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UNIVERSITÀ DI ROMA

Department of Methods and Models for Economics, Territory and Finance

Ph.D. Program in Models for Economics and Finance

Curriculum in Mathematics for Economic-Financial Applications

A Singular Stochastic Control Problem for Hydropower Generation in Renewable Energy Markets

Ph.D. Thesis

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Academic Year 2017-2018

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Introduction

The integration of green resources in the grid is a reason of technical and economic challenges since the power production is highly unpredictable across any timescale. The fluctuations in output make the renewable-oriented power producer not enough competitive in the electricity market. Energy storage facilities not only store a large amount of energy but also provide an efficient compensation system to mitigate power imbalances and avoid intermittent connection with the grid.

Nowadays the high penetration of renewable energy makes storage utilities economically relevant, and the increasing liberalization of the electricity market gives the opportunity to use storage facilities as trading instruments. The owner of a storage system may both sell or rent his/her facility in the market and implement operative strategies to gain profit from the high volatility of electricity prices.

Recently, the economic-financial valuation of storage facilities has becoming a demanding challenge for several researchers within the mathematical framework of stochastic control. The owner of a storage facility is a sort of "controller" that selects the optimal execution policy of his/her facility in order to maximise the expected discounted cash flow subject to operational costs and physical constrains.

In this thesis we focus on *hydroelectric storage* facilities. Such reserve systems are widely used in alternative energy power generation, thanks to their high capability in quickly responding to the energetic compensation necessities, that so often occur in the scenario of renewable energy production.

The optimal management of energy storage facilities is widely diffused in the framework of control theory and different aspects are taken into consideration by means of various mathematical tools. In particular, Thompson, Davison and Rasmussen [TDR04; TDR09] present numerical result for a partial differential equations approach in classical control theory. Spikes for prices dynamics are considered by adding jumps to the usual mean-reverting process. Particular attention is paid to the operational characteristics of the real storage systems. Zhao and Davison [ZD09] propose an optimization model for pumped-storage hydroelectric facilities exposed to a constant water inflow and deterministic dynamics for the prices. Carmona and Ludkovski [CL10] provide two finite horizon models for natural gas dome storage and hydroelectric pumped storage. They construct an optimal switching model and propose a solution based on multiple stopping problems. They also obtain an efficient simulation-based numerical method for valuation.

Among many studies in singular stochastic control, in which the trajectories of the admissible control strategies are not necessarily absolutely continuous with respect the Lebesgue measure, Harrison and Taylor [HT78] consider a storage system whose fuel fluctuations are driven by a Brownian motion that is controlled by a pair of non-negative processes representing to cumulative injection into the storage facility and the cumulative withdrawn from the facility. Their objective is to find such processes in order to minimize the expected discounted costs over an infinite time horizon, subject to the non-negativity constraint on the state process. These costs are the sum of constant proportional execution costs associated with injection and withdrawn respectively and linear holding running cost. The authors prove that an optimal pair exists and it is composed by two almost surely continuous (but non absolutely continuous processes) and they show that the controlled state process behaves as a Brownian motion instantaneously reflected at the origin and at a positive value, unique solution of a certain algebraic equation. Chiarolla, Ferrari and Stabile [CFS15] consider an inventory model for a storable commodity whose supply purchase is subject to price and demand uncertainty. The spot price of the commodity is driven by a general (positive) exponential Lévy process. They aim to determine an optimal procurement policy in order to maximise the expected discounted return obtained by meeting a future random commodity's demand and facing holding and ordering running costs. They obtain necessary and sufficient first order conditions for optimality and, in the particular case of linear holding cost and exponentially distributed demand, they are able to provide an explicit expression for the control policy and a probabilistic representation for the optimal return.

Our research starts from the work of Shardin and Wunderlich [SW17], *Partially observable stochastic optimal control problems for an energy storage*. They consider a hydroelectric storage system in which the energy production activity can be reversed and the reserve can be refilled by pumping water from an underlying reserve. The intent of the authors is to determine the charging and discharging policies in order to exploit the high unpredictability of energy markets and manage the hydroelectric reserve in order to buy energy from the market and load the reserve when prices are low, and to discharge water to produce energy and sell it, when prices increase. In particular they aim at controlling the head of the water inside the upper reservoir by setting both the pumping and releasing *rates*. At each time $t \geq 0$, the optimal rates depends on the state on the current water head are they are chosen among all the admissible progressively measurable stochastic process satisfying certain state constraints.

We began to shape our model aiming at generalising their approach, overcoming the idea of controlling the storage system through the charging and discharging rates and introducing the possibility of acting directly on the volume of water in the reservoir, by considering control policies whose trajectories are not necessarily absolutely continuous with respect to the Lebesgue measure. Hence we first set the problem in the framework of the bounded variation stochastic control policies, i.e. we assume that, at each time t, the current volume of water inside the reservoir is considered as the difference of two non-decreasing processes representing the cumulative amounts of water charged and discharged up to time t, respectively.

The difficulties in dealing with the above bounded variation model emerged immediately, not only in the attempt to recover the dependence of the admissible controls class on the controlled state, but also in relation to the other characteristics of the problem that we are going to introduce below. For these reasons, we decided to focus the purpose of this thesis on the case of pure generation, i.e. when the owner of a hydroelectric reserve can only produce electricity through a turbine activated by the passage of the flow of water released from an overhead reservoir. That is, we consider the problem of finding an optimal control strategy of hydroelectric power production among a suitable class of monotone (non-decreasing) processes. In particular, for any time $t \geq 0$, we consider the volume of water in the reservoir as

$$Y_t^{y,\nu} = y - \nu_t, \quad t \geq 0$$

where $y \in [\underline{y}, \overline{y}]$ represents the initial amount and, at each time t, ν_t represents the cumulative volume of water discharged up to time t. The instantaneous amount of energy that can be produced by releasing fuel is given by

$$dE_t^{y,\nu} = f(Y_t^{y,\nu})d\nu_t,$$

where f is the *instantaneous marginal productivity function* and it represents the intensity at which energy is generated by discharging from the water head corresponding to the current volume $Y_t^{y,\nu}$.

As it is usual in electricity markets modelling (see e.g. [DFM15; De +17; SW17]), we assume that energy spot market prices are described by a mean reverting process $(X_t^x)_{t\geq 0}$ of the Vasicek type which involves the positive probability that negative energy prices occur. On the other hand, we take into account that the owner of a hydroelectric facility faces instantaneous running holding costs concerning the maintenance of unused water. Considering such costs described by a function h that depends on the current volume at time t, our objective is to maximise the discounted expected cash flow

$$J(x,y;\nu)=\mathbb{E}\Big[\int_0^{+\infty} e^{-rt}X_t^x f(Y_t^{y,\nu})d\nu_t^c+\sum_{t\geq 0} e^{-rt}[X_t^x]^+ f(Y_t^{y,\nu})\Delta\nu_t+ \\ -\sum_{t\geq 0} e^{-rt}[X_t^x]^- \int_0^{\Delta\nu_t} f(Y_t^{y,\nu}-z)dz-\int_0^{+\infty} e^{-rt}h(Y_t^{y,\nu})dt\Big],$$

by selecting an optimal control in the class of admissible policies $\mathcal{A}(y)$, that is the family of all non-decreasing, left-continuous, adapted stochastic processes $(\nu_t)_{t\geq 0}$ such that $0\leq \nu_t\leq y-\underline{y}$, for any $t\geq 0$. That is, our scope is to find a control policy solution of the optimization problem

$$v(x,y):=\sup_{\nu\in\mathcal{A}(y)} J(x,y;\nu), \tag{OC}$$

where v is the *value function* of stochastic control problem (OC).

Our model belongs to the wide class of *singular stochastic optimal control problems.* In particular we deal with a finite-fuel, two-dimensional degenerate control problem on $\mathbb{R}\times[\underline{y},\overline{y}]$, where the control policy ν does not affect the diffusion dynamics $(X_t^x)_{t\geq 0}$. Notice that the power productivity is a function of the controlled state process $Y_t^{y,\nu}$ hence the instantaneous "revenue intensity" $X_t^x f(Y_t^{y,\nu})$ is *state-dependent.* It may be positive or negative, according to the sign of X_t^x. So the marginal yield fails to be monotone. The novel structure of our performance J shows that the system may react in two different ways whenever a jump of the control processes occurs. In fact, if the price is positive and it is convenient to discharge a quantity $\Delta\nu_t$ instantaneously, then the system will act with a *single impulse* at the highest available productivity level $f(Y_t^{y,\nu})$. Whereas, if the energy price is negative but it is still preferable to release water rather than pay the cost of maintaining that level, then our model allows the system to act less efficiently in order to contain the loss (negative revenue), by producing less energy with the same discharged quantity of water. In fact, rather than releasing the quantity $\Delta\nu_t$ in a single impulse, the strategy splits that quantity into sufficiently small portions, which are discharged one after the other in an infinitesimal time interval. Such strategy is a sort of *chattering policy*, as noticed by Alvarez [Alv00].

The aforementioned features of our model, including the non-monotone marginal revenue, is a novelty in the field of singular stochastic control, even compared to other state-dependent problems treated in literature. Indeed, [Alv00] considers chattering policies in the ambit of optimal harvesting problems and takes into account an instantaneous monotone (non-increasing) marginal revenue depending on the current state of the system. The author shows that the optimal strategy should be a reflection of the process at a given optimal boundary by means of a chattering policy, but such control is not in the set of the admissible controls. Song, Stockbridge and Zhu [SSZ11] consider the harvesting problem in a random environment by introducing a switching-regime through a continuous-time Markov chain with finite states space. They prove a verification theorem and explicitly construct a chattering type control process that turns out to be ε-optimal for their harvesting model. Again the monotonicity of their marginal yield allows them to show that an admissible optimal control policy might not exist. Similar results are obtained by Alvarez, Lungu and Øksendal [ALØ16] for their multi-dimensional stochastic harvesting framework, involving interaction among different populations. Even in their model, if the price per unit of each population is state-dependent and decreasing, an admissible optimal control may not exist, but a chattering policy might be optimal. Købila [Kob93] faces a decision problem between hydro and thermal power generation. In a scenario with stochastic demand, they consider the alternative between the two different generation possibilities by representing the cost of introducing new hydro resources as an irreversible capital investment. The authors aim at maximise the expected total discounted profit, where the marginal cost of investment on hydroelectric power is considered state-dependent. Such a dependence is partially overcome by approximating the problem through a sequence of absolutely continuous controls and reformulating the profit functional in terms of such processes. Although the problem is explicitly solved, the "chattering behaviour" seems to be excluded from their model.

Unlike [Alv00; ALØ16; SSZ11], in our model the non-admissibility of chattering policies is overcome by the structure of the hydroelectric production system that is allowed to respond by an immediate release of water implemented as a chattering policy. The possibility of acting in such an unconventional manner, as far as we know, appears here for the first time in literature. By exploiting the properties of the performance functional J we provide some *a priori* estimates for the value function $v(x, y)$, that allow us to highlight the regularity properties of v. As it is usual in singular stochastic control, the strip $\mathbb{R} \times [\underline{y}, \overline{y}]$, which our problem is defined on, splits in two subsets: the inaction region $\mathcal{C}$ where it is never optimal exerting the control and its complement, the action region $\mathcal{D}$, where it is optimal to act instantaneously. Heuristic arguments allow us to guess the corresponding Hamilton-Jacobi-Bellman (HJB) equation, which turns out to be a variational inequality with a state-dependent gradient constraint.

We prove a Verification Theorem providing sufficient conditions that characterise the value function v among the solutions of the HJB equation. We obtain such result without any hypothesis of monotonicity on the marginal revenue, condition that instead is crucial in [Alv00; ALØ16; SSZ11]. A similar result in a very general setting is obtained by Davis and Zervos [DZ98], although they explicitly solve the problem only in two specific cases which do not involve state-dependence. The Verification Theorem sheds light on the structure of the optimal control process. In particular we show that the optimal control is *purely discontinuous* and, at the first time of action, it *exerts all the available fuel with a single instantaneous jump*. The hydroelectric system generates power with its highest productivity if the price is positive at exerting time, otherwise it behaves as if the control could be exerted in a chattering manner.

In order to explicitly solve the stochastic control problem (OC) it is necessary to determine the first time at which discharging all the available water allows us to obtain the maximum profit. Such optimal acting time is the solution of a suitable associated optimal stopping problem. In particular, we consider a *family of optimal stopping problems* (OS_y), parametrized by the initial reserve volume y, aiming to show that, for each $(x, y) \in \mathbb{R} \times [\underline{y}, \overline{y}]$, the value function of the optimal control problem (OC) is such that

$$v(x,y) = u(x,y) - \frac{1}{r}h(y), \qquad (v_{con})$$

where $x \mapsto u(x, y)$ is the *optimal reward function* associated to the optimal stopping time τ_y^* of problem (OS_y). The existence of a connection between stochastic control problems and optimal stopping problems is a peculiarity of many results of singular stochastic control problems, see e.g. [BK96; BC67; CH00; CH09; KS84; KS85; KS86], among many others. However, in most works in the literature, the connection between the value function of the optimal control problem and the optimal reward of the associated optimal stopping problem is of "differential" type, i.e. the first derivative of the value function in the direction of the controlled state variable coincides with the optimal reward of the associated optimal stopping problem. In particular, as first noticed in [BC67], the inaction and action regions of the optimal control problem coincide, respectively, with the continuation and stopping regions of the associated stopping problem. Moreover, often for problems that exhibit such type of connection, the "principle of smooth fit" holds true, i.e. the optimal reward of the optimal stopping time is continuously-differentiable across the boundary separating the action and inaction regions. In many cases such property is crucial to construct the optimal control policy.

Instead, under particular hypotheses, our model gives rise to a novel connection between the value function v of the optimal control problem and the optimal reward u of the associated optimal stopping problem. In fact, we prove (see (v_{con})) that the value function of (OC) coincides with the difference between the optimal reward related to (OS_y) and the cost of doing nothing perpetually. As far as we know, this kind of characterisation is new in literature of works featuring state-dependence. Somehow similar results may be found in [DFM15; DFM18; De +17] for models not displaying state-dependence and in specific cases. In particular, the authors of [DFM18] consider a two-dimensional degenerate non-convex singular stochastic control problem for storage-consumption in markets where the price of energy is simply model by a Brownian motion which allows for tractable fundamental solutions ψ and φ of the characteristic equation related to such diffusion. They prove that the first derivative of the value function in the direction of the controlled process can be characterised as the optimal reward of an associated optimal stopping problem for initial inventory levels above a certain critical value. When the parameter is below that threshold, then the differential connection fails and, instead, the value function of the control problem is characterised by the optimal reward of the associated family of optimal stopping problems parametrised by the value of the initial inventory level.

We now describes the steps of our approach to the solution of (OC). For each value of the initial reserve amount y, we solve the optimal stopping problem (OS_y) by borrowing the geometric method of Dayanik and Karatzas [DK03]. Such approach allows us to *graphically* determine the optimal reward function $u(\cdot, y)$ through the construction of the smallest non-negative concave majorant of a suitably defined function. In particular, for each (OS_y), we are able to identify both the stopping $\mathcal{S}_y$ and continuation $\mathcal{C}_y$ regions, as well as the optimal stopping time. Then, under suitable assumptions, we establish the aforementioned connection (v_{con}).

We assume monotonicity properties, with respect to the initial amount of water y, for the ratio between the instantaneous holding cost and the amount of energy that may be produced by a sudden release of the available water, in both the system's reaction modalities (impulsive and chattering). In particular, we study two different cases, (R_b) and (R_∞), for the behaviour of such ratios when y approaches the lowest available level $\underline{y}$. Under assumption (R_b), we have that both ratios are decreasing and bounded above. In this case we completely solve the problem (OC). In fact, for any $y \in (\underline{y}, \overline{y}]$, we show that there exists a unique positive boundary point $\gamma^+(y)$ separating the continuation interval $\mathcal{C}_y$ and the stopping interval $\mathcal{S}_y$ of problem (OS_y) and we prove that it is obtained as the unique solution to a certain fixed point problem. We show that the function $y \mapsto \gamma^+(y)$ is positive, increasing and continuously-differentiable on $(\underline{y}, \overline{y})$. Afterwards, we use our Verification Theorem to prove that the function $u - \frac{1}{r}h$ actually identifies the value function of the optimal control problem (OC). We show that such a function is a solution to the HJB equation and also that the moving boundary separating the action and inaction regions coincides with γ^+. Hence, we find that, for any $(x, y) \in \mathbb{R} \times (\underline{y}, \overline{y}]$, the optimal control policy consists in discharging instantaneously all the available water reserve if the initial price x is above $\gamma^+(y)$. Otherwise, it is optimal doing nothing until the price process X_t^x reaches the optimal positive threshold $\gamma^+(y)$, then suddenly releasing all the available fuel.

On the contrary, when hypothesis (R_b) is replaced by (R_∞), we face the case in which the power produced by discharging instantaneously all the available water resource decreases to 0 faster then the holding cost, when y is sufficiently close to the lowest available level $\underline{y}$. This means that the ratio between holding cost and power produced becomes infinitely large as y approaches $\underline{y}$. In such case the scenario becomes particularly challenging. By exploiting the aforementioned geometric method, we study the family of optimal stopping problem (OS_y), considering three different ranges for the parameter y, showing when the solutions can be traced back to those found under (R_b) and when the solutions may exhibit multiple stopping intervals. In particular, we show that for certain values of the parameter y, the stopping region $\mathcal{S}_y$ is defined as the union of two disjoint intervals $[\gamma^-(y), \gamma_1(y)]$ and $[\gamma_2(y), +\infty)$, where $\gamma^-(y) < \gamma_1(y) < 0$ and $\gamma_2(y) > 0$. Although the function $y \mapsto \gamma^-(y)$ still represents a negative, increasing and continuously-differentiable function, in general an analogous result cannot be proved for $y \mapsto \gamma_1(y)$ and $y \mapsto \gamma_2(y)$. More specifically, the high generality of our model characteristics and the tricky structure of the fundamental solutions ψ and φ (associated to the infinitesimal operator $\mathcal{L}$ of the Vasicek's process X) make extremely difficult the identification of γ_1 and γ_2 as well-defined, continuously-differentiable functions with suitable monotonicity properties. Despite that, in order to depict the framework under (R_∞), we provide some intuitions on the geometry of the action and inaction regions. In particular, based on our Verification Theorem, we make *conjectures* on the structure of the optimal control policy. We show that a portion of the action region $\mathcal{D}$ has to be contained within the "negative half-strip" $\{(x, y) \in \mathbb{R}^2 : x < 0,\ y \in (\underline{y}, \overline{y}]\}$. This means that there is a (understandably small) value y_0 under which it is convenient to exercise control even if the market exhibits a negative price. Indeed, since near the minimum level $\underline{y}$ the capacity of the system to produce energy decreases faster than the cost of inaction, it may be preferable to discharge the volume of water available rather than waiting for successive periods of positive prices.

The rigorous identification of the optimal control policy and the characterisation of the value function in terms of the optimal reward u of (OS_y) remains an open problem. In particular, the state-dependence affecting both the performance functional J and the gradient constraint of the HJB equation makes difficult to prove that $u - \frac{1}{r}h$ still satisfies the gradient constraint inequality inside the inaction region $\mathcal{C}$. However, the relevant results obtained under (R_b) lay the foundations for a broader theory on the connection between state-dependent singular optimal control problems and optimal stopping problems. In particular, one of the main aims of our future research will be to determine conditions on the fundamental characteristics of the model so as to extend the application of the Verification Theorem, when the structure of both action and inaction regions exhibit additional complexities. Furthermore, the continuation of our research will also aim to generalize the approach developed in this thesis to the wider context of the bounded variation singular stochastic control problems, reintroducing in our model of hydroelectric production the possibility of restoring the water level within the reserve.

This thesis is organized as follows. In Chapter 1 we introduce the general formulation of the problem, describing the starting point of our research and presenting the market context where our hydroelectric production system is set. In Chapter 2 we obtain a priori properties of the value function v of the optimal stochastic control problem (OC) and, following heuristic arguments, we determine the associated HJB equation. Then we prove the Verification Theorem that characterise v among the solutions of the HJB equation. In Chapter 3 we define the family of optimal stopping problems (OS_y) associated with the control problem (OC) and, through a geometric approach we determine the solution of such stopping problems. Successively we introduce two different assumptions on the model that highlight two cases. In the first one the connection (OC)-(OS_y) may be rigorously proved, whereas in the second one the connection can only be conjectured. The thesis is completed by Appendices A, B, C and D containing some calculations and some useful results mentioned throughout this work.

Chapter 1

Model Formulation

A hydroelectric storage system usually is composed by a sufficiently large dam which is able to store water. Electricity is produced by releasing a specific amount of water from the reservoir and converting it into energy power by means of a generator, activated by the passage of the water flow through a turbine. The action of the turbine transforms the potential energy stored in the reservoir into electricity power when a certain quantity of water is released.

In this chapter we outline the main features of our model, specifying the initial objectives and subsequent adaptations to a wider general theoretical context.

From now on we will consider a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ where the filtration $(\mathcal{F}_t)_{t\geq 0}$ is assumed to satisfy the usual conditions, i.e. $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous and $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$.

1.1 Recalling Shardin & Wunderlich's model

Our research has been inspired by the increasingly fascinating challenge of building mathematical models to represent the peculiarities of hydroelectric storage systems and make them functional in the context of energy markets. Our starting point has been the paper by Anton Shardin and Ralf Wunderlich, *Partially observable stochastic optimal control problems for an energy storage* [SW17]. They consider an energy storage facility with limited capacity that faces the uncertainty of highly fluctuating energy prices. They focus on a pumping hydroelectric system which, besides producing energy by releasing water, has two reserves placed at different heights in such a way it is possible to pump water from the lower reservoir to fill the overlying reservoir by means of a pump powered by electricity. This energy is purchased in the market when prices are sufficiently low, with the prospect of using the stored resource to produce energy in subsequent periods of higher prices.

The authors of [SW17] consider the water head Q, i.e. the effective height $[m]$ of the water above the system turbine/pump, which is regulated by the action of a classical stochastic control process $(\xi_t)_{t\in[0,T]}$. Such a process describes the flow rate of the water and it measured in $[m^3/s]$. In particular, if $\xi_t > 0$, then at time t some quantity of water is discharged from the upper reservoir to the lower one, if $\xi_t < 0$ then water is pumped following the opposite direction, whereas if $\xi_t = 0$, then no operations occur. They assume the following ordinary differential equation

$$dQ_t = -\frac{c_T}{A(Q_t)}\xi_t dt, \tag{1.1}$$

for the dynamics for the water head, where $A(Q_t)$ describes the cross-sectional area of the upper reservoir at level Q_t and c_T is a time conversion factor. The water level is assumed to satisfy the state constraint $Q_t \in [\underline{q}, \overline{q}]$, where $\overline{q} > \underline{q} > 0$ represent the minimal and maximal water head, respectively. The rate flow process ξ_t, at any time t, has to lie in a set of achievable values depending on the current water head, i.e.

$$\xi_t \in [\underline{\xi}(Q_t), \overline{\xi}(Q_t)], \tag{1.2}$$

with $\underline{\xi}(Q_t) \leq 0$ and $\overline{\xi}(Q_t) \geq 0$ are the minimum and maximum pumping and releasing rate, respectively.

At any time t, the instantaneous power produced by discharging water at rate ξ_t from the height Q_t is given as in [TDR04; ZD09] by the *power function*

$$E_t = c_0\eta(Q_t, \xi_t)(Q_t - Q^+)\xi_t, \qquad \xi_t \in (0, \overline{\xi}(Q_t)], \tag{1.3}$$

where c_0 is a constant depending on the water density and gravitational acceleration g, Q^+ is a constant representing friction losses and η is the turbine efficiency function. If instead, $\xi_t \in [\underline{\xi}(Q_t), 0)$ then the instantaneous energy consumed by pumping water into the overlying reservoir is given by

$$c_1(Q_t + Q^-)\eta_p^{-1}\xi_t, \tag{1.4}$$

where c_1 is a constant involving c_0 and other conversion factors, Q^- is a constant representing friction losses and η_P denotes the efficiency of the pump that the authors suppose to be constant.

Shardin and Wunderlich assume that the spot price dynamics is described by a mean reverting process $(X_t)_{t\geq 0}$ and, at each time t, the instantaneous running reward describing the profits obtained by selling power or the cost faced purchasing energy is given by

$$\Pi(X_t, Q_t, \xi_t) := \begin{cases} c_0 X_t \eta(Q_t, \xi_t)(Q_t - Q^+)\xi_t, & \xi_t \in (0, \overline{\xi}(Q_t)] \\ 0, & \xi_t = 0 \\ c_1 X_t (Q_t + Q^-)\eta_p^{-1}\xi_t, & \xi_t \in [\underline{\xi}(Q_t), 0). \end{cases} \tag{1.5}$$

The class $\mathcal{M}$ of the admissible control processes is the set of all progressively measurable control policies of Markov type $\xi_t = m(t, X_t, Q_t)$ for any $t \in [0, T]$ with m being a measurable function, satisfying the constraint (1.2). The authors aim to maximise the expected total discounted cash flow over the finite[1] time interval $[t, T]$, defined as

$$L(t, x, q; \xi) := \mathbb{E}_{txq}\left[\int_t^T e^{-r(s-t)}\Pi(X_s, Q_s, \xi_s)ds + e^{-r(T-t)}\Phi(X_T, Q_T)\right], \tag{1.6}$$

where $r > 0$ denotes the discounted rate, $\Phi(X_T, Q_T)$ is a final reward function and $\mathbb{E}_{txq}[\cdot]$ is the conditional expectation given that at time t the spot price is $X_t = x$ and the water head is $Q_t = q$. The value function of their optimization problem is

$$V(t, x, q) := \sup_{\xi \in \mathcal{M}} L(t, x, q; \xi). \tag{1.7}$$

As is usual in classical stochastic control theory, the authors apply the dynamic programming principle and derive a suitable Hamilton-Jacobi-Bellman equation for the value function V. Such differential equation turns out to be degenerate, since the diffusion part of the state process (X_t, Q_t) involves only the price component X_t and hence it is not uniformly elliptic. They construct a candidate optimal Markov control policy ξ^* of threshold type, finding certain levels of the price at which the owner of the storage facility changes his/her operational strategy. They notice that the state process (X, Q^*) associated with the candidate optimal policy ξ^* has a *discontinuous* drift. Additionally, since their problem is degenerate the results of classical stochastic control theory cannot be applied and thorny problems concerning the admissibility of ξ^* arise. The authors employ some regularisation techniques, approximating the degenerate problem by a sequence of completely solvable problems. Despite this, the solutions associated to the regularised problems turn out to be only nearly optimal for their original problem.

Although the model proposed in [SW17] includes several technical aspects of hydroelectric production systems, their control problem is limited to the set of classical control processes, i.e. absolutely continuous processes with respect to time. We aim

[1]Here we consider the finite-horizon problem to reproduce as closely as possible the model presented in [SW17]. In the following, the infinite horizon will be considered.

to extend their model by considering a wider class of admissible controls, so as to overcome the admissibility issues arisen from their classical optimization model and to go beyond the absolutely continuous relation in (1.1), capturing further aspects that are not highlighted in [SW17]. Furthermore, we will introduce the maintenance costs that the owner of a hydroelectric power plant must face and which are not taken into consideration in the model proposed by Shardin and Wunderlich.

1.2 Proposal for a bounded variation approach

We intend to extend the model in [SW17], by enlarging the class of the feasible hydroelectric production strategies, allowing also that the operations of the storage system can occur with *instantaneous jumps* of the water amount in the reservoirs.

Let $\mathcal{A}$ denote the class of non-decreasing, left-continuous and adapted process $(\zeta_t)_{t\geq 0}$ with $\zeta_0 = 0$ almost surely and let us consider the class of *bounded variation* processes

$$\mathcal{B} := \{\zeta : \zeta = \zeta^+ - \zeta^- \text{ with } \zeta^\pm \in \mathcal{A}\} \tag{1.8}$$

and its subclass of *finite-fuel* bounded variation processes

$$\mathcal{B}(z) := \{\zeta \in \mathcal{B} : \check{\zeta}_\infty := \zeta^+_\infty + \zeta^-_\infty \leq z \text{ a.s.}\}, \tag{1.9}$$

with $0 \leq z < \infty$. We recall that $\zeta_t = \zeta^+_t - \zeta^-_t$ represents the minimal decomposition of a process in $\zeta \in \mathcal{B}$ and $\check{\zeta}_t := \zeta^+_t + \zeta^-_t$ is defined as its total variation. Such classes of stochastic processes are extensively treated in the literature of stochastic control. For more details and classical examples we refer the reader to [BSW80; KS86], among many others.

The first step is to describe the dynamics of the power generated by the turbine action related to the variation of water volume in the reserve. Denoting by $\underline{y}$ and $\overline{y}$ the minimal and maximal available capacity of the upper reservoir, respectively, and by $y \in [\underline{y}, \overline{y}]$ the initial volume, at each time t, the volume of water into the upper reservoir is

$$Y^{y,\nu}_t = y - \nu_t = y - (\nu^+_t - \nu^-_t), \qquad t \geq 0, \tag{1.10}$$

where $\nu \in \mathcal{B}(\overline{y} - \underline{y})$, and ν^+_t and ν^-_t represent the cumulative amount of water released and the cumulative amount of water pumped up to time t, respectively. We define the instantaneous variation of power exchanged in the system as

$$d\tilde{E}_t = f(Y^{y,\nu}_t)d\nu^+_t - p(Y^{y,\nu}_t)d\nu^-_t, \tag{1.11}$$

where f and p are defined as the *instantaneous marginal productivity functions*. In particular, at each time t $f(Y_t^{y,\nu})$ represents the instantaneous intensity at which energy power is generated by depleting a certain amount of water from the current volume level $Y_t^{y,\nu}$. Conversely, $p(Y_t^{y,\nu})$ is the instantaneous intensity at which power must be produced in order to increase the present volume $Y_t^{y,\nu}$, by pumping water from an underlying reservoir.

We consider the spot price dynamics described by a mean-reverting diffusion process $(X_t^x)_{t\geq 0}$, whose characteristics will be specified in the sequel. The expected discounted net present value associated with the charging/discharging storage operations is given by

$$\tilde{J}(x,y;\nu^+,\nu^-) = \mathbb{E}\bigg[\int_0^{+\infty} e^{-rt}X_t^x f(Y_t^{y,\nu})d\nu_t^+ - \int_0^{+\infty} e^{-rt}X_t^x p(Y_t^{y,\nu})d\nu_t^- \\ - \int_0^{+\infty} e^{-rt}h(Y_t^{y,\nu})dt\bigg], \tag{1.12}$$

where $r > 0$ is the discount rate and h is the cost function. At each time t the quantity $h(Y_t^{y,\nu})$ represents the instantaneous running cost that the owner of a hydroelectric facility incur to maintain the store at the level $Y_t^{y,\nu}$.

Since the beginning of our research, the maximisation of functional $\tilde{J}$ has shown many technical difficulties. In fact, we deal with a problem in which both the marginal productivity functions f and p depend on the state of the system and they affect the performance functional $\tilde{J}$ in a non-standard way. As far as know, such kind of models is an absolute novelty in the field of stochastic control theory, also in the wider class of problems in which the control policies are allowed to be not absolutely continuous with respect to time.

In this thesis we want to lay the foundations for the development of a theory that can determine the suitable tools to solve this type of unconventional problems. For this reason, we consider the case of a *pure generating* hydroelectric facility, allowing the owner to only produce energy power by discharging water from the upper reservoir, without the possibility to refill it by pumping water from an underlying resource. We focus on the stochastic control problem for hydroelectric power production in which the control strategies belong to a suitable subclass of finite-fuel monotone stochastic processes. As will be clear in the following, although this problem is more tractable than the bounded variation problem above, it exhibits several complexities and its solution turns out to be particularly challenging.

1.3 The model for a pure generating hydroelectric facility

From now on, to avoid confusion with previous notations, we will always denote by $(\nu_t)_{t\geq 0}$ a control policy in $\mathcal{A}$, i.e. a $\mathcal{F}_t$-adapted stochastic process ν whose paths $t \mapsto \nu_t(\omega)$ are left-continuous with finite right limits[2] and non-decreasing with $\nu_0 = 0$, for almost every $\omega \in \Omega$. In particular, we stress that now the process

$$Y_t^{y,\nu} = y - \nu_t, \quad t \geq 0, \tag{1.13}$$

denotes the current amount of water in the reservoir at time t, ν_t the cumulative volume of water released up to time t and y the initial amount of water into the reservoir. We have that y is between $\underline{y}$ and $\overline{y}$ which are the minimal and maximal available reservoir capacity, respectively.

Remark 1.1. In order to recover the total activity of the pumping-storage system, one may consider the possibility of recharging water after that the reserve has gone down below a certain threshold R and it might be assumed that this process takes place at a deterministic (potentially constant) rate, properly defined according to the physical-technical constraints. Therefore, the dynamics in (1.13) that describes the current volume of water inside the reservoir may be suitably re-defined taking into consideration such feature. Obviously, such dynamics would no longer be non-increasing and these kind of problem cannot easily be adapted to the optimal control context we are going to develop in this thesis. In particular, the monotonicity property of the process $Y_t^{y,\nu}$ will have a crucial role in the results we will obtain. In the perspective of future research, these kind of problems could be consider an interesting intermediate step between our pure generating model and the more general bounded variation context, including also recharging activity.

1.3.1 The power dynamics

At each time t, the instantaneous variation of power produced by suddenly depleting an amount of water $d\nu_t$ from the current volume $Y_t^{y,\nu}$ is given by

$$dE_t = f(Y_t^{y,\nu})d\nu_t. \tag{1.14}$$

We recall that f is defined as the instantaneous marginal productivity function or -more briefly- productivity function and that $f(Y_t^{y,\nu})$ represents the instantaneous intensity of power generated with respect to the change of water volume.

[2] Often in the literature of stochastic analysis such processes are called by the French acronym *càglàd*, i.e. "continue à gauche, limite à droite".

The form of the productivity function varies according to the reservoirs shape and especially with respect to the efficiency of the turbine that is employed in the power production system. In general we may assume that the level of productivity is higher for higher levels of stored water and that energy production is not allowed when the reserve is at its minimum capacity level. We consider the following basic hypotheses on the productivity function.

Assumption 1.2. The productivity function $f \in C^1((\underline{y}, \overline{y}))$ is non-negative, strictly increasing on $[\underline{y}, \overline{y}]$ and it is such that $f(\underline{y}) = 0$. Moreover, there exists $M > 0$ such that

$$f'(y) \leq M, \qquad \forall\, y \in [\underline{y}, \overline{y}].$$

In the literature of hydropower modelling there exist models that consider concavity assumption for the productivity function, often justified by the analysis of empirical data (see e.g [Vie+15]). As noticed above, the analytical structure of productivity function strictly depends on physical and technical features of the hydroelectric facility. Hence, for the sake of generality, we abstain from assumptions about concavity/convexity property for f. Further hypotheses will be specified in the sequel of the thesis and these will be taken in relation to the other characteristics composing our model.

1.3.2 The prices dynamics

Let us assume now that the owner of a hydroelectric pure-generating storage facility faces the energy markets. As properly highlighted in [Aïd15], although the peculiarity of each electricity market among different countries, some common factors may be identified and a unique substructure can be defined. In particular three different sub-markets can be recognised. In the *day-ahead market* the exchanges take place one day before the delivery, and prices and quantities are settled through public auctions. After the spot price has been established, the final schedule for each generator is fixed. Whenever a participant in the market is unable to deliver the specified amount in the spot market, then his/her surplus/deficit must be adjusted by balancing mechanisms. These operations take place in the imbalance market where exchanges occur between transport system operators (TSO) and market players in order to ensure suitable power level in real-time. Even though the main role of buyer/seller is played by TSOs, exchanges among market agents themselves are also allowed, they occur in the *intraday market* where, according to any new update of the generation prediction, the participants take care of their level of production and implement their strategies in order to avoid penalties at the delivery time. The third sub-sector of energy markets is represented by the *forward markets* within the markets participants can trade electricity for future periods. These markets share many features with those related to other storable commodities, however presenting significant differences in the term structure. Because of the hour-to-hour activity in the day-ahead market, agents need financial contracts for every hour of the year so as to guarantee the hedging of their generation from the risks of electricity production.

Since the financial assets often have very short maturities in the intra-day market, this can be thought of as the actual spot price market. However, the reference price at which the electricity will be sold/bought is that established on the day-ahead market. Therefore, in our model we consider the day-ahead prices as the spot structure and, according to the existing literature (see e.g [SW17; DFM15]), we consider a Vasicek's process $(X_t^x)_{t\geq 0}$ for electricity spot prices that is the unique solution of the following stochastic differential equation

$$\begin{cases} dX_t^x = a(b - X_t^x)dt + \sigma dW_t, \quad t \geq 0 \\ X_0^x = x, \end{cases} \tag{1.15}$$

where $x \in \mathbb{R}$, $(W_t)_{t\geq 0}$ is a standard Brownian motion, σ is the volatility of the spot prices, b and a are positive constants representing the long-term average price and the convergence rate at which the price process reverts to its average b, respectively.

The process $(X_t^x)_{t\geq 0}$ captures some of the principal aspects of energy prices, for instance the mean-reverting behaviour and first of all the tendency of energy markets to allow for negative values of spot prices. The reasons that lead to the presence of negative prices in the electricity market are many and vary from area to area. In [Bar+14] are collected two different aspects that may lead electricity markets to allow for negative prices. The first aspect can be represented by the generator decision of producing power even if there is a current low demand, in view of a later period of higher demand. Such a strategy may be considered profitable when the round-trip of shutting down and restarting a power plant involves costs greater than the price paid in order to induce another participant to get the oversupply generated power. The second aspect concerns the possibility for the power generator to obtain consistent subsides and tax breaks that a renewable-oriented generator may collect by producing green energy. The producer can be prepared to generate and sell power at negative prices however receiving subsidies and covering the losses due to negative revenue.

For the sake of completeness, we recall that in the literature of energy markets modelling there exist very general models taking into account other peculiarities of electricity markets, for instance the presence of *spikes* for energy prices. Such a feature is due to the occurrence of one (or more) upward jump quickly followed by a downward jump in prices dynamics, mainly caused by interruptions of transmission, lack in power generation and/or extreme meteorological phenomena. Barlow [Bar02] considers a non linear transformation for a suitable Vasicek process which is able to reproduce spikes for energy price dynamics, even if the model remains based on a pure diffusion process. In several other studies we can find the presence of jump component in electricity price dynamics in order to describe the "spiky nature" of the price. Among these, Geman and Roncoroni [GR06] consider a positive process for prices considered in natural logarithmic scale and they define it as the unique solution of a stochastic differential equation of the form

$$dP_t = \eta'(t)dt + \theta[\eta(t) - P_{t-}]dt + \tilde{\sigma} dW_t + s(P_{t-})d\tilde{J}_t, \tag{1.16}$$

where are considered the deterministic seasonal trend $\eta(t)$ of the price dynamics around which spot prices fluctuate, the mean reversion behaviour to this trend and a discontinuous part reproducing the effect of spikes. The jump process is defined as

$$\tilde{J}_t = \sum_{i=1}^{N_t} \tilde{J}_i, \tag{1.17}$$

where N_t is a counting process that specifies the number of jumps that occur up to time t and J_i are independent and identically distributed random variables with a probability density chosen within the exponential family.

Despite the presence of spikes for energy prices is a particularly interesting feature, the intent to incorporate it into a model of stochastic optimal control goes beyond the intentions of this thesis. Our model generalise the optimal energy production problem when it takes place in a context where the price is described by a purely diffusive process that capture some of the main characteristics of energy prices, first among these the positive probability that negative prices will occur.

1.3.3 The admissible strategies

Let us now introduce one of the main features of our power production system that, as far as we know, characterises our model as a novelty among the literature of continuous-time stochastic optimisation of hydropower production. We endow our power generation system with the possibility to respond to the action of a production strategy through two different ways, depending on whether negative rather than positive prices occur.

Among the left-continuous, non-decreasing, adapted stochastic processes in $\mathcal{A}$, we are allowed to consider as feasible control policies those processes satisfying the physical constraints of the reservoir.

Definition 1.3. A process $\nu \in \mathcal{A}$ is an *admissible control process* if

$$0 \leq \nu_t \leq y - \underline{y}, \quad \forall t \geq 0, \quad \text{a.s.} \tag{1.18}$$

and the set $\mathcal{A}(y) \subset \mathcal{A}$ denotes the class of all admissible control processes.

We observe that each control in $\mathcal{A}(y)$ can be decomposed into its continuous part and purely discontinuous part, i.e.

$$\nu_t = \nu_t^c + \sum_{0 \leq s < t} \Delta\nu_t, \qquad \Delta\nu_t := \nu_{t+} - \nu_t, \qquad t \geq 0. \tag{1.19}$$

At each time t, when it is convenient to suddenly discharge a certain volume of water $\Delta\nu_t$, we assume that the power generation system can react in two different ways. The first modality consists in release all the water mass $\Delta\nu_t$ by exploiting the maximum productivity available at the current water head and the resultant energy output is given by

$$\Delta E_t^i = f(Y_t^{y,\nu})\Delta\nu_t. \tag{1.20}$$

This strategy leads to a *impulsive* control of the dam. Alternatively, the same amount of fuel $\Delta\nu_t$ can be discharged infinitely fast but only releasing *a small amount many times in a sufficiently small interval of time.* In order to understand such a unusual control strategy, let us consider at time t a jump $\Delta\nu_t$ and an equally spaced partition $Y_t^{y,\nu} = y_0 \leq \cdots \leq y_n = Y_t^{y,\nu} - \Delta\nu_t$, whose any subinterval has length $\delta = \Delta\nu_t/n$. The amount of energy produced by implementing the above alternative strategy is given by sum of $n+1$ contributions obtained releasing instantaneously a portion δ starting from each partition's node. That is,

$$\Delta E_t^n = \sum_{i=0}^{n} f(y_i)\delta = \frac{\Delta\nu_t}{n}\sum_{i=0}^{n} f(Y_t^{y,\nu} - i\delta). \tag{1.21}$$

Since the function f is regular enough, letting the partition get finer we obtain that above sum converges to the Riemann integral

$$\Delta E_t^c = \int_0^{\Delta\nu_t} f(Y_t^{y,\nu} - u)du. \tag{1.22}$$

According to the terminology of the existing literature (see e.g. [Alv00; SSZ11]) we call *chattering policy* the above strategy. Since f is a strictly increasing function, it straightforward notice that, for fixed $t \geq 0$

$$\int_0^{\Delta\nu_t} f(Y_t^{y,\nu} - u)du \leq f(Y_t^{y,\nu})\Delta\nu_t, \tag{1.23}$$

that is, the amount of energy produced by means of a chattering policy is lower than that produced by an impulsive release of water. This fact may be interpretable in terms of efficiency. When a quantity of water is instantaneously and impulsively released from the highest available level of the reserve, the efficiency of the turbine at that level might be considered maximal. On the other hand, when a chattering policy is implemented, turbine efficiency may decrease from one level to the one immediately below. The total contribution is lower than that obtained with a single impulse and the system is affected in terms of productivity. Taking into consideration such double effect of an instantaneous leap in hydroelectric generation, it seems convenient to apply a chattering policy when a negative price occurs but it is still preferable to act.

1.3.4 The performance functional

The owner of a hydroelectric storage system is subject to running holding costs concerning the maintenance, at each time t, of the unused water resource. In our model we consider such costs depending on the water volume into the reservoir and, in particular we suppose that the running holding costs increase with respect to the current available amount of water. Such costs are measured by the function h, satisfying the following hypotheses.

Assumption 1.4. The holding cost function $h \in C^1((\underline{y}, \overline{y}))$ is non-negative, strictly-increasing on $[\underline{y}, \overline{y}]$ and such that $h(\underline{y}) = 0$. Moreover, there exists a constant $\tilde{M} > 0$ such that

$$h'(y) \leq \tilde{M}, \qquad \forall\, y \in [\underline{y}, \overline{y}].$$

An example of holding cost function can be found in [CFS15]. The authors consider a strictly increasing, continuously differentiable, convex holding cost function. As they notice, it is reasonable to assume that there are no holding costs when the reserve reaches its lowest level. Even if a cost occurs when the available reserve is finished, the same hypothesis can be maintained without loss of generality. In fact, it would be enough to consider the function $\tilde{h}(y) := h(y) - h(\underline{y})$ rather than h and the the optimization problem that we are going to introduce would remain unchanged, up to an additive constant. We highlight that contrarily to [CFS15], we do not assume any convexity property for our cost function h.

The presence of holding costs has a significant effect on the choice of the production strategy. Indeed, if there are no holding costs then it would be totally inconvenient to produce energy in periods of negative prices. Whereas in presence of a holding cost, it may be convenient to discharge water to produce energy even if there are negative prices, especially if the cost of maintaining the reserve at a certain current level is greater than the expected revenue in successive periods of positive prices. For this reason it may be convenient to structure the generation system such that it can respond "less efficiently" to an instantaneous control action in case of negative prices.

To any control strategy ν in $\mathcal{A}(y)$ is associated a *performance* described by the functional

$$J(x, y; \nu) = \mathbb{E}\bigg[\int_0^{+\infty} e^{-rt} X_t^x f(Y_t^{y,\nu}) d\nu_t^c + \sum_{t \geq 0} e^{-rt} [X_t^x]^+ f(Y_t^{y,\nu}) \Delta\nu_t + \\ - \sum_{t \geq 0} e^{-rt} [X_t^x]^- \int_0^{\Delta\nu_t} f(Y_t^{y,\nu} - z) dz - \int_0^{+\infty} e^{-rt} h(Y_t^{y,\nu}) dt\bigg], \quad (1.24)$$

representing the expected discounted cash flow over the infinite horizon, obtained by adopting the control policy ν, given the initial price $x \in \mathbb{R}$ and the starting water amount $y \in [\underline{y}, \overline{y}]$. We notice that the performance functional J is the pure generating "version" of the functional $\tilde{J}$ in (1.12). Further, the definition of J takes into account the peculiarities of the production system when jumps in releasing water occur. The Stieltjes integral

$$\int_0^{+\infty} e^{-rt} X_t^x f(Y_t^{y,\nu}) d\nu_t^c \quad (1.25)$$

represents the discounted revenue from the sale of energy produced by continuously discharging water from the reserve. Whereas, the series

$$\sum_{t \geq 0} e^{-rt} [X_t^x]^+ f(Y_t^{y,\nu}) \Delta\nu_t, \qquad \text{and} \qquad \sum_{t \geq 0} e^{-rt} [X_t^x]^- \int_0^{\Delta\nu_t} f(Y_t^{y,\nu} - z) dz, \quad (1.26)$$

are the discounted revenues obtained by a sudden releasing of water when the hydroelectric production system reacts in impulsive and chattering way, respectively.

We aim to maximise the performance functional J over all the admissible control policies in $\mathcal{A}(y)$ and to find the *value function*

$$v(x, y) := \sup_{\nu \in \mathcal{A}(y)} J(x, y; \nu), \quad \text{(OC)}$$

which represents the best energy production performance that can be obtain, for given initial price $x \in \mathbb{R}$ and starting water amount y. A control $\bar{\nu} \in \mathcal{A}(y)$ that maximises the functional J is called *optimal* for the stochastic control problem (OC).

Since among the class of admissible control policies we allows also for strategies whose paths $t \mapsto \nu_t(\omega)$ are not necessarily absolutely continuous with respect to the Lebesgue measure, the continuous-time stochastic optimisation problem (OC) belongs to the wide class of *singular stochastic control problems*. Moreover, due to the dependence of marginal revenues on the current state of the system, we deal with a *state-dependent* singular stochastic control problem.

Contrary to some other state-dependent problems (see e.g. [Alv00; SSZ11; ALØ16]), the instantaneous marginal yield $X^x f(Y^{y,\nu})$ in our problem does not satisfy any monotonicity property with respect to the controlled process $Y^{y,\nu}$, since the sign of the price dynamics X^x may change. For the same reason the performance criterion J does not display any monotonicity and concavity property in the controlled variable. We share this feature with other singular stochastic control problems (see e.g. [DFM15; DFM18; De +17]), although these models do not exhibit any state-dependence.

In conclusion, we highlight that our model falls within the class of *finite-fuel two-dimensional degenerate* singular stochastic control problem, since the diffusive part of the pair $(X^x_t, Y^{y,\nu}_t)$ is not uniformly elliptic. Indeed the diffusion acts only in the direction of the first component and the second one remains always bounded.

Chapter 2

The Value Function of the Stochastic Control Problem

In this chapter we first establish some *a priori* results about the analytical features of the value function v, in particular we obtain preliminary information about the regularity properties of v. Afterwards, by means of heuristic arguments we find out a Hamilton-Jacobi-Bellman (HJB) equation associated with our stochastic control problem (OC). Such partial differential equation has a variational inequality formulation and lead us to a partition of the strip $\mathbb{R} \times [\underline{y}, \overline{y}]$ in two subsets: the inaction and action region, respectively. The main result of the chapter is the Verification Theorem that allows us to characterise v among the solutions of the HJB equation as well as to clearly identify the structure of the optimal control policy.

2.1 Preliminary results

Let us recall some properties of Vasicek's diffusion that we will exploit continuously in the sequel. The solution of the stochastic differential equation (1.15) is represented by the diffusion X whose explicit expression is given by

$$X_t^x = e^{-at}(x-b) + b + \sigma \int_0^t e^{-a(t-s)} dW_s. \tag{2.1}$$

For any fixed $t \geq 0$, X_t^x is a Gaussian random variable whose expectation and variance are given by

$$\mathbb{E}[X_t^x] = (x-b)e^{-at} + b, \quad \mathbb{V}[X_t^x] = \frac{\sigma^2}{2a}(1 - e^{-2at}), \tag{2.2}$$

respectively. Moreover, if we consider its infinitesimal generator

$$\mathcal{L} := \frac{1}{2}\sigma^2 \frac{d^2}{dx^2} + a(b-x)\frac{d}{dx}. \tag{2.3}$$

we have that, for any $r > 0$, the characteristic equation

$$[\mathcal{L} - r]\theta(x) = 0, \qquad x \in \mathbb{R}, \tag{2.4}$$

admits two linearly independent solutions φ and ψ, defined as

$$\varphi(x) = e^{\frac{a(x-b)^2}{2\sigma^2}} D_{-\frac{r}{a}}\left(\frac{(x-b)}{\sigma}\sqrt{2a}\right), \qquad \psi(x) = e^{\frac{a(x-b)^2}{2\sigma^2}} D_{-\frac{r}{a}}\left(-\frac{(x-b)}{\sigma}\sqrt{2a}\right). \tag{2.5}$$

These functions are positive, continuously differentiable, decreasing and increasing, respectively. Moreover, their analytic expressions can be represented in terms of the parabolic cylinder function D_α of index α, whose integral representation (see e.g. [GR14]) is given by

$$D_\alpha(z) = \frac{e^{-\frac{z^2}{2}}}{\Gamma(-\alpha)} \int_0^{+\infty} e^{-xz-\frac{x^2}{2}} x^{-\alpha-1} dx, \quad \text{when} \quad Re(\alpha) < 0,$$

where $\Gamma(\cdot)$ is the Euler's Gamma function.

Moreover, if we consider the diffusion X^x starting from x at time 0 and its *first hitting time*

$$\Theta_z^x := \inf\{t > 0 : X_t^x = z\} \tag{2.6}$$

of $z \in \mathbb{R}$, we have that

$$\mathbb{P}(\Theta_z^x < \infty) = 1, \quad \text{and} \quad \mathbb{E}[\Theta_z^x] < \infty, \quad \forall\, x, z \in \mathbb{R}. \tag{2.7}$$

We recall (see e.g. [BS02]) that each diffusion satisfying $(2.7)_1$ is called *recurrent.* If in addition also $(2.7)_2$ is satisfied, then such diffusion is said *positively recurrent.*

Now we consider some straightforward results that will be useful in the sequel. In the proofs of some of these results we will use the following inequality: given $m \in \mathbb{N}$ and $l \in \mathbb{N}$,

$$(|z_1 + ... + z_l|)^m \leq l^{m-1}(|z_1|^m + ... + |z_l|^m), \quad z_i \in \mathbb{R},\ i = 1, ..., l. \tag{2.8}$$

Lemma 2.1. *The family of r.v.'s $(X_t^x)_{t\geq 0}$ is uniformly integrable.*

Proof. Thanks to Proposition A.3 and Remark A.4 in Appendix A, in order to prove that the process $(X_t^x)_{t\geq 0}$ is uniformly integrable, it suffices to show that there exists $p > 1$ such that $(X_t^x)_{t\geq 0}$ is bounded on $L^p(\Omega, \mathbb{P})$. Recalling the expression for X_t^x in (2.1) and inequality (2.8), we obtain

$$\begin{aligned}\mathbb{E}[|X_t^x|^2] &\leq 2\mathbb{E}\left[|e^{-at}(x-b)+b|^2 + \sigma^2\left(\int_0^t e^{-a(t-s)}dW_s\right)^2\right] = \\ &= 2\mathbb{E}\left[|e^{-at}(x-b)+b|^2 + \sigma^2\int_0^t e^{-2a(t-s)}ds\right] = \\ &= 2|e^{-at}(x-b)+b|^2 + \frac{\sigma^2}{a}(1-e^{-2at}),\end{aligned}$$

where the first equality follows from the well-known isometry property of the Brownian stochastic integral (cf. e.g. pag. 188 of [Bal17]). It follows easily that we can find a positive constant $C = C(a, b, \sigma, x)$ such that

$$\sup_{t\geq 0} \mathbb{E}[|X_t^x|^2] < C.$$

Hence X_t^x turns out to be bounded on $L^2(\Omega, \mathbb{P})$ and the uniform integrability property follows. □

Lemma 2.2. *Given a real number $R > 0$, define*

$$\tau_R := \inf\{t \geq 0 : |X_t^x| \geq R\}, \tag{2.9}$$

i.e. the exit time of the X^x from the interval $(-R, R)$. Then, for any $T > 0$, there exists a positive constant $K = K(T, a, b, \sigma)$ such that, for every $p \geq 2$

$$\mathbb{P}(\tau_R < T) \leq \frac{K(1+|x|^p)}{R^p}, \tag{2.10}$$

that is, $\mathbb{P}(\tau_R < T) \to 0$ when R converges to $+\infty$.

Proof. The process X_t^x is the unique solution to the stochastic differential equation (1.15) with linear drift and constant diffusion coefficient. The thesis easily follows as an immediate application of Theorem 9.1 of [Bal17]. In particular, as stated in Remark 9.3 of [Bal17], the result is a by-product of the proof of Theorem 9.1. □

Lemma 2.3. *Given $(x, y) \in \mathbb{R} \times [\underline{y}, \overline{y}]$ and $\nu \in \mathcal{A}(y)$, the stochastic integral*

$$I_t^{x,y,\nu} := \int_0^t e^{-rs} X_s^x f(Y_s^{y,\nu}) d\nu_s^c, \tag{2.11}$$

is uniformly bounded in $L^2(\Omega, \mathbb{P})$*, for every* $t \geq 0$*. In particular, the family of r.v.'s* $(I_t^{x,y,\nu})_{t\geq 0}$ *is uniformly integrable and*

$$\lim_{t\to+\infty} \mathbb{E}[I_t^{x,y,\nu}] = \mathbb{E}\left[\int_0^{+\infty} e^{-rs} X_s^x f(Y_s^{y,\nu}) d\nu_s^c\right] < \infty. \tag{2.12}$$

Proof. Fixed $y \in [\underline{y}, \overline{y}]$ and $\nu \in \mathcal{A}(y)$, consider the process $G_t^{y,\nu} := \int_0^t f(Y_s^{y,\nu}) d\nu_s^c$. We observe that $G_t^{y,\nu} \geq 0$ and there exists a constant $L > 0$ such that $G_t^{y,\nu} \leq L$, for any $t \geq 0$. Integrating by parts in (2.11), for any $t \geq 0$, we get

$$I_t^{x,y,\nu} = \bar{I}_t + \tilde{I}_t + \hat{I}_t, \qquad \text{a.s.,} \tag{2.13}$$

with

$$\bar{I}_t = \bar{I}_t(x, y, \nu) := e^{-rt} X_t^x G_t^{y,\nu}, \tag{2.14}$$

$$\tilde{I}_t = \tilde{I}_t(x, y, \nu) := \int_0^t e^{-rs} G_s^{y,\nu}[(a + r)X_s^x - ab]ds, \tag{2.15}$$

$$\hat{I}_t = \hat{I}_t(x, y, \nu) := -\int_0^t e^{-rs} \sigma G_s^{y,\nu} dW_s. \tag{2.16}$$

Each of the above processes are uniformly integrable, since $G_t^{y,\nu}$ is bounded for any $t \geq 0$ and $(X_t^x)_{t\geq 0}$ is uniformly integrable. In particular, there exist a constant $C = C(a, b, r, \sigma, L, x)$ such that

$$\sup_{t\geq 0} \mathbb{E}[|\bar{I}_t|^2] \leq L^2 \sup_{t\geq 0} \mathbb{E}[|X_t^x|^2] < C. \tag{2.17}$$

Moreover, applying twice the Jensen's inequality and considering again the inequality (2.8), we get

$$\begin{aligned} \mathbb{E}[|\tilde{I}_t|^2] &\leq \mathbb{E}\left[\int_0^t e^{-2rs} (G_s^{y,\nu})^2 |(a + r)X_s^x - ab|^2 ds\right] \leq \\ &\leq \mathbb{E}\left[2L^2 \int_0^t e^{-2rs}[(a + r)^2 |X_s^x|^2 + (ab)^2] ds\right] \leq \\ &\leq C_1, \end{aligned} \tag{2.18}$$

for some $C_1 = C_1(a, b, r, \sigma, L.x)$. In conclusion, observing that $\hat{I}_t$ is a square integrable martingale with zero mean and variance

$$\mathbb{E}[\hat{I}_t^2] \leq \frac{\sigma^2}{2r} L^2 = C_2, \tag{2.19}$$

and exploiting again the inequality (2.8), we obtain that $(I_t^{x,y,\nu})_{t\geq 0}$ is bounded in $L^2(\Omega,\mathbb{P})$ and

$$\mathbb{E}[|I_t^{x,y,\nu}|^2] \leq 3(\mathbb{E}[|\bar{I}_t|^2 + |\tilde{I}_t|^2 + |\hat{I}_t|^2]) \leq C_3, \tag{2.20}$$

for $C_3 = \max\{C, C_1, C_2\}/3$. Therefore, thanks to Proposition A.3 and Remark A.4 in Appendix, it follows that $(I_t^{x,y,\nu})_{t\geq 0}$ is uniforly integrable. Moreover, by means of the convergence result stated in Theorem A.5 we obtain also (2.12). □

Lemma 2.4. *Given $(x,y) \in \mathbb{R} \times [\underline{y}, \overline{y}]$ and $\nu \in \mathcal{A}(y)$, the sums*

$$\overline{S}_t^{x,y,\nu} := \sum_{0\leq s\leq t} e^{-rs}[X_s^x]^+ f(Y_s^{y,\nu})\Delta\nu_s, \tag{2.21}$$

$$\underline{S}_t^{x,y,\nu} := \sum_{0\leq s\leq t} e^{-rs}[X_s^x]^- \int_0^s f(Y_s^{y,\nu} - u)du, \tag{2.22}$$

$$S_t^{x,y,\nu} := \sum_{0\leq s\leq t} e^{-rs} X_s^x f(Y_s^{y,\nu}) d\nu_s^c, \tag{2.23}$$

are finite in $L^1(\Omega,\mathbb{P})$. In particular, it follows that

$$\lim_{t\to+\infty} \mathbb{E}[\overline{S}_t^{x,y,\nu}] = \mathbb{E}\Bigg[\sum_{t\geq 0} e^{-rt}[X_t^x]^+ f(Y_t^{y,\nu})\Delta\nu_s\Bigg] < \infty, \tag{2.24}$$

$$\lim_{t\to+\infty} \mathbb{E}[\underline{S}_t^{x,y,\nu}] = \mathbb{E}\Bigg[\sum_{t\geq 0} e^{-rt}[X_t^x]^- \int_0^t f(Y_t^{y,\nu} - u)du\Bigg] < \infty, \tag{2.25}$$

$$\lim_{t\to+\infty} \mathbb{E}[S_t^{x,y,\nu}] := \mathbb{E}\Bigg[\sum_{t\geq 0} e^{-rt} X_t^x f(Y_t^{y,\nu}) d\nu_t^c\Bigg] < \infty. \tag{2.26}$$

Proof. The result can be proved by exploiting similar arguments to those used in the Lemma 2.3. □

Let us now consider an equivalent formulation for the performance functional J, as stated in the following result.

Proposition 2.5. *For any $(x,y) \in \mathbb{R} \times [\underline{y}, \overline{y}]$ and $\nu \in \mathcal{A}(y)$, we have*

$$\begin{aligned} J(x,y;\nu) = xF(y) + \mathbb{E}\Bigg[\int_0^{+\infty} & e^{-rt}\{[\mathcal{L} - r](X_t^x F(Y_t^{y,\nu})) - h(Y_t^{y,\nu})\}dt + \\ & + \sum_{t\geq 0} e^{-rt}[X_t^x]^+ \Bigg\{ f(Y_t^{y,\nu})\Delta\nu_t - \int_0^{\Delta\nu_t} f(y-u)du \Bigg\}\Bigg], \end{aligned} \tag{2.27}$$

where $F(y) := \int_{\underline{y}}^{y} f(u)du$.

Proof. Fixed $(x, y) \in \mathbb{R} \times [\underline{y}, \overline{y}]$ and $\nu \in \mathcal{A}(y)$, we take $R > 0$ such that $|x| < R$ and we consider τ_R as in Lemma 2.2. For some $T > 0$, we apply the Itô's bidimensional formula for semimartingale (see e.g. [Pro05]) to $e^{-rt} X_t^x F(Y_t^{y,\nu})$ on the time interval $[0, T_R]$, with $T_R = T \wedge \tau_R$. We obtain,

$$\begin{aligned} e^{-rT_R} X_{T_R}^x F(Y_{T_R}^{y,\nu}) = xF(y) \quad &+ \int_0^{T_R} e^{-rt} [\mathcal{L} - r](X_t^x F(Y_t^{y,\nu}))dt + \\ &+ \int_0^{T_R} e^{-rt} X_t^x F(Y_t^{y,\nu}) dY_t^{y,\nu} \\ &+ \int_0^{T_R} e^{-rt} \sigma F(Y_t^{y,\nu}) dW_t + \\ &+ \sum_{0 \le t \le T_R} e^{-rt} X_t^x \{F(Y_{t+}^{y,\nu}) - F(Y_t^{y,\nu})\} \\ &- \sum_{0 \le t \le T_R} e^{-rt} X_t^x f(Y_t^{y,\nu}) \Delta Y_t^{y,\nu}. \end{aligned} \tag{2.28}$$

We now recall that

$$dY_t^{y,\nu} = -d\nu_t, \quad a.s., \quad \forall t \ge 0, \tag{2.29}$$

and notice also that

$$F(Y_{t+}^{y,\nu}) - F(Y_t^{y,\nu}) = -\int_0^{\Delta\nu_t} f(Y_t^{y,\nu} - u)du, \quad a.s., \quad \forall t \ge 0. \tag{2.30}$$

Hence, using the decomposition (1.19), we write

$$\begin{aligned} e^{-rT_R} X_{T_R}^x F(Y_{T_R}^{y,\nu}) = xF(y) \quad &+ \int_0^{T_R} e^{-rt} [\mathcal{L} - r](X_t^x F(Y_t^{y,\nu}))dt + \\ &- \int_0^{T_R} e^{-rt} X_t^x F(Y_t^{y,\nu}) d\nu_t^c \\ &+ \int_0^{T_R} e^{-rt} \sigma F(Y_t^{y,\nu}) dW_t + \\ &- \sum_{0 \le t \le T_R} e^{-rt} X_t^x \int_0^{\Delta\nu_t} f(Y_t^{y,\nu} - u)du. \end{aligned} \tag{2.31}$$

Since f is bounded on $[\underline{y}, \overline{y}]$, also F remains bounded on the same set and the process

$$M_t := \int_0^t e^{-rs} \sigma F(Y_s^{y,\nu}) dW_s, \qquad t \in [0, T_R], \tag{2.32}$$

is a zero mean square integrable martingale. Hence, taking the expectation in (2.31), we obtain

$$\begin{aligned}\mathbb{E}[e^{-rT_R}X^x_{T_R}F(Y^{y,\nu}_{T_R})] = xF(y) \quad &+ \quad \mathbb{E}\Big[\int_0^{T_R} e^{-rt}[\mathcal{L}-r](X^x_tF(Y^{y,\nu}_t))dt + \\ &- \quad \int_0^{T_R} e^{-rt}X^x_tF(Y^{y,\nu}_t)d\nu^c_t \\ &+ \quad \int_0^{T_R} e^{-rt}\sigma F(Y^{y,\nu}_t)dW_t + \\ &- \quad \sum_{0\le t\le T_R} e^{-rt}X^x_t\int_0^{\Delta\nu_t} f(Y^{y,\nu}_t-u)du\Big].\end{aligned} \tag{2.33}$$

When R converges to $+\infty$, thanks to Lemma 2.2, T_R tends to T and since

$$[\mathcal{L}-r](X^x_tF(Y^{y,\nu}_t)) = (ab-(a+r)X^x_t)F(Y^{y,\nu}_t) \tag{2.34}$$

is uniformly integrable, we get

$$\begin{aligned}\mathbb{E}[e^{-rT}X^x_TF(Y^{y,\nu}_T)] = xF(y) \quad &+ \quad \mathbb{E}\Big[\int_0^{T} e^{-rt}[\mathcal{L}-r](X^x_tF(Y^{y,\nu}_t))dt + \\ &- \quad \int_0^{T} e^{-rt}X^x_tF(Y^{y,\nu}_t)d\nu^c_t \\ &+ \quad \int_0^{T} e^{-rt}\sigma F(Y^{y,\nu}_t)dW_t + \\ &- \quad \sum_{0\le t\le T} e^{-rt}X^x_t\int_0^{\Delta\nu_t} f(Y^{y,\nu}_t-u)du\Big].\end{aligned} \tag{2.35}$$

Then, adding the quantity

$$\mathbb{E}\Big[\sum_{0\le t\le T} e^{-rt}\Big\{[X^x_t]^+f(Y^{y,\nu}_t)\Delta\nu_t - [X^x_t]^-\int_0^{\Delta\nu_t} f(Y^{y,\nu}_t-u)du\Big\} - \int_0^T e^{-rt}h(Y^{y,\nu}_t)dt\Big]$$

to both the sides in the previous equation and rearranging properly each term, we have

$$\begin{aligned}
\mathbb{E}\Big[\int_0^T e^{-rt} X_t^x F(Y_t^{y,\nu}) d\nu_t^c \quad &+ \sum_{0\le t\le T} e^{-rt}[X_t^x]^+ f(Y_t^{y,\nu})\Delta\nu_t + \\
&- \sum_{0\le t\le T} e^{-rt}[X_t^x]^- \int_0^{\Delta\nu_t} f(Y_t^{y,\nu} - u)du + \\
&- \int_0^T e^{-rt} h(Y_t^{y,\nu})dt\Big] = \\
&= xF(y) - \mathbb{E}\Big[e^{-rT} X_T^x F(Y_T^{y,\nu}) + \\
&+ \int_0^T e^{-rt}[\mathcal{L} - r](X_t^x F(Y_t^{y,\nu}))dt + \\
&- \int_0^T e^{-rt} h(Y_t^{y,\nu})dt. \\
&+ \sum_{0\le t\le T} e^{-rt}[X_t^x]^+ f(Y_t^{y,\nu})\Delta\nu_t + \\
&- \sum_{0\le t\le T} e^{-rt}[X_t^x]^+ \int_0^{\Delta\nu_t} f(Y_t^{y,\nu} - u)du\Big]. \quad (2.36)
\end{aligned}$$

In conclusion, letting T converges to $+\infty$, by means of Lemma (2.2), Lemma (2.3) and monotone convergence theorem we obtain the equivalent formula (2.27) for J. □

2.2 Some properties of the value function v

In this section we find out some regularity properties for the value function thanks to some *a priori* estimates that we are able to deduce from the value function definition and from the structure of the performance functional J.

Proposition 2.6. *There exists a constant $C > 0$ such that, for any $(x, y) \in \mathbb{R} \times [\underline{y}, \overline{y}]$, it holds*

$$|J(x, y; \nu)| \le C(1 + |x|), \tag{2.37}$$

for any $\nu \in \mathcal{A}(y)$.

Proof. Let us consider the equivalent formulation for the functional J stated in Proposition 2.5. We observe that

$$|xF(y) - \mathbb{E}[e^{-rT} X_T^x F(Y_T^{y,\nu})]| \le f(\overline{y})(\overline{y} - \underline{y})(|x| + \mathbb{E}[e^{-rT}|X_T^x|]). \tag{2.38}$$

Moreover, if $Z_T := \mathbb{E}\Big[\int_0^T e^{-rt}|X_t^x|dt\Big]$, from (2.34) it follows that

$$\begin{aligned}&\left|\mathbb{E}\left[\int_0^T e^{-rt}\{[\mathcal{L}-r](X_t^x F(Y_t^{y,\nu})) - h(Y_t^{y,\nu})\}dt\right]\right| \leq \\ &\leq \frac{1}{r}(1-e^{-rT})[abf(\overline{y})(\overline{y}-\underline{y}) + h(\overline{y})] + f(\overline{y})(\overline{y}-\underline{y})(a+r)Z_T\end{aligned} \tag{2.39}$$

and further, if $W_T := \mathbb{E}[\sum_{0\leq t\leq T} e^{-rt}|X_t^x|]$, we have

$$\left|\mathbb{E}\left[\sum_{0\leq t\leq T} e^{-rt}[X_t^x]^+\{f(Y_t^{y,\nu})\Delta\nu_t - \int_0^{\Delta\nu_t} f(Y_t^{y,\nu}-u)du\}\right]\right| \leq 2f(\overline{y})(\overline{y}-\underline{y})W_T. \tag{2.40}$$

By means of straightforward calculations we obtain that

$$\begin{aligned}\mathbb{E}[|X_t^x|] =&\sigma_x(t)\sqrt{\frac{2}{\pi}}e^{-\frac{\mu_x^2(t)}{2\sigma_x^2(t)}} + \mu_x(t)\left[2\Phi\left(\frac{\mu_x(t)}{\sqrt{2}\sigma_x(t)}\right)-1\right] \leq \\ \leq&\sigma_x(t)\sqrt{\frac{2}{\pi}} + |\mu_x(t)|,\end{aligned} \tag{2.41}$$

where $\mu_x(t) := \mathbb{E}[X_t^x]$, $\sigma_x(t) := \mathbb{V}[X_t^x]$ and Φ is the cumulative distribution function of a standard normal random variable. From (2.2) we notice that both $\mu_x(t)$ and $\sigma_x(t)$ are uniformly bounded with respect to t and

$$|\mu_x(t)| \leq |x| + 2b, \qquad \sigma_x(t) \leq \frac{\sigma^2}{2a}. \tag{2.42}$$

Hence it follows that there exist two positive constants C_0 and C_1 such that

$$Z_T \leq C_0(1+|x|), \quad W_T \leq C_1(1+|x|). \tag{2.43}$$

Therefore, the right-hand side of (2.36) in proof of Proposition 2.5 can be bounded (uniformly in time) by $C(1+|x|)$ for a suitable positive constant C. Hence, when T converges to $+\infty$ in (2.36), we obtain (2.37). ☐

Proposition 2.7. *Given $y \in [\underline{y}, \overline{y}]$, the function $x \mapsto v(x,y)$ is non-decreasing and convex. Moreover, there exist three constants $C > 0$, $C_1 > 0$ and $C_2 > 0$ such that,* *for any $x, x' \in \mathbb{R}$ and $y \in [\underline{y}, \overline{y}]$, the following hold true*

$$-\frac{1}{r}h(y) \leq v(x,y) \leq C(1+|x|); \tag{2.44}$$

$$|v(x+x',y) - v(x,y)| \leq C_1|x'|; \tag{2.45}$$

$$|v(x,y+y') - v(x,y)| \leq C_2|y'|(1+|x|), \qquad y' \in [\underline{y}-y, \overline{y}-y]. \tag{2.46}$$

Proof. Given $x_1, x_2 \in \mathbb{R}$ with $x_1 \leq x_2$, clearly

$$X_t^{x_1} \leq X_t^{x_2}, \quad [X_t^{x_1}]^+ \leq [X_t^{x_2}]^+, \quad [X_t^{x_1}]^- \geq [X_t^{x_2}]^-, \tag{2.47}$$

hold true for any $t \geq 0$, almost surely. Therefore, since f is non-negative, for $y \in [\underline{y}, \overline{y}]$, we have $J(x_1, y; \nu) \leq J(x_2, y; \nu)$ for any admissible control $\nu \in \mathcal{A}(y)$. Taking the supremum over all $\nu \in \mathcal{A}(y)$, it follows that $v(\cdot, y)$ is a non-decreasing function on $\mathbb{R}$.

Convexity property for the value function in its first variable follows from the convexity of performance functional with respect to x. Such property for J can be established through its equivalent formulation (2.27). Indeed, given $x_1, x_2 \in \mathbb{R}$ and $\lambda \in [0,1]$, we have $X_t^{x_\lambda} = \lambda X_t^{x_1} + (1-\lambda)X_t^{x_2}$ and $[X_t^{x_\lambda}]^+ \leq \lambda[X_t^{x_1}]^+ + (1-\lambda)[X_t^{x_2}]^+$, where $x_\lambda = \lambda x_1 + (1-\lambda)x_2$. Therefore, fixed $y \in [\underline{y}, \overline{y}]$, we have

$$\begin{aligned}
J(x_\lambda, y; \nu) \quad &\leq \quad x_\lambda F(y) + \lambda \mathbb{E}\Big[\int_0^{+\infty} e^{-rt}\{[\mathcal{L}-r](X_t^{x_1}F(Y_t^{y,\nu})) - h(Y_t^{y,\nu})\}dt\Big] + \\
&+ \quad (1-\lambda)\mathbb{E}\Big[\int_0^{+\infty} e^{-rt}\{[\mathcal{L}-r](X_t^{x_2}F(Y_t^{y,\nu})) - h(Y_t^{y,\nu})\}dt\Big] + \\
&+ \quad \lambda \mathbb{E}\Big[\sum_{t\geq 0} e^{-rt}[X_t^{x_1}]^+\Big\{f(Y_t^{y,\nu})\Delta\nu_t - \int_0^{\Delta\nu_t} f(y-u)du\Big\}\Big] + \\
&+ \quad (1-\lambda)\mathbb{E}\Big[\sum_{t\geq 0} e^{-rt}[X_t^{x_2}]^+\Big\{f(Y_t^{y,\nu})\Delta\nu_t - \int_0^{\Delta\nu_t} f(y-u)du\Big\}\Big] = \\
&= \quad \lambda J(x_1, y; \nu) + (1-\lambda)J(x_2, y; \nu),
\end{aligned} \tag{2.48}$$

for any control policy $\nu \in \mathcal{A}(y)$. Again, taking the supremum over all $\nu \in \mathcal{A}(y)$, we obtain that $v(\cdot, y)$ is a convex function on $\mathbb{R}$.

The left-hand inequality of estimate (2.44) follows by considering the control policy $\nu^0 \equiv 0$. Indeed, such a control is obviously admissible and for $x \in \mathbb{R}$ and $y \in [\underline{y}, \overline{y}]$ and one has

$$v(x,y) \geq J(x,y;\nu^0) = -\int_0^{+\infty} e^{-rt}h(y)dt = -\frac{1}{r}h(y). \tag{2.49}$$

Whereas, the right-hand estimation of (2.44) follows directly from (2.37) since it holds true for any admissible control in $\mathcal{A}(y)$.

Now, in order to prove (2.45) we consider $x' \in \mathbb{R}$ and observe that for $t \geq 0$,

$$X_t^{x+x'} = e^{-at}x' + X_t^x, \quad \text{a.s.,} \tag{2.50}$$

hence, fixed $x \in \mathbb{R}$ and $y \in [\underline{y}, \overline{y}]$, we have

$$\left| \int_0^{+\infty} e^{-rt}\{[\mathcal{L}-r](X_t^{x+x'}) - [\mathcal{L}-r](X_t^x)\}dt \right| \leq r|x'| \int_0^{+\infty} e^{-(a+r)t}dt = \frac{r|x'|}{a+r},$$

and, since $|[X_t^x + e^{-at}x']^+ - [X_t^x]^+| \leq e^{-at}|x'|$,

$$\left| \sum_{t\geq 0} e^{-rt}([X_t^x + e^{-at}x']^+ - [X_t^x]^+) \right| \leq |x'| \sum_{t\geq 0} e^{-(a+r)t} = \frac{|x'|}{1-e^{-(a+r)}}.$$

Therefore, by using the equivalent formulation (2.27), it follows that

$$|J(x+x', y; \nu) - J(x, y; \nu)| \leq C_1(1 + |x'|), \tag{2.51}$$

with $C_1 = f(\overline{y})(\overline{y} - \underline{y}) \max\{1, r/(a+r), (1/(1-e^{-(a+r)}))\}$. Property (2.45) follows easily taking the supremum over all $\nu \in \mathcal{A}(y)$.

Now, fixed $x \in \mathbb{R}$ and given $y \in [\underline{y}, \overline{y}]$, we first consider $y' \in [0, \overline{y} - y]$ and we notice that $\mathcal{A}(y) \subseteq \mathcal{A}(y+y')$. Hence we write

$$\begin{aligned}
v(x, y+y') - v(x, y) &= \sup_{\nu' \in \mathcal{A}(y+y')} J(x, y+y'; \nu') - \sup_{\nu \in \mathcal{A}(y)} J(x, y; \nu) = \\
&= \sup_{\nu' \in \mathcal{A}(y+y')} \inf_{\nu \in \mathcal{A}(y)} \big[J(x, y+y'; \nu') - J(x, y; \nu)\big] \geq \\
&= \inf_{\nu \in \mathcal{A}(y)} \big[J(x, y+y'; \nu) - J(x, y; \nu)\big] = \\
&= \inf_{\nu \in \mathcal{A}(y)} \mathbb{E}\bigg[\int_0^{+\infty} e^{-rt}\{[\mathcal{L}-r](X_t^x(F(Y_t^{y+y',\nu}) - F(Y_t^{y,\nu})))\}dt + \\
&+ \int_0^{+\infty} e^{-rt}\{h(Y_t^{y,\nu}) - h(Y_t^{y+y',\nu})\}dt + \\
&+ \sum_{t\geq 0} e^{-rt}[X_t^x]^+ \bigg\{ (f(Y_t^{y+y',\nu}) - f(Y_t^{y,\nu}))\Delta\nu_t + \\
&- \int_0^{\Delta\nu_t} (f(Y_t^{y+y',\nu} - u) - f(Y_t^{y,\nu} - u))du \bigg\} \bigg].
\end{aligned} \tag{2.52}$$

We recall that both f and h are continuously differentiable on $(\underline{y}, \overline{y})$ with their first derivatives bounded on $[\underline{y}, \overline{y}]$. In particular these functions are both Lipschitz-continuous on $[\underline{y}, \overline{y}]$ and there exist two positive constants L^f and L^h such that

$$|f(Y_t^{y+y',\nu}) - f(Y_t^{y,\nu}))| \leq L^f y', \qquad |h(Y_t^{y+y',\nu}) - h(Y_t^{y,\nu}))| \leq L^h y'.$$

Furthermore, we have

$$F(Y_t^{y+y',\nu}) - F(Y_t^{y,\nu}) = \int_{Y_t^{y,\nu}}^{Y_t^{y+y',\nu}} f(z)dz = \int_0^{y'} f(Y_t^{y,\nu} + u)du \leq f(\overline{y})y',$$

and considering (2.34), we write

$$[\mathcal{L} - r](X_t^x(F(Y_t^{y+y',\nu}) - F(Y_t^{y,\nu}))) \geq -f(\overline{y})y'|ab - (a+r)X_t^x|.$$

Hence, exploiting again the integrability property of $|X_t^x|$ and (2.41)-(2.42), we can find a positive constant C_2' such that

$$v(x, y + y') - v(x, y) \geq -C_2'(1 + |x|)y' \tag{2.53}$$

On the other hand, given $\varepsilon > 0$, there exists a control policy $\nu'^{\varepsilon} \in \mathcal{A}(y + y')$ such that

$$v(x, y + y') \leq J(x, y + y'; \nu'^{\varepsilon}) + \varepsilon, \tag{2.54}$$

and

$$\begin{aligned} v(x, y + y') - v(x, y) &\leq J(x, y + y'; \nu'^{\varepsilon}) - \sup_{\nu \in \mathcal{A}(y)} J(x, y, \nu) + \varepsilon = \\ &= \inf_{\nu \in \mathcal{A}(y)} \left[J(x, y + y'; \nu'^{\varepsilon}) - J(x, y; \nu) \right] + \varepsilon \leq \\ &\leq J(x, y + y'; \nu'^{\varepsilon}) - J(x, y; \bar{\nu}^{\varepsilon}) + \varepsilon, \end{aligned} \tag{2.55}$$

where $\bar{\nu}^{\varepsilon}$ is defined as

$$\bar{\nu}_t^{\varepsilon} := \begin{cases} 0 & t \leq \tau_y(\varepsilon), \\ \nu_t'^{\varepsilon} - y' & t > \tau_y(\varepsilon) \end{cases} \tag{2.56}$$

with $\tau_y(\varepsilon) := \inf\{t \geq 0 \ : \ \nu_t'^{\varepsilon} \geq y'\}$, with the convention that $\tau_y(\varepsilon) = +\infty$ if $\{t \geq 0 \ : \ \nu_t'^{\varepsilon} \geq y'\} = \emptyset$. We easily observe that $\bar{\nu}^{\varepsilon}$ belongs to the class of admissible controls $\mathcal{A}(y)$. Indeed the process $\bar{\nu}^{\varepsilon}$ is non-negative, càglàd and it is such that $\bar{\nu}_t^{\varepsilon} \leq y - \underline{y}$, for any $t \geq 0$. Hence, in (2.55) we have

$$
\begin{aligned}
v(x,y+y') - v(x,y) &\leq \mathbb{E}\bigg[\int_0^{\tau_y(\varepsilon)} e^{-rt}\{[\mathcal{L}-r](X_t^x(F(Y_t^{y+y',\nu'^{\varepsilon}})-F(y))\}dt + \\
&+ \int_0^{\tau_y(\varepsilon)} e^{-rt}\{h(y) - h(Y_t^{y+y',\nu'^{\varepsilon}})\}dt + \\
&+ \sum_{t\leq\tau_y(\varepsilon)} e^{-rt}[X_t^x]^+\bigg\{(f(Y_t^{y+y',\nu'^{\varepsilon}})\Delta\nu'^{\varepsilon}_t + \\
&- \int_0^{\Delta\nu'^{\varepsilon}_t} f(Y_t^{y+y',\nu'^{\varepsilon}} - u)du\bigg\}\bigg] + \varepsilon \leq \\
&\leq \mathbb{E}\bigg[y' f(\overline{y})\int_0^{+\infty} e^{-rt}\{ab + (a+r)|X_t^x|\}dt + \\
&+ L^h\int_0^{\tau_y(\varepsilon)} e^{-rt}|\nu'^{\varepsilon}_t - y'|dt + 2f(\overline{y})y'\sum_{t\geq 0} e^{-rt}[X_t^x]^+\bigg] + \varepsilon \leq \\
&\leq y' f(\overline{y})\mathbb{E}\bigg[\int_0^{+\infty} e^{-rt}\{ab + (a+r)|X_t^x|\}dt + \\
&+ 2y'\mathbb{E}\bigg[\frac{L^h}{r}(1 - e^{-r\tau_y(\varepsilon)}) + f(\overline{y})'\sum_{t\geq 0} e^{-rt}|X_t^x|\bigg] + \varepsilon. \qquad (2.57)
\end{aligned}
$$

Hence, thanks to (2.41)-(2.42), we find $C_2'' > 0$ such that

$$
v(x,y+y') - v(x,y) \leq C_2''(1+|x|)y' + \varepsilon. \qquad (2.58)
$$

Observing that (2.58) holds for any $\varepsilon > 0$ and combining it with the inequality (2.53), we can find a constant $C_2 > 0$ such that (2.46) holds for $y \in [\underline{y}, \overline{y}]$ and $y' \in [0, \overline{y} - y]$.

The same argument can be used to prove the previous estimates when y' belongs to $[\underline{y} - y', 0]$. We simply switch the role of the class of admissible control $\mathcal{A}(y)$ with that of $\mathcal{A}(y+y')$ in the above computations. □

We recall that, given an open set $\mathcal{O} \subset \mathbb{R}^2$, the functions space $W^{1,\infty}_{loc}(\mathcal{O})$ denotes the Sobolev space of the functions with weak derivatives locally bounded on $\mathcal{O}$ (see Appendix B). The previous estimates provide us with a first result about the regularity of the value function v.

Proposition 2.8. *The function v belongs to the Sobolev space $W^{1,\infty}_{loc}(\mathbb{R} \times (\underline{y}, \overline{y}))$. In particular, there exist two positive constants C_1 and C_2 such that*

$$
|v_x(x,y)| \leq C_1, \qquad |v(x,y)| + |v_y(x,y)| \leq C_2(1+|x|), \qquad (2.59)
$$

a.e. in $\mathbb{R} \times (\underline{y}, \overline{y})$.

Proof. From properties (2.45)-(2.46) in Proposition 2.7 easily follows that v is locally Lipschitz continuous on $\mathbb{R} \times (\underline{y}, \overline{y})$. Hence, thanks to the characterisation provided by Proposition B.2 in Appendix B, we have that $v \in W^{1,\infty}_{loc}(\mathbb{R} \times (\underline{y}, \overline{y}))$ and both its first weak derivatives v_x and v_y are locally bounded. Moreover, thanks to Theorem B.3 in Appendix B, we have that v is a.e. differentiable and its first weak derivatives coincide with the classic ones, almost everywhere. Therefore, letting $|x'|$ and y' converge to 0 in (2.45) and (2.46), we obtain that (2.59) hold almost everywhere in $\mathbb{R} \times (\underline{y}, \overline{y})$. □

Remark 2.9. The results previously proved are intended to provide an idea of the regularity properties that can be expected for the value function. Nevertheless, these also have a value from the application point of view. For instance, Proposition 2.6 guarantees that, fixed the initial position (x, y), the performance associated to each admissible energy production strategy is bounded. In particular, property (2.44) in Proposition 2.7 says that the value associated at the best performance is finite and, as shown in (2.49), it is at least equal to the perpetual inaction. Furthermore, properties (2.45) in Proposition 2.7 says that the variation of the value associated to the best performance with respect to the variation of the initial price is always bounded and, similarly, property (2.46) guarantees that the change in value of the optimal production policy with respect to a variation of the initial amount of water into the reservoir increases at most linearly with respect the initial price x.

2.3 The Hamilton-Jacobi-Bellman equation

The model studied in this thesis consists in a singular stochastic control problem and, as it is usual for this kind of problems, we expect that the value function v solves a suitable partial differential equation of Hamilton-Jacobi-Bellman type. We present an heuristic discussion in order to obtain the equation associated with our stochastic control problem (OC).

Fixed $(x, y) \in \mathbb{R} \times [\underline{y}, \overline{y}]$ and $\nu \in \mathcal{A}(y)$, by assuming that v satisfies sufficient regularity properties to apply the Itô's formula to $e^{-rt}v(X_t^x, Y_t^{y,\nu})$, we formally obtain

$$
\begin{aligned}
v(x,y) &= \mathbb{E}\bigg[e^{-rT}v(X_T^x, Y_T^{y,\nu}) - \int_0^T e^{-rt}[\mathcal{L}-r]v(X_t^x, Y_t^{y,\nu})dt + \\
&- \int_0^T e^{-rt}v_y(X_t^x, Y_t^{y,\nu})dY_t^{y,\nu} + \\
&- \sum_{0\le t\le T} e^{-rt}\{v(X_t^x, Y_{t+}^{y,\nu}) - v(X_t^x, Y_t^{y,\nu}) - v_y(X_t^x, Y_t^{y,\nu})\Delta Y_t^{y,\nu}\}\bigg].
\end{aligned}
$$

Then, using the decomposition (1.19), writing

$$
v(X_t^x, Y_{t+}^{y,\nu}) - v(X_t^x, Y_t^{y,\nu}) = -\int_0^{\Delta\nu_t} v_y(X_t^x, Y_t^{y,\nu} - u)du, \quad a.s., \quad \forall t \ge 0, \tag{2.60}
$$

and assuming that $e^{-rT}v(X_T^x, Y_T^{y,\nu})$ converges to 0 when $T \to +\infty$, we get

$$
\begin{aligned}
v(x,y) &= \mathbb{E}\bigg[-\int_0^{+\infty} e^{-rt}[\mathcal{L}-r]v(X_t^x, Y_t^{y,\nu})dt + & (2.61)\\
&+ \int_0^{+\infty} e^{-rt}v_y(X_t^x, Y_t^{y,\nu})d\nu_t^c + & (2.62)\\
&+ \sum_{t\ge 0} e^{-rt}\int_0^{\Delta\nu_t} v_y(X_t^x, Y_t^{y,\nu} - u)du\bigg]. & (2.63)
\end{aligned}
$$

Therefore, if v is such that

$$
[\mathcal{L}-r]v(x,y) - h(y) \le 0, \tag{2.64}
$$

$$
xf(y) + [x]^+ f'(y)(y - \underline{y}) - v_y(x,y) \le 0, \tag{2.65}
$$

for any $(x,y) \in \mathbb{R} \times [\underline{y}, \overline{y}]$, it follows that

$$
\begin{aligned}
v(x,y) &\ge \mathbb{E}\bigg[-\int_0^{+\infty} e^{-rt}h(Y_t^{y,\nu})dt + \\
&+ \int_0^{+\infty} e^{-rt}X_t^x f(Y_t^{y,\nu})d\nu_t^c + \\
&+ \int_0^{+\infty} e^{-rt}[X_t^x]^+ f'(Y_t^{y,\nu})(Y_t^{y,\nu} - \underline{y})d\nu_t^c + \\
&+ \sum_{t\ge 0} e^{-rt}X_t^x \int_0^{\Delta\nu_t} f(Y_t^{y,\nu} - u)du + \\
&+ \sum_{t\ge 0} e^{-rt}[X_t^x]^+ \int_0^{\Delta\nu_t} f'(Y_t^{y,\nu} - u)(Y_t^{y,\nu} - \underline{y} - u)du\bigg]. \qquad (2.66)
\end{aligned}
$$

Integrating by parts the last term,

$$
\begin{aligned}
v(x,y) \quad \geq \quad & \mathbb{E}\Big[-\int_0^{+\infty} e^{-rt} h(Y_t^{y,\nu})dt + \\
+ \quad & \int_0^{+\infty} e^{-rt} X_t^x f(Y_t^{y,\nu}) d\nu_t^c + \\
+ \quad & \int_0^{+\infty} e^{-rt} [X_t^x]^+ f'(Y_t^{y,\nu})(Y_t^{y,\nu} - \underline{y}) d\nu_t^c + \\
+ \quad & \sum_{t\geq 0} e^{-rt} X_t^x \int_0^{\Delta\nu_t} f(Y_t^{y,\nu} - u)du + \\
+ \quad & \sum_{t\geq 0} e^{-rt} [X_t^x]^+ \{f(Y_t^{y,\nu})(Y_t^{y,\nu} - \underline{y}) - f(Y_{t+}^{y,\nu})(Y_{t+}^{y,\nu} - \underline{y})\} + \\
- \quad & \sum_{t\geq 0} e^{-rt} [X_t^x]^+ \int_0^{\Delta\nu_t} f(Y_t^{y,\nu} - u)\Big].
\end{aligned}
\tag{2.67}
$$

Now, rearranging properly the terms we get

$$
\begin{aligned}
v(x,y) \quad \geq \quad & \mathbb{E}\Big[-\int_0^{+\infty} e^{-rt} h(Y_t^{y,\nu})dt + && (2.68)\\
+ \quad & \int_0^{+\infty} e^{-rt} X_t^x f(Y_t^{y,\nu}) d\nu_t^c + && (2.69)\\
+ \quad & \int_0^{+\infty} e^{-rt} [X_t^x]^+ f'(Y_t^{y,\nu})(Y_t^{y,\nu} - \underline{y}) d\nu_t^c + && (2.70)\\
- \quad & \sum_{t\geq 0} e^{-rt} [X_t^x]^- \int_0^{\Delta\nu_t} f(Y_t^{y,\nu} - u)du + && (2.71)\\
+ \quad & \sum_{t\geq 0} e^{-rt} [X_t^x]^+ f(Y_t^{y,\nu})\Delta\nu_t + && (2.72)\\
+ \quad & \sum_{t\geq 0} e^{-rt} [X_t^x]^+ \{(Y_{t+}^{y,\nu} - \underline{y})(f(Y_t^{y,\nu}) - f(Y_{t+}^{y,\nu}))\}\Big] \geq && (2.73)\\
\geq \quad & J(x,y;\nu), && (2.74)
\end{aligned}
$$

since (2.70) and (2.73) are non-negative.

We observe that if ν is an optimal control, i.e. $v(x,y) = J(x,y;\nu)$, we must have that both integrals in (2.70) and (2.73) vanish. Since the function f is positive, continuously differentiable and strictly increasing on $(\underline{y}, \overline{y}]$, the integrand in (2.70) is positive and the integral equals 0 if and only if either the control ν is purely discontinuous, i.e. $\nu^c \equiv 0$ a.s., or $Y_t^{y,\nu} = \underline{y}$ for any $t > 0$, which means that ν remains constant after an initial instantaneous jump $\Delta\nu_0 = y - \underline{y}$. Furthermore we notice that the sum in (2.73) equals 0 if, for any time $t \geq 0$, either $f(Y_t^{y,\nu}) = f(Y_{t+}^{y,\nu})$ which means that the control

ν doesn't act, or if $Y^{y,\nu}_{t+} = \underline{y}$, i.e. at time t the control acts with a single jump using all the possible fuel and pushing the process $Y^{y,\nu}$ to its lower bound.

Let us now consider the strip $\mathbb{R} \times [\underline{y}, \overline{y}]$ divided in two disjoint regions: the *inaction (continuation) region* $\mathcal{C}$, within the optimal strategy is doing nothing and its complement $\mathcal{D}$, the *action (discharging) region*, where it is convenient to activate the control policy, generating power and selling it in the energy market.

Fixed $(x, y) \in \mathcal{D}$, in light of previous arguments the optimal policy is discharging instantaneously all the possible water, obtaining the value

$$v(x, y) = [x]^+ f(y)(y - \underline{y}) - [x]^- \int_0^{y-\underline{y}} f(y - u)du. \tag{2.75}$$

Considering the first derivative with respect to y, we have

$$v_y(x, y) = xf(y) + [x]^+ f'(y)(y - \underline{y}), \tag{2.76}$$

and it follows that in (2.65) we have equality.

On the other hand, if we consider (x, y) in the inaction region $\mathcal{C}$, we can pick a sufficiently small $\delta > 0$ in order to remain within the continuation region at least during the time interval $(0, \delta)$. Therefore, exploiting the approach of dynamic programming principle, the optimal value associated with doing nothing before leaving $\mathcal{C}$ can be formally written as

$$v(x, y) = \mathbb{E}\left[e^{-r\delta} v(X^x_\delta, y) - \int_0^\delta e^{-rt} h(y)dt\right]. \tag{2.77}$$

Applying the Itô's formula to $e^{-rt}v(X^x_t, y)$ on $(0, \delta)$, we get

$$v(x, y) = \mathbb{E}\left[v(x, y) + \int_0^\delta e^{-rt}[\mathcal{L} - r]v(X^x_t, y)dt - \int_0^\delta e^{-rt} h(y)dt\right], \tag{2.78}$$

dividing by δ and letting δ converge to 0 we obtain

$$[\mathcal{L} - r]v(x, y) - h(y) = 0, \tag{2.79}$$

hence in (2.64) we have equality.

Therefore, getting together condition (2.64)-(2.65) and (2.79)-(2.76), we expect that the value function v solves of the Hamilton-Jacobi-Bellman equation

$$\max\{[\mathcal{L} - r]v(x, y) - h(y),\ xf(y) + [x]^+ f'(y)(y - \underline{y}) - v_y(x, y)\} = 0, \tag{HJB}$$

for a.e. $(x, y) \in \mathbb{R} \times (\underline{y}, \overline{y}]$, associated with the boundary condition

$$v(x, \underline{y}) = 0, \quad \forall x \in \mathbb{R}. \tag{2.80}$$

We notice that the partial differential equation (HJB) turns out to be a variational inequality involving a degenerate second order elliptic equation subject to a state-dependent gradient constraint.

2.4 The Verification Theorem

In this section we establish a *verification approach* that rigorously confirm what we guessed through the previous heuristic arguments. In particular, we provide some sufficient conditions which allows us to identify the value function v among the solutions of the equation (HJB). Moreover, the Verification Theorem that we prove below gives us fundamental information about the structure of the optimal control policy. Such information will be particularly useful in sequel to explicitly solve our singular stochastic control problem (OC).

To the best of our knowledge, general verification results are rare in the literature of state-dependent singular optimal control whose marginal revenue does not satisfy any monotonicity condition. In fact, Song, Stockbridge and Zhu [SSZ11] provide sufficient conditions to identify the value function of their optimal harvesting problem among the solutions of a suitable HJB equation by exploiting a crucial monotonicity property of their instantaneous marginal yield. A similar result is obtained by Alvarez, Lungu and Øksendal [ALØ16] in a multidimensional harvesting problem with interaction between different populations. They consider a multidimensional state-dependent marginal yield whose each component is non-increasing with respect to each component of the controlled state variable. Davis and Zervos [DZ98] prove a very general verification result for state-dependent singular control problems, by employing the generalised Meyer-Itô change of variable formula with local times, although they explicitly apply such result to solve two special cases which do not exhibit state-dependence of the instantaneous marginal yield.

Theorem 2.10 (Verification Theorem)**.** *Let* $w : \mathbb{R} \times [\underline{y}, \overline{y}] \to \mathbb{R}$ *be a function in* $C^2(\mathbb{R} \times [\underline{y}, \overline{y}])$ *solution to (HJB) and such that*

$$w(x, \underline{y}) = 0, \quad \forall x \in \mathbb{R}, \tag{2.81}$$

and

$$\lim_{t\to+\infty}\mathbb{E}[e^{-rt}w(X_t^x,Y_t^{y,\nu})]=0. \tag{2.82}$$

Then, it follows that $w(x,y)\geq v(x,y)$ for all $(x,y)\in\mathbb{R}\times[\underline{y},\overline{y}]$.

Moreover, consider the inaction region

$$\mathcal{C}=\{(x,y)\in\mathbb{R}\times(\underline{y},\overline{y}]:[\mathcal{L}-r]w(x,y)-h(y)=0\}, \tag{2.83}$$

and assume that there exists a control $\bar\nu\in\mathcal{A}(y)$ such that satisfies the following property almost surely:

$$\bar\nu_t^c=0,\quad \forall t\geq 0\quad (\bar\nu \text{ purely discontinuous}), \tag{2.84}$$

$$(X_t^x,Y_t^{y,\bar\nu})\in\mathcal{C}\cup(\mathbb{R}\times\{\underline{y}\}),\quad \text{for almost every } t\geq 0, \tag{2.85}$$

and for any $t\geq 0$ such that $\Delta\bar\nu_t\neq 0$, holds

$$w(X_t^x,Y_t^{y,\bar\nu})=w(X_t^x,Y_{t+}^{y,\bar\nu})+[X_t^x]^+f(Y_t^{y,\bar\nu})\Delta\bar\nu_t-[X_t^x]^-\int_0^{\Delta\bar\nu_t}f(Y_t^{y,\bar\nu}-u)du, \tag{2.86}$$

Then, it follows that the control $\bar\nu$ is optimal for the stochastic control problem (OC) and $w(x,y)=v(x,y)$, for all $(x,y)\in\mathbb{R}\times[\underline{y},\overline{y}]$.

Proof. Fixed $(x,y)\in\mathbb{R}\times[\underline{y},\overline{y}]$ and $\nu\in\mathcal{A}(y)$, we consider $R>0$ such that $|x|<R$ and define τ_R ad in Lemma 2.2. For some $T>0$, we apply the bidimensional Itô's formula for semimartingale to the function $e^{-rt}w(X_t^x,Y_t^{y,\nu})$ on the time interval $[0,T_R]$, with $T_R=T\wedge\tau_R$. Hence,

$$\begin{aligned} w(x,y) &= e^{-rT_R}w(X_{T_R}^x,Y_{T_R}^{y,\nu})-\int_0^{T_R}e^{-rt}[\mathcal{L}-r]w(X_t^x,Y_t^{y,\nu})dt+\\ &- \int_0^{T_R}e^{-rt}w_y(X_t^x,Y_t^{y,\nu})dY_t^{y,\nu}-\int_0^{T_R}\sigma e^{-rt}w_x(X_t^x,Y_t^{y,\nu})dW_t+\\ &- \sum_{0\leq t\leq T_R}e^{-rt}\{w(X_t^x,Y_{t+}^{y,\nu})-w(X_t^x,Y_t^{y,\nu})-w_y(X_t^x,Y_t^{y,\nu})\Delta Y_t^{y,\nu}\}. \end{aligned} \tag{2.87}$$

Recalling (2.60), we get

$$\begin{aligned} w(x,y) &= e^{-rT_R}w(X_{T_R}^x,Y_{T_R}^{y,\nu})-\int_0^{T_R}e^{-rt}[\mathcal{L}-r]w(X_t^x,Y_t^{y,\nu})dt+\\ &+ \int_0^{T_R}e^{-rt}w_y(X_t^x,Y_t^{y,\nu})d\nu_t-\int_0^{T_R}\sigma e^{-rt}w_x(X_t^x,Y_t^{y,\nu})dW_t+\\ &- \sum_{0\leq t\leq T_R}e^{-rt}\Big\{-\int_0^{\Delta\nu_t}w_y(X_t^x,Y_t^{y,\nu}-u)du+w_y(X_t^x,Y_t^{y,\nu})\Delta\nu_t\Big\}. \end{aligned}$$

and, since the function w_x is locally bounded on $\mathbb{R} \times [\underline{y}, \overline{y}]$, it follows that the process

$$M_t := \int_0^{t\wedge\tau_R} \sigma e^{-rs} w_x(X_s^x, Y_s^{y,\nu})dW_s \tag{2.88}$$

is a square integrable martingale. Since w is a solution of (HJB), taking the expectation and using the decomposition (1.19), we obtain

$$\begin{aligned} w(x,y) &= \mathbb{E}\Big[e^{-rT_R} w(X_{T_R}^x, Y_{T_R}^{y,\nu}) - \int_0^{T_R} e^{-rt}[\mathcal{L} - r]w(X_t^x, Y_t^{y,\nu})dt + \\ &+ \int_0^{T_R} e^{-rt} w_y(X_t^x, Y_t^{y,\nu})d\nu_t^c + \sum_{0\le t\le T_R} e^{-rt}\int_0^{\Delta\nu_t} w_y(X_t^x, Y_t^{y,\nu} - u)du\Big] \ge \\ &\ge \mathbb{E}\Big[e^{-rT_R} w(X_{T_R}^x, Y_{T_R}^{y,\nu}) - \int_0^{T_R} e^{-rt} h(Y_t^{y,\nu})dt + \int_0^{T_R} e^{-rt} X_t^x f(Y_t^{y,\nu})d\nu_t^c + \\ &+ \int_0^{T_R} e^{-rt}[X_t^x] f'(Y_t^{y,\nu})(Y_t^{y,\nu} - \underline{y})d\nu_t^c + \\ &+ \sum_{0\le t\le T_R} e^{-rt} X_t^x \int_0^{\Delta\nu_t} f(Y_t^{y,\nu} - u)du + \\ &+ \sum_{0\le t\le T_R} e^{-rt}[X_t^x]^+ \int_0^{\Delta\nu_t} f'(Y_t^{y,\nu} - u)(Y_t^{y,\nu} - \underline{y} - u)du\Big]. \end{aligned}$$

Thanks to Lemma 2.2, passing to the limit for $R \to +\infty$, we have that $T_R = T \wedge \tau_R$ approaches T and

$$\begin{aligned} w(x,y) &\ge \mathbb{E}\Big[e^{-rT} w(X_T^x, Y_T^{y,\nu}) - \int_0^{T} e^{-rt} h(Y_t^{y,\nu})dt + \int_0^{T} e^{-rt} X_t^x f(Y_t^{y,\nu})d\nu_t^c + \\ &+ \int_0^{T} e^{-rt}[X_t^x] f'(Y_t^{y,\nu})(Y_t^{y,\nu} - \underline{y})d\nu_t^c + \\ &+ \sum_{0\le t\le T} e^{-rt} X_t^x \int_0^{\Delta\nu_t} f(Y_t^{y,\nu} - u)du + \\ &+ \sum_{0\le t\le T} e^{-rt}[X_t^x]^+ \int_0^{\Delta\nu_t} f'(Y_t^{y,\nu} - u)(Y_t^{y,\nu} - \underline{y} - u)du\Big]. \end{aligned} \tag{2.89}$$

Hence, integrating by parts as in the heuristic discussion and letting T converges to $+\infty$, thanks to (2.82) and Lemma (2.4), we obtain

$$w(x,y) \ge J(x,y;\nu). \tag{2.90}$$

Moreover, due to the arbitrariness of $\nu \in \mathcal{A}(y)$, we get $w(x,y) \ge v(x,y)$ for any

$(x,y) \in \mathbb{R} \times [\underline{y}, \overline{y}]$.

On the other hand, fixed $(x,y) \in \mathbb{R} \times (\underline{y}, \overline{y}]$ and $T > 0$, we now consider a control $\bar{\nu} \in \mathcal{A}(y)$ satisfying the properties (2.84)-(2.86). Hence, considering again the decomposition (1.19) and letting R tends to $+\infty$ in (2.87), we obtain

$$\begin{aligned} w(x,y) &= \mathbb{E}\bigg[e^{-rT} w(X_T^x, Y_T^{y,\bar{\nu}}) - \int_0^T e^{-rt}[\mathcal{L}-r]w(X_t^x, Y_t^{y,\bar{\nu}})dt + \\ &+ \int_0^T e^{-rt} w_y(X_t^x, Y_t^{y,\bar{\nu}}) d\bar{\nu}_t^c + \sum_{0 \le t \le T} e^{-rt}\{w(X_t^x, Y_t^{y,\bar{\nu}}) - w(X_t^x, Y_{t+}^{y,\bar{\nu}})\}\bigg] = \\ &= \mathbb{E}\bigg[e^{-rT} w(X_T^x, Y_T^{y,\bar{\nu}}) - \int_0^T e^{-rt}[\mathcal{L}-r]w(X_t^x, Y_t^{y,\bar{\nu}})dt + \\ &+ \sum_{0 \le t \le T} e^{-rt}\{[X_t^x]^+ f(Y_t^{y,\bar{\nu}})\Delta\bar{\nu}_t - [X_t^x]^- \int_0^{\Delta\bar{\nu}_t} f(Y_t^{y,\bar{\nu}} - u)du\}\bigg], \end{aligned} \tag{2.91}$$

since the $\bar{\nu}$ is purely discontinuous (cf. (2.84)) and the condition (2.86) holds. Moreover, we observe that the complementary set of $\mathcal{C} \cup (\mathbb{R} \times \{\underline{y}\})$ coincides with discharging region $\mathcal{D}$ and, from condition (2.85), it follows that

$$\lambda(\{t \in [0,T] : (X_t^x, Y_t^{y,\bar{\nu}}) \in \mathcal{D}\}) = 0, \quad \forall T > 0, \tag{2.92}$$

where $\lambda(\cdot)$ represents the Lebesgue measure on $(0,+\infty)$.Therefore, recalling also that w satisfies the boundary condition (2.81) and that $h(\underline{y}) = 0$, we get

$$\int_0^T e^{-rt}[\mathcal{L}-r]w(X_t^x, Y_t^{y,\bar{\nu}})dt = \tag{2.93}$$

$$= \int_0^T e^{-rt} \mathbb{1}_{\{t \in [0,T] : (X_t^x, Y_t^{y,\bar{\nu}}) \in \mathcal{C}\}}[\mathcal{L}-r]w(X_t^x, Y_t^{y,\bar{\nu}})dt + \tag{2.94}$$

$$+ \int_0^T e^{-rt} \mathbb{1}_{\{t \in [0,T] : (X_t^x, Y_t^{y,\bar{\nu}}) \in (\mathbb{R} \times \{\underline{y}\})\}}[\mathcal{L}-r]w(X_t^x, \underline{y})dt = \tag{2.95}$$

$$= \int_0^T e^{-rt} \mathbb{1}_{\{t \in [0,T] : (X_t^x, Y_t^{y,\bar{\nu}}) \in \mathcal{C}\}} h(Y_t^{y,\bar{\nu}})dt + \tag{2.96}$$

$$+ \int_0^T e^{-rt} \mathbb{1}_{\{t \in [0,T] : (X_t^x, Y_t^{y,\bar{\nu}}) \in (\mathbb{R} \times \{\underline{y}\})\}} h(\underline{y})dt = \tag{2.97}$$

$$= \int_0^T e^{-rt} h(Y_t^{y,\bar{\nu}})dt. \tag{2.98}$$

Hence, in (2.91) we have

$$\begin{aligned} w(x,y) &= \mathbb{E}\Big[e^{-rT}w(X_T^x, Y_T^{y,\bar\nu}) - \int_0^T e^{-rt}h(Y_t^{y,\bar\nu})dt + \\ &+ \sum_{0\le t\le T} e^{-rt}\{[X_t^x]^+ f(Y_t^{y,\bar\nu})\Delta\bar\nu_t - [X_t^x]^- \int_0^{\Delta\bar\nu_t} f(Y_t^{y,\bar\nu}-u)du\}\Big]. \end{aligned} \tag{2.99}$$

Letting T converges to $+\infty$, we obtain

$$\begin{aligned} w(x,y) &= \mathbb{E}\Big[\sum_{t\ge 0} e^{-rt}\{[X_t^x]^+ f(Y_t^{y,\bar\nu})\Delta\bar\nu_t - [X_t^x]^- \int_0^{\Delta\bar\nu_t} f(Y_t^{y,\bar\nu}-u)du\} + \\ &- \int_0^{+\infty} e^{-rt}h(Y_t^{y,\bar\nu})dt\Big] = J(x,y;\bar\nu). \end{aligned} \tag{2.100}$$

Hence, we have that the control $\bar\nu$ is such that $w(x,y) = v(x,y)$, i.e. it is optimal for optimal control problem (OC). □

Remark 2.11. In general, it is not always guaranteed that the value function v associated to the an optimal control problem is a C^2 solution of the equation (HJB). However, the conditions of Verification Theorem can be weakened. In particular, as in Chapter VIII (section VIII.4) of [FS06], Theorem 2.10 holds for functions w satisfying the following more general conditions,

$$w \in C_p(\mathbb{R}\times[\underline{y},\overline{y}]) \cap C^1(\mathbb{R}\times(\underline{y},\overline{y})), \quad w \in C^2(\mathcal{C}) \quad \text{and} \quad w_x \in W^{1,\infty}_{loc}(\mathbb{R}\times(\underline{y},\overline{y})),$$

where C_p is the set of polynomial growth functions. Indeed, under these conditions one can define a sequence $\{w_n\}_{n\in\mathbb{N}}$ such that $w_n \in C^\infty(\mathbb{R}^2)$ and converges to w uniformly on the compact subsets of $\mathbb{R}^2$ as well as its first derivatives $(w_n)_x$ and $(w_n)_y$ converge uniformly to w_x and w_y. Therefore, one can apply the Itô's rule for semimartingale to the regular function w_n obtaining the result. Afterwards, thanks to the conditions required for w, one obtains the same conclusions when $n \to +\infty$.

Chapter 3

The Associated Optimal Stopping Problem

The existence of a connection between optimal control problems and optimal stopping problems is a feature highlighted by numerous studies in the wide theory of singular stochastic optimal control. In particular, Bather and Chernoff [BC67] were the first to establish such correspondence for a specific problem of spaceship controlling. They noticed that the region of inaction of the control problem should coincide with the continuation region of the stopping problem. In particular, the authors found out that the optimal reward function for stopping problem should be the first derivative of the value function associated with the optimal control.

The equivalence *optimal control - optimal stopping* and the aforementioned "differential" connection between value function and optimal stopping reward was later formalised in a one-dimensional context by Karatzas and Shreve [KS84; KS85] and widely developed by many others for singular stochastic control problems which, among various technical features, satisfy some convexity/concavity properties of the performance functional with respect to the controlled variable. In particular, in [KS84], the authors faced the problem of optimally tracking a Brownian motion by a non-decreasing process and they found that the optimal control policy is pushing instantaneously the state process at the boundary separating action and inaction regions, whenever the starting position is within the action region. Afterwards, the optimal process acts only when the controlled state is on the boundary, exerting only the sufficient control to avoid a crossing of the controlled Brownian motion inside the interior of the action region. This means that the optimal control policy behaves like the *local time* of the controlled process at the optimal boundary.

The Verification Theorem proved in Chapter 2 suggests that in our framework the aforementioned control policy does not be optimal. Contrarily, the best performance is obtained through a purely discontinuous control of *bang-bang* type, i.e. whenever the optimal control policy acts, it exerts all the available reserve through an instantaneous single jump. Therefore, the problem that naturally arise is establishing *when* to exert the control and these lead us to believe that a connection between optimal control problem and optimal stopping still holds also for non-convex singular stochastic control problem which exhibit *state-dependence* of the instantaneous marginal revenue, even if such connection cannot be thought in its classical differential meaning.

In this chapter we properly define a parametrised family of optimal stopping problem (OS_y), associated with our singular stochastic control problem (OC) and we show when the latter can be completely solved by considering such connection. We prove that, under particular assumptions on the ratio between instantaneous holding costs and power produced by immediate water release, the value function of the optimal control can be obtained by the optimal reward associated with (OS_y). Contrarily, although we are always able to solve the associated stopping problems (OS_y), the connection between the solutions of such problems with the solution of (OC) cannot be always rigorously proved. In particular, if alternative hypothesis for above ratio are considered, (OS_y) may exhibit disconnected stopping regions and the characterisation of the corresponding moving boundaries for (OC) can be only conjectured, due to the high generality of our model and the complex structure of (OS_y). For such open problem, by providing some considerations about the shape of the action and inaction regions and constructing a candidate optimal control process, we aim at doing the groundwork for future research whose first scope will be establishing some suitable conditions to confirm the actual optimality of such control and to characterise the associated value in terms of the optimal reward of (OS_y).

3.1 A family of optimal stopping problems (OS_y)

Let us consider the parametric family of one-dimensional optimal stopping problems: given $y \in (\underline{y}, \overline{y}]$, find $\tau_y^* = \tau_y^*(x, y) \geq 0$ such that

$$\mathbb{E}[e^{-r\tau_y^*} g(X^x_{\tau_y^*}, y)] = \sup_{\tau \geq 0} \mathbb{E}[e^{-r\tau} g(X^x_\tau, y)], \tag{OS_y}$$

where the supremum is taken over all the $\mathcal{F}$-stopping times τ and the *running reward* function $g : \mathbb{R} \times (\underline{y}, \overline{y}] \to \mathbb{R}$ is defined as

$$g(x, y) := xF(y) + [x]^+ L(y) + \frac{1}{r} h(y), \tag{3.1}$$

recalling that $F(y) = \int_{\underline{y}}^{y} f(z)dz$ and defining $L(y) := \int_{\underline{y}}^{y} f'(z)(z - \underline{y})dz$.

When τ_y^* there exists, we call it an *optimal stopping time* for (OS_y) and we define

$$u(x, y) := \mathbb{E}[e^{-r\tau_y^*} g(X^x_{\tau_y^*}, y)] \tag{3.2}$$

as the *optimal expected reward* associated with (OS_y) and corresponding to τ_y^*.

Before dealing with the stopping problems (OS_y), we observe that $F(y)$ represents the energy power produced by discharging instantaneously all the available water through a chattering policy. Whereas,

$$\begin{aligned} F(y) + L(y) &= \int_{\underline{y}}^{y} [f(z) + f'(z)(z - \underline{y})]dz = \\ &= \int_{\underline{y}}^{y} [f(z)(z - \underline{y})]' dz = f(y)(y - \underline{y}), \end{aligned} \tag{3.3}$$

is the energy power obtained releasing by instantaneously all the available fuel by means of an impulsive strategy.

Let us introduce the following further assumptions for the cost and productivity functions.

Assumption 3.1. Consider the productivity function f and the holding cost function h such that, for every

$$\frac{h'(y)}{h(y)} < \frac{f(y)}{F(y)}, \qquad y \in (\underline{y}, \overline{y}], \tag{3.4}$$

and

$$\frac{h'(y)}{h(y)} < \frac{f(y) + l(y)}{F(y) + L(y)}, \qquad y \in (\underline{y}, \overline{y}], \tag{3.5}$$

where $l(y) := L'(y) = f'(y)(y - \underline{y})$.

Remark 3.2. We highlight that assumptions (3.4) and (3.5) are equivalent to say that, at each initial amount y, the infinitesimal percentage variation in holding costs is always lower than that related to the amount of energy produced by discharging instantaneously all the available water resource, for both the response mode of the electric production system (chattering or single impulse). We note that these assumption are sufficient to say that the ratios

$$\frac{h(y)}{F(y)} \qquad \text{and} \qquad \frac{h(y)}{F(y)+L(y)}, \tag{3.6}$$

between holding costs and the power produced are both decreasing.

In the following we will consider two different behaviours as the water reserve approaches its minimal physical boundary. We will first assume that the ratio between the instantaneous holding cost and the amount of energy that can be produced instantaneously remains bounded on $[\underline{y}, \overline{y}]$. In particular, we assume that there exists a constant $0 < C \leq ab$ such that

$$\lim_{y \to \underline{y}} \frac{h(y)}{F(y)} = C. \tag{R_b}$$

Afterwards, we will face an alternative situation, in which f and h are such that the amount of energy produced by discharging completely the water reserve vanishes faster then than holding costs, so as both the ratios become infinitely large near $\underline{y}$, i.e.

$$\lim_{y \to \underline{y}} \frac{h(y)}{F(y)} = \lim_{y \to \underline{y}} \frac{h(y)}{F(y)+L(y)} = +\infty, \tag{R_∞}$$

The influence of the above different assumptions on the problem's nature is closely related to the following function $A^-, A^+ : (\underline{y}, \overline{y}] \mapsto \mathbb{R}$, that we define as

$$A^-(y) := \frac{1}{(a+r)}\left[ab - \frac{h(y)}{F(y)}\right] \quad \text{and} \quad A^+(y) := \frac{1}{(a+r)}\left[ab - \frac{h(y)}{F(y)+L(y)}\right]. \tag{3.7}$$

In particular, the following result is a direct consequence of the previous hypotheses.

Lemma 3.3. *Let f and h satisfy Assumptions 1.2, 1.4 and 3.1. Then, the functions A^+ and A^- are both increasing. Moreover,*

(i) if (R_b) holds, then

$$A^+(y) > A^-(y) > 0, \quad \forall y \in (\underline{y}, \overline{y}] \quad \text{and} \quad \lim_{y \to \underline{y}} A^+(y) \geq \lim_{y \to \underline{y}} A^-(y) \geq 0, \tag{3.8}$$

(ii) if (R_∞) holds, then

$$A^+(y) > A^-(y), \quad \forall y \in (\underline{y}, \overline{y}] \quad \text{and} \quad \lim_{y \to \underline{y}} A^-(y) = \lim_{y \to \underline{y}} A^+(y) = -\infty. \tag{3.9}$$

Proof. Considering the derivatives of A^- and A^+ we get

$$\frac{d}{dy}A^-(y) = -\frac{1}{(a+r)}\left[\frac{h'(y)F(y) - h(y)f(y)}{F^2(y)}\right]$$

and

$$\frac{d}{dy}A^+(y) = -\frac{1}{(a+r)}\left[\frac{h'(y)[F(y)+L(y)] - h(y)[f(y)+l(y)]}{[F(y)+L(y)]^2}\right]$$

and since (3.4) and (3.5) hold true for any $y \in (\underline{y}, \overline{y}]$, it follows that A^+ and A^- are both strictly increasing. Hence, (3.8) and (3.9) immediately follow from the definition of A^+ and A^- and from (R_b) and (R_∞), respectively. □

Remark 3.4. Consider condition (R_b). The dependence of the choice for the constant C on the parameters a and b could appears particularly restrictive. As shown by Lemma 3.3, this condition is *sufficient* to guarantee that both A^- and A^+ are positive on $[\underline{y}, \overline{y}]$ and this fact will be crucial to completely solve the problem (OC). Indeed, if one consider $C > ab$, the problem may be traced back to the more challenging context concerning the connection (OC)-(OS_y) under the hypothesis (R_∞).

3.2 A geometric approach

Dealing with the parametrized family of optimal stopping problems defined in the previous section is particularly non-trivial since, as the parameter y changes, the nature of the solutions of the associated optimal stopping problem varies significantly.

We solve the optimal stopping problem using a geometric approach inspired by the results of Dayanik and Karatzas [DK03]. In particular, for any problem (OS_y), we construct the associated optimal reward function $u(\cdot, y)$ by considering the smallest non-negative concave majorant of a certain function $H(\cdot, y)$ that we properly define below.

Now we show some properties of the optimal reward function $u(\cdot, y)$ that can be deduced directly from the analytic properties of $g(\cdot, y)$ and from the features of X. Moreover, in order to set the geometric method we resort to some results proved in [DK03] and listed in Appendix C, adapting their notations to our parametrised optimal stopping problem.

Proposition 3.5. *Given $y \in (\underline{y}, \overline{y}]$, we have $u(x, y) < +\infty$ for all $x \in \mathbb{R}$. Moreover, the function $x \mapsto u(x, y)$ is convex on $\mathbb{R}$.*

Proof. In order to prove that the optimal reward function is finite it suffices to show that hypotheses of Proposition 5.10 in [DK03] (see Proposition C.1 in Appendix C) are satisfied, i.e. it is sufficient to show that, for any $y \in (\underline{y}, \overline{y}]$, both the limits

$$l^y_{-\infty} := \lim_{x \to -\infty} \frac{[g(x,y)]^+}{\varphi(x)}, \qquad l^y_{+\infty} := \lim_{x \to +\infty} \frac{[g(x,y)]^+}{\psi(x)} \tag{3.10}$$

are finite.

Fixed $y \in (\underline{y}, \overline{y}]$, we observe that $[g(x,y)]^+$ grows linearly when x increases to $+\infty$ and also that it is identically zero for $x \leq -\frac{h(y)}{rF(y)}$. Moreover, thanks to the asymptotic estimates for parabolic cylinder functions stated in [GST06], we obtain that

$$\psi(x) \approx e^{\frac{x^2}{2}} \left[\frac{\sqrt{2\pi}}{\Gamma(r/a)} x^{\frac{r}{a}-1} + \cos\left(\frac{r\pi}{a}\right) e^{-\frac{x^2}{2}} x^{-\frac{r}{a}} \right] (1 + O(x^{-2})), \tag{3.11}$$

for $x \to +\infty$ and

$$\varphi(x) \approx e^{\frac{x^2}{2}} \left[\frac{\sqrt{2\pi}}{\Gamma(r/a)} (-x)^{\frac{r}{a}-1} + \cos\left(\frac{r\pi}{a}\right) e^{-\frac{x^2}{2}} (-x)^{-\frac{r}{a}} \right] (1 + O((-x)^{-2})), \tag{3.12}$$

when $x \to -\infty$. Hence, $\psi(x)$ and $\varphi(x)$ increase exponentially to $+\infty$ when x approaches respectively $\pm\infty$ and we obtain that

$$l^y_{-\infty} = l^y_{+\infty} = 0. \tag{3.13}$$

Furthermore, for any fixed y, the function $g(\cdot, y)$ is convex since it is represented by the union of two straight lines joined at origin with different slopes. In particular the slope of $g(\cdot, y)$ on $(-\infty, 0]$ is lower than its slope on the positive axis. Hence, convexity of the optimal reward $u(\cdot, y)$ follows immediately from convexity of running reward $g(\cdot, y)$. Indeed, if $x_1, x_2 \in \mathbb{R}$, $\lambda \in (0,1)$ and $x_\lambda = \lambda x_1 + (1-\lambda)x_2$, we recall that $X^{x_\lambda}_t = \lambda X^{x_1}_t + (1-\lambda)X^{x_2}_t$ and, if $\tau \geq 0$ is an arbitrary stopping time, we get

$$\mathbb{E}[e^{-r\tau} g(X^{x_\lambda}_t, y)] \leq \lambda \mathbb{E}[e^{-r\tau} g(X^{x_1}_t, y)] + (1-\lambda)\mathbb{E}[e^{-r\tau} g(X^{x_2}_t, y)] \tag{3.14}$$

$$\leq \lambda u(x_1, y) + (1-\lambda) u(x_2, y). \tag{3.15}$$

We obtain the convexity property of the optimal reward $u(\cdot, y)$ just taking the supremum over all the non-negative stropping time τ. □

Let us now consider the function

$$G(x) := -\frac{\varphi(x)}{\psi(x)}, \quad x \in \mathbb{R}. \tag{3.16}$$

In light of the properties of ψ and φ, the function G is negative, continuously differentiable and strictly increasing on $\mathbb{R}$. As a consequence, its inverse function G^{-1} is well-defined, differentiable and increasing. Hence, if we consider the function

$$H(z,y) := \begin{cases} \frac{g(G^{-1}(z),y)}{\psi(G^{-1}(z))} & z < 0, \\ 0, & z = 0, \end{cases} \tag{3.17}$$

we can restate Proposition 5.12 of [DK03] (see Proposition C.2 in Appendix C) as follows.

Proposition 3.6. *Fixed $y \in (\underline{y}, \overline{y}]$, let $W(\cdot, y)$ be the smallest non-negative concave majorant of $H(\cdot, y)$ on $(-\infty, 0]$. Then the optimal reward function for (OS$_y$) is given by*

$$u(x,y) = \psi(x) W(G(x), y), \quad x \in \mathbb{R}. \tag{3.18}$$

Furthermore, $W(0,y) = l^y(+\infty) = 0$ and $W(\cdot, y)$ is continuous at 0.

Moreover, if we consider the *continuation* and the *stopping* regions for each optimal stopping problem (OS$_y$), defined as

$$\mathcal{C}_y = \{x \in \mathbb{R} : u(x,y) > g(x,y)\} \quad \text{and} \quad \mathcal{S}_y = \{x \in \mathbb{R} : u(x,y) = g(x,y)\}, \tag{3.19}$$

respectively, the following result provides that the stopping region for the optimal stopping problem (OS$_y$) is the inverse image under G of the set of contact points of W and H.

Proposition 3.7. *Fixed $y \in (\underline{y}, \overline{y}]$, we have $\mathcal{S}_y = G^{-1}(\tilde{\mathcal{S}}_y)$, where $\tilde{\mathcal{S}}_y := \{z \in (-\infty, 0] : W(z,y) = H(z,y)\}$ and the stopping time*

$$\tau^*_y(x) := \inf\{t \geq 0 : X^x_t \in \mathcal{S}_y\} \tag{3.20}$$

is optimal for the problem (OS$_y$).

Proof. The result follows immediately from Remark 5.2 and Proposition 5.13 of [DK03] (see Remark C.3 and Proposition C.4 in Apprendix C) since, for any $y \in (\underline{y}, \overline{y}]$, the running reward $g(\cdot, y)$ is continuous and (3.13) holds true. □

Thanks to the previous result, solving the optimal stopping problem is equivalent to finding the smallest non-negative concave majorant of $H(\cdot, y)$. In literature there are many applications of the geometric approach of [DK03]. In the paper itself the authors face with some optimal stopping problems showing the equivalence between the results obtained by their theory and those previously achieved by means of other techniques. Another example is given in [DFM18]. The authors study the problem mentioned at the beginning of the chapter, exploiting the geometric approach to solve problem of optimal stopping for a Brownian motion, for which the inverse of the auxiliary function G has a closed-form expression.

Contrary to the above examples, in our problem the inverse function G^{-1} cannot be explicitly calculated and finding the solutions to (OS_y) turns out to be more challenging. We have to exploit the (implicit) properties of both running reward $g(\cdot, y)$ and function G in order to find out the analytics features of $H(\cdot, y)$. In particular, fixed $y \in (\underline{y}, \overline{y}]$, if $x \neq 0$ and $z = G(x)$, then $g(\cdot, y)$ is twice-continuously differentiable at x, $H(\cdot, y)$ is twice-continuously differentiable at z and, as calculated in Appendix D.1, we have

$$H_z(z, y) = \frac{1}{G'(x)} \cdot \left[\left(\frac{g}{\psi} \right)_x (x, y) \right] \tag{3.21}$$

and

$$H_{zz}(z, y) = \frac{1}{G'(x)} \cdot \frac{2\psi(x)}{\sigma^2 W(\psi, \varphi) S'(x)} \cdot [\mathcal{L} - r] g(x, y), \tag{3.22}$$

respectively. The function $S'(\cdot)$ is the density of the scale function and the constant $W(\psi, \varphi)$ is the Wronskian associated with the diffusion X. Since G is increasing and $\psi(\cdot)$, $S'(\cdot)$ and $W(\psi, \varphi)$ are positive, it follows that

$$H_z(z, y) \cdot \left(\frac{g}{\psi} \right)_x (x, y) \geq 0 \quad \text{and} \quad H_{zz}(z, y) \cdot [\mathcal{L} - r] g(x, y) \geq 0 \tag{3.23}$$

whether $z = G(x)$.

The geometry of H is closely related to the values of A^+ and A^- and, in order to understand completely the properties of $H(\cdot, y)$ and to identify the structure of its non-negative concave majorant $W(\cdot, y)$, we repeatedly resort to $(3.23)_1$ and $(3.23)_2$. In particular, fixed $y \in (\underline{y}, \overline{y}]$, let us consider the function

$$\Phi(x, y) := \left(\frac{g}{\psi} \right)_x (x, y) = \frac{g_x(x, y)\psi(x) - g(x, y)\psi'(x)}{\psi^2(x)}, \tag{3.24}$$

and notice that $\Phi(\cdot, y)$ is continuously differentiable everywhere except at 0, where Φ has a jump discontinuity. Obviously, the sign of $\Phi(\cdot, y)$ depends on the sign of its numerator

$$N\Phi(x,y) := \begin{cases} F(y)\psi(x) - [F(y)x + \frac{h(y)}{r}]\psi'(x), & x<0, \\ (F(y)+L(y))\psi(x) - [(F(y)+L(y))x + \frac{h(y)}{r}]\psi'(x), & x>0, \end{cases} \tag{3.25}$$

and it is positive for $x \leq -\frac{h(y)}{rF(y)}$, since $g(x,y)$ is non-positive for such values of x and ψ is positive and increasing on $\mathbb{R}$. In addition, we also notice that $N\Phi(\cdot,y)$ is decreasing for $x > -\frac{h(y)}{rf(y)}$, since its first derivative is given by

$$(N\Phi)_x(x,y) := \begin{cases} -[F(y)x + \frac{h(y)}{r}]\psi''(x), & x<0, \\ -[(F(y)+L(y))x + \frac{h(y)}{r}]\psi''(x), & x>0, \end{cases} \tag{3.26}$$

and $\psi'' > 0$ on $\mathbb{R}$.

Moreover, adding to the asymptotic approximation for ψ given by (3.11), the approximation for its first derivative

$$\psi'(x) \approx e^{\frac{x^2}{2}}\left[\frac{\sqrt{2\pi}}{\Gamma(\frac{r}{a}+1)}x^{\frac{r}{a}} + \cos\left(\frac{(r+a)\pi}{a}\right)e^{-\frac{x^2}{2}}x^{-\frac{r}{a}-1}\right](1+O(x^{-2})), \tag{3.27}$$

we obtain that

$$\lim_{x\to+\infty} N\Phi(x,y) = \lim_{x\to+\infty}(F(y)+L(y))\psi'(x)\left[\frac{\psi(x)}{\psi'(x)} - \left(x + \frac{h(y)}{r(F(y)+L(y))}\right)\right] = -\infty, \tag{3.28}$$

since

$$\lim_{x\to+\infty}\psi'(x) = +\infty \quad \text{and} \quad \lim_{x\to+\infty}\frac{\psi(x)}{\psi'(x)} = 0. \tag{3.29}$$

Therefore, it is clear that the structure of the sets within $H(\cdot,y)$ is increasing/decreasing and the existence of local maxima/minima depends on the sign of $N\Phi(\cdot,y)$ close to the origin 0. Let us consider the right-hand and left-hand limits of $N\Phi(\cdot,y)$ at 0

$$N\Phi(0-,y) := \lim_{x\to0^-} N\Phi(x,y) = F(y)\psi(0) - \frac{h(y)}{r}\psi'(0) \tag{3.30}$$

and

$$N\Phi(0+,y) := \lim_{x\to0^+} N\Phi(x,y) = (F(y)+L(y))\psi(0) - \frac{h(y)}{r}\psi'(0). \tag{3.31}$$

Hence, recalling that ψ solves $[\mathcal{L}-r]\psi(x)=0$ for any $x\in\mathbb{R}$, we have

$$\begin{aligned} N\Phi(0-,y) &= \frac{F(y)}{r}\left[r\psi(0)-\frac{h(y)}{F(y)}\psi'(0)\right]= \\ &= \frac{F(y)}{r}\left[\frac{1}{2}\sigma^2\psi''(0)+ab\psi'(0)-\frac{h(y)}{F(y)}\psi'(0)\right]= \\ &= \frac{F(y)}{r}\left[\frac{1}{2}\sigma^2\psi''(0)+(a+r)A^-(y)\psi'(0)\right]. \end{aligned} \tag{3.32}$$

and also that

$$\begin{aligned} N\Phi(0+,y) &= \frac{F(y)+L(y)}{r}\left[r\psi(0)-\frac{h(y)}{F(y)+L(y)}\psi'(0)\right]= \\ &= \frac{F(y)+L(y)}{r}\left[\frac{1}{2}\sigma^2\psi''(0)+ab\psi'(0)-\frac{h(y)}{F(y)+L(y)}\psi'(0)\right]= \\ &= \frac{F(y)+L(y)}{r}\left[\frac{1}{2}\sigma^2\psi''(0)+(a+r)A^+(y)\psi'(0)\right]. \end{aligned} \tag{3.33}$$

That is, the behaviour of $N(\cdot,y)$ around the origin depends on both A^- and A^+. In the following, by studying different cases, we explicitly show such dependence.

The convexity and concavity sets of $H(\cdot,y)$ can be identified by means of $(3.23)_2$. Indeed, if $x\neq 0$ and $z=G(x)$ we have that $H(\cdot,y)$ is twice-differentiable at z and $H_{zz}(z,y)>0$ if and only if

$$[\mathcal{L}-r]g(x,y)>0. \tag{3.34}$$

By means of immediate calculations, we notice that for $x<0$,

$$\begin{aligned} [\mathcal{L}-r]g(x,y) &= a(b-x)F(y)-r[xF(y)+\frac{1}{r}h(y)]= \\ &= F(y)[ab-(a+r)x]-h(y)= \\ &= (a+r)F(y)[A^-(y)-x], \end{aligned} \tag{3.35}$$

whereas, for $x>0$,

$$\begin{aligned} [\mathcal{L}-r]g(x,y) &= a(b-x)(F(y)+L(y))-r[x(F(y)+L(y))+\frac{1}{r}h(y)]= \\ &= (F(y)+L(y))[ab-(a+r)x]-h(y)= \\ &= (a+r)(F(y)+L(y))[A^+(y)-x], \end{aligned} \tag{3.36}$$

and it follows that $[\mathcal{L}-r]g(x,y)>0$ if and only if $x<\min\{A^-(y),0\}$ or $0<x<\max\{A^+(y),0\}$. Moreover, introducing the following definitions that will be widely used in the sequel

$$z^y_{-k} := G(-k(y)), \quad z_0 := G(0), \quad z^y_{A^-} := G(A^-(y)), \quad z^y_{A^+} := G(A^+(y)), \tag{3.37}$$

we notice that

$$H(z,y) < 0, \quad z < z^y_{-k} \quad \text{and} \quad H(z,y) \geq 0, \quad z \geq z^y_{-k}. \tag{3.38}$$

and we can summarise the information obtained from above calculation as follows:

$$H(\cdot, y) \text{ is strictly convex on } (-\infty, \min\{z^y_{A^-}, z_0\}) \text{ and } (z_0, \max\{z_0, z^y_{A^+}\}) \tag{3.39}$$

$$H(\cdot, y) \text{ is strictly concave on } (\min\{z^y_{A^-}, z_0\}, z_0) \text{ and } (\max\{z_0, z^y_{A^+}\}, 0]. \tag{3.40}$$

In the following sections we will study the optimal stopping (OS$_y$) under the two different hypotheses (R_b) and (R_∞). Moreover, under (R_∞), we will consider a further subdivision on sub-cases and, for each of them, we will describe in detail the analytic properties of the function $H(\cdot, y)$ by exploiting the above calculations.

3.3 The solution of (OS$_y$) under (R_b)

In this section, by means of the geometric approach previously presented, we completely solve the optimal stopping problem (OS$_y$) under the assumption (R_b). Afterwards, resorting to Verification Theorem, we identify the connection between the parametrized family (OS$_y$) and the optimal control problem (OC). In particular, starting from the optimal stopping time for (OS$_y$), we construct the optimal control policy for (OC) and we prove that the associate value function v coincides with

$$w(x,y) = \begin{cases} u(x,y) - \frac{1}{r}h(y), & \text{on } \mathbb{R} \times (\underline{y}, \overline{y}], \\ 0, & \text{on } \mathbb{R} \times \{\underline{y}\} \end{cases} \tag{3.41}$$

Moreover, we show that there exists a unique moving boundary separating the action and inaction regions for (OC) and that it is identified by means of the optimal boundaries for (OS$_y$).

As stated in Lemma 3.3, under the additional hypothesis (R_b), we have that $A^+(y) > A^-(y) > 0$ and, recalling the notation indroduced in (3.37), we have $z_0 < z^y_{A^-} < z^y_{A^+}$. Hence, from (3.39)-(3.40), we obtain that $H(\cdot, y)$ is strictly convex on the intervals $(-\infty, z_0)$ and $(z_0, z^y_{A^+})$ and strictly concave on $(z^y_{A^+}, 0]$.

Recalling the expressions (3.32) and (3.33) for the left-side and right-side limits of $N\Phi(\cdot, y)$ at 0, we notice that

$$N\Phi(0+, y) > N\Phi(0-, y) \geq 0. \tag{3.42}$$

Therefore, $N\Phi(\cdot, y)$ remains positive on $(-\infty, 0)$, the function $\Phi(\cdot, y)$ is positive on the same set and $H(\cdot, y)$ is increasing on $(-\infty, z_0)$. On the other hand, we also have that $N\Phi(\cdot, y)$ is positive within a right-hand neighbourhood of 0 and decreases to $-\infty$ when x approaches $+\infty$. Hence, $H(\cdot, y)$ increases in such right-hand neighbourhood of 0 until it reaches its maximum in a certain point $z^y_{\gamma^+} > z_0$ after which $H(\cdot, y)$ decreases to 0 when z approaches 0. We easily observe that

$$\begin{aligned} N\Phi(A^+(y), y) &= \frac{F(y)+L(y)}{r}\left[r\psi(A^+(y)) - \left(rA^+(y) + \frac{h(y)}{F(y)+L(y)}\right)\psi'(A^+(y))\right] = \\ &= \frac{F(y)+L(y)}{r}\left[r\psi(A^+(y)) - a(b - A^+(y))\psi'(A^+(y))\right] = \\ &= \frac{\sigma^2}{2}\frac{F(y)+L(y)}{r}\psi''(A^+(y)) > 0, \end{aligned} \tag{3.43}$$

and it follows that $z^y_{\gamma^+} > z^y_{A^+}$, since from (3.43) we have $H_z(z^y_{A^+}, y) > 0$. Moreover, we observe that the maximum point $z^y_{\gamma^+}$ is unique, since $H(\cdot, y)$ is strictly concave on $(z^y_{A^+}, 0)$ and, as depicted on Figure 3.1, the smallest non-negative concave majorant $W(\cdot, y)$ for $H(\cdot, y)$ is given by

$$W(z, y) = \begin{cases} H(z^y_{\gamma^+}, y), & z < z^y_{\gamma^+}, \\ H(z, y), & z^y_{\gamma^+} \leq z \leq 0, \end{cases} \tag{3.44}$$

that is, $W(\cdot, y)$ coincides with $H(\cdot, y)$ on $(z^y_{A^+}, 0]$, whereas it is constant and equal to $H(z^y_{\gamma^+}, y)$ on $(-\infty, z^y_{\gamma^+}]$.

In particular, if for any fixed $y \in (\underline{y}, \overline{y}]$ we define

$$\gamma^+(y) := G^{-1}(z^y_{\gamma^+}), \tag{3.45}$$

thanks to Proposition 3.6, we have that the optimal reward for the stopping problem (OS$_y$) is given by

$$u(x, y) = \begin{cases} \frac{g(\gamma^+(y), y)}{\psi(\gamma^+(y))}\psi(x), & x < \gamma^+(y), \\ g(x, y), & x \geq \gamma^+(y). \end{cases} \tag{3.46}$$

and for γ^+ we have the following result,

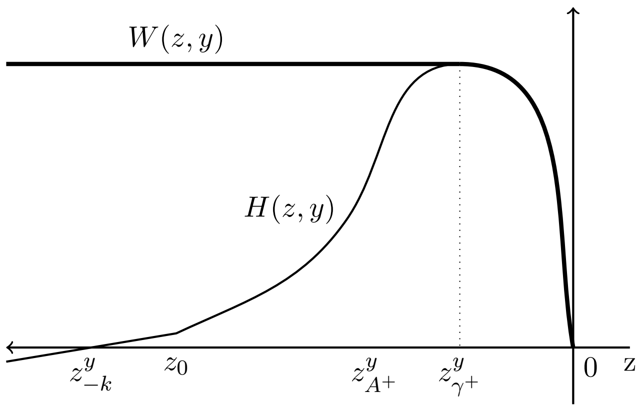

Figure 3.1: Picture of $H(\cdot, y)$ and its concave majorant $W(\cdot, y)$ under (R_b)

Proposition 3.8. *Given $y \in (\underline{y}, \overline{y}]$, $\gamma^+(y)$ is the unique solution to the fixed point problem,*

$$\frac{\psi(x)}{\psi'(x)} - x = \frac{h(y)}{r(F(y)+L(y))}, \qquad x \geq A^+(y). \tag{FP$^y_+$}$$

Moreover, the function $y \mapsto \gamma^+(y)$ is positive, increasing and belongs to $C^1((\underline{y}, \overline{y}))$.

Proof. First of all, we observe that for any $(x, y) \in (0, +\infty) \times (\underline{y}, \overline{y}]$ we have

$$N\Phi(x, y) = (F(y) + L(y))\psi(x) - [(F(y) + L(y))x + \frac{1}{r}h(y)]\psi'(x), \tag{3.47}$$

it is continuously differentiable with respect to both its variables. Morever, fixed $y \in (\underline{y}, \overline{y}]$, as argued in previous calculations, we have that $\gamma^+(y)$ is the unique solution of $N\Phi(x, y) = 0$, i.e.

$$\begin{aligned} N\Phi(\gamma^+(y), y) &= (F(y) + L(y))\left[\psi(\gamma^+(y)) - \left(\gamma^+(y) + \frac{h(y)}{r(F(y)+L(y))}\right)\psi'(\gamma^+(y))\right] = \\ &= (F(y) + L(y))\psi'(\gamma^+(y))\left[\frac{\psi(\gamma^+(y))}{\psi'(\gamma^+(y))} - \left(\gamma^+(y) + \frac{h(y)}{r(F(y)+L(y))}\right)\right] = 0, \end{aligned}$$

and immediately it follows that $\gamma^+(y)$ solves (FP$^y_+$), since $F + L$ is positive and ψ increasing. Moreover, recalling that $N\Phi(\cdot, y)$ is decreasing, we have that $N\Phi_x(\gamma^+(y), y) < 0$ and thanks to the implicit function theorem, the function $y \mapsto \gamma(y)$ is continuously differentiable on $(\underline{y}, \overline{y})$.
The sign of γ^+ easily follows, since G^{-1} is increasing and $z_0 < z^y_{A^+} < z^y_{\gamma^+}$ for any

$y \in (\underline{y}, \overline{y}]$. Moreover, γ^+ is increasing on $y \in [\underline{y}, \overline{y}]$. Indeed, we have that

$$(\gamma^+)'(y) = -\frac{N\Phi_y(\gamma^+(y), y)}{N\Phi_x(\gamma^+(y), y)} > 0, \tag{3.48}$$

since $N\Phi_x(\gamma^+(y), y) < 0$ and

$$\begin{aligned} N\Phi_y(\gamma^+(y), y) &= (f(y) + l(y))\psi'(\gamma^+(y))\left[\frac{\psi(\gamma^+(y))}{\psi'(\gamma^+(y))} - \left(\gamma^+(y) + \frac{h'(y)}{r(f(y) + l(y))}\right)\right] > \\ &> (f(y) + l(y))\psi'(\gamma^+(y))\left[\frac{\psi(\gamma^+(y))}{\psi'(\gamma^+(y))} - \left(\gamma^+(y) + \frac{h(y)}{r(F(y) + L(y))}\right)\right] = 0, \end{aligned}$$

where the inequality follows from (3.5) in Assumption 3.1. □

Now, thanks to Proposition 3.7, we can identify the stopping region for (OS$_y$) as the inverse image of the contact set of $W(\cdot, y)$, i.e. $\mathcal{S}_y = G^{-1}([z^y_{\gamma^+}, 0]) = [\gamma^+(y), +\infty)$ and the optimal stopping time for (OS$_y$) is given by the first entry time of the process X inside $\mathcal{S}_y$, i.e.

$$\tau^*_y = \inf\{t \geq 0 : X^x_t \geq \gamma^+(y)\}. \tag{3.49}$$

We observe that, due to the recurrence property of the process X, the stopping time $\tau^*_y < \infty$, a.s..

3.3.1 The solution of (OC) under (R_b)

We now show that the solutions of the family of optimal stopping problems (OS$_y$) allow us to identify the solution of the optimal control problem (OC). We start providing the following regularity result for the function w in (3.41).

Proposition 3.9. *The function $w \in C^1(\mathbb{R} \times (\underline{y}, \overline{y})) \cap C(\mathbb{R} \times [\underline{y}, \overline{y}])$ and $w_{xx} \in L^\infty_{loc}(\mathbb{R} \times [\underline{y}, \overline{y}])$. Moreover, there exists a constant $C > 0$ such that*

$$|w(x, y)| \leq C(1 + |x|), \qquad \forall (x, y) \in \mathbb{R} \times [\underline{y}, \overline{y}]. \tag{3.50}$$

Proof. The continuity of w on $\mathbb{R} \times (\underline{y}, \overline{y}]$ follows directly from the continuity of u and h. Moreover, for any $x \geq \gamma^+(y)$ we have that w coincides with $g - \frac{1}{r}h$ and in particular one has

$$\lim_{y \to \underline{y}} w(x, y) = \lim_{y \to \underline{y}} \left[g(x, y) - \frac{1}{r}h(y)\right] = 0. \tag{3.51}$$

On the other hand, for $x < \gamma^+(y), y \in (\underline{y}, \overline{y}]$, we have

$$w(x,y) = \frac{g(\gamma^+(y),y)}{\psi(\gamma^+(y))}\psi(x) - \frac{1}{r}h(y), \tag{3.52}$$

and we obtain

$$-\frac{1}{r}h(y) \leq w(x,y) \leq g(\gamma^+(y),y) - \frac{1}{r}h(y) = \gamma^+(y)(F(y)+L(y)), \tag{3.53}$$

where the right-hand bound follows from the monotonicity of ψ and the left-hand inequality follows since u is non-negative. Hence, letting y converges to $\underline{y}$, we obtain (3.51) and the continuity of w is extended to the whole $\mathbb{R} \times [\underline{y}, \overline{y}]$.

The regularity properties of w follows from regularity of u and h. Indeed, as said above, for any $x \geq \gamma^+(y), y \in (\underline{y}, \overline{y})$, w coincides with $g - \frac{1}{r}h$ and both g and h are therein continuously differentiable.

On the other hand, for $x < \gamma^+(y), y \in (\underline{y}, \overline{y})$, the expression for w is (3.52) and its everywhere differentiable with respect to both its variables, since g is differentiable sufficiently far from the origin, $\gamma^+ \in C^1((\underline{y}, \overline{y}))$ and positive and $\psi \in C^2(\mathbb{R})$.

We notice also that both w_x and w_y are continuous everywhere and also along the boundary $\gamma^+(y)$. Indeed,

$$w_x(x,y) = \begin{cases} \frac{g(\gamma^+(y),y)}{\psi(\gamma^+(y))}\psi'(x), & x < \gamma^+(y), y \in (\underline{y}, \overline{y}] \\ F(y) + L(y), & x \geq \gamma^+(y), y \in (\underline{y}, \overline{y}] \end{cases} \tag{3.54}$$

and the equality

$$\frac{g(\gamma^+(y),y)}{\psi(\gamma^+(y))}\psi'(\gamma^+(y)) = F(y) + L(y),$$

easily follows since $\gamma^+(y)$ solves (FP$^y_+$) and $g(\gamma^+(y)) = (F(y)+L(y))\gamma^+(y) + \frac{1}{r}h(y)$. For the first partial derivatives with respect to y we have

$$w_y(x,y) = x(f(y) + l(y)) + \frac{1}{r}h'(y), \qquad x \geq \gamma^+(y), y \in (\underline{y}, \overline{y}]$$

whereas, for $x < \gamma^+(y), y \in (\underline{y}, \overline{y}]$,

$$w_y(x,y) = \psi(x)\left[\frac{g_x(\gamma^+(y),y)\psi(\gamma^+(y)) - g(\gamma^+(y),y)\psi'(\gamma^+(y)}{\psi^2(\gamma^+(y))}(\gamma^+)'(y) + \frac{g_y(\gamma^+(y),y)}{\psi(\gamma^+(y))}\right],$$

and, again thanks to (FP$^y_+$),

$$\begin{aligned} &g_x(\gamma^+(y),y)\psi(\gamma^+(y)) - g(\gamma^+(y),y)\psi'(\gamma^+(y)) = \\ &= (F(y)+L(y))\psi'(\gamma^+(y))\left[\frac{\psi(\gamma^+(y))}{\psi'(\gamma^+(y))} - \left(\gamma^+(y) + \frac{h(y)}{r(F(y)+L(y))}\right)\right] = 0. \end{aligned}$$

Hence,

$$w_y(x,y) = \psi(x)\frac{g_y(\gamma^+(y),y)}{\psi(\gamma^+(y))} = \frac{\psi(x)}{\psi(\gamma^+(y))}\left[(f(y)+l(y))\gamma^+(x) + \frac{1}{r}h'(y)\right] \tag{3.55}$$

and continuity for w_y at the boundary $\gamma^+(y)$ easily follows.

Furthermore, we note that for any y, the map $x \mapsto w(x,y)$ is linear for $x \geq \gamma^+(y)$, hence $w_{xx} \equiv 0$. On the other hand, we have that the second derivative is

$$w_{xx}(x,y) = \frac{g(\gamma^+(y),y)}{\psi(\gamma^+(y))}\psi''(x), \qquad x < \gamma^+(y),\ y \in (\underline{y}, \overline{y}] \tag{3.56}$$

and it is continuous and bounded on every compact subsets of $\mathbb{R} \times [\underline{y}, \overline{y}]$.

In order to show the sublinear growth (3.50), it suffices noticing that $x \mapsto w(\cdot, y)$ is a straight line for $x \geq \gamma^+(y),\ y \in (\underline{y}, \overline{y}]$. Whereas, if $x < \gamma^+(y),\ y \in (\underline{y}, \overline{y}]$, the chain of inequalities (3.53) holds true. Hence, recalling that γ^+, F, L and h are all continuous on $[\underline{y}, \overline{y}]$, we find a constant $C_1 > 0$ such that $|w(x,y)| \leq C_1$ for all $x < \gamma^+(y),\ y \in (\underline{y}, \overline{y}]$ and (3.50) easily follows for any $(x,y) \in \mathbb{R} \times [\underline{y}, \overline{y}]$. □

The previous proposition guarantees the sufficient regularity to show that w is a solution of (HJB) in the weak sense of Remark 2.11. In particular, we have the following result.

Proposition 3.10. *The function w solves the Hamilton-Jacobi-Bellman equation (HJB). Moreover, $w(x, \underline{y}) = 0$, for any $x \in \mathbb{R}$.*

Proof. Fix $x \geq \gamma^+(y)$ and $y \in (\underline{y}, \overline{y}]$. We have that $w(x,y) = x(F(y)+L(y))$ and recalling the definition of A^+ in (3.7), we obtain

$$\begin{aligned} [\mathcal{L} - r]w(x,y) &= (F(y)+L(y))[\mathcal{L} - r](x) = \\ &= (F(y)+L(y))(a(b-x) - rx) = \\ &= (F(y)+L(y))(ab - (a+r)x) = \\ &= (a+r)(F(y)+L(y))(A^+(y) - x) + h(y) < h(y), \end{aligned}$$

since $x \geq \gamma^+(y) > A^+(y)$ for any $y \in [\underline{y}, \overline{y}]$. It follows that

$$[\mathcal{L} - r]w(x,y) - h(y) < 0, \quad x \geq \gamma^+(y),\ y \in (\underline{y}, \overline{y}]. \tag{3.57}$$

On the other hand we have that $w_y(x,y) = x(f(y) + l(y))$. Hence, recalling that $F'(y) = f(y)$, $L'(y) = l(y) = f'(y)(y - \underline{y})$ and $x \geq \gamma^+(y) > 0$, we obtain

$$w_y(x,y) = x(f(y) + f'(y)(y - \underline{y})) = xf(y) + [x]^+ f'(y)(y - \underline{y}). \tag{3.58}$$

Therefore, from (3.57) and (3.58) we obtain that w satisfies the (HJB), for a.e. $x \geq \gamma^+(y),\ y \in (\underline{y}, \overline{y}]$.

On the other hand, consider $x < \gamma^+(y)$ and $y \in (\underline{y}, \overline{y}]$. We have

$$w(x,y) = \frac{g(\gamma^+(y), y)}{\psi(\gamma^+(y))}\psi(x) - \frac{1}{r}h(y).$$

Recalling that ψ solves $[\mathcal{L} - r]\psi(x) = 0$, we directly obtain that

$$[\mathcal{L} - r]w(x,y) - h(y) = 0. \tag{3.59}$$

Now it remains only to prove that w satisfies

$$w_y(x,y) > xf(y) + [x]^+ f'(y)(y - \underline{y}). \tag{3.60}$$

From (3.55) we recall that

$$w_y(x,y) = \psi(x)\frac{g_y(\gamma^+(y), y)}{\psi(\gamma^+(y))} = \frac{\psi(x)}{\psi(\gamma^+(y))}\left[(f(y) + l(y))\gamma^+(x) + \frac{1}{r}h'(y)\right], \tag{3.61}$$

and it is clear that to prove (3.60), it is sufficient to show that, for any $y \in (\underline{y}, \overline{y}]$, the function $M(\cdot, y)$, defined as

$$M(x,y) = \frac{g_y(x,y)}{\psi(x)} \tag{3.62}$$

is increasing on $(-\infty, \gamma^+(y)]$. Indeed, in this case we would obtain

$$w_y(x,y) > \psi(x)\frac{g_y(x,y)}{\psi(x)} = xf(y) + [x]^+ f'(y)(y - \underline{y}), \tag{3.63}$$

for any $x < \gamma^+(y)$ $y \in (\underline{y}, \overline{y}]$. We notice that $M(\cdot, y)$ is continuous on $\mathbb{R}$ and it is everywhere differentiable except at the origin 0. In particular its derivative

$$M_x(x,y) = \frac{g_{yx}(x,y)\psi(x) - g_y(x,y)\psi'(x)}{\psi^2(x)}, \tag{3.64}$$

has a jump discontinuity at 0. Fixed $y \in (\underline{y}, \overline{y}]$, we consider the numerator of $M_x(\cdot, y)$,

$$N(x,y) := \begin{cases} f(y)\psi(x) - [f(y)x + \frac{h'(y)}{r}]\psi'(x), & x < 0, \\ (f(y)+l(y))\psi(x) - [(f(y)+l(y))x + \frac{h'(y)}{r}]\psi'(x), & x > 0, \end{cases} \tag{3.65}$$

and we observe that it is non-increasing, since its first derivative

$$N_x(x,y) := \begin{cases} -[f(y)x + \frac{h'(y)}{r}]\psi''(x), & x < 0, \\ -[(f(y)+l(y))x + \frac{h'(y)}{r}]\psi''(x), & x > 0, \end{cases} \tag{3.66}$$

is everywhere non-positive. In particular we that $N(x,y) > 0$ for $x \leq -\frac{h'(y)}{rf(y)}$ and also that

$$\begin{aligned} N(0+,y) > N(0-,y) &= \frac{f(y)}{r}\left[r\psi(0) - \frac{h'(y)}{f(y)}\psi'(0)\right] = \\ &= \frac{f(y)}{r}\left[\frac{1}{2}\sigma^2\psi''(0) + ab\psi'(0) - \frac{h'(y)}{f(y)}\psi'(0)\right] = \\ &= \frac{f(y)}{r}\left[\frac{1}{2}\sigma^2\psi''(0) + \left(ab - \frac{h'(y)}{f(y)}\right)\psi'(0)\right], \end{aligned} \tag{3.67}$$

where $N(0+,y)$ and $N(0+,y)$ are the right-hand and left-hand limit of $N(\cdot, y)$ at 0, respectively. We observe that, by virtue of (3.4) in Assumption 3.1, we have

$$\begin{aligned} N(0+,y) > N(0-,y) &> \frac{f(y)}{r}\left[\frac{1}{2}\sigma^2\psi''(0) + \left(ab - \frac{h(y)}{F(y)}\right)\psi'(0)\right] = \\ &= \frac{f(y)}{r}\left[\frac{1}{2}\sigma^2\psi''(0) + (a+r)A^-(y)\psi'(0)\right] > 0. \end{aligned} \tag{3.68}$$

Moreover also $N(\gamma^+(y), y) > 0$. Indeed, since $\gamma^+(y)$ solves (FP$^y_+$), it follows that

$$(F(y)+L(y))\psi(\gamma^+(y)) - \left[(F(y)+L(y))\gamma^+(y) + \frac{h(y)}{r}\right]\psi'(\gamma^+(y)) = 0. \tag{3.69}$$

Differentiating (3.69) with respect to y, we obtain

$$(f(y)+l(y))\psi(\gamma^+(y)) - \left[(f(y)+l(y))\gamma^+(y) + \frac{h'(y)}{r}\right]\psi'(\gamma^+(y))$$
$$-\left[(F(y)+L(y))\gamma^+(y) + \frac{h(y)}{r}\right]\psi''(\gamma^+(y))(\gamma^+)'(y) = 0$$

Hence, recalling the definition of $N(\cdot, y)$, we get

$$\begin{aligned} N(\gamma^+(y), y) &= (f(y)+l(y))\psi(\gamma^+(y)) - \left[(f(y)+l(y))\gamma^+(y) + \frac{h'(y)}{r}\right]\psi'(\gamma^+(y)) = \\ &= \left[(F(y)+L(y))\gamma^+(y) + \frac{h(y)}{r}\right]\psi''(\gamma^+(y))(\gamma^+)'(y) > 0, \end{aligned} \tag{3.70}$$

since all the quantities within the square bracket are positive, $\psi'' > 0$ and $\gamma^+ > 0$ and increasing. Collecting all the information above, we obtain that the numerator $N(x, y)$ is positive for all $x \le \gamma^+(y)$, hence the function $M(x, y)$ is increasing for $x \le \gamma^+(y)$ and (3.60) follows.

Therefore, summing up the expressions (3.57), (3.58), (3.59) and (3.60), we can confirm that the function w solves

$$\max\{[\mathcal{L} - r]w(x,y) - h(y),\ xf(y) + [x]^+ f'(y)(y - \underline{y}) - w_y(x,y)\} = 0, \tag{3.71}$$

for a.e. $(x, y) \in \mathbb{R} \times (\underline{y}, \overline{y}]$. The boundary condition $w(x, \underline{y}) = 0$ is guaranteed by the definition and the continuity of w. □

In conclusion, we provide an optimal control policy and we show that w actually identifies the value function v of the optimal control problem (OC).

Proposition 3.11. *For any $y \in (\underline{y}, \overline{y}]$, recall the stopping time τ_y^* in (3.49) and consider the control process ν^* defined as*

$$\nu_t^* := \begin{cases} 0, & t \le \tau_y^*, \\ y - \underline{y}, & t > \tau_y^*. \end{cases} \tag{3.72}$$

Then $\nu^* \in \mathcal{A}(y)$ is optimal for the control problem (OC) and

$$v(x,y) = w(x,y), \qquad \forall\, (x,y) \in \mathbb{R} \times [\underline{y}, \overline{y}], \tag{3.73}$$

i.e., the function w identifies the value function v associated with (OC).

Proof. The results proved in Proposition 3.9 and Proposition 3.10 provide that the function w in (3.41) solves (HJB) and satisfies the boundary condition $w(x, \underline{y}) = 0$. Moreover, the sublinear growth property guarantees that condition (2.82) in Verification Theorem holds true. Indeed,

$$\lim_{t\to+\infty} \mathbb{E}[e^{-rt}|w(X_t^x, Y_t^y)|] \leq \lim_{t\to+\infty} \mathbb{E}[e^{-rt}C(1+|X_t^x|)] = \mathbb{E}[\lim_{t\to+\infty} e^{-rt}C(1+|X_t^x|)] = 0,$$

since $(X_t^x)_{t\geq 0}$ is uniformly integrable. Therefore, in order to prove that ν^* is optimal for (OC) and also that the associated value function v is identified by w, it is sufficient to show that ν^* satisfies the condition (2.84)-(2.86) of Verification Theorem.

From the definition of ν^* follows that it is admissible and purely discontinuous, indeed $\nu_0^* = 0$, and it increases only at τ_y^*, through a single jump and exerting all the available fuel. To prove condition (2.85), we observe that given $(x, y) \in \mathbb{R} \times (\underline{y}, \overline{y}]$,

$$(X_t^x, Y_t^{y,\nu^*}) = (X_t^x, y), \qquad \forall\, t \leq \tau_y^*. \tag{3.74}$$

Moreover, applying Dynkin's formula[1] we obtain

$$\mathbb{E}[e^{-r\tau_y^*} w(X_{\tau_y^*}, y)] = w(x, y) + \mathbb{E}\left[\int_0^{\tau_y^*} e^{-rt}[\mathcal{L} - r]w(X_t^x, y)dt\right], \tag{3.75}$$

Hence,

$$\mathbb{E}\left[e^{-r\tau_y^*}\left(u(X_{\tau_y^*}, y) - \frac{1}{r}h(y)\right)\right] = u(x, y) - \frac{1}{r}h(y) + \mathbb{E}\left[\int_0^{\tau_y^*} e^{-rt}[\mathcal{L} - r]w(X_t^x, y)dt\right].$$

Rearranging properly the terms involving h, we write

$$\mathbb{E}\left[\frac{1}{r}h(y)(e^{-r\tau_y^*} - 1)\right] = -\mathbb{E}\left[\int_0^{\tau_y^*} e^{-rt}h(y)dt\right] \tag{3.76}$$

and recalling that $u(X_{\tau_y^*}, y) = g(X_{\tau_y^*}, y)$ a.s., we obtain

$$\mathbb{E}[e^{-r\tau_y^*} g(X_{\tau_y^*}, y)] = u(x, y) + \mathbb{E}\left[\int_0^{\tau_y^*} e^{-rt}\{[\mathcal{L} - r]w(X_t^x, y) - h(y)\}dt\right]. \tag{3.77}$$

As shown in Proposition 3.10, w solves (HJB) and in general one has $[\mathcal{L} - r]w(x, y) - h(y) \leq 0$. But, since τ_y^* is optimal for (OS_y) and $u(\cdot, y)$ is the associated optimal

[1]We recall that Dynkin's formula holds since X is positively recurrent and $\mathbb{E}[\tau_y^*] < \infty$.

reward, it necessarily holds

$$[\mathcal{L} - r]w(X_t^x, y) - h(y) = 0, \quad \text{a.s.}, \tag{3.78}$$

for almost every $t \leq \tau_y^*$. In conclusion, noticing also that

$$(X_t^x, Y_t^{y,\nu^*}) = (X_t^x, \underline{y}), \qquad \forall t > \tau_y^*, \tag{3.79}$$

we obtain that $(X_t^x, Y_t^{y,\nu^*}) \in \mathcal{C} \cap (\mathbb{R} \times \{\underline{y}\})$, for almost every $t \geq 0$. Furthermore, we have that the control $\Delta\nu_t^* \neq 0$ if and only if $t = \tau_y^*$ and to prove that (2.86) holds true it suffices to show that

$$w(X_{\tau_y^*}, y) = [X_{\tau_y^*}]^+ f(y)(y - \underline{y}) - [X_{\tau_y^*}]^- \int_0^{y-\underline{y}} f(y-u)du, \quad \text{a.s.} \tag{3.80}$$

This follows easily from the definition of w. Indeed, recalling that $\tau_y^* < \infty$, a.s., it follows that $X_{\tau_y^*}^x = \gamma^+(y) > 0$ and for any $y \in (\underline{y}, \overline{y}]$, we have

$$w(\gamma^+(y), y) = g(\gamma^+(y), y) - \frac{1}{r}h(y) = \gamma^+(y)(F(y) + L(y)), \tag{3.81}$$

and, simply observing that

$$\begin{aligned} F(y) + L(y) &= \int_{\underline{y}}^{y} \{f(u) + f'(u)(u - \underline{y})\}du = \\ &= \int_{\underline{y}}^{y} [f(u)(u - \underline{y})]'du = f(y)(y - \underline{y}), \end{aligned}$$

we obtain (3.80). □

The previous result shows that, given the initial position (x, y), the optimal control policy consists in instantly discharging the entire available water reserve as soon as the price process X^x hits the positive threshold $\gamma^+(y)$. Additionally, the identification between the value function $v(x, y)$ related to such optimal control with $w(x, y)$, implies that v satisfies

$$\begin{cases} [\mathcal{L} - r]v(x, y) - h(y) = 0, & x < \gamma^+(y), y \in (\underline{y}, \overline{y}], \\ xf(y) + [x]^+ f'(y)(y - \underline{y}) - v_y(x, y) < 0, & x < \gamma^+(y), y \in (\underline{y}, \overline{y}], \\ [\mathcal{L} - r]v(x, y) - h(y) < 0, & x \geq \gamma^+(y), y \in (\underline{y}, \overline{y}], \\ xf(y) + [x]^+ f'(y)(y - \underline{y}) - v_y(x, y) = 0, & x \geq \gamma^+(y), y \in (\underline{y}, \overline{y}]. \end{cases} \tag{3.82}$$

Hence, as depicted on Figure 3.2, we obtain that the action (discharging) and inaction region are given by

$$\mathcal{D} = \{(x, y) \in \mathbb{R} \times (\underline{y}, \overline{y}] : x \geq \gamma^+(y)\}, \tag{3.83}$$

$$\mathcal{C} = \{(x, y) \in \mathbb{R} \times (\underline{y}, \overline{y}] : x < \gamma^+(y)\}, \tag{3.84}$$

respectively and the function $y \mapsto \gamma^+(y)$ represents the boundary separating these regions.

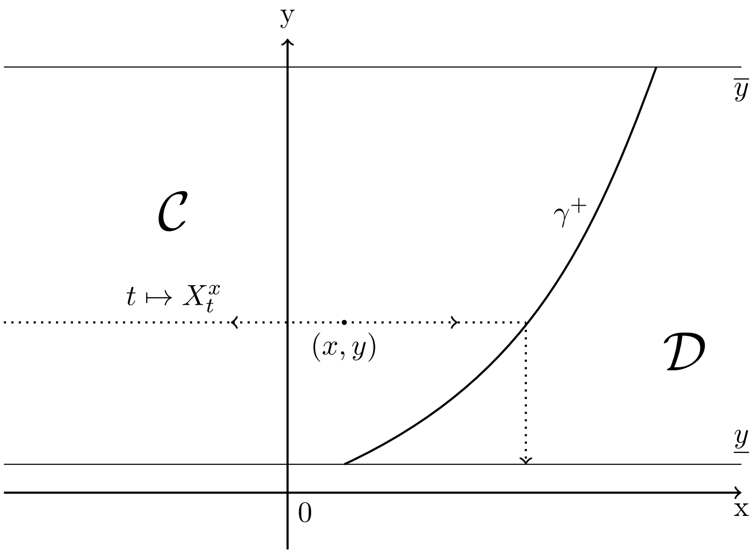

Figure 3.2: An example of optimal boundary for (OC) under (R_b). Starting from (x, y), the process X_t^x diffuses along the horizontal line until reaching $\gamma^+(y)$. At that hitting time, the control $\bar{\nu}$ is exerted by a jump $\Delta\bar{\nu} = y - \underline{y}$, leading the state process $Y_t^{y,\bar{\nu}}$ at the lower bound $\underline{y}$.

3.4 The solution of (OS_y) under (R_∞)

In this section we face the parametrized family of optimal stopping problems (OS_y) assuming that condition (R_∞) is verified, instead of (R_b).

The structure of the solutions of the optimal stopping problems changes drastically according to the values assumed by the parameter y. In particular we show that the connection between the optimal reward $u(\cdot, y)$ and the value function v for optimal control problem (OC) can no longer easily be proved. We highlight that, for certain values of the parameter y, the stopping problem (OS_y) exhibits disconnected stopping intervals $\mathcal{S}_y$ and the identification of the optimal moving boundaries of (OC), starting from those of (OS_y) becomes more challenging.

By means of the geometrical approach, we study case-by-case all the possible geometric configurations for $H(\cdot, y)$ and, taking into account the relevance of $A^+(y)$ and $A^-(y)$, we *graphically* identify its smallest non-negative concave majorant $W(\cdot, y)$. We consider the values $y_1,\ y_2,\ y_*$ and y^* in $(\underline{y}, \overline{y}]$ such that

$$A^+(y_1) = A^-(y_2) = -\frac{\sigma^2}{2(a+r)}\frac{\psi''(0)}{\psi'(0)} \quad \text{and} \quad A^+(y_*) = A^-(y^*) = 0, \tag{3.85}$$

respectively.
It is worth noticing that a priori it is not clear if each of such points there exists. In order to present a general framework as detailed as possible, we can firstly consider the upper level $\overline{y}$ such that

$$\frac{h(\overline{y})}{F(\overline{y})} < ab. \tag{3.86}$$

This condition guarantees that $A^+(\overline{y}) > A^-(\overline{y}) > 0$ and the points defined above exist within $(\underline{y}, \overline{y})$ and, due to the monotonicity of both A^- and A^+, they are unique. In particular and we have that

$$\underline{y} < y_1 < y_* < y_2 < y^* < \overline{y}. \tag{3.87}$$

Any alternative situation can be easily traced back to such more detailed framework.

In the following subsections we distinguish three cases. First of all, we start by solving the optimal stopping problems (OS_y) when these are associated with the initial reserve level y belonging to the overhead range $[y_2, \overline{y}]$. After, we consider the parameter y within an intermediate range (y_1, y_2), highlighting similarities and differences between solutions obtained in this case and those in the previous one. In conclusion we show the peculiarities of the problem when the starting level of the water reserve is in the lower range $(\underline{y}, y_1]$.

3.4.1 Case 1

Let us consider the starting amount of water reserve y within the interval $[y_2, \overline{y}]$. For such values of y, the solution to each optimal stopping problem (OS_y) is analogous to the one obtained under condition (R_b). Indeed, for any fixed $y \in [y_2, \overline{y}]$, we have

$$A^+(y) > A^-(y) \geq -\frac{\sigma^2}{2(a+r)}\frac{\psi''(0)}{\psi(0)}. \tag{3.88}$$

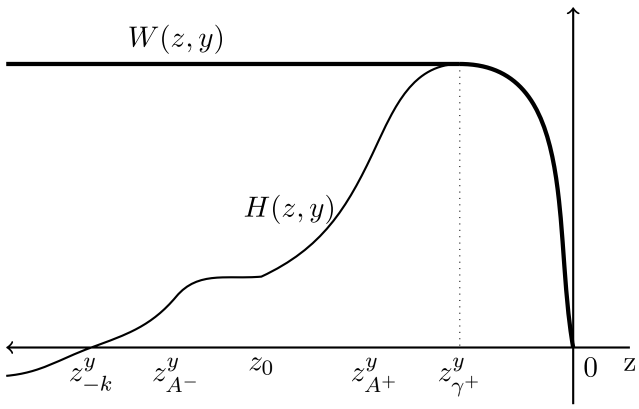

Figure 3.3: Picture of $H(\cdot, y)$ and its concave majorant $W(\cdot, y)$ for $y \in [y_2, y^*)$.

In particular, recalling the definition of y_2 and y^* in (3.85), we get

$$A^-(y) < 0 < A^+(y) \ \text{ if } y \in [y_2, y^*) \quad \text{and} \quad A^+(y) > A^-(y) \geq 0 \ \text{ if } y \in [y^*, \overline{y}], \tag{3.89}$$

and also

$$z^y_{A-} < z_0 < z^y_{A+}, \ \text{ if } y \in [y_2, y^*) \quad \text{and} \quad z_0 \leq z^y_{A-} < z^y_{A+}, \ \text{ if } y \in [y^*, \overline{y}]. \tag{3.90}$$

As when (R_b) holds, (3.88) implies that both $N\Phi(0-, y)$ and $N\Phi(0+, y)$ are positive and there exist a point $z^y_{\gamma+} > z^y_{A+}$ such that $H(\cdot, y)$ is increasing on $(-\infty, z^y_{\gamma+})$ and decreasing on $(z^y_{\gamma+}, 0]$.

Concerning the convexity/concavity intervals, we have two possible configurations. Indeed, from (3.39)-(3.40), if $y \in [y^*, \overline{y}]$ then $H(\cdot, y)$ is strictly convex on the intervals $(-\infty, z_0)$ and (z_0, z^y_{A+}) and strictly concave on $(z^y_{A+}, 0]$. Then, the structure is totally analogous to the situation under (R_b) and it is depicted in Figure 3.1.
Otherwise, if $[y_2, y^*)$ then $H(\cdot, y)$ is strictly convex on $(-\infty, z^y_{A-})$ and (z_0, z^y_{A+}) and strictly concave on (z^y_{A-}, z_0) and $(z^y_{A+}, 0]$. This situation is represented in Figure 3.3. Obviously, both the situations exhibit strict concavity of $H(\cdot, y)$ on $(z^y_{A+}, 0]$ and the maximum point $z^y_{\gamma+}$ is unique.

Hence, for any fixed $y \in [y_2, \overline{y}]$, as depicted on Figure 3.1 and Figure 3.3, the smallest non-negative concave majorant $W(\cdot, y)$ is

$$W(z, y) = \begin{cases} H(z^y_{\gamma+}, y), & z < z^y_{\gamma+}, \\ H(z, y), & z^y_{\gamma+} \leq z \leq 0, \end{cases} \tag{3.91}$$

and both optimal reward function $u(\cdot, y)$ and optimal stopping time τ_y^* are identified by (3.46) and (3.49), respectively. In particular, for any $y \in [y_2, \overline{y}]$, can be identify the boundary point $\gamma^+(y) = G^{-1}(z^y_{\gamma+})$ as the unique solution of (FP^y_+) and $\gamma^+(\cdot)$ is a positive, increasing and once-continuously differentiable function on $(y_2, \overline{y})$.

3.4.2 Case 2

Now we deal with the case $y \in (y_1, y_2)$. For such values of y the quantities $A^-(y)$ and $A^+(y)$ are such that

$$A^-(y) < -\frac{\sigma^2}{2(a+r)}\frac{\psi''(0)}{\psi(0)} < A^+(y). \tag{3.92}$$

In particular, the convexity/concavity sets of $H(\cdot, y)$ are specifically identified by considering the following facts

$$A^-(y) < A^+(y) < 0, \text{ if } y \in (y_1, y_*), \quad \text{and} \quad A^-(y) < 0 < A^+(y) \text{ if } y \in [y_*, y_2), \tag{3.93}$$

and also that

$$z^y_{A-} < z^y_{A+} < z_0 \text{ if } y \in (y_1, y_*) \quad \text{and} \quad z^y_{A-} < z_0 < z^y_{A+} \text{ if } y \in [y_*, y_2). \tag{3.94}$$

Indeed, from (3.39)-(3.40), if $y \in [y_*, y_2)$ then $H(\cdot, y)$ is strictly convex on $(-\infty, z^y_{A-})$ and (z_0, z^y_{A+}) and strictly concave on (z^y_{A-}, z_0) and $(z^y_{A+}, 0]$. Otherwise, if (y_1, y_*) then $H(\cdot, y)$ is strictly convex on $(-\infty, z^y_{A-})$ and and strictly concave on (z^y_{A-}, z_0) and $(z_0, 0]$.
Contrary to the previous case, considering facts in (3.92) and the expressions (3.32) and (3.33), we observe that

$$N\Phi(0-, y) < 0 < N\Phi(0+, y), \tag{3.95}$$

hence $N\Phi(\cdot, y)$ is positive for $x \leq -k(y)$, it decreases and achieves negative value within a left-hand neighbourhood of 0. Additionally, since $N\Phi(0+, y) > 0$, in a right-hand neighbourhood of 0, $N\Phi(\cdot, y)$ is positive and monotonically decreases to $-\infty$ for large value of x. As a consequence, there exist two values $z^y_{\gamma-}$ and $z^y_{\gamma+}$, with $z^y_{\gamma-} < z_0 < z^y_{\gamma+}$, such that $H(\cdot, y)$ is increasing on $(-\infty, z^y_{\gamma-}) \cup (z_0, z^y_{\gamma+})$ and decreasing on $(z^y_{\gamma-}, z_0) \cup (z^y_{\gamma+}, 0)$. It is therefore clear that $H(\cdot, y)$ has two local maxima at $z^y_{\gamma-}$ and $z^y_{\gamma+}$, respectively. Furthermore, through a totally similar argument to the one used in (3.43), we have that $z^y_{\gamma-} > z^y_{A-}$ and $z^y_{\gamma+} > z^y_{A+}$ and these local maximum points are unique since $H(\cdot, y)$ is strictly concave on both (z^y_{A-}, z_0) and $(z^y_{A+}, 0]$. Moreover $\gamma^+(y)$ still solves the fixed point problem (FP^y_+) and for

$$\gamma^-(y) := G^{-1}(z^y_{\gamma-}), \qquad y \in (y_1, y_2), \tag{3.96}$$

holds the following result that is analogous to Proposition 3.8 and can be proved similarly.

Proposition 3.12. *Given $y \in (y_1, y_2)$, $\gamma^-(y)$ is the unique solution to the fixed point problem,*

$$\frac{\psi(x)}{\psi'(x)} - x = \frac{h(y)}{rF(y)}, \qquad A^-(y) \le x < 0. \tag{FP^y_-}$$

Moreover, the function $y \mapsto \gamma^-(y)$ is negative, increasing and belongs to $C^1((y_1, y_2))$.

The specific geometry of the concave majorant $W(\cdot, y)$ depends on the values of the local maxima of $H(\cdot, y)$. Indeed, as depicted in Figure 3.4a and Figure 3.5a, when the local maximum $H(z^y_{\gamma+}, y)$ is greater than the local maximum $H(z^y_{\gamma-}, y)$, it is clear that the function $W(\cdot, y)$ still maintains the same geometric structure of cases previously studied. In particular, $W(\cdot, y)$ coincides with $H(\cdot, y)$ on $(z^y_{\gamma+}, 0]$ and it is constant and equal to $H(z^y_{\gamma+}, y)$ for $z < z^y_{\gamma+}$. It follows that the optimal reward function $u(\cdot, y)$ is still represented by (3.46) and the optimal stopping τ^*_y is given by (3.49).

Alternatively, if $y \in (y_1, y_2)$ is such that $H(z^y_{\gamma-}, y) \ge H(z^y_{\gamma+}, y)$, the shape of the smallest non-negative concave majorant $W(\cdot, y)$ changes significantly. Indeed, if for any fixed parameter $y \in (y_1, y_2)$ we define

$$L^\xi(z, y) := H(\xi, y) + H_z(\xi, y)(z - \xi), \qquad z \le 0, \tag{3.97}$$

as the tangent line of $H(\cdot, y)$ at $\xi \in (-\infty, z_0) \cup (z_0, 0]$, then there exist $z^y_{\gamma_1} \in (z^y_{\gamma-}, z_0)$ and $z^y_{\gamma_2} \in (z^y_{\gamma+}, 0]$ such that the straight line $L^{z^y_{\gamma_1}}(z, y) := H(z^y_{\gamma_1}, y) + H_z(z^y_{\gamma_1}, y)(z - z^y_{\gamma_1})$ is tangent to $H(\cdot, y)$ at both $z^y_{\gamma_1}$ and $z^y_{\gamma_2}$ (cf. Figure 3.6a and Figure 3.7a). In particular, the pair $(z^y_{\gamma_1}, z^y_{\gamma_2})$ solves the equations system

$$\begin{cases} \frac{H(z_2,y)-H(z_1,y)}{z_2-z_1} = H_z(z_2, y), \\ \frac{H(z_2,y)-H(z_1,y)}{z_2-z_1} = H_z(z_2, y), \end{cases} \tag{3.98}$$

with $z_1 \in (z^y_{A-}, z_0)$ and $z_2 \in (z^y_{A+}, 0]$. Moreover, due to the strict concavity of $H(\cdot, y)$ on both (z^y_{A-}, z_0) and $(z^y_{A+}, 0]$, $z^y_{\gamma_1}$ and $z^y_{\gamma_2}$ are unique on the sub-intervals within these

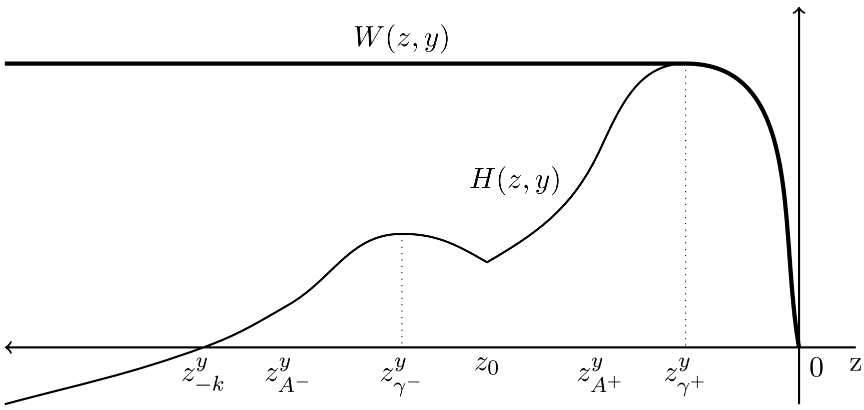

Figure 3.4a: Picture of $H(\cdot,y)$ and its concave majorant $W(\cdot,y)$ when $y \in [y_*, y_2)$ is such that $H(z^y_{\gamma^-}, y) < H(z^y_{\gamma^+}, y)$.

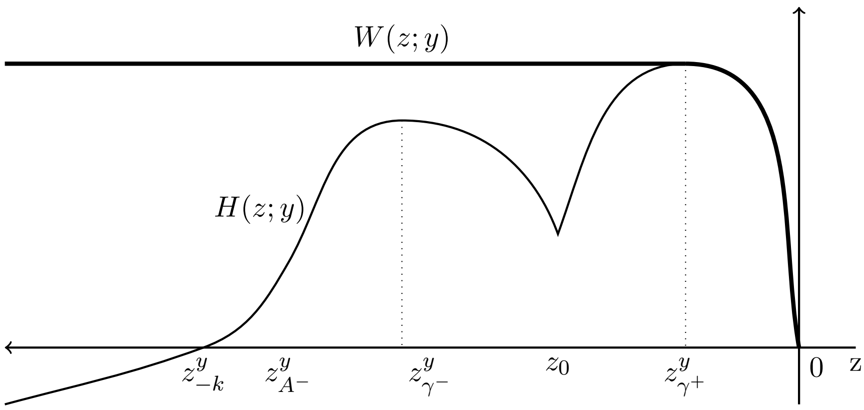

Figure 3.5a: Picture of $H(\cdot,y)$ and its concave majorant $W(\cdot,y)$ when $y \in (y_1, y_*)$ is such that $H(z^y_{\gamma^-}, y) < H(z^y_{\gamma^+}, y)$.

points are respectively identified.
Therefore, as depicted in Figure 3.6a and Figure 3.7a, the smallest non-negative concave majorant for $H(\cdot, y)$ is given by

$$W(z,y) = \begin{cases} H(z^y_{\gamma^-}, y), & z < z^y_{\gamma^-}, \\ H(z,y), & z^y_{\gamma^-} \le z \le z^y_{\gamma_1}, \\ L^{z^y_{\gamma_1}}(z,y), & z^y_{\gamma_1} < z < z^y_{\gamma_2}, \\ H(z,y), & z^y_{\gamma_2} \le z \le 0, \end{cases} \tag{3.99}$$

i.e., $W(\cdot, y)$ coincides with $H(\cdot, y)$ on $[z^y_{\gamma^-}, z^y_{\gamma_1}] \cup [z^y_{\gamma^+}, 0]$, it is equal to the tangent line segment $L^{z^y_{\gamma_1}}(z, y)$ on $(z^y_{\gamma_1}, z^y_{\gamma_2})$ and it is constant on $(-\infty, z^y_{\gamma^-})$. Now, if we rewrite the equation of the straight line $L^{z^y_{\gamma_1}}(z, y)$ by considering the tangency condition (3.98), we obtain

$$L^{z^y_{\gamma_1}}(z,y) = H(z^y_{\gamma_2}, y)\frac{z - z^y_{\gamma_1}}{z^y_{\gamma_2} - z^y_{\gamma_1}} + H(z^y_{\gamma_1}, y)\frac{z^y_{\gamma_2} - z}{z^y_{\gamma_2} - z^y_{\gamma_1}}. \tag{3.100}$$

Defining

$$\gamma_1(y) := G^{-1}(z^y_{\gamma_1}), \quad \text{and} \quad \gamma_2(y) := G^{-1}(z^y_{\gamma_2}), \tag{3.101}$$

thanks to Proposition 3.6, for any fixed $y \in (y_1, y_2)$ such that $H(z^y_{\gamma^-}, y) \ge H(z^y_{\gamma^+}, y)$, we get that the optimal reward for the stopping problem (OS$_y$) is $u(\cdot, y) \equiv \tilde{u}(\cdot, y)$, where $\tilde{u}(\cdot, y)$ is defined by

$$\tilde{u}(x,y) = \begin{cases} \frac{g(\gamma^-(y),y)}{\psi(\gamma^-(y))}\psi(x), & x < \gamma^-(y), \\ g(x,y), & \gamma^-(y) \le x \le \gamma_1(y), \\ g(\gamma_1(y),y)\frac{\Lambda_1(\gamma_2(y),x)}{\Lambda_1(\gamma_2(y),\gamma_1(y))} + g(\gamma_2(y),y)\frac{\Lambda_1(x,\gamma_1(y))}{\Lambda_1(\gamma_2(y),\gamma_1(y))}, & \gamma_1(y) < x < \gamma_2(y), \\ g(x,y), & x \ge \gamma_2(y), \end{cases} \tag{3.102}$$

where the function $\Lambda_1, \ \Lambda_2 : \mathbb{R}^2 \to \mathbb{R}$ are defined as

$$\Lambda_1(\xi,\eta) = \psi(\xi)\varphi(\eta) - \psi(\eta)\varphi(\xi), \quad \text{and} \quad \Lambda_2(\xi,\eta) = \psi'(\xi)\varphi(\eta) - \psi(\eta)\varphi'(\xi). \tag{3.103}$$

Moreover, thanks to Proposition 3.7, the stopping region is given by

$$\mathcal{S}_y = G^{-1}([z^y_{\gamma^-}, z^y_{\gamma_1}] \cup [z^y_{\gamma_2}, 0]) = [\gamma^-(y), \gamma_1(y)] \cup [\gamma_2(y), +\infty), \tag{3.104}$$

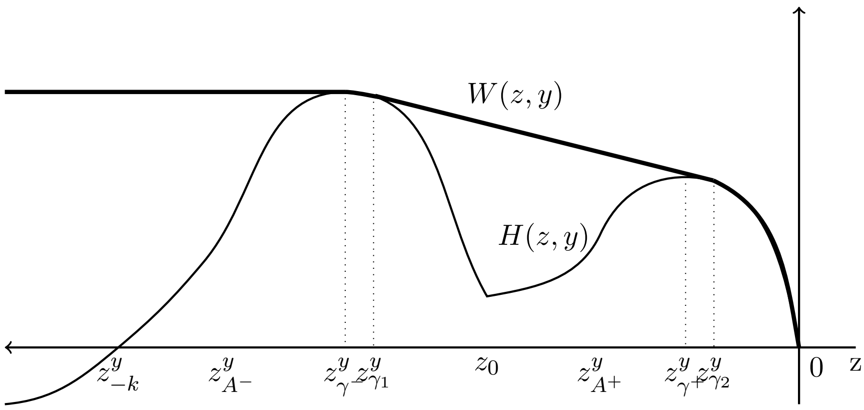

Figure 3.6a: Picture of $H(\cdot, y)$ and its concave majorant $W(\cdot, y)$ for $y \in [y_*, y_2)$ such that $H(z^y_{\gamma^-}, y) \geq H(z^y_{\gamma^+}, y)$

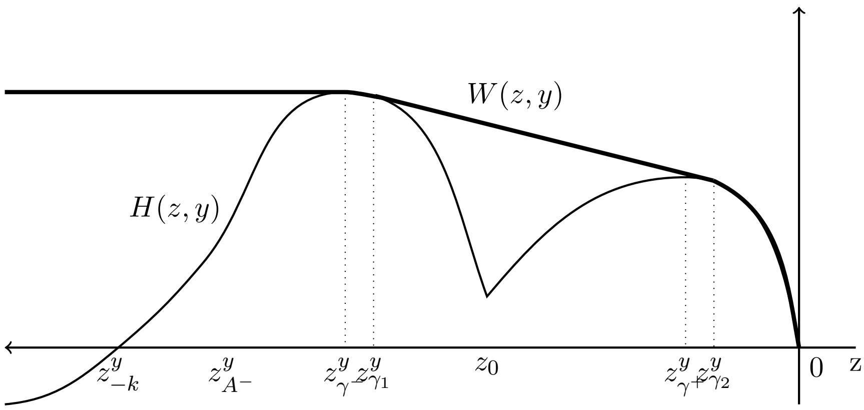

Figure 3.7a: Picture of $H(\cdot, y)$ and its concave majorant $W(\cdot, y)$ for $y \in (y_1, y_*)$ such that $H(z^y_{\gamma^-}, y) \geq H(z^y_{\gamma^+}, y)$.

and, if we consider the stopping times

$$\tau_y^2 := \inf\{t \geq 0 : X_t^x \geq \gamma_2(y)\} \quad \text{and} \quad \tau_y^- := \inf\{t \geq 0 : X_t^x \in [\gamma^-(y), \gamma_1(y)]\}, \tag{3.105}$$

then the first entry time of the process X inside $\mathcal{S}_y$, i.e.

$$\sigma_y^* := \tau_y^2 \wedge \tau_y^-, \tag{3.106}$$

is optimal for stopping problem (OS$_y$).

3.4.3 Case 3

In this subsection we investigate the last case, considering the starting water reserve within the lowest range $y \in (\underline{y}, y_1]$. For such values of y we have,

$$A^-(y) < A^+(y) \leq -\frac{\sigma^2}{2(a+r)} \frac{\psi''(0)}{\psi(0)}. \tag{3.107}$$

In particular, $A^-(y)$ and $A^+(y)$ are both negative and $z^y_{A-} < z^y_{A+} < z_0$ holds. In light of Lemma (3.39)-(3.40), $H(\cdot, y)$ is strictly convex on $(-\infty, z^y_{A-})$ and strictly concave on the intervals (z^y_{A-}, z_0) and $(z_0, 0]$. Moreover, since (3.107) holds, we have that

$$N\Phi(0-, y) < N\Phi(0+, y) \leq 0, \tag{3.108}$$

hence $N\Phi(\cdot, y)$ is positive for $x \leq -k(y)$, it decreases and achieves negative value within a sufficiently small left-hand neighbourhood of 0. In addition, contrary to the cases studied in previous sections, also $N\Phi(0+, y) < 0$ and in a right-hand neighbourhood of 0 we have that $N\Phi(\cdot, y)$ is negative and monotonically decreases to $-\infty$ for large value of x. Therefore, there exists $z^y_{\gamma^-} < z_0$ such $H(\cdot, y)$ is increasing on $(-\infty, z^y_{\gamma^-})$ and decreasing on $(z^y_{\gamma^-}, 0]$. Furthermore, as for the other cases, we can show that the local maximum point $z^y_{\gamma^-} < z_0$ is such that $z^y_{A-} < z^y_{\gamma^-} < z_0$ and it is also the unique, since the $H(\cdot, y)$ is strictly concave on (z^y_{A-}, z_0).

As shown in Figure 3.8, even though $H(\cdot, y)$ is concave on both the intervals (z^y_{A-}, z_0) and $(z_0, 0]$, this property is not preserved on the union of these sets. Indeed, given $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, we always have that $H_z(z_0 - \varepsilon_1, y) < H_z(z_0 + \varepsilon_2, y) < 0$, hence the slopes of tangent lines at the point on the left-hand side of z_0 is steeper than ones of the tangent lines on the right-side and a "convex gap" clearly occurs around z_0. Hence, in order to define the concave majorant $W(\cdot, y)$ we have to remove this

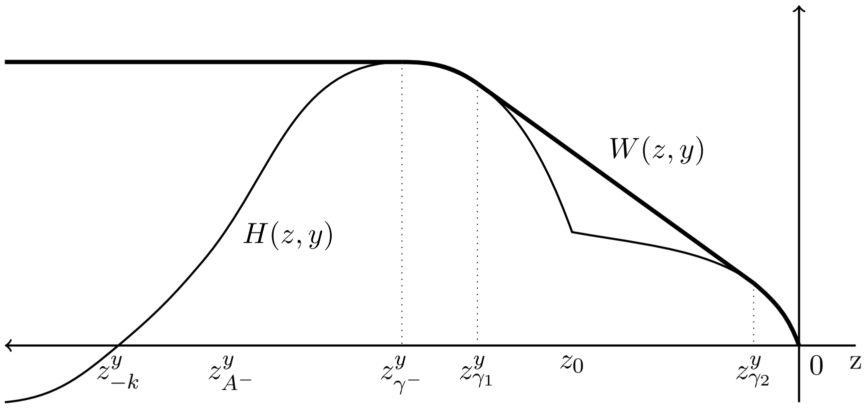

Figure 3.8: Picture of $H(\cdot, y)$ and its concave majorant $W(\cdot, y)$ for $y \in (\underline{y}, y_1]$.

gap again identifying a unique $z^y_{\gamma_1} \in (z^y_{A^-}, z_0)$ and a unique $z^y_{\gamma_2} \in (z_0, 0]$ such that the straight line connecting $(z^y_{\gamma_1}, H(z^y_{\gamma_1}, y))$ and $(z^y_{\gamma_2}, H(z^y_{\gamma_2}, y))$ is tangent to $H(\cdot, y)$ at both $z^y_{\gamma_1}$ and $z^y_{\gamma_2}$. Therefore, as depicted in Figure (3.8), the majorant $W(\cdot, y)$ has the same geometrical structure of latter situation in Case 2. Moreover, for any fixed $y \in (\underline{y}, y_1]$, the optimal stopping time for (OS_y) is given by σ^*_y defined in (3.106) and the associated optimal reward function is $\tilde{u}(\cdot, y)$ in (3.102).

3.5 A discussion on the solution of (OC) under (R_∞)

In previous section, for any $y \in (\underline{y}, \overline{y}]$, exploiting the geometric approach, we found out the solution to each corresponding optimal stopping problem (OS_y), when hypothesis (R_∞) holds, i.e. when we assume that the ratio between the instantaneous holding costs and the power produced by instantaneous water reserve depletion becomes infinitely large close to $\underline{y}$.

We essentially depicted two different structures for the solutions to (OS_y). When the parameter y is sufficiently far from $\underline{y}$, we always identify a unique maximum point for $H(\cdot, y)$ and, by means of the inverse image G^{-1} of the contact set $\{z \leq 0 : H(z, y) = W(z, y)\}$, we are able to recognise a single connected (unbounded above) stopping region $\mathcal{S}_y$. In particular, as when (R_b) holds, we characterise the left-hand *positive* boundary point $\gamma^+(y)$ as the unique solution of the fixed point problem (FP^y_+) and we find out a well-defined continuously differentiable increasing function $y \mapsto \gamma^+(y)$.

Conversely, when $y < y_2$, there exist two local maximum points $z^y_{\gamma^-}$ and $z^y_{\gamma^+}$ for $H(\cdot, y)$ and, depending on the values of $H(\cdot, y)$ at these points, the solutions to (OS_y) may exhibit disconnected stopping regions. In particular, for y such that $H(z^y_{\gamma^-}, y) \geq$

$H(z^y_{\gamma^+}, y)$, we discover the existence of three values $\gamma^-(y) < \gamma_1(y) < 0$ and $\gamma_2(y) > 0$ such that

$$\mathcal{S}_y = [\gamma^-(y), \gamma_1(y)] \cup [\gamma_2(y), +\infty), \tag{3.109}$$

i.e. the stopping region associated with (OS_y) is identified as the union of two disjoint connected components.

The first problem that naturally arise is the identification of the range of parameters y within the stopping set of the corresponding problem (OS_y) is disconnected. We showed that for any $y \in (\underline{y}, y_1]$, due to the "convexity gap" depicted on Figure 3.8, we always obtain the existence of two disjoint connected components for $\mathcal{S}_y$ and $\mathcal{C}_y$, respectively. Moreover, we observe that

$$\lim_{y \to y_2^-} N\Phi(0-, y) = \lim_{y \to y_2^-} \frac{F(y)}{r}\left[\frac{1}{2}\sigma^2 \psi''(0) + (a+r)A^-(y)\psi'(0)\right] = 0, \tag{3.110}$$

and $N\Phi(0+, y_2) > 0$. Hence, in light of arguments previously used, when y is sufficiently close to y_2, the function $H(\cdot, y)$ has a unique positive global maximum point and, for such values of y, we identify a unique connected stopping region $\mathcal{S}_y$. Therefore, we expect that the "transition" from two stopping intervals regime for (OS_y) to single interval regime takes place inside the range (y_1, y_2). Nevertheless, we can only *conjecture* that such transition occurs "monotonically", i.e. that there exists a unique value $y_0 \in (y_1, y_2)$ such that

$$H(z^{y_0}_{\gamma^-}, y_0) = H(z^{y_0}_{\gamma^+}, y_0) \tag{3.111}$$

and

$$\begin{aligned} H(z^y_{\gamma^-}, y) &< H(z^y_{\gamma^+}, y), \quad y_0 < y < y_2, \\ H(z^y_{\gamma^-}, y) &> H(z^y_{\gamma^+}, y), \quad y_1 < y < y_0. \end{aligned} \tag{3.112}$$

Clearly, if do not exist sufficient conditions that rigorously confirm the above conjecture, it becomes particularly difficult identifying the action region $\mathcal{D}$ and the inaction region $\mathcal{C}$ related to optimal control problem (OC) starting from the stopping/continuation regions of (OS_y).

Additionally, even if there exists a unique $y_0 \in (y_1, y_2)$ such that (3.111) and (3.112) hold, in general it is not possible to determine the differentiability and monotonicity properties for the boundaries of $\mathcal{D}$. Indeed, for any $y \in (\underline{y}, y_0]$, the boundary points $\gamma_1(y)$ and $\gamma_2(y)$ for (OS_y) are identified taking into account the tangency condition

(3.98). In particular, as explicitly calculated in Appendix D.2, $\gamma_1(y) < 0$ and $\gamma_2(y) > 0$ solve the equations system

$$\begin{cases} g(x_1,y)\frac{\Lambda_2(x_1,x_2)}{\Lambda_1(x_1,x_2)} - g(x_2,y)\frac{\Lambda_2(x_1,x_1)}{\Lambda_1(x_1,x_2)} = g_x(x_1,y), \\ g(x_1,y)\frac{\Lambda_2(x_2,x_2)}{\Lambda_1(x_1,x_2)} - g(x_2,y)\frac{\Lambda_2(x_2,x_1)}{\Lambda_1(x_1,x_2)} = g_x(x_2,y), \end{cases} \tag{S_{12}^y}$$

and managing Λ_1 and Λ_2 is particularly tricky, since these functions are defined in terms of the fundamental solutions ψ and φ whose tractability is challenging (also numerically), due to their intricate analytic expressions. Therefore, in the general framework within our model is inserted, it cannot be rigorously proved that system (S_{12}^y) implicitly defines two continuously differentiable functions $y \mapsto \gamma_1(y)$ and $y \mapsto \gamma_2(y)$.

By virtue of above considerations, establishing a rigorous connection between the parametrized family of optimal control (OS_y) and the optimal control problem (OC) remains an open problem when multiple boundaries may occur. In particular, the future research will be focused on identifying some suitable conditions for both the productivity function f and the holding costs function h, in order to completely depict the geometry of both action region and inaction region for (OC) and also to confirm the characterisation of the value function v in terms of the optimal reward $\tilde{u}$.

However, in order to provide an idea of (OC)-(OS_y) connection also under (R_∞), we assume that the functions γ_1 and γ_2 there exist and that these are continuously differentiable on $(\underline{y}, y_0)$. We observe that for stopping problem (OS_{y_0}), one has that (3.111) holds true, therefore the tangent line to the graph of $H(\cdot, y_0)$ at both $z_{\gamma_1}^{y_0}$ and $z_{\gamma_2}^{y_0}$ is a horizontal segment. It follows that the tangency points $z_{\gamma_1}^{y_0}$ and $z_{\gamma_2}^{y_0}$ must coincide with the local maximum points for $H(\cdot, y_0)$ on $(z_{A^-}^{y_0}, z_0)$ and $(z_{A^+}^{y_0}, 0)$, respectively. It means that

$$\gamma^-(y_0) = \gamma_1(y_0), \qquad \text{and} \qquad \gamma^+(y_0) = \gamma_2(y_0). \tag{3.113}$$

On the other hand, when y approaches $\underline{y}$, the graphs of function $H(\cdot, y)$ tend to flatten out on the z-axis and the quantity

$$\Delta N\Phi(0, y) := N\Phi(0+, y) - N\Phi(0-, y) = L(y)\psi(0)$$

vanishes. This fact lead us to guess that the "convex gap" tends to disappear close to $\underline{y}$ and, as a consequence, it follows that

$$\lim_{y \to \underline{y}} \gamma_1(y) = \lim_{y \to \underline{y}} \gamma_2(y) = 0. \tag{3.114}$$

Therefore, we are sufficiently allowed to believe that γ_1 *decreases* on $(\underline{y}, y_0)$ until it meets γ^- at y_0 and γ_2 *increases* on the same interval until it encounters γ^+ at y_0. Regarding the behaviour of γ^-, when y approaches $\underline{y}$ we observe that

$$\lim_{y \to \underline{y}} \gamma^-(y) = -\infty. \tag{3.115}$$

Indeed, recalling that $\gamma^-(y)$ solves (FP_-^y) and also that (R_∞) holds, one has

$$\lim_{y \to \underline{y}} \left[\frac{\psi(\gamma^-(y))}{\psi'(\gamma^-(y))} - \gamma^-(y) \right] = \lim_{y \to \underline{y}} \frac{h(y)}{rF(y)} = +\infty, \tag{3.116}$$

and we would obtain a contradiction, if the limit of $\gamma^-(y)$ for y converging to $\underline{y}$ were finite.

Therefore, among all reasonable shapes for $\mathcal{D}$ and $\mathcal{C}$ for (OC), in Figure 3.9 we draft a possible configuration for the partition of the strip $\mathbb{R} \times [\underline{y}, \overline{y}]$ into the inaction (continuation) $\mathcal{C}$ and the action (discharging) $\mathcal{D}$ regions, respectively obtained from the continuation $\mathcal{C}_y$ and stopping interval $\mathcal{S}_y$ associated with the family of optimal stopping problems (OS_y).

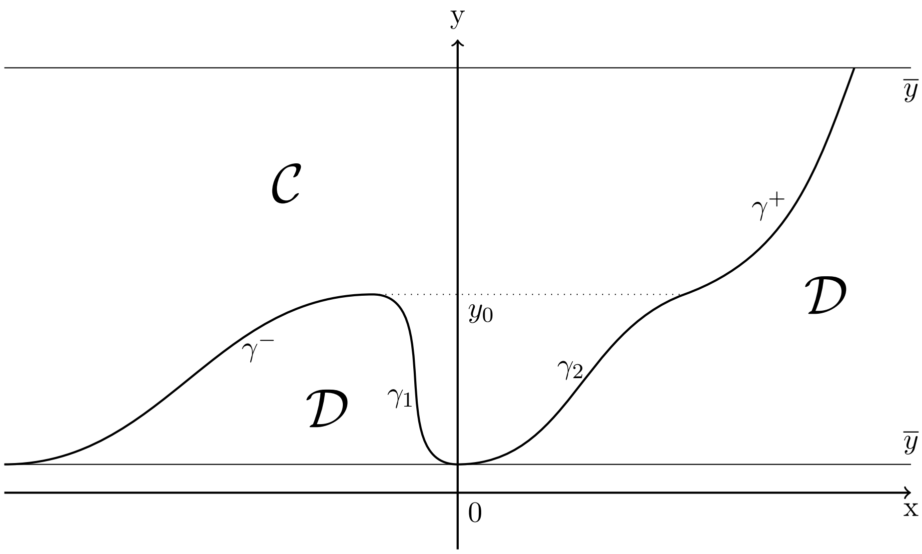

Figure 3.9: A picture of the conjectured partition of $\mathbb{R} \times [\underline{y}, \overline{y}]$ into the action region $\mathcal{D}$ and inaction region $\mathcal{C}$. Notice how disconnected stopping interval for (OS_y) may lead to multiple optimal moving boundaries for (OC).

We notice that, contrarily to the case (R_b), a portion of discharging region is contained on the negative half-strip. This can be explained as follows: whenever the productivity capability decreases faster then the inaction costs, it is natural expect that, for an initial amount y of water reserve sufficiently low ($y < y_0$) and a starting price x within the negative range $[\gamma^-(y), \gamma_1(y)]$, its preferable to release instantaneously such reserve and sell produced power at negative price rather than waiting for a future positive price. The production system responds to such control policy as if it was exerted *chattering* way, i.e. releasing a small amount of water many times in a sufficiently small interval of time. This power generation modality allows the producer to reduce the loss caused by selling power at negative price.

In general, taking in mind the configuration depicted on Figure 3.9, if we define

$$\tilde{\gamma}(y) := \begin{cases} \gamma_2(y), & y \in [\underline{y}, y_0], \\ \gamma^+(y), & y \in (y_0, \overline{y}], \end{cases} \tag{3.117}$$

and generalise the values of γ^- and γ_1 to the extended real line $\mathbb{R} \cup \{\pm\infty\}$, i.e.

$$\gamma^-(y) = \gamma_1(y) = -\infty, \qquad y \in (y_0, \overline{y}), \tag{3.118}$$

then, the natural *candidate* to be the optimal control policy for (OC) is given by

$$\bar{\nu}_t := \begin{cases} 0, & t \le \tau^*, \\ y - \underline{y}, & t > \tau^*, \end{cases} \tag{3.119}$$

where $\tau^* := \tau^- \wedge \tau^+$ and

$$\tau^- := \inf\{t \ge 0 : X_t^x \in [\gamma^-(y), \gamma_1(y)]\} \quad \text{and} \quad \tau^+ := \inf\{t \ge 0 : X_t^x \ge \tilde{\gamma}(y)\}, \tag{3.120}$$

considering the convention $\inf \emptyset = +\infty$. In conclusion, even though we naturally expect that the *candidate* value function v associated to the above optimal policy is given by

$$v(x, y) = \tilde{u}(x, y) - \frac{1}{r} h(y), \tag{3.121}$$

the objective of future research will be also to establish whether such function still solves the Hamilton-Jacobi-Bellman equation (HJB). In particular, it will be necessary to identify the general conditions on the model that guarantee the validity of the above heuristic discussion and consequently confirm the connection between the optimal state-dependent singular stochastic control problem (OC) and the parametrized family of optimal stopping problems (OS_y).

Conclusions

In this thesis, we considered a singular stochastic control problem for hydroelectric power production in an energy market that allows for prices that may reach negative values with positive probability.

We proposed a hydroelectric production system that is able to react in two different modes when it is convenient to produce energy through an instantaneous release of water. In particular, we endowed the system with the possibility of producing "less efficiently" when negative prices appear in the market but it is still preferable to produce instantaneously rather than waiting for positive prices.

We defined a novel optimisation problem whose performance functional that we aim to maximise among a suitable class of non-decreasing control policies, exhibits a state-dependent instantaneous marginal revenue whose sign is directly affected by the sign of the prices dynamics.

We proved the Verification Theorem, allowing to characterise the value function of our singular stochastic control problem (OC) among the solutions of the associated variational inequality with state-dependent gradient constraint (HJB). The Verification Theorem provided us also with a clear description of the optimal control as a purely discontinuous process that, at the first time of action, exerts all the available fuel with a single instantaneous jump.

Under assumption (R_b), we identified the value function of (OC) in terms of the optimal reward function of the associated family of optimal stopping problems (OS_y), explicitly solved by a geometric approach. We identified a unique positive boundary γ^+, separating the action and inaction regions and we showed that the optimal strategy consists in completely discharge the water reservoir as soon as the price dynamics reaches values grater or equal such optimal threshold. Under assumption (R_∞), we highlighted the difficulties that arise in this case and, based on the solutions of (OS_y), we provided some intuitions on the tricky structure of the action and inaction regions as well as on the nature of the candidate optimal control policy.

Our future research will start from these difficulties, trying to determine the sufficient conditions on our model in order to overcome these complexities and to be able to apply the verification approach even in this more complicated case. In particular, we will focus on the properties that the characteristics of the model should satisfied in order to prove that the connection (OC)-(OS_y) still holds.

The results obtained in this thesis represents a springboard to the generalisation of our approach to the bounded variation formulation, which aims at reproducing the features of hydroelectric power systems endowed with the double production-storage functionality that makes these systems extremely competitive in the markets of renewable resources.

Appendix

A Uniform integrability

In this first appendix, we report a list of some definitions and well-known results that are used in the development of the thesis. Further details can be found in [Bal17] and references therein. In particular, for the proofs of the following results one can refer to [Nev64].

Definition A.1. Given $p \geq 1$, a stochastic process $(X_t)_{t\geq 0}$ is *bounded* in the Lebesgue's space $L^p(\Omega, \mathbb{P})$ if

$$\sup_{t\geq 0} \mathbb{E}[|X_t|^p] < \infty. \tag{A.1}$$

Definition A.2. A family of random variables $\mathcal{H}$ is *uniformly integrable* if

$$\lim_{c\to+\infty} \sup_{X\in\mathcal{H}} \mathbb{E}[|X|\mathbb{1}_{\{|X|>c\}}] = 0. \tag{A.2}$$

For a uniformly integral family of r.v.'s the following characterisation holds.

Proposition A.3. *A family $\mathcal{H} \subset L^1(\Omega, \mathbb{P})$ is uniformly integrable if and only if there exists a positive increasing convex function $m : [0, +\infty) \to \mathbb{R}$ such that*

$$\lim_{x\to+\infty} \frac{m(x)}{x} = +\infty \quad \text{and} \quad \sup_{X\in\mathcal{H}} \mathbb{E}[m(|X|)] < \infty. \tag{A.3}$$

Remark A.4. If $\mathcal{H}$ is bounded in $L^p(\Omega, \mathbb{P})$ for some $p > 1$ then it is uniformly integrable. It suffices to consider $m(x) = |x|^p$ in Proposition A.3.

The concept of uniform integrability provides the following extension of the Lebesgue's theorem of dominated convergence.

Theorem A.5. *Let $(X_n)_n$ be a sequence of r.v.'s converging a.s. to X. The r.v. X is integrable and the convergence takes place in $L^1(\Omega, \mathbb{P})$ if and only if $(X_n)_n$ is uniformly integrable.*

B The Sobolev space $W^{1,\infty}_{loc}$

In this appendix we just mention the notion of weak derivative and the definition of the Sobolev space of functions with locally bounded first weak derivatives. For general definitions and further details one can refer to [Bre10; EG15; Eva10], among many others books on measure theory and functional analysis.

Definition B.1. Let $\mathcal{O} \subset \mathbb{R}^2$ an open set. A function $v : \mathcal{O} \to \mathbb{R}$ belongs to the Sobolev space $W^{1,\infty}_{loc}(\mathcal{O})$ if $v \in L^\infty(\mathcal{O})$ and there exist two functions $v_x, v_y \in L^\infty_{loc}(\mathcal{O})$ such that

$$\int_{\mathcal{O}} v(x,y)\xi_x(x,y)dxdy = -\int_{\mathcal{O}} v_x(x,y)\xi(x,y)dxdy, \tag{B.1}$$

and

$$\int_{\mathcal{O}} v(x,y)\xi_y(x,y)dxdy = -\int_{\mathcal{O}} v_y(x,y)\xi(x,y)dxdy, \tag{B.2}$$

for every $\xi \in C^\infty_c(\mathcal{O})$, i.e. ξ in the space of infinitely differentiable functions with compact support in $\mathcal{O}$. The functions u_x and u_y are called the weak partial derivatives of u and $Dv = (v_x, v_y)$ denotes the weak gradient of v.

We recall the following characterisation of $W^{1,\infty}_{loc}$, proved in Theorem 4.5 of [EG15].

Proposition B.2. *Let $\mathcal{O} \subset \mathbb{R}^2$ an open set. A function $v \in W^{1,\infty}_{loc}(\mathcal{O})$ if and only if v is locally Lipschitz continuous on $\mathcal{O}$.*

Moreover, the following result shows the connection between weak partial derivatives and the partial derivatives in the usual sense. For a proof see page 295 of Evans' book [Eva10].

Theorem B.3 (Differentiability almost everywhere)**.** *Assume $v \in W^{1,\infty}_{loc}(\mathcal{O})$. Then u is differentiable a.e. in $\mathcal{O}$, and its gradient equals its weak gradient a.e.*

We recall that the above result is also known as Rademacher's Theorem.

C Some results of [DK03]

The geometric method widely exploited in solving the optimal stopping problems (OS_y) is based on the approach developed in *On the optimal stopping problem for one-dimensional diffusion* by Dayanik and Karatzas [DK03]. The authors offer a new characterisation of excessive functions for one-dimensional regular diffusion processes in terms of a generalised concept of concavity. In particular, they provide a characterisation for the expected optimal reward function of the optimal stopping problem as the smallest nonnegative concave majorant of the running reward function.

Consider a diffusion X with state space $\mathcal{I} = (a, b)$ with $-\infty \leq a < b \leq +\infty$ and assume that X is regular in (a, b), i.e. if the process X starts from x, then it reaches a point y with positive probability, for any x and y in (a, b) (cf. e.g. [BS02]). Define the optimal expected reward function

$$u(x) := \sup_{\tau \geq 0} \mathbb{E}[e^{-rt} g(X^x_\tau)], \tag{C.1}$$

with $\tau \geq 0$ an $\mathcal{F}$-stopping time, r a positive constant and $g(\cdot)$ the running reward function.

In this appendix we collect part of the results of Subsection 5.2 of [DK03], where both the boundaries a and b are assumed to be *natural* for X, i.e. a and b cannot be reached in finite time (cf. e.g. [BS02]). Assume also that g is bounded on every compact subset of (a, b). The following results hold true.

Proposition C.1 (Proposition 5.10 of [DK03]). *We have either $u \equiv +\infty$ in (a, b), or $u(x) < +\infty$ for all $x \in (a, b)$. Moreover, $u(x) < \infty$ for every $x \in (a, b)$, if and only if*

$$l_a := \limsup_{x \to a^+} \frac{[g(x)]^+}{\varphi(x)} \qquad \text{and} \qquad l_b := \limsup_{x \to b^-} \frac{[g(x)]^+}{\psi(x)} \tag{C.2}$$

are both finite.

Assuming that the quantities l_a and l_b are finite and considering the functions

$$\tilde{G}(x) := \frac{\psi(x)}{\varphi(x)} \qquad \text{and} \qquad G(x) := -\frac{\varphi(x)}{\psi(x)}, \tag{C.3}$$

then one has the following characterisation.

Proposition C.2 (Proposition 5.12 of [DK03]). *Let $\tilde{W} : [0, +\infty) \to \mathbb{R}$ and $W : (-\infty, 0] \to \mathbb{R}$ be the smallest non-negative concave majorants of the functions*

$$\tilde{H}(z) := \begin{cases} \frac{g(\tilde{G}^{-1}(z))}{\varphi(\tilde{G}^{-1}(z))}, & z > 0, \\ l_a, & z = 0, \end{cases} \tag{C.4}$$

and

$$H(z) := \begin{cases} \frac{g(G^{-1}(z))}{\varphi(G^{-1}(z))}, & z < 0, \\ l_b, & z = 0, \end{cases} \tag{C.5}$$

respectively. Then $u(x) = \varphi(x)\tilde{W}(\tilde{G}(x)) = \psi(x)W(G(x))$*, for every* $x \in (a,b)$*. Furthermore,* $\tilde{W}(0) = l_a$*,* $W(0) = l_b$ *and* $\tilde{W}(\cdot)$ *and* $W(\cdot)$ *are continuous at* 0*.*

Remark C.3 (cf. Remark 5.2 of [DK03]). Given W and H as in Proposition C.2, defined on $(-\infty, 0]$. If $\mathcal{S} := \{x \in (a,b) : u(x) = g(x)\}$ and $\tilde{\mathcal{S}} := \{z \in (-\infty, 0) : W(z) = H(z)\}$, then $\mathcal{S} = G^{-1}(\tilde{\mathcal{S}})$.

Furthermore, if we consider the stopping time

$$\tau^* := \inf\{t \geq 0 : X_t \in \mathcal{S}\}, \tag{C.6}$$

then we have the following result.

Proposition C.4 (Proposition 5.13 of [DK03])**.** *The optimal reward function* u *is continuous on* (a,b)*. If* $g : (a,b) \to \mathbb{R}$ *is continuous, and* $l_a = l_b = 0$*, then* τ^* *is an optimal stopping time.*

D Some calculations in Chapter 3

In this appendix we report some calculations that we omitted in Chapter 3. In the following we exploit some concepts related to diffusion processes (e.g. scale function, Wronskian etc.). For the sake of brevity we leave out details that are extensively treated in Chapter 2 of [BS02].

D.1 The derivatives $H_z(\cdot, y)$ and $H_{zz}(\cdot, y)$

Fixed $y \in (\underline{y}, \overline{y}]$, we have

$$H(z,y) := \frac{g(G^{-1}(z), y)}{\psi(G^{-1}(z))}, \quad z \in G(\mathbb{R}) \tag{D.1}$$

where G as in (C.3) and ψ, φ are the linearly independent fundamental solutions defined in (2.5). We have

$$G'(x) = -\frac{\varphi'(x)\psi(x) - \varphi(x)\psi(x)}{\psi^2(x)} = \frac{S'(x)W(\psi,\varphi)}{\psi^2(x)}, \tag{D.2}$$

where $S'(x)$ are the density of the scale function associated with the diffusion X and

$$W(\psi,\varphi) := \frac{\psi'(x)}{S'(x)}\varphi(x) - \frac{\varphi'(x)}{S'(x)}\psi(x). \tag{D.3}$$

is the positive constant Wronskian of ψ and φ. If $g(\cdot,y)$ is differentiable in x and $z = G(x)$, we have that $H(\cdot,y)$ is differentiable in z and

$$H_z(z,y) = \frac{1}{G'(x)}\cdot\left(\frac{g}{\psi}\right)_x(x,y), \quad \text{and} \quad H_{zz}(z,y) = \frac{1}{G'(x)}\cdot\left[\frac{1}{G'(x)}\cdot\left(\frac{g}{\psi}\right)_x(x,y)\right]_x,$$

with

$$\left(\frac{g}{\psi}\right)_x(x,y) = \frac{g_x(x,y)\psi(x) - g(x,y)\psi'(x)}{\psi^2(x)}. \tag{D.4}$$

Moreover,

$$\begin{aligned}
\left[\frac{1}{G'(x)}\cdot\left(\frac{g}{\psi}\right)_x(x,y)\right]_x &= \left[\frac{g_x(x,y)\psi(x) - g(x,y)\psi'(x)}{S'(x)W(\psi,\varphi)}\right]_x = \\
&= \frac{1}{W(\psi,\varphi)(S'(x))^2}[S'(x)(g_{xx}(x,y)\psi(x) - g(x,y)\psi''(x))] + \\
&- \frac{1}{W(\psi,\varphi)(S'(x))^2}[S''(x)(g_x(x,y)\psi(x) - g(x,y)\psi'(x))].
\end{aligned}$$

Recalling that the functions S and ψ satisfy

$$\begin{aligned}
&\mathcal{L}S(x) = \frac{1}{2}\sigma^2 S''(x) + a(b-x)S'(x) = 0, \quad x \in \mathbb{R} \\
&[\mathcal{L} - r]\psi(x) = \frac{1}{2}\sigma^2\psi''(x) + a(b-x)\psi'(x) - r\psi(x) = 0, \quad x \in \mathbb{R}
\end{aligned}$$

then we can write

$$\begin{aligned}
\left[\frac{1}{G'(x)}\cdot\left(\frac{g}{\psi}\right)_x(x,y)\right]_x &= \frac{2}{\sigma^2 W(\psi,\varphi)S'(x)}\left[\frac{1}{2}\sigma^2(g_{xx}(x,y)\psi(x) - g(x,y)\psi''(x))\right] + \\
&+ \frac{2}{\sigma^2 W(\psi,\varphi)S'(x)}\left[a(b-x)(g_x(x,y)\psi(x) - g(x,y)\psi'(x))\right] = \\
&= \frac{2\psi(x)}{\sigma^2 W(\psi,\varphi)S'(x)}\left[\frac{1}{2}\sigma^2 g_{xx}(x,y) + a(b-x)g_x(x,y) - rg(x,y)\right] = \\
&= \frac{2\psi(x)}{\sigma^2 W(\psi,\varphi)S'(x)}[\mathcal{L} - r]g(x,y). \qquad \text{(D.5)}
\end{aligned}$$

In conclusion, for any $x \in \mathbb{R}$ and $z = G(x)$, we have

$$H_{zz}(z,y) = \frac{1}{G'(x)}\cdot\frac{2\psi(x)}{\sigma^2 W(\psi,\varphi)S'(x)}[\mathcal{L} - r]g(x,y). \tag{D.6}$$

D.2 The boundary points $\gamma_1(y)$ and $\gamma_2(y)$

Recalling (3.21), we have that

$$H_z(z^y_{\gamma_i}, y) = \frac{g_x(\gamma_i(y), y)\psi(\gamma_i(y)) - g(\gamma_i(y), y)\psi'(\gamma_i(y))}{\psi'(\gamma_i(y))\varphi(\gamma_i(y)) - \psi(\gamma_i(y))\varphi'(\gamma_i(y))}, \quad i = 1, 2 \tag{D.7}$$

and observing that

$$z^y_{\gamma_2} - z^y_{\gamma_1} = G(\gamma_2(y)) - G(\gamma_1(y)) = \frac{\psi(\gamma_2(y))\varphi(\gamma_1(y)) - \psi(\gamma_1(y))\varphi(\gamma_2(y))}{\psi(\gamma_2(y))\psi(\gamma_1(y))}, \tag{D.8}$$

it follows that

$$\frac{H_z(z^y_{\gamma_2}, y) - H_z(z^y_{\gamma_1}, y)}{z^y_{\gamma_2} - z^y_{\gamma_1}} = \frac{\psi(\gamma_2(y))\varphi(\gamma_1(y)) - \psi(\gamma_1(y))\varphi(\gamma_2(y))}{\psi(\gamma_2(y))\varphi(\gamma_1(y)) - \psi(\gamma_1(y))\varphi(\gamma_2(y))}. \tag{D.9}$$

Hence, solving system (3.98) is equivalent to find $\gamma_1(y) < 0$ and $\gamma_2(y) > 0$ such that simultaneously solve

$$\frac{\psi(\gamma_2(y))\varphi(\gamma_1(y)) - \psi(\gamma_1(y))\varphi(\gamma_2(y))}{\psi(\gamma_2(y))\varphi(\gamma_1(y)) - \psi(\gamma_1(y))\varphi(\gamma_2(y))} = \frac{g_x(\gamma_1(y), y)\psi(\gamma_1(y)) - g(\gamma_1(y), y)\psi'(\gamma_1(y))}{\psi'(\gamma_1(y))\varphi(\gamma_1(y)) - \psi(\gamma_1(y))\varphi'(\gamma_1(y))},$$

and

$$\frac{\psi(\gamma_2(y))\varphi(\gamma_1(y)) - \psi(\gamma_1(y))\varphi(\gamma_2(y))}{\psi(\gamma_2(y))\varphi(\gamma_1(y)) - \psi(\gamma_1(y))\varphi(\gamma_2(y))} = \frac{g_x(\gamma_2(y), y)\psi(\gamma_2(y)) - g(\gamma_2(y), y)\psi'(\gamma_2(y))}{\psi'(\gamma_2(y))\varphi(\gamma_2(y)) - \psi(\gamma_2(y))\varphi'(\gamma_2(y))}.$$

Now, by means of easy algebra and recalling the definition of Λ_1 and Λ_2 in (3.103), we obtain that the pair $(\gamma_1(y), \gamma_1(y))$ solves both the following equations

$$g(\gamma_1(y), y)\frac{\Lambda_2(\gamma_1(y), \gamma_2(y))}{\Lambda_1(\gamma_1(y), \gamma_2(y))} - g(\gamma_2(y), y)\frac{\Lambda_2(\gamma_1(y), \gamma_1(y))}{\Lambda_1(\gamma_1(y), \gamma_2(y))} = g_x(\gamma_1(y), y), \tag{D.10}$$

and

$$g(\gamma_1(y), y)\frac{\Lambda_2(\gamma_2(y), \gamma_2(y))}{\Lambda_1(\gamma_1(y), \gamma_2(y))} - g(\gamma_2(y), y)\frac{\Lambda_2(\gamma_2(y), \gamma_1(y))}{\Lambda_1(\gamma_1(y), \gamma_2(y))} = g_x(\gamma_2(y), y). \tag{D.11}$$

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Ringraziamenti

È molto difficile trovare le parole adatte ad esprimere tutta la mia gratitudine verso la Prof.ssa Maria Chiarolla, supervisor di questa tesi e guida infaticabile di tutto il mio percorso di dottorato. La ringrazio per tutto il tempo che mi ha dedicato e per la piena e costante fiducia nei miei confronti. Con immenso affetto e profonda riconoscenza custodirò tutti gli insegnamenti che hanno contribuito alla mia formazione professionale ma anche, e soprattutto, alla mia crescita personale.

Ringrazio la Scuola di Dottorato in Economia e il suo direttore Prof. Maurizio Franzini, il Collegio del corso di Dottorato in Modelli per l'Economia e la Finanza e il suo coordinatore Prof. Brunero Liseo, sempre pronto e disponibile ad accogliere e sostenere le esigenze dei dottorandi.

Voglio ringraziare l'intero Dipartimento di Metodi e Modelli per l'Economia, il Territorio e la Finanza, in particolar modo la Sezione di Matematica e il coordinatore del curriculum di dottorato in Matematica per le Applicazioni Economico-Finanziarie Prof. Raimondo Manca, per l'attenzione e la cura rivolta ai giovani ricercatori.

Un sincero ringraziamento va ai miei colleghi Alessandra, Armando, Dario, Fabrizio e Stefano con i quali ho condiviso, non soltanto l'ufficio, ma molti dei momenti e dei sentimenti che più hanno caratterizzato il mio percorso di dottorato. Li ringrazio di cuore per la stima e l'affetto che mi hanno sempre dimostrato.

Ringrazio gli amici e parenti che, come sempre, anche in questa occasione non hanno mancato di esprimermi la loro fiducia e il loro profondo apprezzamento. Per mia fortuna sono molti e sarebbe impossibile citarli tutti, senza dimenticarne qualcuno.

Nessun riconoscimento potrà mai essere sufficiente a rendere merito all'amore e al sostegno di Giulia, dei miei genitori Americo e Morena e di mio fratello Fabrizio. Senza di loro né questa tesi né null'altro sarebbe mai stato possibile.

Infine, ringrazio il Gruppo Telecom Italia che ha finanziato interamente la mia borsa di dottorato e che ha sempre garantito totale libertà e autonomia alla mia esperienza di ricerca.