h)(j,k)}$ for $(i,h) \geq (j,k)$. Hence we can write $A_{\mathcal{I}^c\mathcal{J}^c} = \mathbb{1}_{N-r} + N$, where $N \in \mathrm{Mat}_{N-r\times N-r} U(\mathfrak{g})[z]$ is a strictly upper triangular matrix.

Remark 2.1.1. Note that the ordering given in Equation 2.1.7 is equivalent to the numbering from 1 to N of the boxes of the pyramid starting from the right bottom corner and moving bottom to top and right to left.

As a consequence, we can expand the inverse $(A_{\mathcal{I}^c\mathcal{J}^c})^{-1}$ matrix as geometric power series and get, for every $(i_\alpha, h_\alpha), (i_\beta, h_\beta) \in \mathcal{J}^c$,

$$\begin{aligned} (A_{\mathcal{I}^c\mathcal{J}^c})^{-1}_{(i_\alpha,h_\alpha)(i_\beta,h_\beta)} &= \delta_{(i_\alpha,h_\alpha)(i_\beta,h_\beta)} + \sum_{l>0} (-1)^l \sum_{(i_0,h_0)\dots(i_l,h_l)\in\mathcal{J}^c} \delta_{(i_0,h_0)(i_\alpha,h_\alpha)} \delta_{(i_l,h_l)(i_\beta,h_\beta)} \\ &\times (\delta_{(i_0,h_0+1)(i_1,h_1)} z + \pi_{\mathfrak{p}} \tilde{e}_{(i_1,h_1)(i_0,h_0+1)}) \cdots (\delta_{(i_{l-1},h_{l-1}+1)(i_l,h_l)} z + \pi_{\mathfrak{p}} \tilde{e}_{(i_l,h_l)(i_{l-1},h_{l-1}+1)}) \,. \end{aligned} \tag{2.1.8}$$

Regarding the other submatrices of A we have, for $(i,1) \in \mathcal{I}$, $(j,p_j) \in \mathcal{J}$, $(i,h),(j,k) \in \mathcal{J}^c$:

$$\begin{aligned} (A_{\mathcal{I}\mathcal{J}})_{(i,1)(j,p_j)} &= \delta_{(i,1)(j,p_j)} z + \tilde{e}_{(j,p_j)(i,1)} \,, \\ (A_{\mathcal{I}\mathcal{J}^c})_{(i,1)(j,k)} &= \delta_{(i,1)(j,k)} z + \pi_{\mathfrak{p}} \tilde{e}_{(j,k)(i,1)} \,, \\ (A_{\mathcal{I}^c\mathcal{J}})_{(i,h)(j,p_j)} &= \delta_{(i,h+1)(j,p_j)} z + \pi_{\mathfrak{p}} \tilde{e}_{(j,p_j)(i,h+1)} \,. \end{aligned} \tag{2.1.9}$$

Combining (2.1.9) and (2.1.6), and by the bijection (2.1.2), we get for every $1 \leq i,j \leq r$,

$$\begin{aligned} T_{ij}(z) &= \delta_{(i,1)(j,p_j)} z + \tilde{e}_{(j,p_j)(i,1)} - \sum_{l\geq 0} (-1)^l \sum_{(i_0,h_0),\dots,(i_l,h_l)\in\mathcal{J}^c} (\delta_{(i,1)(i_0,h_0)} z + \pi_{\mathfrak{p}} \tilde{e}_{(i_0,h_0)(i,1)}) \\ &\times (\delta_{(i_0,h_0+1)(i_1,h_1)} z + \pi_{\mathfrak{p}} \tilde{e}_{(i_1,h_1)(i_0,h_0+1)}) \cdots (\delta_{(i_{l-1},h_{l-1}+1)(i_l,h_l)} z + \pi_{\mathfrak{p}} \tilde{e}_{(i_l,h_l)(i_{l-1},h_{l-1}+1)}) \\ &\times (\delta_{(i_l,h_l+1)(j,p_j)} z + \pi_{\mathfrak{p}} \tilde{e}_{(j,p_j)(i_l,h_l+1)}) \,. \end{aligned}$$

□

Remark 2.1.2. By Equation (2.1.8), Proposition 2.1.1 also proves that the quasideterminant $T(z)$ exists in the algebra $\mathrm{Mat}_{r\times r} U(\mathfrak{g})((z^{-1}))$.

As a corollary, we obtain

Corollary 2.1.1. *Gathering the powers of z in the RHS of (2.1.5), a straightforward computation shows that*

$$\begin{aligned} T_{ij}(z) &= -(-z)^{p_j} \delta_{ij} - \sum_{k\geq 0} (-z)^k \sum_{s\geq 1} (-1)^s \sum_{\substack{n_i\geq 0 \\ n_0+\dots+n_s=k}} \sum_{\substack{(i_0,h_0),\dots,(i_{s-1},h_{s-1})\in\mathcal{J}^c \\ (i_s,h_s)\in\mathcal{T}}} \delta_{(i_0,h_0)(i,1)} \\ &\times \delta_{(i_s,h_s+n_s)(j,p_j)} \pi_{\mathfrak{p}} \tilde{e}_{(i_1,h_1)(i_0,h_0+n_0)} \pi_{\mathfrak{p}} \tilde{e}_{(i_2,h_2)(i_1,h_1+1+n_1)} \cdots \pi_{\mathfrak{p}} \tilde{e}_{(i_s,h_s)(i_{s-1},h_{s-1}+\delta_{s>1}+n_{s-1})} \,. \end{aligned} \tag{2.1.10}$$

The presence of the projections $\pi_{\mathfrak{p}}$ makes sure that $T_{ij}(z)$ has polynomial form. In fact, since $\mathfrak{p} \subseteq \mathfrak{g}_{\leq \frac{1}{2}}$, we must have $\deg e_{(i_t,h_t)(i_{t-1},h_{t-1}+\delta_{t>1}+n_{t-1})} \leq \frac{1}{2}$ *for any* $e_{(i_t,h_t)(i_{t-1},h_{t-1}+\delta_{t>1}+n_{t-1})}$, $1 \leq t \leq s$, *such that*

$$\pi_{\mathfrak{p}} e_{(i_t,h_t)(i_{t-1},h_{t-1}+\delta_{t>1}+n_{t-1})} \neq 0\,.$$

As a consequence, we obtain the following upper bound on k: $k \leq \frac{3}{2} - x(j,p_j) + x(i,1)$ (which reduces to $k \leq 1 - x(j,p_j) + x(i,1)$ in the case of an even $\frac{1}{2}\mathbb{Z}$-grading Γ).

When $p_1 > 1$ and ${}'p$ exists we are able to describe the polynomials $T_{ij} \in U(\mathfrak{g})[z]$ recursively as follows.

Proposition 2.1.2. *Let $p_1 > 1$. Suppose $t_1 \leq s_1$ and suppose moreover that we may remove the leftmost column of p. Then the following recursive formula holds:*

$$T_{ij}(z) = \begin{cases} \sigma_l(T_{ij}^{'p}(z))\,, & j \geq t_1 + 1 \\ [e_{(j,p_1)(j,p_1-1)}, \sigma_l(T_{ij}^{'p}(z))] - \sum_{h=1}^{t_1} \sigma_l(T_{ih}^{'p}(z))(\delta_{hj}z + \tilde{e}_{(j,p_1)(h,p_1)})\,, & j \leq t_1\,. \end{cases} \tag{2.1.11}$$

Proof. Let us first suppose $j \geq t_1 + 1$, in this case $p_j^{'p} = p_j$.

We define the $r \times r$ matrix $T^{'p}(z)$ for the reduced pyramid ${}'p$ as in Equation (2.1.3). By Equation (2.1.5) for $T_{ij}^{'p}(z)$, we have

$$\begin{aligned} T_{ij}^{'p}(z) = \delta_{(i,1)(j,p_j)}z + \tilde{e}_{(j,p_j)(i,1)} - \sum_{l\geq 0}(-1)^l \sum_{(i_0,h_0),\ldots,(i_l,h_l)\in(\mathcal{J}^{'p})^c} (\delta_{(i,1)(i_0,h_0)}z + \pi_{\mathfrak{p}}\tilde{e}_{(i_0,h_0)(i,1)}) \\ \times (\delta_{(i_0,h_0+1)(i_1,h_1)}z + \pi_{\mathfrak{p}}\tilde{e}_{(i_1,h_1)(i_0,h_0+1)}) \cdots (\delta_{(i_{l-1},h_{l-1}+1)(i_l,h_l)}z + \pi_{\mathfrak{p}}\tilde{e}_{(i_l,h_l)(i_{l-1},h_{l-1}+1)}) \\ \times (\delta_{(i_l,h_l+1)(j,p_j)}z + \pi_{\mathfrak{p}}\tilde{e}_{(j,p_j)(i_l,h_l+1)})\,, \end{aligned}$$

where

$$(\mathcal{J}^{'p})^c = \{(j,k) \in \mathcal{T} \mid 1 \leq j \leq t_1, 1 \leq k < p_1 - 1\} \sqcup \{(j,k) \in \mathcal{T} \mid t_1 < j \leq r, 1 \leq k < p_j\}\,.$$

By 1.1.24, σ_l commutes with the projection $\pi_{\mathfrak{p}}$. It is then sufficient to show that the sum in the RHS of Equation (2.1.5) actually runs over $(\mathcal{J}^{'p})^c$ as well. Note that $x(i,h) > -\frac{d}{2} + 1$ for $(i,h) \in (\mathcal{J}^{'p})^c$ and that $x(i,h) = -\frac{d}{2} + 1$ for $(i,h) \in \mathcal{J}^c \smallsetminus (\mathcal{J}^{'p})^c = \{(j,p_1 - 1) \mid 1 \leq j \leq t_1\}$.

Consider the last factor $\pi_{\mathfrak{p}} e_{(j,p_j)(i_l,h_l+1)}$ in (2.1.5). We claim that $\pi_{\mathfrak{p}} e_{(j,p_j)(i_l,h_l+1)} = 0$ for every $(i_l,h_l) \in \mathcal{J}^c \smallsetminus (\mathcal{J}^{'p})^c$. In fact, $j \geq t_1 + 1$ we have $x(j,p_j) \geq -\frac{d}{2} + 1$ and for $(i_l,h_l) \in \mathcal{J}^c \smallsetminus (\mathcal{J}^{'p})^c$ we have

$$\deg\, e_{(j,p_j)(i_l,h_l+1)} = x(j,p_j) - x(i_l,h_l) + 1 = x(j,p_j) + \frac{d}{2} - 1 + 1 \geq -\frac{d}{2} + 1 + \frac{d}{2} \geq 1\,.$$

Therefore, $(i_l,h_l) \in (\mathcal{J}^{'p})^c$.

Consider now $\pi_{\mathfrak{p}}\, e_{(i_s,h_s)(i_{s-1},h_{s-1}+1)}$, for $1 \leq s \leq l$ and $(i_s,h_s) \in (\mathcal{J}^{'p})^c$. For $(i_{s-1},h_{s-1}) \in \mathcal{J}^c \smallsetminus (\mathcal{J}^{'p})^c$ we have

$$\deg\, e_{(i_s,h_s)(i_{s-1},h_{s-1}+1)} = x(i_s,h_s) - x(i_{s-1},h_{s-1}) + 1 = x(i_s,h_s) + \frac{d}{2} > 1\,.$$

For $\pi_{\mathfrak{p}}\, e_{(i_s,h_s)(i_{s-1},h_{s-1}+1)}$ to be nonzero, we therefore have $(i_{s-1},h_{s-1}) \in (\mathcal{J}^{'p})^c$ for all $1 \leq s \leq l$. As a consequence,

$$\begin{aligned} T_{ij}(z) = \delta_{(i,1)(j,p_j)}z + \sigma_l(\tilde{e}_{(j,p_j)(i,1)}) - \sum_{l\geq 0}(-1)^l \sum_{(i_0,h_0),\ldots,(i_l,h_l)\in(\mathcal{J}^{'p})^c} (\delta_{(i,1)(i_0,h_0)}z + \pi_{\mathfrak{p}}\sigma_l(\tilde{e}_{(i_0,h_0)(i,1)})) \\ \times (\delta_{(i_0,h_0+1)(i_1,h_1)}z + \pi_{\mathfrak{p}}\sigma_l(\tilde{e}_{(i_1,h_1)(i_0,h_0+1)})) \cdots (\delta_{(i_{l-1},h_{l-1}+1)(i_l,h_l)}z + \pi_{\mathfrak{p}}\sigma_l(\tilde{e}_{(i_l,h_l)(i_{l-1},h_{l-1}+1)})) \\ \times (\delta_{(i_l,h_l+1)(j,p_j)}z + \pi_{\mathfrak{p}}\sigma_l(\tilde{e}_{(j,p_j)(i_l,h_l+1)})) = \sigma_l(T_{ij}^{'p}(z))\,. \end{aligned}$$

Suppose now that $j \leq t_1$. In this case $p_j^{'p} = p_j - 1 = p_1 - 1$. Let us rewrite Equation (2.1.5) as $T_{ij}(z) = A + B + C$, with

$$
\begin{aligned}
A &= \delta_{(i,1)(j,p_1)} z + \tilde{e}_{(j,p_1)(i,1)} = e_{(j,p_1)(i,1)}\,,\\
B &= -\sum_{l\geq 0} (-1)^l \sum_{\substack{(i_0,h_0),\dots,(i_{l-1},h_{l-1})\in\mathcal{J}^c\\ (i_l,h_l)\in(\mathcal{J}^{'p})^c}} \big(\delta_{(i,1)(i_0,h_0)} z + \pi_{\mathfrak{p}}\tilde{e}_{(i_0,h_0)(i,1)}\big)\big(\delta_{(i_0,h_0+1)(i_1,h_1)} z + \pi_{\mathfrak{p}}\tilde{e}_{(i_1,h_1)(i_0,h_0+1)}\big)\\
&\quad \times \cdots \times \big(\delta_{(i_{l-1},h_{l-1}+1)(i_l,h_l)} z + \pi_{\mathfrak{p}}\tilde{e}_{(i_l,h_l)(i_{l-1},h_{l-1}+1)}\big) e_{(j,p_1)(i_l,h_l+1)}\,,\\
C &= -\sum_{l\geq 0} (-1)^l \sum_{(i_0,h_0),\dots,(i_{l-1},h_{l-1})\in\mathcal{J}^c} \big(\delta_{(i,1)(i_0,h_0)} z + \pi_{\mathfrak{p}}\tilde{e}_{(i_0,h_0)(i,1)}\big)\big(\delta_{(i_0,h_0+1)(i_1,h_1)} z + \pi_{\mathfrak{p}}\tilde{e}_{(i_1,h_1)(i_0,h_0+1)}\big)\\
&\quad \times \cdots \times \sum_{m=1}^{t_1} \big(\delta_{(i_{l-1},h_{l-1}+1)(m,p_1-1)} z + \pi_{\mathfrak{p}}\tilde{e}_{(m,p_1-1)(i_{l-1},h_{l-1}+1)}\big)\big(\delta_{mj} z + \pi_{\mathfrak{p}}\tilde{e}_{(j,p_1)(m,p_1)}\big)\,.
\end{aligned}
$$

Then we have

$$
A = [e_{(j,p_1)(j,p_1-1)}, e_{(j,p_1-1)(i,1)}] = [e_{(j,p_1)(j,p_1-1)}, \delta_{(i,1)(j,p_1-1)} z + \sigma_l(\tilde{e}_{(j,p_1-1)(i,1)})]\,,
$$

whereas

$$
\begin{aligned}
B &= -\sum_{l\geq 0} (-1)^l \sum_{\substack{(i_0,h_0),\dots,(i_{l-1},h_{l-1})\in\mathcal{J}^c\\ (i_l,h_l)\in(\mathcal{J}^{'p})^c}} \big(\delta_{(i,1)(i_0,h_0)} z + \pi_{\mathfrak{p}}\tilde{e}_{(i_0,h_0)(i,1)}\big)\big(\delta_{(i_0,h_0+1)(i_1,h_1)} z + \pi_{\mathfrak{p}}\tilde{e}_{(i_1,h_1)(i_0,h_0+1)}\big)\\
&\quad \times \cdots \times \big(\delta_{(i_{l-1},h_{l-1}+1)(i_l,h_l)} z + \pi_{\mathfrak{p}}\tilde{e}_{(i_l,h_l)(i_{l-1},h_{l-1}+1)}\big)[e_{(j,p_1)(j,p_1-1)}, (\delta_{ij} z + \pi_{\mathfrak{p}}\sigma_l(\tilde{e}_{(j,p_1)(i_l,h_l+1)}))]\,,
\end{aligned}
$$

since

$$
e_{(j,p_1)(i_l,h_l+1)} = [e_{(j,p_1)(j,p_1-1)}, \delta_{ij} z + \pi_{\mathfrak{p}}\sigma_l(\tilde{e}_{(j,p_1-1)(i_l,h_l+1)})]\,.
$$

Moreover, as in the case $j \geq t_1 + 1$, $(i_l, h_l) \in (\mathcal{J}^{'p})^c$ implies $(i_s, h_s) \in (\mathcal{J}^{'p})^c$ for all $0 \leq s \leq l-1$. Therefore,

$$
\begin{aligned}
B &= -\sum_{l\geq 0} (-1)^l \sum_{(i_0,h_0),\dots,(i_{l-1},h_{l-1})\in(\mathcal{J}^{'p})^c} [e_{(j,p_1)(j,p_1-1)}, \big(\delta_{(i,1)(i_0,h_0)} z + \pi_{\leq 0}\tilde{e}_{(i_0,h_0)(i,1)}\big)\big(\delta_{(i_0,h_0+1)(i_1,h_1)} z + \pi_{\leq 0}\tilde{e}_{(i_1,h_1)(i_0,h_0+1)}\big)\\
&\quad \times \cdots \times \big(\delta_{(i_{l-1},h_{l-1}+1)(i_l,h_l)} z + \pi_{\leq 0}\tilde{e}_{(i_l,h_l)(i_{l-1},h_{l-1}+1)}\big)\big(\delta_{ij} z + \pi_{\leq 0}\sigma_l(\tilde{e}_{(j,p_1)(i_l,h_l+1)})\big)]
\end{aligned}
$$

since

$$
\begin{aligned}
&[e_{(j,p_1)(j,p_1-1)}, \pi_{\mathfrak{p}}\sigma_l(\tilde{e}_{(i_0,h_0)(i,1)})] = 0, \qquad \text{for } (i_0,h_0) \in (\mathcal{J}^{'p})^c\,,\\
&[e_{(j,p_1)(j,p_1-1)}, \pi_{\mathfrak{p}}\sigma_l(\tilde{e}_{(i_s,h_s)(i_{s-1},h_{s-1}+1)})] = 0, \qquad \text{for } (i_s,h_s) \in (\mathcal{J}^{'p})^c,\ 1 \leq s \leq l-1\,.
\end{aligned}
$$

As a consequence, $A + B = [e_{(j,p_1)(j,p_1-1)}, \sigma_l(T_{ij}^{'p}(z))]$.
Let us analyze C. We can rewrite it as

$$
\begin{aligned}
C &= -\sum_{m=1}^{t_1} \big(\delta_{(i,1)(m,p_1-1)} z + \pi_{\mathfrak{p}}\tilde{e}_{(m,p_1-1)(i,1)}\big)\big(\delta_{mj} z + \pi_{\mathfrak{p}}\tilde{e}_{(j,p_1)(m,p_1)}\big)\\
&\quad - \sum_{l\geq 1} (-1)^l \sum_{(i_0,h_0),\dots,(i_{l-1},h_{l-1})\in\mathcal{J}^c} \big(\delta_{(i,1)(i_0,h_0)} z + \pi_{\mathfrak{p}}\tilde{e}_{(i_0,h_0)(i,1)}\big)\big(\delta_{(i_0,h_0+1)(i_1,h_1)} z + \pi_{\mathfrak{p}}\tilde{e}_{(i_1,h_1)(i_0,h_0+1)}\big)\\
&\quad \times \cdots \times \sum_{m=1}^{t_1} \big(\delta_{(i_{l-1},h_{l-1}+1)(m,p_1-1)} z + \pi_{\mathfrak{p}}\tilde{e}_{(m,p_1-1)(i_{l-1},h_{l-1}+1)}\big)\big(\delta_{mj} z + \pi_{\mathfrak{p}}\tilde{e}_{(j,p_1)(m,p_1)}\big)\,.
\end{aligned}
$$

Let us consider first the term $\pi_{\mathfrak{p}}\tilde{e}_{(j,p_1)(m,p_1)}$: for $(i_{l-1}, h_{l-1}) \in \mathcal{J}^c \setminus (\mathcal{J}^{'p})^c$ we have

$$
\deg e_{(m,p_1-1)(i_{l-1},h_{l-1}+1)} = -\frac{d}{2} + 1 - (-\frac{d}{2} + 1) + 1 > 1\,,
$$

therefore it must be $(i_{l-1}, h_{l-1}) \in (\mathcal{J}'^p)^c$. Then , as in the case $j \geq t_1+1$, we can conclude that $(i_s, h_s) \in (\mathcal{J}'^p)^c$ for all $0 \leq s \leq l-1$. We obtain,

$$\begin{aligned}
C = &- \sum_{m=1}^{t_1} (\delta_{(i,1)(m,p_1-1)} z + \pi_\mathfrak{p} \tilde{e}_{(m,p_1-1)(i,1)})(\delta_{mj} z + \pi_\mathfrak{p} \tilde{e}_{(j,p_1)(m,p_1)}) \\
&+ \sum_{q \geq 0} (-1)^q \sum_{(i_0,h_0),\dots,(i_q,h_q) \in (\mathcal{J}'^p)^c} (\delta_{(i,1)(i_0,h_0)} z + \pi_\mathfrak{p} \tilde{e}_{(i_0,h_0)(i,1)})(\delta_{(i_0,h_0+1)(i_1,h_1)} z + \pi_\mathfrak{p} \tilde{e}_{(i_1,h_1)(i_0,h_0+1)}) \\
&\times \cdots \times \sum_{m=1}^{t_1} (\delta_{(i_q,h_q+1)(m,p_1-1)} z + \pi_\mathfrak{p} \tilde{e}_{(m,p_1-1)(i_q,h_q+1)})(\delta_{mj} z + \pi_\mathfrak{p} \tilde{e}_{(j,p_1)(m,p_1)}) \\
&= - \sum_{m=1}^{t_1} \sigma_l(T'^p_{im}(z))(\delta_{mj} z + \pi_\mathfrak{p} \tilde{e}_{(j,p_1)(m,p_1)}) \,.
\end{aligned}$$

Combining A, B and C we get, for $j \leq t_1$,

$$T_{ij}(z) = [e_{(j,p_1)(j,p_1-1)}, \sigma_l(T'^p_{ij}(z))] - \sum_{h=1}^{t_1} \sigma_l(T'^p_{ih}(z))(\delta_{hj} z + \tilde{e}_{(j,p_1)(h,p_1)}) \,,$$

as claimed. ☐

Proposition 2.1.3. *Let $p_1 > 1$. Suppose $s_1 \leq t_1$ and suppose moreover that we may remove the rightmost column of p. Then the following recursive formula holds:*

$$T_{ij}(z) = \begin{cases} \sigma_r(T^{p'}_{ij}(z)) \,, & i \geq s_1 + 1 \\ [\sigma_r(T^{p'}_{ij}(z)) \,, e_{(i,2)(i,1)}] - \sum_{h=1}^{s_1} (\delta_{hi} z + e_{(h,1)(i,1)}) \sigma_r(T^{p'}_{hj}(z)), & i \leq s_1 \,. \end{cases} \tag{2.1.12}$$

Proof. Starting from Equation (2.1.5), the proof is analogous to the proof of Proposition 2.1.2. Here, we define the $r \times r$ matrix $T^{p'}(z)$ for the reduced pyramid p' as in Equation (2.1.3). ☐

We now address the issue of the recursions introduced in Propositions 2.1.2 and 2.1.3 to be well-defined. Namely, we want to check that in the case that it is possible to remove both the leftmost and the rightmost column of p and moreover $t_1 = s_1$, the choice of the recursion is irrelevant.

Proposition 2.1.4. *Let $p_1 > 1$ and $t_1 = s_1$. Suppose moreover that it is possible to remove both the leftmost and the rightmost column of p. Then applying the recursions in the two different orders gives the same result. We are supposing $\sigma_r \circ \sigma_l$, $\sigma_l \circ \sigma_r : U(\mathfrak{gl}_{N-t_1-s_1}) \longrightarrow U(\mathfrak{g})$.*

Proof. For $p_1 = 2$ (p is therefore a rectangle and $t_1 = s_1 = r$), the claim is obvious, since by application of the left recursion we obtain

$$\begin{aligned}
T_{ij}(z) &= [e_{(j,2)(j,1)}, \delta_{ij} z + e_{(j,1)(i,1)}] - \sum_{h=1}^{t_1} (\delta_{ih} z + e_{(h,1)(i,1)})(\delta_{hj} z + \tilde{e}_{(j,2)(h,2)}) \\
&= e_{(j,2)(i,1)} - \sum_{h=1}^{t_1} (\delta_{ih} z + e_{(h,1)(i,1)})(\delta_{hj} z + e_{(j,2)(h,2)} - \delta_{hj} t_1)
\end{aligned}$$

while by application of the right recursion

$$\begin{aligned}
T_{ij}(z) &= [\delta_{ij} z + e_{(j,1+\delta_{j \leq s_1})(i,1+\delta_{i \leq s_1})}, e_{(i,2)(i,1)}] - \sum_{h=1}^{s_1} (\delta_{hi} z + e_{(h,1)(i,1)})(\delta_{hj} z + e_{(j,1+\delta_{j \leq s_1})(h,1+\delta_{h \leq s_1})} - \delta_{hj} s_1) \\
&= e_{(j,2)(i,1)} - \sum_{h=1}^{s_1} (\delta_{hi} z + e_{(h,1)(i,1)})(\delta_{hj} z + e_{(j,2)(h,2)} - \delta_{hj} s_1) \,,
\end{aligned}$$

and the two expressions clearly coincide.

Suppose now $p_1 > 2$. Applying first the right recursion (Proposition 2.1.3) and then the left recursion (Proposition 2.1.2) we get

$$T_{ij}(z) = \begin{cases} \sigma_r(\sigma_l(T_{ij}^{'p'}(z)))\,, & i \geq s_1+1,\, j \geq t_1+1 \\ [e_{(j,p_1)(j,p_1-1)}, \sigma_r(\sigma_l(T_{ij}^{'p'}(z)))] - \sum_{t=1}^{t_1} \sigma_r(\sigma_l(T_{it}^{'p'}(z)))(\delta_{tj}z + \tilde{e}_{(j,p_1)(t,p_1)})\,, & i \geq s_1+1,\, j \leq t_1 \\ [\sigma_r(\sigma_l(T_{ij}^{'p'}(z))), e_{(i,2)(i,1)}] - \sum_{h=1}^{s_1}(\delta_{hi}z + e_{(h,1)(i,1)})\sigma_r(\sigma_l(T_{hj}^{'p'}(z)))\,, & i \leq s_1,\, j \geq t_1+1 \\ [[e_{(j,p_1)(j,p_1-1)}, \sigma_r(\sigma_l(T_{ij}^{'p'}(z)))] - \sum_{t=1}^{t_1} \sigma_r(\sigma_l(T_{ih}^{'p'}(z)))(\delta_{hj}z + \tilde{e}_{(j,p_1)(h,p_1)}), e_{(i,2)(i,1)}] & \\ \quad - \sum_{h=1}^{s_1}(\delta_{hi}z + e_{(h,1)(i,1)})\Big([e_{(j,p_1)(j,p_1-1)}, \sigma_r(\sigma_l(T_{hj}^{'p'}(z)))] & \\ \quad - \sum_{t=1}^{t_1} \sigma_r(\sigma_l(T_{ht}^{'p'}(z)))(\delta_{tj}z + \tilde{e}_{(j,p_1)(t,p_1)})\Big)\,, & i \leq s_1,\, j \leq t_1\,. \end{cases}$$

On the other hand, applying first the left recursion and then the right recursion we get

$$T_{ij}(z) = \begin{cases} \sigma_l\sigma_r(T_{ij}^{'p'}(z))\,, & i \geq s_1+1,\, j \geq t_1+1 \\ [\sigma_l\sigma_r(T_{ij}^{'p'}(z)), e_{(i,2)(i,1)}] - \sum_{h=1}^{s_1}(\delta_{hi}z + e_{(h,1)(i,1)})\sigma_l\sigma_r(T_{hj}^{'p'}(z))\,, & i \leq s_1,\, j \geq t_1+1 \\ [e_{(j,p_1)(j,p_1-1)}, \sigma_l\sigma_r(T_{ij}^{'p'}(z))] - \sum_{t=1}^{t_1} \sigma_l\sigma_r(T_{it}^{'p'}(z))(\delta_{tj}z + \tilde{e}_{(j,p_1)(t,p_1)})\,, & i \geq s_1+1,\, j \leq t_1 \\ [e_{(j,p_1)(j,p_1-1)}, \Big([\sigma_l\sigma_r(T_{ij}^{'p'}(z)), e_{(i,2)(i,1)}] - \sum_{h=1}^{s_1}(\delta_{hi}z + e_{(h,1)(i,1)})\sigma_l\sigma_r(T_{hj}^{'p'}(z))\Big)] & \\ \quad - \sum_{t=1}^{t_1}\Big([\sigma_l\sigma_r(T_{it}^{'p'}(z)), e_{(i,2)(i,1)}] & \\ \quad - \sum_{h=1}^{s_1}(\delta_{hi}z + e_{(h,1)(i,1)})\sigma_l\sigma_r(T_{ht}^{'p'}(z))\Big)(\delta_{tj}z + \tilde{e}_{(j,p_1)(t,p_1)})\,, & i \leq s_1,\, j \leq t_1\,. \end{cases}$$

The result follows because $\sigma_l \circ \sigma_r = \sigma_r \circ \sigma_l$, even for the non-obvious case when $j \leq t_1$ and $i \leq s_1$. ☐

2.2 Definition of the matrix $L(z)$

We will now introduce the second main object of our construction, the matrix $L(z)$, that will play a key role in Conjecture 3.1.1. In the case of Γ a Dynkin grading and $\mathfrak{l} = 0$, the matrix $L(z)$ has already been introduced in [DSKV16c].

Recall that $r_1 = \min(t_1, s_1)$ is the height of the maximal rectangular block at the bottom of p. Let us introduce the matrices

$$I_1 = \sum_{i=1}^{r_1} E_{(i,1)i} \in \mathrm{Mat}_{N\times r_1}\mathbb{C}\,, \qquad J_1 = \sum_{i=1}^{r_1} E_{i(i,p_1)} \in \mathrm{Mat}_{r_1\times N}\mathbb{C}\,, \tag{2.2.1}$$

corresponding to the index subsets of $\mathcal{T}$ (of cardinality r_1)

$$\mathcal{I}_1 = \{(i,1) \mid 1 \leq i \leq r_1\}\,, \qquad \mathcal{J}_1 = \{(i,p_i) \mid 1 \leq i \leq r_1\}\,. \tag{2.2.2}$$

Definition 2.2.1. Define the matrix

$$\widetilde{L}(z) = |z\mathbb{1}_N + F + \pi_{\mathfrak{p}}E + D_{\mathfrak{l}}|_{I_1J_1} \in \mathrm{Mat}_{r_1\times r_1}U(\mathfrak{g})((z^{-1}))\,. \tag{2.2.3}$$

Proposition 2.2.1. *The quasideterminant $\widetilde{L}(z)$ exists in the algebra $Mat_{r_1\times r_1}U(\mathfrak{g})((z^{-1}))$. Namely, the matrix $z\mathbb{1}_N + F + \pi_{\mathfrak{p}}E + D_{\mathfrak{l}}$ is invertible in the algebra $Mat_{N\times N}U(\mathfrak{g})((z^{-1}))$ and the matrix $J_1(z\mathbb{1}_N + F + \pi_{\mathfrak{p}}E + D_{\mathfrak{l}})^{-1}I_1$ is invertible in the algebra $Mat_{r_1\times r_1}U(\mathfrak{g})((z^{-1}))$.*

Proof. The proof is similar to the proof of [DSKV16c, Proposition 4.1]. ☐

As a consequence of (the proof of) Proposition 2.2.1, we have:

Corollary 2.2.1. *$\widetilde{L}(z) = (\widetilde{L}_{ij}(z))_{i,j=1}^{r_1}$ is a matrix whose entries are formal Laurent series in z^{-1} of degree p_1. Moreover, by the hereditary property of quasideterminants (1.5.5),*

$$\widetilde{L}(z) = ||z\mathbb{1}_N + F + \pi_{\mathfrak{p}}E + D_{\mathfrak{l}}|_{IJ}|_{I_1J_1} =: |T(z)|_{I_{rr_1}J_{r_1r}}\,, \tag{2.2.4}$$

where I, J are the matrices

$$I = \sum_{i=1}^{r} E_{(i,1)i} \in Mat_{N\times r}\mathbb{C}\,, \qquad J = \sum_{i=1}^{r} E_{i(i,p_i)} \in Mat_{r\times N}\mathbb{C}\,, \tag{2.2.5}$$

corresponding to the subsets $\mathcal{I}$, $\mathcal{J}$ in 2.1.1 (such that $T(z) = |z\mathbb{1}_N + F + \pi_{\mathfrak{p}}E + D_{\mathfrak{l}}|_{IJ}$, according to Definition 1.5.3), and I_{rr_1}, J_{r_1r} are the matrices

$$I_{rr_1} = (\delta_{ij})_{\substack{i\in\{1,\dots,r\}\\ j\in\{1,\dots,r_1\}}} \in Mat_{r\times r_1}\mathbb{C}\,, \qquad J_{r_1r} = (\delta_{ji})_{\substack{i\in\{1,\dots,r\}\\ j\in\{1,\dots,r_1\}}} \in Mat_{r_1\times r}\mathbb{C}\,. \tag{2.2.6}$$

corresponding to the subsets $\mathcal{I}_{rr_1} = \mathcal{J}_{r_1r} = \{1,\dots,r_1\}$ of $\{1,\dots,r\}$ (cf. Section 1.5).

Suppose that $p_1 > 1$, namely that the pyramid p doesn't consist of a single column, suppose that $r_1 = t_1 \le s_1$ and that moreover it is possible to remove the leftmost column of p. We obtain a pyramid $'p$ of size $N - r_1$ which uniquely defines a good $\frac{1}{2}\mathbb{Z}$-grading for the Lie algebra $\mathfrak{gl}_{N-r_1}$ and the nilpotent element $f^{'p}$ associated with the partition $(p_1-1,\dots,p_{r_1}-1,p_{r_1+1},\dots,p_r)$ of N. From these data we can define a matrix $T^{'p}(z)$ as in Definition 2.1.1 and a matrix $\widetilde{L}^{'p}(z)$ as in Definition 2.2.3. Note that while $T^{'p}(z)$ is still a $r\times r$ matrix, $\widetilde{L}^{'p}(z) = |T^{'p}(z)|_{I_{rr_2}J_{r_2r}}$ is a $r_2\times r_2$ matrix, where $r_2 = \min(t_2, s_1) \ge r_1$ is the second minimal column height in p and I_{rr_2}, J_{r_2r} are matrices as in (2.2.6) corresponding to the subsets $\mathcal{I}_{rr_2} = \mathcal{J}_{r_2r} = \{1,\dots,r_2\}$.

As we did for the matrix $T(z)$ in Section 2.1, it is possible to construct $\widetilde{L}(z)$ recursively through the matrix $\widetilde{L}^{'p}(z)$.

Proposition 2.2.2. *Let $p_1 > 1$. Suppose $r_1 = t_1 \le s_1$ and suppose moreover that it is possible to remove the leftmost column of p. Then the following recursive formula holds for every $1 \le i, j \le r_1$:*

$$\widetilde{L}_{ij}(z) = [e_{(j,p_1)(j,p_1-1)}, \sigma_l((|\widetilde{L}^{'p}(z)|_{I_{r_2r_1}J_{r_1r_2}})_{ij})] - \sum_{h=1}^{r_1} \sigma_l((|\widetilde{L}^{'p}(z)|_{I_{r_2r_1}J_{r_1r_2}})_{ih})(\delta_{jh}z + \tilde{e}_{(j,p_1)(h,p_1)})\,. \tag{2.2.7}$$

Note that the matrices $I_{r_2r_1}$, $J_{r_2r_1}$ for the quasideterminant $|\widetilde{L}^{'p}(z)|_{I_{r_2r_1}J_{r_1r_2}}$ are as in (2.2.6), corresponding to the subset $\{1,\dots,r_1\}$ of $\{1,\dots,r_2\}$.

Proof. First of all note that by the hereditary property of quasideterminants (1.5.5),

$$|\widetilde{L}^{'p}(z)|_{I_{r_2r_1}J_{r_1r_2}} = ||T^{'p}(z)|_{I_{rr_2}J_{r_2r}}|_{I_{r_2r_1}J_{r_1r_2}} = |T^{'p}(z)|_{I_{rr_1}J_{r_1r}}\,. \tag{2.2.8}$$

By (2.2.4), the definition of quasideterminant[1] and Proposition 2.1.2, we have

$$\begin{aligned}
\widetilde{L}_{ij}(z) &= T_{ij}(z) - \sum_{h,k\ge r_1+1} T_{ik}(z)((T(z)_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c})^{-1})_{kh}T_{hj}(z)\\
&= T_{ij}(z) - \sum_{h,k\ge r_1+1} \sigma_l(T^{'p}_{ik}(z))((\sigma_l(T^{'p}(z)_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}))^{-1})_{kh}T_{hj}(z)\\
&= [e_{(j,p_1)(j,p_1-1)}, \sigma_l(T^{'p}_{ij}(z))] - \sum_{m=1}^{r_1} \sigma_l(T^{'p}_{im}(z))(\delta_{mj}z + \tilde{e}_{(j,p_1)(m,p_1)})\\
&- \sum_{h,k\ge r_1+1} \sigma_l(T^{'p}_{ik}(z))((\sigma_l(T^{'p}(z)_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}))^{-1})_{kh}[e_{(j,p_1)(j,p_1-1)}, \sigma_l(T^{'p}_{hj}(z))]\\
&+ \sum_{h,k\ge r_1+1} \sigma_l(T^{'p}_{ik}(z))((\sigma_l(T^{'p}(z)_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}))^{-1})_{kh} \sum_{m=1}^{r_1} \sigma_l(T^{'p}_{hm}(z))(\delta_{mj}z + \tilde{e}_{(j,p_1)(m,p_1)})\\
&= A + B + C + D\,.
\end{aligned}$$

[1] In order to simplify the notation we number the rows and columns of the matrix $(T^{'p}(z)_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c})^{-1}$ from r_1+1 to r instead of the natural numbering from 1 to $r - r_1$.

Then,

$$
\begin{aligned}
B+D &= -\sum_{m=1}^{r_1}\Big(\sigma_l(T'^p_{im}(z)) - \sum_{h,k\geq r_1+1}\sigma_l(T'^p_{ik}(z))((\sigma_l(T'^p(z)_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}))^{-1})_{kh}\sigma_l(T'^p_{hm}(z))\Big)(\delta_{mj}z+\tilde{e}_{(j,p_1)(m,p_1)}) \\
&= -\sum_{m=1}^{r_1}\sigma_l(|T'^p(z)|_{\mathcal{I}_{rr_1}\mathcal{J}_{r_1r}})_{im}(\delta_{mj}z+\tilde{e}_{(j,p_1)(m,p_1)}) \\
&= -\sum_{m=1}^{r_1}\sigma_l(|\widetilde{L}'^p(z)|_{I_{r_2r_1}J_{r_1r_2}})_{im}(\delta_{mj}z+\tilde{e}_{(j,p_1)(m,p_1)})\,,
\end{aligned}
$$

where for the last equality we have used Equation (2.2.8). Obviously, this holds provided that $(\sigma_l(T'^p(z)_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}))^{-1} = \sigma_l((T'^p(z)_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c})^{-1})$ in $\mathrm{Mat}_{r-r_1\times r-r_1}U(\mathfrak{g})[[z^{-1}]]$. This is clear by the definition of σ_l (cf. (1.1.20)), since the inverse of $T'^p(z)_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}$ can be expanded as geometric power series (cf. (2.1.10)). Note that the existence of this inverse is also ensured by the existence of the generalized quasideterminant $|T'^p(z)|_{I_{rr_1}J_{r_1r}}$, thanks to the hereditary property (1.5.5).

On the other hand,

$$
\begin{aligned}
A+C &= [e_{(j,p_1)(j,p_1-1)},\sigma_l(T'^p_{ij}(z)) - \sum_{h,k\geq r_1+1}\sigma_l(T'^p_{ik}(z))((\sigma_l(T'^p(z)_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}))^{-1})_{kh},\sigma_l(T'^p_{hj}(z))] \\
&- \sum_{h,k\geq r_1+1}[e_{(j,p_1)(j,p_1-1)},\sigma_l(T'^p_{ik}(z))((\sigma_l(T'^p(z)_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}))^{-1})_{kh}]\sigma_l(T'^p_{hj}(z)) \\
&= [e_{(j,p_1)(j,p_1-1)},\sigma_l(|T'^p(z)|_{I_{rr_1}J_{r_1r}})_{ij}] - \sum_{h,k\geq r_1+1}[e_{(j,p_1)(j,p_1-1)},\sigma_l(T'^p_{ik}(z))((\sigma_l(T'^p(z)_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}))^{-1})_{kh}]\sigma_l(T'^p_{hj}(z)) \\
&= [e_{(j,p_1)(j,p_1-1)},\sigma_l(|\widetilde{L}'^p(z)|_{I_{r_2r_1}J_{r_1r_2}})_{ij}] - \sum_{h,k\geq r_1+1}[e_{(j,p_1)(j,p_1-1)},\sigma_l(T'^p_{ik}(z))((\sigma_l(T'^p(z)_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}))^{-1})_{kh}]\sigma_l(T'^p_{hj}(z))\,,
\end{aligned}
$$

where for the last equality we have used Equation (2.2.8).

By Lemma 2.2.1 below it is therefore sufficient to show that $[e_{(j,p_1)(j,p_1-1)},\sigma_l(T'^p_{ik}(z))] = 0$ for every $k\geq r_1+1$. Recalling Equation (2.1.5), the latter holds because we have

$$
\begin{aligned}
&[e_{(j,p_1)(j,p_1-1)},\pi_{\mathfrak{p}}\sigma_l(e_{(k,p_k)(i,1)})] = 0\,, \\
&[e_{(j,p_1)(j,p_1-1)},\pi_{\mathfrak{p}}\sigma_l(e_{(i_0,h_0)(i,1)})] = 0\,, \\
&[e_{(j,p_1)(j,p_1-1)},\pi_{\mathfrak{p}}\sigma_l(e_{(i_q,h_q)(i_{q-1},h_{q-1}+1)})] = 0\,, \\
&[e_{(j,p_1)(j,p_1-1)},\pi_{\mathfrak{p}}\sigma_l(e_{(k,p_k)(i_l,h_l+1)})] = 0\,,
\end{aligned}
$$

where $(i_q,h_q)\in\mathcal{J}'^{p,c}$ for all $0\leq q\leq l$. □

Lemma 2.2.1. *[DSKV16c, Lemma 2.8] Let $\mathcal{U}$ be a unital associative algebra and let $A\in Mat_{N\times N}\mathcal{U}$ be invertible. For every $a\in\mathcal{U}$ and $i,j\in\{1,\dots,N\}$, we have*

$$
[a,(A^{-1})_{ij}] = -\sum_{h,k=1}^{N}(A^{-1})_{ih}[a,A_{hk}](A^{-1})_{kj}\,.
$$

Remark 2.2.1. When the nilpotent element f is rectangular, namely when it is associated with the partition $(p_1^{r_1})$ of N, then $r_1=r_2=r$ and Equation (2.2.7) reduces to

$$
\widetilde{L}_{ij}(z) = [e_{(j,p_1)(j,p_1-1)},\sigma_l(\widetilde{L}'^p_{ij}(z))] - \sum_{h=1}^{r_1}\sigma_l(\widetilde{L}'^p_{ih}(z))(\delta_{jh}z+\tilde{e}_{(j,p_1)(h,p_1)})\,.
$$

Note that in this case the corresponding pyramid p is a rectangle of size $p_1\times r_1$.

Similarly, let $r_1 = s_1 \le r_1$ and suppose that it is possible to remove the rightmost column of p. We obtain a pyramid p' of size $N - r_1$ which uniquely defines a good $\frac{1}{2}\mathbb{Z}$-grading for the Lie algebra $\mathfrak{gl}_{N-r_1}$ and the nilpotent element $f^{p'}$ associated with the partition $(p_1 - 1, \dots, p_{r_1} - 1, p_{r_1+1}, \dots, p_r)$ of N. From this data we can define a matrix $T^{p'}(z)$ as in Definition 2.1.1 and a matrix $\widetilde{L}^{p'}(z)$ as in Definition 2.2.3. Note that while $T^{p'}(z)$ is still a $r \times r$ matrix, $\widetilde{L}^{p'}(z) = |T^{p'}(z)|_{I_{rr_2} J_{r_2 r}}$ is a $r_2 \times r_2$ matrix, where $r_2 = \min(t_1, s_2) \ge s_1$ is the second minimal column height in p and $I_{rr_2}, J_{r_2 r}$ are matrices as in (2.2.6) corresponding to the subset $\{1, \dots, r_2\}$.

Using the right recursion for $T_{ij}(z)$ we can prove a result analogous to the one of Proposition 2.2.2:

Proposition 2.2.3. *Let $p_1 > 1$. Suppose $r_1 = s_1 \le r_1$ and suppose that it is possible to remove the rightmost column of p. Then the following recursive formula holds for every $1 \le i, j \le r_1$:*

$$\widetilde{L}_{ij}(z) = [\sigma_r((|\widetilde{L}^{p'}(z)|_{I_{r_2 r_1} J_{r_1 r_2}})_{ij}), e_{(i,2)(i,1)}] - \sum_{h=1}^{s_1} (z\delta_{hi} + e_{(h,1)(i,1)})\sigma_r((|\widetilde{L}^{p'}(z)|_{I_{r_2 r_1} J_{r_1 r_2}})_{hj}) .$$

2.2.1 Main properties of the matrix $L(z)$

Let $L(z) = \widetilde{L}(z)\,\bar{1}_{\mathfrak{l}} \in \mathrm{Mat}_{r_1 \times r_1} M_{\mathfrak{l}}((z^{-1}))$ be the image of $\widetilde{L}(z)$ in the quotient module $M_{\mathfrak{l}}$ (in other words, $L_{ij}(z)$ coincides with the image of $\widetilde{L}_{ij}(z)$ under the quotient map $\rho_{\mathfrak{l}} : U(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})/I_{\mathfrak{l}} = M_{\mathfrak{l}}$, cf. (1.1.14)). We can prove the following:

Theorem 2.2.1. *The matrix $L(z)$ lies in $Mat_{r_1 \times r_1} W(\mathfrak{g}, f, \Gamma, \mathfrak{l})((z^{-1}))$.*

Theorem 2.2.2. *$L(z)$ is an operator of Yangian type for the algebra $W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$.*

By Theorems 2.2.1 and 2.2.2, the matrix $L(z)$ is the quantum analogue of the matrix pseudodifferential operator of Adler type $L(\partial)$ introduced in the classical affine case (see [DSKV16b]).

Analogous results have already been proved in [DSKV16c, Theorems 4.2, 4.3] in the special case of a Dynkin $\frac{1}{2}\mathbb{Z}$-grading and the isotropic subspace $\mathfrak{l} = 0$. With Theorems 2.2.1 and 2.2.2 we are able to generalize these results for any good $\frac{1}{2}\mathbb{Z}$-grading Γ and for any isotropic subspace $\mathfrak{l}$. To this purpose, it is more useful to adopt the same notation as in [DSKV17b]. It would be natural to extend this coordinate-free approach to the previous sections, but so far we were unable to do so.

We still begin with a good $\frac{1}{2}\mathbb{Z}$-grading for $\mathfrak{g}$, $\Gamma : \mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$, which is given by the adjoint action of a certain element x, and a nilpotent element $f \in \mathfrak{g}_{-1}$. Chosen an isotropic subspace $\mathfrak{l} \subset \mathfrak{g}_{\frac{1}{2}}$ define $\mathfrak{m}$, $\mathfrak{n}$ in Section 1.1.1, and let $\pi_{\mathfrak{p}} : \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{m}$ be the quotient map. The definition of W-algebra is clearly unaltered.

Let V be a vector space of dimension N over $\mathbb{C}$, and let $\varphi : \mathfrak{g} \longrightarrow \mathrm{End}\, V$ be a faithful representation of $\mathfrak{g}$. In analogy in the notation of Section 1.1.1, which correspond to the choice $V = \mathbb{C}^N$ and φ the standard representation, we will denote with uppercase letters the images in $\mathrm{End}\, V$ through φ of the corresponding element in $\mathfrak{g}$: $A = \varphi(a)$ for each $a \in \mathfrak{g}$. Thus, $X = \varphi(x)$ is a semisimple endomorphism of V, and the corresponding X-eigenspace decomposition is $V = \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} V[k]$. As in Section 1.1.1, $\frac{d}{2}$ is the largest X-eigenvalue for V and d is the largest $\mathrm{ad}X$-eigenvalue for $\mathrm{End}\, V = \bigoplus_{k \in \frac{1}{2}\mathbb{Z}} (\mathrm{End}\, V)[k]$. We will use the shorthands $V[> k] = \bigoplus_{j > k} V[j]$ and $V[< k] = \bigoplus_{j < k} V[j]$.

We shall denote, for $k \in \frac{1}{2}\mathbb{Z}$, the maps

$$\begin{aligned} &\Psi_k : V[k] \hookrightarrow V\,, && \Pi_k : V \twoheadrightarrow V[k]\,, \\ &\Psi_{>k} : V[> k] \hookrightarrow V\,, && \Pi_{>k} : V \twoheadrightarrow V[> k]\,, \\ &\Psi_{<k} : V[< k] \hookrightarrow V\,, && \Pi_{<k} : V \twoheadrightarrow V[< k]\,. \end{aligned} \tag{2.2.9}$$

Recall that $r_1 = \min(t_1, s_1)$ is the minimal column length, which coincides with the height of the maximal rectangular block at the bottom of the pyramid p. Each box of the pyramid corresponds to a basis element of V, and the x-coordinates of the center of each box is the corresponding Γ-degree. We can then decompose $V = V_{r_1} \oplus V_{>r_1}$, where V_{r_1} is the subspace of V generated by the basis elements corresponding to the boxes of the pyramid in the maximal rectangular block at the bottom of p, and $V_{>r_1}$ is the subspace of V generated by the basis elements that correspond to the boxes of the pyramid above the maximal rectangular block. We shall denote, for $k \in \frac{1}{2}\mathbb{Z}$,

$$V[k]_{r_1} := V[k] \cap V_{r_1}\,, \qquad V[k]_{>r_1} := V[k] \cap V_{>r_1}\,. \tag{2.2.10}$$

Combining (2.2.9) and (2.2.10), for $k \in \frac{1}{2}\mathbb{Z}$ we can define the maps

$$\begin{aligned}
&\Psi_{k,r_1} : V[k]_{r_1} \hookrightarrow V[k] \overset{\Psi_k}{\hookrightarrow} V\,,\\
&\Psi_{k,>r_1} : V[k]_{>r_1} \hookrightarrow V[k] \overset{\Psi_k}{\hookrightarrow} V\,,\\
&\Pi_{k,r_1} : V \overset{\Pi_k}{\twoheadrightarrow} V[k] \twoheadrightarrow V[k]_{r_1}\,,\\
&\Pi_{k,>r_1} : V \overset{\Pi_k}{\twoheadrightarrow} V[k] \twoheadrightarrow V[k]_{>r_1}\,,
\end{aligned}$$

We therefore have short exact sequences

$$\begin{aligned}
&\chi_1 : 0 \longrightarrow V\big[>-\frac{d}{2}\big] \oplus V\big[-\frac{d}{2}\big]_{>r_1} \overset{\Psi_{>-\frac{d}{2},r_1}}{\longrightarrow} V \overset{\Pi_{-\frac{d}{2},r_1}}{\longrightarrow} V\big[-\frac{d}{2}\big]_{r_1} \longrightarrow 0\\
&\chi_2 : 0 \longrightarrow V\big[\frac{d}{2}\big]_{r_1} \overset{\Psi_{\frac{d}{2},r_1}}{\longrightarrow} V \overset{\Pi_{<\frac{d}{2},r_1}}{\longrightarrow} V\big[<\frac{d}{2}\big] \oplus V\big[\frac{d}{2}\big]_{>r_1} \longrightarrow 0\,.
\end{aligned} \tag{2.2.11}$$

where $\Psi_{>-\frac{d}{2},r_1} = \Psi_{>-\frac{d}{2}} + \Psi_{-\frac{d}{2},>r_1}$ and $\Pi_{<\frac{d}{2},r_1} = \Pi_{<\frac{d}{2}} + \Pi_{\frac{d}{2},>r_1}$.

Moreover, let $\{u_i\}_{i\in I}$ be a basis of $\mathfrak{g}$ compatible with the $\operatorname{ad} x$-eigenspace decomposition, i.e. $I = \sqcup_k I_k$ where $\{u_i\}_{i\in I_k}$ is a basis of $\mathfrak{g}_k$. Let $\{u^i\}_{i\in I}$ be the basis of $\mathfrak{g}$ dual to $\{u_i\}_{i\in I}$ with respect to the bilinear form $(\cdot|\cdot)$. We shall denote by $U_i = \varphi(u_i)$ and $U^i = \varphi(u^i)$ the corresponding elements in $\operatorname{End} V$. In the setting of Section 1.1.1, $\{u_i\}_{i\in I} = \{e_{(i,h)(j,k)}\}_{(i,h),(j,k)\in\mathcal{T}}$ and $\{U_i\} = \{E_{(i,h)(j,k)}\}_{(i,h),(j,k)\in\mathcal{T}}$; in this case, the dual element is just the transpose.

We shall also denote by $I_{\mathfrak{l}}$ and $I_{\mathfrak{l}^c}$ the indexing sets for the subspaces $\mathfrak{l}$ and $\mathfrak{l}^c$ and, without loss of generality, we may assume that $I_{\frac{1}{2}} = I_{\mathfrak{l}} \sqcup I_{\mathfrak{l}^c}$. In particular, $\{U_i\}_{i\in I_{\mathfrak{l}}}$ and $\{U_i\}_{i\in I_{\mathfrak{l}^c}}$ are bases for $\varphi(\mathfrak{l})$ and $\varphi(\mathfrak{l}^c)$ respectively.

With this notation, we can redefine the matrices E and $D_{\mathfrak{l}}$ from Equations (1.1.15), (1.1.16) as

$$E := \sum_{i\in I} u_i U^i \in \mathfrak{g} \otimes \operatorname{End} V\,, \tag{2.2.12}$$

$$D_{\mathfrak{l}} := -\sum_{j\in I_{\mathfrak{l}}\sqcup I_{\geq 1}} U^j U_j \in \big(\operatorname{End} V\big)[0]\,. \tag{2.2.13}$$

Note that to simplify notation we are omitting the tensor product sign. Moreover,

$$F = \varphi(f) = \sum_{i\in I_1} (f \mid u_i) U^i \in (\operatorname{End} V)[-1]\,. \tag{2.2.14}$$

We shall denote by $\delta(u_i)$ the eigenvalue of $\operatorname{ad} x$ on u_i, namely we will have $u_i \in I_{\delta(u_i)}$. For an index $i \in I$ let $\delta(i) := \delta(u_i)$, and we shall use the following convention on summations, where $F(i)$ denotes any expression depending on i:

$$\sum_{h\leq\delta(i)\leq k} F(i) = \sum_{h\leq j\leq k}\sum_{i\in I_j} F(i)\,. \tag{2.2.15}$$

With this notation, we shall write

$$z^{-\Delta} u_i = z^{\delta(i)-1} u_i\,, \quad \text{for each} \quad i \in I\,.$$

Note that

$$\pi_{\mathfrak{p}} E = \sum_{i\in I} \pi_{\mathfrak{p}}(u_i) U^i = \sum_{i\in I_{\mathfrak{l}^c}\sqcup I_{\leq\frac{1}{2}}} u_i U^i\,.$$

We will also make use of the following operators:

$$z^{-\Delta}E := \sum_{i\in I} z^{\delta(i)-1}u_iU^i, \qquad \pi_{\mathfrak{p}}z^{-\Delta}E := \sum_{i\in I_{\mathfrak{l}^c}\sqcup I_{\leq\frac{1}{2}}} z^{\delta(i)-1}u_iU^i\,, \tag{2.2.16}$$

both lying in $\mathcal{R}U(\mathfrak{g})\otimes \operatorname{End}V$, where $\mathcal{R}U(\mathfrak{g})$ is the Rees algebra of $U(\mathfrak{g})$ (1.4.1) defined in Section 1.4.

Finally, note that in the notation of Definition 1.5.4, the quasideterminant in Equation 2.2.3 becomes

$$\widetilde{L}(z) = |z\mathbb{1}_V + F + \pi_{\mathfrak{p}}E + D_{\mathfrak{l}}|_{\Psi_{\frac{d}{2},r_1}\Pi_{-\frac{d}{2},r_1}} \in U(\mathfrak{g})((z^{-1}))\otimes \operatorname{Hom}(V\big[-\frac{d}{2}\big]_{r_1}, V\big[\frac{d}{2}\big]_{r_1})\,. \tag{2.2.17}$$

With this notation, we can therefore restate Theorem 2.2.1 as follows, in analogy with [DSKV17b, Theorem 4.9]:

Theorem 2.2.3. *Let Γ be a good $\frac{1}{2}\mathbb{Z}$-grading for f and let $\mathfrak{l}\subset\mathfrak{g}_{\frac{1}{2}}$ be an isotropic subspace. Then,*

$$L(z) = |z\mathbb{1}_V + F + \pi_{\mathfrak{p}}E + D_{\mathfrak{l}}|_{\Psi_{\frac{d}{2},r_1}\Pi_{-\frac{d}{2},r_1}}\ \bar{1}_{\mathfrak{l}} \in W(\mathfrak{g},f,\Gamma,\mathfrak{l})((z^{-1}))\otimes \operatorname{Hom}(V\big[-\frac{d}{2}\big]_{r_1}, V\big[\frac{d}{2}\big]_{r_1})\,.$$

The proof is similar to the proof of [DSKV17b, Theorem 4.9] where, however, thanks to the symmetry properties of a Dynkin grading, $V\big[-\frac{d}{2}\big]_{r_1} = V\big[-\frac{d}{2}\big]$ and $V\big[\frac{d}{2}\big]_{r_1} = V\big[\frac{d}{2}\big]$. The exact sequences in (2.2.11) and the morphisms $\Psi_\bullet$ and $\Pi_\bullet$ need to be modified accordingly:

$$\begin{aligned} &\Psi_{\frac{d}{2},r_1}\mapsto\Psi_{\frac{d}{2}}: V\big[\frac{d}{2}\big]\hookrightarrow V\,, && \Psi_{>-\frac{d}{2},r_1}\mapsto\Psi_{>-\frac{d}{2}}: V\big[>-\frac{d}{2}\big]\hookrightarrow V\,,\\ &\Pi_{-\frac{d}{2},r_1}\mapsto\Pi_{-\frac{d}{2}}: V\twoheadrightarrow V\big[-\frac{d}{2}\big]\,, && \Pi_{<\frac{d}{2},r_1}\mapsto\Pi_{<\frac{d}{2}}: V\twoheadrightarrow V\big[<\frac{d}{2}\big]\,. \end{aligned}$$

This is essentially due to the fact that the largest rectangular block at the bottom of the pyramid is invariant with respect to any chosen good $\frac{1}{2}\mathbb{Z}$-grading of $\mathfrak{g}$; it has size $p_1\times r_1$, largest x-coordinate $\frac{d}{2}$ and smallest x-coordinate $-\frac{d}{2}$. Thus, the largest X-eigenvalues of V is also independent of the good $\frac{1}{2}\mathbb{Z}$-grading.

The main ingredient of the proof is Lemma 2.2.2 below, which will also be crucial in the proof of our results in Chapter 4.

Lemma 2.2.2. *The generalized quasideterminant $|\mathbb{1}_V + z^{-\Delta}E|_{\Psi_{\frac{d}{2},r_1}\Pi_{-\frac{d}{2},r_1}}$ exists in the space $\mathcal{R}_\infty U(\mathfrak{g})\otimes$ Hom$(V\big[-\frac{d}{2}\big]_{r_1}, V\big[\frac{d}{2}\big]_{r_1})$, and the following identity holds in $\mathcal{R}M_{\mathfrak{l}}\otimes$ Hom$(V\big[-\frac{d}{2}\big]_{r_1}, V\big[\frac{d}{2}\big]_{r_1})$:*

$$|\mathbb{1}_V + z^{-\Delta}E|_{\Psi_{\frac{d}{2},r_1}\Pi_{-\frac{d}{2},r_1}}\ \bar{1}_{\mathfrak{l}} = z^{-d-1}|z\mathbb{1}_V + F + \pi_{\mathfrak{p}}E + D_{\mathfrak{l}}|_{\Psi_{\frac{d}{2},r_1}\Pi_{-\frac{d}{2},r_1}}\ \bar{1}_{\mathfrak{l}}\,. \tag{2.2.18}$$

Proof. This result generalizes [DSKV17b, Lemma 5.5]. The argument concerning the existence of the quasideterminant $|\mathbb{1}_V + z^{-\Delta}E|_{\Psi_{\frac{d}{2},r_1}\Pi_{-\frac{d}{2},r_1}}$ follows the proof of [DSKV17b, Lemma 5.5] (see [DSKV17b, Lemmas 5.7 and 5.9] for all details), and since no significant change appears we will not report it.

However, significant differences appear in the proof of Equation 2.2.18, since we are now allowing the isotropic subspace $\mathfrak{l}$ to be non-zero. For the sake of completeness, and to highlight the differences with the case $\mathfrak{l}=0$, we present a proof of Equation (2.2.18).

Consider the semisimple endomorphism

$$z^X = \sum_{k\in\frac{1}{2}\mathbb{Z}} z^k\mathbb{1}_{V[k]} \in (\operatorname{End}V)[z^{\pm\frac{1}{2}}]\,,$$

where $\mathbb{1}_{V[k]} := \Psi_k\Pi_k \in \operatorname{End}V$ is the projection onto $V[k]\subset V$. It is an invertible element of the algebra $\mathbb{C}[z^{\pm\frac{1}{2}}]\otimes\operatorname{End}V \subset U(\mathfrak{g})((z^{-\frac{1}{2}}))\otimes\operatorname{End}V$. Its adjoint action on $\operatorname{End}V$ is given by $z^{-X}Az^X = z^{-k}A$ for $A\in(\operatorname{End}V)[k]$.

By definition of this adjoint action, we have

$$z^{-X}Ez^X = \sum_{i\in I} u_iz^{-X}U^iz^X = \sum_{i\in I} z^{\delta(i)}u_iU^i = z^{1-\Delta}E\,.$$

We can also easily deduce the following identities

$$z^{-X}\pi_{\mathfrak p}Ez^X=\pi_{\mathfrak p}z^{1-\Delta}E\ ,\quad z^{-X}Fz^X=zF\,,$$
$$z^{-X}\mathbb{1}_Vz^X=\mathbb{1}_V\ ,\quad z^{-X}D_{\mathfrak l}z^X=D_{\mathfrak l}\,,$$

from which we obtain the following identity:

$$z\mathbb{1}_V+F+\pi_{\mathfrak p}E+D_{\mathfrak l}=z^{1+X}(\mathbb{1}_V+F+\pi_{\mathfrak p}z^{-\Delta}E+z^{-1}D_{\mathfrak l})z^{-X}\,. \tag{2.2.19}$$

By (1.5.8), taking the $(\Psi_{\frac d2,r_1},\Pi_{-\frac d2,r_1})$-quasideterminant of both sides of (2.2.19) we get,

$$\begin{aligned}
&|z\mathbb{1}_V+F+\pi_{\mathfrak p}E+D_{\mathfrak l}|_{\Psi_{\frac d2,r_1},\Pi_{-\frac d2,r_1}}\\
&\quad=\big(\Pi_{-\frac d2,r_1}z^X(\mathbb{1}_V+F+\pi_{\mathfrak p}z^{-\Delta}E+z^{-1}D_{\mathfrak l})^{-1}z^{-1-X}\Psi_{\frac d2,r_1}\big)^{-1}\\
&\quad=z^{1+d}\big(\Pi_{-\frac d2,r_1}(\mathbb{1}_V+F+\pi_{\mathfrak p}z^{-\Delta}E+z^{-1}D_{\mathfrak l})^{-1}\Psi_{\frac d2,r_1}\big)^{-1}\\
&\quad=z^{1+d}|\mathbb{1}_V+F+\pi_{\mathfrak p}z^{-\Delta}E+z^{-1}D_{\mathfrak l}|_{\Psi_{\frac d2,r_1},\Pi_{-\frac d2,r_1}}\ ,
\end{aligned} \tag{2.2.20}$$

since $\Pi_{-\frac d2,r_1}z^X=z^{-\frac d2}\Pi_{-\frac d2,r_1}$ and $z^{-X}\Psi_{\frac d2,r_1}=z^{-\frac d2}\Psi_{\frac d2,r_1}$. In view of (2.2.20), Equation (2.2.18) becomes

$$|\mathbb{1}_V+z^{-\Delta}E|_{\Psi_{\frac d2,r_1},\Pi_{-\frac d2,r_1}}\overline{1}_{\mathfrak l}=|\mathbb{1}_V+F+\pi_{\mathfrak p}z^{-\Delta}E+z^{-1}D_{\mathfrak l}|_{\Psi_{\frac d2,r_1},\Pi_{-\frac d2,r_1}}\overline{1}_{\mathfrak l}\,. \tag{2.2.21}$$

To complete the proof of Lemma 2.2.2 it is therefore sufficient to prove Equation (2.2.21).

Let us compute the quasideterminants in the LHS and the RHS of Equation (2.2.21) using Formula (1.5.9) for the quasideterminants with the short exact sequences χ_1,χ_2 in (2.2.11). For the LHS, we have

$$\begin{aligned}
&|\mathbb{1}_V+z^{-\Delta}E|_{\Psi_{\frac d2,r_1},\Pi_{-\frac d2,r_1}}=\Psi^{-1}_{\frac d2,r_1}\Big(\mathbb{1}_V+z^{-\Delta}E\\
&\quad-(\mathbb{1}_V+z^{-\Delta}E)\Psi_{>-\frac d2,r_1}\big(\Pi_{<\frac d2,r_1}(\mathbb{1}_V+z^{-\Delta}E)\Psi_{>-\frac d2,r_1}\big)^{-1}\Pi_{<\frac d2,r_1}(\mathbb{1}_V+z^{-\Delta}E)\Big)\Pi^{-1}_{-\frac d2,r_1}\,.
\end{aligned} \tag{2.2.22}$$

By hypothesis, the expression in the RHS is well-defined, i.e. the operator in parenthesis induces a well-defined map from $V\big[-\frac d2\big]_{r_1}$ to $V\big[\frac d2\big]_{r_1}$. Hence, we can replace $\Psi^{-1}_{\frac d2,r_1}$ and $\Pi^{-1}_{-\frac d2,r_1}$ in the RHS with $\Pi_{\frac d2,r_1}$ and $\Psi_{-\frac d2,r_1}$ respectively (cf. (2.2.11)). Note also that

$$\Pi_{\frac d2,r_1}\mathbb{1}_V\Psi_{-\frac d2,r_1}=0,\qquad \Pi_{\frac d2,r_1}z^{-\Delta}E\Psi_{-\frac d2,r_1}=z^{-1-d}\Pi_{\frac d2,r_1}E\Psi_{-\frac d2,r_1}\,.$$

Hence, we can rewrite (2.2.22) as

$$\begin{aligned}
&|\mathbb{1}_V+z^{-\Delta}E|_{\Psi_{\frac d2,r_1},\Pi_{-\frac d2,r_1}}=z^{-1-d}\Pi_{\frac d2,r_1}E\Psi_{-\frac d2,r_1}-\Pi_{\frac d2,r_1}(\mathbb{1}_V+z^{-\Delta}E)\Psi_{>-\frac d2,r_1}\\
&\quad\times\big(\Pi_{<\frac d2,r_1}(\mathbb{1}_V+z^{-\Delta}E)\Psi_{>-\frac d2,r_1}\big)^{-1}\Pi_{<\frac d2,r_1}(\mathbb{1}_V+z^{-\Delta}E)\Psi_{-\frac d2,r_1}\,.
\end{aligned} \tag{2.2.23}$$

Similarly, we use formula (1.5.9) to compute the quasideterminant in the RHS of (2.2.21). We have

$$\begin{aligned}
&|\mathbb{1}_V+F+\pi_{\mathfrak p}z^{-\Delta}E+z^{-1}D_{\mathfrak l}|_{\Psi_{\frac d2,r_1},\Pi_{-\frac d2,r_1}}\\
&\quad=\Pi_{\frac d2,r_1}(\mathbb{1}_V+F+\pi_{\mathfrak p}z^{-\Delta}E+z^{-1}D_{\mathfrak l})\Psi_{-\frac d2,r_1}-\Pi_{\frac d2,r_1}(\mathbb{1}_V+F+\pi_{\mathfrak p}z^{-\Delta}U+z^{-1}D_{\mathfrak l})\Psi_{>-\frac d2,r_1}\\
&\quad\times\big(\Pi_{<\frac d2,r_1}(\mathbb{1}_V+F+\pi_{\mathfrak p}z^{-\Delta}E+z^{-1}D_{\mathfrak l})\Psi_{>-\frac d2,r_1}\big)^{-1}\Pi_{<\frac d2,r_1}(\mathbb{1}_V+F+\pi_{\leq\frac12}z^{-\Delta}E+z^{-1}D_{\mathfrak l})\Psi_{-\frac d2,r_1}\\
&\quad=z^{-1-d}\Pi_{\frac d2,r_1}E\Psi_{-\frac d2,r_1}-\Pi_{\frac d2,r_1}(\mathbb{1}_V+z^{-\Delta}E)\Psi_{>-\frac d2,r_1}\\
&\quad\times\big(\Pi_{<\frac d2,r_1}(\mathbb{1}_V+F+\pi_{\mathfrak p}z^{-\Delta}E+z^{-1}D_{\mathfrak l})\Psi_{>-\frac d2,r_1}\big)^{-1}\Pi_{<\frac d2,r_1}(\mathbb{1}_V+z^{-\Delta}E+z^{-1}D_{\mathfrak l})\Psi_{-\frac d2,r_1}\,,
\end{aligned} \tag{2.2.24}$$

where we used, for the second equality, the obvious identities

$$\begin{aligned}
&\Pi_{\frac d2,r_1}\mathbb{1}_V\Psi_{-\frac d2,r_1}=0\ ,\quad \Pi_{\frac d2,r_1}F=F\Psi_{-\frac d2,r_1}=0\,,\\
&\Pi_{\frac d2,r_1}D_{\mathfrak l}=0\ ,\quad D_{\mathfrak l}\Psi_{\frac d2,r_1}=0\,,\\
&\Pi_{\frac d2,r_1}\pi_{\mathfrak p}z^{-\Delta}E\Psi_{-\frac d2,r_1}=z^{-1-d}\Pi_{\frac d2,r_1}E\Psi_{-\frac d2,r_1}\,,\\
&\Pi_{\frac d2,r_1}\pi_{\mathfrak p}z^{-\Delta}E=\Pi_{\frac d2,r_1}z^{-\Delta}E\ ,\quad \pi_{\mathfrak p}z^{-\Delta}E\Psi_{-\frac d2,r_1}=z^{-\Delta}E\Psi_{-\frac d2,r_1}\,.
\end{aligned}$$

In view of (2.2.23) and (2.2.24), Equation (2.2.21) reduces to the following equation

$$\begin{aligned}&\left(\Pi_{<\frac{d}{2},r_1}(\mathbb{1}_V+z^{-\Delta}E)\Psi_{>-\frac{d}{2},r_1}\right)^{-1}\Pi_{<\frac{d}{2},r_1}(\mathbb{1}_V+z^{-\Delta}E)\Psi_{-\frac{d}{2},r_1}\bar{1}_{\mathfrak{l}}\\&\quad=\left(\Pi_{<\frac{d}{2},r_1}(\mathbb{1}_V+F+\pi_{\mathfrak{p}}z^{-\Delta}E+z^{-1}D_{\mathfrak{l}})\Psi_{>-\frac{d}{2},r_1}\right)^{-1}\Pi_{<\frac{d}{2},r_1}(\mathbb{1}_V+z^{-\Delta}E+z^{-1}D_{\mathfrak{l}})\Psi_{-\frac{d}{2},r_1}\bar{1}_{\mathfrak{l}}\,,\end{aligned}\tag{2.2.25}$$

which we are left to prove.

To simplify notation, we introduce the operators $A,B\in\mathcal{R}U(\mathfrak{g})\otimes\operatorname{Hom}\left(V[>-\frac{d}{2}]\oplus V[-\frac{d}{2}]_{>r_1},V[<\frac{d}{2}]\oplus V[\frac{d}{2}]_{>r_1}\right)$ and $v,w\in\mathcal{R}U(\mathfrak{g})\otimes\operatorname{Hom}\left(V[-\frac{d}{2}]_{r_1},V[<\frac{d}{2}]\oplus V[\frac{d}{2}]_{>r_1}\right)$, defined as follows

$$\begin{aligned}A&:=\Pi_{<\frac{d}{2},r_1}(\mathbb{1}_V+F+\pi_{\mathfrak{p}}z^{-\Delta}E+z^{-1}D_{\mathfrak{l}})\Psi_{>-\frac{d}{2},r_1}\,,\\B&:=\Pi_{<\frac{d}{2},r_1}(\mathbb{1}_V+z^{-\Delta}E)\Psi_{>-\frac{d}{2},r_1}-A\\&=\sum_{i\in I_{\geq1}\cup I_{\mathfrak{l}}}(z^{-\Delta}u_i-(f|u_i))\,\Pi_{<\frac{d}{2},r_1}U^i\Psi_{>-\frac{d}{2},r_1}-z^{-1}\Pi_{<\frac{d}{2},r_1}D_{\mathfrak{l}}\Psi_{>-\frac{d}{2},r_1}\,,\\v&:=\Pi_{<\frac{d}{2},r_1}(\mathbb{1}_V+z^{-\Delta}E+z^{-1}D_{\mathfrak{l}})\Psi_{-\frac{d}{2},r_1}\,,\\w&:=\Pi_{<\frac{d}{2},r_1}(\mathbb{1}_V+z^{-\Delta}E)\Psi_{-\frac{d}{2},r_1}-v=-z^{-1}\Pi_{<\frac{d}{2},r_1}D_{\mathfrak{l}}\Psi_{-\frac{d}{2},r_1}\,.\end{aligned}\tag{2.2.26}$$

Using notation (2.2.26), Equation (2.2.25) can be rewritten as follows

$$(A+B)^{-1}(v+w)\bar{1}_{\mathfrak{l}}=A^{-1}v\bar{1}_{\mathfrak{l}}\in\mathcal{R}M_{\mathfrak{l}}\otimes\operatorname{Hom}\left(V[-\frac{d}{2}]_{r_1},V[>-\frac{d}{2}]\oplus V[-\frac{d}{2}]_{>r_1}\right).\tag{2.2.27}$$

For every $i\in I_{\geq1}\cup I_{\mathfrak{l}}$, we shall denote

$$X_i:=(z^{-\Delta}u_i-(f|u_i))A^{-1}v\bar{1}_{\mathfrak{l}}\in\mathcal{R}M_{\mathfrak{l}}\otimes\operatorname{Hom}\left(V[-\frac{d}{2}]_{r_1},V[>-\frac{d}{2}]\oplus V[-\frac{d}{2}]_{>r_1}\right).\tag{2.2.28}$$

We also let $X_i=0$ for $i\in I_{\leq0}\cup I_{\mathfrak{l}^c}$.

Lemma 2.2.3. *For every $i\in I_{\geq1}\cup I_{\mathfrak{l}}$ we have,*

$$\begin{aligned}&X_i+z^{-1}\sum_{\substack{1\leq\delta(j)\leq\delta(i)+\frac12\\\sqcup j\in I_{\mathfrak{l}}}}A^{-1}\Pi_{<\frac{d}{2},r_1}[U^j,U_i]\Psi_{>-\frac{d}{2},r_1}X_j\\&\quad-z^{-1}\sum_{\delta(j)=\delta(i)+\frac12}A^{-1}\Pi_{<\frac{d}{2},r_1}\Pi_{\mathfrak{l}^*}\left([U^j,U_i]\right)\Psi_{>-\frac{d}{2},r_1}X_j\\&\quad=-z^{-1}\Pi_{>-\frac{d}{2},r_1}U_i\left(\Psi_{>-\frac{d}{2},r_1}A^{-1}v-\Psi_{-\frac{d}{2},r_1}\right)\bar{1}_{\mathfrak{l}}\\&\quad+z^{-2}A^{-1}\Pi_{<\frac{d}{2},r_1}[D_{\mathfrak{l}},U_i]\left(\Psi_{>-\frac{d}{2},r_1}A^{-1}v-\Psi_{-\frac{d}{2},r_1}\right)\bar{1}_{\mathfrak{l}}\,.\end{aligned}\tag{2.2.29}$$

Remark 2.2.2. Let $\mathfrak{l}\subset\mathfrak{g}_{\frac12}$ be an isotropic subspace, and let $\mathfrak{l}^*\subset\mathfrak{g}_{-\frac12}$ be the subspace dual to $\mathfrak{l}$ with respect to the bilinear form $(\cdot|\cdot)$. We shall denote by $(\mathfrak{l}^{\perp})^c\subset\mathfrak{g}_{\frac12}$ and $(\mathfrak{l}^*)^c\subset\mathfrak{g}_{-\frac12}$ a choice of complementary subspaces to $\mathfrak{l}^{\perp}$ and $\mathfrak{l}^*$ respectively.

The bijection $\operatorname{ad}f:\mathfrak{g}_{\frac12}\longrightarrow\mathfrak{g}_{-\frac12}$ provides direct sum decompositions

$$\mathfrak{g}_{-\frac12}=[f,\mathfrak{l}]\oplus[f,\mathfrak{l}^c]\,,\qquad\mathfrak{g}_{-\frac12}=[f,\mathfrak{l}^{\perp}]\oplus[f,(\mathfrak{l}^{\perp})^c]\,.$$

By this direct sum decomposition, and by isotropicity of $\mathfrak{l}$, $\mathfrak{l}^*=[f,(\mathfrak{l}^{\perp})^c]$ because $\mathfrak{l}^{\perp}$ is the maximal subspace with the property $(f|[\mathfrak{l},\mathfrak{l}^{\perp}])=0$. As a consequence, we may identify $(\mathfrak{l}^*)^c=[f,\mathfrak{l}^{\perp}]$. On the other hand, by non-degeneracy of the bilinear form ω we have $(\mathfrak{l}^c|[f,\mathfrak{l}^{\perp}])\neq0$. Hence, $[f,\mathfrak{l}^{\perp}]\subseteq(\mathfrak{l}^c)^*$ and the equality holds by the following dimension argument:

$$\dim\,(\mathfrak{l}^c)^*=\dim\,\mathfrak{l}^c=\dim\,\mathfrak{g}_{\frac12}-\dim\,\mathfrak{l}=\dim\,\mathfrak{g}_{-\frac12}-\dim\,\mathfrak{l}^*=\dim\,(\mathfrak{l}^*)^c=\dim\,[f,\mathfrak{l}^{\perp}]\,.$$

Therefore, $(\mathfrak{l}^c)^*=(\mathfrak{l}^*)^c=[f,\mathfrak{l}^{\perp}]$.

Let $\{U^i\}_{i\in I_{\mathfrak{l}}}$ and $\{U^i\}_{i\in I_{\mathfrak{l}^c}}$ be bases for the subspaces $\varphi(\mathfrak{l}^*)$ and $\varphi((\mathfrak{l}^c)^*)$ of $\operatorname{End} V$ respectively. We shall denote by

$$\Pi_{\mathfrak{l}^*}:\operatorname{End} V\big[-\frac{1}{2}\big]\longrightarrow\varphi(\mathfrak{l}^*),\qquad \Pi_{(\mathfrak{l}^c)^*}:\operatorname{End} V\big[-\frac{1}{2}\big]\longrightarrow\varphi((\mathfrak{l}^c)^*)\,,$$

the projections with respect to the direct sum decomposition $\operatorname{End} V\big[-\frac{1}{2}\big]=\varphi((\mathfrak{l}^c)^*)\oplus\varphi(\mathfrak{l}^*)$. Note that $\Pi_{(\mathfrak{l}^c)^*}=\mathbb{1}_{V\left[-\frac{1}{2}\right]}-\Pi_{\mathfrak{l}^*}$.

Proof of Lemma 2.2.3. Recall that $(z^{-\Delta}u_i-(f|u_i))\bar{1}_{\mathfrak{l}}=0$ in $\mathcal{R}M_{\mathfrak{l}}$ for every $i\in I_{\geq 1}\sqcup I_{\mathfrak{l}}$. Hence,

$$\begin{aligned}X_i&=-A^{-1}[z^{-\Delta}u_i,A]A^{-1}v\bar{1}_{\mathfrak{l}}+A^{-1}[z^{-\Delta}u_i,v]\bar{1}_{\mathfrak{l}}\\&=-\sum_{j\in I_{\leq 0}\sqcup I_{\mathfrak{l}^c}}A^{-1}\big[z^{-\Delta}u_i,z^{-\Delta}u_j\big]\Pi_{<\frac{d}{2},r_1}U^j\Psi_{>-\frac{d}{2},r_1}A^{-1}v\bar{1}_{\mathfrak{l}}\\&\quad+\sum_{j\in I}A^{-1}\big[z^{-\Delta}u_i,z^{-\Delta}u_j\big]\Pi_{<\frac{d}{2},r_1}U^j\Psi_{-\frac{d}{2},r_1}\bar{1}_{\mathfrak{l}}\,.\end{aligned}\tag{2.2.30}$$

By the definition of conformal weight, we have

$$\big[z^{-\Delta}u_i,z^{-\Delta}u_j\big]=z^{-1-\Delta}[u_i,u_j]\,.$$

Lemma 2.2.4. *We have the following identities*

$$\sum_{j\in I}[u_i,u_j]\,U^j=\sum_{j\in J}u_j[U^j,U_i]\,,$$

and

$$\sum_{j\in I_{\leq 0}\sqcup I_{\mathfrak{l}^c}}[u_i,u_j]\,U^j=\sum_{\delta(j)\leq\delta(i)+\frac{1}{2}}u_j[U^j,U_i]-\sum_{\delta(j)=\delta(i)+\frac{1}{2}}u_j\Pi_{\mathfrak{l}^*}\big([U^j,U_i]\big)\,.\tag{2.2.31}$$

Proof. Both identities follow by the completeness relations

$$\sum_{k=1}^{M}(w^k|u)u_k=u\ ,\quad\sum_{k=1}^{M}(w|u_k)w^k=w\quad\text{for all }u\in U,\,w\in W\,,\tag{2.2.32}$$

where $\{u_k\}_{k=1}^M$ and $\{w^k\}_{k=1}^M$ are dual bases of a pair of M-dimensional vector spaces U and W respectively, the pairing being with respect to a non-degenerate bilinear form $(\cdot|\cdot)$ (namely, $(w^k|u_h)=\delta_{h,k}$ for any $1\leq h,k\leq M$).

However, the RHS of Equation (2.2.31) is due to the fact that $\{U^j\}_{j\in I_{\mathfrak{l}^c}}$ is a basis for $\varphi((\mathfrak{l}^c)^*)$, and not for the whole $\operatorname{End} V\big[-\frac{1}{2}\big]$. In fact,

$$\begin{aligned}&\sum_{j\in I_{\leq 0}\sqcup I_{\mathfrak{l}^c}}[u_i,u_j]\,U^j=\sum_{j\in I_{\leq 0}}[u_i,u_j]\,U^j+\sum_{j\in I_{\mathfrak{l}^c}}[u_i,u_j]\,U^j=\sum_{j\in I_{\leq 0}}\sum_{k\leq\delta(i)}([u_i,u_j]|u^k)\,u_kU^j\\&+\sum_{j\in I_{\mathfrak{l}^c}}\sum_{k=\delta(i)+\frac{1}{2}}([u_i,u_j]|u^k)\,u_kU^j=\sum_{\delta(j)\leq\delta(i)}u_j[U^j,U_i]+\sum_{\delta(j)=\delta(i)+\frac{1}{2}}u_j\Pi_{(\mathfrak{l}^c)^*}\big([U^j,U_i]\big)\\&=\sum_{\delta(j)\leq\delta(i)+\frac{1}{2}}u_j[U^j,U_i]-\sum_{\delta(j)=\delta(i)+\frac{1}{2}}u_j\Pi_{\mathfrak{l}^*}\big([U^j,U_i]\big)\,.\end{aligned}$$

∎

Hence, (2.2.30) gives

$$\begin{aligned}X_i=&-z^{-1}\sum_{\delta(j)\leq\delta(i)+\frac{1}{2}}A^{-1}(z^{-\Delta}u_j)\Pi_{<\frac{d}{2},r_1}\big[U^j,U_i\big]\Psi_{>-\frac{d}{2},r_1}A^{-1}v\bar{1}_{\mathfrak{l}}\\&+z^{-1}\sum_{\delta(j)=\delta(i)+\frac{1}{2}}A^{-1}(z^{-\Delta}u_j)\Pi_{<\frac{d}{2},r_1}\Pi_{\mathfrak{l}^*}\big(\big[U^j,U_i\big]\big)\Psi_{>-\frac{d}{2},r_1}A^{-1}v\bar{1}_{\mathfrak{l}}\\&+z^{-1}\sum_{j\in I}A^{-1}(z^{-\Delta}u_j)\Pi_{<\frac{d}{2},r_1}\big[U^j,U_i\big]\Psi_{-\frac{d}{2},r_1}\bar{1}_{\mathfrak{l}}\,.\end{aligned}\tag{2.2.33}$$

Since, by assumption, $i \in I_{\geq 1} \sqcup I_{\mathfrak{l}}$, we have $\operatorname{Im} U_i \subset V\big[>-\frac{d}{2}\big] \oplus V\big[-\frac{d}{2}\big]_{>r_1}$ and $V\big[\frac{d}{2}\big]_{r_1} \subset \ker U_i$. As a consequence, we have the following identities (cf. (2.2.9))

$$U_i = \Psi_{>-\frac{d}{2},r_1}\Pi_{>-\frac{d}{2},r_1}U_i \quad \text{and} \quad U_i = U_i\Psi_{<\frac{d}{2},r_1}\Pi_{<\frac{d}{2},r_1}\,. \tag{2.2.34}$$

We can therefore rewrite (2.2.33) as follows

$$\begin{aligned} X_i = -z^{-1} &\sum_{\delta(j)\leq\delta(i)+\frac{1}{2}} A^{-1}(z^{-\Delta}u_j)\Pi_{<\frac{d}{2},r_1}U^j\Psi_{>-\frac{d}{2},r_1}\Pi_{>-\frac{d}{2},r_1}U_i\Psi_{>-\frac{d}{2},r_1}A^{-1}v\bar{1}_{\mathfrak{l}} \\ &+ z^{-1}\sum_{\delta(j)\leq\delta(i)+\frac{1}{2}} A^{-1}(z^{-\Delta}u_j)\Pi_{<\frac{d}{2},r_1}U_i\Psi_{<\frac{d}{2},r_1}\Pi_{<\frac{d}{2},r_1}U^j\Psi_{>-\frac{d}{2},r_1}A^{-1}v\bar{1}_{\mathfrak{l}} \\ &+ z^{-1}\sum_{\delta(j)=\delta(i)+\frac{1}{2}} A^{-1}(z^{-\Delta}u_j)\Pi_{<\frac{d}{2},r_1}\Pi_{\mathfrak{l}^*}([U^j,U_i])\Psi_{>-\frac{d}{2},r_1}A^{-1}v\bar{1}_{\mathfrak{l}} \\ &+ z^{-1}\sum_{j\in I} A^{-1}(z^{-\Delta}u_j)\Pi_{<\frac{d}{2},r_1}U^j\Psi_{>-\frac{d}{2},r_1}\Pi_{>-\frac{d}{2},r_1}U_i\Psi_{-\frac{d}{2},r_1}\bar{1}_{\mathfrak{l}} \\ &- z^{-1}\sum_{j\in I} A^{-1}(z^{-\Delta}u_j)\Pi_{<\frac{d}{2},r_1}U_i\Psi_{<\frac{d}{2},r_1}\Pi_{<\frac{d}{2},r_1}U^j\Psi_{-\frac{d}{2},r_1}\bar{1}_{\mathfrak{l}}\,. \end{aligned} \tag{2.2.35}$$

Recalling the definitions (2.2.26) of A and v, we have the following identities:

- $\displaystyle\sum_{\delta(j)\leq\delta(i)+\frac{1}{2}} (z^{-\Delta}u_j)\Pi_{<\frac{d}{2},r_1}U^j\Psi_{>-\frac{d}{2},r_1}$
  $\displaystyle= A + \sum_{\substack{1\leq\delta(j)\leq\delta(i)+\frac{1}{2}\\ \sqcup j\in I_{\mathfrak{l}}}} (z^{-\Delta}u_j - (f|u_j))\Pi_{<\frac{d}{2},r_1}U^j\Psi_{>-\frac{d}{2},r_1} - \Pi_{<\frac{d}{2},r_1}(\mathbb{1}_V + z^{-1}D_{\mathfrak{l}})\Psi_{>-\frac{d}{2},r_1}\,,$
- $\displaystyle\sum_{j\in I}(z^{-\Delta}u_j)\Pi_{<\frac{d}{2},r_1}U^j\Psi_{>-\frac{d}{2},r_1}$
  $\displaystyle= A + \sum_{j\in I_{\geq 1}\sqcup j\in I_{\mathfrak{l}}} (z^{-\Delta}u_j - (f|u_j))\Pi_{<\frac{d}{2},r_1}U^j\Psi_{>-\frac{d}{2},r_1} - \Pi_{<\frac{d}{2},r_1}(\mathbb{1}_V + z^{-1}D_{\mathfrak{l}})\Psi_{>-\frac{d}{2},r_1}\,,$
- $\displaystyle\sum_{j\in I}(z^{-\Delta}u_j)\Pi_{<\frac{d}{2},r_1}U^j\Psi_{-\frac{d}{2},r_1} = v - \Pi_{<\frac{d}{2},r_1}(\mathbb{1}_V + z^{-1}D_{\mathfrak{l}})\Psi_{-\frac{d}{2},r_1}\,.$

Hence, the first term in the RHS of (2.2.35) can be rewritten as

$$\begin{aligned} &-z^{-1}\sum_{\substack{1\leq\delta(j)\leq\delta(i)+\frac{1}{2}\\ \sqcup j\in I_{\mathfrak{l}}}} A^{-1}\Pi_{<\frac{d}{2},r_1}U^jU_i\Psi_{>-\frac{d}{2},r_1}X_j - z^{-1}\Pi_{>-\frac{d}{2},r_1}U_i\Psi_{>-\frac{d}{2},r_1}A^{-1}v\bar{1}_{\mathfrak{l}} \\ &\quad + z^{-1}A^{-1}\Pi_{<\frac{d}{2},r_1}(\mathbb{1}_V + z^{-1}D_{\mathfrak{l}})U_i\Psi_{>-\frac{d}{2},r_1}A^{-1}v\bar{1}_{\mathfrak{l}}\,, \end{aligned} \tag{2.2.36}$$

the second term in the RHS of (2.2.35) becomes

$$\begin{aligned} &+z^{-1}\sum_{\substack{1\leq\delta(j)\leq\delta(i)+\frac{1}{2}\\ \sqcup j\in I_{\mathfrak{l}}}} A^{-1}\Pi_{<\frac{d}{2},r_1}U_iU^j\Psi_{>-\frac{d}{2},r_1}X_j + z^{-1}A^{-1}\Pi_{<\frac{d}{2},r_1}U_i\Psi_{<\frac{d}{2},r_1}v\bar{1}_{\mathfrak{l}} \\ &\quad - z^{-1}A^{-1}\Pi_{<\frac{d}{2},r_1}U_i(\mathbb{1}_V + z^{-1}D_{\mathfrak{l}})\Psi_{>-\frac{d}{2},r_1}A^{-1}v\bar{1}_{\mathfrak{l}}\,, \end{aligned} \tag{2.2.37}$$

the fourth term in the RHS of (2.2.35) becomes

$$z^{-1}\Pi_{>-\frac{d}{2},r_1}U_i\Psi_{-\frac{d}{2},r_1}\bar{1}_{\mathfrak{l}} - z^{-1}A^{-1}\Pi_{<\frac{d}{2},r_1}(\mathbb{1}_V + z^{-1}D_{\mathfrak{l}})U_i\Psi_{-\frac{d}{2},r_1}\bar{1}_{\mathfrak{l}}\,, \tag{2.2.38}$$

and the last term in the RHS of (2.2.35) becomes

$$-z^{-1}A^{-1}\Pi_{<\frac{d}{2},r_1}U_i\Psi_{<\frac{d}{2},r_1}v\bar{1}_{\mathfrak{l}} + z^{-1}A^{-1}\Pi_{<\frac{d}{2},r_1}U_i(\mathbb{1}_V + z^{-1}D_{\mathfrak{l}})\Psi_{-\frac{d}{2},r_1}\bar{1}_{\mathfrak{l}}\,. \tag{2.2.39}$$

For the third term we need a different approach. Since $\delta(j) = \delta(i) + \frac{1}{2} \geq 1$, it becomes

$$\begin{aligned}
&z^{-1} \sum_{\delta(j)=\delta(i)+\frac{1}{2}} A^{-1}\Pi_{<\frac{d}{2},r_1}\Pi_{\mathfrak{l}^*}\big([U^j,U_i]\big)\Psi_{>-\frac{d}{2},r_1}X_j \\
&+ z^{-1}\delta_{i\in I_{\mathfrak{l}}} \sum_{\delta(j)=\delta(i)+\frac{1}{2}} A^{-1}\Pi_{<\frac{d}{2},r_1}\Pi_{\mathfrak{l}^*}\big([U^j,U_i]\big)\Psi_{>-\frac{d}{2},r_1}(f|u_j)A^{-1}v\bar{1}_{\mathfrak{l}} \\
&\quad = z^{-1} \sum_{\delta(j)=\delta(i)+\frac{1}{2}} A^{-1}\Pi_{<\frac{d}{2},r_1}\Pi_{\mathfrak{l}^*}\big([U^j,U_i]\big)\Psi_{>-\frac{d}{2},r_1}X_j\,.
\end{aligned} \tag{2.2.40}$$

The last equality is due to the following identity, by duality of the bases $\{u_j\}$, $\{u^j\}$ and the invariance of the trace form:

$$\begin{aligned}
&\delta_{i\in I_{\mathfrak{l}}} \sum_{\delta(j)=\delta(i)+\frac{1}{2}} \Pi_{\mathfrak{l}^*}\big([U^j,U_i]\big)(f|u_j) = \delta_{i\in I_{\mathfrak{l}}} \sum_{\delta(j)=1}\sum_{k\in I_{\mathfrak{l}}}([U^j,U_i]|U_k)U^k(f|u_j) \\
&\quad = \delta_{i\in I_{\mathfrak{l}}} \sum_{j\in I}\sum_{k\in I_{\mathfrak{l}}}(u^j|[u_i,u_k])U^k(f|u_j) = \delta_{i\in I_{\mathfrak{l}}} \sum_{k\in I_{\mathfrak{l}}} U^k(f|[u_i,u_k]) = 0\,,
\end{aligned}$$

where $(f|[u_i,u_k]) = 0$ because of the isotropicity of $\mathfrak{l}$. Combining (2.2.36)-(2.2.39) and (2.2.40), we get (2.2.29). ☐

Lemma 2.2.5. *The unique solution of Equation* (2.2.29) *is (for $i \in I_{\geq 1} \sqcup I_{\mathfrak{l}}$):*

$$X_i = -z^{-1}\Pi_{>-\frac{d}{2},r_1}U_i\big(\Psi_{>-\frac{d}{2},r_1}A^{-1}v - \Psi_{-\frac{d}{2},r_1}\big)\bar{1}_{\mathfrak{l}}\,. \tag{2.2.41}$$

Proof. First, we prove that (2.2.41) solves Equation (2.2.29). Note that the first term in the LHS of (2.2.29) equals, by (2.2.41), the first term in the RHS of (2.2.29). We hence need to prove that the second terms in the LHS and RHS of (2.2.29) coincide:

$$\begin{aligned}
&-z^{-2} \sum_{\substack{1\leq\delta(j)\leq\delta(i)+\frac{1}{2}\\ \sqcup j\in I_{\mathfrak{l}}}} A^{-1}\Pi_{<\frac{d}{2},r_1}[U^j,U_i]\Psi_{>-\frac{d}{2},r_1}\Pi_{>-\frac{d}{2},r_1}U_j\big(\Psi_{>-\frac{d}{2},r_1}A^{-1}v - \Psi_{-\frac{d}{2},r_1}\big)\bar{1}_{\mathfrak{l}} \\
&+ z^{-2} \sum_{\delta(j)=\delta(i)+\frac{1}{2}} A^{-1}\Pi_{<\frac{d}{2},r_1}\Pi_{\mathfrak{l}^*}\big([U^j,U_i]\big)\Psi_{>-\frac{d}{2},r_1}\Pi_{>-\frac{d}{2},r_1}U_j\big(\Psi_{>-\frac{d}{2},r_1}A^{-1}v - \Psi_{-\frac{d}{2},r_1}\big)\bar{1}_{\mathfrak{l}} \\
&= z^{-2}A^{-1}\Pi_{<\frac{d}{2},r_1}[D_{\mathfrak{l}},U_i]\big(\Psi_{>-\frac{d}{2},r_1}A^{-1}v - \Psi_{-\frac{d}{2},r_1}\big)\bar{1}_{\mathfrak{l}}\,.
\end{aligned} \tag{2.2.42}$$

Recalling the first equation of (2.2.34), Equation (2.2.42) is established once we prove the following identity:

$$\sum_{\substack{1\leq\delta(j)\leq\delta(i)+\frac{1}{2}\\ \sqcup j\in I_{\mathfrak{l}}}} [U^j,U_i]U_j - \sum_{\delta(j)=\delta(i)+\frac{1}{2}} \Pi_{\mathfrak{l}^*}\big([U^j,U_i]\big)U_j = -[D_{\mathfrak{l}},U_i]\,. \tag{2.2.43}$$

By the definition (2.2.13) of the shift matrix $D_{\mathfrak{l}}$ and the Leibniz rule, we have

$$-[D_{\mathfrak{l}},U_i] = \sum_{j\in I_{\geq 1}} \big([U^j,U_i]U_j + U^j[U_j,U_i]\big) + \sum_{j\in I_{\mathfrak{l}}} \big([U^j,U_i]U_j + U^j[U_j,U_i]\big)\,. \tag{2.2.44}$$

On the other hand, by the duality of the bases $\{U_j\}$, $\{U^j\}$ and the invariance of the trace form, we have

$$\begin{aligned}
&\sum_{j\in I_{\geq 1}} U^j[U_j,U_i] = \sum_{j\in I_{\geq 1}}\sum_{k\in I}([U_j,U_i]|U^k)U^jU_k \\
&\quad = -\sum_{\delta(k)\geq\delta(i)+1}\sum_{j\in I}(U_j|[U^k,U_i])U^jU_k = -\sum_{\delta(k)\geq\delta(i)+1}[U^k,U_i]U_k\,.
\end{aligned} \tag{2.2.45}$$

and

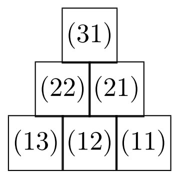
$$\begin{aligned}
&\sum_{j\in I_{\mathfrak{l}}} U^j[U_j,U_i] = \sum_{j\in I_{\mathfrak{l}}}\sum_{k\in I}([U_j,U_i]|U^k)U^jU_k = -\sum_{j\in I_{\mathfrak{l}}}\sum_{k\in I}(U_j|[U^k,U_i])U^jU_k \\
&\quad = -\sum_{\delta(k)=\delta(i)+\frac{1}{2}} \Pi_{\mathfrak{l}^*}([U^k,U_i])U_k\,.
\end{aligned} \tag{2.2.46}$$

Combining (2.2.44), (2.2.45) and (2.2.46), we get Equation (2.2.43).

The uniqueness of the solution of Equation (2.2.29) is clear. Indeed, Equation (2.2.29) has the matrix form $(\mathbb{1}+z^{-1}M)X=Y$, where X is the column vector $(X_i)_{i\in I_{\geq 1}\sqcup I_{\mathfrak{l}}}$, with entries in the vector space $V=\mathcal{R}M_{\mathfrak{l}}\otimes\mathrm{Hom}\left(V\left[-\frac{d}{2}\right]_{r_1},V\left[>-\frac{d}{2}\right]\oplus V\left[-\frac{d}{2}\right]_{>r_1}\right)$, Y is the analogous column vector defined by the RHS of (2.2.29), and M is some matrix with entries in $\mathcal{R}U(\mathfrak{g})\otimes\mathrm{Hom}\left(V\left[>-\frac{d}{2}\right]\oplus V\left[-\frac{d}{2}\right]_{>r_1},V\left[>-\frac{d}{2}\right]\oplus V\left[-\frac{d}{2}\right]_{>r_1}\right)$, which is an algebra acting on the vector space V. But then the matrix $\mathbb{1}+z^{-1}M$ can be inverted by geometric series expansion. ☐

Before continuing, we shall give an example of how the matrix $D_{\mathfrak{l}}$ changes in accordance with the choice of a different $\mathfrak{l}$ for the same pyramid p and the same grading.

Example 2.2.1. Let p be the pyramid of size $N=6$ and shape $\lambda=(3,2,1)$ as follows:



Then, for $\mathfrak{l}=0$ we have

$$D_0=-\sum_{j\in I_{\geq 1}}U^jU_j=-E_{(1,2)(1,2)}-4E_{(1,3)(1,3)}-2E_{(2,2)(2,2)}-E_{(3,1)(3,1)}\,.$$

For $\mathfrak{l}_1=\mathbb{C}e_{(2,1)(1,2)}+\mathbb{C}e_{(2,2)(1,3)}+\mathbb{C}e_{(3,1)(2,2)}$ we have

$$D_{\mathfrak{l}_1}=-\sum_{j\in I_{\mathfrak{l}_1}\sqcup I_{\geq 1}}U^jU_j=-2E_{(1,2)(1,2)}-5E_{(1,3)(1,3)}-3E_{(2,2)(2,2)}-E_{(3,1)(3,1)}\,,$$

whereas for $\mathfrak{l}_2=\mathbb{C}e_{(2,1)(1,2)}+\mathbb{C}e_{(2,2)(1,3)}+\mathbb{C}e_{(2,1)(3,1)}$ we have

$$D_{\mathfrak{l}_2}=-\sum_{j\in I_{\mathfrak{l}_1}\sqcup I_{\geq 1}}U^jU_j=-2E_{(1,2)(1,2)}-5E_{(1,3)(1,3)}-2E_{(2,2)(2,2)}-2E_{(3,1)(3,1)}\,.$$

In all cases, it is easily seen that they provide solutions to the corresponding Equation (2.2.43).

Corollary 2.2.2. *We have*

$$BA^{-1}v\bar{1}_{\mathfrak{l}}=w\bar{1}_{\mathfrak{l}}\,. \tag{2.2.47}$$

Proof. By the definitions (2.2.26) of B, the definition (2.2.28) of X_i and its formula (2.2.41), we have

$$\begin{aligned}BA^{-1}v\bar{1}_{\mathfrak{l}}&=\sum_{i\in I_{\geq 1}\sqcup I_{\mathfrak{l}}}\Pi_{<\frac{d}{2},r_1}U^i\Psi_{>-\frac{d}{2},r_1}X_i-z^{-1}\Pi_{<\frac{d}{2},r_1}D_{\mathfrak{l}}\Psi_{>-\frac{d}{2},r_1}A^{-1}v\bar{1}_{\mathfrak{l}}\\&=-z^{-1}\sum_{i\in I_{\geq 1}\sqcup I_{\mathfrak{l}}}\Pi_{<\frac{d}{2},r_1}U^i\Psi_{>-\frac{d}{2},r_1}\Pi_{>-\frac{d}{2},r_1}U_i\big(\Psi_{>-\frac{d}{2},r_1}A^{-1}v-\Psi_{-\frac{d}{2},r_1}\big)\bar{1}_{\mathfrak{l}}-z^{-1}\Pi_{<\frac{d}{2},r_1}D_{\mathfrak{l}}\Psi_{>-\frac{d}{2},r_1}A^{-1}v\bar{1}_{\mathfrak{l}}\\&=z^{-1}\Pi_{<\frac{d}{2},r_1}D_{\mathfrak{l}}\big(\Psi_{>-\frac{d}{2},r_1}A^{-1}v-\Psi_{-\frac{d}{2},r_1}\big)\bar{1}_{\mathfrak{l}}-z^{-1}\Pi_{<\frac{d}{2},r_1}D_{\mathfrak{l}}\Psi_{>-\frac{d}{2},r_1}A^{-1}v\bar{1}_{\mathfrak{l}}\\&=-z^{-1}\Pi_{<\frac{d}{2},r_1}D_{\mathfrak{l}}\Psi_{-\frac{d}{2},r_1}\bar{1}_{\mathfrak{l}}=w\bar{1}_{\mathfrak{l}}\,,\end{aligned} \tag{2.2.48}$$

where, for the third equality, we used (2.2.34) and the definition (2.2.13) of the shift matrix $D_{\mathfrak{l}}$. ☐

The operators A,B in (2.2.26) lie in $\mathcal{R}U(\mathfrak{g})\otimes\mathrm{Hom}\left(V\left[>-\frac{d}{2}\right]\oplus V\left[-\frac{d}{2}\right]_{>r_1},V\left[<\frac{d}{2}\right]\oplus V\left[\frac{d}{2}\right]_{>r_1}\right)$, and, by the definition of B and the definition of the homomorphism $\epsilon:\mathcal{R}U(\mathfrak{g})\to\mathbb{F}$, we have $\epsilon(B)=0$ (where ϵ here is acting on the first factor of the tensor product $\mathcal{R}U(\mathfrak{g}) \otimes \operatorname{Hom}\left(V\left[>-\frac{d}{2}\right] \oplus V\left[-\frac{d}{2}\right]_{>r_1}, V\left[<\frac{d}{2}\right] \oplus V\left[\frac{d}{2}\right]_{>r_1}\right)\right)$. It then follows by Proposition 1.4.2 (f) that

$$\mathbb{1}_{V[<\frac{d}{2}]_{r_1}} + BA^{-1}$$

is an invertible element of $\mathcal{R}_\infty U(\mathfrak{g}) \otimes \operatorname{End}\left(V\left[<\frac{d}{2}\right] \oplus V\left[\frac{d}{2}\right]_{>r_1}\right)$. Moreover, by Corollary 2.2.2, we have

$$(\mathbb{1} + BA^{-1})v\bar{1}_{\mathfrak{l}} = (v+w)\bar{1}_{\mathfrak{l}}\,.$$

We then have:

$$A^{-1}v\bar{1}_{\mathfrak{l}} = A^{-1}(\mathbb{1}+BA^{-1})^{-1}(\mathbb{1}+BA^{-1})v\bar{1}_{\mathfrak{l}} = (A+B)^{-1}(v+w)\bar{1}_{\mathfrak{l}}\,.$$

□

Note that in the notation of Chapter 1 and Section 2.2, we can restate Lemma 2.2.2 as

Lemma 2.2.6. *[DSKV16c, Lemma 6.1] The generalized quasideterminant $|\mathbb{1}_N + z^{-\Delta}E|_{I_1J_1}$ exists in the space $Mat_{r_1\times r_1}\mathcal{R}_\infty U(\mathfrak{g})$, and the following identity holds in $Mat_{r_1\times r_1}\mathcal{R}M_{\mathfrak{l}}$:*

$$|\mathbb{1}_N + z^{-\Delta}E|_{I_1J_1}\bar{1}_{\mathfrak{l}} = z^{-d-1}|z\mathbb{1}_N + F + \pi_{\mathfrak{p}}E + D_{\mathfrak{l}}|_{I_1J_1}\bar{1}_{\mathfrak{l}}\,. \tag{2.2.49}$$

Proof of Theorem 2.2.3. After all the techniques we have developed, we can now prove Theorem 2.2.3 in a similar manner as the analogous result for classical affine W-algebras, presented in [DSKV17a, Section 4]. By Lemma 2.2.2, the operator $|\mathbb{1}_V + z^{-\Delta}E|_{\Psi_{\frac{d}{2},r_1},\Pi_{-\frac{d}{2},r_1}}$ is an invertible element of $\mathcal{R}_\infty U(\mathfrak{g}) \otimes \operatorname{Hom}\left(V\left[-\frac{d}{2}\right]_{r_1}, V\left[\frac{d}{2}\right]_{r_1}\right)$, and Equation (2.2.18) holds. Hence, in view of Proposition 1.4.3, Theorem 2.2.3 holds provided that

$$\left[a, |\mathbb{1}_V + z^{-\Delta}E|_{\Psi_{\frac{d}{2},r_1},\Pi_{-\frac{d}{2},r_1}}\right]\bar{1}_{\mathfrak{l}} = 0 \quad \text{for all} \quad a \in \mathfrak{n}\,. \tag{2.2.50}$$

By the invertibility of $|\mathbb{1}_V + z^{-\Delta}E|_{\Psi_{\frac{d}{2},r_1},\Pi_{-\frac{d}{2},r_1}}$ in order to prove Equation (2.2.50) it suffices to prove that

$$\left[a, \left(|\mathbb{1}_V + z^{-\Delta}E|_{\Psi_{\frac{d}{2},r_1},\Pi_{-\frac{d}{2},r_1}}\right)^{-1}\right] = 0\,. \tag{2.2.51}$$

By Definition 1.5.4 for the generalized quasideterminant, we have

$$\begin{aligned}\left[a, \left(|\mathbb{1}_V + z^{-\Delta}E|_{\Psi_{\frac{d}{2},r_1},\Pi_{-\frac{d}{2},r_1}}\right)^{-1}\right] &= \Pi_{-\frac{d}{2},r_1}\left[a, (\mathbb{1}_V + z^{-\Delta}E)^{-1}\right]\Psi_{\frac{d}{2},r_1} \\ &= -\Pi_{-\frac{d}{2},r_1}(\mathbb{1}_V + z^{-\Delta}E)^{-1}\left[a, z^{-\Delta}E\right](\mathbb{1}_V + z^{-\Delta}E)^{-1}\Psi_{\frac{d}{2},r_1}\,.\end{aligned} \tag{2.2.52}$$

Recalling the definition (2.2.16) of the operator $z^{-\Delta}E$, we have

$$\begin{aligned}\left[a, z^{-\Delta}E\right] &= \sum_{i\in I} z^{\delta(i)-1}[a,u_i]U^i = \sum_{i,k\in I} z^{\delta(i)-1}([a,u_i]|u^k)u_kU^i \\ &= \sum_{i,k\in I} z^{\delta(k)-\delta(a)-1}(u_i|[u^k,a])u_kU^i = z^{-\delta(a)}\sum_{k\in I} z^{\delta(k)-1}u_k[U^k,\varphi(a)] \\ &= z^{-\delta(a)}[z^{-\Delta}E,\varphi(a)] = z^{-\delta(a)}[\mathbb{1}_V + z^{-\Delta}E,\varphi(a)]\,.\end{aligned} \tag{2.2.53}$$

Using (2.2.53), we can rewrite the RHS of (2.2.52) as

$$\begin{aligned}&- z^{-\delta(a)}\Pi_{-\frac{d}{2},r_1}(\mathbb{1}_V + z^{-\Delta}E)^{-1}[\mathbb{1}_V + z^{-\Delta}E,\varphi(a)](\mathbb{1}_V + z^{-\Delta}E)^{-1}\Psi_{\frac{d}{2},r_1} \\ &\quad = -z^{-\delta(a)}\Pi_{-\frac{d}{2},r_1}\varphi(a)(\mathbb{1}_V + z^{-\Delta}E)^{-1}\Psi_{\frac{d}{2},r_1} + z^{-\delta(a)}\Pi_{-\frac{d}{2},r_1}(\mathbb{1}_V + z^{-\Delta}E)^{-1}\varphi(a)\Psi_{\frac{d}{2},r_1}\,,\end{aligned} \tag{2.2.54}$$

Since, by assumption, $a \in \mathfrak{n} \subseteq \mathfrak{g}_{\geq\frac{1}{2}}$, we have $\varphi(a) \in (\operatorname{End} V)[\geq \frac{1}{2}]$, and therefore $\Pi_{-\frac{d}{2},r_1}\varphi(a) = 0$, $\varphi(a)\Psi_{\frac{d}{2},r_1} = 0$. Hence, the RHS of (2.2.54) vanishes, proving (2.2.51). □

The proof of Theorem 2.2.2 is the same as in [DSKV16c, Theorem 4.3]. It is based on Lemma 2.2.2, Theorem 2.2.3 and the properties of generalized quasideterminants.

2.3 Relation with the results in [BK06, BK08a]

In [BK06, BK08a] Brundan and Kleshchev provide a presentation for the W-algebra $W(\mathfrak{g}, f, \Gamma, 0)$ associated with a nilpotent element f and an even $\frac{1}{2}\mathbb{Z}$-grading Γ. Let $(p_1^{r_1} > \ldots > p_s^{r_s})$ be a partition of N, with $r = r_1 + \ldots + r_s$. They define a surjective filtered algebra homomorphism between the *shifted* Yangian for $\mathfrak{gl}_r$, $Y_r(\sigma)$ and the W-algebra $W(\mathfrak{gl}_N, f, \Gamma, 0)$. See [Mo07] for an extended treatment of Yangians and their properties.

Definition 2.3.1. The *Yangian for* $\mathfrak{gl}_n$ is a unital associative $\mathbb{C}$-algebra $Y_n := Y(\mathfrak{gl}_n)$ with countably many generators $T_{ij}^{(r)}$ and defining relations

$$[T_{ij}^{(r)}, T_{hk}^{(s)}] = \sum_{a=0}^{\min(r,s)-1} T_{hj}^{(a)} T_{ik}^{(r+s-a-1)} - T_{hj}^{(r+s-a-1)} T_{ik}^{(a)}\,, \tag{2.3.1}$$

where $1 \leq i,\, j,\, h,\, k \leq n$ and $r,\, s \geq 0$. We set $T_{ij}^{(0)} = \delta_{ij}$.

Introducing the formal generating series $T_{ij}(z) = \sum_{r\geq 0} T_{ij}^{(r)} z^{-r} \in Y_n[[z^{-1}]]$ we can rewrite Equation (2.3.1) as

$$(z-w)[T_{ij}(z), T_{hk}(w)] = T_{hj}(z)T_{ik}(w) - T_{hj}(w)T_{hk}(z)\,. \tag{2.3.2}$$

Note that Equation (2.3.2) is, up to an overall sign, equal to Identity (1.5.10). It is also convenient to introduce the matrix $T(z) = (T_{ij}(z))_{i,j=1}^n \in \mathrm{Mat}_{n\times n} Y_n[[z^{-1}]]$; this provides us with the so-called RTT presentation of the Yangian Y_n. In [BK05] Brundan and Kleshchev introduce presentations for Y_n which are parametrized by tuples of positive integers summing to n, the parabolic presentations.

In fact, given a tuple $(\nu_1, \ldots, \nu_m)$ of non-negative integers summing to n, we can factor the matrix $T(z)$ via Gauss factorization as

$$T(z) = E(z)D(z)F(z) \tag{2.3.3}$$

for unique block matrices

$$\begin{gathered} D(z) = \begin{pmatrix} D_1(z) & 0 & \ldots & 0 \\ 0 & D_2(z) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & D_m(z) \end{pmatrix}, \\ F(z) = \begin{pmatrix} I_{\nu_1} & 0 & \ldots & 0 \\ F_{21}(z) & I_{\nu_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_{m1}(z) & F_{m2}(z) & \ldots & I_{\nu_m} \end{pmatrix}, \quad E(z) = \begin{pmatrix} I_{\nu_1} & E_{12}(z) & \ldots & E_{1m}(z) \\ 0 & I_{\nu_2} & \ldots & E_{2m}(z) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I_{\nu_m} \end{pmatrix}, \end{gathered} \tag{2.3.4}$$

where $D_a(z) = (D_{a;ij}(z))_{1\leq i,j\leq \nu_a}$, $E_{ab}(z) = (E_{ab;ij}(z))_{1\leq i\leq \nu_a, 1\leq j\leq \nu_b}$ and $F_{ba}(z) = (F_{ba;ij}(z))_{1\leq i\leq \nu_b, 1\leq j\leq \nu_a}$ are $\nu_a \times \nu_a$, $\nu_a \times \nu_b$ and $\nu_b \times \nu_a$ matrices respectively. Define also the $\nu_a \times \nu_a$ matrix $\widetilde{D}_a(z) = (\widetilde{D}_{a;ij}(z))_{1\leq i,j\leq \nu_a}$ by $\widetilde{D}_a(u) = -D_a(u)^{-1}$. The entries of these matrices define power series

$$\begin{aligned} D_{a;ij}(z) &= \sum_{r\geq 0} D_{a;ij}^{(r)} z^{-r}\,, & E_{ab;ij}(z) &= \sum_{r\geq 1} E_{ab;ij}^{(r)} z^{-r}\,, \\ \widetilde{D}_{a;ij}(z) &= \sum_{r\geq 0} \widetilde{D}_{a;ij}^{(r)} z^{-r}\,, & F_{ba;ij}(z) &= \sum_{r\geq 1} F_{ba;ij}^{(r)} z^{-r}\,. \end{aligned}$$

The following holds:

Theorem 2.3.1. *[BK05, Theorem A] The algebra Y_n is generated by the elements*

$$\begin{aligned} &\{D_{a;ij}^{(r)},\, \widetilde{D}_{a;ij}^{(r)}\}_{1\leq a\leq m,\, 1\leq i,j\leq \nu_a,\, r\geq 0}\,, \\ &\{E_{a,a+1;ij}^{(r)}\}_{1\leq a< m,\, 1\leq i\leq \nu_a,\, 1\leq j\leq \nu_{a+1},\, r\geq 1}\,, \\ &\{F_{a+1,a;ij}^{(r)}\}_{1\leq a< m,\, 1\leq i\leq \nu_{a+1},\, 1\leq j\leq \nu_a,\, r\geq 1}\,, \end{aligned} \tag{2.3.5}$$

subject to the relations (1.1) – (1.14) *in [BK05].*

Note that by Equation (2.3.3) we obtain, for every $1 \le i, j \le m$,

$$T_{ij}(z) = \sum_{k=\max(i,j)}^{m} E_{ik}(z) D_k(z) F_{kj}(z) .$$

Remark 2.3.1. We highlight the following remarkable cases: for $\nu = (n)$ we obtain the RTT presentation for Y_n, while for $\nu = (1^n)$ we obtain a variation of Drinfeld's presentation for Y_n introduced in [Dr88].

After writing $T(z)$ in block form as

$$T(z) = \begin{pmatrix} T^{\nu}_{11}(z) & \dots & T^{\nu}_{1m}(z) \\ \vdots & \ddots & \vdots \\ T^{\nu}_{m1}(z) & \dots & T^{\nu}_{mm}(z) \end{pmatrix} , \tag{2.3.6}$$

where $T^{\nu}_{ab}(z)$ is a $\nu_a \times \nu_b$ matrix, we can describe explicitly the matrices $D_a(z)$, $E_{ab}(z)$, $F_{ba}(z)$ from (2.3.4) in terms of quasideterminants as follows:

$$D_a(z) = \begin{vmatrix} \boxed{T^{\nu}_{aa}(z)} & T^{\nu}_{a,a+1}(z) & \cdots & T^{\nu}_{am}(z) \\ T^{\nu}_{a+1,a}(z) & T^{\nu}_{a+1,a+1}(z) & \cdots & T^{\nu}_{a+1m}(z) \\ \vdots & \vdots & \ddots & \vdots \\ T^{\nu}_{ma}(z) & T^{\nu}_{ma+1}(z) & \cdots & T^{\nu}_{mm}(z) \end{vmatrix} , \tag{2.3.7}$$

$$E_{ab}(z) = \begin{vmatrix} \boxed{T^{\nu}_{ab}(z)} & T^{\nu}_{ab+1}(z) & \cdots & T^{\nu}_{am}(z) \\ T^{\nu}_{b+1b}(z) & T^{\nu}_{b+1b+1}(z) & \cdots & T^{\nu}_{b+1m}(z) \\ \vdots & \vdots & \ddots & \vdots \\ T^{\nu}_{mb}(z) & T^{\nu}_{mb+1}(z) & \cdots & T^{\nu}_{mm}(z) \end{vmatrix} D_b(z)^{-1} , \tag{2.3.8}$$

$$F_{ba}(z) = D_b(z)^{-1} \begin{vmatrix} \boxed{T^{\nu}_{ba}(z)} & T^{\nu}_{bb+1}(z) & \cdots & T^{\nu}_{bm}(z) \\ T^{\nu}_{b+1a}(z) & T^{\nu}_{b+1b+1}(z) & \cdots & T^{\nu}_{b+1,m}(z) \\ \vdots & \vdots & \ddots & \vdots \\ T^{\nu}_{ma}(z) & T^{\nu}_{mb+1}(z) & \cdots & T^{\nu}_{mm}(z) \end{vmatrix} . \tag{2.3.9}$$

We are using following notation for the quasideterminant: suppose that A, B, C and D are $m \times n$, $m \times q$, $q \times n$, $q \times q$ matrices respectively, with entries in an a ring R. Assuming that the matrix D is invertible, then

$$\begin{vmatrix} \boxed{A} & B \\ C & D \end{vmatrix} := A - BD^{-1}C . \tag{2.3.10}$$

Brundan and Kleshchev in [BK06] introduce particular subalgebras of Y_n called shifted Yangians, that depend on the choice of a shift matrix σ.

Definition 2.3.2. A matrix $\sigma = (s_{ij})_{i,j=1}^n \in \mathrm{Mat}_{n\times n}\mathbb{Z}_+$ is a *shift matrix* if $s_{ij} = s_{ik} + s_{kj}$ whenever $|i-j| = |i-k| + |k-j|$.

Let $\nu = (\nu_1, \dots, \nu_m)$ be a tuple of nonnegative integers summing to n that is *admissible* for σ, meaning that $s_{ij} = 0$ for all $\nu_1 + \ldots + \nu_{a-1} + 1 \le i, j \le \nu_1 + \ldots + \nu_a$ and $1 \le a \le m$. Then the (parabolic presentation for the) shifted Yangian is defined as follows:

Definition 2.3.3. The *shifted Yangian* associated to the shift matrix σ is the algebra $Y_n(\sigma)$ over $\mathbb{C}$ defined by generators

$$\begin{aligned} &\{D^{(r)}_{a;ij}, \widetilde{D}^{(r)}_{a;ij}\}_{1\le a\le m,\, 1\le i,j\le \nu_a,\, r\ge 0} , \\ &\{E^{(r)}_{a,a+1;ij}\}_{1\le a< m,\, 1\le i\le \nu_a,\, 1\le j\le \nu_{a+1},\, r> s_{a,a+1}(\nu)} , \\ &\{F^{(r)}_{a+1,a;ij}\}_{1\le a< m,\, 1\le i\le \nu_{a+1},\, 1\le j\le \nu_a,\, r> s_{a+1,a}(\nu)} , \end{aligned} \tag{2.3.11}$$

subject to the relations (3.3) – (3.14) in [BK06]. We use the shorthand $s_{ab}(\nu) := s_{\nu_1+\ldots+\nu_a, \nu_1+\ldots+\nu_b}$.

Note that the usual (non parabolic) presentation for the shifted Yangian $Y_n(\sigma)$ is obtained for $\nu = (1^n)$, that is always admissible. Also, there is a canonical homomorphism $Y_n(\sigma) \longrightarrow Y_n$ mapping the generators $D^{(r)}_{a;ij}$, $E^{(r)}_{a,a+1;ij}$, $F^{(r)}_{a+1,a;ij}$ of $Y_n(\sigma)$ to the elements of Y_n with the same names. This homomorphism is actually injective and its image is independent from the particular tuple ν ([BK06, Section 3]). We can therefore identify $Y_n(\sigma)$ with a subalgebra of Y_n; these algebras clearly coincide for $\sigma = 0$.

For an integer $l \geq s_{1,n} + s_{n,1}$ called *level*, it is possible to define the following

Definition 2.3.4. The *shifted Yangian of level* l, denoted $Y_{n,l}(\sigma)$, is the quotient of $Y_n(\sigma)$ by the two-sided ideal generated by the elements $\{D^{(r)}_{m;ij}\}_{1\leq i,j\leq \nu_1, r\geq l-s_{1n}-s_{n1}}$.

In the case $\sigma = 0$, then $Y_{n,l}(\sigma) =: Y_{n,l}$ coincides with the Yangian of level l introduced by Cherednik [Ch87].

Let σ be the shift matrix $\sigma = (s_{ij})^r_{ij=1}$ given by the elements s_{ij} as in (1.1.2), and let $Y_{r,p_1}(\sigma)$ be the quotient of $Y_r(\sigma)$ by the two-sided ideal generated by the elements $\{D^{(k)}_{m;ij}\}_{1\leq i,j\leq \nu_1}$ for $k \geq p_1 - s_{1r} - s_{r1} = p_r$. In [BK06], Brundan and Kleshchev describe an isomorphism of filtered[2] algebras between this truncated shifted Yangian of level p_1 and the W-algebra $W(\mathfrak{g}, f, \Gamma, 0)$ in the case when Γ is an even $\frac{1}{2}\mathbb{Z}$-grading:

$$\kappa : Y_{r,p_1}(\sigma) \longrightarrow W(\mathfrak{g}, f, \Gamma, 0)\,. \tag{2.3.12}$$

To establish the isomorphism κ, Brundan and Kleshchev describe a matrix ${}^{\mathrm{BK}}T(z) = (T_{ij;0})^r_{i,j=1} \in \mathrm{Mat}_{r\times r} U(\mathfrak{g}_{\leq 0})[z^{-1}]$ such that, for every $1 \leq i, j \leq r$

$$T_{ij;0}(z) = \delta_{ij} + \sum_{k=1}^{\min(p_i,p_j)+s_{ij}} T^{(k)}_{ij;0} z^{-k}\,.$$

Using the same setup and in particular the same basis for $\mathfrak{gl}_N$ as in Section 1.1.1, the coefficients $T^{(k)}_{ij;0}$ are computed as follows:

$$T^{(k)}_{ij;0} = \sum_{s=1}^{k} (-1)^{k-s} \sum_{\substack{(i_1,h_1),\ldots,(i_s,h_s)\\(j_1,k_1),\ldots,(j_s,k_s)}} \tilde{e}_{(i_1,h_1)(j_1,k_1)} \cdots \tilde{e}_{(i_s,h_s)(j_s,k_s)}\,, \tag{2.3.13}$$

where $(i_1, h_1), \ldots, (i_s, h_s)$, $(j_1, k_1), \ldots, (j_s, k_s) \in \mathcal{T}$ and

(i) $-\big(x(i_1, h_1) - x(j_1, k_1) + \ldots + x(i_s, h_s) - x(j_s, k_s)\big) + s = k$;

(ii) $x(i_m, h_m) \leq x(j_m, k_m)$, $\forall m = 1, \ldots, s$;

(iii) $j_1 = i$, $i_s = j$;

(iv) $j_{m+1} = i_m$, $\forall m = 1, \ldots, s$;

(v) $k_{m+1} > h_m$, $\forall m = 1, \ldots s - 1$.

Remark 2.3.2. It is however important to remark that the Laurent polynomials $T_{ij;0}(z)$ introduced in [BK06, Section 9] are not exactly the same as in (2.3.13). This is due to the fact that in [BK06] the authors use a different combinatorial setup for the pyramid attached to a good $\frac{1}{2}\mathbb{Z}$-grading Γ and the numbering of their boxes. Moreover, their construction is adapted to the W-algebra associated with the nilpotent element $e = \sum_{\substack{(i,h)\in\mathcal{T}\\h<p_i}} e_{(i,h)(i,h+1)}$, that in the case of an even grading is a subalgebra of $U(\mathfrak{g}_{\geq 0})$.

Consistently, differences arise also in the parabolic presentations for the shifted Yangian $Y_n(\sigma)$ and for the truncated shifted Yangian $Y_{n,l}(\sigma)$; for instance, the Gauss factorization of $T(z)$ in [BK06] becomes $T(z) = F(z)D(z)E(z)$, where $D(z)$, $E(z)$, $F(z)$ are matrices as in (2.3.4).

[2] The W-algebra is equipped with the Kazhdan filtration, while the truncated shifted Yangian is equipped with the *canonical filtration* which is defined by declaring that the elements $D^{(r)}_{a;ij}$, $E^{(r)}_{ab;ij}$ and $F^{(r)}_{ba;ij}$ are all of degree r.

Fixed a tuple $(\nu_1, \ldots, \nu_m)$ summing to r, by a process of Gauss factorization of ${}^{\mathrm{BK}}T(z)$ as in (2.3.3), we obtain matrices

$$\begin{aligned} D(z) &= (D_a(z))_{1\le a\le m} = ((D_{a;ij}(z))_{1\le i,j\le \nu_a})_{1\le a\le m} = \Big(\big(\sum_{k\ge 0} D^{(k)}_{a;ij} z^{-k}\big)_{1\le i,j\le \nu_a}\Big)_{1\le a\le m}\,, \\ E(z) &= (E_{ab}(z))_{1\le a,b\le m} = ((E_{ab;ij}(z))_{\substack{1\le i\le \nu_a\\ 1\le j\le \nu_b}})_{1\le a,b\le m} = \Big(\big(\sum_{k>0} E^{(k)}_{ab;ij} z^{-k}\big)_{\substack{1\le i\le \nu_a\\ 1\le j\le \nu_b}}\Big)_{1\le a,b\le m}\,, \\ F(z) &= (F_{ba}(z))_{1\le a,b\le m} = ((F_{ba;ij}(z))_{\substack{1\le i\le \nu_b\\ 1\le j\le \nu_a}})_{1\le a,b\le m} = \Big(\big(\sum_{k>0} F^{(k)}_{ba;ij} z^{-k}\big)_{\substack{1\le i\le \nu_b\\ 1\le j\le \nu_a}}\Big)_{1\le a,b\le m}\,. \end{aligned} \tag{2.3.14}$$

These matrices can be described explicitly as in (2.3.7) - (2.3.9). Fixed an admissible tuple $(\nu_1, \ldots, \nu_m)$ summing to r, and chosen l and σ as before, the isomorphism κ therefore sends the generators

$$\begin{aligned} &\{D^{(r)}_{a;ij}\}_{1\le a\le m,\,1\le i,j\le \nu_a,\,r\ge 0}\,, \\ &\{E^{(r)}_{a,a+1;ij}\}_{1\le a<m,\,1\le i\le \nu_a,\,1\le j\le \nu_{a+1},\,r>s_{a,a+1}(\nu)}\,, \\ &\{F^{(r)}_{a+1,a;ij}\}_{1\le a<m,\,1\le i\le \nu_{a+1},\,1\le j\le \nu_a,\,r>s_{a+1,a}(\nu)}\,, \end{aligned} \tag{2.3.15}$$

of $Y_{n,l}(\sigma)$ to the elements of $U(\mathfrak{g}_{\le 0})$ with the same name. These elements are proved to be invariant under the adjoint action of the nilpotent subalgbra $\mathfrak{g}_{\ge 1}$ of $\mathfrak{g}$, and hence belong to $W(\mathfrak{g}, f, \Gamma, 0)$.

To explain the connection with our work, remember that the polynomials $T_{ij}(z)$ introduced in Definition 2.1.1 can be described by (2.1.5) or equivalently by (2.1.10).

Using Equation (2.1.10) in the case of an even $\frac{1}{2}\mathbb{Z}$-grading it is easy to show that, up to a sign change and a shift in the powers of z, the polynomials $T_{ij}(z)$ described there coincide with the elements with the same name defined by Equation 2.3.13. In fact, from Equation (2.1.10) we obtain

$$T_{ij}(z) = -(-z)^{p_i}\delta_{ij} + \sum_{k=0}^{s_{ij}+\min(p_i,p_j)-1} T^{(k)}_{ij}(-z)^k, \tag{2.3.16}$$

where s_{ij} is as in (1.1.2). Moreover, the coefficient of $(-z)^k$ in $T_{ij}(z)$ is, for every $0 \le k \le s_{ij} + \min(p_i, p_j) - 1$,

$$\begin{aligned} T^{(k)}_{ij} = &- \sum_{s=1}^{-k+s_{ij}+\min(p_i,p_j)} (-1)^s \sum_{\substack{(i_0,h_0),\ldots,(i_{s-1},h_{s-1})\in J^c\\ (i_s,h_s)\in\mathcal{T}}} \sum_{\substack{n_i\ge 0\\ n_0+\ldots+n_s=k}} \delta_{(i_0,h_0)(i,1)}\delta_{(i_s,h_s+n_s)(j,p_j)} \\ &\pi_{\le 0}\tilde{e}_{(i_1,h_1)(i_0,h_0+n_0)}\pi_{\le 0}\tilde{e}_{(i_2,h_2)(i_1,h_1+1+n_1)}\cdots\pi_{\le 0}\tilde{e}_{(i_s,h_s)(i_{s-1},h_{s-1}+\delta_{s>1}+n_{s-1})}\,. \end{aligned} \tag{2.3.17}$$

We can reformulate (2.3.17) as

$$T^{(k)}_{ij} = -\sum_{s=1}^{-k+s_{ij}+\min(p_i,p_j)} (-1)^s \sum_{\substack{(i_1,h_1),\ldots,(i_s,h_s)\\ (j_1,k_1),\ldots,(j_s,k_s)}} \tilde{e}_{(i_1,h_1)(j_1,k_1)}\cdots\tilde{e}_{(i_s,h_s)(j_s,k_s)}\,, \tag{2.3.18}$$

where $(i_1,h_1),\ldots,(i_s,h_s)$, $(j_1,k_1),\ldots,(j_s,k_s) \in \mathcal{T}$ and

(i) $x(i_m,h_m) \le x(j_m,k_m)$, $\forall\, m = 1,\ldots,s$;

(ii) $-\big(x(i_1,h_1) - x(j_1,k_1) + \ldots + x(i_s,h_s) - x(j_s,k_s)\big) + s = s_{ij} + \min(p_i,p_j) - k$;

(iii) $k_{m+1} > h_m$, $\forall\, m = 1,\ldots s-1$;

(iv) $j_1 = i$, $i_s = j$;

(v) $j_{m+1} = i_m$, $\forall\, m = 1,\ldots,s$.

Comparing (2.3.13) and (2.3.18) we clearly have

$$T_{ij;0}^{(k)} = (-1)^{k-1} T_{ij}^{(s_{ij}+\min(p_i,p_j)-k)}, \qquad \text{and} \qquad -(-z)^{\min(p_i,p_j)+s_{ij}} T_{ij;0}(z) = T_{ij}(z). \tag{2.3.19}$$

Taken the admissible shape $\nu = (r_1, \dots, r_s)$, we can compute quasideterminants of $T(z)$ and ${}^{\mathrm{BK}}T(z)$ with respect to the top leftmost $r_1 \times r_1$ block, namely

$$|T(z)|_{I_{rr_1} J_{r_1 r}} = \begin{vmatrix} \boxed{T_{11}^{\nu}(z)} & \dots & T_{1m}^{\nu}(z) \\ \vdots & \ddots & \vdots \\ T_{m1}^{\nu}(z) & \dots & T_{mm}^{\nu}(z) \end{vmatrix}, \quad |{}^{\mathrm{BK}}T(z)|_{I_{rr_1} J_{r_1 r}} = \begin{vmatrix} \boxed{T_{11;0}^{\nu}(z)} & \dots & T_{1m;0}^{\nu}(z) \\ \vdots & \ddots & \vdots \\ T_{m1;0}^{\nu}(z) & \dots & T_{mm;0}^{\nu}(z) \end{vmatrix}.$$

By (2.3.19) and a straightforward computation we get $|T(z)|_{I_{rr_1} J_{r_1 r}} = -(-z)^{p_1} |{}^{\mathrm{BK}}T(z)|_{I_{rr_1} J_{r_1 r}}$.

By (2.3.7) we also get $|{}^{\mathrm{BK}}T(z)|_{I_{rr_1} J_{r_1 r}} = D_1(z)$, whereas by Definition 2.2.3, $|T(z)|_{I_{rr_1} J_{r_1 r}} = \widetilde{L}(z)$. Combining these results, we obtain

$$\widetilde{L}(z) = -(-z)^{p_1} D_1(z). \tag{2.3.20}$$

This is significant not only because it provides another abstract connection between our results and the work of Brundan and Kleshchev, but also because the commutation relations for $D_1(z)$ given in [BK06, Equation (3.3), Section 3] show that $D_1(z)$ itself is an operator of Yangian type. The same theorem [BK06, Theorem 10.1] also proves the coefficients of $D_1(z)$ belong to $W(\mathfrak{g}, f, \Gamma, 0)$. We have therefore obtained a second and indirect proof for the properties of the matrix operator $L(z)$ that we proved in Theorems 2.2.1 and 2.2.2.

As it was already highlighted in [DSKV17b, Remark 6.17]), note that as a consequence of Theorem 2.2.1 we obtain an algebra homomorphism $Y_{r_1} \longrightarrow W(\mathfrak{g}, f, \Gamma, 0)$, sending the generator matrix $T_{r_1}(z) \in Y_{r_1}[[z^{-1}]]$ to $L(z) \in \mathrm{Mat}_{r_1 \times r_1} W(\mathfrak{g}, f, \Gamma, 0)((z^{-1}))$. This homomorphism is the restriction of the surjective homomorphism $Y_r(\sigma) \longrightarrow W(\mathfrak{g}, f, \Gamma, 0)$ (cf. (2.3.12)) to the subalgebra $Y_{r_1} \subset Y_r(\sigma)$ corresponding to $D_1(z)$.

Chapter 3

A finite set of generators for $W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$

3.1 A conjecture on generators

Fix a subspace $U \subset \mathfrak{g}$ complementary to $[f, \mathfrak{g}]$ and compatible with the $\frac{1}{2}\mathbb{Z}$-grading. For example, we could take $U = \mathfrak{g}^e$, the Slodowy slice.

Note that by the non-degeneracy of $(\cdot\,|\,\cdot)$ the orthocomplement to $[f, \mathfrak{g}]$ is $\mathfrak{g}^f$. Hence, after identifying $\mathfrak{g}^* \cong \mathfrak{g}$, the direct sum decomposition dual to $\mathfrak{g} = [f, \mathfrak{g}] \oplus U$ has the form

$$\mathfrak{g} = U^{\perp} \oplus \mathfrak{g}^f \,, \tag{3.1.1}$$

where $U^{\perp}$ is the orthocomplement to U with respect to the form $(\cdot\,|\,\cdot)$. As a consequence of (3.1.1) we have the decomposition in a direct sum of subspaces

$$S(\mathfrak{g}) = S(\mathfrak{g}^f) \oplus S(\mathfrak{g})U^{\perp},$$

and we denote by

$$\eta^f : S(\mathfrak{g}) \twoheadrightarrow S(\mathfrak{g}^f) \tag{3.1.2}$$

the surjective algebra homomorphism defined by $\eta^f(a) = \pi^f(a) + (f|a)$, where $\pi^f : \mathfrak{g} \twoheadrightarrow \mathfrak{g}^f$ is the projection with kernel $U^{\perp}$. Recall that we can identify $\mathrm{gr}\, I_{\mathfrak{l}} \subset \mathrm{gr}\, U(\mathfrak{g})$ with the two-sided ideal $S(\mathfrak{g})\langle b - (f|b)\rangle_{b\in\mathfrak{m}}$ of $\mathrm{gr}\, U(\mathfrak{g}) \cong S(\mathfrak{g})$. By assumption the subspace U is compatible with the grading, therefore since $\mathfrak{g}^f \subset \mathfrak{g}_{\leq 0}$, we have $\pi^f(\mathfrak{g}_{\geq \frac{1}{2}}) = 0$, and it follows that $\mathrm{gr}\ I_{\mathfrak{l}} \subset \mathrm{Ker}\, \eta^f$. Let us denote by the same letter η^f be the unique algebra homomorphism such that the following diagram commutes:

$$\begin{array}{ccc} S(\mathfrak{g}) & \xrightarrow{\ \eta^f\ } & S(\mathfrak{g}^f) \\ \downarrow & \nearrow_{\eta^f} & \\ S(\mathfrak{g})/\mathrm{gr}\, I_{\mathfrak{l}} & & \end{array} \tag{3.1.3}$$

In [Pr02], Premet gives a description of a PBW basis for $W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$, with $\mathfrak{l} \subset \mathfrak{g}_{\frac{1}{2}}$ Lagrangian and Γ the Dynkin grading associated with f:

Theorem 3.1.1. *[Pr02, Theorem 4.6] There exists a (non-unique) linear map*

$$w : \mathfrak{g}^f \longrightarrow W(\mathfrak{g}, f, \Gamma, \mathfrak{l}) \tag{3.1.4}$$

such that $w(x) \in F_{\Delta}W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$ and $\eta^f(gr_{\Delta}(w(x)) = x$, for every $x \in \mathfrak{g}^f_{1-\Delta}$. Moreover, if $\{x_i\}_{i=1}^t, t =$ dim $\mathfrak{g}^f$, is an ordered basis of $\mathfrak{g}^f$ consisting of ad$x$-eigenvectors $x_i \in \mathfrak{g}^f_{1-\Delta_i}$, then the monomials

$$\{w(x_{i_1}) \cdots w(x_{i_k}) \mid k \in \mathbb{Z}_+, 1 \leq i_1 \leq \cdots \leq i_k \leq t, \Delta_1 + \ldots + \Delta_k \leq \Delta\}$$

form a basis of $F_{\Delta}W(\mathfrak{g}, f, \Gamma, \mathfrak{l}), \Delta \geq 0$.

For an analogous of Theorem 3.1.1 in the classical case, see [DSKV14, Theorem 2.4]; it is in fact proved that the classical affine W-algebra is an algebra of differential polynomials, parametrized by $\mathfrak{g}^f$.

For our purposes, given a pyramid p, we fix U to be the following subspace of $\mathfrak{g}$:

$$U = \mathrm{Span}_{\mathbb{C}}\{e_{(j,1+k)(i,p_i)} \mid 1 \le i,j \le r, 0 \le k \le \min(p_i,p_j)-1\}\,. \tag{3.1.5}$$

It is proved as in [DSKV16b] that for this choice of U, the equality $\mathfrak{g} = [f,\mathfrak{g}] \oplus U$ holds. For the choice of U as in (3.1.5), it is easy to compute that its orthocomplement $U^\perp$ is the following subspace:

$$\begin{aligned} U^\perp = &\mathrm{Span}_{\mathbb{C}}\{e_{(j,k)(i,h)} \mid 1 \le i,j \le r\,, 1 \le k \le p_j - 1\,, 1 \le h \le p_i\} \\ &\oplus \mathrm{Span}_{\mathbb{C}}\{e_{(j,p_j)(i,h)} \mid 1 \le i < j \le r\,, p_j + 1 \le h \le p_i\}\,. \end{aligned} \tag{3.1.6}$$

We also fix the basis of $\mathfrak{g}^f$ that is dual to the basis of U in (3.1.5). This is given by the set

$$\{f_{ij;k} \mid 1 \le i,j \le r,\ 0 \le k \le \min(p_i,p_j)-1\}\,, \tag{3.1.7}$$

where

$$f_{ij;k} = \sum_{h=0}^{k} e_{(i,p_i+h-k)(j,h+1)}\,. \tag{3.1.8}$$

In [DSKV16c] the following conjecture is introduced with the purpose of giving a (finite) free set of generators for $W(\mathfrak{g},f,\Gamma,0)$ (where Γ is the Dynkin grading),

$$\{W_{ij;k} = w(f_{ji;k}) \mid 1 \le i,j \le r, 0 \le k \le \min(p_i,p_j)-1\} \tag{3.1.9}$$

which satisfies the conditions of Theorem 3.1.1. Note that, since the collection of elements $f_{ij;k}$ as in (3.1.7) constitutes a basis of $\mathfrak{g}^f$, the assignment

$$f_{ji;k} \mapsto w(f_{ji;k}) = W_{ij;k}\,,$$

is sufficient to uniquely determine a linear map $w : \mathfrak{g}^f \to W(\mathfrak{g},f,\Gamma,0)$.

Conjecture 3.1.1. *[DSKV16c, Conjecture 8.2] There exists a unique set of generators $W_{ij;k} = w(f_{ji;k})$, $1 \le i,j \le r$, and $0 \le k \le min(p_i,p_j)-1$, of $W(\mathfrak{g},f,\Gamma,0)$, for which the following identity holds*

$$L(z)\,\bar{1} = |z\mathbb{1}_N + F + \pi_{\le\frac{1}{2}}E + D|_{I_1J_1}\,\bar{1} = |-(-z)^p\mathbb{1}_r + W(z)|_{I_{rr_1}J_{r_1r}}\,, \tag{3.1.10}$$

where

$$W(z) = (W_{ij}(z))_{1\le i,j\le r}\,, \qquad W_{ij}(z) = \sum_{k=0}^{min(p_i,p_j)-1} W_{ij;k}(-z)^k \in W(\mathfrak{g},f,\Gamma,0)[z]\,, \tag{3.1.11}$$

and

$$(-z)^p\mathbb{1}_r = \begin{pmatrix} (-z)^{p_1} & & \\ & \ddots & \\ & & (-z)^{p_r} \end{pmatrix}.$$

Moreover, I_{rr_1}, J_{r_1r} are as in (2.2.6) corresponding to the subsets $\mathcal{I} = \mathcal{J} = \{1,\dots,r_1\}$. In this case, the linear map $w : \mathfrak{g}^f \longrightarrow W(\mathfrak{g},f,\Gamma,0)$ defined by (3.1.9) satisfies all the conditions of Premet's Theorem 3.1.1.

Note that Conjecture 3.1.1 is the finite counterpart of an analogue result that holds for the classical affine W-algebra case (see [DSKV16b]).

3.2 Definition of the matrix $W(z)$

We shall therefore define a suitable matrix $W(z) \in \mathrm{Mat}_{r\times r} W(\mathfrak{g}, f, \Gamma, 0)[z]$ that, as we will later show, provides a solution to Conjecture 3.1.1. Unfortunately, we cannot give a compact formulation via a quasideterminant as in Definitions 2.1.1 and 2.2.1. Here the definition is recursive, and it depends on the number of columns of the pyramid p. The recursive nature of this definition, however, allows us to determine completely the matrix $W(z)$ only in the case of an even $\frac{1}{2}\mathbb{Z}$-grading, namely an even pyramid p.

Definition 3.2.1. Let Γ be an even good $\frac{1}{2}\mathbb{Z}$-grading for a nilpotent element f associated with the partition $(p_1 \geq p_2 \geq \ldots \geq p_r)$. If $p_1 = 1$, namely if the pyramid p consists of a single column, define

$$W_{ij}(z) := \delta_{ij} z + e_{(j,1)(i,1)} \,. \tag{3.2.1}$$

If $p_1 > 1$, we let $W_{ij}(z) = \widetilde{W}_{ij}(z)\bar{1}_{\mathfrak{l}}$, where $\widetilde{W}_{ij}(z) \in U(\mathfrak{g})[z]$ and $\bar{1}_{\mathfrak{l}}$ is the image of 1 in the quotient $U(\mathfrak{g})/I_{\mathfrak{l}}$ (in other words, $W_{ij}(z)$ coincides with the image of $\widetilde{W}_{ij}(z)$ under the quotient map $\rho_{\mathfrak{l}} : U(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})/I_{\mathfrak{l}}$, cf. (1.1.14)). Denote, as in Section 1.1.3, by ${}'p$ (resp. p') the pyramid obtained from p by removing the leftmost (resp. rightmost) column. Assume by induction that the matrix $\widetilde{W}^{\prime p}(z) \in \mathrm{Mat}_{r\times r} U(\mathfrak{g}^{\prime p})[z]$ (resp. $\widetilde{W}^{p'}(z) \in \mathrm{Mat}_{r\times r} U(\mathfrak{g}^{p'})[z]$) has been defined. Then, we define $\widetilde{W}(z)$ via the following recursive formulas, where we have to possible ways to proceed, depending on which is the shortest column of p.

If $t_1 \leq s_1$, namely if the leftmost column of the pyramid is the shortest, define

$$\widetilde{W}_{ij}(z) = \begin{cases} \sigma_l(\widetilde{W}^{\prime p}_{ij}(z))\,, & j \geq t_1 + 1 \\ [e_{(j,p_1)(j,p_1-1)}, \sigma_l(\widetilde{W}^{\prime p}_{ij}(z))] - \displaystyle\sum_{h=1}^{t_1} \sigma_l(\widetilde{W}^{\prime p}_{ih}(z))(\delta_{hj} z + \tilde{e}_{(j,p_1)(h,p_1)}) & \\ \qquad + \displaystyle\sum_{h=t_1+1}^{r} \sigma_l(\widetilde{W}^{\prime p}_{ih}(z)) \sigma_l(\widetilde{W}^{\prime p}_{hj;p_h-1})\,, & j \leq t_1\,. \end{cases} \tag{3.2.2}$$

Whereas if $s_1 \leq t_1$, namely if the rightmost column of the pyramid is the shortest, define

$$\widetilde{W}_{ij}(z) = \begin{cases} \sigma_r(\widetilde{W}^{p'}_{ij}(z))\,, & i \geq s_1 + 1 \\ [\sigma_r(\widetilde{W}^{p'}_{ij}(z)), e_{(i,2)(i,1)}] - \displaystyle\sum_{h=1}^{s_1} (\delta_{ih} z + \tilde{e}_{(h,1)(i,1)}) \sigma_r(\widetilde{W}^{p'}_{hj}(z)) & \\ \qquad + \displaystyle\sum_{h=s_1+1}^{r} \sigma_r(\widetilde{W}^{p'}_{ih;p_h-1}) \sigma_r(\widetilde{W}^{p'}_{hj}(z))\,, & i \leq s_1\,. \end{cases} \tag{3.2.3}$$

Note that we denote by $\widetilde{W}^{\prime p}_{hj;p_h-1}$ (resp. $\widetilde{W}^{p'}_{ih;p_h-1}$) the coefficient of the maximum power of $(-z)$ in $\widetilde{W}^{\prime p}_{hj}(z)$ (resp. $\widetilde{W}^{p'}_{ih}(z)$).

We shall denote by *left recursion* the recursion defined by Equation (3.2.2), and by *right recursion* the recursion defined by Equation (3.2.3).

Example 3.2.1. Let $p_1 = 2$ and suppose $t_1 \leq s_1$. In view of Definition 1.1.6, p is in this case aligned to the right. Then,

$$\widetilde{W}_{ij}(z) = \begin{cases} \delta_{ij} z + e_{(j,1)(i,1)}\,, & j \geq t_1 + 1 \\ -\delta_{ij} z^2 - \delta_{i\leq t_1}(\tilde{e}_{(j,2)(i,2)} + e_{(j,1)(i,1)}) z + e_{(j,2)(i,1)} + \displaystyle\sum_{h\geq t_1+1} e_{(h,1)(i,1)} e_{(j,1)(h,1)} & \\ \qquad - \displaystyle\sum_{h=1}^{t_1} e_{(h,1)(i,1)} \tilde{e}_{(j,2)(h,2)}\,, & j \leq t_1\,. \end{cases}$$

Example 3.2.2. Let $p_1 = 3$ and suppose that p is right-aligned (it implies $t_1 \leq t_2 \leq r = s_1$). Then,

$$\widetilde{W}_{ij}(z) = \begin{cases} \delta_{ij}z + e_{(j,1)(i,1)}\,, & j \geq t_2+1 \\ -\delta_{ij}z^2 - \delta_{i\leq t_2}(\tilde{e}_{(j,2)(i,2)} + e_{(j,1)(i,1)})z + e_{(j,2)(i,1)} + \sum_{h\geq t_2+1} e_{(h,1)(i,1)}e_{(j,1)(h,1)} & \\ \qquad\qquad - \sum_{h=1}^{t_2} e_{(h,1)(i,1)}\tilde{e}_{(j,2)(h,2)}\,, & t_1+1 \leq j \leq t_2 \\ e_{(j,3)(i,1)} - \sum_{t=1}^{t_2}(\delta_{it}z + e_{(t,1)(i,1)})e_{(j,3)(t,2)} & \\ \quad - \sum_{h=1}^{t_1}(e_{(h,2)(i,1)} - \sum_{t=1}^{t_2}(\delta_{it}z + e_{(t,1)(i,1)})(\delta_{th}z + \tilde{e}_{(h,2)(t,2)}) & \\ \quad + \sum_{t\geq t_2+1}(\delta_{it}z + e_{(t,1)(i,1)})e_{(h,1)(t,1)}))(\delta_{hj}z + \tilde{e}_{(j,3)(h,3)}) & \\ \quad + \sum_{h=t_1+1}^{t_2}(e_{(h,2)(i,1)} - \sum_{t=1}^{t_2}(\delta_{it}z + e_{(t,1)(i,1)})(\delta_{tj}z + \tilde{e}_{(h,2)(t,2)}) & \\ \quad + \sum_{t\geq t_2+1}(\delta_{it}z + e_{(t,1)(i,1)})e_{(h,1)(t,1)}))(\tilde{e}_{(j,2)(h,2)} + e_{(j,1)(h,1)}) & \\ \quad + \sum_{h\geq t_2+1}(\delta_{hi}z + e_{(h,1)(i,1)})(e_{(j,2)(h,1)} - \sum_{t=1}^{t_2} e_{(t,1)(h,1)}\tilde{e}_{(j,2)(t,2)}) & \\ \quad + \sum_{t\geq t_2+1} e_{(t,1)(h,1)}e_{(j,1)(t,1)})\,, & j \leq t_1\,. \end{cases}$$

Remark 3.2.1. When the pyramid p reduces to a rectangle, namely when the partition is $(p_1, \dots, p_1) = (p_1^{r_1})$, the matrices $T(z)$ and $\widetilde{W}(z)$ coincide. This is easily checked proceeding by induction on the number of columns of the pyramid p, the base case $p_1 = 1$ being obvious by comparing (2.1.4) and (3.2.1). The inductive step is given by the fact that for a rectangular p, $r_1 = s_1 = r$. As a consequence, the recursion in Equation (3.2.2) (resp. (3.2.3)) coincides with the recursion in Equation (2.1.11) (resp. (2.1.12)).

Example 3.2.3 (Principal nilpotent). Let $p = (N)$ be a partition of $N > 1$, namely the partition whose associated nilpotent element f is principal nilpotent and has nilpotent Jordan block form consisting of a single block

$$f = \begin{pmatrix} 0 & \dots & \dots & 0 \\ 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix} = \sum_{h=1}^{N-1} e_{h+1,h}\,.$$

where for simplicity we are numbering the blocks (and the corresponding basis of $\mathfrak{gl}_N$) with the numbers $1, \dots, N$ from right to left. The associated pyramid is composed of a single row of length N, the corresponding grading is even and we can write $\deg e_{ij} = j - i$. As a consequence, we can identify $W(\mathfrak{g}, f, \Gamma, 0)$ with a subalgebra of $U(\mathfrak{g}_{\leq 0})$ (cf. Remark 1.2.1). From Remark 3.2.1, and by Definitions 2.1.1 and 2.2.1, we have

$$W(z) = T(z) = L(z) = |z\mathbb{1}_N + F + \pi_{\leq 0}E + D|_{1N} \in U(\mathfrak{g}_{\leq 0})[z]\,, \tag{3.2.4}$$

where we simply write $E = \sum_{i,j=1}^{N} e_{ji}E_{ij}$ and $D = -\sum_{i=1}^{N}(i-1)E_{ii}$. Note that in this case the quasideterminant $L(z)$ has polynomial form. We can expand the quasideterminant and get (cf. (2.1.5))

$$\begin{gathered} W(z) = T(z) = L(z) = e_{N1} - \sum_{l=0}^{N-2}(-1)^l \sum_{1\leq h_0 < \dots < h_l \leq N-1} \\ (\delta_{h_0,1}z + \tilde{e}_{h_0,1})(\delta_{h_1,h_0+1}z + \tilde{e}_{h_1,h_0+1}) \cdots (\delta_{h_l,h_{l-1}+1}z + \tilde{e}_{h_l,h_{l-1}+1})(\delta_{N,h_l+1}z + \tilde{e}_{N,h_l+1})\,, \end{gathered} \tag{3.2.5}$$

where $\tilde{e}_{ij} = e_{ij} - \delta_{ij}(i-1)$. We can rewrite Equation (3.2.5) as

$$W(z) = L(z) = -(-z)^N + \sum_{k=0}^{N-1} W_k(-z)^k, \tag{3.2.6}$$

for unique elements $W_k \in W(\mathfrak{g}, f, \Gamma, 0)$ (cf. Theorem 2.2.1). These results agree with Conjecture 3.1.1. The elements W_k can be computed as in (2.3.17):

$$W_k = -\sum_{s=1}^{N-k}(-1)^s \sum_{\substack{1\leq h_0,\dots,h_{s-1}\leq N-1 \\ 1\leq h_s\leq N}} \sum_{\substack{n_i\geq 0 \\ n_0+\dots+n_s=k}} \delta_{h_0,1}\delta_{h_s+n_s,N}\pi_{\leq 0}\tilde{e}_{h_1,h_0+n_0}\pi_{\leq 0}\tilde{e}_{h_2,h_1+1+n_1}\cdots\pi_{\leq 0}\tilde{e}_{h_s,h_{s-1}+\delta_{s>1}+n_{s-1}}\,. \tag{3.2.7}$$

For instance, $W_{N-1} = e_{11} + \ldots + e_{NN} - \frac{N(N-1)}{2}$. Isolating, for each $0 \le k \le N-1$, the term in (3.2.7) corresponding to $s = 1$ we obtain

$$\sum_{n_0=0}^{k} \pi_{\le 0} \tilde{e}_{N-k+n_0,1+n_0} = \sum_{n_0=0}^{k} \tilde{e}_{N-k+n_0,1+n_0} - \delta_{k+1,N} n_0$$

confirming the conjectural dependence $W_k = w(f_k)$, where $f_k \in \mathfrak{g}^f$ is defined as in 3.1.8.

By Theorem 2.2.2, $[L(z), L(w)] = 0$, which implies that $[W_k, W_h] = 0$ for all $0 \le h, k \le N-1$. Therefore, we can identify $W(\mathfrak{g}, f, \Gamma, 0)$ with the polynomial algebra $\mathbb{C}[W_0, \ldots, W_{N-1}]$ (cf. Example 1.2.2). These commutation relations also allow us to identify $W(\mathfrak{g}, f, \Gamma, 0)$ with the truncated Yangian $Y_{1,N}$ (cf. Equation 2.3.1 and Equation 2.3.12).

Note that the quasideterminant in Equation (3.2.4) coincides up to a sign (see Proposition 3.2.1 below) with the row determinant of the same matrix $z\mathbb{1}_N + F + \pi_{\le 0} E + D$:

$$W(z) = |z\mathbb{1}_N + F + \pi_{\le 0}E + D|_{1N} = \pm \mathrm{rdet} \begin{pmatrix} z + e_{11} & e_{21} & e_{31} & \ldots & e_{N1} \\ 1 & e_{22} + z - 1 & e_{32} & \ldots & e_{N2} \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ldots & \ddots & \ddots & \vdots \\ 0 & & & 1 & e_{NN} + z - N + 1 \end{pmatrix}.$$

We have thus obtained the same explicit formula for the generators of the W-algebra for $\mathfrak{gl}_N$ and a principal nilpotent element that was presented as a special case by Brundan and Kleshchev in [BK06, Section 12][1].

A simple computation shows that

Proposition 3.2.1. *Let R be a unital associative algebra and let $A \in Mat_{N\times N} R$ be a matrix of the form*

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1N} \\ 1 & a_{22} & \ldots & a_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & a_{NN} \end{pmatrix}.$$

Then,

$$(-1)^{N-1} |A|_{1N} = \mathit{rdet}\ A = \mathit{cdet}\ A\,,$$

where rdet denotes the row determinant and cdet the column determinant of A.

Note that for a generic $A \in \mathrm{Mat}_{N\times N} R$ the row determinant and the column determinant do not coincide.

Example 3.2.4 (Rectangular nilpotent). Let $(p_1, \ldots, p_1) = (p_1^{r_1})$ be a partition of N consisting of $r_1 = r$ equal parts of size $p_1 > 1$. It corresponds to the so-called rectangular nilpotent element $f = \sum_{\substack{1\le i\le r_1 \\ 1\le h<p_1}} e_{(i,h+1)(i,h)}$. The corresponding grading is even, with deg $e_{(i,h)(j,k)} = k - h$ and the corresponding pyramid consists of a rectangle of size $p_1 \times r_1$.

In this case it ease useful to identify

$$\mathrm{Mat}_{N\times N}\mathbb{C} \cong \ \mathrm{Mat}_{p_1\times p_1}\mathbb{C} \otimes \mathrm{Mat}_{r_1\times r_1}\mathbb{C} \tag{3.2.8}$$

by mapping $E_{(i,h)(j,k)} \mapsto E_{hk} \otimes E_{ij}$. Under this identification we have

$$\begin{aligned} \mathbb{1}_N &\mapsto \mathbb{1}_{p_1} \otimes \mathbb{1}_{r_1}\,, \\ F + \pi_{\le 0} E &\mapsto \sum_{k=1}^{p_1 - 1} E_{k+1,k} \otimes \mathbb{1}_{r_1} + \sum_{i,j=1}^{r_1} \sum_{1\le h\le k\le p_1} e_{(j,k)(i,h)} E_{hk} \otimes E_{ij}\,, \end{aligned}$$

[1]See Remark 2.3.2

$$D=-\sum_{(i,h)\in\mathcal{T}} r_1(h-1)E_{(i,h)(i,h)} \mapsto -\sum_{h=1}^{p_1} r_1(h-1)E_{hh}\otimes \mathbb{1}_{r_1}\,.$$

As for the principal nilpotent case, we can identify $W(\mathfrak{g},f,\Gamma,0)$ with a subalgebra of $U(\mathfrak{g}_{\leq 0})$ (cf. Remark 1.2.1). By Remark 3.2.1, together with Definitions 2.1.1 and 2.2.1, we have

$$W(z)=T(z)=L(z)=|z\big(\mathbb{1}_{p_1}\otimes\mathbb{1}_{r_1}\big)+\sum_{k=1}^{p_1-1}E_{k+1,k}\otimes\mathbb{1}_{r_1}+\sum_{i,j=1}^{r_1}\sum_{1\leq h\leq k\leq p_1} e_{(j,k)(i,h)}E_{hk}\otimes E_{ij}$$
$$-\sum_{h=1}^{p_1} r_1(h-1)E_{hh}\otimes\mathbb{1}_{r_1}|_{I_1J_1}\in \mathrm{Mat}_{r_1\times r_1}U(\mathfrak{g}_{\leq 0})[z]\,,$$

where $I_1\,J_1$ are as in (2.2.1).

Let $\tilde{e}_{(jk),(ih)}=e_{(jk),(ih)}-\delta_{(i,h)(j,k)}r_1(h-1)$. We have

$$\begin{aligned}
&|z\big(\mathbb{1}_{p_1}\otimes\mathbb{1}_{r_1}\big)+\sum_{k=1}^{p_1-1}E_{k+1,k}\otimes\mathbb{1}_{r_1}+\sum_{i,j=1}^{r_1}\sum_{1\leq h\leq k\leq p_1}\tilde{e}_{(j,k)(i,h)}E_{hk}\otimes E_{ij}|_{I_1J_1}\\
&\quad=\sum_{i,j=1}^{r_1} e_{(j,p_1)(i,1)}E_{ij}-\left(\sum_{i,j=1}^{r_1}\big(\delta_{ij}z+e_{(j,1)(i,1)}\quad e_{(j,2)(i,1)}\quad\dots\quad e_{(j,p_1-1)(i,1)}\big)\otimes E_{ij}\right)\\
&\quad\left(\mathbb{1}_{p_1-1}\otimes\mathbb{1}_{r_1}+\sum_{i,j=1}^{r_1}\begin{pmatrix}0 & \delta_{ij}z+\tilde{e}_{(j,2)(i,2)} & \dots & e_{(j,p_1-1)(i,2)}\\ \vdots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \delta_{ij}z+\tilde{e}_{(j,p_1-1)(i,p_1-1)}\\ 0 & \dots & \dots & 0\end{pmatrix}\otimes E_{ij}\right)^{-1}\\
&\quad\left(\sum_{i,j=1}^{r_1}\begin{pmatrix} e_{(j,p_1)(i,2)}\\ \vdots\\ e_{(j,p_1)(i,p_1-1)}\\ \delta_{ij}z+\tilde{e}_{(,jp_1)(i,p_1)}\end{pmatrix}\otimes E_{ij}\right).
\end{aligned}\tag{3.2.9}$$

By expanding the inverse matrix in the RHS of (3.2.9) in geometric power series, we have, for every $1\leq i,j\leq r_1$,

$$\begin{aligned}
&W_{ij}(z)=L_{ij}(z)=e_{(j,p_1)(i,1)}+\sum_{s=1}^{p_1-1}(-1)^s\sum_{i_1,\dots,i_s=1}^{r_1}\sum_{1\leq h_1<\dots<h_s\leq p_1-1}\\
&\big(\delta_{(i_1,h_1)(i,1)}z+\tilde{e}_{(i_1,h_1)(i,1)}\big)\big(\delta_{(i_2,h_2)(i_1,h_1+1)}z+\tilde{e}_{(i_2,h_2)(i_1,h_1+1)}\big)\\
&\quad\times\dots\times\big(\delta_{(i_s,h_s)(i_{s-1},h_{s-1}+1)}z+\tilde{e}_{(i_s,h_s)(i_{s-1},h_{s-1}+1)}\big)\big(\delta_{(i_s,h_s+1)(j,p_1)}z+\tilde{e}_{(j,p_1)(i_s,h_s+1)}\big)\,.
\end{aligned}\tag{3.2.10}$$

Note that $L_{ij}(z)=T_{ij}(z)=W_{ij}(z)$ has polynomial form

$$L_{ij}(z)=-(-z)^{p_1}\delta_{ij}+\sum_{k=0}^{p_1-1}W_{ij;k}(-z)^k\,,\tag{3.2.11}$$

for unique elements $W_{ij;k}\in W(\mathfrak{g},f,\Gamma,0)$ (cf. Theorem 2.2.1). These results agree with Conjecture 3.1.1.

We can compute the coefficients $W_{ij;k}$ as in (2.3.17). Isolating the term corresponding to $s=1$ we obtain

$$\sum_{n_0=0}^{k}\pi_{\leq 0}\tilde{e}_{(j,p_j+n_0-k)(i,1+n_0)}=\sum_{n_0=0}^{k}e_{(j,p_j+n_0-k)(i,1+n_0)}-\delta_{ij}\delta_{k,p_1-1}r_1n_0=f_{ji;k}-\delta_{ij}\delta_{k,p_1-1}r_1n_0\,,$$

which agrees with Conjecture 3.1.1.

Combining Theorem 2.2.2 and Equation (3.2.11) we can explicitly compute the commutation relations among the generators $W_{ij;k}$, for $1 \le i, j \le r_1$, $0 \le k \le p_1 - 1$:

$$[W_{ij;r}, W_{hk;s}] = \sum_{a=0}^{\min(p_1-1-r,s)} W_{hj;r+a-1} W_{ik;s-a} - W_{hj;s-a} W_{ik;r+a+1} \,, \tag{3.2.12}$$

where we set $W_{ij;p_1} = \delta_{ij}$.

In analogy with (3.2.8), we shall identify $\mathfrak{gl}_N \cong \mathfrak{gl}_{p_1} \otimes \mathfrak{gl}_{r_1}$, by mapping $e_{(j,k)(i,h)} \mapsto e_{kh} \otimes e_{ji}$. Then, by (3.2.10), for every $1 \le i, j \le r_1$,

$$W_{ij}(z) = T_{ij}(z) = L_{ij}(z) = \mathsf{T}_{ij} \left(\begin{vmatrix} z + e_{11} & e_{21} & e_{31} & \dots & e_{p_1 1} \\ 1 & e_{22} + z - 1 & e_{32} & \dots & e_{p_1 2} \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \dots & \ddots & \ddots & \vdots \\ 0 & & & 1 & z + e_{p_1 p_1} - p_1 + 1 \end{vmatrix}_{1N} \right),$$

where $\mathsf{T}_{ij} : T(\mathfrak{gl}_{p_1}) \longrightarrow U(\mathfrak{gl}_N)$ is the map as in [BK06, Section 12] and $T(\mathfrak{gl}_{p_1})$ is the tensor algebra of $\mathfrak{gl}_{p_1}$. As a consequence, the generators $W_{ij;k}$ coincide[2], up to a sign, with the generators obtained at the end of [BK06, Section 12], in the special case of a rectangular nilpotent element f.

Example 3.2.5 (**Short nilpotent**). As a particular case of Example 3.2.4 we find the short nilpotent f, whose associated partition is $(2, \dots, 2) = 2^{r_1}$ (this only holds for even N). The corresponding $\frac{1}{2}\mathbb{Z}$-grading reduces to $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

In this case,

$$W_{ij}(z) = L_{ij}(z) = -z^2 \delta_{ij} - W_{ij;1} z + W_{ij;0} \,. \tag{3.2.13}$$

By Equation (2.3.17) we obtain the following formulas for the generators:

$$\begin{aligned} W_{ij;1} &= e_{(j,2)(i,2)} + e_{(j,1)(i,1)} - \delta_{ij} r_1 \,, \\ W_{ij;0} &= e_{(j,2)(i,1)} - \sum_{h=1}^{r_1} e_{(h,1)(i,1)} (e_{(j,2)(h,2)} - \delta_{hj} r_1) \,, \end{aligned} \tag{3.2.14}$$

for every $1 \le i, j \le r_1$. By (3.1.8), $f_{ji;1} = e_{(j,1)(i,1)} + e_{(j,2)(i,2)}$ and $f_{ji;0} = e_{(j,2)(i,1)}$; therefore $W_{ij;1} = w(f_{ji;1})$ and $W_{ij;0} = w(f_{ji;0})$, as conjectured in Conjecture 3.1.1. By Theorem 2.2.2 and Equation (3.2.11) we can compute the commutation relations between the generators. For any $1 \le i, j, h, k, \le r_1$ we obtain:

$$\begin{aligned} [W_{ij;0}, W_{hk;0}] &= W_{hj;1} W_{ik;0} - W_{hj;0} W_{ik;1} \,, \qquad & [W_{ij;1}, W_{hk;0}] &= \delta_{hj} W_{ik;0} - \delta_{ik} W_{hj;0} \,, \\ [W_{ij;0}, W_{hk;1}] &= \delta_{hj} W_{ik;0} - \delta_{ik} W_{hj;0} \,, & [W_{ij;1}, W_{hk;1}] &= \delta_{hj} W_{ik;1} - \delta_{ik} W_{hj;1} \,. \end{aligned} \tag{3.2.15}$$

As in Proposition 2.1.4, we are now addressing the issue of the matrix $W(z)$ being well-defined.

Proposition 3.2.2. *Let $p_1 > 1$ and $t_1 = s_1$. Suppose moreover that it is possible to remove both the leftmost and the rightmost column of p. Then, the matrix $W(z)$ is well-defined. Namely, the result of the application of recursion (3.2.2) followed by recursion (3.2.3) coincides with the result of the application of recursion (3.2.3) followed by recursion (3.2.2).*

Proof. When $p_1 = 2$, the pyramid reduces to a rectangle and the recursion defining the matrix $\widetilde{W}(z)$ agrees with the recursion defining the matrix $T(z)$. The result therefore follows from Proposition 2.1.4.

Suppose now $p_1 > 2$. Since $\sigma_r \circ \sigma_l$, $\sigma_l \circ \sigma_r : U(\mathfrak{g}^{'p'}) := U(\mathfrak{gl}_{N-t_1-s_1}) \longrightarrow U(\mathfrak{g})$, the claim follows almost immediately as in Proposition 2.1.4 by writing down the explicit formulas in both cases. However, when both

[2] See Remark 2.3.2

$i,j \le t_1 = s_1$ the difference between the expression obtained applying recursion (3.2.2) followed by (3.2.3) and the expression obtained applying recursion (3.2.3) followed by (3.2.2) is not trivial:

$$-\sum_{t=t_1+1}^{r}[e_{(j,p_1)(j,p_1-1)},\sigma_r\sigma_l(\widetilde{W}'^{p'}_{it;p_t-1})]\sigma_r\sigma_l(\widetilde{W}'^{p'}_{tj}(z))+\sum_{h=t_1+1}^{r}\sigma_l\sigma_r(\widetilde{W}'^{p'}_{ih}(z))[\sigma_l\sigma_r(\widetilde{W}'^{p'}_{hj;p_h-1}),e_{(i,2)(i,1)}]\,. \tag{3.2.16}$$

We will therefore proceed with an analysis of this case. For $p_1 = 3$, $\sigma_r\sigma_l(\widetilde{W}'^{p'}_{it;p_t-1}) = \sigma_r\sigma_l(e_{(t,1)(i,1)}) = e_{(t,1)(i,2)}$ and $\sigma_l\sigma_r(\widetilde{W}'^{p'}_{hj;p_h-1}) = \sigma_l\sigma_r(e_{(j,1)(h,1)}) = e_{(j,2)(h,1)}$. Both brackets in (3.2.16) are therefore clearly zero.

We can therefore suppose $p_1 > 3$. Consider the first summand in (3.2.16); if $t_2 \le s_2$ we can apply the left recursion a second time and obtain

$$\sigma_r\sigma_l(\widetilde{W}'^{p'}_{it}(z)) = \begin{cases} \sigma_r\sigma_l^2(\widetilde{W}^{2p'}_{it}(z))\,, & t > t_2 \\ [e_{(t,p_1-2)(t,p_1-3)},\sigma_r\sigma_l^2(\widetilde{W}^{2p'}_{it}(z))] - \sum_{k=1}^{t_2}\sigma_r\sigma_l^2(\widetilde{W}^{2p'}_{ik}(z))(\delta_{kt}z+\tilde{e}_{(t,p_1-2)(k,p_1-2+\delta_{k\le s_1})}) & \\ \qquad + \sum_{k\ge t_2+1}\sigma_r\sigma_l^2(\widetilde{W}^{2p'}_{ik}(z))\sigma_r\sigma_l^2(\widetilde{W}^{2p'}_{ikt;p_k-1})\,, & t_1 < t \le t_2\,, \end{cases} \tag{3.2.17}$$

where for each $1 \le i,j \le r$ we denote by $\widetilde{W}^{2p'}_{ij}(z) \in U(\mathfrak{g}^{2p'})(z)$ the polynomials as in Definition 3.2.1, where the pyramid corresponding to $\mathfrak{g}^{2p'}$ is obtained from p by removing the two leftmost and the rightmost columns. More generally, $\widetilde{W}^{hp^k}_{ij}(z) \in U(\mathfrak{g}^{hp^k})(z)$ are the polynomials as in Definition 3.2.1, where the pyramid corresponding to $\mathfrak{g}^{hp^k}$ is obtained from p by removing the h leftmost and the k rightmost columns. The commutator of (3.2.17) with $e_{(j,p_1)(j,p_1-1)}$ is clearly (term by term) zero.

On the other hand, if $t_2 > s_2$, we can apply the right recursion a second time and obtain (by hypothesis, $i \le s_1 \le s_2$)

$$\begin{aligned} \sigma_r\sigma_l(\widetilde{W}'^{p'}_{it}(z)) &= \sigma_r\sigma_l([\sigma_r(\widetilde{W}'^{p^2}_{it}(z),e_{(i,2)(i,1)}]) - \sum_{k=1}^{s_2}\sigma_r\sigma_l(\delta_{ik}z+\tilde{e}_{(k,1)(i,1)})\sigma_r\sigma_l\sigma_r(\widetilde{W}'^{p^2}_{kt}(z)) \\ &+ \sum_{k\ge s_2+1}\sigma_r\sigma_l\sigma_r(\widetilde{W}'^{p^2}_{ik;p_k-1})\sigma_r\sigma_l\sigma_r(\widetilde{W}'^{p^2}_{kt}(z)) \\ &= [\sigma_l\sigma_r^2(\widetilde{W}'^{p^2}_{it}(z),e_{(i,3)(i,2)}]) - \sum_{k=1}^{s_2}(\delta_{ik}z+\tilde{e}_{(k,1+\delta_{k\le s_1})(i,2)})\sigma_l\sigma_r^2(\widetilde{W}'^{p^2}_{kt}(z)) + \sum_{k\ge s_2+1}\sigma_l\sigma_r^2(\widetilde{W}'^{p^2}_{ik;p_k-1})\sigma_l\sigma_r^2(\widetilde{W}'^{p^2}_{kt}(z))\,. \end{aligned}$$

Note that if $p_1 = 4$

$$[\sigma_l\sigma_r^2(\widetilde{W}'^{p^2}_{it}(z)),[e_{(j,p_1)(j,p_1-1)},e_{(i,3)(i,2)}]] = [\sigma_l\sigma_r^2(\widetilde{W}'^{p^2}_{it}(z)),\delta_{ij}e_{(j,4)(i,2)}] = 0$$

because $x(j4) = -\frac{d}{2} = -\frac{3}{2}$ while $x(i2) = \frac{d}{2} - 1 = \frac{1}{2}$. In any other case $[e_{(j,p_1)(j,p_1-1)},e_{(i,3)(i,2)}] = 0$.

Thus, to compute $[e_{(j,p_1)(j,p_1-1)},\sigma_r\sigma_l(\widetilde{W}'^{p'}_{it;p_t-1})]$ we reduce to compute $[e_{(j,p_1)(j,p_1-1)},\sigma_l\sigma_r^2(\widetilde{W}'^{p^2}_{\gamma t}(z))]$, $t \ge t_1+1$, and then take the coefficient of the correct power of $(-z)$.

If $t_2 \le s_3$, we compute $\widetilde{W}'^{p^2}_{\gamma t}(z)$ using the left recursion, and the result follows from (3.2.17). Otherwise, we iterate this process k times, namely we keep applying the right recursion until either we reach a k such that $t_2 \le s_{3+k}$ and we can finally apply the left recursion (then the result follows from (3.2.17) above), or until we reach k such that $k+3 = p_1-2$ (this is the case if $t_2 = r$ and there is only one column of maximal length). In this case, $[e_{(j,p_1)(j,p_1-1)},\sigma_r\sigma_l\sigma_r^{2+k}(\widetilde{W}'^{p^{3+k}}_{\gamma t}(z))] = [e_{(j,p_1)(j,p_1-1)},\sigma_r\sigma_l\sigma_r^{2+k}(e_{(t,1)(\gamma,1)})] = [e_{(j,p_1)(j,p_1-1)},e_{(t,\le p_1-2)(\gamma,\le p_1-1)}] = 0$.

A similar argument, swapping the roles of the right and left recursion, shows that $[\sigma_l\sigma_r(\widetilde{W}'^{p'}_{hj;p_h-1}),e_{(i,2)(i,1)}] = 0$, proving Proposition 3.2.2.

□

It is fundamental to observe that the associative algebra injections σ_l and σ_r, although inducing well defined quotient maps as illustrated by Diagrams 1.1.21 and 1.1.22, do not extend to maps between the corresponding W-algebras: $W(\mathfrak{g}^{'p}, f^{'p}, \Gamma^{'p}, \mathfrak{l}^{'p}) \xrightarrow{\sigma_l} W(\mathfrak{g}, f, \Gamma, \mathfrak{l}) \xleftarrow{\sigma_r} W(\mathfrak{g}^{p'}, f^{p'}, \Gamma^{p'}, \mathfrak{l}^{p'})$. In Example 3.2.6, we show an instance of this fact.

Example 3.2.6 (Counterexample)**.** Let $(3,2,1)$ be a partition of $N=6$, and suppose that the corresponding pyramid p is right-aligned. By Definition 3.2.1 we can compute

$$\widetilde{W}^{'p}_{21}(z) = e_{(12)(21)} - e_{(11)(21)}(z+\tilde{e}_{(12)(12)}) - (z+e_{(21)(21)})e_{(12)(22)} + e_{(31)(21)}e_{(11)(31)}\,.$$

We can also directly compute

$$\mathfrak{g}^{'p}_{\geq 1} = \mathrm{Span}_{\mathbb{C}}\{e_{(1,1)(1,2)}, e_{(1,1)(2,2)}, e_{(2,1)(1,2)}, e_{(2,1)(2,2)}, e_{(3,1)(1,2)}, e_{(3,1)(2,2)}\}\,.$$

Therefore it is a straightforward computation to check that $\rho([\mathfrak{g}^{'p}_{\geq 1}, \sigma_l(\widetilde{W}^{'p}_{21}(z))]) = 0$. However, once we consider $e_{(12)(13)} \in \mathfrak{g}_{\geq 1}$ but not in $\sigma_l(\mathfrak{g}^{'p}_{\geq 1})$ we have

$$\rho([e_{(12)(13)}, \sigma_l(\widetilde{W}^{'p}_{21}(z))]) = e_{(11)(21)} \neq 0\,.$$

Proposition 3.2.3. *For $1 \leq i,j \leq r$ the polynomial $\widetilde{W}_{ij}(z) \in U(\mathfrak{g})[z]$ has the following form:*

$$\widetilde{W}_{ij}(z) = -\delta_{ij}(-z)^{p_i} + \sum_{k=0}^{min(p_i,p_j)-1} \widetilde{W}_{ij;k}(-z)^k. \tag{3.2.18}$$

Proof. The proof works by induction on the number of columns of the pyramid p. For the base case $p_1 = 1$ the claim is obvious, since by Definition 3.2.1 we have

$$\widetilde{W}_{ij}(z) = \delta_{ij} z + e_{(j,1)(i,1)} = -(-z)^1\delta_{ij} + \widetilde{W}_{ij;0}(-z)^0\,.$$

For $p_1 > 1$ we need to make use of the recursive part of the definition of $W(z)$. Suppose $t_1 \leq s_1$, the other case being analogous. Thus, for $j \geq t_1+1$ by the induction hypothesis we have

$$\begin{aligned}\widetilde{W}_{ij}(z) = \sigma_l(\widetilde{W}^{'p}_{ij}(z)) &= -(-z)^{p_j}\delta_{ij} + \sum_{k=0}^{\min(p^{'p}_i,p_j)-1} \sigma_l(\widetilde{W}^{'p}_{ij;k})(-z)^k \\ &= -(-z)^{p_j}\delta_{ij} + \sum_{k=0}^{\min(p_i,p_j)-1} \sigma_l(\widetilde{W}^{'p}_{ij;k})(-z)^k. \end{aligned}\tag{3.2.19}$$

Last identity is due to the fact that if $p^{'p}_i \neq p_i$ then $p_i = p_1$, and $\min(p^{'p}_i, p_j) = p_j = \min(p_i,p_j)$. Formula (3.2.18) then holds with $\widetilde{W}_{ij;k} = \sigma_l(\widetilde{W}^{'p}_{ij;k})$.

For $j \leq t_1$, by the (3.2.2) and from the induction hypothesis we get

$$\begin{aligned}\widetilde{W}_{ij}(z) = {}&[e_{(j,p_1)(j,p_1-1)}, \sum_{k=0}^{p^{'p}_i-1} \sigma_l(\widetilde{W}^{'p}_{ij;k})(-z)^k] - \sum_{h=1}^{t_1}\Big(-(-z)^{p_1-1}\delta_{ih} + \sum_{k=0}^{\min(p^{'p}_i,p_1-1)-1} \sigma_l(\widetilde{W}_{ih;k})(-z)^k\Big)(\delta_{hj}z + \tilde{e}_{(j,p_1)(h,p_1)}) \\ &+ \sum_{h=t_1+1}^{r}\Big(-(-z)^{p_h}\delta_{ih} + \sum_{k=0}^{\min(p^{'p}_i,p_h)-1} \sigma_l(\widetilde{W}_{ih;k})(-z)^k\Big)\sigma_l(\widetilde{W}^{'p}_{hj,p_h-1}) \\ &= \sum_{k=0}^{p^{'p}_i-1} [e_{(j,p_1)(j,p_1-1)}, \sigma_l(\widetilde{W}^{'p}_{ij;k})](-z)^k - (-z)^{p_1}\delta_{ij} + \delta_{i\leq t_1}\tilde{e}_{(j,p_1)(i,p_1)}(-z)^{p_1-1} \\ &+ \sum_{k=1}^{p^{'p}_i} \sigma_l(\widetilde{W}^{'p}_{ih;k-1})(-z)^k - \sum_{h=1}^{t_1}\sum_{k=0}^{\min(p^{'p}_i,p_1-1)-1} \sigma_l(\widetilde{W}_{ih;k})\tilde{e}_{(j,p_1)(h,p_1)}(-z)^k\end{aligned}$$

$$
\begin{aligned}
&- \delta_{i\geq t_1+1}\sigma_l(\widetilde{W}'^{p}_{ij,p_i-1})(-z)^{p_i} + \sum_{h=t_1+1}^{r}\sum_{k=0}^{\min(p'^{p}_i,p_h)-1}\sigma_l(\widetilde{W}'^{p}_{ih;k})\sigma_l(\widetilde{W}'^{p}_{hj,p_h-1})(-z)^k \qquad (3.2.20)\\
&= \sum_{k=0}^{p_i-\delta_{i\leq t_1}-1}[e_{(j,p_1)(j,p_1-1)},\sigma_l(\widetilde{W}'^{p}_{ij;k})](-z)^k - (-z)^{p_1}\delta_{ij} + \delta_{i\leq t_1}\tilde{e}_{(j,p_1)(i,p_1)}(-z)^{p_1-1}\\
&+ \sum_{k=1}^{p_i-1}\sigma_l(\widetilde{W}'^{p}_{ih;k-1})(-z)^k - \sum_{h=1}^{t_1}\sum_{k=0}^{p_i-\delta_{i\leq t_1}-1}\sigma_l(\widetilde{W}_{ih;k})\tilde{e}_{(j,p_1)(h,p_1)}(-z)^k\\
&+ \sum_{h=t_1+1}^{r}\sum_{k=0}^{\min(p_i-\delta_{i\leq t_1},p_h)-1}\sigma_l(\widetilde{W}'^{p}_{ih;k})\sigma_l(\widetilde{W}'^{p}_{hj,p_h-1})(-z)^k .
\end{aligned}
$$

Thus, $\widetilde{W}_{ij}(z)$ has the required form since $\min(p_i,p_j)-1=p_i-1$ for $j\leq t_1$. For the last equality we have moreover used the following decomposition

$$
\begin{aligned}
\sum_{k=1}^{p'^{p}_i}\sigma_l(\widetilde{W}'^{p}_{ih;k-1})(-z)^k &= \delta_{i\geq t_1+1}\sum_{k=1}^{p_i}\sigma_l(\widetilde{W}'^{p}_{ih;k-1})(-z)^k + \delta_{i\leq t_1}\sum_{k=1}^{p_i-1}\sigma_l(\widetilde{W}'^{p}_{ih;k-1})(-z)^k\\
&= \delta_{i\geq t_1+1}\sigma_l(\widetilde{W}'^{p}_{ih;p_i-1})(-z)^{p_i} + \sum_{k=1}^{p_i-1}\sigma_l(\widetilde{W}'^{p}_{ih;k-1})(-z)^k .
\end{aligned}
$$

□

3.3 Main properties of the matrix $W(z)$

3.3.1 Invariance of the coefficients of $W(z)$

Theorem 3.3.1. *Suppose that the pyramid p is aligned to the right. Then*

$$
W(z)\in Mat_{r\times r}W(\mathfrak{g},f,\Gamma,0)[z] . \qquad (3.3.1)
$$

Namely, $\rho([a,\widetilde{W}_{ij}(z)])=0$ in $\big(U(\mathfrak{g})/I_{\mathfrak{l}}\big)[z]$ for every $a\in\mathfrak{g}_{\geq 1}$, $1\leq i,j,\leq r$.

Proof. We work by induction on the number of columns of p. The base case is $p_1=1$, where

$$
W(z)=z\mathbb{1}_N+E\in \mathrm{Mat}_{N\times N}U(\mathfrak{g})[z]=\mathrm{Mat}_{N\times N}W(\mathfrak{g},f,\Gamma,0)[z]
$$

because this case corresponds to the nilpotent element $f=0$, when $W(\mathfrak{g},f,\Gamma,0)=U(\mathfrak{g})$. Since we will use a two step induction, let us now consider the case $p_1=2$. In this case, $\mathfrak{g}_{\geq 1}=\mathfrak{g}_1=\mathrm{Span}\{e_{(a,1)(b,2)}\mid 1\leq a\leq r\,,1\leq b\leq t_1\}$. In this case, we can write $W_{ij}(z)$ explicitly as in Example 3.2.1: for $j\geq t_1+1$ we have

$$
\rho([e_{(a,1)(b,2)},\widetilde{W}_{ij}(z)])=\rho([e_{(a,1)(b,2)},e_{(j,1)(i,1)}])=-\rho(\delta_{ai}e_{(j,1)(b,2)})=-\delta_{ai}\delta_{jb}=0\,, \qquad (3.3.2)
$$

since $1\leq b\leq t_1$ while $j\geq t_1+1$. On the other hand, for $j\leq t_1$ we have

$$
\begin{aligned}
\rho([e_{(a,1)(b,2)},\widetilde{W}_{ij}(z)]) &= -\delta_{i\leq t_1}(\delta_{bj}\rho(e_{(a,1)(i,2)})-\delta_{ai}\rho(e_{(j,1)(b,2)}))z+\rho(\delta_{bj}e_{(a,1)(i,1)}-\delta_{ai}e_{(j,2)(b,2)})\\
&- \delta_{ai}\sum_{h\geq t_1+1}\rho(e_{(h,1)(b,2)}e_{(j,1)(h,1)})-\delta_{a\geq t_1+1}\rho(e_{(a,1)(i,1)}e_{(j,1)(b,2)})\\
&+ \delta_{ai}\sum_{h=1}^{t_1}\rho(e_{(h,1)(b,2)}\tilde{e}_{(j,2)(h,2)})-\delta_{bj}\sum_{h=1}^{t_1}\rho(e_{(h,1)(i,1)}e_{(a,1)(h,2)})\\
&= -\delta_{i\leq t_1}(\delta_{bj}\delta_{ai}-\delta_{ai}\delta_{jb})z+\delta_{bj}e_{(a,1)(i,1)}-\delta_{ai}e_{(j,2)(b,2)}-\delta_{ai}\sum_{h\geq t_1+1}\rho([e_{(h,1)(b,2)},e_{(j,1)(h,1)}]) \qquad (3.3.3)
\end{aligned}
$$

$$- \delta_{a\geq t_1+1}\delta_{bj}e_{(a,1)(i,1)} + \delta_{ai}(e_{(j,2)(b,2)} - \delta_{bj}r) + \delta_{ai}\sum_{h=1}^{t_1}\rho([e_{(h,1)(b,2)}\tilde{e}_{(j,2)(h,2)}]) - \delta_{a\leq t_1}\delta_{bj}e_{(a,1)(i,1)}$$

$$= \delta_{bj}e_{(a,1)(i,1)} - \delta_{ai}e_{(j,2)(b,2)} + \delta_{ai}\delta_{bj}(r-t_1) - \delta_{a\geq t_1+1}\delta_{bj}e_{(a,1)(i,1)}$$

$$+ \delta_{ai}(e_{(j,2)(b,2)} - \delta_{bj}r) + \delta_{ai}\delta_{bj}t_1 - \delta_{a\leq t_1}\delta_{bj}e_{(a,1)(i,1)} = 0\,. \tag{3.3.4}$$

Next, suppose $p_1 > 2$ and suppose that the (3.3.1) holds for ${}'p$; namely suppose that

$$0 = \rho^{'p}([\mathfrak{g}^{'p}_{\geq 1}, \widetilde{W}^{'p}_{ij}(z)]) \in \big(U(\mathfrak{g}^{'p})/U(\mathfrak{g}^{'p})\langle b - (f^{'p}|b)\rangle_{b\in\mathfrak{g}^{'p}_{\geq 1}}\big)[z]\,,$$

for every $1 \leq i, j \leq r$.

We can decompose $\mathfrak{g}_{\geq 1} = \sigma_l(\mathfrak{g}^{'p}_{\geq 1}) \oplus \mathfrak{g}^{p_1}_{\geq 1}$, where $\mathfrak{g}^{p_1}_{\geq 1}$ is the complementary subspace to $\sigma_l(\mathfrak{g}^{'p}_{\geq 1})$ in $\mathfrak{g}_{\geq 1}$ consisting of all elementary matrices $e_{(i,h)(j,k)}$ such that $x(jk) = -\frac{d}{2}$.

First, consider the commutator between $\widetilde{W}_{ij}(z)$ and $\sigma_l(\mathfrak{g}^{'p}_{\geq 1})$; we shall distinguish two cases. If $j \geq t_1 + 1$, then $\widetilde{W}_{ij}(z) = \sigma_l(\widetilde{W}^{'p}_{ij}(z))$ and $\rho([\sigma_l(\mathfrak{g}^{'p}_{\geq 1}), \widetilde{W}_{ij}(z)]) = 0$ because of the induction hypothesis (cf. (1.1.21)). If $j \leq t_1$, from (3.2.2) we have the following identity in $\big(U(\mathfrak{g})/I_{\mathfrak{l}}\big)[z]$:

$$\rho([e_{(a,q)(b,k)}, \widetilde{W}_{ij}(z)]) = \rho([e_{(a,q)(b,k)}, [e_{(j,p_1)(j,p_1-1)}, \sigma_l(\widetilde{W}^{'p}_{ij}(z))]]) - \sum_{h=1}^{t_1}\rho([e_{(a,q)(b,k)}, \sigma_l(\widetilde{W}^{'p}_{ih}(z))(\delta_{hj}z + \tilde{e}_{(j,p_1)(h,p_1)})])$$
$$+ \sum_{h=t_1+1}^{r}\rho([e_{(a,q)(b,k)}, \sigma_l(\widetilde{W}^{'p}_{ih}(z))]\sigma_l(\widetilde{W}^{'p}_{hj;p_h-1})) + \sum_{h=t_1+1}^{r}\rho(\sigma_l(\widetilde{W}^{'p}_{ih}(z))[e_{(a,q)(b,k)}, \sigma_l(\widetilde{W}^{'p}_{hj;p_h-1})])$$
$$= \rho([e_{(a,q)(b,k)}, [e_{(j,p_1)(j,p_1-1)}, \sigma_l(\widetilde{W}^{'p}_{ij}(z))]])$$

for any $e_{(a,q)(b,k)} \in \sigma_l(\mathfrak{g}^{'p}_{\geq 1})$. This holds thanks to the fact that $[e_{(a,q)(b,k)}, e_{(j,p_1),(h,p_1)}] = 0$ for (a,q), $(b,k) \in \mathcal{T}^{'p}$, and by using the induction hypothesis. Moreover, to show that the third term in the RHS vanishes we used the fact that by the induction hypothesis $\widetilde{W}^{'p}_{hj;p_h-1} \in \widetilde{W}'$ (cf. equation (1.2.2)) and $I'_{\mathfrak{l}} \subset \widetilde{W}'$ is a bilateral ideal. By the Jacobi identity,

$$\rho([e_{(a,q)(b,k)}, [e_{(j,p_1)(j,p_1-1)}, \sigma_l(\widetilde{W}^{'p}_{ij}(z))]]) = \rho([[e_{(a,q)(b,k)}, e_{(j,p_1)(j,p_1-1)}], \sigma_l(\widetilde{W}^{'p}_{ij}(z))])$$
$$+ \rho([e_{(j,p_1)(j,p_1-1)}, [e_{(a,q)(b,k)}, \sigma_l(\widetilde{W}^{'p}_{ij}(z))]])\,,$$

where $[e_{(a,q)(b,k)}, e_{(j,p_1)(j,p_1-1)}] = -\delta_{(j,p_1-1)(a,q)}e_{(j,p_1)(b,k)} = 0$ for degree reasons. Through the induction hypothesis we can similarly check that

$$[e_{(j,p_1)(j,p_1-1)}, [e_{(a,q)(b,k)}, \sigma_l(\widetilde{W}^{'p}_{ij}(z))]] \in U(\mathfrak{g})\langle b - (f|b)\rangle_{b\in\mathfrak{g}_{\geq 1}}\,.$$

We shall now consider the elements of $\mathfrak{g}^{p_1}_{\geq 1}$. By Lemma 3.3.1 below, it is sufficient to consider elements of the form $e_{(a,p_1-1)(b,p_1)}$, with $1 \leq a \leq t_2$ and $1 \leq b \leq t_1$. In fact, for each $1 \leq h < p_1 - 1$, $e_{(a,h)(b,p_1)} = [e_{(a,h)(a,p_1-1)}, e_{(a,p_1-1)(b,p_1)}]$, with $e_{(a,h)(a,p_1-1)} \in \sigma_l(\mathfrak{g}^{'p}_{\geq 1})$.

Lemma 3.3.1. *Let $x, y \in \mathfrak{g}_{\geq 1}$ be such that $[x, \widetilde{W}_{ij}(z)], [y, \widetilde{W}_{ij}(z)] \in I[z]$, where $I = U(\mathfrak{g})\langle b - (f|b)\rangle_{b\in\mathfrak{g}_{\geq 1}}$. Then,*

$$[[x,y], \widetilde{W}_{ij}(z)] \in I[z]\,. \tag{3.3.5}$$

Proof. Use the Jacobi Identity on the commutator $[[x,y], \widetilde{W}_{ij}(z)]$. □

We shall distinguish between three cases. Each time we will apply the recursion in (3.2.2) twice. First, if $j \geq t_2 + 1$

$$\rho([e_{(a,p_1-1)(b,p_1)}, \widetilde{W}_{ij}(z)]) = \rho([e_{(a,p_1-1)(b,p_1)}, \sigma_l^2(\widetilde{W}^{''p}_{ij}(z)]) = 0$$

because $[e_{(a,p_1-1)(b,p_1)}, \sigma_l^2(\mathfrak{g}^{''p})] = 0$.

Second, if $t_1+1\le j\le t_2$ the two applications of the recursion give

$$\rho([e_{(a,p_1-1)(b,p_1)},\widetilde{W}_{ij}(z)])=\rho([e_{(a,p_1-1)(b,p_1)},\sigma_l(\widetilde{W}'^{p}_{ij}(z))])=\rho([e_{(a,p_1-1)(b,p_1)},[e_{(j,p_1-1)(j,p_1-2)},\sigma_l^2(\widetilde{W}^{2p}_{ij}(z))])$$
$$-\rho([e_{(a,p_1-1)(b,p_1)},\sum_{h=1}^{t_2}\sigma_l^2(\widetilde{W}^{2p}_{ih}(z))(\delta_{hj}z+\tilde{e}_{(j,p_1-1)(h,p_1-1)}])+\rho([e_{(a,p_1-1)(b,p_1)},\sum_{h\ge t_2+1}\sigma_l^2(\widetilde{W}^{2p}_{ih}(z))\sigma_l^2(\widetilde{W}^{2p}_{hj,p_h-1})])$$
$$=-\rho(\sum_{h=1}^{t_2}\sigma_l^2(\widetilde{W}^{2p}_{ih}(z))(-\delta_{ah}e_{(j,p_1-1)(b,p_1)})=\sigma_l^2(\widetilde{W}^{2p}_{ia}(z))\delta_{bj}=0\,.$$

Finally, if $j\le t_1$ the application of the recursion gives us at first

$$\rho([e_{(a,p_1-1)(b,p_1)},\widetilde{W}_{ij}(z)])=\rho([e_{(a,p_1-1)(b,p_1)},[e_{(j,p_1)(j,p_1-1)},\sigma_l(\widetilde{W}'^{p}_{ij}(z))])$$
$$-\rho([e_{(a,p_1-1)(b,p_1)},\sum_{h=1}^{t_1}\sigma_l(\widetilde{W}'^{p}_{ih}(z))](\delta_{hj}z+\tilde{e}_{(j,p_1)(h,p_1)})-\rho(\sum_{h=1}^{t_1}\sigma_l(\widetilde{W}'^{p}_{ih}(z))[e_{(a,p_1-1)(b,p_1)},e_{(j,p_1)(h,p_1)}])$$
$$+\rho(\sum_{h\ge t_1+1}[e_{(a,p_1-1)(b,p_1)},\sigma_l(\widetilde{W}'^{p}_{ih}(z))]\sigma_l(\widetilde{W}'^{p}_{hj,p_h-1}))+\rho(\sum_{h\ge t_1+1}\sigma_l(\widetilde{W}'^{p}_{ih}(z))[e_{(a,p_1-1)(b,p_1)},\sigma_l(\widetilde{W}'^{p}_{hj,p_h-1})])$$
$$=A+B+C+D+E\,.$$

Remark 3.3.1. $\sigma_l(\widetilde{W}'^{p}_{hj,p_h-1})$ is the coefficient of $(-z)^{p_h-1}$ in $\sigma_l(\widetilde{W}'^{p}_{hj}(z))$. We can compute it explicitly:

$$\sigma_l(\widetilde{W}'^{p}_{hj,p_h-1})=\begin{cases}[e_{(j,p_1-1)(j,p_1-2)},\sigma_l^2(\widetilde{W}^{2p}_{hj,p_h-1})]+\sigma_l^2(\widetilde{W}^{2p}_{hj,p_h-2})-\sum_{k=1}^{t_2}\sigma_l^2(\widetilde{W}^{2p}_{hk,p_h-1})e_{(j,p_1-1)(k,p_1-1)} & \\ \quad+\sum_{k\ge t_2+1}\sigma_l^2(\widetilde{W}^{2p}_{hk,p_h-1})\sigma_l^2(\widetilde{W}^{2p}_{kj,p_k-1})\,, & h\ge t_2+1\\ \tilde{e}_{(j,p_1-1)(h,p_1-1)}+\sigma_l^2(\widetilde{W}^{2p}_{hj,p_h-2})\,, & t_1+1\le h\le t_2\\ \tilde{e}_{(j,p_1-1)(h,p_1-1)}+\sigma_l^2(\widetilde{W}^{2p}_{hj,p_h-3})\,, & h\le t_1\,.\end{cases}$$

After a second application of the recursion we obtain the following:

$$A=[e_{(a,p_1-1)(b,p_1-2)},\sigma_l^2(\widetilde{W}^{2p}_{ib}(z))]-\sum_{s=1}^{t_2}\sigma_l^2(\widetilde{W}^{2p}_{is}(z))e_{(a,p_1-1)(s,p_1-1)}\delta_{bj}+\sigma_l^2(\widetilde{W}^{2p}_{ia}(z))e_{(j,p_1)(b,p_1)}\,,$$
$$B=-\sigma_l^2(\widetilde{W}^{2p}_{ia}(z))\delta_{bj}z-\sigma_l^2(\widetilde{W}^{2p}_{ia}(z))e_{(j,p_1)(b,p_1)}+\sigma_l^2(\widetilde{W}^{2p}_{ia}(z))\delta_{bj}(N-t_1)-\delta_{bj}\sigma_l^2(\widetilde{W}^{2p}_{ia}(z))t_1\,,$$
$$C=-\delta_{a\le t_1}\delta_{bj}\Big([e_{(a,p_1-1)(a,p_1-2)},\sigma_l^2(\widetilde{W}^{2p}_{ia}(z))]-\sigma_l^2(\widetilde{W}^{2p}_{ia}(z))(z-N+t_1+r)-\sum_{l=1}^{t_2}\sigma_l^2(\widetilde{W}^{2p}_{il}(z))e_{(a,p_1-1)(l,p_1-1)}$$
$$+\sum_{l\ge t_2+1}\sigma_l^2(\widetilde{W}^{2p}_{il}(z))\sigma_l^2(\widetilde{W}^{2p}_{la,p_l-1})\Big)\,,$$
$$D=-\delta_{bj}\sum_{h=t_1+1}^{t_2}\sigma_l^2(\widetilde{W}^{2p}_{ia}(z))=-\delta_{bj}(t_2-t_1)\sigma_l(\widetilde{W}^{2p}_{ia}(z))\,,$$
$$E=-\delta_{a\ge t_1+1}\delta_{bj}\Big([e_{(a,p_1-1)(a,p_1-2)},\sigma_l^2(\widetilde{W}^{2p}_{ia}(z))]-\sigma_l^2(\widetilde{W}^{2p}_{ia}(z))(z-N+t_1+t_2)-\sum_{l=1}^{t_2}\sigma_l^2(\widetilde{W}^{2p}_{il}(z))e_{(a,p_1-1)(l,p_1-1)}$$
$$+\sum_{l\ge t_2+1}\sigma_l^2(\widetilde{W}^{2p}_{il}(z))\sigma_l^2(\widetilde{W}^{2p}_{la,p_l-1})\Big)+\delta_{bj}\sum_{l\ge t_2+1}\sigma_l^2(\widetilde{W}^{2p}_{il}(z))\sigma_l^2(\widetilde{W}^{2p}_{la,p_l-1})\,.$$

These terms almost completely cancel out, it only remains to prove the following identity:

$$[e_{(a,p_1-1)(b,p_1-2)},\sigma_l^2(\widetilde{W}^{2p}_{ib}(z))]=[e_{(a,p_1-1)(a,p_1-2)},\sigma_l^2(\widetilde{W}^{2p}_{ia}(z))]\,.$$

or, equivalently,

$$[e_{(a,p_1)(b,p_1-1)},\sigma_l(\widetilde{W}'^{p}_{ib}(z))]=[e_{(a,p_1)(a,p_1-1)},\sigma_l(\widetilde{W}'^{p}_{ia}(z))]\,. \tag{3.3.6}$$

It is sufficient to show that the LHS does not depend on b, $1 \leq b \leq t_2$. Applying the recursion for $\sigma_l(\widetilde{W}'^p_{ib}(z))$ we get

$$
\begin{aligned}
&[e_{(a,p_1)(b,p_1-1)}, \sigma_l(\widetilde{W}'^p_{ib}(z))] = [e_{(a,p_1)(b,p_1-1)}, [e_{(b,p_1-1)(b,p_1-2)}, \sigma_l^2(\widetilde{W}^{2}_{ib}{}^{p}(z))]] \\
&\quad - \sum_{h=1}^{t_2} \sigma_l^2(\widetilde{W}^{2}_{ih}{}^{p}(z))[e_{(a,p_1)(b,p_1-1)}, e_{(b,p_1-1)(h,p_1-1)}] \\
&\quad = [e_{(a,p_1)(b,p_1-2)}, \sigma_l^2(\widetilde{W}^{2}_{ib}{}^{p}(z))] - \sum_{h=1}^{t_2} \sigma_l^2(\widetilde{W}^{2}_{ih}{}^{p}(z)) e_{(a,p_1)(h,p_1-1)} \,.
\end{aligned}
$$

Note that the second term does not depend on b. Iterating, we get

$$
[e_{(a,p_1)(b,p_1-k)}, \sigma_l^k(\widetilde{W}^{k}_{ib}{}^{p}(z))] = [e_{(a,p_1)(b,p_1-k-1)}, \sigma_l^{k+1}(\widetilde{W}^{k+1}_{ib}{}^{p}(z))] + \text{something not depending on } b.
$$

Keep iterating until $p_1 - k - 1 = 1$. At this point we obtain $\sigma_l^{k+1}(\widetilde{W}^{k+1}_{ib}{}^{p}(z)) = \sigma_l^{p_1-1}(\widetilde{W}^{p_1-1}_{ib}{}^{p}(z)) = \delta_{ib} z + e_{(b,1)(i,1)}$. Hence

$$
[e_{(a,p_1)(b,p_1-k-1)}, \sigma_l^{k+1}(\widetilde{W}^{k+1}_{ib}{}^{p}(z))] = e_{(a,p_1)(i,1)}
$$

that does not depend on b either. □

Theorem 3.3.2. *Suppose that the pyramid p is aligned to the left. Then*

$$
W(z) \in Mat_{r\times r} W(\mathfrak{g}, f, \Gamma, 0)[z] \,. \tag{3.3.7}
$$

Namely, $\rho([a, \widetilde{W}_{ij}(z)]) = 0$ *in* $\big(U(\mathfrak{g})/I_{\mathfrak{l}}\big)[z]$ *for every* $a \in \mathfrak{g}_{\geq 1}$, $1 \leq i, j, \leq r$.

Proof. We proceed by induction on the number of columns of p, analogously to the proof of Theorem 3.3.1, paying however attention to the fact that σ_r is not the identity map anymore.

In the base case, when $p_1 = 1$, we have

$$
W(z) = z\mathbb{1}_N + E \in \mathrm{Mat}_{N\times N} U(\mathfrak{g})[z] = \mathrm{Mat}_{N\times N} W(\mathfrak{g}, f, \Gamma, 0)[z]
$$

because this case corresponds to the nilpotent element $f = 0$, when $W(\mathfrak{g}, f, \Gamma, 0) = U(\mathfrak{g})$. Since we will use a two step induction, let us now consider the case $p_1 = 2$. In this case, $\mathfrak{g}_{\geq 1} = \mathfrak{g}_1 = \mathrm{Span}\{e_{(a,1)(b,1+\delta_{b\leq s_1})} \mid 1 \leq a \leq s_1 \,, 1 \leq b \leq r\}$. In this case, for $i \geq s_1 + 1$ we have

$$
\begin{aligned}
\rho([e_{(a,1)(b,1+\delta_{b\leq s_1})}, \widetilde{W}_{ij}(z)]) &= \rho([e_{(a,1)(b,1+\delta_{b\leq s_1})}, e_{(j,1+\delta_{j\leq s_1})(i,1)}]) = \rho(\delta_{bj} e_{(a,1)(i,1)} - \delta_{ai} e_{(j,1+\delta_{j\leq s_1})(b,1+\delta_{b\leq s_1})}) \\
&= \delta_{bj} \rho(e_{(a,1)(i,1)}) = 0 \,,
\end{aligned} \tag{3.3.8}
$$

since $1 \leq a \leq s_1$ while $i \geq s_1 + 1$. On the other hand, for $i \leq s_1$ by (3.2.3) we have

$$
\begin{aligned}
\widetilde{W}_{ij}(z) &= e_{(j,1+\delta_{j\leq s_1})(i,1)} - \sum_{h=1}^{s_1} (\delta_{ih} z + e_{(h,1)(i,1)})(\delta_{hj} z + \tilde{e}_{(j,1+\delta_{j\leq s_1})(h,2)}) \\
&\quad + \sum_{h\geq s_1+1} e_{(h,1)(i,1)}(\delta_{hj} z + \tilde{e}_{(j,1+\delta_{j\leq s_1})(h,1)}) \\
&\quad = -\delta_{ij} z^2 - z\delta_{j\leq s_1}(\tilde{e}_{(j,2)(i,2)} + e_{(j,1)(i,1)}) + e_{(j,1+\delta_{j\leq s_1})(i,1)} \\
&\quad - \sum_{h=1}^{s_1} e_{(h,1)(i,1)} \tilde{e}_{(j,1+\delta_{j\leq s_1})(h,2)} + \sum_{h\geq s_1+1} e_{(h,1)(i,2)} \tilde{e}_{(j,1+\delta_{j\leq s_1})(h,1)}
\end{aligned} \tag{3.3.9}
$$

and therefore

$$\begin{aligned}
&\rho([e_{(a,1)(b,1+\delta_{b\le s_1})},\widetilde{W}_{ij}(z)]) = -z\delta_{j\le s_1}\rho(\delta_{bj}e_{(a,1)(i,2)}-\delta_{ai}e_{(j,1)(b,1+\delta_{b\le s_1})})+\delta_{bj}\rho(e_{(a,1)(i,1)})\\
&-\delta_{ai}\rho(e_{(j,1+\delta_{j\le s_1})(b,1+\delta_{b\le s_1})})+\sum_{h=1}^{s_1}\delta_{ai}\rho(e_{(h,1)(b,1+\delta_{b\le s_1})}\tilde{e}_{(j,1+\delta_{j\le s_1})(h,2)})-\sum_{h=1}^{s_1}\delta_{bj}\rho(e_{(h,1)(i,1)}e_{(a,1)(h,2)})\\
&+\sum_{h\ge s_1+1}\delta_{bh}\rho(e_{(a,1)(i,2)}\tilde{e}_{(j,1+\delta_{j\le s_1})(h,1)})+\sum_{h\ge s_1+1}\delta_{bj}\rho(e_{(h,1)(i,2)}e_{(a,1)(h,1)})\\
&=-z\delta_{j\le s_1}(\delta_{bj}\delta_{ai}-\delta_{ai}\delta_{bj})+\delta_{bj}e_{(a,1)(i,1)}-\delta_{ai}e_{(j,1+\delta_{j\le s_1})(b,1+\delta_{b\le s_1})}+\delta_{ai}\delta_{b\le s_1}\big(e_{(j,1+\delta_{j\le s_1})(b,2)}-\delta_{jb}s_1\big)\\
&+\delta_{ai}\delta_{bj}s_1-\delta_{bj}e_{(a,1)(i,1)}+\sum_{b\ge s_1+1}\delta_{ai}\big(e_{(j,1+\delta_{j\le s_1})(b,1)}-\delta_{bj}s_1\big)=0\,.
\end{aligned}\tag{3.3.10}$$

Next, suppose $p_1>1$ and suppose that equation (3.3.7) holds for p', namely

$$0=\rho^{p'}([\mathfrak{g}^{p'}_{\ge1},\widetilde{W}^{p'}_{ij}(z)])\in U(\mathfrak{g}^{p'})/U(\mathfrak{g}^{p'})\langle b-(f^{p'}|b)\rangle_{b\in\mathfrak{g}^{p'}_{\ge1}}[z]\,,$$

for every $1\le i,j\le r$.

We can decompose $\mathfrak{g}_{\ge1}=\sigma_r(\mathfrak{g}^{p'}_{\ge1})\oplus\mathfrak{g}^1_{\ge1}$, where $\mathfrak{g}^1_{\ge1}$ is the complementary subspace to $\sigma_r(\mathfrak{g}^{p'}_{\ge1})$ in $\mathfrak{g}_{\ge1}$ consisting of all complementary matrices $e_{(i,h)(j,k)}$ such that $x(ih)=\frac{d}{2}$.

First, consider the commutator between $\widetilde{W}_{ij}(z)$ and $\sigma_r(\mathfrak{g}^{p'}_{\ge1})$. If $i\ge s_1+1$, from (3.2.3) and the induction hypothesis (cf. (1.1.22)) we have the following identity in $\big(U(\mathfrak{g})/I_{\mathfrak{l}}\big)[z]$:

$$\rho([\sigma_r(\mathfrak{g}^{p'}_{\ge1}),\widetilde{W}_{ij}(z)])=\rho([\sigma_r(\mathfrak{g}^{p'}_{\ge1}),\sigma_r(\widetilde{W}^{p'}_{ij}(z))])=\sigma_r(\rho^{p'}([\mathfrak{g}^{p'}_{\ge1},\widetilde{W}^{p'}_{ij}(z)]))=0\,.$$

On the other hand, if $i\le s_1$, we have

$$\begin{aligned}
&\rho([\sigma_r(\mathfrak{g}^{p'}_{\ge1}),\widetilde{W}_{ij}(z)])=\rho([\sigma_r(\mathfrak{g}^{p'}_{\ge1}),[\sigma_r(\widetilde{W}^{p'}_{ij}(z)),e_{(i,2)(i,1)}]])\\
&-\rho([\sigma_r(\mathfrak{g}^{p'}_{\ge1}),\sum_{h=1}^{s_1}(\delta_{ih}z+e_{(h,1)(i,1)})\sigma_r(\widetilde{W}^{p'}_{hj}(z))])+\rho([\sigma_r(\mathfrak{g}^{p'}_{\ge1}),\sum_{h\ge s_1+1}\sigma_r(\widetilde{W}^{p'}_{ih;p_h-1})\sigma_r(\widetilde{W}^{p'}_{hj}(z))])\\
&=\rho([\sigma_r(\widetilde{W}^{p'}_{ij}(z)),[\sigma_r(\mathfrak{g}^{p'}_{\ge1}),e_{(i,2)(i,1)}]])+\rho([[\sigma_r(\mathfrak{g}^{p'}_{\ge1}),\sigma_r(\widetilde{W}^{p'}_{ij}(z))],e_{(i,2)(i,1)}])\,,
\end{aligned}$$

where for the last equality we have used the induction hypothesis. However, for degree reasons we necessarily have

$$[\sigma_r(e_{(a,q)(b,k)}),e_{(i,2)(i,1)}]=0\,,$$

for every $e_{(a,q)(b,k)}\in\mathfrak{g}^{p'}_{\ge1}$. Finally, $[\sigma_r(\mathfrak{g}^{p'}_{\ge1}),\sigma_r(\widetilde{W}^{p'}_{ij}(z))]\in U(\sigma_r(\mathfrak{g}^{p'}))\langle b-(f^{p'}|b)\rangle_{b\in\sigma_r(\mathfrak{g}^{p'}_{\ge1})}$ from the induction hypothesis and since $[e_{(i,2)(i,1)},\sigma_r(\mathfrak{g}^{p'}_{\ge1})]=0$, by the Leibniz rule we have $[[\sigma_r(\mathfrak{g}^{p'}_{\ge1}),\sigma_r(\widetilde{W}^{p'}_{ij}(z))],e_{(i,2)(i,1)}]\in U(\mathfrak{g})\langle\sigma_r(b)-(f|b)\rangle_{b\in\mathfrak{g}_{\ge1}}$.

We shall now check what happens to the commutator with elements of $\mathfrak{g}^1_{\ge1}$. Again, by Lemma 3.3.1 it is sufficient to consider elements of the form $e_{(a,1)(b,1+\delta_{b\le s_1})}$, with $1\le a\le s_1$, $1\le b\le s_2$ (namely, with $x(a,1)=\frac{d}{2}$ and $x(b,1)=\frac{d}{2}-1$), since for each $e_{(a,1)(c,k)}\in\mathfrak{g}^1_{\ge1}$ with $2+\delta_{c\le s_1}\le k\le p_c$ (namely, with $x(c,k)<\frac{d}{2}-1$) we have

$$e_{(a,1)(c,k)}=[e_{(a,1)(b,1+\delta_{b\le s_1})},e_{(b,1+\delta_{b\le s_1})(c,k)}]$$

and $e_{(b,1+\delta_{b\le s_1})(c,k)}\in\sigma_r(\mathfrak{g}^{p'}_{\ge1})$.

As in Theorem 3.3.1 we will need to apply the (right) recursion twice. First, let us suppose $i\ge s_2+1$; then

$$[e_{(a,1)(b,1+\delta_{b\le s_1})},\widetilde{W}_{ij}(z)]=[e_{(a,1)(b,1+\delta_{b\le s_1})},\sigma_r^2(\widetilde{W}^{p^2}_{ij}(z))]=0\,.$$

Remark 3.3.2. Be careful with the recursive formula for $\widetilde{W}^{p'}_{ij}(z)$, $1\le i\le s_2$:

$$\widetilde{W}^{p'}_{ij}(z)=[\sigma_r(\widetilde{W}^{p^2}_{ij}(z)),e_{(i,2)(i,1)}]-\sum_{h=1}^{s_2}(\delta_{ih}z+e_{(h,1)(i,1)})\sigma_r(\widetilde{W}^{p^2}_{hj}(z))+\sum_{h\ge s_2+1}\sigma_r(\widetilde{W}^{p^2}_{ih,p_h-1})\sigma_r(\widetilde{W}^{p^2}_{hj}(z))\,.$$

In fact, after applying σ_r it gives

$$\begin{aligned}\sigma_r(\widetilde{W}^{p'}_{ij}(z)) &= [\sigma_r^2(\widetilde{W}^{p^2}_{ij}(z)), e_{(i,2+\delta_{i\le s_1})(i,1+\delta_{i\le s_1})}] - \sum_{h=1}^{s_2}(\delta_{ih}z + e_{(h,1+\delta_{h\le s_1})(i,1+\delta_{i\le s_1})})\sigma_r^2(\widetilde{W}^{p^2}_{hj}(z)) \\ &+ \sum_{h\ge s_2+1} \sigma_r^2(\widetilde{W}^{p^2}_{ih,p_h-1})\sigma_r^2(\widetilde{W}^{p^2}_{hj}(z))\,. \end{aligned} \tag{3.3.11}$$

Using (3.3.11) and the induction hypothesis, we can conclude that $\rho([e_{(a,1)(b,1+\delta_{b\le s_1})}, \widetilde{W}_{ij}(z)]) = 0$ for any $s_1+1 \le i \le s_2$.

Finally, for $i \le s_1$ we have

$$\begin{aligned}\rho([e_{(a,1)(b,1+\delta_{b\le s_1})}, \widetilde{W}_{ij}(z)]) &= \rho([e_{(a,1)(b,1+\delta_{b\le s_1})}, [\sigma_r(\widetilde{W}^{p'}_{ij}(z)), e_{(i,2)(i,1)}]]) \\ &- \sum_{h=1}^{s_1}\rho([e_{(a,1)(b,1+\delta_{b\le s_1})}, e_{(h,1)(i,1)}]\sigma_r(\widetilde{W}^{p'}_{hj}(z)) - \sum_{h=1}^{s_1}(\delta_{ih}z + e_{(h,1)(i,1)})\rho([e_{(a,1)(b,1+\delta_{b\le s_1})}, \sigma_r(\widetilde{W}^{p'}_{hj}(z))]) \\ &+ \sum_{h\ge s_1+1}\rho([e_{(a,1)(b,1+\delta_{b\le s_1})}, \sigma_r(\widetilde{W}^{p'}_{ih,p_h-1})])\sigma_r(\widetilde{W}^{p'}_{hj}(z)) + \sum_{h\ge s_1+1}\sigma_r(\widetilde{W}^{p'}_{ih,p_h-1})\rho([e_{(a,1)(b,1+\delta_{b\le s_1})}, \sigma_r(\widetilde{W}^{p'}_{hj}(z))]) \\ &= A+B+C+D+E\,. \end{aligned}$$

Note that in the case $1 \le i \le s_1$ it is

$$\sigma_r(\widetilde{W}^{p'}_{ih,p'_h-1}) = \begin{cases} \tilde{e}_{(h,2)(i,2)} + \sigma_r^2(\widetilde{W}^{p^2}_{ih,p_h-3})\,, & h \le s_1 \\ \tilde{e}_{(h,1)(i,2)} + \sigma_r^2(\widetilde{W}^{p^2}_{ih,p_h-2})\,, & s_1+1 \le h \le s_2 \\ [\sigma_r^2(\widetilde{W}^{p^2}_{ih,p_h-1}), e_{(i,3)(i,2)}] + \sigma_r^2(\widetilde{W}^{p^2}_{ih,p_h-2}) & \\ \qquad - \sum_{k=1}^{s_2}\tilde{e}_{(k,1+\delta_{k\le s_1})(i,2)}\sigma_r^2(\widetilde{W}^{p^2}_{kh,p_h-1}) & \\ \qquad + \sum_{k\ge s_2+1}\sigma_r^2(\widetilde{W}^{p^2}_{kh,p_h-1})\sigma_r^2(\widetilde{W}^{p^2}_{ik,p_k-1})\,, & h \ge s_2+1\,. \end{cases}$$

Applying the recursion (3.2.3) a second time we obtain the following

$$\begin{aligned} A &= -\delta_{ia}[\sigma_r^2(\widetilde{W}^{p^2}_{aj}(z)), e_{(a,3)(b,1+\delta_{b\le s_1})}] - e_{(a,1)(h,1)}\sigma_r^2(\widetilde{W}^{p^2}_{bj}(z)) + \delta_{ai}\sum_{h=1}^{s_2} e_{(h,1+\delta_{h\le s_1})(b,1+\delta_{b\le s_1})}\sigma_r^2(\widetilde{W}^{p^2}_{hj}(z))\,, \\ B &= -\delta_{ia}\sigma_r^2(\widetilde{W}^{p^2}_{bj}(z))s_1 + \delta_{ai}\delta_{b\le s_1}\Big([\sigma_r^2(\widetilde{W}^{p^2}_{bj}(z)), e_{(b,3)(b,2)}] - z\sigma_r^2(\widetilde{W}^{p^2}_{bj}(z)) - \sum_{h=1}^{s_2} e_{(h,1+\delta_{h\le s_1})(b,2)}\sigma_r^2(\widetilde{W}^{p^2}_{hj}(z)) \\ &\quad - s_1\sigma_r^2(\widetilde{W}^{p^2}_{bj}(z)) + \sum_{h\ge s_2+1}\sigma_r^2(\widetilde{W}^{p^2}_{bh;p_h-1})\sigma_r^2(\widetilde{W}^{p^2}_{hj}(z))\Big)\,, \\ C &= \delta_{ai}z\sigma_r^2(\widetilde{W}^{p^2}_{bj}(z)) + e_{(a,1)(i,1)}\sigma_r^2(\widetilde{W}^{p^2}_{bj}(z))\,, \\ D &= -\delta_{ai}\sum_{h\ge s_2+1}\sigma_r^2(\widetilde{W}^{p^2}_{bh;p_h-1})\sigma_r^2(\widetilde{W}^{p^2}_{hj}(z)) + \delta_{ai}\delta_{b\ge s_1+1}\Big([\sigma_r^2(\widetilde{W}^{p^2}_{bj}(z)), e_{(b,2)(b,1)}] - z\sigma_r^2(\widetilde{W}^{p^2}_{bj}(z)) \\ &\quad - \sum_{h=1}^{s_2} e_{(h,1+\delta_{h\le s_1})(b,1)}\sigma_r^2(\widetilde{W}^{p^2}_{hj}(z)) + s_1\sigma_r^2(\widetilde{W}^{p^2}_{bj}(z)) + \sum_{h\ge s_2+1}\sigma_r^2(\widetilde{W}^{p^2}_{bh;p_h-1})\sigma_r^2(\widetilde{W}^{p^2}_{hj}(z))\Big)\,, \\ E &= 0\,. \end{aligned}$$

These terms almost completely cancel out, it only remains to prove the following right-aligned analogue of the identity (3.3.6):

$$[\sigma_r(\widetilde{W}^{p^2}_{aj}(z)), e_{(a,2)(b,1)}] = [\sigma_r(\widetilde{W}^{p^2}_{bj}(z)), e_{(b,2)(b,1)}]\,, \quad \text{for each } 1 \le b \le s_2\,. \tag{3.3.12}$$

This is proved in a similar fashion, using the fact that, for each $1 \le k \le p_1 - 1$,

$$[\sigma_r^k(\widetilde{W}^{p^{k+1}}_{aj}(z)), e_{(a,k+1)(b,1)}] = [\sigma_r^{k+1}(\widetilde{W}^{p^{k+2}}_{aj}(z)), e_{(a,k+2)(b,1)}] + \text{ something not depending on } a\,.$$

□

3.3.2 Dependence on the elements of the centralizer $\mathfrak{g}^f$

Let f be a nilpotent element associated with the partition $(p_1 \geq p_2 \geq \ldots \geq p_r)$. In Section 3.1 we have introduced the following basis for $\mathfrak{g}^f$:

$$\{f_{ij;k} \mid 1 \leq i,j \leq r,\, 0 \leq k \leq \min(p_i,p_j)-1\}\,,$$

where, as in (3.1.8),

$$f_{ij;k} = \sum_{h=0}^{k} e_{(i,p_i+h-k)(j,h+1)}\,.$$

Let us assume that the grading Γ is even. We now want to prove that for the elements $W_{ijk} \in W(\mathfrak{g},f,\Gamma,0)$ obtained as the coefficients of the polynomials $W_{ij}(z)$ from Definition 3.2.1 (cf. also Proposition 3.2.3), the linear map $w : \mathfrak{g}^f \to W(\mathfrak{g},f,\Gamma,0)$ given by

$$f_{ji;k} \mapsto W_{ij;k}\,,$$

satisfies the conditions of Premet's Theorem 3.1.1.

Proposition 3.3.1 (Premet's form). *Let Γ be an even good $\frac{1}{2}\mathbb{Z}$-grading for a nilpotent element f, and let $U \subset \mathfrak{g}$ be the complementary subspace to $[f,\mathfrak{g}]$ as in (3.1.5).*

Then, for every $1 \leq i,j \leq r$ and $f_{ji;k} \in \mathfrak{g}_{1-\Delta_{ji;k}}$ (namely, $\Delta_{ji;k} := \Delta(f_{ji;k})$), we have

$$\widetilde{W}_{ij;k} \in f_{ji;k} + F_{\Delta_{ji;k},2}U(\mathfrak{g})\,, \tag{3.3.13}$$

where we are using the refinement of the Kazhdan filtation (1.3.7). As a consequence, $W_{ij;k} \in F_{\Delta_{ji;k}}W(\mathfrak{g},f,\Gamma,0)$. Moreover,

$$\eta^f(gr_{\Delta_{ji;k}}(W_{ij;k})) = f_{ji;k}\,, \tag{3.3.14}$$

where $\eta^f : S(\mathfrak{g})/\operatorname{gr} I_{\mathfrak{l}} \twoheadrightarrow S(\mathfrak{g}^f)$ is the projection defined in Section 3.1 with kernel the subspace $U^\perp$ as in (3.1.6).

Remark 3.3.3. We recall the following properties of the elements $f_{ji;k}$, $1 \leq i,j \leq r$, $0 \leq k \leq \min(p_i,p_j)-1$, which clearly follow from their definition (3.1.8):

- deg $(f_{ji;k}) = s_{1i} - s_{1j} - p_j + k + 1$;
- $\Delta_{ji;k} = \Delta(f_{ji;k}) = s_{1j} - s_{1i} + p_j - k$;
- $f_{ji;k}(-z)^k \in F_{\Delta_{ji;k}+k,1}(U(\mathfrak{g})[z])$, with respect to the extended Kazhdan filtration (1.3.8).

We want to rewrite equations (3.3.13) and (3.3.14) in terms of the extended Kazhdan filtration of equation (1.3.8). In order to do so, we need the following preliminary results.

Lemma 3.3.2. *Let $t_1 \leq s_1$, and suppose that the pyramid ${}'p$ exists. Then, for every $1 \leq i,j \leq r$, $0 \leq k \leq$ min$(p_i,p_j)-1$ the following holds*

$$f_{ji;k} = \begin{cases} \sigma_l(f^{'p}_{ji;k})\,, & j \geq t_1+1 \\ \begin{cases} [e_{(j,p_1)(j,p_1-1)}, \sigma_l(f^{'p}_{ji;0})]\,, & k=0 \\ [e_{(j,p_1)(j,p_1-1)}, \sigma_l(f^{'p}_{ji;k})] + \sigma_l(f^{'p}_{ji;k-1})\,, & 1 \leq k \leq min(p_i,p_1-1)-1 \\ \sigma_l(f^{'p}_{ji;k-1}) + e_{(j,p_1)(i,p_1)}\,, & k = p_1-1 \end{cases}, & j \leq t_1\,. \end{cases} \tag{3.3.15}$$

Proof. Using Formula (3.1.8) compute both LHS and RHS of Equation (3.3.15). □

Note that Lemma 3.3.2 holds not only for an even $\frac{1}{2}\mathbb{Z}$-grading, but in the more general case when it is possible to remove the leftmost column of p, namely if ${}'p$ exists. The same is true for Lemma 3.3.3 below, provided this time that it is possible to remove the rightmost column of p, namely if p' exists.

We can analogously prove the following.

Lemma 3.3.3. *Let $s_1 \leq t_1$, and suppose that the pyramid p' exists. Then, for every $1 \leq i, j \leq r$, $0 \leq k \leq$ $min(p_i, p_j) - 1$ the following holds*

$$f_{ji;k} = \begin{cases} \sigma_r(f^{p'}_{ji;k}) \,, & i \geq s_1 + 1 \\ \begin{cases} [\sigma_r(f^{p'}_{ji;0}), e_{(i2)(i1)}] \,, & k = 0 \\ [\sigma_r(f^{p'}_{ji;k}), e_{(i2)(i1)}] + \sigma_r(f^{p'}_{ji;k-1}) \,, & 1 \leq k \leq min(p_j, p_1 - 1) - 1 \\ \sigma_r(f^{p'}_{ji;k-1}) + e_{(j,1)(i,1)} \,, & k = p_1 - 1 \end{cases} & i \leq s_1 \,. \end{cases}$$

We would like to comprise the results of Lemma 3.3.2 and Lemma 3.3.3 in a more compact form. This will be done in Lemmas 3.3.4 and 3.3.5 below.

Definition 3.3.1. Define the polynomial

$$f_{ji}(z) = \sum_{k=0}^{\min(p_i,p_j)-1} f_{ji;k}(-z)^k \in \mathfrak{g}^f[z] \,.$$

Let $\Delta_{ji} := \Delta(f_{ji}(z)) = \Delta(f_{ji;k}) + k = p_j + s_{1j} - s_{1i}$ for any $0 \leq k \leq \min(p_i, p_j) - 1$. Then, in terms of the extended Kazhdan filtration (1.3.8), we have $f_{ji}(z) \in F_{\Delta_{ji}}(U(\mathfrak{g})[z])$.

Lemma 3.3.4. *Let $t_1 \leq s_1$, and suppose that the pyramid ${}'p$ exists. Then, for every $1 \leq i, j \leq r$ the following recursion holds*

$$f_{ji}(z) = \begin{cases} \sigma_l(f^{'p}_{ji}(z)) \,, & j \geq t_1 + 1 \\ [e_{(j,p_1)(j,p_1-1)}, \sigma_l(f^{'p}_{ji}(z))] - \sigma_l(f^{'p}_{ji}(z))z - \delta_{i \geq t_1+1}\sigma_l(f^{'p}_{ji;p_i-1})(-z)^{p_i} & \\ +\delta_{i \leq t_1} e_{(j,p_1)(i,p_1)}(-z)^{p_1-1} \,, & j \leq t_1 \,. \end{cases}$$

Proof. The case $j \geq t_1 + 1$ is an immediate consequence of Lemma 3.3.2 and Definition 3.3.1, since

$$\sigma_l(f^{'p}_{ji}(z)) = \sum_{k=0}^{\min(p^{'p}_i, p^{'p}_j)-1} \sigma_l(f^{'p}_{ji;k})(-z)^k$$

and $\min(p^{'p}_i, p^{'p}_j) = \min(p^{'p}_i, p_j) = \min(p_i, p_j)$, where the first equality is due to the fact that $p^{'p}_j = p_j$ for $j \geq t_1 + 1$ and last one is due to the fact that $p^{'p}_i \neq p_i$ means $p_i = p_1$, and in this case $\min(p_1 - 1, p_j) = \min(p_1, p_j) = p_j$.

The case $j \leq t_1$ is also obtained from Lemma 3.3.2 and Definition 3.3.1, as follows

$$\begin{aligned} f_{ji}(z) = [e_{(j,p_1)(j,p_1-1)}, \sigma_l(f^{'p}_{ji;0})] + \sum_{k=1}^{\min(p_i,p_1-1)-1} ([e_{(j,p_1)(j,p_1-1)}, \sigma_l(f^{'p}_{ji;k})] + \sigma_l(f^{'p}_{ji;k-1}))(-z)^k \\ + \delta_{i \leq t_1}(\sigma_l(f^{'p}_{ji;p_1-2}) + e_{(j,p_1)(i,p_1)})(-z)^{p_1-1} \\ = \sum_{k=0}^{\min(p_i,p_1-1)-1} [e_{(j,p_1)(j,p_1-1)}, \sigma_l(f^{'p}_{ji;k})](-z)^k + \sum_{k=1}^{\min(p_i,p_1)-1} \sigma_l(f^{'p}_{ji;k-1})(-z)^k + \delta_{i \leq t_1} e_{(j,p_1)(i,p_1)}(-z)^{p_1-1} \\ = [e_{(j,p_1)(j,p_1-1)}, \sigma_l(f^{'p}_{ji}(z))] + \sum_{h=0}^{\min(p_i-1,p_1-1)-1} \sigma_l(f^{'p}_{ji;h})(-z)^{h+1} + \delta_{i \leq t_1} e_{(j,p_1)(i,p_1)}(-z)^{p_1-1} \\ = [e_{(j,p_1)(j,p_1-1)}, \sigma_l(f^{'p}_{ji}(z))] - \sigma_l(f^{'p}_{ji}(z))z - \delta_{i \geq t_1+1}\sigma_l(f^{'p}_{ji;p_i-1})(-z)^{p_i} + \delta_{i \leq t_1} e_{(j,p_1)(i,p_1)}(-z)^{p_1-1} \,. \end{aligned}$$

Here, we have used the following facts, that can be easily checked:

(i) $\min(p_i, p_j - 1) - 1 = \min(p^{'p}_i, p^{'p}_j) - 1$;

(ii) $\{1, \ldots, \min(p_i, p_1 - 1) - 1\} \cup \delta_{i \leq t_1}\{p_1 - 1\} = \{1, \ldots, \min(p_i, p_1) - 1\}$;

(iii) $\min(p_i-1,p_j-1)-1 = \min(p_i^{\prime p},p_j^{\prime p})-1$ if $i\leq t_1$, whereas $\{0,\dots,\min(p_i-1,p_j-1)-1\} = \{0,\dots,\min(p_i^{\prime p},p_j^{\prime p})-1\}\setminus\{p_i-1\}$ if $i\geq t_1+1$.

□

An analogous proof holds for the following

Lemma 3.3.5. *Let $s_1\leq t_1$, and suppose that the pyramid p' exists. Then, for every $1\leq i,j\leq r$ the following recursion holds*

$$f_{ji}(z)=\begin{cases}\sigma_r(f_{ji}^{p'}(z))\,, & i\geq s_1+1\\ [\sigma_r(f_{ji}^{p'}(z)),e_{(i2)(i1)}]-\sigma_r(f_{ji}^{p'}(z))z-\delta_{j\geq s_1+1}\sigma_l(f_{ji;p_j-1}^{p'})(-z)^{p_j} & \\ +\delta_{j\leq s_1}e_{(j1)(i1)}(-z)^{p_1-1}\,, & i\leq s_1\,.\end{cases}$$

We are finally ready for the proof of Proposition 3.3.1.

Proof of Prop. 3.3.1. Let us first prove Equation (3.3.13). In terms of the extended Kazhdan filtration of (1.3.8), proving (3.3.13) is equivalent to proving that (cf. (3.2.18))

$$\widetilde{W}_{ij}(z)\in-(-z)^{p_i}\delta_{ij}+f_{ji}(z)+F_{\Delta_{ji},2}\big(U(\mathfrak{g})[z]\big)\,, \tag{3.3.16}$$

which we need to prove.

We proceed by induction on the number of columns of p. The base case $p_1=1$ is obvious, since $\mathfrak{g}^f=\mathfrak{g}=\mathfrak{g}_0$, hence $\Delta_{ji}=1$ for all $1\leq i,j,\leq r$, and by definition we have

$$\widetilde{W}_{ij}(z)=z\delta_{ij}+e_{(j,1)(i,1)}=-(-z)\delta_{ij}+f_{ji}(z)\,,$$

proving (3.3.16).

Next, suppose that $p_1>1$. Without loss of generality, we may assume that $t_1\leq s_1$, the case when $s_1\leq t_1$ having a similar proof (which uses the results of Lemmas 3.3.3 and 3.3.5 instead). By the induction hypotesis, we have

$$\widetilde{W}_{ij}^{\prime p}(z)\in-(-z)^{p_i^{\prime p}}\delta_{ij}+f_{ji}^{\prime p}(z)+F_{\Delta_{ji}^{\prime p},2}(U(\mathfrak{g}^{\prime p})[z])\,,$$

where $\Delta_{ji}^{\prime p}=\Delta(f_{ji;k}^{\prime p})+k=p_j^{\prime p}+s_{1j}-s_{1i}$. Note that, since the grading on $\mathfrak{g}^{\prime p}$ is compatible with that of $\mathfrak{g}$ (through the map σ_l), the (extended) Kazhdan filtration of $U(\mathfrak{g}')$ is compatible with that of $U(\mathfrak{g})$:

$$\sigma_l\big(F_{\Delta,n}(U(\mathfrak{g}^{\prime p})[z])\big)\subseteq F_{\Delta,n}(U(\mathfrak{g})[z])\,. \tag{3.3.17}$$

By Equation (3.2.2), in the case $j\geq t_1+1$, by the recursive definition (3.2.2) of $\widetilde{W}_{ij}(z)$ and the induction hypothesis, we have

$$\widetilde{W}_{ij}(z)=\sigma_l(\widetilde{W}_{ij}^{\prime p}(z))\in-(-z)^{p_j^{\prime p}}\delta_{ij}+\sigma_l(f_{ji}^{\prime p}(z))+\sigma_l(F_{\Delta_{ji}^{\prime p},2}(U(\mathfrak{g}^{\prime p})[z]))\subseteq-(-z)^{p_j}\delta_{ij}+f_{ji}(z)+F_{\Delta_{ji},2}(U(\mathfrak{g})[z])\,,$$

where for the last inclusion we have used Lemma (3.3.4), the inclusion (3.3.17), and the fact that $p_j^{\prime p}=p_j$ (and therefore $\Delta_{ji}^{\prime p}=\Delta_{ji}$) for $j\geq t_1+1$. Hence, (3.3.16) holds in this case. On the other hand, for $j\leq t_1$ by the recursive definition (3.2.2) of $\widetilde{W}_{ij}(z)$ and from the induction hypothesis we have

$$\begin{aligned}\widetilde{W}_{ij}(z)\in&[e_{(j,p_1)(j,p_1-1)},\sigma_l(f_{ji}^{\prime p}(z))+\sigma_l(F_{\Delta_{ji}^{\prime p},2}(U(\mathfrak{g}^{\prime p})[z]))]+\sum_{h=0}^{t_1}(-z)^{p_i^{\prime p}}\delta_{ih}(\delta_{hj}z+\tilde{e}_{(j,p_1)(h,p_1)})\\&-\sum_{h=0}^{t_1}\big(\sigma_l(f_{hi}^{\prime p}(z))+\sigma_l(F_{\Delta_{hi}^{\prime p},2}(U(\mathfrak{g}^{\prime p})[z]))\big)(\delta_{hj}z+\tilde{e}_{(j,p_1)(h,p_1)})\\&+\sum_{h\geq t_1+1}\big(-(-z)^{p_i^{\prime p}}\delta_{ih}+\sigma_l(f_{hi}^{\prime p}(z))+\sigma_l(F_{\Delta_{hi}^{\prime p},2}(U(\mathfrak{g}^{\prime p})[z]))\big)\big(\sigma_l(f_{jh;p_h-1}^{\prime p})+\sigma_l(F_{\Delta_{jh;p_h-1}^{\prime p},2}(U(\mathfrak{g}^{\prime p})))\big)\,.\end{aligned} \tag{3.3.18}$$

Note that we clearly have

$$\Delta(e_{(j,p_1)(j,p_1-1)}) = 2\,, \quad \Delta(e_{(j,p_1)(h,p_1)}) = 1\,, \tag{3.3.19}$$

and

$$\Delta_{hi}^{'p} = \Delta_{hi} - 1\,, \quad \text{for } h \le t_1\,, \tag{3.3.20}$$

while

$$\Delta_{hi}^{'p} = \Delta_{hi} \text{ and } \Delta_{hi}^{'p} + \Delta_{jh;p_h-1}^{'p} = s_{1j} - s_{1i} + p_j = \Delta_{ji}\,, \quad \text{for } h \ge t_1 + 1\,. \tag{3.3.21}$$

Hence, by (3.3.17), (3.3.19), (3.3.20) and the property (1.3.10) of the (extended) Kazhdan filtration, we have

$$[e_{(j,p_1)(j,p_1-1)}, \sigma_l(F_{\Delta_{ji}^{'p},2}(U(\mathfrak{g}^{'p})[z]))] \subseteq [F_{2,1}(U(\mathfrak{g})[z]), F_{\Delta_{ji}-1,2}(U(\mathfrak{g})[z])] \subset F_{\Delta_{ji},2}(U(\mathfrak{g})[z])\,.$$

Moreover, by (3.3.19)-(3.3.21), the induction hypothesis and the property (1.3.9) of the (extended) Kazhdan filtration, we have

$$\big(\sigma_l(f_{hi}^{'p}(z)) + \sigma_l(F_{\Delta_{hi}^{'p},2}(U(\mathfrak{g}^{'p})[z]))\big)\tilde{e}_{(j,p_1)(h,p_1)} \in F_{\Delta_{hi},2}(U(\mathfrak{g})[z]) = F_{\Delta_{ji},2}(U(\mathfrak{g})[z])\,, \tag{3.3.22}$$

when $h \le t_1$, and

$$\big(\sigma_l(f_{hi}^{'p}(z)) + \sigma_l(F_{\Delta_{hi}^{'p},2}(U(\mathfrak{g}^{'p})[z]))\big)\big(\sigma_l(f_{jh;p_h-1}^{'p}) + \sigma_l(F_{\Delta_{jh;p_h-1}^{'p},2}(U(\mathfrak{g}^{'p})))\big) \subseteq F_{\Delta_{ji},2}(U(\mathfrak{g})[z]))\,, \tag{3.3.23}$$

when $h \ge t_1 + 1$.

Combining (3.3.18) with Lemma 3.3.4, (3.3.17), and (3.3.22)-(3.3.23), we can conclude that

$$\widetilde{W}_{ij}(z) \in -(-z)^{p_1}\delta_{ij} + f_{ji}(z) + F_{\Delta_{ji},2}(U(\mathfrak{g})[z])\,.$$

Thus, (3.3.16) holds also in the case when $j \le t_1$.

Next, let us prove equation (3.3.14). We may assume that $t_1 \le s_1$, the case $s_1 \le t_1$ having a similar proof (which uses Lemmas 3.3.3 and 3.3.5 instead). We proceed by induction on the number of columns of p, the base case $p_1 = 1$ being immediate, since

$$W_{ij;k} = W_{ij;0} = e_{(j1)(i1)} = f_{ji;0}\,.$$

Let us now consider the case when $p_1 > 1$. In terms of the extended Kazhdan filtration of (1.3.8), and by equation (3.2.18), proving (3.3.14) is equivalent to proving that

$$\eta^f\left(\operatorname{gr}_{\Delta_{ji}}\left(W_{ij}(z)\right)\right) = -(-z)^{p_j}\delta_{ij} + f_{ji}(z)\,, \tag{3.3.24}$$

By the way the map $\eta^f : \operatorname{gr} W(\mathfrak{g}, f, \Gamma, 0) \to S(\mathfrak{g}^f)$ is defined (cf. Section 3.1), we can equivalently rewrite (3.3.24) as

$$\eta^f\left(\operatorname{gr}_{\Delta_{ji}}\left(\widetilde{W}_{ij}(z)\right)\right) = -(-z)^{p_j}\delta_{ij} + f_{ji}(z)\,, \tag{3.3.25}$$

where the projection $\eta^f : \operatorname{gr} U(\mathfrak{g}) \cong S(\mathfrak{g}) \to S(\mathfrak{g}^f)$ is as in (3.1.2), which we need to prove.

First, note that for the choice of the subspace $U^\perp$ as in (3.1.6), the following properties hold:

(i) $\sigma_l(U^{'p\perp}) \subset U^\perp$;

(ii) $[e_{(j,p_1)(j,p_1-1)}, \sigma_l(U^{'p\perp})] = 0$, for $j \le t_1$;

(iii) $\sigma_l(f_{ji;k}^{'p}) \in U^\perp$, for $1 \le j \le t_1$ and $0 \le k \le p_i^{'p} - 1 = \min(p_i^{'p}, p_j^{'p}) - 1$.

In fact, by (3.1.6), given a pyramid p, we have

$$\begin{aligned} U^{'p\perp} &= \operatorname{Span}_{\mathbb{C}}\{e_{(j,k)(i,h)} \mid 1 \le i,j \le r\,, 1 \le k \le p_j^{'p} - 1\,, 1 \le h \le p_i^{'p}\} \\ &\oplus \operatorname{Span}_{\mathbb{C}}\{e_{(j,p_j^{'p})(i,h)} \mid 1 \le i < j \le r\,, p_j^{'p} + 1 \le h \le p_i^{'p}\}\,. \end{aligned}$$

Since $p_i^{'p} \leq p_i$, for every $1 \leq i \leq r$, and since $p_i^{'p} = p_1 - 1$ if and only if $i \leq t_1$, it is clear that $\sigma_l(U^{'p\perp}) \subset U^\perp$, proving (i). For (ii) observe that given $e_{(a,k)(b,h)} \in U^{'p\perp}$, with $1 \leq k \leq p_a^{'p} - 1$ and $1 \leq h \leq p_b^{'p}$, the commutator

$$[e_{(j,p_1)(j,p_1-1)}, e_{(a,k)(b,h)}] = \delta_{aj}\delta_{k,p_1-1}e_{(j,p_1)(b,h)} - \delta_{bj}\delta_{h,p_1}e_{(a,k)(j,p_1-1)}$$

clearly vanishes, since $\delta_{k,p_1-1} = 0$ and $\delta_{h,p_1} = 0$. Moreover, given $e_{(a,p_a^{'p})(b,h)} \in U^{'p\perp}$ with $1 \leq b < a \leq r$ and $p_b^{'p} + 1 \leq h \leq p_a^{'p}$, the commutator

$$[e_{(j,p_1)(j,p_1-1)}, e_{(a,p_a^{'p})(b,h)}] = \delta_{aj}\delta_{p_a^{'p},p_1-1}e_{(j,p_1)(b,h)} - \delta_{bj}\delta_{h,p_1}e_{(a,p_a^{'p})(j,p_1-1)}$$

also vanishes since $\delta_{h,p_1} = 0$, and since for $a = j \leq t_1$ we must also have $b \leq t_1$ and therefore $p_1 \leq h \leq p_1 - 1$ is not possible. Finally, for (iii), note that by equation (3.1.8), for $1 \leq j \leq t_1$, we have

$$\sigma_l(f_{ji;k}^{'p}) = \sigma_l\big(\sum_{h=0}^{k} e_{(j,p_1-1+h-k)(i,h+1)}\big)\,.$$

Since $p_1 - 1 + h - k \leq p_1 - 1 = p_j - 1$, we clearly have $e_{(j,p_1-1+h-k)(i,h+1)} \in U^\perp$ for each $0 \leq h \leq k$, proving (iii). Note that similar properties hold in the case when $s_1 \leq t_1$ and the pyramid p' exists.

Next, let us start by considering the case $j \geq t_1 + 1$. By the recursive definition of $W_{ij}(z)$, we get,

$$\mathrm{gr}_{\Delta_{ji}}\big(\widetilde{W}_{ij}(z)\big) = \mathrm{gr}_{\Delta_{ji}}\big(\sigma_l(\widetilde{W}_{ij}^{'p}(z))\big) = \sigma_l\big(\mathrm{gr}_{\Delta_{ji}^{'p}}\big(\widetilde{W}_{ij}^{'p}(z)\big)\big)\,, \tag{3.3.26}$$

since $\Delta_{ji} = \Delta_{ji}^{'p}$ for $j \geq t_1 + 1$ and, moreover, $\mathrm{gr}_{\Delta_{ji}} \circ \sigma_l = \sigma_l \circ \mathrm{gr}_{\Delta_{ji}^{'p}}$, due to the fact that the grading on $\mathfrak{g}^{'p}$ is compatible with that of $\mathfrak{g}$ (and therefore the same holds for the corresponding Kazhdan filtrations).

By the induction hypothesis, we have

$$\mathrm{gr}_{\Delta_{ji}^{'p}}\big(\widetilde{W}_{ij}^{'p}(z)\big) = -(-z)^{p_j}\delta_{ij} + f_{ji}^{'p}(z) + Y_{ij}^{'p}(z)\,,$$

where $Y_{ij}^{'p}(z)$ is an element of $S(\mathfrak{g}^{'p})U^{'p\perp}[z]$ with (extended) Kazhdan degree equal to $\Delta_{ji}^{'p}$. Hence, by (3.3.26), we have

$$\begin{aligned}
&\eta^f\big(\mathrm{gr}_{\Delta_{ji}}\big(\widetilde{W}_{ij}(z)\big)\big) = \eta^f\big(\sigma_l\big(\mathrm{gr}_{\Delta_{ji}^{'p}}\big(\widetilde{W}_{ij}^{'p}(z)\big)\big)\big) = -(-z)^{p_j}\delta_{ij} + \eta^f\big(\sigma_l\big(f_{ji}^{'p}(z) + Y_{ij}^{'p}(z)\big)\big)\\
&= -(-z)^{p_j}\delta_{ij} + \eta^f\big(\sigma_l\big(\eta^{f^{'p}}\big(f_{ji}^{'p}(z) + Y_{ij}^{'p}(z)\big)\big)\big) = -(-z)^{p_j}\delta_{ij} + \eta^f\big(\sigma_l\big(f_{ji}^{'p}(z)\big)\big)\\
&= -(-z)^{p_j}\delta_{ij} + f_{ji}(z)\,.
\end{aligned}$$

The third equality is due to the fact that, thanks to property (i) of $U^\perp$, when restricting to $U(\mathfrak{g}^{'p})$ we have

$$\eta^f \circ \sigma_l = \eta^f \circ \sigma_l \circ \eta^{f^{'p}}\,. \tag{3.3.27}$$

Moreover, in the fourth equality we used the induction hypothesis and in the fifth equality we used Lemma 3.3.4. Hence, equation (3.3.25) holds in the case when $j \geq t_1 + 1$.

Next, let us consider the case $j \leq t_1$, By the recursive definition of $W_{ij}(z)$, we get

$$\begin{aligned}
&\mathrm{gr}_{\Delta_{ji}}\big(\widetilde{W}_{ij}(z)\big) = \mathrm{gr}_{\Delta_{ji}}\big([e_{(j,p_1)(j,p_1-1)}, \sigma_l(\widetilde{W}_{ij}^{'p}(z))]\big) - \sum_{h=1}^{t_1}\mathrm{gr}_{\Delta_{ji}}\big(\sigma_l(\widetilde{W}_{ih}^{'p}(z))(\delta_{jh}z + \tilde{e}_{(j,p_1)(h,p_1)})\big)\\
&+ \sum_{h\geq t_1+1}\mathrm{gr}_{\Delta_{ji}}\big(\sigma_l(\widetilde{W}_{ih}^{'p}(z))\sigma_l(\widetilde{W}_{hj;p_h-1}^{'p})\big)\\
&= [e_{(j,p_1)(j,p_1-1)}, \sigma_l\big(\mathrm{gr}_{\Delta_{ji}^{'p}}\big(\widetilde{W}_{ij}^{'p}(z)\big)\big)] - \sum_{h=1}^{t_1}\sigma_l\big(\mathrm{gr}_{\Delta_{hi}^{'p}}\big(\widetilde{W}_{ih}^{'p}(z)\big)\big)(\delta_{jh}z + e_{(j,p_1)(h,p_1)})\\
&+ \sum_{h\geq t_1+1}\sigma_l\big(\mathrm{gr}_{\Delta_{hi}^{'p}}\big(\widetilde{W}_{ih}^{'p}(z)\big)\big)\sigma_l\big(\mathrm{gr}_{\Delta_{jh;p_h-1}^{'p}}\big(\widetilde{W}_{hj;p_h-1}^{'p}\big)\big)
\end{aligned} \tag{3.3.28}$$

Here, we have used the properties (1.3.9) and (1.3.10) of the (extended) Kazhdan filtration, together with (3.3.19) and the facts that $\Delta'^{p}_{hi} = \Delta_{hi}-1$ for $h \leq t_1$, while $\Delta'^{p}_{hi} = \Delta_{hi}$ and $\Delta'^{p}_{hi}+\Delta'^{p}_{jh;p_h-1} = s_{1j}-s_{1i}+p_j = \Delta_{ji}$ for $h \geq t_1+1$.

By the induction hypothesis, for every $1 \leq i,j \leq r$, we have

$$\operatorname{gr}_{\Delta'^{p}_{ji}}\big(\widetilde{W}'^{p}_{ij}(z)\big) = -(-z)^{p'^{p}_{j}}\delta_{ij} + f'^{p}_{ji}(z) + Y'^{p}_{ij}(z)\,, \tag{3.3.29}$$

and moreover, for $h \geq t_1+1$, we have

$$\operatorname{gr}_{\Delta'^{p}_{jh;p_h-1}}\big(\widetilde{W}'^{p}_{hj;p_h-1}\big) = f'^{p}_{jh;p_h-1} + Y'^{p}_{hj;p_h-1}\,, \tag{3.3.30}$$

where $Y'^{p}_{ij}(z)$ is an element of $S(\mathfrak{g}'^{p})U'^{p\perp}[z]$ with (extended) Kazhdan degree equal to Δ'^{p}_{ji}, and $Y'^{p}_{hj;p_h-1}$ is an element of $S(\mathfrak{g}'^{p})U'^{p\perp}$ with Kazhdan degree equal to $\Delta'^{p}_{jh;p_h-1}$. Hence, by (3.3.28) we have

$$\begin{aligned}
&\eta^f\big(\operatorname{gr}_{\Delta_{ji}}\big(\widetilde{W}_{ij}(z)\big)\big) = \eta^f\big([e_{(j,p_1)(j,p_1-1)},\sigma_l\big(\operatorname{gr}_{\Delta'^{p}_{ji}}\big(\widetilde{W}'^{p}_{ij}(z)\big)\big)]\big)\\
&-\sum_{h=1}^{t_1}\eta^f\big(\sigma_l\big(\operatorname{gr}_{\Delta'^{p}_{hi}}\big(\widetilde{W}'^{p}_{ih}(z)\big)\big)(\delta_{jh}z+e_{(j,p_1)(h,p_1)})\big)\\
&+\sum_{h\geq t_1+1}\eta^f\big(\sigma_l\big(\operatorname{gr}_{\Delta'^{p}_{hi}}\big(\widetilde{W}'^{p}_{ih}(z)\big)\big)\sigma_l\big(\operatorname{gr}_{\Delta'^{p}_{jh;p_h-1}}\big(\widetilde{W}'^{p}_{hj;p_h-1}\big)\big)\big)\\
&=\eta^f\big([e_{(j,p_1)(j,p_1-1)},\sigma_l\big(f'^{p}_{ji}(z)+Y'^{p}_{ij}(z)\big)]\big)-\delta_{ij}(-z)^{p_1}+\delta_{i\leq t_1}(-z)^{p_1-1}e_{(j,p_1)(i,p_1)}\\
&-\sum_{h=1}^{t_1}\eta^f\big(\sigma_l\big(f'^{p}_{hi}(z)+Y'^{p}_{ih}(z)\big)(\delta_{jh}z+e_{(j,p_1)(h,p_1)})\big)\\
&-\delta_{i\geq t_1+1}(-z)^{p_i}\sigma_l\big(f'^{p}_{ji;p_h-1}+Y'^{p}_{ij;p_h-1}\big)+\sum_{h\geq t_1+1}\eta^f\big(\sigma_l\big(f'^{p}_{hi}(z)+Y'^{p}_{ih}(z)\big)\sigma_l\big(f'^{p}_{jh;p_h-1}+Y'^{p}_{hj;p_h-1}\big)\big)
\end{aligned} \tag{3.3.31}$$

Here, for the second equality we have used the induction hypotheses (3.3.29) and (3.3.30). By Lemma 3.3.4 and equation (3.3.27), we can rewrite the RHS of (3.3.31) as

$$\begin{aligned}
&-\delta_{ij}(-z)^{p_1}+f_{ji}(z)+\eta^f\big([e_{(j,p_1)(j,p_1-1)},\sigma_l\big(Y'^{p}_{ij}(z)\big)]\big)-\sum_{h=1}^{t_1}\eta^f\big(\sigma_l\big(f'^{p}_{hi}(z)\big)e_{(j,p_1)(h,p_1)}\big)\\
&+\sum_{h\geq t_1+1}\eta^f\big(\sigma_l\big(f'^{p}_{hi}(z)\big)\sigma_l\big(f'^{p}_{jh;p_h-1}\big)\big)\\
&=-\delta_{ij}(-z)^{p_1}+f_{ji}(z)\,,
\end{aligned} \tag{3.3.32}$$

where for the last equality we have used properties (ii) and (iii) of $U^{\perp}$. Hence, equation (3.3.25) holds also in the case $j \leq t_1$, thus completing the proof of Proposition 3.3.1. $\square$

Remark 3.3.4. Given a pyramid p, the choice of a subspace $U \subset \mathfrak{g}$ complementary to $[f,\mathfrak{g}]$ (and therefore of its orthocomplement $U^{\perp}$) is not unique. We can for instance choose, as in [DSKV16b],

$$U = \operatorname{Span}_{\mathbb{C}}\{e_{(j,1)(i,p_i-k)} \mid 1 \leq i,j \leq r, 0 \leq k \leq \min(p_i,p_j)-1\}\,,$$

and its orthocomplement

$$\begin{aligned}
U^{\perp} = {}&\operatorname{Span}_{\mathbb{C}}\{e_{(j,k)(i,h)} \mid 1 \leq i,j \leq r\,, 1 \leq k \leq p_j\,, 2 \leq h \leq p_i\}\\
&\oplus \operatorname{Span}_{\mathbb{C}}\{e_{(j,k)(i,1)} \mid 1 \leq j < i \leq r\,, 1 \leq k \leq p_j - p_i\}\,.
\end{aligned}$$

Although the properties (ii) and (iii) of $U^{\perp}$ above fail for this choice of $U^{\perp}$, we can still prove Proposition 3.3.1, through some similar properties that do hold instead.

3.3.3 Remarks on the recursion when the pyramid p is not left/right-aligned

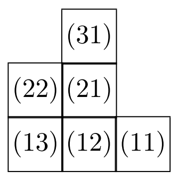
Since Definition 3.2.1 makes sense for a generic even grading Γ, but Theorems 3.3.1 and 3.3.2 only work when the associated pyramid is either left or right-aligned, we shall now address the issue of understanding what happens when the pyramid p is still even but no longer left/right-aligned, and what are the limits of our approach.

(i) The assumption of p being right-aligned in Theorem 3.3.1 (resp. left-aligned in Theorem 3.3.2) makes computations significantly easier. It in fact allows us to apply the same recursion multiple times, consequently obtaining nice formulas to work with. This is especially advantageous to show the vanishing of some commutators, or to deal with iterating procedures as in the case of (3.3.6) (resp. (3.3.12) in Theorem 3.3.2). In the case of a general even pyramid p this is no longer possible, making computations definitely harder.

(ii) We therefore wonder whether it is possible to use the recursions arbitrarily, namely proceeding by arbitrarily removing the leftmost or the rightmost column of the pyramid, despite of which one is the shortest.

This is not possible.

Example 3.3.1 (Counterexample)**.** Let $(3,2,1)$ be a partition of 6 and suppose $t_1 = 2, r = 3, s_1 = 1$. Namely, we are considering the following pyramid:



Applying the recursions in the correct order (namely, the (3.2.3) first and then (3.2.2)), we get

$$\begin{aligned}\widetilde{W}_{11}(z) =& e_{(13)(11)} - e_{(12)(11)}(z+\tilde{e}_{(13)(13)}) - e_{(21)(11)}e_{(13)(22)} + e_{(31)(11)}e_{(12)(31)} \\ &- (z+\tilde{e}_{(11)(11)})(e_{(13)(12)} - (z+\tilde{e}_{(12)(12)})(z+\tilde{e}_{(13)(13)}) - e_{(21)(12)}e_{(13)(22)} \\ &+ e_{(31)(12)}e_{(12)(31)}) + e_{(31)(12)}(e_{(13)(31)} - e_{(12)(31)}(z+\tilde{e}_{(13)(13)}) \\ &- e_{(21)(31)}e_{(13)(22)} + (z+\tilde{e}_{(31)(31)})e_{(12)(31)}) + (e_{(22)(13)} + e_{(21)(12)})(e_{(13)(21)} \\ &- e_{(12)(21)}(z+\tilde{e}_{(13)(13)}) - (z+\tilde{e}_{(21)(21)})e_{(13)(22)} + e_{(31)(21)}e_{(12)(31)}) \,.\end{aligned}$$

Applying necessary changes to (3.2.2), in order for it to keep making sense, we get

$$\overline{W}_{ij}(z) = \begin{cases} \sigma_l(\overline{W}'^{p}_{ij}(z))\,, & j \geq t_1+1 \\ [e_{(j,p_1-s_{1j})(j,p_1-1-s_{1j})}, \sigma_l(\overline{W}'^{p}_{ij}(z))] - \displaystyle\sum_{h=1}^{t_1} \sigma_l(\overline{W}'^{p}_{ih}(z))(\delta_{ij}z + \tilde{e}_{(j,p_1-s_{1j})(h,p_1-s_{1h})}) & \\ \qquad + \displaystyle\sum_{h=t_1+1}^{r} \sigma_l(\overline{W}'^{p}_{ih}(z))\sigma_l(\overline{W}'^{p}_{hj,p_h-1}(z))\,, & j \leq t_1\,, \end{cases}$$

and then proceeding by applying the recursions in the reverse order, we get

$$\begin{aligned}\overline{W}_{11}(z) =& e_{(13)(11)} - (z+\tilde{e}_{(11)(11)})e_{(13)(12)} + e_{(21)(12)}e_{(13)(21)} + e_{(31)(12)}e_{(13)(31)} \\ &- (e_{(12)(11)} - (z+\tilde{e}_{(11)(11)})(z+\tilde{e}_{(12)(12)}) + e_{(21)(12)}e_{(12)(21)} \\ &+ e_{(31)(12)}e_{(12)(31)})(z+\tilde{e}_{(13)(13)}) - (e_{(21)(11)} - (z+\tilde{e}_{(11)(11)})e_{(21)(12)} \\ &+ e_{(21)(12)}(z+\tilde{e}_{(21)(21)}) + e_{(31)(12)}e_{(21)(31)})e_{(13)(22)} + (e_{(31)(11)} \\ &- (z+\tilde{e}_{(11)(11)})e_{(31)(12)} + e_{(21)(12)}e_{(31)(21)} + e_{(31)(12)}(z+\tilde{e}_{(31)(31)}))e_{(12)(31)}\,.\end{aligned}$$

Then,

$$\widetilde{W}_{11}(z) = \overline{W}_{11}(z) + e_{(22)(13)}(e_{(13)(21)} - e_{(12)(21)}(z+\tilde{e}_{(13)(13)}) - (z+\tilde{e}_{(21)(21)})e_{(13)(22)} + e_{(31)(21)}e_{(12)(31)}) \,.$$

A direct computation shows that $\overline{W}_{11}(z)\bar{1}_0 \notin W(\mathfrak{g}, f, \Gamma, 0)$, since for instance for $e_{(12)(22)} \in \mathfrak{g}_{\geq 1}$ we have

$$\rho([e_{(12)(22)}, \overline{W}_{11}(z)]) = -e_{(13)(21)} + e_{(12)(21)}(z+\tilde{e}_{(13)(13)}) + (z+\tilde{e}_{(21)(21)})e_{(13)(22)} - e_{(31)(21)}e_{(12)(31)} \neq 0 \,.$$

(iii) We have however some positive results. When $p_1 = 3$ and the pyramid p is even but not aligned, we can prove by direct computation that $W_{ij}(z) \in W(\mathfrak{g}, f, \Gamma, 0)$ using the recursions from Definition 3.2.1 in the right order. We just need to write down the explicit expression for $W_{ij}(z)$ in the four possible cases (matching $j \leq t_1$, $j \geq t_1 + 1$ with $i \leq s_1$, $i \geq s_1 + 1$) and then apply the map ρ to the commutator with elements of $\mathfrak{g}_1$:

$$\begin{aligned} &e_{(a,1+\delta_{a\leq s_1})(b,3)}, \qquad 1 \leq a \leq r,\, 1 \leq b \leq t_1 \,, \\ &e_{(b,1)(a,1+\delta_{a\leq s_1})}, \qquad 1 \leq a \leq r,\, 1 \leq b \leq s_1 \,. \end{aligned}$$

(iv) When $p_1 > 3$, we have analogously tried a direct approach, which combines the left and right recursion in the correct order (assuming first that the shortest column is the leftmost). It is possible in some special cases, depending on the range of the first index i, to produce a proof similar to the one of Theorem 3.3.1. However, even in these cases we have the issues with the analogue of the identity (3.3.6).

3.4 A complete example

Let us now illustrate all the concepts and properties introduced so far through the example of $W(\mathfrak{gl}_N, f, \Gamma, \mathfrak{l})$ when $N = 3$ with partition $(2, 1)$, namely when the f is a minimal nilpotent element. Associated with this partition we have three possible pyramids/gradings:

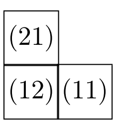


Figure 3.1: Γ_1

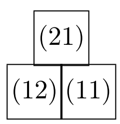


Figure 3.2: Γ_2

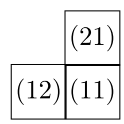


Figure 3.3: Γ_3

The matrices encoding the different gradings $\Gamma_i = \sum_{(i,h),(j,k)\in\mathcal{T}} \left(\deg_{\Gamma_i} e_{(j,k)(i,h)}\right) E_{(i,h)(j,k)}$ are

$$\Gamma_1 = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 0 & -1 & -\frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}, \qquad \Gamma_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

Figure 3.4: Matrices encoding the gradings Γ_i, $i = 1, 2, 3$.

while the corresponding shift matrices $D_i = \sum_{(i,h)\in\mathcal{T}} d_{(i,h)} E_{(i,h)(i,h)}$ are

$$D_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad D_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The nilpotent element associated with $(2, 1)$ is $f = e_{(1,2)(1,1)}$ and we can compute explicitly its centralizer

$$\begin{aligned} \mathfrak{g}^f &= \mathrm{Span}_{\mathbb{C}}\langle e_{(1,2)(1,1)}, e_{(1,1)(1,1)} + e_{(1,2)(1,2)}, e_{(1,2)(2,1)}, e_{(2,1)(1,1)}, e_{(2,1)(2,1)}\rangle \\ &\underset{\text{cf.(3.1.8)}}{=} \mathrm{Span}_{\mathbb{C}}\langle f_{11;0}, f_{11;1}, f_{12;0}, f_{21;0}, f_{22;0}\rangle \,. \end{aligned} \tag{3.4.1}$$

According to the three different gradings we have (choosing $\mathfrak{l}=0$ for Γ_2)

$$z\mathbb{1}_3+F+\pi^{\Gamma_1}_{\leq 0}E+D_1=\begin{pmatrix} z+e_{(1,1)(1,1)} & e_{(1,2)(1,1)} & e_{(2,1)(1,1)} \\ 1 & z+e_{(1,2)(1,2)}-1 & e_{(2,1)(1,2)} \\ 0 & e_{(1,2)(2,1)} & z+e_{(2,1)(2,1)}-1 \end{pmatrix},$$

$$z\mathbb{1}_3+F+\pi^{\Gamma_2}_{\leq \frac{1}{2}}E+D_2=\begin{pmatrix} z+e_{(1,1)(1,1)} & e_{(1,2)(1,1)} & e_{(2,1)(1,1)} \\ 1 & z+e_{(1,2)(1,2)}-1 & e_{(2,1)(1,2)} \\ e_{(1,1)(2,1)} & e_{(1,2)(2,1)} & z+e_{(2,1)(2,1)} \end{pmatrix},$$

$$z\mathbb{1}_3+F+\pi^{\Gamma_3}_{\leq 0}E+D_3=\begin{pmatrix} z+e_{(1,1)(1,1)} & e_{(1,2)(1,1)} & e_{(2,1)(1,1)} \\ 1 & z+e_{(1,2)(1,2)}-2 & 0 \\ e_{(1,1)(2,1)} & e_{(1,2)(2,1)} & z+e_{(2,1)(2,1)} \end{pmatrix}.$$

We compute the quasideterminants $L_k(z)=|z\mathbb{1}_3+F+\pi^{\Gamma_k}_{\leq\frac{1}{2}}E+D_k|_{12}\,\bar{1}_{\Gamma_k}$, where $\bar{1}_{\Gamma_k}$ is the image of $1\in U(\mathfrak{g})$ in the quotient $U(\mathfrak{g})/U(\mathfrak{g})\langle b-(f|b)\rangle_{b\in\mathfrak{g}^{\Gamma_k}_{\geq 1}}$ and obtain:

$$\begin{aligned}
L_1(z)&=e_{(1,2)(1,1)}\bar{1}_{\Gamma_1}-\big(z+e_{(1,1)(1,1)}\quad e_{(2,1)(1,1)}\big)\begin{pmatrix}1 & e_{(2,1)(1,2)}\\ 0 & z+e_{(2,1)(2,1)}-1\end{pmatrix}^{-1}\begin{pmatrix}z+e_{(1,2)(1,2)}-1\\ e_{(1,2)(2,1)}\end{pmatrix}\bar{1}_{\Gamma_1}\\
&=e_{(1,2)(1,1)}\bar{1}_{\Gamma_1}-(z+e_{(1,1)(1,1)})(z+e_{(1,2)(1,2)}-1)\bar{1}_{\Gamma_1}\\
&-\big(e_{(2,1)(1,1)}-(z+e_{(1,1)(1,1)})e_{(2,1)(1,2)}\big)(z+e_{(2,1)(2,1)}-1)^{-1}e_{(1,2)(2,1)}\bar{1}_{\Gamma_1}\\
&=W^1_{11}(z)-W^1_{12}(z)\big(W^1_{22}(z)\big)^{-1}W^1_{21}(z),
\end{aligned}$$

$$\begin{aligned}
L_2(z)&=e_{(1,2)(1,1)}\bar{1}_{\Gamma_2}-\big(z+e_{(1,1)(1,1)}\quad e_{(2,1)(1,1)}\big)\begin{pmatrix}1 & e_{(2,1)(1,2)}\\ e_{(1,1)(2,1)} & z+e_{(2,1)(2,1)}\end{pmatrix}^{-1}\begin{pmatrix}z+e_{(1,2)(1,2)}-1\\ e_{(1,2)(2,1)}\end{pmatrix}\bar{1}_{\Gamma_2}\\
&=e_{(1,2)(1,1)}\bar{1}_{\Gamma_2}-(z+e_{(1,1)(1,1)})(z+e_{(1,2)(1,2)}-1)\bar{1}_{\Gamma_2}-\big((z+e_{(1,1)(1,1)})e_{(2,1)(1,2)}-e_{(2,1)(1,1)}\big)\\
&\times\big((z+e_{(2,1)(2,1)})-e_{(1,1)(2,1)}e_{(2,1)(1,2)}\big)^{-1}\big(e_{(1,1)(2,1)}(z+e_{(1,2)(1,2)}-1)-e_{(1,2)(2,1)}\big)\bar{1}_{\Gamma_2}\\
&=W^2_{11}(z)-W^2_{12}(z)\big(W^2_{22}(z)\big)^{-1}W^2_{21}(z),
\end{aligned}$$

and

$$\begin{aligned}
L_3(z)&=e_{(1,2)(1,1)}\bar{1}_{\Gamma_3}-\big(z+e_{(1,1)(1,1)}\quad e_{(2,1)(1,1)}\big)\begin{pmatrix}1 & 0\\ e_{(1,1)(2,1)} & z+e_{(2,1)(2,1)}\end{pmatrix}^{-1}\begin{pmatrix}z+e_{(1,2)(1,2)}-2\\ e_{(1,2)(2,1)}\end{pmatrix}\bar{1}_{\Gamma_3}\\
&=e_{(1,2)(1,1)}\bar{1}_{\Gamma_3}-(z+e_{(1,1)(1,1)})(z+e_{(1,2)(1,2)}-2)\bar{1}_{\Gamma_3}\\
&-e_{(2,1)(1,1)}(z+e_{(2,1)(2,1)})^{-1}\big(e_{(1,2)(2,1)}-e_{(1,1)(2,1)}(z+e_{(1,2)(1,2)}-2)\big)\bar{1}_{\Gamma_3}\\
&=W^3_{11}(z)-W^3_{12}(z)\big(W^3_{22}(z)\big)^{-1}W^3_{21}(z),
\end{aligned}$$

where $W^k_{ij}(z)=\widetilde{W}^k_{ij}(z)\bar{1}_{\Gamma_k}$, with $1\leq i,j,\leq 2$, are the generators for the corresponding W-algebras $W(\mathfrak{g},f,\Gamma_k,0)$ below. For $W(\mathfrak{g},f,\Gamma_1,0)$:

$$\begin{aligned}
\widetilde{W}^1_{11}(z)&=-z^2-z(\boldsymbol{e_{(1,1)(1,1)}+e_{(1,2)(1,2)}}-1)+\boldsymbol{e_{(1,2)(1,1)}}-e_{(1,1)(1,1)}(e_{(1,2)(1,2)}-1)+e_{(2,1)(1,2)}e_{(1,2)(2,1)}\\
\widetilde{W}^1_{12}(z)&=\boldsymbol{e_{(2,1)(1,1)}}-e_{(1,1)(1,1)}e_{(2,1)(1,2)}+e_{(2,1)(1,2)}(e_{(2,1)(2,1)}-1)\\
\widetilde{W}^1_{21}(z)&=\boldsymbol{e_{(1,2)(2,1)}}\\
\widetilde{W}^1_{22}(z)&=z+\boldsymbol{e_{(2,1)(2,1)}}-1\,.
\end{aligned}\tag{3.4.2}$$

For $W(\mathfrak{g}, f, \Gamma_2, 0)$:

$$\begin{aligned}
\widetilde{W}^2_{11}(z) &= -z^2 - z(\boldsymbol{e_{(1,1)(1,1)}} + \boldsymbol{e_{(1,2)(1,2)}} + e_{(2,1)(1,2)}e_{(1,1)(2,1)} - 1) \\
&\quad + \boldsymbol{e_{(1,2)(1,1)}} - e_{(1,1)(1,1)}(e_{(1,2)(1,2)} - 1) + e_{(2,1)(1,2)}\widetilde{W}^2_{21}(z) + \widetilde{W}^2_{12}(z)e_{(1,1)(2,1)} - e_{(2,1)(1,2)}\widetilde{W}^2_{22;0}e_{(1,1)(2,1)} \\
\widetilde{W}^2_{12}(z) &= \boldsymbol{e_{(2,1)(1,1)}} - e_{(1,1)(1,1)}e_{(2,1)(1,2)} + e_{(2,1)(1,2)}\widetilde{W}^2_{22;0} \\
\widetilde{W}^2_{21}(z) &= \boldsymbol{e_{(1,2)(2,1)}} - e_{(1,1)(2,1)}(e_{(1,2)(1,2)} - 1) + \widetilde{W}^2_{22;0}e_{(1,1)(2,1)} \\
\widetilde{W}^2_{22}(z) &= z + \boldsymbol{e_{(2,1)(2,1)}} - e_{(1,1)(2,1)}e_{(2,1)(1,2)} = z + \widetilde{W}^2_{22;0}\,.
\end{aligned} \tag{3.4.3}$$

And finally, for $W(\mathfrak{g}, f, \Gamma_3, 0)$:

$$\begin{aligned}
\widetilde{W}^3_{11}(z) &= -z^2 - z(\boldsymbol{e_{(1,1)(1,1)}} + \boldsymbol{e_{(1,2)(1,2)}} - 2) + \boldsymbol{e_{(1,2)(1,1)}} - e_{(1,1)(1,1)}(e_{(1,2)(1,2)} - 2) + e_{(2,1)(1,1)}e_{(1,1)(2,1)} \\
\widetilde{W}^3_{12}(z) &= \boldsymbol{e_{(2,1)(1,1)}} \\
\widetilde{W}^3_{21}(z) &= \boldsymbol{e_{(1,2)(2,1)}} - e_{(1,1)(2,1)}(e_{(1,2)(1,2)} - 2) + e_{(2,1)(2,1)}e_{(1,1)(2,1)} \\
\widetilde{W}^3_{22}(z) &= z + \boldsymbol{e_{(2,1)(2,1)}}\,.
\end{aligned} \tag{3.4.4}$$

In bold we have shown the dependence of the generators on the element of the centralizer $\mathfrak{g}^f$ in (3.4.1). The generators for $W(\mathfrak{g}, f, \Gamma_1, 0)$ (resp. for $W(\mathfrak{g}, f, \Gamma_3, 0)$) can be equivalently computed using the recursion (3.2.3) (resp. using the recursion (3.2.2)). However, we would need to show directly that the polynomials $W^2_{ij}(z)$ belong to $W(\mathfrak{g}, f, \Gamma_2, 0)$. In [DSKV16c, Section 9.3] explicit formulas are described for the generators of the W-algebras for a generic minimal nilpotent f and a Dynkin grading. This agrees with Conjecture 3.1.1.

Note moreover that from Figure 3.4 above it is clear that for the choice of $\mathfrak{l}_1 = \mathbb{C}e_{(1,1)(2,1)}$ and $\mathfrak{l}_3 = \mathbb{C}e_{(2,1)(1,2)}$ we have $W(\mathfrak{g}, f, \Gamma_2, \mathfrak{l}_1) = W(\mathfrak{g}, f, \Gamma_1, 0)$ and $W(\mathfrak{g}, f, \Gamma_2, \mathfrak{l}_3) = W(\mathfrak{g}, f, \Gamma_3, 0)$. This is an instance of a more general result, which we will explain in Chapter 4 (see Lemma 4.2.2).

A straightforward computation shows that under the maps

$$\begin{aligned}
\rho_{0,\mathfrak{l}_1} &: W(\mathfrak{g}, f, \Gamma_2, 0) \longrightarrow W(\mathfrak{g}, f, \Gamma_2, \mathfrak{l}_1) = W(\mathfrak{g}, f, \Gamma_1, 0) \\
\rho_{0,\mathfrak{l}_3} &: W(\mathfrak{g}, f, \Gamma_2, 0) \longrightarrow W(\mathfrak{g}, f, \Gamma_2, \mathfrak{l}_3) = W(\mathfrak{g}, f, \Gamma_3, 0)
\end{aligned}$$

the following holds

$$\begin{aligned}
\rho_{0,\mathfrak{l}_1}(W^2_{11;1}) &= W^1_{11;1}\,, & \rho_{0,\mathfrak{l}_3}(W^2_{11;1}) &= W^3_{11;1}\,, \\
\rho_{0,\mathfrak{l}_1}(W^2_{11;0}) &= W^1_{11;0}\,, & \rho_{0,\mathfrak{l}_3}(W^2_{11;0}) &= W^3_{11;0}\,, \\
\rho_{0,\mathfrak{l}_1}(W^2_{12;0}) &= W^1_{12;0}\,, & \rho_{0,\mathfrak{l}_3}(W^2_{12;0}) &= W^3_{12;0}\,, \\
\rho_{0,\mathfrak{l}_1}(W^2_{21;0}) &= W^1_{21;0}\,, & \rho_{0,\mathfrak{l}_3}(W^2_{21;0}) &= W^3_{21;0}\,, \\
\rho_{0,\mathfrak{l}_1}(W^2_{22;0}) &= W^1_{22;0}\,, & \rho_{0,\mathfrak{l}_3}(W^2_{22;0}) &= W^3_{22;0}\,.
\end{aligned}$$

As a consequence, $\rho_{0,\mathfrak{l}_1}(L_2(z)) = L_1(z)$ and $\rho_{0,\mathfrak{l}_3}(L_2(z)) = L_3(z)$. Again, this is an instance of a more general result that we will introduce in Chapter 4 (see Theorem 4.4.2).

Finally, we can compute commutators between the generators directly using (3.4.2) - (3.4.3). For $k = 1, 2, 3$ we obtain:

$$\begin{aligned}
[W_{22;0}, W_{12;0}] &= W_{12;0}\,, \\
[W_{22;0}, W_{21;0}] &= -W_{21;0}\,, \\
[W_{12;0}, W_{21;0}] &= -W_{11;0} - W_{22;0}W_{11;1} + W_{22;0}W_{22;0}\,, \\
[W_{12;0}, W_{11;1}] &= W_{12;0}\,, \\
[W_{12;0}, W_{11;0}] &= -W_{12;0}W_{11;1} + W_{12;0}W_{22;0}\,, \\
[W_{21;0}, W_{11;1}] &= -W_{21;0}\,, \\
[W_{21;0}, W_{11;0}] &= -W_{22;0}W_{21;0} + W_{11;1}W_{21;0}\,.
\end{aligned}$$

Chapter 4

About Conjecture 3.1.1

4.1 The case of a left/right aligned pyramid

Theorem 4.1.1. *Suppose that the pyramid p is either left or right-aligned. Then, for the matrix $W(z) \in Mat_{r\times r}W(\mathfrak{g},f,\Gamma,0)[z]$ the quasideterminant $|W(z)|_{I_{rr_1}J_{r_1r}}$, where I_{rr_1}, J_{r_1r} are as in (2.2.6), exists and the following identity holds:*

$$|W(z)|_{I_{rr_1}J_{r_1r}} = L(z) \in Mat_{r_1\times r_1}W(\mathfrak{g},f,\Gamma,0)((z^{-1}))\,. \tag{4.1.1}$$

Proof. First observe that since $W(z) \in \mathrm{Mat}_{r\times r}\, W(\mathfrak{g},f,\Gamma,0)[z]$ (Theorems 3.3.1 and 3.3.2), we have

$$|W(z)|_{I_{rr_1}J_{r_1r}} = |\widetilde{W}(z)|_{I_{rr_1}J_{r_1r}}\bar{1}_{\mathfrak{l}}\,.$$

Thus, in order to prove equation (4.1.1), it is sufficient to prove that

$$|\widetilde{W}(z)|_{I_{rr_1}J_{r_1r}} = \widetilde{L}(z) \in \mathrm{Mat}_{r_1\times r_1}U(\mathfrak{g})((z^{-1}))\,. \tag{4.1.2}$$

First, let us prove that the quasideterminant $|\widetilde{W}(z)|_{I_{rr_1}J_{r_1r}}$ exists. By Definition 1.5.2, it is sufficient to prove that the matrix $\widetilde{W}(z)_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}$ is invertible, where $\mathcal{I}_{rr_1}$ and $\mathcal{J}_{r_1r}$ are the index sets corresponding to the matrices I_{rr_1} and J_{r_1r}. By Proposition 3.2.3, we have

$$\widetilde{W}(z)_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c} = -(-z)^q\mathbb{1}_{r-r_1} + \Big(\sum_{k=0}^{\min(p_i,p_j)-1}\widetilde{W}_{ij;k}(-z)^k\Big)_{r_1<i,j\le r}\,, \tag{4.1.3}$$

where $(-z)^q\mathbb{1}_{r-r_1}$ is the $r-r_1\times r-r_1$ matrix

$$(-z)^q\mathbb{1}_{r-r_1} = \begin{pmatrix} (-z)^{\overline{p}_2}\mathbb{1}_{\overline{r}_2} & & \\ & \ddots & \\ & & (-z)^{\overline{p}_s}\mathbb{1}_{\overline{r}_s}\end{pmatrix}\,. \tag{4.1.4}$$

corresponding to the partition $q = (p_1 \ge \ldots \ge p_r) = (\overline{p}_1^{\overline{r}_1} > \ldots > \overline{p}_s^{\overline{r}_s})$ (note that $\overline{r}_1 = r_1$). Then, $(-z)^q\mathbb{1}_{r-r_1}$ is clearly invertible with inverse $(-z)^{-q}\mathbb{1}_{r-r_1} \in \mathrm{Mat}_{r-r_1,r-r_1}\left(\mathbb{C}[z^{-1}]\right)$ and we can rewrite

$$\widetilde{W}(z)_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c} = -(-z)^q\mathbb{1}_{r-r_1}\Big(\mathbb{1}_{r-r_1} - \Big(\sum_{k=0}^{\min(p_i,p_j)-1}\widetilde{W}_{ij;k}(-z)^{-p_i+k}\Big)_{r_1<i,j\le r}\Big)\,.$$

Note that the matrix $\left(\sum_{k=0}^{\min(p_i,p_j)-1}\widetilde{W}_{ij;k}(-z)^{-p_i+k}\right)_{r_1<i,j\le r}$ lies in $\mathrm{Mat}_{r-r_1,r-r_1}\left(U(\mathfrak{g})[z^{-1}]\right)$. Therefore, $\mathbb{1}_{r-r_1} - \left(\sum_{k=0}^{\min(p_i,p_j)-1}\widetilde{W}_{ij;k}(-z)^{-p_i+k}\right)_{r_1<i,j\le r}$ is invertible in $\mathrm{Mat}_{r-r_1,r-r_1}\left(U(\mathfrak{g})[[z^{-1}]]\right)$, and its inverse

can be computed by geometric series expansion. Thus, $\widetilde{W}(z)_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}$ is invertible and the quasideterminant $|\widetilde{W}(z)|_{I_{rr_1}J_{r_1r}}$ exists.

Next, let us prove equation (4.1.2). We may assume that $r_1 = t_1 \leq s_1$. We proceed by induction on the number of columns. For the base case $p_1 = 1$, we have $|\widetilde{W}(z)|_{I_{rr_1}J_{r_1r}} = \widetilde{W}(z) = z\mathbb{1}_N + E = \widetilde{L}(z)$ (cf. Equation (2.2.3)), hence Equation (4.1.2) clearly holds. For the case when $p_1 > 1$ we will then prove that $(|\widetilde{W}(z)|_{I_{rr_1}J_{r_1r}})_{ij}$, $1 \leq i,j \leq r_1$, satisfies a recursive formula that coincides with the one proved for $\widetilde{L}_{ij}(z)$ in Proposition 2.2.2. Namely, we shall prove that, for every $1 \leq i,j \leq r_1$, we have

$$(|\widetilde{W}(z)|_{I_{rr_1}J_{r_1r}})_{ij} = [e_{(j,p_1)(j,p_1-1)}, \sigma_l(|\widetilde{W}'^p(z)|_{I_{rr_1}J_{r_1r}})_{ij}] - \sum_{h=1}^{r_1} \sigma_l(|\widetilde{W}'^p(z)|_{I_{rr_1}J_{r_1r}})_{ih}(\delta_{hj}z + \tilde{e}_{(j,p_1)(h,p_1)}). \tag{4.1.5}$$

Note that, by the hereditary property of quasideterminants, we have

$$|\widetilde{W}'^p(z)|_{I_{rr_1}J_{r_1r}} = ||\widetilde{W}'^p(z)|_{I_{rr_2}J_{r_2r}}|_{I_{r_2r_1}J_{r_1r_2}}\,, \tag{4.1.6}$$

where $|\widetilde{W}'^p(z)|_{I_{rr_2}J_{r_2r}}$ is the correct quasideterminant to consider when we restrict from p to the pyramid with its leftmost column removed $'p$, and $r_2 = t_2 \geq t_1$ is the height of the second shortest column of p. Hence, substituting (4.1.6) in (4.1.5), it is clear that (4.1.5) actually is a recursive formula, and that it moreover coincides with the recursive formula for $\widetilde{L}(z)$ in Proposition 2.2.2. Finally note that the same argument used for the quasideterminant $|\widetilde{W}(z)|_{I_{rr_1}J_{r_1r}}$ can be used to show that the quasideterminant $|\widetilde{W}'^p(z)|_{I_{rr_1}J_{r_1r}}$ exists.

Now, let us prove the recursive formula (4.1.5). By definition of quasideterminant[1] and by (3.2.3) we have

$$\begin{aligned}
(|\widetilde{W}(z)|_{I_{rr_1}J_{r_1r}})_{ij} &= \widetilde{W}_{ij}(z) - \sum_{\alpha,\beta=r_1+1}^{r} \widetilde{W}_{i\alpha}(z)((\widetilde{W}_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}(z))^{-1})_{\alpha\beta}\widetilde{W}_{\beta j}(z) \\
&= \widetilde{W}_{ij}(z) - \sum_{\alpha,\beta=r_1+1}^{r} \sigma_l(\widetilde{W}'^p_{i\alpha}(z))((\sigma_l(\widetilde{W}'^p_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}(z)))^{-1})_{\alpha\beta}\widetilde{W}_{\beta j}(z) \\
&= [e_{(j,p_1)(j,p_1-1)}, \sigma_l(\widetilde{W}'^p_{ij}(z))] - \sum_{h=1}^{r_1}\sigma_l(\widetilde{W}'^p_{ih}(z))(\delta_{hj}z + \tilde{e}_{(j,p_1)(h,p_1)}) + \sum_{h=r_1+1}^{r}\sigma_l(\widetilde{W}'^p_{ih}(z))\sigma_l(\widetilde{W}'^p_{hj,p_h-1}(z)) \\
&\quad - \sum_{\alpha,\beta=r_1+1}^{r} \sigma_l(\widetilde{W}'^p_{i\alpha}(z))((\sigma_l(\widetilde{W}'^p_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}(z)))^{-1})_{\alpha\beta}\Big([e_{(j,p_1)(j,p_1-1)}, \sigma_l(\widetilde{W}'^p_{\beta j}(z))] \\
&\quad - \sum_{h=1}^{r_1}\sigma_l(\widetilde{W}'^p_{\beta h}(z))(\delta_{hj}z + \tilde{e}_{(j,p_1)(h,p_1)}) + \sum_{h=r_1+1}^{r}\sigma_l(\widetilde{W}'^p_{\beta h}(z))\sigma_l(\widetilde{W}'^p_{hj,p_h-1}(z))\Big) \\
&= [e_{(j,p_1)(j,p_1-1)}, \sigma_l(\widetilde{W}'^p_{ij}(z))] - \sum_{\alpha,\beta=r_1+1}^{r} \sigma_l(\widetilde{W}'^p_{i\alpha}(z))((\sigma_l(\widetilde{W}'^p_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}(z)))^{-1})_{\alpha\beta}[e_{(j,p_1)(j,p_1-1)}, \sigma_l(\widetilde{W}'^p_{\beta j}(z))] \\
&\quad - \sum_{h=1}^{r_1}\Big(\sigma_l(\widetilde{W}'^p_{ih}(z)) + \sum_{\alpha,\beta=r_1+1}^{r} \sigma_l(\widetilde{W}'^p_{i\alpha}(z))((\sigma_l(\widetilde{W}'^p_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}(z)))^{-1})_{\alpha\beta}\sigma_l(\widetilde{W}'^p_{\beta h}(z))\Big)(\delta_{hj}z + \tilde{e}_{(j,p_1)(h,p_1)})\,,
\end{aligned} \tag{4.1.7}$$

where the last equality is due to the fact that $((\sigma_l(\widetilde{W}_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}(z)))^{-1})_{\alpha\beta} \cdot \sigma_l(\widetilde{W}_{\beta h}(z)) = \delta_{\alpha h}$, and therefore for every $r_1 + 1 \leq h \leq r$ we have

$$\sigma_l(\widetilde{W}'^p_{ih}(z))\sigma_l(\widetilde{W}'^p_{hj,p_h-1}(z)) - \sum_{\alpha,\beta=r_1+1}^{r} \sigma_l(\widetilde{W}'^p_{i\alpha}(z))((\sigma_l(\widetilde{W}_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}(z)))^{-1})_{\alpha\beta}\sigma_l(\widetilde{W}'^p_{\beta h}(z))\sigma_l(\widetilde{W}'^p_{hj,p_h-1}(z)) = 0.$$

[1] In order to simplify the notation we number the rows and columns of the matrix $(\widetilde{W}'^p(z)_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c})^{-1}$ from $r_1 + 1$ to r instead of the natural numbering from 1 to $r - r_1$.

By definition of generalized quasideterminant, the RHS of (4.1.7) becomes

$$[e_{(j,p_1)(j,p_1-1)}, \sigma_l(|\widetilde{W}'^p(z)|_{I_{rr_1}J_{r_1r}})_{ij}] - \sum_{h=1}^{r_1} \sigma_l(|\widetilde{W}'^p(z)|_{I_{rr_1}J_{r_1r}})_{ih}(\delta_{hj}z + \tilde{e}_{(j,p_1)(h,p_1)})$$
$$+ \sum_{\alpha,\beta=r_1+1}^{r} [e_{(j,p_1)(j,p_1-1)}, \sigma_l(\widetilde{W}'^p_{i\alpha}(z))((\sigma_l(\widetilde{W}'^p_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}(z)))^{-1})_{\alpha\beta}]\sigma_l(\widetilde{W}'^p_{\beta j}(z)).$$

We are therefore left to show that $[e_{(j,p_1)(j,p_1-1)}, \sigma_l(\widetilde{W}'^p_{i\alpha}(z))] = [e_{(j,p_1)(j,p_1-1)}, ((\sigma_l(\widetilde{W}'^p_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}(z)))^{-1})_{\alpha\beta}] = 0$. It is sufficient to prove that

$$[e_{(j,p_1)(j,p_1-1)}, \sigma_l(\widetilde{W}'^p_{i\alpha}(z))] = 0, \qquad \text{for } \alpha \geq r_1+1,\ 1 \leq i \leq r. \tag{4.1.8}$$

We proceed by induction on the number of columns of p, the base case being $p_1 = 2$. In this case,

$$\widetilde{W}'^p_{i\alpha}(z) = \delta_{i\alpha}z + e_{(\alpha,1)(i,1)},$$

and $[e_{(j,p_1)(j,p_1-1)}, e_{(\alpha,1)(i,1)}] = \delta_{\alpha,j}e_{(j,p_1)(i1)} = 0$.

If $p_1 > 2$, by the recursive formula we get

$$\widetilde{W}^p_{i\alpha}(z) = \sigma_l(\widetilde{W}'^p_{i\alpha}(z)) = \begin{cases} \sigma_l^2(\widetilde{W}''^p_{i\alpha}(z)), & \alpha > t_2 \\ [e_{(\alpha,p_1-1)(\alpha,p_1-2)}, \sigma_l^2(\widetilde{W}''^p_{i\alpha}(z))] - \sum_{h=1}^{t_2}\sigma_l^2(\widetilde{W}''^p_{ih}(z))(\delta_{h\alpha}z + \tilde{e}_{(\alpha,p_1-1)(h,p_1-1)}) & \\ \qquad + \sum_{h=t_2}^{r}\sigma_l^2(\widetilde{W}''^p_{ih}(z))\sigma_l^2(\widetilde{W}''^p_{h\alpha,p_h-1}(z)), & t_1+1 \leq \alpha \leq t_2. \end{cases} \tag{4.1.9}$$

Clearly, the commutator of (4.1.9) with $e_{(j,p_1)(j,p_1-1)}$ is also zero.

Combining (4.1.5) and (4.1.6) and comparing with (2.2.7), the claim follows.

The proof is analogous in the case $s_1 \leq t_1$, using recursion (3.2.3) instead. □

Remark 4.1.1. Note that with some additional computation it is possible to prove Theorem 4.1.1 even in the more general case of the pyramid p being even but not aligned. However, we are interested so far only in the left/right aligned cases, namely the only ones for which we can directly prove that the matrix $W(z)$ from Definition 3.2.1 has coefficients in the W-algebra.

Remark 4.1.2. Write $W(z)$ in block form

$$W(z) = \begin{pmatrix} -(-z)^{p_1}\mathbb{1}_{r_1} + W_1(z) & W_2(z) \\ W_3(z) & -(-z)^q\mathbb{1}_{r-r_1} + W_4(z) \end{pmatrix}, \tag{4.1.10}$$

where $W_1(z)$, $W_2(z)$, $W_3(z)$, $W_4(z)$ are block matrices of sizes $r_1 \times r_1$, $r_1 \times (r-r_1)$, $(r-r_1) \times r_1$ and $(r-r_1) \times (r-r_1)$ respectively, while $-(-z)^q\mathbb{1}_{r-r_1}$ is as in (4.1.4).

As a consequence of Theorem (4.1.1),

$$L(z) = \begin{pmatrix} \boxed{W_1(z)} & W_2(z) \\ W_3(z) & W_4(z) \end{pmatrix} = -(-z)^{p_1}\mathbb{1}_{r_1} + W_1(z) - W_2(z)(-(-z)^q\mathbb{1}_{r-r_1} + W_4(z))^{-1}W_3(z), \tag{4.1.11}$$

as in Conjecture [DSKV16c, Conjecture 8.2].

4.2 Properties of gradings and Lagrangian subspaces, and the corresponding W-algebras

Definition 3.2.1 allows us to recursively construct the matrix $W(z)$ only in the case of an even $\frac{1}{2}\mathbb{Z}$-grading. Moreover, we can only prove that its entries $W_{ij}(z)$ have coefficients in the W-algebra in the particular case of a left or right aligned pyramid (cf. Theorems 3.3.1 and 3.3.2). Therefore, for a left or right aligned grading, combining Proposition 3.3.1 and Theorem 4.1.1, we can prove Conjecture 3.1.1. We would like to extend these results to an arbitrary good $\frac{1}{2}\mathbb{Z}$-grading, whose associated pyramid is not necessarily aligned. Even more, we would like to allow the grading to be odd, as it is for instance often the case with a Dynkin grading.

A key ingredient for this purpose is the result about the isomorphism of W-algebras associated with different isotropic subspaces described in [GG02]:

Theorem 4.2.1. *[GG02, Theorem 4.1] Let $\mathfrak{l}_1$, $\mathfrak{l}_2 \subseteq \mathfrak{g}_{\frac{1}{2}}$ be isotropic subspaces with respect to the bilinear form ω. Suppose moreover that $\mathfrak{l}_1 \subset \mathfrak{l}_2$. We have the natural linear map between quotients*

$$\begin{aligned}\overline{\rho_{\mathfrak{l}_1,\mathfrak{l}_2}} : U(\mathfrak{g})/U(\mathfrak{g})\langle b-(f|b)\rangle_{b\in\mathfrak{l}_1\oplus\mathfrak{g}_{\geq 1}} &\longrightarrow U(\mathfrak{g})/U(\mathfrak{g})\langle b-(f|b)\rangle_{b\in\mathfrak{l}_2\oplus\mathfrak{g}_{\geq 1}}\\ \overline{1}_{\mathfrak{l}_1} &\mapsto \overline{1}_{\mathfrak{l}_2}.\end{aligned} \tag{4.2.1}$$

Then, the restriction of the map $\overline{\rho_{\mathfrak{l}_1,\mathfrak{l}_2}}$ to the corresponding W-algebras is an isomorphism of Kazhdan filtered algebras

$$\rho_{\mathfrak{l}_1,\mathfrak{l}_2} : W(\mathfrak{g},f,\Gamma,\mathfrak{l}_1) \xrightarrow{\sim} W(\mathfrak{g},f,\Gamma,\mathfrak{l}_2). \tag{4.2.2}$$

Thus, for every $\mathfrak{l} \subseteq \mathfrak{g}_{\frac{1}{2}}$ isotropic we have an isomorphism $\rho_{0,\mathfrak{l}} : W(\mathfrak{g},f,\Gamma,0) \xrightarrow{\sim} W(\mathfrak{g},f,\Gamma,\mathfrak{l})$ of filtered algebras.

As a consequence, given arbitrary isotropic subspaces $\mathfrak{l}_1,\mathfrak{l}_2$ such that neither $\mathfrak{l}_1 \subset \mathfrak{l}_2$ nor $\mathfrak{l}_2 \subset \mathfrak{l}_1$, we have an isomorphism of filtered algebras

$$\rho_{0,\mathfrak{l}_2} \circ (\rho_{0,\mathfrak{l}_1})^{-1} : W(\mathfrak{g},f,\Gamma,\mathfrak{l}_1) \xrightarrow{\sim} W(\mathfrak{g},f,\Gamma,\mathfrak{l}_2)\,. \tag{4.2.3}$$

Another key ingredient is the notion of adjacent $\frac{1}{2}\mathbb{Z}$-gradings for $\mathfrak{g}$, introduced by Brundan and Goodwin in [BG05]:

Definition 4.2.1. [BG05] Let $\Gamma_1 : \mathfrak{g} = \bigoplus_{i\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_i^{(1)}$ and $\Gamma_2 : \mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_j^{(2)}$ be good $\frac{1}{2}\mathbb{Z}$-gradings for f. We say that they are *adjacent* if

$$\mathfrak{g} = \bigoplus_{i-\frac{1}{2}\leq j\leq i+\frac{1}{2}} \mathfrak{g}_i^{(1)} \cap \mathfrak{g}_j^{(2)}. \tag{4.2.4}$$

Clearly, if (4.2.4) holds then

$$\mathfrak{g} = \bigoplus_{j-\frac{1}{2}\leq i\leq j+\frac{1}{2}} \mathfrak{g}_i^{(1)} \cap \mathfrak{g}_j^{(2)},$$

also holds. Equivalently, (4.2.4) implies that, for each $i \in \frac{1}{2}\mathbb{Z}$, we can decompose $\mathfrak{g}_i^{(1)} = \bigoplus_{|j-i|\leq\frac{1}{2}} \mathfrak{g}_i^{(1)} \cap \mathfrak{g}_j^{(2)}$. We also remark that the definition in [BG05] is slightly different, since we needed to adapt it to our definition of good $\frac{1}{2}\mathbb{Z}$-gradings.

Adjacent good $\frac{1}{2}\mathbb{Z}$-gradings Γ_1,Γ_2 have the nice following property

Lemma 4.2.1. *[BG05, Lemma 26] Let $\Gamma_1 : \mathfrak{g} = \bigoplus_{i\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_i^{(1)}$ and $\Gamma_2 : \mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_j^{(2)}$ be adjacent good $\frac{1}{2}\mathbb{Z}$-gradings for f. Then, there exist Lagrangian subspaces $\mathfrak{l}_1 \subseteq \mathfrak{g}_{\frac{1}{2}}^{(1)}$ and $\mathfrak{l}_2 \subseteq \mathfrak{g}_{\frac{1}{2}}^{(1)}$ (both with respect to the form ω) such that*

$$\mathfrak{l}_1 \oplus \bigoplus_{i\geq 1} \mathfrak{g}_i^{(1)} = \mathfrak{l}_2 \oplus \bigoplus_{j\geq 1} \mathfrak{g}_j^{(2)}. \tag{4.2.5}$$

Proof. We shall include the proof because it describes how to construct such a pair of Lagrangian subspaces. First, let us decompose

$$\begin{aligned}\mathfrak{g}_{\frac{1}{2}}^{(1)} &= ((\mathfrak{g}_{\frac{1}{2}}^{(1)} \cap \mathfrak{g}_{\leq 0}^{(2)}) \oplus (\mathfrak{g}_{\frac{1}{2}}^{(1)} \cap \mathfrak{g}_{\geq 1}^{(2)})) \perp (\mathfrak{g}_{\frac{1}{2}}^{(1)} \cap \mathfrak{g}_{\frac{1}{2}}^{(2)})\\ \mathfrak{g}_{\frac{1}{2}}^{(2)} &= ((\mathfrak{g}_{\leq 0}^{(1)} \cap \mathfrak{g}_{\frac{1}{2}}^{(2)}) \oplus (\mathfrak{g}_{\geq 1}^{(1)} \cap \mathfrak{g}_{\frac{1}{2}}^{(2)})) \perp (\mathfrak{g}_{\frac{1}{2}}^{(1)} \cap \mathfrak{g}_{\frac{1}{2}}^{(2)})\,,\end{aligned}$$

where with $\mathfrak{g} = A \perp B$ we mean that $\mathfrak{g} = A \oplus B$ and moreover $B = A^{\perp}$, the orthogonal complement being with respect to the bilinear form ω. Therefore, the restriction of ω to $\mathfrak{g}_{\frac{1}{2}}^{(1)} \cap \mathfrak{g}_{\frac{1}{2}}^{(2)}$ is non-degenerate. After choosing a Lagrangian subspace $\mathfrak{l} \subseteq \mathfrak{g}_{\frac{1}{2}}^{(1)} \cap \mathfrak{g}_{\frac{1}{2}}^{(2)}$, in view of the definition of adjacency,

$$\mathfrak{l}_1 := \mathfrak{l} \oplus (\mathfrak{g}_{\frac{1}{2}}^{(1)} \cap \mathfrak{g}_{\geq 1}^{(2)}) \subset \mathfrak{g}_{\frac{1}{2}}^{(1)}, \qquad \mathfrak{l}_2 := \mathfrak{l} \oplus (\mathfrak{g}_{\geq 1}^{(1)} \cap \mathfrak{g}_{\frac{1}{2}}^{(2)}) \subset \mathfrak{g}_{\frac{1}{2}}^{(2)}$$

is a pair of Lagrangian subspaces such that (4.2.5) holds. ∎

The Lagrangianity condition is necessary in order to obtain, as a consequence, that the corresponding W-algebras coincide. In fact, let $\mathfrak{m}_1 := \mathfrak{l}_1 \oplus \mathfrak{g}^{(1)}_{\geq 1} = \mathfrak{l}_2 \oplus \mathfrak{g}^{(2)}_{\geq 1} =: \mathfrak{m}_2$, then by Lagrangianity, in the notation of Section 1.1.2, $\mathfrak{n}_1^{\perp} = \mathfrak{m}_1 = \mathfrak{n}_2 = \mathfrak{m}_2^{\perp}$, and therefore

$$W(\mathfrak{g}, f, \Gamma_1, \mathfrak{l}_1) = \Big(U(\mathfrak{g})/U(\mathfrak{g})\langle b - (f|b)\rangle_{b\in\mathfrak{m}_1}\Big)^{\mathrm{ad}\mathfrak{m}_1} = \Big(U(\mathfrak{g})/U(\mathfrak{g})\langle b - (f|b)\rangle_{b\in\mathfrak{m}_2}\Big)^{\mathrm{ad}\mathfrak{m}_2} = W(\mathfrak{g}, f, \Gamma_2, \mathfrak{l}_2)\,.$$

However, when one of the $\frac{1}{2}\mathbb{Z}$-gradings is even, then the corresponding $\frac{1}{2}$-space is $\mathfrak{g}_{\frac{1}{2}} = 0$ and consequently $\mathfrak{m} = \mathfrak{g}_{\geq 1} = \mathfrak{n}$. We thus obtain the following generalized version of Lemma 4.2.1:

Lemma 4.2.2. *Let $\Gamma_1 : \mathfrak{g} = \bigoplus_{i\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}^{(1)}_i$ and $\Gamma_2 : \mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}^{(2)}_j$ be adjacent good $\frac{1}{2}\mathbb{Z}$-gradings for f. Suppose moreover that Γ_2 is even. Then there exists a Lagrangian subspace $\mathfrak{l}_1 \subseteq \mathfrak{g}^{(1)}_{\frac{1}{2}}$ (with respect to the form ω) such that*

$$\mathfrak{l}_1 \oplus \bigoplus_{i\geq 1} \mathfrak{g}^{(1)}_i = \bigoplus_{j\geq 1} \mathfrak{g}^{(2)}_j. \tag{4.2.6}$$

Proof. The argument is the same of Lemma 4.2.1, although because of Γ_2 being even we have

$$\begin{aligned} \mathfrak{l} &= 0, \\ \mathfrak{l}_1 &= \mathfrak{g}^{(1)}_{\frac{1}{2}} \cap \mathfrak{g}^{(2)}_{\geq 1}, \\ \mathfrak{l}_2 &= 0\,. \end{aligned}$$

As a consequence, we obtain the following equality:

$$W(\mathfrak{g}, f, \Gamma_1, \mathfrak{l}_1) = W(\mathfrak{g}, f, \Gamma_2, 0)\,.$$

□

Lemma 4.2.3. *Let $\Gamma_1 : \mathfrak{g} = \bigoplus_{i\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}^{(1)}_i$ and $\Gamma_2 : \mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}^{(2)}_j$ be adjacent good $\frac{1}{2}\mathbb{Z}$-gradings for f, and let $\mathfrak{l}_1 \subset \mathfrak{g}^{(1)}_{\frac{1}{2}}$, $\mathfrak{l}_2 \subset \mathfrak{g}^{(2)}_{\frac{1}{2}}$ be a pair of Lagrangian subspaces as in (4.2.5). Suppose that ${}'p$ (resp. p') exists for both gradings Γ_1 and Γ_2. Then,*

$$\mathfrak{l}_1^{'p} \oplus \bigoplus_{i\geq 1} \mathfrak{g}_i^{'p,(1)} = \mathfrak{l}_2^{'p} \oplus \bigoplus_{j\geq 1} \mathfrak{g}_j^{'p,(2)}, \qquad \Big(\textit{resp. } \mathfrak{l}_1^{p'} \oplus \bigoplus_{i\geq 1} \mathfrak{g}_i^{p',(1)} = \mathfrak{l}_2^{p'} \oplus \bigoplus_{j\geq 1} \mathfrak{g}_j^{p',(2)}\Big) \tag{4.2.7}$$

Proof. First, remember that $\mathfrak{l}_i = \sigma_l(\mathfrak{l}_i^{'p}) \subset \sigma_l(\mathfrak{g}^{'p,(i)})$ in the case ${}'p$ exists, and $\mathfrak{l}_i = \sigma_r(\mathfrak{l}_i^{p'}) \subset \sigma_r(\mathfrak{g}^{p',(i)})$ in the case p' exists. We shall illustrate the proof for $\mathfrak{g}^{'p}$, the case of $\mathfrak{g}^{p'}$ being analogous. Since the gradings on the Lie algebras $\mathfrak{g}^{'p,(i)}$ are induced by the gradings on $\mathfrak{g}^{(i)}$, we can write (after applying σ_l)

$$\begin{aligned} \mathfrak{l}_1 \oplus \bigoplus_{i\geq 1} \sigma_l(\mathfrak{g}_i^{'p,(1)}) &= \mathfrak{l}_1 \oplus \bigoplus_{i\geq 1} \big(\mathfrak{g}_i^{(1)} \cap \sigma_l(\mathfrak{g}^{'p,(1)})\big) \\ \mathfrak{l}_2 \oplus \bigoplus_{j\geq 1} \sigma_l(\mathfrak{g}_j^{'p,(2)}) &= \mathfrak{l}_2 \oplus \bigoplus_{j\geq 1} \big(\mathfrak{g}_j^{(2)} \cap \sigma_l(\mathfrak{g}^{'p,(2)})\big)\,, \end{aligned}$$

and we can moreover decompose $\mathfrak{g}^{(i)}_{\geq 1} = \sigma_l(\mathfrak{g}^{'p,(i)}_{\geq 1}) \oplus \mathfrak{g}^{p_1,(i)}_{\geq 1}$, where $\mathfrak{g}^{p_1,(i)}_{\geq 1}$ is the vector subspace consisting of the matrices $e_{(a,b)(c,p_1)} \in \mathfrak{g}^{(i)}_{\geq 1}$ such that $1 \leq c \leq t_1^{(1)}$ ($t_1^{(1)}$ being the height of the leftmost column of p with grading Γ_1).

Since we are assuming that $\mathfrak{l}_1 \oplus \mathfrak{g}^{(1)}_{\geq 1} = \mathfrak{l}_2 \oplus \mathfrak{g}^{(2)}_{\geq 1}$, it is sufficient to show that

$$\mathfrak{g}^{p_1,(1)}_{\geq 1} = \mathfrak{g}^{p_1,(2)}_{\geq 1}\,.$$

Let $e_{(a,b)(c,p_1)} \in \mathfrak{g}^{p_1,(1)}_{\geq 1}$. By adjacency of Γ_1 and Γ_2 and by the hypothesis that ${}'p$ exists for both gradings, then $e_{(a,b)(c,p_c)} \in \mathfrak{g}^{(2)}_{\geq 1}$, and moreover force $e_{(a,b)(c,p_1)} \in \mathfrak{g}^{p_1,(2)}_{\geq 1}$. The same clearly holds with the role of $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$ reversed. The result then holds by injectivity of the map σ_l.

As a consequence,

$$W(\mathfrak{g}'^p, f'^p, \Gamma_1'^p, \mathfrak{l}_2) = W(\mathfrak{g}'^p, f'^p, \Gamma_2'^p, \mathfrak{l}_2). \tag{4.2.8}$$

□

Remark 4.2.1. Let $\Gamma_1 : \mathfrak{g} = \bigoplus_{i\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_i^{(1)}$ and $\Gamma_2 : \mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_j^{(2)}$ be adjacent good $\frac{1}{2}\mathbb{Z}$-gradings for f. Then only one of the following holds:

$$x^{\Gamma_2}(ih) = \begin{cases} x^{\Gamma_1}(ih) \\ x^{\Gamma_1}(ih) + \frac{1}{2} \end{cases} \quad \forall (i,h) \in \mathcal{T}, \qquad (4.2.9) \qquad x^{\Gamma_2}(ih) = \begin{cases} x^{\Gamma_1}(ih) \\ x^{\Gamma_1}(ih) - \frac{1}{2} \end{cases} \quad \forall (i,h) \in \mathcal{T}. \qquad (4.2.10)$$

We denote by $x^{\Gamma_i}(ih)$ the x-coordinate of the pair (i,h) with respect to the grading Γ_i. In fact, suppose on the contrary that there exist $(i,h),(j,k) \in \mathcal{T}$ such that $x^{\Gamma_2}(ih) = x^{\Gamma_1}(ih) + \frac{1}{2}$ and $x^{\Gamma_2}(jk) = x^{\Gamma_1}(jk) - \frac{1}{2}$. Then, $\deg_{\Gamma_2}(e_{(i,h)(j,k)}) = \deg_{\Gamma_1}(e_{(i,h)(j,k)}) + 1$, contradicting the hypothesis of adjacency. Graphically, it means that if Γ_1 and Γ_2 are adjacent good $\frac{1}{2}\mathbb{Z}$-gradings for f, then Γ_2 can be obtained by Γ_1 by moving of $\frac{1}{2}$ space some blocks of the pyramid either all to the right (when $x^{\Gamma_2}(ih) = x^{\Gamma_1}(ih) + \frac{1}{2}$) or all to the left (when $x^{\Gamma_2}(ih) = x^{\Gamma_1}(ih) - \frac{1}{2}$).

We shall now introduce some special pairs of adjacent good $\frac{1}{2}\mathbb{Z}$-gradings, that are particularly useful for our purposes.

Definition 4.2.2. Let $\Gamma_1 : \mathfrak{g} = \bigoplus_{i\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_i^{(1)}$ and $\Gamma_2 : \mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_j^{(2)}$ be good $\frac{1}{2}\mathbb{Z}$-gradings for f. We say that Γ_1 and Γ_2 are *strictly adjacent* if they are adjacent (as in Definition 4.2.1), and moreover one of the following holds

$$s_{1i}^{\Gamma_1} \in \frac{1}{2} + \mathbb{N} \text{ for all } (i,h) \in \mathcal{T} \text{ such that } x^{\Gamma_2}(ih) \neq x^{\Gamma_1}(ih) \tag{4.2.11}$$

or

$$s_{1i}^{\Gamma_1} \in \mathbb{N} \text{ for all } (i,h) \in \mathcal{T} \text{ such that } x^{\Gamma_2}(ih) \neq x^{\Gamma_1}(ih) \tag{4.2.12}$$

where we denote by $s_{1j}^{\Gamma_i}$, $1 \leq j \leq r$, the semi-integer s_{ij} as in (1.1.2) with respect to the grading Γ_i.

Graphically, the gradings Γ_1 and Γ_2 are strictly adjacent if they are adjacent and Γ_2 is obtained by Γ_1 moving only either (some of the) integer or semi-integer rows (cf. Definition 1.1.5).

Note that if Γ_1 and Γ_2 are strictly adjacent, they also are strictly adjacent when we reverse the roles of Γ_1 and Γ_2 in the definition.

Example 4.2.1. Let $\Gamma_1 : \mathfrak{g} = \bigoplus_{i\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_i^{(1)}$ and $\Gamma_2 : \mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_j^{(2)}$ be adjacent good $\frac{1}{2}\mathbb{Z}$-gradings for f. If one of the gradings, say Γ_2, is even then they clearly are strictly adjacent.

Lemma 4.2.4. *Let $\Gamma_1 : \mathfrak{g} = \bigoplus_{i\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_i^{(1)}$ and $\Gamma_2 : \mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}_j^{(2)}$ be strictly adjacent good $\frac{1}{2}\mathbb{Z}$-gradings for f. Then, we can choose a Lagrangian subspaces $\mathfrak{l} = \mathfrak{g}_{\frac{1}{2}}^{(1)} \cap \mathfrak{g}_{\frac{1}{2}}^{(2)}$ such that for the Lagrangian subspaces $\mathfrak{l}_1$ and $\mathfrak{l}_2$ as in Lemma 4.2.1,*

$$\mathfrak{l}_1 \oplus \mathfrak{g}_{\geq 1}^{(1)} = \mathfrak{l}_2 \oplus \mathfrak{g}_{\geq 1}^{(2)}$$

and moreover they are of the following form:

$$\mathfrak{l}_i = \{e_{(i,h)(j,k)} \mid deg_{\Gamma_i}(e_{(i,h)(j,k)}) = \frac{1}{2}, s_{1i}^{\Gamma_i} \in \frac{1}{2} + \mathbb{N}\}, \quad i = 1,2 \tag{4.2.13}$$

or

$$\mathfrak{l}_i = \{e_{(i,h)(j,k)} \mid deg_{\Gamma_i}(e_{(i,h)(j,k)}) = \frac{1}{2}, s_{1i}^{\Gamma_i} \in \mathbb{N}\}, \quad i = 1,2 \tag{4.2.14}$$

Note that in both cases, $\mathfrak{l}_i$ is an abelian (Lagrangian) subspace of $\mathfrak{g}_{\frac{1}{2}}^{\Gamma_i}$.

Before giving the proof of Lemma 4.2.4, we illustrate through an example the reason why we need to add the hypothesis of the gradings Γ_1 and Γ_2 being strictly adjacent. The advantage of having Lagrangian subspaces as in Lemma 4.2.4 will be clear later.

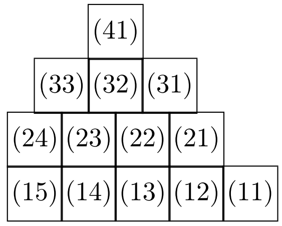


Figure 4.1: Γ_1

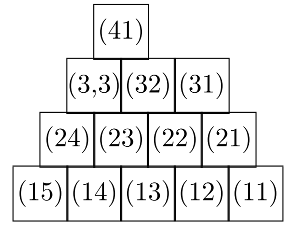


Figure 4.2: Γ_2

Example 4.2.2. Let $N = 13$ with partition $(5, 4, 3, 1)$, and let Γ_1 and Γ_2 be the following good $\frac{1}{2}\mathbb{Z}$-gradings for f associated with this partition:

The $\frac{1}{2}\mathbb{Z}$-gradings Γ_1 and Γ_2 are clearly adjacent but not strictly adjacent, because Γ_2 is obtained from Γ_1 by moving to the right row 2, which is integer, and row 3, which is semi-integer. Note

$$\mathfrak{g}_{\frac{1}{2}}^{\Gamma_1} \cap \mathfrak{g}_1^{\Gamma_2} = \{e_{(3,1)(1,3)}, e_{(2,2)(5,1)}, e_{(3,2)(1,4)}, e_{(3,3)(1,5)}\}$$

hence $\mathfrak{l}_1 = \mathfrak{l} \oplus \left(\mathfrak{g}_{\frac{1}{2}}^{\Gamma_1} \cap \mathfrak{g}_1^{\Gamma_2}\right)$ is neither of the form (4.2.13) nor of the form (4.2.14), because $s_{12}^{\Gamma_1} = 2 \in \mathbb{N}$ while $s_{13}^{\Gamma_2} = \frac{3}{2} \in \frac{1}{2} + \mathbb{N}$.

Proof of Lemma 4.2.4. By Lemma 4.2.1 and by the adjacency property, the Lagrangian subspaces will be $\mathfrak{l}_1 = \mathfrak{l} \oplus \left(\mathfrak{g}_{\frac{1}{2}}^{(1)} \cap \mathfrak{g}_1^{(2)}\right)$ and $\mathfrak{l}_2 = \mathfrak{l} \oplus \left(\mathfrak{g}_1^{(1)} \cap \mathfrak{g}_{\frac{1}{2}}^{(2)}\right)$ for a Lagrangian subspace $\mathfrak{l} \subset \mathfrak{g}_{\frac{1}{2}}^{(1)} \cap \mathfrak{g}_{\frac{1}{2}}^{(2)}$.

We may assume that (4.2.11) holds, namely that $s_{1i}^{\Gamma_1} \in \frac{1}{2} + \mathbb{N}$ for all $(i, h) \in \mathcal{T}$ such that $x^{\Gamma_2}(ih) \neq x^{\Gamma_1}(ih)$, and moreover that (4.2.9) holds, namely that Γ_2 is obtained by Γ_1 by moving some of the (semi-integers) boxes to the right. For the other cases see Remark 4.2.2.

We have

$$\begin{aligned}
\mathfrak{g}_{\frac{1}{2}}^{(1)} \cap \mathfrak{g}_1^{(2)} &= \{e_{(i,h)(j,k)} \mid s_{1j}^{\Gamma_1} + k - s_{1i}^{\Gamma_1} - h = \frac{1}{2},\, s_{1j}^{\Gamma_2} + k - s_{1i}^{\Gamma_2} - h = 1\} \\
&= \{e_{(i,h)(j,k)} \mid -s_{1j}^{\Gamma_1} + s_{1i}^{\Gamma_1} = -s_{1j}^{\Gamma_2} + s_{1i}^{\Gamma_2} + \frac{1}{2}\}. \qquad (4.2.15)
\end{aligned}$$

By (4.2.9), $s_{1i}^{\Gamma_2} = s_{1i}^{\Gamma_1} - \frac{1}{2}$, $s_{1j}^{\Gamma_2} = s_{1j}^{\Gamma_1}$ is the only possible solution for (4.2.15). By (4.2.11) we moreover must have $s_{1i}^{\Gamma_1} \in \frac{1}{2} + \mathbb{N}$, while $s_{1i}^{\Gamma_2}, s_{1j}^{\Gamma_1}, s_{1j}^{\Gamma_2} \in \mathbb{N}$.

Therefore we can rewrite (4.2.15) as

$$\mathfrak{g}_{\frac{1}{2}}^{(1)} \cap \mathfrak{g}_1^{(2)} = \{e_{(i,h)(j,k)} \mid \deg_{\Gamma_1}(e_{(i,h)(j,k)}) = \frac{1}{2},\, \deg_{\Gamma_2}(e_{(i,h)(j,k)}) = 1,\, s_{1i}^{\Gamma_1} \in \frac{1}{2} + \mathbb{N}\}. \qquad (4.2.16)$$

Similarly,

$$\begin{aligned}
\mathfrak{g}_1^{(1)} \cap \mathfrak{g}_{\frac{1}{2}}^{(2)} &= \{e_{(i,h)(j,k)} \mid s_{1j}^{\Gamma_1} + k - s_{1i}^{\Gamma_1} - h = 1,\, s_{1j}^{\Gamma_2} + k - s_{1i}^{\Gamma_2} - h = \frac{1}{2}\} \\
&= \{e_{(i,h)(j,k)} \mid -s_{1j}^{\Gamma_1} + s_{1i}^{\Gamma_1} = -s_{1j}^{\Gamma_2} + s_{1i}^{\Gamma_2} - \frac{1}{2}\}. \qquad (4.2.17)
\end{aligned}$$

By (4.2.9), $s_{1j}^{\Gamma_2} = s_{1j}^{\Gamma_1} - \frac{1}{2}$, $s_{1i}^{\Gamma_2} = s_{1i}^{\Gamma_1}$ is the only possible solution for (4.2.17). By (4.2.11), we moreover must have $s_{1j}^{\Gamma_1}, s_{1i}^{\Gamma_1}, s_{1i}^{\Gamma_2} \in \frac{1}{2} + \mathbb{N}$, while $s_{1j}^{\Gamma_2} \in \mathbb{N}$.

Therefore we can rewrite (4.2.17) as

$$\mathfrak{g}_1^{(1)} \cap \mathfrak{g}_{\frac{1}{2}}^{(2)} = \{e_{(i,h)(j,k)} \mid \deg_{\Gamma_1}(e_{(i,h)(j,k)}) = 1,\, \deg_{\Gamma_2}(e_{(i,h)(j,k)}) = \frac{1}{2},\, s_{1i}^{\Gamma_2} \in \frac{1}{2} + \mathbb{N}\}. \qquad (4.2.18)$$

Since

$$\mathfrak{g}_{\frac{1}{2}}^{(1)} = \{e_{(i,h)(j,k)} \mid \deg_{\Gamma_1}(e_{(i,h)(j,k)}) = \frac{1}{2},\, s_{1i}^{\Gamma_1} \in \frac{1}{2} + \mathbb{N}\} \oplus \{e_{(i,h)(j,k)} \mid \deg_{\Gamma_1}(e_{(i,h)(j,k)}) = \frac{1}{2},\, s_{1i}^{\Gamma_1} \in \mathbb{N}\}$$

$$= A_{\Gamma_1} \oplus B_{\Gamma_1} \tag{4.2.19}$$

and

$$\begin{aligned}\mathfrak{g}^{(2)}_{\frac{1}{2}} =& \{e_{(i,h)(j,k)} \mid \deg_{\Gamma_2}(e_{(i,h)(j,k)}) = \frac{1}{2},\, s^{\Gamma_2}_{1i} \in \frac{1}{2} + \mathbb{N}\} \oplus \{e_{(i,h)(j,k)} \mid \deg_{\Gamma_2}(e_{(i,h)(j,k)}) = \frac{1}{2},\, s^{\Gamma_2}_{1i} \in \mathbb{N}\} \\ =& A_{\Gamma_2} \oplus B_{\Gamma_2},\end{aligned}$$

we can decompose

$$\begin{aligned}\mathfrak{g}^{(1)}_{\frac{1}{2}} \cap \mathfrak{g}^{(2)}_{\frac{1}{2}} =& \{e_{(i,h)(j,k)} \mid \deg_{\Gamma_1}(e_{(i,h)(j,k)}) = \deg_{\Gamma_2}(e_{(i,h)(j,k)}) = \frac{1}{2},\, s^{\Gamma_1}_{1i} = s^{\Gamma_2}_{1i} \in \frac{1}{2} + \mathbb{N}\} \\ &\oplus \{e_{(i,h)(j,k)} \mid \deg_{\Gamma_1}(e_{(i,h)(j,k)}) = \deg_{\Gamma_2}(e_{(i,h)(j,k)}) = \frac{1}{2},\, s^{\Gamma_1}_{1i} = s^{\Gamma_2}_{1i} \in \mathbb{N}\} \\ &= (A_{\Gamma_1} \cap A_{\Gamma_2}) \oplus (B_{\Gamma_1} \cap B_{\Gamma_2}) = A \oplus B,\end{aligned}$$

where $s^{\Gamma_1}_{1i} = s^{\Gamma_2}_{1i}$ follows from Remark 4.2.1 and from the strict adjacency condition. Namely, these conditions imply that if $x^{\Gamma_1}(ih) - x^{\Gamma_1}(jk) = x^{\Gamma_2}(ih) - x^{\Gamma_2}(jk)$, rows i and j necessarily are one integer and one semi-integer, it must be $x^{\Gamma_1}(ih) = x^{\Gamma_2}(ih)$ and $x^{\Gamma_1}(jk) = x^{\Gamma_2}(jk)$ and therefore $s^{\Gamma_1}_{1i} = s^{\Gamma_2}_{1i}$.

By the definition of pyramid, $\dim A_{\Gamma_1} = \dim B_{\Gamma_1} = \frac{1}{2}\dim \mathfrak{g}^{(1)}_{\frac{1}{2}}$ and $\dim A_{\Gamma_2} = \dim B_{\Gamma_2} = \frac{1}{2}\dim \mathfrak{g}^{(2)}_{\frac{1}{2}}$. In fact, for each box labeled by $(i,h) \in \mathcal{T}$ such that $s_{1i} \in \frac{1}{2} + \mathbb{N}$, then must exist $(a,b),(c,d) \in \mathcal{T}$ such that both $e_{(i,h)(a,b)}, e_{(c,d)(i,h)} \in \mathfrak{g}^{(i)}_{\frac{1}{2}}$.

By a similar argument, $\dim A = \dim B = \frac{1}{2}\dim(\mathfrak{g}^{(1)}_{\frac{1}{2}} \cap \mathfrak{g}^{(2)}_{\frac{1}{2}})$. Moreover, under the usual bracket product, A and B are abelian subspaces of $\mathfrak{g}^{(1)}_{\frac{1}{2}} \cap \mathfrak{g}^{(2)}_{\frac{1}{2}}$. Therefore, both A and B are Lagrangian subspaces of $\mathfrak{g}^{(1)}_{\frac{1}{2}} \cap \mathfrak{g}^{(2)}_{\frac{1}{2}}$ with respect to the form ω.

Choosing $\mathfrak{l} := A$ we get

$$\begin{aligned}\mathfrak{l}_1 =& \{e_{(i,h)(j,k)} \mid \deg_{\Gamma_1}(e_{(i,h)(j,k)}) = \deg_{\Gamma_2}(e_{(i,h)(j,k)}) = \frac{1}{2},\, s^{\Gamma_1}_{1i} \in \frac{1}{2} + \mathbb{N}\} \\ &\oplus \{e_{(i,h)(j,k)} \mid \deg_{\Gamma_1}(e_{(i,h)(j,k)}) = \frac{1}{2},\, \deg_{\Gamma_2}(e_{(i,h)(j,k)}) = 1,\, s^{\Gamma_1}_{1i} \in \frac{1}{2} + \mathbb{N}\}.\end{aligned} \tag{4.2.20}$$

and

$$\begin{aligned}\mathfrak{l}_2 =& \{e_{(i,h)(j,k)} \mid \deg_{\Gamma_1}(e_{(i,h)(j,k)}) = \deg_{\Gamma_2}(e_{(i,h)(j,k)}) = \frac{1}{2},\, s^{\Gamma_2}_{1i} \in \frac{1}{2} + \mathbb{N}\} \\ &\oplus \{e_{(i,h)(j,k)} \mid \deg_{\Gamma_1}(e_{(i,h)(j,k)}) = 1,\, \deg_{\Gamma_2}(e_{(i,h)(j,k)}) = \frac{1}{2},\, s^{\Gamma_2}_{1i} \in \frac{1}{2} + \mathbb{N}\},\end{aligned} \tag{4.2.21}$$

which are of the form (4.2.13) since by (4.2.9) and (4.2.11) the following holds:

$$\begin{aligned}&\{e_{(i,h)(j,k)} \mid \deg_{\Gamma_1}(e_{(i,h)(j,k)}) = \frac{1}{2},\, \deg_{\Gamma_2}(e_{(i,h)(j,k)}) = 0,\, s^{\Gamma_1}_{1i} \in \frac{1}{2} + \mathbb{N}\} = \emptyset \\ &\{e_{(i,h)(j,k)} \mid \deg_{\Gamma_1}(e_{(i,h)(j,k)}) = 0,\, \deg_{\Gamma_2}(e_{(i,h)(j,k)}) = \frac{1}{2},\, s^{\Gamma_2}_{1i} \in \frac{1}{2} + \mathbb{N}\} = \emptyset.\end{aligned} \tag{4.2.22}$$

□

Remark 4.2.2. Assuming instead (4.2.10) (under the hypothesis (4.2.11)) we would get

$$\begin{aligned}\mathfrak{g}^{\Gamma_1}_{\frac{1}{2}} \cap \mathfrak{g}^{\Gamma_2}_{1} &= \{e_{(i,h)(j,k)} \mid \deg_{\Gamma_1}(e_{(i,h)(j,k)}) = \frac{1}{2},\, \deg_{\Gamma_2}(e_{(i,h)(j,k)}) = 1,\, s^{\Gamma_1}_{1j} \in \frac{1}{2} + \mathbb{N}\} \\ \mathfrak{g}^{\Gamma_1}_{1} \cap \mathfrak{g}^{\Gamma_2}_{\frac{1}{2}} &= \{e_{(i,h)(j,k)} \mid \deg_{\Gamma_1}(e_{(i,h)(j,k)}) = 1,\, \deg_{\Gamma_2}(e_{(i,h)(j,k)}) = \frac{1}{2},\, s^{\Gamma_2}_{1j} \in \frac{1}{2} + \mathbb{N}\}\end{aligned}$$

and Equation (4.2.14) holds for $\mathfrak{l} := B$.

Assuming (4.2.12) instead, Equation (4.2.14) holds for $\mathfrak{l} := B$ under the hypothesis (4.2.9), and Equation (4.2.13) holds for $\mathfrak{l} := A$ under the hypothesis (4.2.10).

Corollary 4.2.1. *Let $\mathfrak{l}_i \subset \mathfrak{g}^{(i)}_{\frac{1}{2}}$ be a Lagrangian subspace as in Equation (4.2.13) or (4.2.14), $i = 1, 2$. Then, we can choose a complementary subspace $\mathfrak{l}^c_i$ to $\mathfrak{l}_i$ in $\mathfrak{g}^{(i)}_{\frac{1}{2}}$ in a way that it is still an abelian Lagrangian subspace. Moreover, both $\mathfrak{l}_i$ and $\mathfrak{l}^c_i$ are $\mathfrak{g}^{(i)}_0$-modules.*

Proof. Suppose that $\mathfrak{l}_i = \{e_{(i,h)(j,k)} \mid \deg_{\Gamma_i}(e_{(i,h)(j,k)}) = \frac{1}{2}, s^{\Gamma_i}_{1i} \in \frac{1}{2}+\mathbb{N}\}$. By the decomposition (4.2.19), then we can choose

$$\mathfrak{l}^c_i = \{e_{(i,h)(j,k)} \mid \deg_{\Gamma_i}(e_{(i,h)(j,k)}) = \frac{1}{2}, s^{\Gamma_i}_{1i} \in \mathbb{N}\}$$

that by Lemma 4.2.4 is an abelian Lagrangian subspace of $\mathfrak{g}^{\Gamma_i}_{\frac{1}{2}}$. On the other hand, for $\mathfrak{l}_i$ as in (4.2.14) the same holds with $\mathfrak{l}^c_i$ as in (4.2.13).

Moreover, given $e_{(a,b)(c,d)} \in \mathfrak{g}^{(i)}_0$, then we must have either $s^{\Gamma_i}_{1a}, s^{\Gamma_i}_{1c} \in \frac{1}{2} + \mathbb{N}$ or $s^{\Gamma_i}_{1a}, s^{\Gamma_i}_{1c} \in \mathbb{N}$. As a consequence, for $\mathfrak{l}_i$ as in (4.2.13) or (4.2.14) we get

$$[\mathfrak{g}^{(i)}_0, \mathfrak{l}_i] \subset \mathfrak{l}_i, \qquad [\mathfrak{g}^{(i)}_0, \mathfrak{l}^c_i] \subset \mathfrak{l}^c_i.$$

Namely, both $\mathfrak{l}_i$ and $\mathfrak{l}^c_i$ are $\mathfrak{g}^{\Gamma_i}_0$-modules. □

4.3 An attempt to extend the recursive definition of $W(z)$

Here, an attempt to generalize the results of the previous sections to a more arbitrary good $\frac{1}{2}\mathbb{Z}$-grading Γ.

Theorem 4.3.1. *Let $\Gamma_1 : \mathfrak{g} = \bigoplus_{i\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}^{(1)}_i$ be a good $\frac{1}{2}\mathbb{Z}$-grading for f. Suppose that it is possible to remove the leftmost (resp. rightmost) column of p and suppose that there exist a choice of generators $\{W^{'p}_{ij}(z)\}_{1\le i,j\le r}$ for $W(\mathfrak{g}^{'p}, f^{'p}, \Gamma^{'p}_1, 0)[z]$ (resp. $\{W^{p'}_{ij}(z)\}_{1\le i,j\le r}$ for $W(\mathfrak{g}^{p'}, f^{p'}, \Gamma^{p'}_1, 0)[z]$) and $\{W^{p}_{ij}(z)\}_{1\le i,j\le r}$ for $W(\mathfrak{g}, f, \Gamma_1, 0)[z]$ such that they are related by the left recursion (3.2.2) (resp. right recursion (3.2.3)). Suppose moreover that we are given another good $\frac{1}{2}\mathbb{Z}$-grading $\Gamma_2 : \mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}} \mathfrak{g}^{(2)}_j$ for f such that, with respect to this grading, it is still possible to remove the leftmost (resp. rightmost) column.*

By [BG05], there exist associative algebra isomorphisms (see Section 4.4.1 below for an explicit description)

$$\begin{aligned}\Phi_p : W(\mathfrak{g}, f, \Gamma_1, 0)[z] &\xrightarrow{\sim} W(\mathfrak{g}, f, \Gamma_2, 0)[z]\,,\\ \Phi_{'p} : W(\mathfrak{g}^{'p}, f^{'p}, \Gamma^{'p}_1, 0)[z] &\xrightarrow{\sim} W(\mathfrak{g}^{'p}, f^{'p}, \Gamma^{'p}_2, 0)[z]\,.\end{aligned} \tag{4.3.1}$$

Then, there exists a choice of generators $\{\Phi_{'p}(W^{'p}_{ij}(z))\}_{1\le i,j\le r}$ for $W(\mathfrak{g}^{'p}, f^{'p}, \Gamma^{'p}_2, 0)[z]$ and $\{\Phi_p(W^{p}_{ij}(z))\}_{1\le i,j\le r}$ for $W(\mathfrak{g}, f, \Gamma_2, 0)[z]$ for which the same recursion relation (3.2.2) holds.

Note that the possibility to extend the recursive relation from a pair of sets of generators of $W(\mathfrak{g}, f, \Gamma_1, 0)$ and $W(\mathfrak{g}^{'p}, f^{'p}, \Gamma^{'p}_1, 0)$ to the pair of sets of generators of $W(\mathfrak{g}, f, \Gamma_2, 0)$ and $W(\mathfrak{g}^{'p}, f^{'p}, \Gamma^{'p}_2, 0)$ obtained under the action of the isomorphisms Φ_p and $\Phi_{'p}$ is not obvious, not even for the simple case of the recursion (3.2.2), since the associative algebra injection σ_l does not extend to an homomorphism of W-algebras, and therefore we only know that

$$\sigma_l(W(\mathfrak{g}^{'p}, f^{'p}, \Gamma^{'p}_i, 0)) \subset U(\mathfrak{g})/U(\mathfrak{g})\langle b - (f|b)\rangle_{b\in\mathfrak{g}^{(i)}_{\ge 1}}\,.$$

Proof. We will work under the assumption that it is possible to remove the leftmost column of p, the other case being analogous.

It is sufficient to suppose that Γ_1 and Γ_2 are strictly adjacent $\frac{1}{2}\mathbb{Z}$-gradings. This statement will be clarified by the algorithm described in Section 4.4.1.

Remark 4.3.1. The fact that the left recursion holds for a choice of generators $\{W^{'p}_{ij}(z)\}_{1\le,j\le r}$ implies that the leftmost column is the shortest, namely $t^{\Gamma_1}_1 \le s^{\Gamma_1}_1$. The adjacency condition, together with the assumption that even for the grading Γ_2 it is possible to remove the leftmost column, imply that $t^{\Gamma_2}_1 \le s^{\Gamma_2}_1$ and moreover $t^{\Gamma_1}_1 = t^{\Gamma_2}_1$.

By Lemma 4.2.1 there exist Lagrangian subspaces $\mathfrak{l}_1 \subset \mathfrak{g}^{(1)}_{\frac{1}{2}}$ and $\mathfrak{l}_2 \subset \mathfrak{g}^{(2)}_{\frac{1}{2}}$ such that $W(\mathfrak{g}, f, \Gamma_1, \mathfrak{l}_1) = W(\mathfrak{g}, f, \Gamma_2, \mathfrak{l}_2)$ and, by Lemma 4.2.3, $W(\mathfrak{g}'^p, f'^p, \Gamma'^p_1, \mathfrak{l}'^p_1) = W(\mathfrak{g}'^p, f'^p, \Gamma'^p_2, \mathfrak{l}'^p_2)$. By Lemma 4.2.4, we can choose $\mathfrak{l}_1$, $\mathfrak{l}_2$ such they are of either one of the forms (4.2.13) or (4.2.14).

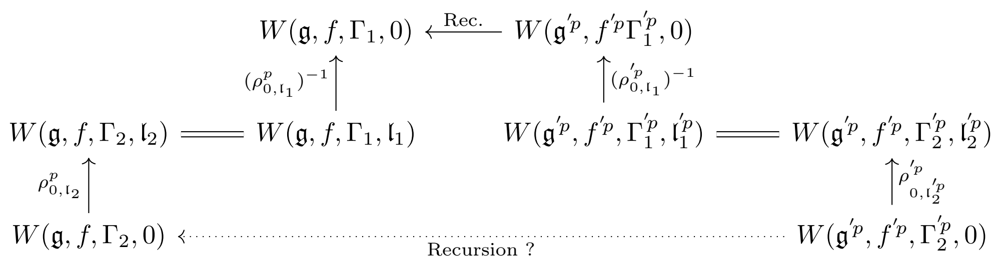
For $i = 1, 2$, let

$$\begin{aligned} \rho^p_{0,\mathfrak{l}_i} &: W(\mathfrak{g}, f, \Gamma_i, 0)[z] \xrightarrow{\sim} W(\mathfrak{g}, f, \Gamma_i, \mathfrak{l}_i)[z] \\ \rho'^p_{0,\mathfrak{l}_i} &: W(\mathfrak{g}'^p, f'^p, \Gamma'^p_i, 0)[z] \xrightarrow{\sim} W(\mathfrak{g}'^p, f'^p, \Gamma'^p_i, \mathfrak{l}'^p_i)[z] \end{aligned} \tag{4.3.2}$$

be the (polynomial extensions of the) isomorphisms as in (4.2.2) induced by the quotient maps

$$\begin{aligned} \overline{\rho}^p_{0,\mathfrak{l}_i} &: U(\mathfrak{g})/U(\mathfrak{g})\langle b - (f|b)\rangle_{b \in \mathfrak{g}^{(i)}_{\geq 1}} \twoheadrightarrow U(\mathfrak{g})/U(\mathfrak{g})\langle b - (f|b)\rangle_{b \in \mathfrak{l}_i \oplus \mathfrak{g}^{(i)}_{\geq 1}}, \\ \overline{\rho}'^p_{0,\mathfrak{l}_i} &: U(\mathfrak{g}'^p)/U(\mathfrak{g}'^p)\langle b - (f'^p|b)\rangle_{b \in \mathfrak{g}'^{p,(i)}_{\geq 1}} \twoheadrightarrow U(\mathfrak{g}'^p)/U(\mathfrak{g}'^p)\langle b - (f'^p|b)\rangle_{b \in \mathfrak{l}'^p_i \oplus \mathfrak{g}'^{p,(i)}_{\geq 1}}. \end{aligned} ,$$

We may assume the maps in (4.3.1) to be $\Phi_p = (\rho^p_{0,\mathfrak{l}_2})^{-1} \circ \rho^p_{0,\mathfrak{l}_1}$ and $\Phi_{'p} = (\rho'^p_{0,\mathfrak{l}_2})^{-1} \circ \rho'^p_{0,\mathfrak{l}_1}$. The situation is represented by the following diagram



(4.3.3)

Let $\overline{W}^p_{ij}(z) := \Phi_p(W^p_{ij}(z)) = (\rho^p_{0,\mathfrak{l}_2})^{-1}(\rho^p_{0,\mathfrak{l}_1}(W^p_{ij}(z)))$, and $\overline{W}'^p_{ij}(z) := \Phi_{'p}(W'^p_{ij}(z)) = (\rho'^p_{0,\mathfrak{l}_2})^{-1}(\rho'^p_{0,\mathfrak{l}_1}(W'^p_{ij}(z)))$. For each $i = 1, 2$, fix PBW bases of $U(\mathfrak{g}'^p)$ and $U(\mathfrak{g})$ as follows:

- Order the elements of $\mathfrak{g}^{(i)}_{\leq 0}$ increasingly with respect to the degree: $\mathfrak{g}^{(i)}_{-d} \cdots \mathfrak{g}^{(i)}_{-\frac{1}{2}}\, \mathfrak{g}^{(i)}_0$, and denote by $x_{-k,1}, \dots, x_{-k,s_{-k}}$ a basis for $\mathfrak{g}_{-k}$, $0 \leq k \leq d$, $s_{-k} \geq 1$. Same for $\mathfrak{g}'^{p,(i)}_{\leq 0}$;
- Order the elements of $\mathfrak{l}^c_i$ in lexicographical order with respect to the order of $\mathcal{T}$ given by Equation (2.1.7):
$$e_{(i_p,h_p)(j_p,k_p)} \geq \cdots \geq e_{(i_1,h_1)(j_1,k_1)},$$
where $e_{(i_1,h_1)(j_1,k_1)} < e_{(i_2,h_2)(j_2,k_2)}$ if either $(i_1, h_1) < (i_2, h_2)$ or if $(i_1, h_1) = (i_2, h_2)$ and $(j_1, k_1) < (j_2, k_2)$. We shortly denote by $\mathcal{B}^c_i = \{\overline{\ell}^{(i)}_1, \dots, \overline{\ell}^{(i)}_{m_i}\}$ such an ordered basis for $\mathfrak{l}^c_i$. Since $\mathfrak{l}^c_i = \sigma_l((\mathfrak{l}'^p_i)^c)$, with a slight abuse of notation, let us choose the same ordered base for $(\mathfrak{l}'^p_i)^c$;
- Order the elements of $\mathfrak{l}_i$ in (inverse) lexicographical order with respect to the order of $\mathcal{T}$ given by Equation (2.1.7):
$$e_{(i_p,h_p)(j_p,k_p)} \geq \cdots \geq e_{(i_1,h_1)(j_1,k_1)},$$
where $e_{(i_1,h_1)(j_1,k_1)} < e_{(i_2,h_2)(j_2,k_2)}$ if either $(j_1, k_1) < (j_2, j_2)$ or if $(j_1, k_1) = (j_2, k_2)$ and $(i_1, h_1) < (i_2, h_2)$. We shortly denote by $\mathcal{B}_i = \{\ell^{(i)}_1, \dots, \ell^{(i)}_{m_i}\}$ such an ordered basis for $\mathfrak{l}_i$. Since $\mathfrak{l}_i = \sigma_l(\mathfrak{l}'^p_i)$, with a slight abuse of notation, let us choose the same ordered base for $\mathfrak{l}'^p_i$;
- Order the elements of $\mathfrak{g}^{(i)}_{\geq 1}$ increasingly with respect to the degree: $\mathfrak{g}^{(i)}_1 \cdots \mathfrak{g}^{(i)}_d$. Same for $\mathfrak{g}'^{p,(i)}_{\geq 1}$;

As in (1.1.13), we shall thus identify the quotients

$$\begin{aligned} U(\mathfrak{g})/U(\mathfrak{g})\langle b - (f|b)\rangle_{b \in \mathfrak{g}^{(i)}_{\geq 1}} &\cong U(\mathfrak{g}^{(i)}_{\leq 0}) \otimes F(\mathfrak{g}^{(i)}_{\frac{1}{2}}), \\ U(\mathfrak{g}'^p)/U(\mathfrak{g}'^p)\langle b - (f'^p|b)\rangle_{b \in \mathfrak{g}'^{p,(i)}_{\geq 1}} &\cong U(\mathfrak{g}'^{p,(i)}_{\leq 0}) \otimes F(\mathfrak{g}'^{p,(i)}_{\frac{1}{2}}), \\ U(\mathfrak{g})/U(\mathfrak{g})\langle b - (f|b)\rangle_{b \in \mathfrak{l}_i \oplus \mathfrak{g}^{(i)}_{\geq 1}} &\cong U(\mathfrak{g}^{(i)}_{\leq 0}) \otimes F(\mathfrak{l}^c_i), \\ U(\mathfrak{g}'^p)/U(\mathfrak{g}'^p)\langle b - (f'^p|b)\rangle_{b \in \mathfrak{l}'^p_i \oplus \mathfrak{g}'^{p,(i)}_{\geq 1}} &\cong U(\mathfrak{g}'^{p,(i)}_{\leq 0}) \otimes F((\mathfrak{l}'^p_i)^c), \end{aligned} \tag{4.3.4}$$

where $F(\mathfrak{g}^{(i)}_{\frac{1}{2}})$, $F(\mathfrak{g}^{'p,(i)}_{\frac{1}{2}})$, $F(\mathfrak{l}_i^c)$ and $F((\mathfrak{l}_i^{'p})^c)$ are the (generalized) Weyl algebras for $\mathfrak{g}^{(i)}_{\frac{1}{2}}$, $\mathfrak{g}^{'p,(i)}_{\frac{1}{2}}$, $\mathfrak{l}_i^c$ and $(\mathfrak{l}_i^{'p})^c$ respectively (cf. Definition 1.1.7).

Analyzing the first column of Diagram 4.3.3, let $X^p_{ij}(z)$ be the unique element in $W(\mathfrak{g}, f, \Gamma_1, \mathfrak{l}_1)[z] = W(\mathfrak{g}, f, \Gamma_2, \mathfrak{l}_2)[z]$ such that $X^p_{ij}(z) := \rho^p_{0,\mathfrak{l}_1}(W^p_{ij}(z))$ for all $1 \leq i, j \leq r$.

Therefore we can write $W^p_{ij}(z) = (\rho^p_{0,\mathfrak{l}_1})^{-1}(X^p_{ij}(z)) = X^p_{ij}(z) + Y^p_{ij}\ell^{(1)}_{ij}(z)$ for every $1 \leq i, j \leq r$, where $Y^p_{ij}\ell^{(1)}_{ij}(z)$ is a polynomial whose coefficients are sums of monomials $Y_{i_1\cdots i_s}\ell^{(1)}_{i_1}\cdots\ell^{(1)}_{i_s}$, $s \geq 1$, with $Y_{i_1\cdots i_s} \in U(\mathfrak{g}^{(1)}_{\leq 0}) \otimes F(\mathfrak{l}_1^c)$ and $\ell^{(1)}_{i_1}, \dots, \ell^{(1)}_{i_s} \in \mathcal{B}_1$.

Proceeding downwards the first column, there exists a unique $\overline{W}^p_{ij}(z) := (\rho^p_{0,\mathfrak{l}_2})^{-1}(\rho^p_{0,\mathfrak{l}_1}(W^p_{ij}(z)))$, $1 \leq i, j \leq r$, such that

$(a.i)$ $(\rho^p_{0,\mathfrak{l}_2})^{-1}(\rho^p_{0,\mathfrak{l}_1}(W^p_{ij}(z))) = X^p_{ij}(z) + Z^p_{ij}\ell^{(2)}_{ij}(z)$, where $Z^p_{ij}\ell^{(2)}_{ij}(z)$ is a polynomial whose coefficients are sums of monomials $Z_{i_1\cdots i_s}\ell^{(2)}_{i_1}\cdots\ell^{(2)}_{i_s}(z)$, $s \geq 1$, with $Z_{i_1\cdots i_s} \in U(\mathfrak{g}^{(2)}_{\leq 0}) \otimes F(\mathfrak{l}_2^c)$ and $\ell^{(2)}_{i_1}, \dots, \ell^{(2)}_{i_s} \in \mathcal{B}_2$;

$(a.ii)$ $[a, X^p_{ij}(z) + Z^p_{ij}\ell^{(2)}_{ij}(z)] \in U(\mathfrak{g})\langle b - (f|b)\rangle_{b \in \mathfrak{g}^{(2)}_{\geq 1}}[z]$, for every $a \in \mathfrak{g}^{(2)}_{\geq \frac{1}{2}}$.

Similarly, analyzing the second column of Diagram 4.3.3, let $X^{'p}_{ij}(z)$ be the unique element in $W(\mathfrak{g}^{'p}, f^{'p}, \Gamma^{'p}_1, \mathfrak{l}^{'p}_1)[z] = W(\mathfrak{g}^{'p}, f^{'p}, \Gamma^{'p}_2, \mathfrak{l}^{'p}_2)[z]$ such that $X^{'p}_{ij}(z) := \rho^{'p}_{0,\mathfrak{l}_1}(W^{'p}_{ij}(z))$ for every $1 \leq i, j \leq r$.

Therefore, for every $1 \leq i, j \leq r$, we can write $W^{'p}_{ij}(z) = (\rho^{'p}_{0,\mathfrak{l}_1})^{-1}(X^{'p}_{ij}(z)) = X^{'p}_{ij}(z) + Y^{'p}_{ij}\ell^{'p,(1)}_{ij}(z)$, where $Y^{'p}_{ij}\ell^{'p,(1)}_{ij}(z)$ is a polynomial whose coefficients are sums of monomials $Y^{'p}_{i_1\cdots i_s}\ell^{(1)}_{i_1}\cdots\ell^{(1)}_{i_s}$, $s \geq 0$, with $Y^{'p}_{i_1\cdots i_s} \in U(\mathfrak{g}^{'p,(1)}_{\leq 0}) \otimes F((\mathfrak{l}^{'p}_1)^c)$ and $\ell^{(1)}_{i_1}, \dots, \ell^{(1)}_{i_s} \in \mathcal{B}_1$.

Therefore, there exists a unique $\overline{W}^{'p}_{ij}(z) := (\rho^{'p}_{0,\mathfrak{l}_2})^{-1}(\rho^{'p}_{0,\mathfrak{l}_1}(W^{'p}_{ij}(z)))$, $1 \leq i, j \leq r$, such that

$(b.i)$ $(\rho^{'p}_{0,\mathfrak{l}_2})^{-1}(\rho^{'p}_{0,\mathfrak{l}_1}(W^{'p}_{ij}(z))) = X^{'p}_{ij}(z) + Z^{'p}_{ij}\ell^{'p,(2)}_{ij}(z)$, where $Z^{'p}_{ij}\ell^{'p,(2)}_{ij}(z)$ is a polynomial whose coefficients are sums of monomials $Z^{'p}_{i_1\cdots i_s}\ell^{(2)}_{i_1}\cdots\ell^{(2)}_{i_s}(z)$, $s \geq 1$, with $Z^{'p}_{i_1\cdots i_s} \in U(\mathfrak{g}^{'p,(2)}_{\leq 0}) \otimes F((\mathfrak{l}^{'p}_2)^c)$ and $\ell^{(2)}_{i_1}, \dots, \ell^{(2)}_{i_s} \in \mathcal{B}_2$;

$(b.ii)$ $[a, X^{'p}_{ij}(z) + Z^{'p}_{ij}\ell^{'p,(2)}_{ij}(z)] \in U(\mathfrak{g}^{'p})\langle b - (f^{'p}|b)\rangle_{b \in \mathfrak{g}^{'p,(2)}_{\geq 1}}[z]$, for every $a \in \mathfrak{g}^{'p,(2)}_{\geq \frac{1}{2}}$.

Let us first approach the simple case of the recursion. Namely, we would like to prove that

$$X^p_{ij}(z) + Z^p_{ij}\ell^{(2)}_{ij}(z) = \overline{W}^p_{ij}(z) \stackrel{?}{=} \sigma_l(\overline{W}^{'p}_{ij}(z)) = \sigma_l(X^{'p}_{ij}(z) + Z^{'p}_{ij}\ell^{'p,(2)}(z)) \tag{4.3.5}$$

holds for every $j \geq r_1 + 1$. By our hypotheses, $W^p_{ij}(z) = \sigma_l(W^{'p}_{ij}(z))$ for all $j \geq r_1 + 1$, hence we already know that

$$X^p_{ij}(z) + Y^p_{ij}\ell^{(1)}_{ij}(z) = \sigma_l(X^{'p}_{ij}(z) + Y^{'p}_{ij}\ell^{'p,(1)}_{ij}(z))$$

Our choice of a PBW basis allows us to conclude that $X^p_{ij}(z) = \sigma_l(X^{'p}_{ij}(z))$. By uniqueness of the generators $\rho^p_{0,\mathfrak{l}_1}(W^p_{ij}(z))$, for (4.3.5) to hold it is therefore sufficient to show that

$$X^p_{ij}(z) + \sigma_l(Z^{'p}_{ij}\ell^{(2)}_{ij}(z)) \in W(\mathfrak{g}, f, \Gamma_2, 0)[z]\,. \tag{4.3.6}$$

By definition, this is the case if and only if condition $(a.ii)$ holds. Since $X^p_{ij}(z) + Z^{'p}_{ij}\ell^{(2)}_{ij}(z) \in W(\mathfrak{g}^{'p}, f^{'p}, \Gamma^{'p}_2, 0)[z]$, we already know that condition $(b.ii)$ holds, namely that

$$[a, X^p_{ij}(z) + Z^{'p}_{ij}\ell^{'p,(2)}_{ij}(z)] \in U(\mathfrak{g}^{'p,(2)})\langle b - (f^{'p}|b)\rangle_{b \in \mathfrak{g}^{'p,(2)}_{\geq 1}}[z]$$

for every $a \in \mathfrak{g}^{'p,(2)}_{\geq \frac{1}{2}}$.

Since $\sigma_l\big(U(\mathfrak{g}^{'p})\langle b - (f^{'p}|b)\rangle_{b \in \mathfrak{g}^{'p,(2)}_{\geq 1}}\big) \subset U(\mathfrak{g})\langle b - (f|b)\rangle_{b \in \mathfrak{g}^{(2)}_{\geq 1}}$, it is sufficient to check what happens for the adjoint action of $a \in \mathfrak{g}^{p_1,(2)}_{\geq \frac{1}{2}}$, which is a complementary subspace to $\sigma_l(\mathfrak{g}^{'p,(2)}_{\geq \frac{1}{2}})$ in $\mathfrak{g}^{(2)}_{\geq \frac{1}{2}}$ made of all those elements $e_{(i,h)(j,k)} \in \mathfrak{g}^{(2)}_{\geq \frac{1}{2}}$ such that $x(jk) = -\frac{d}{2}$ (and also a complementary subspace to $\sigma_l(\mathfrak{g}^{'p,(2)}_{\geq 1})$ in $\mathfrak{g}^{(2)}_{\geq 1}$, as consequence of the hypothesis that $'p$ with respect to the grading Γ_2 exists).

Consider the PBW basis for $U(\mathfrak{g}^{(2)}_{\leq 0}) \otimes F(\mathfrak{g}^{(2)}_{\frac{1}{2}})$ as above. We can decompose each $\mathfrak{g}_{-k}$, $k \geq 0$ as $\sigma_l(\mathfrak{g}^{'p}_{-k}) \oplus \mathfrak{g}^{p_1,(2)}_{-k}$. We obtain a decomposition

$$U(\mathfrak{g}^{(2)}_{\leq 0}) \otimes F(\mathfrak{g}^{(2)}_{\frac{1}{2}}) = \Big(U(\sigma_l(\mathfrak{g}^{'p,(2)}_{\leq 0})) \otimes F(\mathfrak{g}^{(2)}_{\frac{1}{2}}) \Big) \oplus \Big(U(\mathfrak{g}^{p_1,(2)}_{\leq 0}) \otimes F(\mathfrak{g}^{(2)}_{\frac{1}{2}}) \Big)$$

where $U(\sigma_l(\mathfrak{g}^{'p,(2)}_{\leq 0}))$ consists of all monomials whose factors only belong to $\sigma_l(\mathfrak{g}^{'p,(2)}_{\leq 0})$, and $U(\mathfrak{g}^{p_1,(2)}_{\leq 0})$ contains all the other possible monomials. Namely, each monomial in $U(\mathfrak{g}^{p_1,(2)}_{\leq 0})$ contains at least a factor belonging to a $\mathfrak{g}^{p_1,(2)}_{-k}$, for some $k \geq 0$.

We shall split $Z^p_{ij} \ell^{(2)}_{ij}(z) = Z^{(1)\,p}_{ij} \ell^{(2)}_{ij}(z) + Z^{(2)\,p}_{ij} \ell^{(2)}_{ij}(z)$ accordingly, where $Z^{(1)\,p}_{ij} \ell^{(2)}_{ij}(z) \in U(sigma_l(\mathfrak{g}^{'p,(2)}_{\leq 0})) \otimes F(\mathfrak{g}^{(2)}_{\frac{1}{2}})[z]$.

Similarly, we can decompose

$$\begin{aligned} &U(\mathfrak{g}^{(2)})\langle b - (f|b)\rangle_{b \in \mathfrak{g}^{(2)}_{\geq 1}} = U(\mathfrak{g}^{(2)})\langle b - (f^{'p}|b)\rangle_{b \in \sigma_l(\mathfrak{g}^{'p,(2)}_{\geq 1})} \oplus U(\mathfrak{g}^{(2)})\langle b - (f|b)\rangle_{b \in \mathfrak{g}^{p_1,(2)}_{\geq 1}} \\ &= U(\sigma_l(\mathfrak{g}^{'p,(2)}))\langle b - (f^{'p}|b)\rangle_{b \in \sigma_l(\mathfrak{g}^{'p,(2)}_{\geq 1})} \oplus U(\mathfrak{g}^{p_1,(2)})\langle b - (f^{'p}|b)\rangle_{b \in \sigma_l(\mathfrak{g}^{'p,(2)}_{\geq 1})} \oplus U(\mathfrak{g}^{(2)})\langle b - (f|b)\rangle_{b \in \mathfrak{g}^{p_1,(2)}_{\geq 1}}\,. \end{aligned}$$

Thus, for $a \in \sigma_l(\mathfrak{g}^{'p,(2)}_{\geq \frac{1}{2}})$ we have

$$[a, X^p_{ij}(z) + Z^{(1)\,p}_{ij} \ell^{(2)}_{ij}(z)] \in U(\sigma_l(\mathfrak{g}^{'p,(2)}))\langle b - (f^{'p}|b)\rangle_{b \in \sigma_l(\mathfrak{g}^{'p,(2)}_{\geq 1})}[z]\,,$$

because all terms belong to $U(\sigma_l(\mathfrak{g}^{'p,(2)}))[z]$. As a consequence, by uniqueness of the generators $(\rho^{'p}_{0,\mathfrak{l}_2})^{-1}(\rho^{'p}_{0,\mathfrak{l}_1}(W^{'p}_{ij}(z)))$, we have $X^p_{ij}(z) + Z^{(1)\,p}_{ij} \ell^{(2)}_{ij}(z) = X^p_{ij}(z) + \sigma_l(Z^{'p}_{ij} \ell^{'p,(2)}_{ij}(z))$. It implies

$$X^p_{ij}(z) + Z^p_{ij} \ell^{(2)}_{ij}(z) = X^p_{ij}(z) + \sigma_l(Z^{'p}_{ij} \ell^{'p,(2)}_{ij}(z)) + Z^{(2)\,p}_{ij} \ell^{(2)}_{ij}(z).$$

The identities

(i) $[a, X^p_{ij}(z) + \sigma_l(Z^{'p}_{ij} \ell^{'p,(2)}_{ij}(z)) + Z^{(2)\,p}_{ij} \ell^{(2)}_{ij}(z)] \in U(\mathfrak{g}^{(2)})\langle b - (f|b)\rangle_{b \in \mathfrak{g}^{(2)}_{\geq 1}}[z]$ for every $a \in \sigma_l(\mathfrak{g}^{'p,(2)}_{\geq \frac{1}{2}})$;

(ii) $[a, X^p_{ij}(z) + \sigma_l(Z^{'p}_{ij} \ell^{'p,(2)}_{ij}(z))] \in U(\sigma_l(\mathfrak{g}^{'p,(2)}))\langle b - (f^{'p}|b)\rangle_{b \in \sigma_l(\mathfrak{g}^{'p,(2)}_{\geq 1})}[z]$ for every $a \in \sigma_l(\mathfrak{g}^{'p,(2)}_{\geq \frac{1}{2}})$;

also imply

$$[a, Z^{(2)\,p}_{ij} \ell^{(2)}_{ij}(z)] \in U(\mathfrak{g}^{(2)})\langle b - (f|b)\rangle_{b \in \mathfrak{g}^{(2)}_{\geq 1}}[z], \text{ for every } a \in \sigma_l(\mathfrak{g}^{'p,(2)}_{\geq \frac{1}{2}})\,.$$

We shall now apply Lemma 4.3.1 below to $u = Z^{(2)\,p}_{ij} \ell^{(2)}_{ij}(z)$ and $\mathfrak{l} = \mathfrak{l}_2$ to conclude that $Z^{(2)\,p}_{ij} \ell^{(2)}_{ij}(z) = 0$ in $U(\mathfrak{g}^{(2)})/U(\mathfrak{g}^{(2)})\langle b - (f|b)\rangle_{b \in \mathfrak{g}^{(2)}_{\geq 1}}[z]$, proving (4.3.6)

Let us now suppose $j \leq r_1$. In this case, from the recursion (3.2.2) for $W^p_{ij}(z)$, under the identifications (4.3.4), we get[2]

$$\begin{aligned} W^p_{ij}(z) = \mathcal{F}(\sigma_l(W^{'p}_{ij}(z))) &:= [e_{(j,p_1)(j,p_1-1)}, \sigma_l(W^{'p}_{ij}(z))] \\ &- \sum_{h=1}^{r_1} \sigma_l(W^{'p}_{ih}(z))(\delta_{hj} z + \tilde{e}_{(j,p_1)(h,p_1)}) + \sum_{h=r_1+1}^{r} \sigma_l(W^{'p}_{ih}(z)) \sigma_l(W^{'p}_{hj;p_h-1}). \end{aligned}$$

[2]Note that under the hypotheses of Theorem 4.3.1 we have $e_{(j,p_1)(j,p_1-1)} \in \mathfrak{g}^{(1)}_{-1} \cap \mathfrak{g}^{(2)}_{-1}$ and $e_{(j,p_1)(h,p_1)} \in \mathfrak{g}^{(1)}_0 \cap \mathfrak{g}^{(2)}_0$ for every $1 \leq j, h \leq r_1$ (cf. (1.1.8)) and moreover the numeric shifts $\delta_{jh} d_{(jp_1)}$ attached to $e_{(j,p_1)(h,p_1)}$ are the same with respect to both gradings. Also, $[e_{(j,p_1)(h,p_1)}, \sigma_l(\mathfrak{g}^{'p,(i)}_{\frac{1}{2}})] = 0$.

With the notation as before, then $\overline{W}^p_{ij}(z)=\mathcal{F}(\sigma_l(\overline{W}'^p_{ij}(z)))$ holds if and only if

$$(\rho^p_{0,\mathfrak{l}_2})^{-1}(\rho^p_{0,\mathfrak{l}_1}(\mathcal{F}(\sigma_l(W'^p_{ij}(z)))))=\mathcal{F}(\sigma_l((\rho'^p_{0,\mathfrak{l}_2})^{-1}(\rho'^p_{0,\mathfrak{l}_1}(W'^p_{ij}(z)))))\in U(\mathfrak{g}^{(2)}_{\leq 0})\otimes F(\mathfrak{g}^{(2)}_{\frac{1}{2}})[z]\,. \tag{4.3.7}$$

Let us consider the RHS of Equation (4.3.7). We have (with a slight abuse of notation)

$$\begin{aligned}
&\mathcal{F}(\sigma_l((\rho'^p_{0,\mathfrak{l}_2})^{-1}(\rho'^p_{0,\mathfrak{l}_1}(W'^p_{ij}(z)))))=\mathcal{F}(\sigma_l(X'^p_{ij}(z)+Z'^p_{ij}\ell'^{p,(2)}_{ij}(z)))\\
&=[e_{(j,p_1)(j,p_1-1)},\sigma_l(X'^p_{ij}(z)+Z'^p_{ij}\ell'^{p,(2)}_{ij}(z))]-\sum_{h=1}^{r_1}\sigma_l(X'^p_{ih}(z)+Z'^p_{ih}\ell'^{p,(2)}_{ih}(z))(\delta_{hj}z+\tilde{e}_{(j,p_1)(h,p_1)})\\
&+\sum_{h=r_1+1}^{r}\sigma_l(X'^p_{ih}(z)+Z'^p_{ih}\ell'^{p,(2)}_{ih}(z))\sigma_l(X'^p_{hj;p_h-1}+Z'^p_{hj;p_h-1}\ell'^{p,(2)}_{hj;p_h-1})\\
&=\mathcal{F}(\sigma_l(X'^p_{ij}(z)))+[e_{(j,p_1)(j,p_1-1)},\sigma_l(Z'^p_{ij}\ell'^{p,(2)}_{ij}(z))]-\sum_{h=1}^{r_1}\sigma_l(Z'^p_{ih}\ell'^{p,(2)}_{ih}(z))(\delta_{hj}z+\tilde{e}_{(j,p_1)(h,p_1)})\\
&+\sum_{h=r_1+1}^{r}\Big(\sigma_l(X'^p_{ih}(z))\sigma_l(Z'^p_{hj;p_h-1}\ell'^{p,(2)}_{hj;p_h-1})+\sigma_l(Z'^p_{ih}\ell'^{p,(2)}_{ih}(z))\sigma_l(X'^p_{hj;p_h-1})\\
&+\sigma_l(Z'^p_{ih}\ell'^{p,(2)}_{ih}(z))\sigma_l(Z'^p_{hj;p_h-1}\ell'^{p,(2)}_{hj;p_h-1})\Big)\\
&=\mathcal{F}(\sigma_l(X'^p_{ij}(z)))+[e_{(j,p_1)(j,p_1-1)},\sigma_l(Z'^p_{ij}(z))]\sigma_l(\ell'^{p,(2)}_{ij}(z))-\sum_{h=1}^{r_1}\sigma_l(Z'^p_{ih}(z))(\delta_{hj}z+\tilde{e}_{(j,p_1)(h,p_1)})\sigma_l(\ell'^{p,(2)}_{ih}(z))\\
&+\sum_{h=r_1+1}^{r}\Big(\sigma_l(X'^p_{ih}(z))\sigma_l(Z'^p_{hj;p_h-1}\sigma_l(\ell'^{p,(2)}_{hj;p_h-1}))+\sigma_l(Z'^p_{ih}\ell'^{p,(2)}_{ih}(z))\sigma_l(Z'^p_{hj;p_h-1}\ell'^{p,(2)}_{hj;p_h-1})\Big)\\
&+\sigma_l(Z'^p_{ih}(z))\sigma_l(X'^p_{hj;p_h-1})\sigma_l(\ell'^{p,(2)}_{ih}(z))+\sigma_l(Z'^p_{ih}(z))[\sigma_l(\ell'^{p,(2)}_{ih}(z)),\sigma_l(X'^p_{hj;p_h-1})]\,,
\end{aligned}\tag{4.3.8}$$

where the last equality follows from the fact that $[e_{(j,p_1)(j,p_1-1)},\sigma_l(x)]=0=[e_{(j,p_1)(h,p_1)},\sigma_l(x)]$ for each $x\in\mathfrak{l}'^p_2$. Moreover we have

$$\begin{aligned}
&[\sigma_l(\ell'^{p,(2)}_{ih}(z)),\sigma_l(X'^p_{hj;p_h-1})]=[\sigma_l(\ell'^{p,(2)}_{ih}(z)),\sigma_l(W'^p_{hj;p_h-1})]-[\sigma_l(\ell'^{p,(2)}_{ih}(z)),\sigma_l(Z'^p_{hj;p_h-1}\ell'^{p,(2)}_{hj;p_h-1})]\\
&=[\sigma_l(\ell'^{p,(2)}_{ih}(z)),\sigma_l(W'^p_{hj;p_h-1})]-[\sigma_l(\ell'^{p,(2)}_{ih}(z)),\sigma_l(Z'^p_{hj;p_h-1})]\sigma_l(\ell'^{p,(2)}_{hj;p_h-1})-\sigma_l(Z'^p_{hj;p_h-1})[\sigma_l(\ell'^{p,(2)}_{ih}(z)),\sigma_l(\ell'^{p,(2)}_{hj;p_h-1})].
\end{aligned}$$

The first term clearly lies in $U(\mathfrak{g})\langle b-(f|b)\rangle_{b\in\mathfrak{g}^{(2)}_{\geq 1}}$, and the last one vanishes because of Lemma 4.2.4. As a consequence, after reordering accordingly to the PBW basis chosen above,

$$\mathcal{F}(\sigma((\rho'^p_{0,\mathfrak{l}_2})^{-1}(\rho'^p_{0,\mathfrak{l}_1}(W'^p_{ij}(z)))))=\mathcal{F}(\sigma_l(X'^p_{ij}(z)))+\widetilde{Z}^p_{ij}\widetilde{\ell}^{(2)}_{ij}(z)\in U(\mathfrak{g}^{(2)}_{\leq 0})\otimes F(\mathfrak{g}^{(2)}_{\frac{1}{2}})[z]\,,$$

where $\widetilde{Z}^p_{ij}\widetilde{\ell}^{(2)}_{ij}(z)$ is a polynomial with coefficients in $U(\mathfrak{g}^{(2)}_{\leq 0})\otimes F(\mathfrak{g}^{(2)}_{\frac{1}{2}})\mathfrak{l}_2$.

As far as the LHS of Equation (4.3.7) is concerned, we clearly have

$$(\rho^p_{0,\mathfrak{l}_2})^{-1}(\rho^p_{0,\mathfrak{l}_1}(\mathcal{F}(\sigma_l(W'^p_{ij}(z)))))=\rho^p_{0,\mathfrak{l}_1}(\mathcal{F}(\sigma_l(W'^p_{ij}(z))))+\overline{Z}^p_{ij}\overline{\ell}^{(2)}_{ij}(z),$$

where $\overline{Z}^p_{ij}\overline{\ell}^{(2)}_{ij}(z)$ is a polynomial with coefficients in $U(\mathfrak{g}^{(2)}_{\leq 0})\otimes F(\mathfrak{g}^{(2)}_{\frac{1}{2}})\mathfrak{l}_2$.

Expanding $\mathcal{F}(\sigma_l(W'^p_{ij}(z)))$ as in (4.3.8) we obtain $\mathcal{F}(\sigma_l(W'^p_{ij}(z)))=\mathcal{F}(\sigma_l(X'^p_{ij}(z)))+\overline{Y}^p_{ij}\overline{\ell}^{(1)}_{ij}(z)$, where $\overline{Y}^p_{ij}\overline{\ell}^{(1)}_{ij}(z)$ is a polynomial with coefficients in $U(\mathfrak{g}^{(1)}_{\leq 0})\otimes F(\mathfrak{g}^{(1)}_{\frac{1}{2}})\mathfrak{l}_1$.

Since $X'^p_{ij}(z)$ is in $U(\mathfrak{g}'^{p,(1)}_{\leq 0})\otimes F((\mathfrak{l}'^p_1)^c)$, when applying $\mathcal{F}(\sigma_l(X'^p_{ij}(z)))$ we have

- $[\mathfrak{g}^{(1)}_{-1},\sigma_l(X'^p_{ij}(z))]\in U(\mathfrak{g}^{(1)}_{\leq 0})\otimes F(\mathfrak{l}^c_1)[z]$;

- $[\mathfrak{l}_1^c, e_{(j,p_1)(h,p_1)}] = 0$, for every $1 \le j, h \le r_1$;

- $[\mathfrak{l}_1^c, \sigma_l(\mathfrak{g}_0^{'p,(1)})] \subset \mathfrak{l}_1^c$, by Corollary 4.2.1.

Therefore, $\mathcal{F}(\sigma_l(X_{ij}^{'p}(z))) \in U(\mathfrak{g}_{\le 0}^{(1)}) \otimes F(\mathfrak{l}_1^c)[z]$ and

$$\rho_{0,\mathfrak{l}_1}^p(\mathcal{F}(\sigma_l(W_{ij}^{'p}(z)))) = \rho_{0,\mathfrak{l}_1}^p(\mathcal{F}(\sigma_l(X_{ij}^{'p}(z))) + \overline{Y}_{ij}^p \overline{\ell}_{ij}^{(1)}(z)) = \mathcal{F}(\sigma_l(X_{ij}^{'p}(z))).$$

It is now just sufficient to show that

$$[\mathfrak{g}_{\frac{1}{2}}^{(2)}, \mathcal{F}(\sigma((\rho_{0,\mathfrak{l}_2}^{'p})^{-1}(\rho_{0,\mathfrak{l}_1}^{'p}(W_{ij}^{'p}(z)))))] \in U(\mathfrak{g})\langle b-(f|b)\rangle_{b\in\mathfrak{g}_{\ge 1}^{(2)}}[z]. \tag{4.3.9}$$

In fact, if (4.3.9) holds then the following identity holds in $U(\mathfrak{g})/U(\mathfrak{g})\langle b-(f|b)\rangle_{b\in\mathfrak{g}_{\ge 1}^{(2)}}[z]$:

$$\begin{aligned}
0 &\equiv [\mathfrak{g}_{\frac{1}{2}}^{(2)}, \mathcal{F}(\sigma((\rho_{0,\mathfrak{l}_2}^{'p})^{-1}(\rho_{0,\mathfrak{l}_1}^{'p}(W_{ij}^{'p}(z))))) - (\rho_{0,\mathfrak{l}_2}^{p})^{-1}(\rho_{0,\mathfrak{l}_1}^{p}(\mathcal{F}(\sigma_l(W_{ij}^{'p}(z)))))] \\
&= [\mathfrak{g}_{\frac{1}{2}}^{(2)}, \mathcal{F}(\sigma_l(X_{ij}^{'p}(z))) + \widetilde{Z}_{ij}^p \widetilde{\ell}_{ij}^{(2)}(z) - \rho_{0,\mathfrak{l}_1}^p(\mathcal{F}(\sigma_l(W_{ij}^{'p}(z)))) - \overline{Z}_{ij}^p \overline{\ell}_{ij}^{(2)}(z)] \\
&= [\mathfrak{g}_{\frac{1}{2}}^{(2)}, \mathcal{F}(\sigma_l(X_{ij}^{'p}(z))) + \widetilde{Z}_{ij}^p \widetilde{\ell}_{ij}^{(2)}(z) - \mathcal{F}(\sigma_l(X_{ij}^{'p}(z))) - \overline{Z}_{ij}^p \overline{\ell}_{ij}^{(2)}(z)] \\
&= [\mathfrak{g}_{\frac{1}{2}}^{(2)}, \widetilde{Z}_{ij}^p \widetilde{\ell}_{ij}^{(2)}(z) - \overline{Z}_{ij}^p \overline{\ell}_{ij}^{(2)}(z)].
\end{aligned}$$

Applying Lemma 4.3.1 to $\mathfrak{l} = \mathfrak{l}_2 \subseteq \mathfrak{g}_{\frac{1}{2}}^{(2)}$ and $u = \widetilde{Z}_{ij}^p \widetilde{\ell}_{ij}^{(2)}(z) - \overline{Z}_{ij}^p \overline{\ell}_{ij}^{(2)}(z) \in U(\mathfrak{g}_{\le 0}^{(2)}) \otimes F(\mathfrak{g}_{\frac{1}{2}}^{(2)})\mathfrak{l}_2$, Equation (4.3.7) holds.

Given $a \in \mathfrak{g}_{\frac{1}{2}}^{(2)}$ we have

- $$[a, [e_{(j,p_1)(j,p_1-1)}, \sigma_l(X_{ij}^{'p}(z) + Z_{ij}^{'p}\ell_{ij}^{(2)}(z))]] = [\underbrace{[a, e_{(j,p_1)(j,p_1-1)}]}_{=0}, \sigma_l(X_{ij}^{'p}(z) + Z_{ij}^{'p}\ell_{ij}^{(2)}(z))]$$
$$+ [e_{(j,p_1)(j,p_1-1)}, \underbrace{[a, \sigma_l(X_{ij}^{'p}(z) + Z_{ij}^{'p}\ell_{ij}^{(2)}(z))]}_{\in\sigma_l(U(\mathfrak{g}'^p)\langle b-(f'^p|b)\rangle_{b\in\mathfrak{g}_{\ge 1}^{'p,(2)}})[z]}] \in U(\mathfrak{g})\langle b-(f|b)\rangle_{b\in\mathfrak{g}_{\ge 1}^{(2)}}[z]$$

- $$[a, \sum_{h=1}^{r_1} \sigma_l(X_{ih}^{'p}(z) + Z_{ih}^{'p}\ell_{ih}^{(2)}(z))(\delta_{hj}z + \tilde{e}_{(j,p_1)(h,p_1)})] = \sum_{h=1}^{r_1} \underbrace{[a, \sigma_l(X_{ih}^{'p}(z) + Z_{ih}^{'p}\ell_{ih}^{(2)}(z))]}_{\in\sigma_l(U(\mathfrak{g}'^p)\langle b-(f'^p|b)\rangle_{b\in\mathfrak{g}_{\ge 1}^{'p,(2)}})[z]}(\delta_{hj}z + \delta_{hj}d_{(j,p_1)})$$
$$+ \sum_{h=1}^{r_1} \underbrace{[a, \sigma_l(X_{ih}^{'p}(z) + Z_{ih}^{'p}\ell_{ih}^{(2)}(z))]}_{\in\sigma_l(U(\mathfrak{g}'^p)\langle b-(f'^p|b)\rangle_{b\in\mathfrak{g}_{\ge 1}^{'p,(2)}})[z]} \underbrace{e_{(j,p_1)(h,p_1)}}_{\in\mathfrak{g}_0^{(2)}} + \sum_{h=1}^{r_1} \sigma_l(X_{ih}^{'p}(z) + Z_{ih}^{'p}\ell_{ih}^{(2)}(z)) \underbrace{[a, e_{(j,p_1)(h,p_1)}]}_{=0}$$
$$\in U(\mathfrak{g})\langle b-(f|b)\rangle_{b\in\mathfrak{g}_{\ge 1}}[z];$$

- $$[a, \sum_{h\ge r_1+1} \sigma_l(X_{ih}^{'p}(z) + Z_{ih}^{'p}\ell_{ih}^{(2)}(z))\sigma_l(X_{ih;p_h-1}^{'p} + Z_{ih;p_h-1}^{'p}\ell_{ih;p_h-1}^{(2)})]$$
$$\in \sigma_l(U(\mathfrak{g}^{'p})\langle b-(f|b)\rangle_{b\in\mathfrak{g}_{\ge 1}^{'p,(2)}})[z] \subseteq U(\mathfrak{g})\langle b-(f|b)\rangle_{b\in\mathfrak{g}_{\ge 1}^{(2)}}[z];$$

where the last inclusion is due to the fact that $\sigma_l(X_{ih}^{'p}(z) + Z_{ih}^{'p}\ell_{ih}^{(2)}(z))\sigma_l(X_{ih;p_h-1}^{'p} + Z_{ih;p_h-1}^{'p}\ell_{ih;p_h-1}^{(2)})$ is a product of elements in $W(\mathfrak{g}'^p, f'^p, \Gamma_2^{'p}, 0)[z]$. ◻

Corollary 4.3.1. *Let Γ_1 and Γ_2 be a pair of good $\frac{1}{2}\mathbb{Z}$-gradings for f as in Theorem 4.3.1, and let $\mathfrak{l} \subset \mathfrak{g}^{(2)}_{\frac{1}{2}}$ be an isotropic subspace with respect to the bilinear form ω. By [BG05], we have associative algebra isomorphisms*

$$\begin{aligned} \overline{\Phi}_p &: W(\mathfrak{g}, f, \Gamma_1, 0)[z] \longrightarrow W(\mathfrak{g}, f, \Gamma_2, \mathfrak{l})[z] \\ \overline{\Phi}_{'p} &: W(\mathfrak{g}^{'p}, f^{'p}, \Gamma_1^{'p}, 0)[z] \longrightarrow W(\mathfrak{g}^{'p}, f^{'p}, \Gamma_2^{'p}, \mathfrak{l}^{'p})[z]\,. \end{aligned} \tag{4.3.10}$$

Let $\{W^{'p}_{ij}(z)\}_{1\le i,j\le r}$ be a choice of generators for $W(\mathfrak{g}^{'p}, f^{'p}, \Gamma_1^{'p}, 0)[z]$ and $\{W^{p}_{ij}(z)\}_{1\le i,j\le r}$ a choice of generators for $W(\mathfrak{g}, f, \Gamma_1, 0)[z]$ such that they are related by the recursive relation (3.2.2). Then, there exist a choice of generators $\{\overline{\Phi}_{'p}(W^{'p}_{ij}(z))\}_{1\le i,j\le r}$ for $W(\mathfrak{g}^{'p}, f^{'p}, \Gamma_2^{'p}, \mathfrak{l}^{'p})[z]$ and $\{\overline{\Phi}_p(W^{p}_{ij}(z))\}_{1\le i,j\le r}$ for $W(\mathfrak{g}, f, \Gamma_2, \mathfrak{l})[z]$ for which the same recursive relation (3.2.2) holds.

Proof. By Theorem 4.3.1, there exist a choice of generators $\{\overline{W}^{'p}_{ij}(z)\}_{1\le i,j\le r}$ for $W(\mathfrak{g}^{'p}, f^{'p}, \Gamma_2^{'p}, 0)[z]$ and $\{\overline{W}^{p}_{ij}(z)\}_{1\le i,j\le r}$ for $W(\mathfrak{g}, f, \Gamma_2, 0)[z]$ for which the same recursive relation (3.2.2) holds. Therefore we just need to show how to extend the recursion to a generic isotropic subspace $\mathfrak{l}$.

Let $\rho^p_{0,\mathfrak{l}} : W(\mathfrak{g}, f, \Gamma_2, 0) \longrightarrow W(\mathfrak{g}, f, \Gamma_2, \mathfrak{l})$ and $\rho^{'p}_{0,\mathfrak{l}} : W(\mathfrak{g}^{'p}, f^{'p}, \Gamma_2^{'p}, 0) \longrightarrow W(\mathfrak{g}^{'p}, f^{'p}, \Gamma_2^{'p}, \mathfrak{l}^{'p})$ be the associative algebra isomorphisms as in (4.2.2). We shall denote

$$\widehat{W}^{'p}_{ij}(z) := \rho^{'p}_{0,\mathfrak{l}}(\overline{W}^{'p}_{ij}(z)), \qquad \widehat{W}^{p}_{ij}(z) := \rho^{p}_{0,\mathfrak{l}}(\overline{W}^{p}_{ij}(z)), \quad 1 \le i,j \le r.$$

Choose PBW bases for $U(\mathfrak{g})$ and $U(\mathfrak{g}^{'p})$ as in 4.3.1 (denote by $\mathcal{B}$ an ordered basis of $\mathfrak{l}$), and use the identifications as in (4.3.4). Then,

$$\overline{W}^{p}_{ij}(z) = (\rho^{p}_{0,\mathfrak{l}})^{-1}(\widehat{W}^{p}_{ij}(z)) = \widehat{W}^{p}_{ij}(z) + Y^{p}_{ij}\ell^{p}_{ij}(z)$$

where $Y^{p}_{ij}\ell^{p}_{ij}(z)$ is a polynomial whose coefficients are sums of monomials $Y^{p}_{i_1\cdots i_s}\ell^{p}_{i_1}\cdots\ell^{p}_{i_s}$, $s\ge 0$, with $Y^{p}_{i_1\cdots i_s} \in U(\mathfrak{g}^{(2)}_{\le 0}) \otimes F(\mathfrak{l}^c)$ and $\ell^{p}_{i_1},\dots,\ell^{p}_{i_s} \in \mathcal{B}$. Similarly,

$$\overline{W}^{'p}_{ij}(z) = (\rho^{'p}_{0,\mathfrak{l}})^{-1}(\widehat{W}^{'p}_{ij}(z)) = \widehat{W}^{'p}_{ij}(z) + Y^{'p}_{ij}\ell^{'p}_{ij}(z)$$

where $Y^{'p}_{ij}\ell^{'p}_{ij}(z)$ is a polynomial whose coefficients are sums of monomials $Y^{'p}_{i_1\cdots i_s}\ell^{'p}_{i_1}\cdots\ell^{'p}_{i_s}$, $s\ge 0$, with $Y^{'p}_{i_1\cdots i_s} \in U(\mathfrak{g}^{'p,(2)}_{\le 0}) \otimes F((\mathfrak{l}^{'p})^c)$ and $\ell^{'p}_{i_1},\dots,\ell^{'p}_{i_s} \in \mathcal{B}$.

Let us first suppose that $j \ge r_1+1$; in this case by hypothesis we have $\overline{W}^{p}_{ij}(z) = \sigma_l(\overline{W}^{'p}_{ij}(z))$. Then

$$\widehat{W}^{p}_{ij}(z) + Y^{p}_{ij}\ell^{p}_{ij}(z) = \overline{W}^{p}_{ij}(z) = \sigma_l(\overline{W}^{'p}_{ij}(z)) = \sigma_l(\widehat{W}^{'p}_{ij}(z) + Y^{'p}_{ij}\ell^{'p}_{ij}(z))$$

that implies $\widehat{W}^{p}_{ij}(z) = \sigma_l(\widehat{W}^{'p}_{ij}(z))$. Suppose now that $j \le r_1$; in this case $\overline{W}^{p}_{ij}(z) = \mathcal{F}(\sigma_l(\overline{W}^{'p}_{ij}(z)))$. We have

$$\begin{aligned}
&\widehat{W}^{p}_{ij}(z)) + Y^{p}_{ij}\ell^{p}_{ij}(z) = [e_{(j,p_1)(j,p_1-1)}, \sigma_l(\widehat{W}^{'p}_{ij}(z) + Y^{'p}_{ij}\ell^{'p}_{ij}(z))] \\
&- \sum_{h=1}^{r_1} \sigma_l(\widehat{W}^{'p}_{ih}(z) + Y^{'p}_{ih}\ell^{'p}_{ih}(z))(\delta_{hj}z + \tilde{e}_{(j,p_1)(h,p_1)}) + \sum_{h\ge r_1+1} \sigma_l(\widehat{W}^{'p}_{ih}(z) + Y^{'p}_{ih}\ell^{'p}_{ih}(z))\sigma_l(\widehat{W}^{'p}_{hj;p_h-1} \\
&+ Y^{'p}_{hj;p_h-1}\ell^{'p}_{hj;p_h-1}) \\
&= \mathcal{F}(\sigma_l(\widehat{W}^{'p}_{ij}(z))) + [e_{(j,p_1)(j,p_1-1)}, \sigma_l(Y^{'p}_{ij}\ell^{'p}_{ij}(z))] \\
&- \sum_{h=1}^{r_1} \sigma_l(Y^{'p}_{ih}\ell^{'p}_{ih}(z))(\delta_{hj}z + \tilde{e}_{(j,p_1)(h,p_1)}) + \sum_{h\ge r_1+1} \sigma_l(Y^{'p}_{ih}\ell^{'p}_{ih}(z))\sigma_l(\widehat{W}^{'p}_{hj;p_h-1} + Y^{'p}_{hj;p_h-1}\ell^{'p}_{hj;p_h-1}) \\
&+ \sum_{h\ge r_1+1} \sigma_l(\widehat{W}^{'p}_{ih}(z))\sigma_l(Y^{'p}_{hj;p_h-1}\ell^{'p}_{hj;p_h-1})\,.
\end{aligned}$$

After reordering according to our choice of a PBW basis we obtain $\widehat{W}^p_{ij}(z)+Y^p_{ij}\ell^p_{ij}(z)=\mathcal{F}(\sigma_l(\widehat{W}'^p_{ij}(z)))+Z^p_{ij}\widehat{\ell}^p_{ij}(z)$, and we can conclude that $\widehat{W}^p_{ij}(z)=\mathcal{F}(\sigma_l(\widehat{W}'^p_{ij}(z)))$. Note in fact that $\mathcal{F}(\sigma_l(\widehat{W}'^p_{ij}(z)))\in U(\mathfrak{g}^{(2)}_{\leq 0})\otimes F(\mathfrak{l}^c)$. Moreover, $\sigma_l(\ell'^p_{ih}(z))\sigma_l(\widehat{W}'^p_{hj;p_h-1})=\sigma_l(\widehat{W}'^p_{hj;p_h-1})\sigma_l(\ell'^p_{ih}(z))+[\sigma_l(\ell'^p_{ih}(z)),\sigma_l(\widehat{W}'^p_{hj;p_h-1})]$ and by our hypothesis $[\sigma_l(\ell'^p_{ih}(z)),\sigma_l(\widehat{W}'^p_{hj;p_h-1})]\equiv 0$ in $W(\mathfrak{g},f,\Gamma,\mathfrak{l})$. □

Remark 4.3.2. An alternative version of Corollary 4.3.1 can be stated, and proved with a similar method, in the case when we remove the rightmost column of the pyramid.

Lemma 4.3.1. *Let $\mathfrak{l}\subset\mathfrak{g}_{\frac12}$ be a Lagrangian subspace with respect to the bilinear form ω as in (4.2.13) (resp. (4.2.14)). By the decomposition (4.2.19) we get $\mathfrak{l}^c$ as in (4.2.14) (resp. (4.2.13)).*

Let $u\in U(\mathfrak{g}_{\leq 0})\otimes F(\mathfrak{g}_{\frac12})$ such that $[\mathfrak{l}^c,u]\equiv 0$ in the quotient space $U(\mathfrak{g})/U(\mathfrak{g})\langle b-(f|b)\rangle_{b\in\mathfrak{g}_{\geq 1}}$. Decompose

$$u=\sum_{\substack{s\geq 0\\ i_1\leq\ldots\leq i_s}} u_s\ell_{i_1}\cdots\ell_{i_s}, \tag{4.3.11}$$

where $\ell_{i_j}\in\mathfrak{l}$, $u_s\in U(\mathfrak{g}_{\leq 0})\otimes F(\mathfrak{l}^c)$ and suppose moreover that $u_0=0$ (namely, all the summands in u have at least one element of $\mathfrak{l}$). Then, $u\equiv 0$.

Proof. Choose a PBW basis for $U(\mathfrak{g}_{\leq 0})\otimes F(\mathfrak{g}_{\frac12})$ as in Theorem 4.3.1. Then, $u\in U(\mathfrak{g}_{\leq 0})\otimes F(\mathfrak{g}_{\frac12})$ can be written as

$$\begin{aligned} u=\sum_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} C_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}\mathbf{x}^{\boldsymbol{\gamma}}\overline{\boldsymbol{\ell}}^{\boldsymbol{\beta}}\boldsymbol{\ell}^{\boldsymbol{\alpha}} &= \sum_{M\geq 0}\sum_{|\boldsymbol{\alpha}|=M}\sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} C^M_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}\mathbf{x}^{\boldsymbol{\gamma}}\overline{\boldsymbol{\ell}}^{\boldsymbol{\beta}}\ell_1^{\alpha_1}\cdots\ell_m^{\alpha_m}\\ &=\sum_{M\geq 1}\sum_{|\boldsymbol{\alpha}|=M}\sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} C^M_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}\mathbf{x}^{\boldsymbol{\gamma}}\overline{\boldsymbol{\ell}}^{\boldsymbol{\beta}}\ell_1^{\alpha_1}\cdots\ell_m^{\alpha_m}, \end{aligned} \tag{4.3.12}$$

where for the last equality we have used the hypothesis that $u_0=0$, and

- $\mathbf{x}^{\boldsymbol{\gamma}}=x_{-d,1}^{\gamma_{-d,1}}\cdots x_{-d,s_{-d}}^{\gamma_{-d,s_{-d}}}\cdots x_{0,1}^{\gamma_{0,1}}\cdots x_{0,s_0}^{\gamma_{0,s_0}}$, with $\boldsymbol{\gamma}=(\gamma_{-d,1},\ldots,\gamma_{-d,s_{-d}},\ldots,\gamma_{,1},\ldots,\gamma_{0,s_0})\in\mathbb{N}^{s_{-d}+\ldots+s_0}$,
- $\overline{\boldsymbol{\ell}}^{\boldsymbol{\beta}}=\overline{\ell}_1^{\beta_1}\cdots\overline{\ell}_m^{\beta_m}$, with $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_m)\in\mathbb{N}^m$,
- $\boldsymbol{\ell}^{\boldsymbol{\alpha}}=\ell_1^{\alpha_1}\cdots\ell_m^{\alpha_m}$, with $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_m)\in\mathbb{N}^m$,
- $|\boldsymbol{\alpha}|=\alpha_1+\ldots+\alpha_m$, for $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_m)\in\mathbb{N}^m$,
- $C_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}$, $C^M_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}\in\mathbb{C}$.

For $\overline{\ell}\in\mathfrak{l}^c$, we have[3]

$$[\overline{\ell},u]=\sum_{M\geq 1}\sum_{|\boldsymbol{\alpha}|=M}\sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} C^M_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}}\Big([\overline{\ell},x^{\boldsymbol{\gamma}}]\overline{\boldsymbol{\ell}}^{\boldsymbol{\beta}}\boldsymbol{\ell}^{\boldsymbol{\alpha}}+x^{\boldsymbol{\gamma}}\overline{\boldsymbol{\ell}}^{\boldsymbol{\beta}}[\overline{\ell},\boldsymbol{\ell}^{\boldsymbol{\alpha}}]\Big). \tag{4.3.13}$$

Moreover,

$$[\overline{\ell},\boldsymbol{\ell}^{\boldsymbol{\alpha}}]=\sum_{i=1}^m \ell_1^{\alpha_1}\cdots\ell_{i-1}^{\alpha_{i-1}}\Big(\sum_{h=1}^{\alpha_i}\ell_i^{h-1}[\overline{\ell},\ell_i]\ell_i^{\alpha_i-h}\Big)\ell_{i+1}^{\alpha_{i+1}}\cdots\ell_m^{\alpha_m},$$

with $\sum_{h=1}^{\alpha_i}\ell_i^{h-1}[\overline{\ell},\ell_i]\ell_i^{\alpha_i-h}\equiv(f|[\overline{\ell},\ell_i])\alpha_i\ell_i^{\alpha_i-1}$ in the case when $[\overline{\ell},\ell_i]\neq 0$.

Note that, by Corollary 4.2.1, $[\overline{\ell},\mathfrak{g}_{<0}]\subset\mathfrak{g}_{\leq 0}$ and $[\overline{\ell},\mathfrak{g}_0]\subseteq\mathfrak{l}^c$. It implies that $[\overline{\ell},x^{\boldsymbol{\gamma}}]$ does not increase the total amount of factors from $\mathfrak{l}$ appearing.

By Remark 2.2.2, the dual of $\mathfrak{l}$ with respect to the bilinear form $(\cdot|\cdot)$ is $\mathfrak{l}^*=[f,\mathfrak{l}^c]$. As a consequence, for each $\ell\in\mathfrak{l}$ there exists an $\overline{\ell}\in\mathfrak{l}^c$ such that $(f|[\ell,\overline{\ell}])=\pm 1$. We denote this element by ℓ^*.

[3] Note that $[\mathfrak{l}^c,\mathfrak{l}^c]=0$ by Corollary 4.2.1.

Remark 4.3.3. Given $\ell = e_{(i,h)(j,k)} \in \mathfrak{l}$, (i,h), $(j,k) \in \mathcal{T}$, we can explicitly describe ℓ^*. In fact we have

$$(f|[e_{(j,k-1)(i,h)}, e_{(i,h)(j,k)}]) = 1;$$
$$(f|[e_{(j,k)(i,h+1)}, e_{(i,h)(j,k)}]) = -1$$

while $(f|[e_{(a,b)(c,d)}, e_{(i,h)(j,k)}]) = 0$ for any other $e_{(a,b)(c,d)}$. By Lemma 4.2.4, both $e_{(j,k-1)(i,h)}$ and $e_{(j,k)(i,h+1)}$ belong to $\mathfrak{l}^c$. In the following, we will consider $\mathfrak{l}^* = e_{(j,k-1)(i,h)}$ if $\mathfrak{l} = e_{(i,h)(j,k)}$ with $k > 1$, and $\mathfrak{l}^* = e_{(j,k)(i,h+1)}$ otherwise.

First suppose that there exists at least one nonzero coefficient $C^M_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} \in \mathbb{C}$ with $\alpha_m \geq 1$ (i.e. $\boldsymbol{\alpha} = \mathbf{e}_m + \boldsymbol{\alpha}'$, for $\mathbf{e}_m = (0,\dots,0,1)$ and $\boldsymbol{\alpha}' \in \mathbb{N}^m$), and consider the bracket with the element $\bar{\ell} := \ell^*_m \in \mathfrak{l}^c$. We may assume that $(f|[\ell^*_m, \ell_m]) = 1$, By Remark 4.3.4 below there exists at most one other element $\ell_i \in \mathfrak{l}$, with $i \neq m$, such that $(f|[\ell^*_m, \ell_i]) \neq 0$ (and in this case, $(f|[\ell^*_m, \ell_i]) = -(f|[\ell^*_m, \ell_m]) = -1$).

By (4.3.13),

$$\begin{aligned}
[\ell^*_m, u] \equiv &\sum_{M\geq 1} \sum_{\substack{|\boldsymbol{\alpha}|=M \\ \alpha_m \geq 1}} \sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} C^M_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} [\ell^*_m, x^{\boldsymbol{\gamma}}] \bar{\ell}^{\boldsymbol{\beta}} \ell^{\boldsymbol{\alpha}} \\
&+ \sum_{M\geq 1} \sum_{\substack{|\boldsymbol{\alpha}|=M \\ \alpha_m \geq 1}} \sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} C^M_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} x^{\boldsymbol{\gamma}} \bar{\ell}^{\boldsymbol{\beta}} \sum_{i=1}^{m-1} (f|[\ell^*_m, \ell_i]) \alpha_i \ell_1^{\alpha_1} \cdots \ell_{i-1}^{\alpha_{i-1}} \ell_i^{\alpha_i - 1} \ell_{i+1}^{\alpha_{i+1}} \cdots \ell_m^{\alpha_m} \\
&+ \sum_{M\geq 1} \sum_{\substack{|\boldsymbol{\alpha}|=M \\ \alpha_m \geq 1}} \sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} C^M_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} x^{\boldsymbol{\gamma}} \bar{\ell}^{\boldsymbol{\beta}} \alpha_m \ell_1^{\alpha_1} \cdots \ell_{m-1}^{\alpha_{m-1}} \ell_m^{\alpha_m - 1} = 0\,.
\end{aligned} \tag{4.3.14}$$

Let us suppose that $\bar{\imath}$ is the unique $i \neq m$ such that $(f|[\ell^*_m, \ell_i]) \neq 0$. We reorder the terms in $[\ell^*_m, u]$ according to their $\mathfrak{l}$-*degree*, namely the total amount of factors from $\mathfrak{l}$ appearing, and we then proceed by induction on the $\mathfrak{l}$-degree. In degree 0 we get

$$\sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} C^1_{\mathbf{e}_m,\boldsymbol{\beta},\boldsymbol{\gamma}} x^{\boldsymbol{\gamma}} \bar{\ell}^{\boldsymbol{\beta}} \equiv 0\,, \tag{4.3.15}$$

and we can therefore conclude that $C^1_{\mathbf{e}_m,\boldsymbol{\beta},\boldsymbol{\gamma}} \equiv 0$ for all $\boldsymbol{\beta}, \boldsymbol{\gamma}$. Let us now suppose that the $\mathfrak{l}$-degree is $M \geq 1$, and by the induction hypothesis suppose moreover that $C^M_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} \equiv 0$ for every $\boldsymbol{\alpha} = \mathbf{e}_m + \boldsymbol{\alpha}'$, $\boldsymbol{\alpha}' \in \mathbb{N}^m$ with $|\boldsymbol{\alpha}'| = M - 1$. By (4.3.14) we have

$$\begin{aligned}
0 \equiv &\sum_{\substack{|\boldsymbol{\alpha}|=M \\ \alpha_m \geq 1}} \sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} C^M_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} [\ell^*_m, x^{\boldsymbol{\gamma}}] \bar{\ell}^{\boldsymbol{\beta}} \ell^{\boldsymbol{\alpha}} \\
&- \sum_{\substack{|\boldsymbol{\alpha}|=M+1 \\ \alpha_{\bar{\imath}}, \alpha_m \geq 1}} \sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} C^{M+1}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} x^{\boldsymbol{\gamma}} \bar{\ell}^{\boldsymbol{\beta}} \alpha_{\bar{\imath}} \ell_1^{\alpha_1} \cdots \ell_{\bar{\imath}-1}^{\alpha_{\bar{\imath}-1}} \ell_{\bar{\imath}}^{\alpha_{\bar{\imath}} - 1} \ell_{\bar{\imath}+1}^{\alpha_{\bar{\imath}+1}} \cdots \ell_m^{\alpha_m} \\
&+ \sum_{\substack{|\boldsymbol{\alpha}|=M+1 \\ \alpha_m \geq 1}} \sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} C^{M+1}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} x^{\boldsymbol{\gamma}} \bar{\ell}^{\boldsymbol{\beta}} \alpha_m \ell_1^{\alpha_1} \cdots \ell_{m-1}^{\alpha_{m-1}} \ell_m^{\alpha_m - 1}\,,
\end{aligned} \tag{4.3.16}$$

which we can rewrite, by the induction hypothesis, as

$$0 \equiv \sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} \Big(- \sum_{|\boldsymbol{\mu}|=M-1} C^{M+1}_{\boldsymbol{\mu}+\mathbf{e}_{\bar{\imath}}+\mathbf{e}_m,\boldsymbol{\beta},\boldsymbol{\gamma}} x^{\boldsymbol{\gamma}} \bar{\ell}^{\boldsymbol{\beta}} \alpha_{\bar{\imath}} \ell_1^{\mu_1} \cdots \ell_m^{\mu_m} \ell_m + \sum_{|\boldsymbol{\nu}|=M} C^{M+1}_{\boldsymbol{\nu}+\mathbf{e}_m,\boldsymbol{\beta},\boldsymbol{\gamma}} x^{\boldsymbol{\gamma}} \bar{\ell}^{\boldsymbol{\beta}} \alpha_m \ell_1^{\nu_1} \cdots \ell_m^{\nu_m} \Big)\,, \tag{4.3.17}$$

where $\boldsymbol{\mu} = (\mu_1, \dots, \mu_m) = (\alpha_1, \dots, \alpha_{\bar{\imath}-1}, \alpha_{\bar{\imath}} - 1, \alpha_{\bar{\imath}+1}, \dots, \alpha_m - 1)$ and $\boldsymbol{\nu} = (\nu_1, \dots, \nu_m) = (\alpha_1, \dots, \alpha_m - 1)$. First, by considering the monomials in the second term when ℓ_m does not appear, we can easily conclude that $C^{M+1}_{\boldsymbol{\nu}+\mathbf{e}_m,\boldsymbol{\beta},\boldsymbol{\gamma}} \equiv 0$ for every $\boldsymbol{\nu}$ such that $\nu_m = 0$. In particular, $C^{M+1}_{\boldsymbol{\mu}+\mathbf{e}_{\bar{\imath}}+\mathbf{e}_m,\boldsymbol{\beta},\boldsymbol{\gamma}} \equiv 0$ for all $\boldsymbol{\mu}$ such that $\mu_m = 0$. Since each one of these terms is a monomial of degree 1 in ℓ_m, we get as a consequence that $C^{M+1}_{\boldsymbol{\nu}+\mathbf{e}_m,\boldsymbol{\beta},\boldsymbol{\gamma}} \equiv 0$ for every $\boldsymbol{\nu}$ such that $\nu_m = 1$. By repeating this argument we finally conclude that $C^{M+1}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} \equiv 0$ for every $\boldsymbol{\alpha} \in \mathbb{N}^m$

with $\alpha_m \geq 1$. Note that this result would directly follow from (4.3.16) in the case when $(f|[\ell_m^*, \ell_i]) = 0$ for every $i \neq m$.

As a consequence, (4.3.12) becomes

$$u \equiv \sum_{M \geq 1} \sum_{\substack{|\boldsymbol{\alpha}|=M \\ \alpha_m=0}} \sum_{\boldsymbol{\beta},\boldsymbol{\gamma}} C^M_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} \mathbf{x}^{\boldsymbol{\gamma}} \overline{\boldsymbol{\ell}}^{\boldsymbol{\beta}} \ell_1^{\alpha_1} \cdots \ell_{m-1}^{\alpha_{m-1}} .$$

Iterating one by one for $\ell_{m-1}, \dots, \ell_1$ (i.e by considering the commutator with $\ell^*_{m-1}, \dots, \ell^*_1$ respectively) we can finally conclude that $C^M_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}} \equiv 0$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, and therefore $u \equiv 0$.

Note that Remark 4.3.4 holds at every step of the iteration, since at every step we are considering the commutator with ℓ_q^* when ℓ_q is the maximal element of $\mathcal{B}$ appearing as a factor in u. □

Remark 4.3.4. Let $\mathcal{B} = \{\ell_1, \dots, \ell_m\}$ be a basis of $\mathfrak{l}$ as in the proof of Theorem 4.3.1. Let $1 \leq q \leq m$ be the largest integer such that $\ell_q \in \mathcal{B}$ appears as a factor in u. As a consequence, $\ell_q < \ell_i$ for any other $\ell_i \in \mathcal{B}$ appearing as a factor in u, when $i \neq q$. Then, for $\ell_i \in \mathcal{B}$ with $i \neq q$ the only possible non-zero commutators $[\ell_q^*, \ell_i]$ are the following:

$$\begin{aligned}
&0 \neq [e_{(j,k-1)(i,h)}, e_{(i,h)(c,d)}] = e_{(j,k-1)(c,d)} , && \text{but } (f|e_{(j,k-1)(c,d)}) = 0 \text{ for } (c,d) \neq (j,k) , \\
&0 \neq [e_{(j,k-1)(i,h)}, e_{(a,b)(j,k-1)}] = -e_{(a,b)(i,h)} , && \text{but } e_{(a,b)(j,k-1)} < \ell_q , \\
&0 \neq [e_{(j,k)(i,h+1)}, e_{(a,b)(j,k)}] = -e_{(a,b)(i,h+1)} , && \text{but } (f|e_{(a,b)(i,h+1)}) = 0 \text{ for } (a,b) \neq (i,h) ,
\end{aligned}$$

while

$$0 \neq [e_{(j,k)(i,h+1)}, e_{(i,h+1)(c,d)}] = e_{(j,k)(c,d)} , \qquad \text{and } (f|e_{(j,k)(c,d)}) = 1 \text{ for } (c,d) = (j,k+1) .$$

4.4 Extension to a generic good $\frac{1}{2}\mathbb{Z}$-grading

In the case when it is possible to remove the leftmost (resp. rightmost) column of the pyramid p, Theorem 4.3.1 allows to construct a particular choice of generators for the W-algebras $W(\mathfrak{g}, f, \Gamma, 0)$ and $W(\mathfrak{g}'^p, f'^p, \Gamma'^p, 0)$ (resp. $W(\mathfrak{g}^{p'}, f^{p'}, \Gamma^{p'}, 0)$) such that they are related by a recursion like (3.2.2) (resp. (3.2.3)). From Remark 4.1.1 it is possible to extend Theorem 4.1.1 in the case of an even but not necessarily aligned good $\frac{1}{2}\mathbb{Z}$-grading. Together, these results almost prove Conjecture 3.1.1 in the case of an even $\frac{1}{2}\mathbb{Z}$-grading. Unluckily, Proposition 3.3.1 still shows some limitations in the case p is not aligned.

However, this is anyway not sufficient to solve Conjecture 3.1.1 in all generality: the case of an arbitrary odd $\frac{1}{2}\mathbb{Z}$-grading is still missing. In fact, even in the case it is possible to remove either the leftmost or rightmost column for the pyramid associated to such a grading (and therefore apply Theorem 4.3.1), we are still not allowed to perform arbitrary removal of columns on either side, as we have seen it is necessary for many results.

The search for an alternative method that would allow us to prove Conjecture 3.1.1 in this most general case eventually brought to the development of a powerful machinery.

Definition 4.4.1. Let Γ be a good $\frac{1}{2}\mathbb{Z}$-grading for f, not necessarily even. Let

$$\begin{aligned}
a &:= \max\{i \mid s_{1i} \in \mathbb{N}\} , \\
b &:= \max\{i \mid s_{1i} \in \frac{1}{2} + \mathbb{N}\} .
\end{aligned} \tag{4.4.1}$$

We set $b = 0$ if $s_{1i} \in \mathbb{N}$ for every $1 \leq i \leq r$. Then we define the *right distance* of Γ, namely the distance of the pyramid with grading Γ from the right-aligned pyramid Γ_R associated with the same partition of N, as

$$\operatorname{dist}_R(\Gamma) := \delta_{s_{1a}>0}(2s_{1a} + \delta_{b\neq 0}) + \delta_{s_{1a},0}\delta_{b\neq 0} 2s_{1b} , \tag{4.4.2}$$

and the *left distance* of Γ, namely the distance of the pyramid with grading Γ from the left-aligned pyramid Γ_L associated with the same partition of N, as

$$\operatorname{dist}_L(\Gamma) := \delta_{s_{a1}>0}(2s_{a1} + \delta_{b\neq 0}) + \delta_{s_{a1},0}\delta_{b\neq 0} 2s_{b1} . \tag{4.4.3}$$

We also denote by *distance* of Γ the minimum between its left and right distance:

$$\operatorname{dist}(\Gamma) = \min(\operatorname{dist}_L(\Gamma), \operatorname{dist}_R(\Gamma)) . \tag{4.4.4}$$

Remark 4.4.1. Note that $\operatorname{dist}_R(\Gamma_R) = 0$ and $\operatorname{dist}_L(\Gamma_L) = 0$. Moreover, $b \neq 0$ if and only if Γ is odd and, by (4.4.2) and (4.4.3), it is clear that both $\operatorname{dist}_R(\Gamma)$, $\operatorname{dist}_L(\Gamma)$ are odd if Γ is odd and even if Γ is even.

Example 4.4.1. Consider the following pyramids:

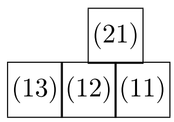


Figure 4.3: Γ_1

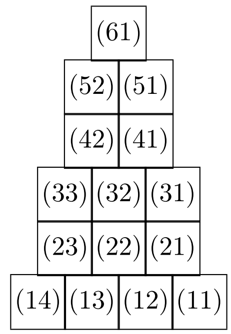


Figure 4.4: Γ_2

Then,

$$\operatorname{dist}_L(\Gamma_1) = 3,\ \operatorname{dist}_R(\Gamma_1) = 1 , \qquad \operatorname{dist}_L(\Gamma_2) = 3,\ \operatorname{dist}_R(\Gamma_2) = 3 .$$

4.4.1 Algorithm for a chain of adjacent gradings

Let Γ be a good $\frac{1}{2}\mathbb{Z}$-grading for f, and let $\mathfrak{l} \subset \mathfrak{g}_{\frac{1}{2}}$ be an isotropic subspace with respect to the bilinear form ω. Suppose that $\operatorname{dist}(\Gamma) = \operatorname{dist}_R(\Gamma) = m \geq 0$. By [BG05] there exists an associative algebra isomorphism

$$\Phi_\Gamma : W(\mathfrak{g}, f, \Gamma, \mathfrak{l}) \xrightarrow{\sim} W(\mathfrak{g}, f, \Gamma_R, 0) . \tag{4.4.5}$$

This isomorphism is obtained by the composition of a chain of associative algebras isomorphisms

$$W(\mathfrak{g}, f, \Gamma, \mathfrak{l}_1) \xrightarrow{\sim} W(\mathfrak{g}, f, \Gamma, \mathfrak{l}_2) ,$$

as in (4.2.3), where $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are a pair of isotropic subspaces of $\mathfrak{g}_{\frac{1}{2}}$, whereas Lemmas 4.2.1 and 4.2.2 regulate the relationship between W-algebras associated with different but adjacent good $\frac{1}{2}\mathbb{Z}$-gradings.

We will now explicitly describe how to construct a choice for such a chain of good $\frac{1}{2}\mathbb{Z}$-gradings for f that is particularly advantageous. The strategy will be to construct a chain of good $\frac{1}{2}\mathbb{Z}$-gradings for f

$$\Gamma_0 = \Gamma,\ \Gamma_1,\ \Gamma_2 \dots$$

such that $\operatorname{dist}_R(\Gamma_i) = \operatorname{dist}_R(\Gamma_{i-1}) - 1$ for each $0 \leq i \leq m$ ($0 < i \leq m+1$ if m is even) and $\Gamma_m = \Gamma_R$ ($\Gamma_{m+1} = \Gamma_R$ if m is even).

First consider the pair $(\Gamma_0, \mathfrak{l}_0) := (\Gamma, \mathfrak{l})$:

Step 1 If $\mathfrak{l}_0 = 0$, go to Step 2. If $\mathfrak{l}_0 \neq 0$, then apply the associative algebra isomorphism $(\rho^{\Gamma_0}_{0,\mathfrak{l}_0})^{-1}$,

$$(\rho^{\Gamma_0}_{0,\mathfrak{l}_0})^{-1} : W(\mathfrak{g}, f, \Gamma_0, \mathfrak{l}_0) \xrightarrow{\sim} W(\mathfrak{g}, f, \Gamma_0, 0) , \tag{4.4.6}$$

to move from the pair $(\Gamma_0, \mathfrak{l}_0)$ to the pair $(\Gamma_0, 0)$.

Step 2 If Γ_0 is even, let $\Gamma_1 := \Gamma_0$ and go to Step 3. Otherwise, move all semi-integers rows of $\frac{1}{2}$ to the right. We obtain a new good $\frac{1}{2}\mathbb{Z}$-grading Γ_1 such that

$$x^{\Gamma_1}(ih) = \begin{cases} x^{\Gamma_0}(ih) , & \text{if } s^{\Gamma_0}_{1i} \in \mathbb{N} \\ x^{\Gamma_0}(ih) + \frac{1}{2} , & \text{if } s^{\Gamma_0}_{1i} \in \frac{1}{2} + \mathbb{N} . \end{cases}$$

and consequently

$$s_{1i}^{\Gamma_1}=\begin{cases} s_{1i}^{\Gamma_0}\,, & \text{if } s_{1i}^{\Gamma_0}\in\mathbb{N} \\ s_{1i}^{\Gamma_0}-\frac{1}{2}\,, & \text{if } s_{1i}^{\Gamma_0}\in\frac{1}{2}+\mathbb{N}\,. \end{cases}$$

By construction, Γ_0 and Γ_1 are strictly adjacent good $\frac{1}{2}\mathbb{Z}$-gradings for f (cf. Definition 4.2.2 and Theorem 1.1.1) and $\mathrm{dist}_R(\Gamma_1)=\mathrm{dist}_R(\Gamma_0)-1$. Moreover, the grading Γ_1 is even.

By Lemma 4.2.1 there exists a Lagrangian subspace $\mathfrak{l}_1\subset\mathfrak{g}^{(0)}_{\frac{1}{2}}$ such that $\mathfrak{l}_1\oplus\mathfrak{g}^{(0)}_{\geq 1}=\mathfrak{g}^{(1)}_{\geq 1}$, and as a consequence $W(\mathfrak{g},f,\Gamma_0,\mathfrak{l}_1)=W(\mathfrak{g},f,\Gamma_1,0)$, and let

$$\rho^{\Gamma_0}_{0,\mathfrak{l}_1}:W(\mathfrak{g},f,\Gamma_0,0)\longrightarrow W(\mathfrak{g},f,\Gamma_0,\mathfrak{l}_1)\,,$$

be the associative algebra isomorphism as in (4.2.2) corresponding to the choices of isotropic subspaces $0\subseteq\mathfrak{l}_1$. Since the grading Γ_1 is even we moreover know that $\mathfrak{l}_1=\mathfrak{g}^{(1)}_1\cap\mathfrak{g}^{(0)}_{\frac{1}{2}}$.

Step 3 Combining the results of Step 1 and Step 2 we have the following associative algebra isomorphism:

$$\Phi_0:W(\mathfrak{g},f,\Gamma_0,\mathfrak{l}_0)\xrightarrow{\sim}W(\mathfrak{g},f,\Gamma_1,0)\,.$$

where

$$\Phi_0:=\delta_{\Gamma_0,\mathrm{odd}}\left(\rho^{\Gamma_0}_{0,\mathfrak{l}_1}\circ(\rho^{\Gamma_0}_{0,\mathfrak{l}_0})^{-1}\right)+\delta_{\Gamma_0,\mathrm{even}}\mathbb{1}_{W(\mathfrak{g},f,\Gamma_0,0)=W(\mathfrak{g},f,\Gamma_1,0)}\,.$$

Note that $(\rho^{\Gamma_0}_{0,\mathfrak{l}_0})^{-1}=\mathbb{1}_{W(\mathfrak{g},f,\Gamma_0,0)}$ in the case when $\mathfrak{l}_0=0$ (cf. (4.2.2)). If $\mathrm{dist}_R(\Gamma_1)=0$, namely if $\Gamma_1=\Gamma_R$, stop (go to Step 5). If $\mathrm{dist}_R(\Gamma_1)\geq 1$, then let $\bar{i}$ be the lowest $i\in\{1,\dots,r\}$ such that $s^{\Gamma_1}_{1i}\neq 0$. Then move row i, for all $i\geq\bar{i}$ of $\frac{1}{2}$ to the right.

We obtain an odd good $\frac{1}{2}\mathbb{Z}$-grading Γ_2 which is strictly adjacent to Γ_1 and such that $\mathrm{dist}_R(\Gamma_2)=\mathrm{dist}_R(\Gamma_1)-1$. For Γ_2:

$$x^{\Gamma_2}(ih)=\begin{cases} x^{\Gamma_1}(ih)\,, & \text{if } h<\bar{i} \\ x^{\Gamma_1}(ih)+\frac{1}{2}\,, & \text{if } h\geq\bar{i}\,. \end{cases}$$

and consequently

$$s^{\Gamma_2}_{1i}=\begin{cases} s^{\Gamma_1}_{1i}\,, & \text{if } h<\bar{i} \\ s^{\Gamma_1}_{1i}-\frac{1}{2}\,, & \text{if } h\geq\bar{i}\,. \end{cases}$$

By Lemma 4.2.1 there exists a Lagrangian subspace $\mathfrak{l}_2\in\mathfrak{g}^{(2)}_{\frac{1}{2}}$ such that $\mathfrak{l}_2\oplus\mathfrak{g}^{(2)}_{\geq 1}=\mathfrak{g}^{(1)}_{\geq 1}$, and as a consequence $W(\mathfrak{g},f,\Gamma_1,0)=W(\mathfrak{g},f,\Gamma_2,\mathfrak{l}_2)$. In particular, we have $\Phi_0:W(\mathfrak{g},f,\Gamma_0,\mathfrak{l}_0)\xrightarrow{\sim}W(\mathfrak{g},f,\Gamma_2,\mathfrak{l}_2)$.

Step 4 Repeat Steps 1 - 3, starting with $(\Gamma_2,\mathfrak{l}_2)$. More generally, for $m>1$ repeat Steps 1 - 2 for all pairs $(\Gamma_i,\mathfrak{l}_i)$ with $i\in\{2,\dots,m\}\cap 2\mathbb{N}$. For each such i, at the end of Step 3 we have an associative algebra isomorphism

$$\Phi_i:=\rho^{\Gamma_i}_{0,\mathfrak{l}_{i+1}}\circ(\rho^{\Gamma_i}_{0,\mathfrak{l}_i})^{-1}:W(\mathfrak{g},f,\Gamma_i,\mathfrak{l}_i)\xrightarrow{\sim}W(\mathfrak{g},f,\Gamma_i,\mathfrak{l}_{i+1})=W(\mathfrak{g},f,\Gamma_{i+1},0)=W(\mathfrak{g},f,\Gamma_{i+2},\mathfrak{l}_{i+2})\,,\tag{4.4.7}$$

where the last equality only appears in the case $\Gamma_{i+1}\neq\Gamma_R$.

Step 5 Composing the isomorphisms Φ_i we obtain

$$\begin{aligned}&W(\mathfrak{g},f,\Gamma_0,\mathfrak{l}_0)\xrightarrow{\Phi_0}W(\mathfrak{g},f,\Gamma_2,\mathfrak{l}_2)\xrightarrow{\Phi_2}W(\mathfrak{g},f,\Gamma_4,\mathfrak{l}_4)\longrightarrow\cdots\\&\cdots\longrightarrow W(\mathfrak{g},f,\Gamma_{m-1},\mathfrak{l}_{m-1})\xrightarrow{\Phi_{m-1}}W(\mathfrak{g},f,\Gamma_{m-1},\mathfrak{l}_m)=W(\mathfrak{g},f,\Gamma_m,0)=W(\mathfrak{g},f,\Gamma_R,0)\end{aligned}\tag{4.4.8}$$

in the case when m is odd, or

$$\begin{aligned}&W(\mathfrak{g},f,\Gamma_0,0)\overset{\Phi_0}{\equiv}W(\mathfrak{g},f,\Gamma_1,0)=W(\mathfrak{g},f,\Gamma_2,\mathfrak{l}_2)\xrightarrow{\Phi_2}W(\mathfrak{g},f,\Gamma_4,\mathfrak{l}_4)\longrightarrow\cdots\\&\cdots\longrightarrow W(\mathfrak{g},f,\Gamma_m,\mathfrak{l}_m)\xrightarrow{\Phi_m}W(\mathfrak{g},f,\Gamma_m,\mathfrak{l}_{m+1})=W(\mathfrak{g},f,\Gamma_{m+1},0)=W(\mathfrak{g},f,\Gamma_R,0)\end{aligned}\tag{4.4.9}$$

in the case when m is even. Therefore,

$$\Phi_\Gamma : W(\mathfrak{g}, f, \Gamma, \mathfrak{l}) \xrightarrow{\sim} W(\mathfrak{g}, f, \Gamma_R, 0) \tag{4.4.10}$$

is given by the composition $\Phi_\Gamma = \Phi_{m-1} \circ \ldots \circ \Phi_0$ if m is odd, and $\Phi_\Gamma = \Phi_m \circ \ldots \circ \Phi_2$ if m is even.

Remark 4.4.2. (i) As a consequence of this algorithm, given any odd good $\frac{1}{2}\mathbb{Z}$-grading Γ it is possible to construct an even good $\frac{1}{2}\mathbb{Z}$-grading Γ' such that Γ and Γ' are strictly adjacent, and therefore there exists a Lagrangian subspace $\mathfrak{l} \subset \mathfrak{g}^\Gamma_{\frac{1}{2}}$ for which $W(\mathfrak{g}, f, \Gamma, \mathfrak{l}) = W(\mathfrak{g}, f, \Gamma', 0)$.

(ii) If $\mathrm{dist}(\Gamma) = \mathrm{dist}_L(\Gamma)$ instead, in Step 2 move the blocks of each semi-integer row of $\frac{1}{2}$ to the left. As a consequence,

$$x^{\Gamma_1}(ih) = \begin{cases} x^{\Gamma_0}(ih)\,, & \text{if } s^{\Gamma_0}_{1i} \in \mathbb{N} \\ x^{\Gamma_0}(ih) - \frac{1}{2}\,, & \text{if } s^{\Gamma_0}_{1i} \in \frac{1}{2} + \mathbb{N}\,. \end{cases}$$

and

$$s^{\Gamma_1}_{1i} = \begin{cases} s^{\Gamma_0}_{1i}\,, & \text{if } s^{\Gamma_0}_{1i} \in \mathbb{N} \\ s^{\Gamma_0}_{1i} + \frac{1}{2}\,, & \text{if } s^{\Gamma_0}_{1i} \in \frac{1}{2} + \mathbb{N}\,. \end{cases}$$

Similarly, in Step 3 move all blocks of row j for $j \geq \bar{j}$ of $\frac{1}{2}$ to the left, where $\bar{j}$ is the minimum $j \in \{1, \ldots, r\}$ such that $s_{j1} \neq 0$. We also get

$$x^{\Gamma_2}(ih) = \begin{cases} x^{\Gamma_1}(ih)\,, & \text{if } h < \bar{j} \\ x^{\Gamma_1}(ih) - \frac{1}{2}\,, & \text{if } h \geq \bar{j}\,, \end{cases}$$

and

$$s^{\Gamma_2}_{1i} = \begin{cases} s^{\Gamma_1}_{1i}\,, & \text{if } h < \bar{j} \\ s^{\Gamma_1}_{1i} + \frac{1}{2}\,, & \text{if } h \geq \bar{j}\,. \end{cases}$$

(iii) Given a pair Γ_1 and Γ_2 of good $\frac{1}{2}\mathbb{Z}$-gradings for f, and a choice of Lagrangian subspaces $\mathfrak{l}_1 \subset \mathfrak{g}^{(1)}_{\frac{1}{2}}$ and $\mathfrak{l}_2 \subset \mathfrak{g}^{(2)}_{\frac{1}{2}}$, we can obtain an isomorphism

$$\overline{\Phi} : W(\mathfrak{g}, f, \Gamma_1, \mathfrak{l}_1) \xrightarrow{\sim} W(\mathfrak{g}, f, \Gamma_2, \mathfrak{l}_2) \tag{4.4.11}$$

as a composition of isomorphisms $\overline{\Phi} = (\Phi_{\Gamma_2})^{-1} \circ \Phi_{\Gamma_1}$ where

$$\Phi_{\Gamma_1} : W(\mathfrak{g}, f, \Gamma_1, \mathfrak{l}_1) \xrightarrow{\sim} W(\mathfrak{g}, f, \Gamma_R, 0)$$

and

$$\Phi_{\Gamma_2} : W(\mathfrak{g}, f, \Gamma_2, \mathfrak{l}_2) \xrightarrow{\sim} W(\mathfrak{g}, f, \Gamma_R, 0)$$

are as in (4.4.10). Alternatively, it is also possible to describe a different algorithm moving from the pyramid associated with Γ_1 to the pyramid associated with Γ_2 directly.

4.4.2 Proof of Conjecture 3.1.1 in the general case

Let $\Gamma : \mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$ be a good $\frac{1}{2}\mathbb{Z}$-grading for f, and let $\mathfrak{l} \subset \mathfrak{g}_{\frac{1}{2}}$ be an isotropic subspace with respect to the bilinear form ω. Suppose moreover that $\mathrm{dist}(\Gamma) = \mathrm{dist}_R(\Gamma)$. Note that the treatment of the case $\mathrm{dist}(\Gamma) = \mathrm{dist}_L(\Gamma)$ is analogous. Let

$$\Phi_\Gamma : W(\mathfrak{g}, f, \Gamma, \mathfrak{l}) \xrightarrow{\sim} W(\mathfrak{g}, f, \Gamma_R, 0) \tag{4.4.12}$$

be the associative algebra isomorphism as in (4.4.5), which we constructed explicitly in (4.4.10).

Let $\{W_{ij}(z)\}_{1 \leq i,j \leq r}$ be the set of polynomials defined through (3.2.2), whose coefficients provide a finite set of generators $\{W_{ij;k} \mid 1 \leq i, j \leq r,\ 0 \leq k \leq \min(p_i, p_j) - 1\}$ for $W(\mathfrak{g}, f, \Gamma_R, 0)$. Let

$$\Big\{ W^\Phi_{ij}(z) := (\Phi_\Gamma)^{-1}(W_{ij}(z)) = -\delta_{ij}(-z)^{p_i} + \sum_{k=0}^{\min(p_i,p_j)-1} (\Phi_\Gamma)^{-1}(W_{ij;k})(-z)^k \Big\}_{1 \leq i,j \leq r} \tag{4.4.13}$$

be the corresponding set of polynomials through the isomorphism Φ_Γ. The coefficients $(\Phi_\Gamma)^{-1}(W_{ij;k})$ provide a finite set of generators for $W(\mathfrak{g},f,\Gamma,\mathfrak{l})$. Let $W^\Phi(z)\in \mathrm{Mat}_{r\times r}W(\mathfrak{g},f,\Gamma,\mathfrak{l})[z]$ be the $r\times r$ matrix whose (i,j)-entry is $W^\Phi_{ij}(z)$. The following result generalizes Theorem 4.1.1.

Theorem 4.4.1. *For the matrix $W^\Phi(z)\in Mat_{r\times r}W(\mathfrak{g},f,\Gamma,\mathfrak{l})[z]$ the quasideterminant $|W^\Phi(z)|_{I_{rr_1}J_{r_1r}}$ exists and the following identity holds:*

$$|W^\Phi(z)|_{I_{rr_1}J_{r_1r}}=L^{\Gamma,\mathfrak{l}}(z)\in Mat_{r_1\times r_1}W(\mathfrak{g},f,\Gamma,\mathfrak{l})((z^{-1}))\,, \tag{4.4.14}$$

where I_{rr_1}, J_{r_1r} are defined as in (2.2.6), *and*

$$L^{\Gamma,\mathfrak{l}}(z)=|z\mathbb{1}_N+F+\pi^\Gamma_{\mathfrak{p}}E+D_{\Gamma,\mathfrak{l}}|_{I_1J_1}\,\bar{1}_{\Gamma,\mathfrak{l}}\in Mat_{r_1\times r_1}W(\mathfrak{g},f,\Gamma,\mathfrak{l})((z^{-1}))\,,$$

is as in (2.2.3).

Here and further, since we will need to consider different good $\frac{1}{2}$-gradings and isotropic subspaces at the same time, we use the notation $L^{\Gamma,\mathfrak{l}}(z)$ for the operator $L(z)$ as in (2.2.3) with respect to the good $\frac{1}{2}$-grading Γ and the isotropic subspace $\mathfrak{l}\subset\mathfrak{g}_{\frac{1}{2}}$ (with respect to the grading Γ).

Proof. The existence of the quasideterminant $|W^\Phi(z)|_{I_{rr_1}J_{r_1r}}$ follows from (4.4.13) and the existence of the quasideterminant $|W(z)|_{I_{rr_1}J_{r_1r}}$, and we have $((\Phi_\Gamma)^{-1}(W_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}(z)))^{-1}=(\Phi_\Gamma)^{-1}((W_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}(z))^{-1})$ since the inverse of $W_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}(z)$ can be computed by geometric series expansion (cf. Theorem 4.1.1).

By definition of quasideterminant we have

$$\begin{aligned}
(|W^\Phi(z)|_{I_{rr_1}J_{r_1r}})_{ij}&=W^\Phi_{ij}(z)-\sum_{\alpha,\beta=r_1+1}^{r}W^\Phi_{i\alpha}(z)((W^\Phi_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}(z))^{-1})_{\alpha\beta}W^\Phi_{\beta j}(z)\\
&=(\Phi_\Gamma)^{-1}(W_{ij}(z))-\sum_{\alpha,\beta=r_1+1}^{r}(\Phi_\Gamma)^{-1}(W_{i\alpha}(z))((\Phi_\Gamma)^{-1}(W_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}(z)))^{-1})_{\alpha\beta}(\Phi_\Gamma)^{-1}(W_{\beta j}(z))\\
&=(\Phi_\Gamma)^{-1}\Big(W_{ij}(z)-\sum_{\alpha,\beta=r_1+1}^{r}W_{i\alpha}(z)((W_{\mathcal{I}_{rr_1}{}^c\mathcal{J}_{r_1r}{}^c}(z))^{-1})_{\alpha\beta}W_{\beta j}(z)\Big)\\
&=(\Phi_\Gamma)^{-1}((|W(z)|_{I_{rr_1}J_{r_1r}})_{ij})\\
&=(\Phi_\Gamma)^{-1}(L^{\Gamma_R,0}_{ij}(z))\,,
\end{aligned}$$

where

$$L^{\Gamma_R,0}(z)=|z\mathbb{1}_N+F+\pi^{\Gamma_R}_{\leq 0}E+D_{\Gamma_R}|_{I_1J_1}\,\bar{1}_{\Gamma_R,0}\in \mathrm{Mat}_{r_1\times r_1}W(\mathfrak{g},f,\Gamma_R,0)((z^{-1}))\,,$$

is the quasideterminant defined by Equation (2.2.3) with respect to the grading Γ_R and the choice of the isotropic subspace 0. The last equality is given by Theorem 4.1.1. We also remark that by Theorem 2.2.3, $L^{\Gamma_R,0}(z)\in\mathrm{Mat}_{r_1\times r_1}W(\mathfrak{g},f,\Gamma_R,0)((z^{-1}))$, therefore we can apply the isomorphism $(\Phi_\Gamma)^{-1}$ and get $(\Phi_\Gamma)^{-1}(L^{\Gamma_R,0}_{ij}(z))\subset W(\mathfrak{g},f,\Gamma,\mathfrak{l})((z^{-1}))$. On the other hand, still in view of Theorem 2.2.3, $L^{\Gamma,\mathfrak{l}}(z)\in \mathrm{Mat}_{r_1\times r_1}W(\mathfrak{g},f,\Gamma,\mathfrak{l})((z^{-1}))$.

We are therefore left to prove that

$$L^{\Gamma,\mathfrak{l}}(z)=(\Phi_\Gamma)^{-1}(L^{\Gamma_R,0}(z))\,. \tag{4.4.15}$$

This will be a consequence of Theorem 4.4.2, that follows. □

Theorem 4.4.2. *Let $\Gamma:\mathfrak{g}=\bigoplus_{i\in\frac{1}{2}\mathbb{Z}}\mathfrak{g}_j$ be a good $\frac{1}{2}\mathbb{Z}$-grading for f, and let $\mathfrak{l}_1,\mathfrak{l}_2\subset\mathfrak{g}_{\frac{1}{2}}$ be a pair of isotropic subspaces with respect to the bilinear form ω. Let*

$$\Phi=\rho_{0,\mathfrak{l}_2}\circ(\rho_{0,\mathfrak{l}_1})^{-1}:W(\mathfrak{g},f,\Gamma,\mathfrak{l}_1)\xrightarrow{\sim}W(\mathfrak{g},f,\Gamma,\mathfrak{l}_2) \tag{4.4.16}$$

be the induced associative algebra isomorphism as in (4.2.3). *Let*

$$\begin{aligned}
L^{\Gamma,\mathfrak{l}_1}(z)&=|z\mathbb{1}_V+F+\pi_{\mathfrak{p}_1}E+D_{\mathfrak{l}_1}|_{\Psi_{\frac{d}{2},r_1}\Pi_{-\frac{d}{2},r_1}}\,\bar{1}_{\Gamma,\mathfrak{l}_1}\,,\\
L^{\Gamma,\mathfrak{l}_2}(z)&=|z\mathbb{1}_V+F+\pi_{\mathfrak{p}_2}E+D_{\mathfrak{l}_2}|_{\Psi_{\frac{d}{2},r_1}\Pi_{-\frac{d}{2},r_1}}\,\bar{1}_{\Gamma,\mathfrak{l}_2}\,,
\end{aligned} \tag{4.4.17}$$

be the (images of the) quasideterminants defined by Equation (2.2.17) with respect to the same grading Γ but different isotropic subspaces $\mathfrak{l}_1, \mathfrak{l}_2$. Then,

$$\Phi(L^{\Gamma,\mathfrak{l}_1}(z)) = L^{\Gamma,\mathfrak{l}_2}(z) \in Mat_{r_1\times r_1}\mathcal{R}W(\mathfrak{g},f,\Gamma,\mathfrak{l}_2)\,. \tag{4.4.18}$$

Example 4.4.2. Before proving Theorem 4.4.2 we will show why Equation (4.4.18) holds in the case of $N=4$ with partition $(3,1)$ and a good $\frac{1}{2}\mathbb{Z}$-grading Γ as in Figure 4.5. We choose Lagrangian subspaces $\mathfrak{l}_1 = 0$, and $\mathfrak{l}_2 = \mathbb{C}e_{(2,1)(1,2)}$. For this choice of $\mathfrak{l} := \mathfrak{l}_2$, we can easily check that $\mathfrak{l}\oplus\mathfrak{g}_{\geq 1} = \mathfrak{g}'_{\geq 1}$ where $\Gamma' : \mathfrak{g} = \bigoplus_{j\in\frac{1}{2}\mathbb{Z}}\mathfrak{g}'_j$ is the even good $\frac{1}{2}\mathbb{Z}$-grading as in Figure 4.6, that is strictly adjacent to the grading Γ. As a consequence, $W(\mathfrak{g},f,\Gamma,\mathfrak{l}) = W(\mathfrak{g},f,\Gamma',0)$.

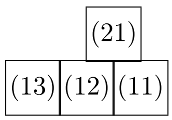


Figure 4.5: Γ

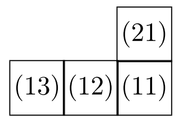


Figure 4.6: Γ'

The shift matrices corresponding to the gradings Γ and Γ' are $D_0 = \mathrm{diag}(0,-1,0,-3)$ and $D_{\mathfrak{l}} = \mathrm{diag}(0,-2,0,-3)$. We can compute the quasideterminants $\widetilde{L}^{\Gamma,\mathfrak{l}}(z)$, $\widetilde{L}^{\Gamma,0}(z) \in U(\mathfrak{g})((z^{-1}))$ (note that $r_1 = 1$ in both cases). Let $\mathrm{M}_{\Gamma,\mathfrak{l}} = U(\mathfrak{g})/U(\mathfrak{g})\langle b-(f|b)\rangle_{b\in\mathfrak{l}\oplus\mathfrak{g}_{\geq 1}} = U(\mathfrak{g})/U(\mathfrak{g})\langle b-(f|b)\rangle_{b\in\mathfrak{g}'_{\geq 1}}$ and $\mathrm{M}_{\Gamma,0} = U(\mathfrak{g})/U(\mathfrak{g})\langle b-(f|b)\rangle_{b\in\mathfrak{g}_{\geq 1}}$ be the corresponding quotient modules; we obtain:

$$\begin{aligned}
L^{\Gamma,\mathfrak{l}}(z) &= \begin{vmatrix} z+e_{(1,1)(1,1)} & e_{(1,2)(1,1)} & e_{(1,3)(1,1)} & e_{(2,1)(1,1)} \\ 1 & z+e_{(1,2)(1,2)}-2 & e_{(1,3)(1,2)} & 0 \\ 0 & 1 & z+e_{(1,3)(1,3)}-3 & 0 \\ e_{(1,1)(2,1)} & e_{(1,2)(2,1)} & e_{(1,3)(2,1)} & z+e_{(2,1)(2,1)} \end{vmatrix}_{13} \bar{1}_{\Gamma,\mathfrak{l}} \\
&= e_{(1,3)(1,1)}\,\bar{1}_{\Gamma,\mathfrak{l}} - \begin{pmatrix} z+e_{(1,1)(1,1)} & e_{(1,2)(1,1)} & e_{(2,1)(1,1)} \end{pmatrix} \begin{pmatrix} 1 & z+e_{(1,2)(1,2)}-2 & 0 \\ 0 & 1 & 0 \\ e_{(1,1)(2,1)} & e_{(1,2)(2,1)} & z+e_{(2,1)(2,1)} \end{pmatrix}^{-1} \\
&\quad\times \begin{pmatrix} e_{(1,3)(1,2)} \\ z+e_{(1,3)(1,3)}-3 \\ e_{(1,3)(2,1)} \end{pmatrix} \bar{1}_{\Gamma,\mathfrak{l}} \in \mathrm{M}_{\Gamma,\mathfrak{l}}((z^{-1}))\,,
\end{aligned} \tag{4.4.19}$$

while

$$\begin{aligned}
L^{\Gamma,0}(z) &= \begin{vmatrix} z+e_{(1,1)(1,1)} & e_{(1,2)(1,1)} & e_{(1,3)(1,1)} & e_{(2,1)(1,1)} \\ 1 & z+e_{(1,2)(1,2)}-1 & e_{(1,3)(1,2)} & e_{(2,1)(1,2)} \\ 0 & 1 & z+e_{(1,3)(1,3)}-3 & 0 \\ e_{(1,1)(2,1)} & e_{(1,2)(2,1)} & e_{(1,3)(2,1)} & z+e_{(2,1)(2,1)} \end{vmatrix}_{13} \bar{1}_{\Gamma,0} \\
&= e_{(1,3)(1,1)}\,\bar{1}_{\Gamma,0} - \begin{pmatrix} z+e_{(1,1)(1,1)} & e_{(1,2)(1,1)} & e_{(2,1)(1,1)} \end{pmatrix} \begin{pmatrix} 1 & z+e_{(1,2)(1,2)}-1 & e_{(2,1)(1,2)} \\ 0 & 1 & 0 \\ e_{(1,1)(2,1)} & e_{(1,2)(2,1)} & z+e_{(2,1)(2,1)} \end{pmatrix}^{-1} \\
&\quad\times \begin{pmatrix} e_{(1,3)(1,2)} \\ z+e_{(1,3)(1,3)}-3 \\ e_{(1,3)(2,1)} \end{pmatrix} \bar{1}_{\Gamma,0} \in \mathrm{M}_{\Gamma,0}((z^{-1}))\,.
\end{aligned} \tag{4.4.20}$$

By (4.4.10), we have an associative algebra isomorphism

$$\Phi = \rho^{\Gamma}_{0,\mathfrak{l}} : W(\mathfrak{g},f,\Gamma,0) \xrightarrow{\sim} W(\mathfrak{g},f,\Gamma,\mathfrak{l}) = W(\mathfrak{g},f,\Gamma',0)\,.$$

By Theorem 2.2.3, $L^{\Gamma,0}(z) \in W(\mathfrak{g},f,\Gamma,0)((z^{-1}))$, therefore we have $\Phi(L^{\Gamma,0}(z)) \subset W(\mathfrak{g},f,\Gamma,\mathfrak{l})((z^{-1}))$.

Comparing (4.4.19) and (4.4.20), in order to prove $\Phi(L^{\Gamma,0}(z)) = L^{\Gamma,\mathfrak{l}}(z)$ it is sufficient to show that

$$\begin{aligned}
&\Phi\left(\begin{pmatrix} 1 & z+e_{(1,2)(1,2)}-1 & e_{(2,1)(1,2)} \\ 0 & 1 & 0 \\ e_{(1,1)(2,1)} & e_{(1,2)(2,1)} & z+e_{(2,1)(2,1)} \end{pmatrix}^{-1} \begin{pmatrix} e_{(1,3)(1,2)} \\ z+e_{(1,3)(1,3)}-3 \\ e_{(1,3)(2,1)} \end{pmatrix} \bar{1}_{\Gamma,0}\right) \\
&= \begin{pmatrix} 1 & z+e_{(1,2)(1,2)}-2 & 0 \\ 0 & 1 & 0 \\ e_{(1,1)(2,1)} & e_{(1,2)(2,1)} & z+e_{(2,1)(2,1)} \end{pmatrix}^{-1} \begin{pmatrix} e_{(1,3)(1,2)} \\ z+e_{(1,3)(1,3)}-3 \\ e_{(1,3)(2,1)} \end{pmatrix} \bar{1}_{\Gamma,\mathfrak{l}}\,.
\end{aligned} \tag{4.4.21}$$

To simplify the notation, let us denote

$$A = \begin{pmatrix} 1 & z+e_{(1,2)(1,2)}-2 & 0 \\ 0 & 1 & 0 \\ e_{(1,1)(2,1)} & e_{(1,2)(2,1)} & z+e_{(2,1)(2,1)} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & e_{(2,1)(1,2)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$v = \begin{pmatrix} e_{(1,3)(1,2)} \\ z+e_{(1,3)(1,3)}-3 \\ e_{(1,3)(2,1)} \end{pmatrix}.$$

Then, Equation (4.4.21) becomes

$$\Phi\big((A+B)^{-1} v\, \bar{1}_{\Gamma,0}\big) = A^{-1} v\, \bar{1}_{\Gamma,\mathfrak{l}}\,. \tag{4.4.22}$$

Note that

$$(A+B)^{-1} = \big((\mathbb{1}_4 + BA^{-1})A\big)^{-1} = A^{-1}\big(\mathbb{1}_4 + BA^{-1}\big)^{-1} = \sum_{n\geq 0} (-1)^n A^{-1} (BA^{-1})^n\,,$$

by expanding $\big(\mathbb{1}_4 + BA^{-1}\big)^{-1}$ in geometric power series. Then,

$$\Phi\big((A+B)^{-1} v\, \bar{1}_{\Gamma,0}\big) = \Phi\Big(\big(A^{-1} + \sum_{n\geq 1} (-1)^n A^{-1} \big(BA^{-1}\big)^n\big) v\, \bar{1}_{\Gamma,0}\Big)\,,$$

and Equation (4.4.22) follows if $\Phi\big(\sum_{n\geq 1} (-1)^n A^{-1} \big(BA^{-1}\big)^n v\, \bar{1}_{\Gamma,0}\big) = 0$.

By definition of the map Φ, for each $n \geq 1$ we have $\Phi\big(A^{-1}(BA^{-1})^n v\, \bar{1}_{\Gamma,0}\big) = A^{-1}\Phi\big(BA^{-1}\Phi((BA^{-1})^{n-1} v\, \bar{1}_{\Gamma,0})\big)$, therefore it is sufficient to show that $\Phi\big(A^{-1}BA^{-1} v\, \bar{1}_{\Gamma,0}\big) = A^{-1}\Phi\big(BA^{-1} v\, \bar{1}_{\Gamma,0}\big) = 0$.

We have

$$A^{-1} = \begin{pmatrix} 1 & (A^{-1})_{12} & 0 \\ 0 & 1 & 0 \\ (A^{-1})_{31} & (A^{-1})_{32} & (A^{-1})_{33} \end{pmatrix},$$

with

$$\begin{aligned}
(A^{-1})_{31} &= -(z+e_{(2,1)(2,1)})^{-1} e_{(1,1)(2,1)}\,, \\
(A^{-1})_{32} &= (z+e_{(2,1)(2,1)})^{-1} \big(e_{(1,1)(2,1)}(z+e_{(1,2)(1,2)}-2) - e_{(1,2)(2,1)}\big)\,, \\
(A^{-1})_{33} &= (z+e_{(2,1)(2,1)})^{-1}\,.
\end{aligned}$$

Note that $(z+e_{(2,1)(2,1)})$ is invertible and $(z+e_{(2,1)(2,1)})^{-1} = \sum_{n\geq 0} (-1)^n z^{-n-1} {e_{(2,1)(2,1)}}^n$ by geometric power series expansion. Hence,

$$
\begin{aligned}
&A^{-1}\Phi\big(BA^{-1}v\,\bar{1}_{\Gamma,0}\big)\\
&= A^{-1}\Phi\begin{pmatrix} e_{(2,1)(1,2)}(A^{-1})_{31}e_{(1,3)(1,2)} + (1+e_{(2,1)(1,2)}A^{-1}_{32})(z+e_{(1,3)(1,3)}-3) + e_{(2,1)(1,2)}(A^{-1})_{33}e_{(1,3)(2,1)} \\ 0 \\ 0 \end{pmatrix}\bar{1}_{\Gamma,0}\\
&= \begin{pmatrix} \Phi\Big(e_{(2,1)(1,2)}(A^{-1})_{31}e_{(1,3)(1,2)} + (1+e_{(2,1)(1,2)}A^{-1}_{32})(z+e_{(1,3)(1,3)}-3) + e_{(2,1)(1,2)}(A^{-1})_{33}e_{(1,3)(2,1)}\Big) \\ 0 \\ (A^{-1})_{31}\Phi\Big(e_{(2,1)(1,2)}(A^{-1})_{31}e_{(1,3)(1,2)} + (1+e_{(2,1)(1,2)}A^{-1}_{32})(z+e_{(1,3)(1,3)}-3) + e_{(2,1)(1,2)}(A^{-1})_{33}e_{(1,3)(2,1)}\Big) \end{pmatrix}\bar{1}_{\Gamma,0}\,.
\end{aligned}
$$

In view of Lemma 2.2.1, we compute

$$
\begin{aligned}
&\Phi\big(e_{(2,1)(1,2)}(A^{-1})_{31}e_{(1,3)(1,2)}\,\bar{1}_{\Gamma,0}\big) = -\Phi\big([e_{(2,1)(1,2)},(z+e_{(2,1)(2,1)})^{-1}]e_{(1,1)(2,1)}e_{(1,3)(1,2)}\,\bar{1}_{\Gamma,0}\\
&-\ (z+e_{(2,1)(2,1)})^{-1}[e_{(2,1)(1,2)},e_{(1,1)(2,1)}e_{(1,3)(1,2)}]\,\bar{1}_{\Gamma,0}\big)\\
&= -\,(z+e_{(2,1)(2,1)})^{-1}\Phi\big(e_{(2,1)(1,2)}(z+e_{(2,1)(2,1)})^{-1}e_{(1,1)(2,1)}e_{(1,3)(1,2)}\,\bar{1}_{\Gamma,0}\big) + (z+e_{(2,1)(2,1)})^{-1}e_{(1,3)(1,2)}\,\bar{1}_{\Gamma,\mathfrak{l}}\,,
\end{aligned}
$$

and therefore

$$
\Phi\big(e_{(2,1)(1,2)}(A^{-1})_{31}e_{(1,3)(2,1)}\,\bar{1}_{\Gamma,0}\big) = (-1+(1-(z+e_{(2,1)(2,1)})^{-1})^{-1})e_{(1,3)(1,2)}\,\bar{1}_{\Gamma,\mathfrak{l}}\,. \tag{4.4.23}
$$

Then,

$$
\begin{aligned}
&\Phi\big((1+e_{(2,1)(1,2)}(A^{-1})_{32})(z+e_{(1,3)(1,3)}-3)\,\bar{1}_{\Gamma,0})\big)\\
&= (z+e_{(1,3)(1,3)}-3)\,\bar{1}_{\Gamma,\mathfrak{l}}\\
&+\Phi\Big([e_{(2,1)(1,2)},(z+e_{(2,1)(2,1)})^{-1}](e_{(1,1)(2,1)}(z+e_{(1,2)(1,2)}-2)-e_{(1,2)(2,1)})(z+e_{(1,3)(1,3)}-3)\,\bar{1}_{\Gamma,0}\\
&+\ (z+e_{(2,1)(2,1)})^{-1}[e_{(2,1)(1,2)},(e_{(1,1)(2,1)}(z+e_{(1,2)(1,2)}-2)-e_{(1,2)(2,1)})(z+e_{(1,3)(1,3)}-3)]\,\bar{1}_{\Gamma,0}\Big)\\
&= (z+e_{(1,3)(1,3)}-3))\,\bar{1}_{\Gamma,\mathfrak{l}}\\
&+\ (z+e_{(1,3)(1,3)}-3)(z+e_{(2,1)(2,1)})^{-1}\Phi\big(e_{(2,1)(1,2)},(z+e_{(2,1)(2,1)})^{-1}(e_{(1,1)(2,1)}(z+e_{(1,2)(1,2)}-2)-e_{(1,2)(2,1)})\,\bar{1}_{\Gamma,0}\big)\\
&-\ (z+e_{(1,3)(1,3)}-3)(z+e_{(2,1)(2,1)})^{-1}(z+e_{(2,1)(2,1)}-1)\,\bar{1}_{\Gamma,\mathfrak{l}}\,,
\end{aligned}
$$

and therefore

$$
\Phi\big(e_{(2,1)(1,2)}(A^{-1})_{32}\,\bar{1}_{\Gamma,0}\big) = (z+e_{(1,3)(1,3)}-3))\,\bar{1}_{\Gamma,\mathfrak{l}} - (z+e_{(1,3)(1,3)}-3))\,\bar{1}_{\Gamma,\mathfrak{l}} = 0\,. \tag{4.4.24}
$$

Finally,

$$
\begin{aligned}
&\Phi\big(e_{(2,1)(1,2)}(A^{-1})_{33}e_{(1,3)(2,1)}\,\bar{1}_{\Gamma,0}\big) = \Phi\big([e_{(2,1)(1,2)},(z+e_{(2,1)(2,1)})^{-1}]e_{(1,3)(2,1)}\,\bar{1}_{\Gamma,0}\\
&+\ (z+e_{(2,1)(2,1)})^{-1}[e_{(2,1)(1,2)},e_{(1,3)(2,1)}]\,\bar{1}_{\Gamma,0}\big)\\
&=\ (z+e_{(2,1)(2,1)})^{-1}\Phi\big(e_{(2,1)(1,2)}(z+e_{(2,1)(2,1)})^{-1}e_{(1,3)(2,1)}\,\bar{1}_{\Gamma,0}\big) - (z+e_{(2,1)(2,1)})^{-1}e_{(1,3)(1,2)}\,\bar{1}_{\Gamma,\mathfrak{l}}\,,
\end{aligned}
$$

and therefore

$$
\Phi\big(e_{(2,1)(1,2)}(A^{-1})_{33}e_{(1,3)(2,1)}\,\bar{1}_{\Gamma,0}\big) = \big(e_{(1,3)(1,2)} - (1-(z+e_{(2,1)(2,1)})^{-1})^{-1}e_{(1,3)(1,2)}\big)\,\bar{1}_{\Gamma,\mathfrak{l}}\,. \tag{4.4.25}
$$

Combining (4.4.23), (4.4.24) and (4.4.25) we get

$$
\Phi\Big(\big(e_{(2,1)(1,2)}(A^{-1})_{31}e_{(1,3)(1,2)} + (1+e_{(2,1)(1,2)}A^{-1}_{32})(z+e_{(1,3)(1,3)}-3) + e_{(2,1)(1,2)}(A^{-1})_{33}e_{(1,3)(2,1)}\big)\,\bar{1}_{\Gamma,0}\Big) = 0\,,
$$

as desired.

We shall now proceed with the proof of Theorem 4.4.2.

Proof of Theorem 4.4.2. By Lemma 2.2.6, the following identities hold in $\mathrm{Mat}_{r_1\times r_1}\mathcal{R}M_{\mathfrak{l}_1}$ and $\mathrm{Mat}_{r_1\times r_1}\mathcal{R}M_{\mathfrak{l}_2}$ respectively:

$$z^{-d-1}|z\mathbb{1}_N + F + \pi_{\mathfrak{p}_1}E + D_{\mathfrak{l}_1}|_{I_1J_1}\,\overline{1}_{\Gamma,\mathfrak{l}_1} = |\mathbb{1}_N + z^{-\Delta}E|_{I_1J_1}\,\overline{1}_{\Gamma,\mathfrak{l}_1}\,, \tag{4.4.26}$$

and

$$z^{-d-1}|z\mathbb{1}_N + F + \pi_{\mathfrak{p}_2}E + D_{\mathfrak{l}_2}|_{I_1J_1}\,\overline{1}_{\Gamma,\mathfrak{l}_2} = |\mathbb{1}_N + z^{-\Delta}E|_{I_1J_1}\,\overline{1}_{\Gamma,\mathfrak{l}_2}\,. \tag{4.4.27}$$

Combining (4.4.26) and (4.4.27), we obtain

$$\begin{aligned} L^{\Gamma,\mathfrak{l}_2}(z) &= |z\mathbb{1}_V + F + \pi_{\mathfrak{p}_2}E + D_{\mathfrak{l}_2}|_{\Psi_{\frac{d}{2},r_1}\Pi_{-\frac{d}{2},r_1}}\,\overline{1}_{\Gamma,\mathfrak{l}_2} \\ &= z^{1+d}|\mathbb{1}_V + z^{-\Delta}E|_{\Psi_{\frac{d}{2},r_1}\Pi_{-\frac{d}{2},r_1}}\,\overline{1}_{\Gamma,\mathfrak{l}_2} \\ &= \Phi(|z\mathbb{1}_V + F + \pi_{\mathfrak{p}_1}E + D_{\mathfrak{l}_1}|_{\Psi_{\frac{d}{2},r_1}\Pi_{-\frac{d}{2},r_1}}\,\overline{1}_{\Gamma,\mathfrak{l}_1}) \\ &= \Phi(L^{\Gamma,\mathfrak{l}_1}(z)) \in \mathrm{Mat}_{r_1\times r_1}\mathcal{R}W(\mathfrak{g},f,\Gamma,\mathfrak{l}_2)\,. \end{aligned}$$

For the third equality we have used the fact that the map $\Phi : W(\mathfrak{g},f,\Gamma,\mathfrak{l}_1) \xrightarrow{\sim} W(\mathfrak{g},f,\Gamma,\mathfrak{l}_2)$ is induced by the map between the corresponding quotient spaces $U(\mathfrak{g})/I_{\mathfrak{l}_1}$ and $U(\mathfrak{g})/I_{\mathfrak{l}_2}$, as in (4.2.1). ☐

Conclusion of the proof of Theorem 4.4.1. Using the algorithm of Section 4.4.1, supposing that $\mathrm{dist}(\Gamma) = \mathrm{dist}_R(\Gamma) = m \geq 0$, we can decompose

$$\Phi_\Gamma : W(\mathfrak{g},f,\Gamma,\mathfrak{l}) \xrightarrow{\sim} W(\mathfrak{g},f,\Gamma_R,0)$$

as $\Phi_\Gamma = \Phi_{m-1}\circ\ldots\circ\Phi_0$ for m odd, or as $\Phi_\Gamma = \Phi_m\circ\ldots\circ\Phi_2$ for m even where, for each $i\geq 0$, Φ_i is an associative algebra isomorphism of the form

$$\Phi_i : \rho^{\Gamma_i}_{0,\mathfrak{l}_{i+1}}\circ(\rho^{\Gamma_i}_{0,\mathfrak{l}_i})^{-1} : W(\mathfrak{g},f,\Gamma_i,\mathfrak{l}_i) \xrightarrow{\sim} W(\mathfrak{g},f,\Gamma_i,\mathfrak{l}_{i+1}) = W(\mathfrak{g},f,\Gamma_{i+1},0) = W(\mathfrak{g},f,\Gamma_{i+2},\mathfrak{l}_{i+2})\,, \tag{4.4.28}$$

where $\Gamma_0 = \Gamma, \Gamma_1, \ldots, \Gamma_m = \Gamma_R$ (or $\Gamma_{m+1} = \Gamma_R$, if m is even) is a chain of adjacent good $\frac{1}{2}\mathbb{Z}$-gradings, $\mathfrak{l}_i, \mathfrak{l}_{i+1}$ are Lagrangian subspaces of $\mathfrak{g}_{\frac{1}{2}}$ (with respect to the grading Γ_i) and $\mathfrak{l}_{i+2}$ is a Lagrangian subspace of $\mathfrak{g}_{\frac{1}{2}}$ (with respect to the grading Γ_{i+2}). Note that the algorithm of Section 4.4.1 allows to choose $\mathfrak{l}_0 = \mathfrak{l}$ to be just isotropic.

We shall now proceed by induction on m. For $m = 0$, $\Gamma = \Gamma_R$ and $\mathfrak{l} = 0$, and the claim is obvious. Without loss of generality we may assume that m is odd. Then, for $m = 1$ we have

$$\Phi_\Gamma = \Phi_0 : W(\mathfrak{g},f,\Gamma,\mathfrak{l}) \xrightarrow{\sim} W(\mathfrak{g},f,\Gamma,\mathfrak{l}_1) = W(\mathfrak{g},f,\Gamma_1,0) = W(\mathfrak{g},f,\Gamma_R,0)\,,$$

and by 4.4.2 we can conclude that $\Phi_\Gamma(L^{\Gamma,\mathfrak{l}}(z)) = L^{\Gamma_R,0}(z)$.

Let us now assume $m > 1$, and that (4.4.15) holds for any Γ' such that $\mathrm{dist}_R(\Gamma') < m$. We can decompose $\Phi_\Gamma = \Phi_{m-1}\circ\big(\Phi_{m-3}\circ\ldots\circ\Phi_0\big)$, where for each i, Φ_i is as in (4.4.28). By the induction hypothesis we have

$$\big(\Phi_{m-3}\circ\ldots\circ\Phi_0\big)(L^{\Gamma,\mathfrak{l}}(z)) = L^{\Gamma_{m-3},\mathfrak{l}_{m-2}}(z) = L^{\Gamma_{m-2},0}(z) = L^{\Gamma_{m-1},\mathfrak{l}_{m-1}}(z)\,.$$

For the last equalities recall that we chose the Lagrangian subspaces $\mathfrak{l}_{i+1}\subset\mathfrak{g}^{\Gamma_i}$, $\mathfrak{l}_{i+2}\subset\mathfrak{g}^{\Gamma_{i+2}}$ such that $\mathfrak{l}_{i+1}\oplus\mathfrak{g}^{\Gamma_i}_{\geq 1} = \mathfrak{g}^{\Gamma_{i+1}} = \mathfrak{l}_{i+1}\oplus\mathfrak{g}^{\Gamma_{i+2}}_{\geq 1}$, and therefore (by the very definition (2.2.3)), the corresponding operators $L^{\Gamma_i,\mathfrak{l}_{i+1}}(z)$, $L^{\Gamma_{i+1},0}(z)$ and $L^{\Gamma_{i+2},\mathfrak{l}_{i+2}}(z)$ coincide. By Theorem 4.4.2, we moreover have $\Phi_{m-1}(L^{\Gamma_{m-1},\mathfrak{l}_{m-1}}(z)) = L^{\Gamma_{m-1},\mathfrak{l}_m}(z) = L^{\Gamma_R,0}(z)$. Composing, we get

$$\Phi_\Gamma(L^{\Gamma,\mathfrak{l}}(z)) = \big(\Phi_{m-1}\circ\big(\Phi_{m-3}\circ\ldots\circ\Phi_0\big)\big)(L^{\Gamma,\mathfrak{l}}(z)) = L^{\Gamma_R,0}(z)\,,$$

as claimed in (4.4.15). ☐

As a corollary of Theorem 4.4.1, we obtain the following:

Corollary 4.4.1. *As we did in Remark 4.1.2 in the case of an even good $\frac{1}{2}\mathbb{Z}$-grading whose corresponding pyramid is either left or right aligned, we can write $W^\Phi(z)$ in block form*

$$W^\Phi(z) = \begin{pmatrix} -(-z)^{p_1}\mathbb{1}_{r_1} + W_1^\Phi(z) & W_2^\Phi(z) \\ W_3^\Phi(z) & -(-z)^q\mathbb{1}_{r-r_1} + W_4^\Phi(z) \end{pmatrix}, \tag{4.4.29}$$

where $W_1^\Phi(z)$, $W_2^\Phi(z)$, $W_3^\Phi(z)$, $W_4^\Phi(z)$ are block matrices of sizes $r_1 \times r_1$, $r_1 \times (r - r_1)$, $(r - r_1) \times r_1$ and $(r - r_1) \times (r - r_1)$ respectively, and $-(-z)^q\mathbb{1}_{r-r_1}$ is as in (4.1.4).

Thus, as a consequence of Theorem 4.4.1, we have

$$L(z) = \begin{pmatrix} \boxed{W_1(z)} & W_2(z) \\ W_3(z) & W_4(z) \end{pmatrix} = -(-z)^{p_1}\mathbb{1}_{r_1} + W_1(z) - W_2(z)(-(-z)^q\mathbb{1}_{r-r_1} + W_4(z))^{-1}W_3(z), \tag{4.4.30}$$

which agrees with Conjecture [DSKV16c, Conjecture 8.2].

Moreover, since for each $1 \leq i, j \leq r_1$ we have

$$L_{ij}^{\Gamma,\mathfrak{l}}(z) = W_{ij}^\Phi(z) - \sum_{\alpha,\beta \geq r_1+1} W_{i\alpha}^\Phi(z)((W^\Phi_{I_{rr_1}{}^c J_{r_1 r}{}^c})^{-1})_{\alpha\beta} W_{\beta j}^\Phi, \tag{4.4.31}$$

it is possible to deduce the commutation relations for $W_{ij}^\Phi(z) \in W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$ from the Yangian identity (1.5.10) for $L^{\Gamma,\mathfrak{l}}(z)$.

Recall the basis $\{f_{ij;k}\}$ of the centarlizer $\mathfrak{g}^f$ given in (3.1.8). We can finally summarize our results with the following theorem:

Theorem 4.4.3. *Let $\Gamma : \mathfrak{g} = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_i$ be a good $\frac{1}{2}\mathbb{Z}$-grading for the nilpotent element $f \in \mathfrak{g}$, associated with the partition $(p_1 \geq \ldots \geq p_r)$ of N, and let $\mathfrak{l} \subset \mathfrak{g}_{\frac{1}{2}}$ be an isotropic subspace with respect to the bilinear form ω.*

There exists a set of generators $W_{ij;k} = w(f_{ji;k})$, $1 \leq i, j \leq r$, and $0 \leq k \leq min(p_i, p_j) - 1$, of $W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$, for which the following identity holds

$$L(z) := |z\mathbb{1}_N + F + \pi_{\mathfrak{p}}E + D_{\mathfrak{l}}|_{I_1 J_1} \bar{1}_{\mathfrak{l}} = |W(z)|_{I_{rr_1} J_{r_1 r}}, \tag{4.4.32}$$

where

$$W(z) = (W_{ji}(z))_{1 \leq i,j \leq r}, \qquad W_{ij}(z) = -\delta_{ij}(-z)^{p_i} + \sum_{k=0}^{min(p_i,p_j)-1} W_{ij;k}(-z)^k \in W(\mathfrak{g}, f, \Gamma, \mathfrak{l})[z], \tag{4.4.33}$$

and $I_{rr_1}, J_{r_1 r}$ are as in (2.2.6) corresponding to the subsets $\mathcal{I} = \mathcal{J} = \{1, \ldots, r_1\}$. In this case, the linear map

$$w : \mathfrak{g}^f \longrightarrow W(\mathfrak{g}, f, \Gamma, \mathfrak{l}), \qquad f_{ji;k} \mapsto W_{ij;k} \tag{4.4.34}$$

satisfies all the conditions of Premet's Theorem 3.1.1.

Proof. In the case of Γ an even grading whose corresponding pyramid is either left or right aligned, use Definition 3.2.1 to build the polynomials $W_{ij}(z)$. By Theorems 3.3.1 and 3.3.2, $W_{ij}(z) \in W(\mathfrak{g}, f, \Gamma, 0)[z]$. Then, the statement is a consequence of Proposition 3.3.1 and Theorem 4.1.1.

For a generic good $\frac{1}{2}\mathbb{Z}$-grading, let

$$\Phi_\Gamma : W(\mathfrak{g}, f, \Gamma, \mathfrak{l}) \xrightarrow{\sim} W(\mathfrak{g}, f, \Gamma_R, 0)$$

be the associative algebra isomorphism constructed explicitly in (4.4.10), and let $W^\Phi(z)$ be the $r \times r$ matrix whose entries are the polynomials $W_{ij}^\Phi(z)$ defined in (4.4.13). By definition, $W^\Phi(z) \in \mathrm{Mat}_{r\times r} W(\mathfrak{g}, f, \Gamma, \mathfrak{l})[z]$. By Theorem 4.4.1, Equation (4.4.32) holds. We are therefore left to prove that the conditions of Premet's Theorem 3.1.1 hold for these generators.

First, we define the linear map $w^\Gamma : \mathfrak{g}^f \longrightarrow W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$ as the composition $w^\Gamma = \Phi_\Gamma^{-1} \circ w^{\Gamma_R}$, where $w^{\Gamma_R} : \mathfrak{g}^f \longrightarrow W(\mathfrak{g}, f, \Gamma_R, 0)$ is the linear map described in Section 3.3.2. We now want to check that w^Γ satisfies the conditions of Theorem 3.1.1.

As in the proof of Theorem 4.4.1, we can decompose

$$\Phi_\Gamma = \Phi_{m-1} \circ \ldots \circ \Phi_0,\ \text{ if } m \text{ odd}, \qquad \Phi_\Gamma = \Phi_m \circ \ldots \circ \Phi_2,\ \text{ if } m \text{ even} \tag{4.4.35}$$

where, for each $s \geq 0$, Φ_s is an associative algebra isomorphism of the form

$$\Phi_s : \rho^{\Gamma_s}_{0,\mathfrak{l}_{s+1}} \circ (\rho^{\Gamma_s}_{0,\mathfrak{l}_s})^{-1} : W(\mathfrak{g}, f, \Gamma_s, \mathfrak{l}_s) \xrightarrow{\sim} W(\mathfrak{g}, f, \Gamma_s, \mathfrak{l}_{s+1}) = W(\mathfrak{g}, f, \Gamma_{s+1}, 0) = W(\mathfrak{g}, f, \Gamma_{s+2}, \mathfrak{l}_{s+2}).$$

Without loss of generality we may assume that m is odd. In this case we have

$$W^\Phi_{ij;k} = (\Phi_\Gamma)^{-1}(W_{ij;k}) = \big((\Phi_0)^{-1} \circ \cdots \circ (\Phi_{m-1})^{-1}\big)(W_{ij;k}). \tag{4.4.36}$$

We can now proceed inductively on m. First, when $m = 1$ we have $\Phi_\Gamma = \Phi_0$, given by the following chain of isomorphisms

$$W(\mathfrak{g}, f, \Gamma, \mathfrak{l}) \overset{(\rho^\Gamma_{0,\mathfrak{l}})^{-1}}{\longrightarrow} W(\mathfrak{g}, f, \Gamma, 0) \overset{\rho^\Gamma_{0,\mathfrak{l}_1}}{\longrightarrow} W(\mathfrak{g}, f, \Gamma, \mathfrak{l}_1) = W(\mathfrak{g}, f, \Gamma_R, 0),$$

where $\mathfrak{l}_1 \subset \mathfrak{g}^\Gamma_{\frac{1}{2}}$ is an isotropic subspace such that $\mathfrak{l}_1 \oplus \mathfrak{g}^\Gamma_{\geq 1} = \mathfrak{g}^{\Gamma_R}_{\geq 1}$. In this case,

$$W^\Phi_{ij;k} = (\Phi_0)^{-1}(W_{ij;k}) = \rho^\Gamma_{0,\mathfrak{l}} \circ (\rho^\Gamma_{0,\mathfrak{l}_1})^{-1}(W_{ij;k}),$$

where $W_{ij;k}$ is a set of generators for $W(\mathfrak{g}, f, \Gamma, \mathfrak{l}_1) = W(\mathfrak{g}, f, \Gamma_R, 0)$ as in Definition 3.2.1. For every $1 \leq i, j \leq r$ and $0 \leq k \leq \min(p_i.p_j) - 1$, let $\Delta_{ij;k}, \Delta'_{ij;k} \in \frac{1}{2}\mathbb{Z}$ such that $f_{ij;k} \in \mathfrak{g}^\Gamma_{1-\Delta'_{ij;k}} \cap \mathfrak{g}^{\Gamma_R}_{1-\Delta_{ij;k}}$. By (3.3.13), then $W_{ij;k} \in f_{ji;k} + F_{\Delta'_{ji;k},2} W(\mathfrak{g}, f, \Gamma, \mathfrak{l}_1)$. Moreover, since by [GG02, Section 5.5], $\rho^\Gamma_{0,\mathfrak{l}}$, $\rho^\Gamma_{0,\mathfrak{l}_1}$ are filtered algebra isomorphisms, we have $\rho^\Gamma_{0,\mathfrak{l}} \circ (\rho^\Gamma_{0,\mathfrak{l}_1})^{-1}(W_{ij;k}) \in F_{\Delta'_{ji;k}} W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$, and and

$$\mathrm{gr}^\Gamma_{\Delta'_{ji;k}}\big(W^\Phi_{ij;k}\big) = \mathrm{gr}^\Gamma_{\Delta'_{ji;k}}\big((\Phi_\Gamma)^{-1}(W_{ij;k})\big) = (\Phi^{\mathrm{gr}}_\Gamma)^{-1}\big(\mathrm{gr}^\Gamma_{\Delta'_{ji;k}}(W_{ij;k})\big),$$

where we denote by $\Phi^{\mathrm{gr}}_\Gamma = \rho^{\Gamma,\mathrm{gr}}_{0,\mathfrak{l}} \circ (\rho^{\Gamma,\mathrm{gr}}_{0,\mathfrak{l}_1})^{-1}$ the same map between the corresponding graded algebras $\mathrm{gr}W(\mathfrak{g}, f, \Gamma, \mathfrak{l}_1) \to \mathrm{gr}W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$. Let us now consider the projections

$$\begin{aligned} &\eta^f_{\Gamma,\mathfrak{l}} : S(\mathfrak{g})/\mathrm{gr}^\Gamma I_\mathfrak{l} \twoheadrightarrow S(\mathfrak{g}^f), \qquad &&\eta^f_{\Gamma,0} : S(\mathfrak{g})/\mathrm{gr}^\Gamma I \twoheadrightarrow S(\mathfrak{g}^f) \\ &\eta^f_{\Gamma,\mathfrak{l}_1} : S(\mathfrak{g})/\mathrm{gr}^\Gamma I_{\mathfrak{l}_1} \twoheadrightarrow S(\mathfrak{g}^f), \qquad &&\eta^f_{\Gamma_R,0} : S(\mathfrak{g})/\mathrm{gr}^{\Gamma_R} I \twoheadrightarrow S(\mathfrak{g}^f) \end{aligned} \tag{4.4.37}$$

induced by the surjective algebra homomorphism $S(\mathfrak{g}) \twoheadrightarrow S(\mathfrak{g}^f)$ as in (3.1.3), with respect to the different gradings for $\mathfrak{g}$ and the different isotropic subspaces. By definition of the isomorphisms $\rho^{\Gamma,\mathrm{gr}}_{0,\mathfrak{l}}$ and $\rho^{\Gamma,\mathrm{gr}}_{0,\mathfrak{l}_1}$, and since $\mathfrak{l}$, $\mathfrak{l}_1 \subset \mathfrak{g}^\Gamma_{\frac{1}{2}} \subset \mathrm{Ker}\, \eta^f_{\Gamma,0}$, then

$$\begin{aligned} &\bullet\ \ \eta^f_{\Gamma,\mathfrak{l}} \circ \rho^{\Gamma,\mathrm{gr}}_{0,\mathfrak{l}} = \eta^f_{\Gamma,0} : S(\mathfrak{g})/\,\mathrm{gr}^\Gamma I \twoheadrightarrow S(\mathfrak{g}^f), \\ &\bullet\ \ \eta^f_{\Gamma,0} \circ (\rho^{\Gamma,\mathrm{gr}}_{0,\mathfrak{l}_1})^{-1} = \eta^f_{\Gamma,\mathfrak{l}_1} : S(\mathfrak{g})/\,\mathrm{gr}^\Gamma I_{\mathfrak{l}_1} \twoheadrightarrow S(\mathfrak{g}^f). \end{aligned} \tag{4.4.38}$$

Therefore,

$$\eta^f_{\Gamma,\mathfrak{l}}(\mathrm{gr}^\Gamma_{\Delta'_{ji;k}}(W^\Phi_{ij;k})) = \eta^f_{\Gamma,\mathfrak{l}}(\rho^{\Gamma,\mathrm{gr}}_{0,\mathfrak{l}} \circ (\rho^{\Gamma,\mathrm{gr}}_{0,\mathfrak{l}_1})^{-1}(\mathrm{gr}^\Gamma_{\Delta'_{ji;k}}(W_{ij;k}))) = \eta^f_{\Gamma,\mathfrak{l}_1}(\mathrm{gr}^\Gamma_{\Delta'_{ji;k}}(W_{ij;k})). \tag{4.4.39}$$

The result then follows from Proposition 3.3.1 once we show that $\eta^f_{\Gamma,\mathfrak{l}_1}(\mathrm{gr}^\Gamma_{\Delta'_{ji;k}}(W_{ij;k})) = \eta^f_{\Gamma_R,0}(\mathrm{gr}^{\Gamma_R}_{\Delta_{ji;k}}(W_{ij;k}))$. This is clear because by construction the ideal $I_{\mathfrak{l}_1}$ (with respect to the grading Γ) and I (with respect to the grading Γ_R) coincide, hence $\mathrm{Ker}\, \eta^f_{\Gamma_R,0} = \mathrm{Ker}\, \eta^f_{\Gamma,\mathfrak{l}_1}$.

Let us now assume $m > 1$, and that the conditions of Premet's Theorem hold for any Γ' such that $\mathrm{dist}_R(\Gamma') < m$. We can decompose

$$W^\Phi_{ij;k} = (\Phi_\Gamma)^{-1}(W_{ij;k}) = (\Phi_0)^{-1} \circ \big((\Phi_2)^{-1} \circ \ldots \circ (\Phi_{m-1})^{-1}\big)(W_{ij;k}),$$

where $\Delta'_{ji;k} \in \frac{1}{2}\mathbb{Z}$ is such that $f_{ji;k} \in \mathfrak{g}^{\Gamma}_{1-\Delta'_{ij;k}}$, and $W^{\Phi}_{ij;k} \in F_{\Delta'_{ji;k}} W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$ for every i, j, k. Let moreover $\Phi'(W_{ij;k}) := (\Phi_2)^{-1} \circ \ldots \circ (\Phi_{m-1})^{-1}(W_{ij;k})$ be a finite set of generators for $W(\mathfrak{g}, f, \Gamma_2, \mathfrak{l}_2)$, and we assume that $\Phi'(W_{ij;k}) \in F_{\Delta''_{ji;k}} W(\mathfrak{g}, f, \Gamma_2, \mathfrak{l}_2)$, where $\Delta''_{ji;k} \in \frac{1}{2}\mathbb{Z}$ is such that $f_{ji;k} \in \mathfrak{g}^{\Gamma_2}_{1-\Delta''_{ij;k}}$. By the inductive hypothesis we have

$$\eta^f_{\Gamma_2,\mathfrak{l}_2}\big(\mathrm{gr}^{\Gamma_2}_{\Delta''_{ji;k}}\,(\Phi'(W_{ij;k}))\big) = f_{ji;k}\,,$$

Next, let $(\Phi_0)^{-1} = \rho^{\Gamma}_{0,\mathfrak{l}} \circ (\rho^{\Gamma}_{0,\mathfrak{l}_1})^{-1} : W(\mathfrak{g}, f, \Gamma_2, \mathfrak{l}_2) \longrightarrow W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$ be the map given by

$$W(\mathfrak{g}, f, \Gamma, \mathfrak{l}) \overset{(\rho^{\Gamma}_{0,\mathfrak{l}})^{-1}}{\longrightarrow} W(\mathfrak{g}, f, \Gamma, 0) \overset{\rho^{\Gamma}_{0,\mathfrak{l}_1}}{\longrightarrow} W(\mathfrak{g}, f, \Gamma, \mathfrak{l}_1) = W(\mathfrak{g}, f, \Gamma_2, \mathfrak{l}_2)\,,$$

and let $\eta^f_{\Gamma,\mathfrak{l}}$, $\eta^f_{\Gamma,0}$, $\eta^f_{\Gamma,\mathfrak{l}_1}$ and $\eta^f_{\Gamma_R,0}$ be the corresponding projections, as in (4.4.37). Clearly, the equalities (4.4.38) still hold in this case. Thus,

$$\begin{aligned}
&\eta^f_{\Gamma,\mathfrak{l}}\big(\mathrm{gr}^{\Gamma}_{\Delta'_{ji;k}}(W^{\Phi}_{ij;k})\big) = \eta^f_{\Gamma,\mathfrak{l}}\big(\mathrm{gr}^{\Gamma}_{\Delta'_{ji;k}}\,((\Phi_\Gamma)^{-1}(W_{ij;k}))\big) = \eta^f_{\Gamma,\mathfrak{l}}\big(\mathrm{gr}^{\Gamma}_{\Delta'_{ji;k}}\,((\Phi_0)^{-1}(\Phi'(W_{ij;k})))\big) \\
&= \eta^f_{\Gamma,\mathfrak{l}}\big((\Phi_0)^{-1}\big(\mathrm{gr}^{\Gamma}_{\Delta'_{ji;k}}(\Phi'(W_{ij;k}))\big)\big) = \eta^f_{\Gamma,\mathfrak{l}_1}\big(\mathrm{gr}^{\Gamma}_{\Delta'_{ji;k}}(\Phi'(W_{ij;k}))\big)\,.
\end{aligned}$$

For the third equality we have used the fact that $(\Phi_0)^{-1} : W(\mathfrak{g}, f, \Gamma, \mathfrak{l}_1) \overset{\sim}{\longrightarrow} W(\mathfrak{g}, f, \Gamma, \mathfrak{l})$ is an isomorphism of filtered algebras, and for for the last equality we have used (4.4.38). By construction,

$$\eta^f_{\Gamma,\mathfrak{l}_1}\big(\mathrm{gr}^{\Gamma}_{\Delta'_{ji;k}}\,(\Phi'(W_{ij;k})))\big) = \eta^f_{\Gamma_2,\mathfrak{l}_2}\big(\mathrm{gr}^{\Gamma_2}_{\Delta''_{ji;k}}\,(\Phi'(W_{ij;k}))\big)\,,$$

and moreover $\eta^f_{\Gamma_2,\mathfrak{l}_2}\big(\mathrm{gr}^{\Gamma_2}_{\Delta''_{ji;k}}\,(\Phi'(W_{ij;k}))\big) = f_{ji;k}$ by the induction hypothesis. The claim then follows.

□

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