DETERMINATION OF ORDER IN LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

IN "FCAA" JOURNAL

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Abstract

The order of fractional differential equations (FDEs) has been proved to be of great importance in an accurate simulation of the system under study. In this paper, the orders of some classes of linear FDEs are determined by using the asymptotic behaviour of their solutions. Specifically, it is demonstrated that the decay rate of the solutions is influenced by the order of fractional derivatives. Numerical investigations are conducted into the proven formulas.

MSC 2010: Primary 34A08; Secondary 26A33, 33E12, 34E05

Key Words and Phrases: fractional differential equations, Mittag-Leffler type functions, asymptotic analysis, order estimation.

1. Introduction

The practical significance of fractional calculus has been recently discerned as a vastly superior method of describing the long-memory processes and developed remarkably over the last few years [3, 5, 8, 15, 20, 23]. In particular, fractional differential equations (FDEs) have been proven extremely important for more accurately modelling of many physical phenomena [2, 6, 14, 19, 22, 25].

Inverse problems to FDEs occur in many branches of science. Such problems have been investigated in, for instance, fractional diffusion equation [7, 12, 18, 24, 27] and inverse boundary value problem for semi-linear fractional telegraph equation [17]. In [12], it has been specifically shown that determination of β, the order of fractional differential operator, is definitely crucial to the appropriate simulation of the anomalous diffusion in

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order to specify that the transport phenomenon exhibits sub-diffusion or super-diffusion (respectively for $\beta < 1$ and $\beta > 1$). The authors of [12] have presented a theorem, the idea behind which is the determination of the order of a fractional diffusion equation. It has been the motivation behind this paper for proving formulas indicating the relationship between the fractional order and the asymptotic behaviour of the solutions to several different classes of linear FDEs.

The paper is organized as follows. Section 2 is allocated for recalling some bases of fractional calculus. In Section 3, a few special cases of linear FDEs are considered for which the fractional orders are determined. In Section 4, two examples are given to illustrate the correctness of the obtained results.

2. Preliminaries

For the sake of convenience, some basic definitions in fractional calculus and some useful properties are reviewed.

2.1. **Fractional integral and derivatives.** the definitions of Riemann-Liouville integral and derivative, and also the Caputo fractional derivative are represented in summary [15, 21].

Definition 2.1. Let $t_0 \in \mathbb{R}$ and $f : (t_0, +\infty) \to \mathbb{R}$ be continuous and integrable in every finite interval (t_0, t). The Riemann-Liouville fractional integral of order $\beta \in \mathbb{C}$ $(\Re(\beta) > 0)$ of the function f is defined by

$$\left(I_{t_0}^{\beta} f\right)(t) = \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (t-\tau)^{\beta-1} f(\tau)\,\mathrm{d}\tau, \quad t > t_0. \tag{2.1}$$

Definition 2.2. Let t_0 be a real number, $\beta \in \mathbb{C}$ $(\Re(\beta) > 0)$, $n = [\Re(\beta)] + 1$, and let the function $f : (t_0, +\infty) \to \mathbb{R}$ be continuous and integrable in every finite interval (t_0, t). The Riemann-Liouville fractional derivative of order β of the function f is defined by

$$\left(D_{t_0}^{\beta} f\right)(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n} \int_{t_0}^{t} (t-\tau)^{n-\beta-1} f(\tau)\,\mathrm{d}\tau, \quad t > t_0. \tag{2.2}$$

Caputo differential operator plays a major role in physical phenomena due to the fact that initial conditions for the FDEs with Caputo derivative are the same as those for integer-order differential equations. Caputo fractional derivative [4] is defined as follows:

Definition 2.3. Let $t_0 \in \mathbb{R}$, $\beta \in \mathbb{C}$ $(\Re(\beta) > 0)$, $n - 1 < \Re(\beta) < n$ $(n \in \mathbb{N})$, and let the function f be continuous and have n continuous derivatives in the interval $(t_0, +\infty)$. The Caputo fractional derivative of order β of the function f is defined by

$$\left(D_t^{\beta} f\right)(t) = \frac{1}{\Gamma(n-\beta)} \int_{t_0}^{t} (t-\tau)^{n-\beta-1} f^{(n)}(\tau)\, d\tau, \quad t > t_0. \tag{2.3}$$

Sequential fractional derivative (see [15, 21]) is defined by

$$\mathcal{D}^{n\beta} u(t) = \underbrace{D^{\beta} D^{\beta} \dots D^{\beta} u(t)}_{n}, \tag{2.4}$$

where D^{β} could be Riemann-Liouville, Caputo, or any other type of fractional derivative not considered here.

2.2. **Mittag-Leffler function and its derivatives.** The two-parameter function of Mittag-Leffler type, which first appeared in an article by Wiman [26] and studied by Agarwal and Humbert [1, 13], is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \quad \alpha, \beta, z \in \mathbb{C}, \quad \Re(\alpha) > 0, \tag{2.5}$$

where $E_{\alpha,1}(z)$ is simply denoted by $E_{\alpha}(z)$. In the case α and β are real and positive numbers, the series converges for all values of the argument z, therefore, the Mittag-Leffler function $E_{\alpha,\beta}(z)$ is an entire function of the order α^{-1}. Detailed information on Mittag-Leffler type functions and their properties can be found in, for instance, [15].

The Mittag-Leffler function satisfies the following differentiation formula [21]:

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(z^{\beta-1} E_{\alpha,\beta}(\lambda z^{\alpha})\right) = z^{\beta-n-1} E_{\alpha,\beta-n}(\lambda z^{\alpha}), \quad n \in \mathbb{N},\ \lambda \in \mathbb{R}, \tag{2.6}$$

and the following practical formulas can be directly derived from (2.6):

$$\frac{\mathrm{d}}{\mathrm{d}z} E_{\alpha}(\lambda z^{\alpha}) = \frac{1}{z} E_{\alpha,0}(\lambda z^{\alpha}) = \lambda z^{\alpha-1} E_{\alpha,\alpha}(\lambda z^{\alpha}), \tag{2.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^{\beta-1} E_{\alpha,\beta}(\lambda z^{\alpha})\right) = z^{\beta-2} E_{\alpha,\beta-1}(\lambda z^{\alpha}), \tag{2.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z E_{\alpha,2}(\lambda z^{\alpha})\right) = E_{\alpha}(\lambda z^{\alpha}). \tag{2.9}$$

2.3. **Asymptotic expansion of Mittag-Leffler function.** The asymptotic behaviour of Mittag-Leffler function $E_{\alpha,\beta}(z)$ $(|z| \to \infty)$ is complicated for $\alpha > 0$ and diverges greatly for $0 < \alpha < 2$ and $\alpha \geq 2$. This topic can be perfectly investigated in [21] with scrupulous attention to detail. Here, the asymptotic behaviour of Mittag-Leffler function is briefly stated for the case $0 < \alpha < 2$.

Let $0 < \alpha < 2$, $\beta, z \in \mathbb{C}$ and μ be an arbitrary real number such that $\frac{\pi}{2}\alpha < \mu < \min(\pi, \pi\alpha)$ then:

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{n} \frac{1}{\Gamma(\beta - k\alpha) z^k} + O\left(|z|^{-n-1}\right), \qquad |z| \to \infty, \quad \mu \leq |\arg(z)| \leq \pi. \tag{2.10}$$

Consequently, the following expansions can be obtained from (2.10):

$$E_{\alpha}(\lambda z^{\alpha}) = -\frac{z^{-\alpha}}{\lambda\Gamma(1-\alpha)} + O\left(\frac{1}{|\lambda|^2 z^{2\alpha}}\right), \quad z \to \infty, \tag{2.11}$$

$$E_{\alpha,\alpha}(\lambda z^{\alpha}) = \frac{\alpha z^{-2\alpha}}{\lambda^2\Gamma(1-\alpha)} + O\left(\frac{1}{|\lambda|^3 z^{3\alpha}}\right), \quad z \to \infty. \tag{2.12}$$

where z is a positive real number, $0 < \alpha < 2$, and $\lambda < 0$.

3. Main results

In this section the fractional orders of several classes of linear FDEs are determined by using the asymptotic behaviour of Mittag-Leffler functions.

THEOREM 3.1. Let $0 < \beta \leq 1$ and $t_0 > 0$. Consider the sequential linear differential equation in the sense of Riemann-Liouville Derivative:

$$\mathcal{D}_{t_0}^{2\beta} u + a_1 \mathcal{D}_{t_0}^{\beta} u + a_0 u = 0, \tag{3.1}$$

with the initial condition $u(t_0) = u_0$ and $\mathcal{D}_{t_0}^{\beta} u(t_0) = u_1$, and let a_0 and a_1 are reals such that r_1 and r_2, the roots of the characteristic equation $r^2 + a_1 r + a_0 = 0$, are distinct and negative real numbers. Then, the following formula holds

$$\beta = -1 - \lim_{t \to \infty} \frac{t u'}{u}. \tag{3.2}$$

P r o o f. The solution of (3.1) is as follows [15]:

$$u(t) = c_1 t^{\beta-1} E_{\beta,\beta}\left(r_1 t^{\beta}\right) + c_2 t^{\beta-1} E_{\beta,\beta}\left(r_2 t^{\beta}\right). \tag{3.3}$$

where c_1 and c_2 depend on the initial conditions. The first derivative of $u(t)$ can be calculated by using (2.8) as below:

$$u'(t) = c_1 t^{\beta-2} E_{\beta,\beta-1}\left(r_1 t^{\beta}\right) + c_2 t^{\beta-2} E_{\beta,\beta-1}\left(r_2 t^{\beta}\right). \tag{3.4}$$

The asymptotic expansions of $u(t)$ and $u'(t)$ can be obtained by using (2.10)

$$u(t) = \frac{-t^{-\beta-1}}{\Gamma(-\beta)}\left(\frac{c_1}{r_1^2} + \frac{c_2}{r_2^2}\right) + c_1 t^{\beta-1} O\left(|r_1|^{-3} t^{-3\beta}\right) + c_2 t^{\beta-1} O\left(|r_2|^{-3} t^{-3\beta}\right), \tag{3.5}$$

$$\begin{aligned} u'(t) = &\frac{(\beta+1)\, t^{-\beta-2}}{\Gamma(-\beta)}\left(\frac{c_1}{r_1^2} + \frac{c_2}{r_2^2}\right) + c_1 t^{\beta-2} O\left(|r_1|^{-3} t^{-3\beta}\right) \\ &+ c_2 t^{\beta-2} O\left(|r_2|^{-3} t^{-3\beta}\right). \end{aligned} \tag{3.6}$$

Thus

$$-\lim_{t\to\infty} \frac{tu'}{u} = -\lim_{t\to\infty} \frac{\frac{(\beta+1)t^{-\beta-1}}{\Gamma(-\beta)}\left(\frac{c_1}{r_1^2} + \frac{c_2}{r_2^2}\right)}{\frac{-t^{-\beta-1}}{\Gamma(-\beta)}\left(\frac{c_1}{r_1^2} + \frac{c_2}{r_2^2}\right)} = \beta + 1. \tag{3.7}$$

□

THEOREM 3.2. Let $0 < \beta < \frac{1}{2}$ and $\gamma, \mu \in \mathbb{R}$ such that $0 < \gamma < \mu^2$, and let $D_t^{\beta} u$ indicate the Caputo differentiation operator. For the initial value problem

$$D_t^{2\beta} u(t) + 2\mu D_t^{\beta} u(t) + \gamma u(t) = 0, \tag{3.8}$$

with the initial condition $u(0) = 1$, and also for sequential linear differential equation of fractional order

$$\mathcal{D}_t^{2\beta} u + 2\mu \mathcal{D}_t^{\beta} u + \gamma u = 0, \tag{3.9}$$

with the initial condition $\mathcal{D}_t^{\beta} u(0) = 0$ and $u(0) = 1$, the following formula holds

$$\beta = -\lim_{t\to\infty} \frac{tu'}{u}. \tag{3.10}$$

REMARK 3.1. If $\mathcal{D}_t^{\beta} u(0) = 0$, then $c_1 r_1 + c_2 r_2 = 0$ and $\mathcal{D}_t^{2\beta} u = D_t^{2\beta} u$. The case of $\mathcal{D}_t^{\beta} u(0) \neq 0$ leads to $c_1 r_1 + c_2 r_2 \neq 0$. Thus, the coefficients c_1 and c_2 are not the same as those represented in the proof of Theorem 3.2 and therefore must be calculated.

P r o o f. The solutions of (3.8) and (3.9) are represented by [9]

$$u(t) = c_1 E_{\beta}\left(r_1 t^{\beta}\right) + c_2 E_{\beta}\left(r_2 t^{\beta}\right), \tag{3.11}$$

where the coefficients c_1 and c_2 are respectively equal to $\frac{1}{2}\left(1+\frac{\mu}{\sqrt{\mu^2-\gamma}}\right)$ and $\frac{1}{2}\left(1-\frac{\mu}{\sqrt{\mu^2-\gamma}}\right)$, and the parameters r_1 and r_2 are equal to $-\mu+\sqrt{\mu^2-\gamma}<0$ and $-\mu-\sqrt{\mu^2-\gamma}<0$ respectively. Equation (3.11) can be rewritten in the following form by using (2.11)

$$u(t)=-\frac{t^{-\beta}}{\Gamma(1-\beta)}\left(\frac{c_1}{r_1}+\frac{c_2}{r_2}\right)+c_1O\left(r_1^{-2}t^{-2\beta}\right)+c_2O\left(r_2^{-2}t^{-2\beta}\right). \quad (3.12)$$

The first derivative of (3.11) can be obtained by referring to (2.7)

$$u'(t)=c_1r_1t^{\beta-1}E_{\beta,\beta}\left(r_1t^\beta\right)+c_2r_2t^{\beta-1}E_{\beta,\beta}\left(r_2t^\beta\right). \quad (3.13)$$

Equation (2.12) is used for applying the asymptotic behaviour of Mittag-Leffler function to (3.13):

$$\begin{aligned}u'(t)=&\frac{\beta t^{-\beta-1}}{\Gamma(1-\beta)}\left(\frac{c_1}{r_1}+\frac{c_2}{r_2}\right)\\&+t^{\beta-1}\left(c_1r_1O\left(r_1^{-3}t^{-3\beta}\right)+c_2r_2O\left(r_2^{-3}t^{-3\beta}\right)\right).\end{aligned} \quad (3.14)$$

Therefore,

$$-\lim_{t\to\infty}\frac{tu'}{u}=-\lim_{t\to\infty}t\frac{\frac{\beta t^{-\beta-1}}{\Gamma(1-\beta)}\left(\frac{c_1}{r_1}+\frac{c_2}{r_2}\right)}{-\frac{t^{-\beta}}{\Gamma(1-\beta)}\left(\frac{c_1}{r_1}+\frac{c_2}{r_2}\right)}=\beta. \quad (3.15)$$

□

THEOREM 3.3. Let $1<\beta<2$, and r be a negative real number. For the fractional differential equation with Caputo derivative

$$D_t^\beta u-ru=0, \quad (3.16)$$

with the initial condition $u(0)=1$ and $u'(0)=1$, the following relationship is held true:

$$\beta=1-\lim_{t\to\infty}\frac{tu'}{u}. \quad (3.17)$$

P r o o f. The solution to (3.16) is

$$u(t)=E_\beta\left(rt^\beta\right)+tE_{\beta,2}\left(rt^\beta\right). \quad (3.18)$$

The first derivative of $u(t)$ could be calculated by referring to (2.7) and (2.9)

$$u'(t)=rt^{\beta-1}E_{\beta,\beta}\left(rt^\beta\right)+E_\beta\left(rt^\beta\right). \quad (3.19)$$

The asymptotic expansions of $u(t)$ and $u'(t)$ are respectively

$$u(t) = -\frac{t^{-\beta}}{r\Gamma(1-\beta)}\left(1+\frac{t}{1-\beta}\right) + O\left(|r|^{-2}t^{-2\beta}\right) + tO\left(|r|^{-2}t^{-2\beta}\right), \tag{3.20}$$

and

$$u'(t) = \frac{t^{-\beta}}{r\Gamma(1-\beta)}\left(\frac{\beta}{t}-1\right) + rt^{\beta-1}O\left(|r|^{-3}t^{-3\beta}\right) + O\left(|r|^{-2}t^{-2\beta}\right), \tag{3.21}$$

therefore,

$$\lim_{t\to\infty}\frac{tu'}{u} = \lim_{t\to\infty}\frac{\frac{t^{-\beta}}{r\Gamma(1-\beta)}(\beta-t)}{-\frac{t^{-\beta}}{r\Gamma(1-\beta)}\left(1+\frac{t}{1-\beta}\right)} = 1-\beta. \tag{3.22}$$

□

Theorem 3.4. Let $0<\beta<1$, and r be a negative real number. For (3.16) with the initial condition $u(0)=1$, the following is held true

$$\beta = -\lim_{t\to\infty}\frac{tu'}{u}. \tag{3.23}$$

P r o o f. The solution to (3.16) is as follows:

$$u(t) = E_\beta\left(rt^\beta\right), \tag{3.24}$$

where $0<\beta<1$ [16]. The first derivative of $u(t)$ could be calculated by referring to (2.7):

$$u'(t) = rt^{\beta-1}E_{\beta,\beta}\left(rt^\beta\right). \tag{3.25}$$

The asymptotic expansions of $u(t)$ and $u'(t)$ are respectively

$$u(t) = -\frac{t^{-\beta}}{r\Gamma(1-\beta)} + O\left(|r|^{-2}t^{-2\beta}\right), \tag{3.26}$$

and

$$u'(t) = \frac{\beta t^{-\beta-1}}{r\Gamma(1-\beta)} + rt^{\beta-1}O\left(|r|^{-3}t^{-3\beta}\right), \tag{3.27}$$

then

$$\lim_{t\to\infty}\frac{tu'}{u} = \lim_{t\to\infty}\frac{\frac{\beta t^{-\beta}}{r\Gamma(1-\beta)}}{-\frac{t^{-\beta}}{r\Gamma(1-\beta)}} = -\beta. \tag{3.28}$$

□

4. Numerical investigation

EXAMPLE 4.1. Consider the initial value problem

$$D_t^{2\beta} u + 2D_t^{\beta} u + 0.7u = 0, \qquad t \geq 0, \qquad 0 < \beta < \frac{1}{2}, \tag{4.1}$$

with the initial condition $u(0) = 1$. The solution to (4.1) is the following:

$$u(t) = c_1 E_\beta \left(r_1 t^\beta \right) + c_2 E_\beta \left(r_2 t^\beta \right), \tag{4.2}$$

where $r_1 = -0.4523$, $r_2 = -1.5477$, $c_1 = 1.4129$, $c_2 = -0.4129$. The graph of $-\frac{tu'}{u}$, which has been evaluated for several different values of β, is shown in Fig. 4.1. It is obvious that $-\frac{tu'}{u}$ tends asymptotically to β, as t goes to infinity. The term $-\frac{tu'}{u}$ is numerically evaluated by means of the MATLAB code "ml.m" [10] which is based on the numerical inversion of the Laplace transform of Mittag-Leffler function [11]. Numerical results are in good agreement with the formula introduced in Theorem 3.2 and the rate of the convergence of $-\frac{tu'}{u}$ is greatly influenced by the value of β.

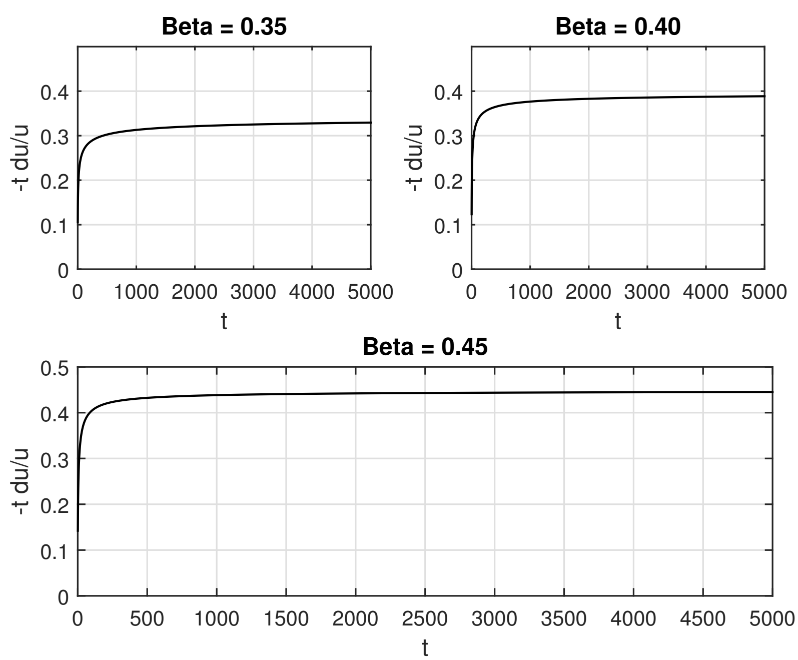

Fig. 4.1. Graph of $-\frac{tu'}{u}$ for $\beta = 0.35$, $\beta = 0.40$, and $\beta = 0.45$.

Example 4.2. Consider the fractional differential equation

$$\begin{cases} D_t^{\beta} u + 2u = 0, \\ u(0) = 1, \\ u'(0) = 1, \end{cases} \tag{4.3}$$

where $1 < \beta < 2$ and $D_t^{\beta} u$ is in the sense of Caputo derivative. The solution is in the form of

$$u(t) = E_{\beta}\left(-2t^{\beta}\right) + tE_{\beta,2}\left(-2t^{\beta}\right). \tag{4.4}$$

According to Theorem 3.3, the term $1 - \frac{tu'}{u}$ tends to β as t goes to the infinity. The numerical evaluation of $1 - \frac{tu'}{u}$ has been conducted by using (4.4) and its derivative. The results are shown in Fig. 4.2. As it can be observed, $1 - \frac{tu'}{u}$ converges to β with a rate affected by the value of β, i.e. the convergence will be faster if the fractional order β tends to 2.

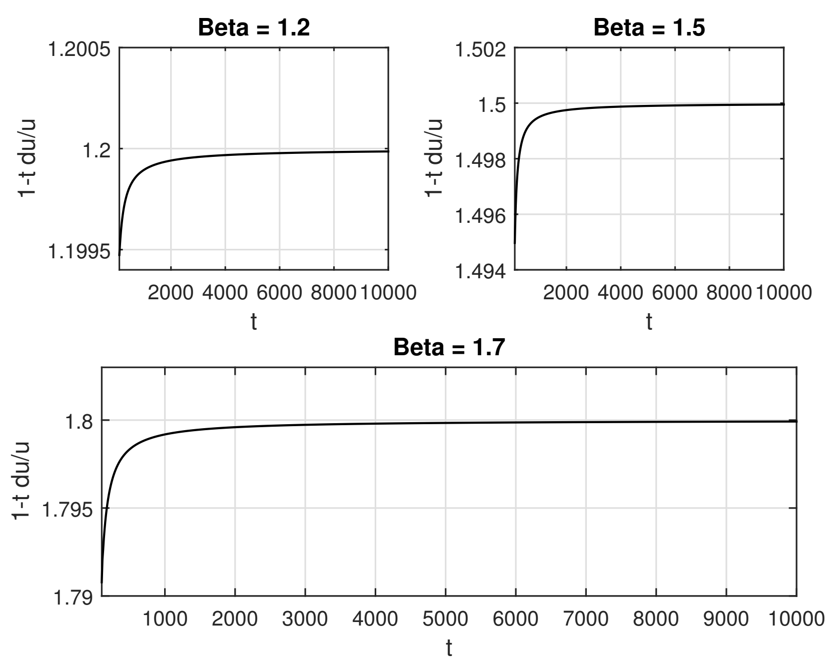

Fig. 4.2. Graph of $1 - \frac{tu'}{u}$ for $\beta = 1.2$, $\beta = 1.5$, and $\beta = 1.7$.

5. Conclusion

Inverse problems occur in many branches of science and have been also examined in the models described by FDEs. For instance, determination of the order of fractional systems has been indicated to be of such crucial importance that it could influence how anomalous diffusion equations must be appropriately simulated. Thus, in this paper, the solutions to several classes of linear FDEs represented, for which the order determination has been demonstrated by using asymptotic behaviour of Mittag-Leffler type functions. The formulas in Theorems 3.1-3.4 have been numerically examined for different values of β. The results show the accuracy of the obtained formulas.

Acknowledgements

The authors are grateful to Professor Masahiro Yamamoto for pointing out the papers [7, 12, 24].

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Received: August 02, 2018