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ABELIAN VERSUS NON-ABELIAN BÄCKLUND CHARTS: SOME REMARKS

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Abstract. Connections via Bäcklund transformations among different nonlinear evolution equations are investigated aiming to compare corresponding Abelian and non Abelian results. Specifically, links, via Bäcklund transformations, connecting Burgers and KdV-type hierarchies of nonlinear evolution equations are studied. Crucial differences as well as notable similarities between Bäcklund charts in the case of the Burgers - heat equation, on one side, and KdV-type equations, on the other, are considered. The Bäcklund charts constructed in [16] and [17], respectively, to connect Burgers and KdV-type hierarchies of operator nonlinear evolution equations show that the structures, in the non-commutative cases, are richer than the corresponding commutative ones.

1. **Introduction.** Bäcklund transformations in soliton theory are well known to represent a key tool both for finding solutions to applicative problems as well as for investigating symmetry and structural properties admitted by nonlinear evolution equations, see [1, 7, 58, 55, 59, 28] where Bäcklund and Darboux Transformations and applications to partial differential equations admitting soliton solutions are studied. The focus of the results in this critical survey is on Bäcklund transformations as a tool to connect different nonlinear evolution equations both in the Abelian as well as in the non-Abelian setting. Non-commutative nonlinear equations are considered wherein the unknown is an operator on a Banach space. The investigation was initiated by Marchenko [44] and, later, further developped in [2, 18].

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The obtained results are part of a research project on operator equations which takes its origin in [12] wherein, further to other results, connections, via Bäcklund transformation, among operator potential Korteweg-de Vries (pKdV), Korteweg-de Vries (KdV) and modified Korteweg-de Vries (mKdV) equations are studied and extended to the corresponding hierarchies. Subsequent developments are comprised in [13] where the special case in which the operator unknowns can be represented as matrices, is considered; solutions are also constructed. Indeed, a great interest on non-Abelian nonlinear evolution equations is testified by a wide bibliography, see [53, 41, 39, 37, 33, 34].

Then, aiming to construct an operator counterpart of the Bäcklund chart in [24], the previous Bäcklund chart is further extended in [15, 17] together with the investigation of properties enjoyed by KdV-type non-Abelian equation and the corresponding hierarchies generated on application of the constructed recursion operators. Some of these hierarchies were already known in the literature, while some other ones are new. In particular, in [17], new non-commutative hierarchies are constructed on the basis of results in previous works [15], motivated by an observation in [3]. These hierarchies, *mirror* to each other, when commutativity is assumed, reduce to the nonlinear equation for the KdV eigenfunction, (*KdV eigenfunction* equation) [38] which takes its origin in the early days of *soliton* theory when the *inverse spectral transform* (IST) method was introduced [49]. It was later studied in [38, 45, 66] together with many further nonlinear evolution equations. Notably, in the commutative case, the Dym equation is connected to all the other KdV-type nonlinear evolution equations, via reciprocal transformation [57, 24]. Indeed, reciprocal transformations, according to the pioneering work [60], are closely connected to the IST method. Investigations aiming to classify *integrable* nonlinear evolution equations are comprised in [4, 5, 46, 47, 67].

The material is organised as follows. The opening Section 2 provides the definition of Bäcklund transformation and few properties needed throughout the subsequent parts. Section 3 is devoted to connections involving the Burgers equation, both in the Abelian as well as in the non-Abelian case. The three subsections are concerned, in turn, to the well known commutative Burgers equation where some of its properties are recalled, to the two non-commutative counterparts of Burgers equation [11, 14, 16] and the last subsection points out differences and similarities between the Abelian and the non-Abelian case. In the next Section 4 the attention is focussed on the wide Bäcklund charts involving KdV-type equations. Also Section 4 is organised in a way similar to the previous one. Thus, the Bäcklund chart in [24], extended in [9], is recalled in the subsection devoted to the classical commutative setting. The following subsection is concerned about the Bäcklund chart, constructed in [12] and later extended [15, 17]. The last subsection is concerned about pointing out differences and analogies between the two cases.

2. **Background notions.**

Some background notions required in the presented study, are collected in this section. A wide literature, such as the books [1, 7, 58, 55, 59, 28], to restrict the list to those ones more closely connected with the present investigation, are concerned about Bäcklund transformations and their applications to *soliton equations*. In this section, for sake of brevity, only definitions, not uniquely given in the literature, are shortly recalled. In particular, the definition of Bäcklund transformation in [21], is adopted. Consider two non linear evolution equations

$$u_t = K(u) \text{ and } v_t = G(v) \text{ where } K, G : M \to TM \ , \ K, G \in C^\infty \tag{1}$$

and the unknown functions u, v depend both on the independent variables x, t and, for fixed t, $u(x,t) \in M$, which denotes a *smooth* manifold, then its generic *fiber* $T_u M$, at $u \in M$, can be identified with M itself. In addition, when *soliton solutions* are considered, M is assumed to coincide, for each fixed t, with the Schwartz space S [1] of *rapidly decreasing functions* on $\mathbb{R}^n$. In detail,

$$u_t \quad = K(u),\ K : M_1 \to TM_1,\ u : (x,t) \in \mathbb{R} \times \mathbb{R} \to u(x,t) \in M_1 \tag{2}$$

$$v_t \quad = G(v),\ G : M_2 \to TM_2,\ v : (x,t) \in \mathbb{R} \times \mathbb{R} \to v(x,t) \in M_2, \tag{3}$$

where $M := M_1 \equiv M_2$, is usually assumed. Then, a Bäcklund transformation can be defined as follows.

Definition 2.1. Given two evolution equations, $u_t = K(u)$ and $v_t = G(v)$, then B (u , v) = 0 represents a Bäcklund transformation between them whenever given two solutions of such equations, respectively, $u(x,t)$ and $v(x,t)$, such that

$$B(u,v)|_{t=0} = 0, \tag{4}$$

then, it follows

$$B(u,v)|_{t=\bar{t}} = 0,\ \forall \bar{t} > 0 \ . \tag{5}$$

The connection between solutions of the two equations via the Bäcklund transformation B can be graphicallly represented as

$$\boxed{u_t = K(u)} \overset{B}{—} \boxed{v_t = G(v)} \ . \tag{6}$$

Among the many properties, note that hereditariness is preserved under Bäcklund transformations. That is, according to [21, 52], let $\Phi(u)$ denote the hereditary recursion operator admitted by one of the nonlinear evolution equations (1), say $u_t = K(u)$, so that it can be written as

$$u_t = \Phi(u) u_x \text{ where } K(u) = \Phi(u) u_x \ , \tag{7}$$

then, also the other equation (1) admits a hereditary recursion operator which can be constructed via the Bäcklund transformation and the recursion operator $\Phi(u)$. Specifically, the recursion operator $\Psi(v)$ admitted by the second equation is given by

$$\Psi(v) = \Pi\, \Phi(u)\, \Pi^{-1} \implies G(v) = \Psi(v)\, v_x \text{ and } v_t = \Psi(v) v_x \tag{8}$$

where

$$\Pi := -B_v^{-1} B_u \ , \ \Pi : TM_1 \to TM_2 \ , \tag{9}$$

and B_u and B_v denote the Frechet derivatives of the Bäcklund transformation $B(u,v)$. Then, according to [21, 22], on subsequent applications of the admitted recursion operators, the result can be extendeed to the two generated hierarchies

$$u_t = [\Phi(u)]^n\, u_x \text{ and } v_t = [\Psi(v)]^n\, v_x \ , \ n \in \mathbb{N} \ ; \tag{10}$$

their *base members* equations, which correspond to $n = 1$, coincide with equations (1). Fixed any $m \in \mathbb{N}$, the two equations $u_t = [\Phi(u)]^m\, u_x$ and $v_t = [\Psi(v)]^m$

[1]Specifically, the Schwartz space S of *rapidly decreasing functions* on $\mathbb{R}^n$, is defined as $S(\mathbb{R}^n) := \{f \in C^\infty(\mathbb{R}^n) : \|f\|_{\alpha,\beta} < \infty, \forall \alpha, \beta \in \mathbb{N}_0^n\}$, where $\|f\|_{\alpha,\beta} := sup_{x \in \mathbb{R}^n} \left|x^\alpha D^\beta f(x)\right|$, and $D^\beta := \partial^\beta / \partial x^\beta$.

are connected via the same Bäcklund Transformation which connects the two base members equations. This extension is graphically represented, $\forall\, n \in \mathbb{N}$, as

$$\boxed{u_t = [\Phi(u)]^n\, u_x} \overset{B}{—} \boxed{v_t \;=\; [\Psi(v)]^n\, v_x}\ . \tag{11}$$

3. **Burgers Bäcklund charts.** This section is devoted to the Burgers equation. It represents the a well known example of nonlinear evolution equation whose linearisation via a Bäcklund transformation, the celebrated Cole-Hopf transformation [20, 35], represented a crucial step. Indeed, the Cole-Hopf transformation opened the way to the study of many applicative problems, such as initial boundary value problems, see for instance [29, 6]. This section is divided in three different subsections which, respectively, are concerned about the Abelian, the non-Abelian Burgers equation and, finally, a comparison between some properties in the two cases.

Specifically, the first subsection, provides a brief survey of some properties enjoyed by the Burgers equation and the corresponding hierarchy, in the Abelian framework; accordingly, the unknown is assumed to be a real valued function[2].

The next subsection is concerned about the non-Abelian Burgers equations and the corresponding hierarchies [11, 14, 16]. The first appearence of a non-commutative Burgers hierarchies is due to Levi, Ragnisco and Bruschi [41] who were concerned about matrix equations. The non-commutative Burgers equations and their properties were studied by Kupershmidt in [39], the hereditariness of the recursion operator of the non-commutative Burgers equation in [11] was proved in [14]. The same recursion operator was independently obtained, via a Lax pair method in [31], by Gürses, Karasu and Turhan [32]. Recently, in [16], also the *mirror* non-Abelian Burgers equation, together with the admitted recursion operator and, hence, the corresponding hierarchy are constructed. The obtained results are compared with previous ones in [11, 14, 32, 39, 41].

In the last subsection there is a concise comparison between the two different cases Abelian versus non-Abelian.

3.1. **Abelian Bäcklund chart.** Burgers equation reads:

$$v_t = v_{xx} + 2vv_x\ . \tag{12}$$

It is linked via the Cole-Hopf transformation [20, 35], which can be represented as the special Bäcklund transformation

$$CH:\ \ B(u,v) = 0\ ,\ \text{ where } B(u,v) = vu - u_x\ , \tag{13}$$

to the linear heat equation

$$u_t = u_{xx}\ . \tag{14}$$

The latter admits the hereditary recursion operator $\Phi(u) = D$, where D denotes derivation with respect to the spatial variable x. Hence, the heat equation reads

$$u_t = \Phi(u)u_x \text{ where } K(u) = \Phi(u)u_x\ ,\ \Phi(u) = D \tag{15}$$

and the corresponding hierarchy, respectively, can be written as

$$u_t = [\Phi(u)]^n u_x \text{ where } \Phi(u) = D\ ,\ n \in \mathbb{N}\ . \tag{16}$$

The link (13) between the heat (14) and the Burgers (12) equations can be depicted, via the following Bäcklund chart

$$\boxed{u_t = u_{xx}} \overset{CH}{—} \boxed{v_t = v_{xx} + 2vv_x}\ . \tag{17}$$

[2]Sometimes, for short, the term *scalar* is used to identify the commutative case.

The link via Cole-Hopf transformation [20, 35], on one side, allows to construct solutions of assigned initial boundary value problems modelled via Burgers equation, as testified by many results among which also [29, 6]. On the other side, indicates how to obtain, on application of formula (8), from the recursion operator of the heat equation, the Burgers recursion operator

$$\Psi(v) = D + v_x D^{-1} + v \ . \tag{18}$$

Furthermore, the Bäcklund chart can extended to the corresponding hierarchies according to (11), wherein $\Phi(u) = D$ and $\Psi(v)$ is given by (18), that is

$$\boxed{u_t = [\Phi(u)]^n u_x} \overset{CH}{—} \boxed{v_t = [\Psi(v)]^n v_x} \ , \ n \in \mathbb{N} \ . \tag{19}$$

3.2. **Non-Abelian Bäcklund chart.** This subsection is concerned about the non-Abelian Burgers equations which both represent the non-Abelian counterparts of the Abelian Burgers equation. For notational convenience, from here on, operator unknowns are denoted via capital letters while lower case letters are used referring to real valued ones so that, for instance, the unknown functions u, in the linear heat and v in the Burgers equations in Subsection 3.1 are denoted by lower case letters, while, in the present subsection capital case letters are used since operator equations are considered.

Consider the linear heat equation

$$U_t = K(U) \ , \ K(U) = U_{xx} \tag{20}$$

where, now, the unknown is an operator [14, 16]. The two different Cole-Hopf transformations, namely the

$$B_1(U, S) = 0 \ , \ \text{where } B_1(U, S) = US - U_x \implies S = U^{-1}U_x, \tag{21}$$

and its *mirror*

$$B_2(U, R) = 0 \ , \ \text{where } B_2(U, R) = RU - U_x \implies R = U_x U^{-1}, \tag{22}$$

give, respectively, the non-Abelian Burgers equation

$$S_t = G_1(S) \ , \ G_1(S) = S_{xx} + 2SS_x, \tag{23}$$

and the non-Abelian Burgers mirror equation

$$R_t = G_2(R) \ , \ G_2(R) = R_{xx} + 2R_x R \ . \tag{24}$$

The recursion operator admitted by the heat equation is represented by the operator $\widehat{\Phi}(U) = D$, those ones admitted by the non-Abelian Burgers and the mirror non-Abelian Burgers equations [14, 16] are, respectively, $\Psi(S)$ and $\Phi(R)$. When L_W and R_W denote, in turn, left and right multiplication by W, for any W, the Burgers and mirror Burgers operators are given as follows

$$\Psi(S) = (D + C_S)(D + L_S)(D + C_S)^{-1}, \ \text{where } C_S := \big[S, \cdot\big], \tag{25}$$

which can also be written as [14]

$$\Psi(S) = D + L_S + R_{S_x}(D + C_S)^{-1}; \tag{26}$$

and

$$\Phi(R) = (D - C_R)(D + R_R)(D - C_R)^{-1}, \ \text{where } C_R := \big[R, \cdot\big], \tag{27}$$

The corresponding hierarchies, namely, in turn, the heat, the non-Abelian Burgers and the *mirror* non-Abelian Burgers hierarchies are represented as

$$U_{t_n} = D^{n-1}U_x, \ S_{t_n} = \Psi(S)^{n-1}S_x, \ R_{t_n} = \Phi(R)^{n-1}R_x, \quad n \geq 1. \tag{28}$$

The lowest order members of the first two hierarchies are, respectively

$$\begin{array}{llll} U_{t_1} & = U_x, & S_{t_1} & = S_x, \\ U_{t_2} & = U_{xx}, & S_{t_2} & = S_{xx} + 2SS_x, \\ U_{t_3} & = U_{xxx}, & S_{t_3} & = S_{xxx} + 3SS_{xx} + 3S_x^2 + 3S^2 S_x. \end{array} \tag{29}$$

while, the first members of the mirror non-Abelian Burgers hierarchy, are

$$\begin{array}{lll} R_{t_1} & = & R_x, \\ R_{t_2} & = & R_{xx} + 2R_x R, \\ R_{t_3} & = & R_{xxx} + 3R_{xx}R + 3R_x^2 + 3R_x R^2. \end{array} \tag{30}$$

Notably, in the case of the non-Abelian Burgers hierarchy, the order is crucial: the two hierarchies (29) and (30) are different from each other, however, they coincide as soon as commutativity is assumed.

The links among them can be summarised in the following Bäcklund charts which, refer, respectively, to the equations, in Fig.1, and is extended to the whole corresponding hierarchies in the subsequent Fig.2.

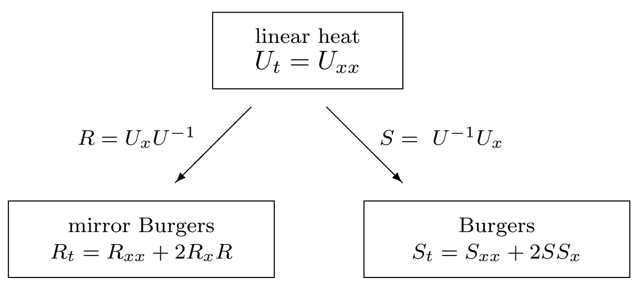

FIGURE 1. Burgers and mirror Burgers equations and their Bäcklund links: the non-commutative case.

The hereditariness guarantees that the Bäcklund chart can be extended to the corresponding hierarchies, as depicted in the following Fig.2.

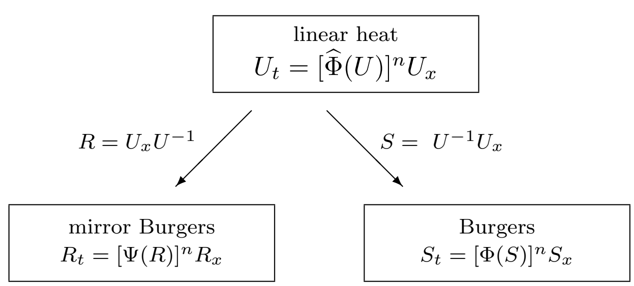

FIGURE 2. Burgers and mirror Burgers hierarchies and their Bäcklund links: the non-commutative case.

3.3. **Non-Abelian versus Abelian Bäcklund charts.** This brief subsection is concerned about differences and similarities between the Abelian and non-Abelian Burgers hierarchies. Indeed, there are two different non-Abelian transformations which generalise the Abelian Cole-Hopf transformation. Hence, when the linear heat equation is considered, if the unknown is a real valued function, say *scalar* for simplicity, the usual (Abelian) Burgers equation (12) is obtained. Conversely, if the unknown in the linear heat equation is an operator, then the two different non-Abelian equations (23) and (24) are obtained. On the other hand, the Cole-Hopf transformation which links the Burgers to the heat equation, both in the non-Abelian as well as in the Abelian case, allows to construct the recursion operator (18) admitted by the (commutative) Burgers equation, or the two different recursion operators, in turn (25) and (27), admitted by the (non-commutative) Burgers and by its mirror equation, in the non-commutative case. Furthermore, the form of the latter ones is more complicated, however, as expected, when commutativity is assumed, both the Burgers and mirror Burgers recursion operators reduce to the usual (commutative) Burgers recursion operator. These operators allow to generate two hierarchies of nonlinear evolution equations: they are both the same ones obtained by Kupershmidt [39] who construct them via a recursive definition of the hierarchies themselves. Also Gürses, Karasu and Turhan [32] obtained the same recursion operator $\Psi(S)$ and the hierarchy it generates via a method [31] based on the Lax pair formulation. For a direct proof of the hereditary property enjoyed by the recursion operators we refer to [14]. Note that it requires more involved computation and *ad hoc* routines [16] to take into account the non-commutativity of products. The following Section 4 shows that non-Abelian KdV-type equations are characterised by an even richer structure.

4. **KdV-type Bäcklund charts.** This Section is devoted to the two different Bäcklund charts connecting KdV-type equations, in turn, in the commutative case and in the non-commutative one. Specifically, to start with, the Bäcklund chart constructed in [24] is given. Then, the operator Bäcklund chart obtained in [12] and its extensions in [15, 17], are considered. Finally, a comparison between the two different situations closes this section.

4.1. **Abelian Bäcklund chart.** In this subsection, the links among the various scalar nonlinear evolution equations are depicted in the Bäcklund chart in [9] wherein, inspired by the non-commutative results in [17], the previous Bäcklund chart constructed in [24] is extended to include also the KdV eigenfunction equation. The links, up to this stage, are summarised in the following Bäcklund chart.

$$\boxed{\text{KdV}(u)} \overset{(a)}{—} \boxed{\text{mKdV}(v)} \overset{(b)}{—} \boxed{\text{KdV eig.}(w)} \overset{(c)}{—} \boxed{\text{KdV sing.}(\varphi)} \overset{(d)}{—} \boxed{\text{int. sol KdV}(s)} \overset{(e)}{—} \boxed{\text{Dym}(\rho)}$$

FIGURE 3. KdV-type Bäcklund chart: the Abelian case.

The third order nonlinear evolution equations in the Bäcklund chart are, respectively

$$u_t = u_{xxx} + 6uu_x \qquad \text{(KdV)},$$

$$v_t = v_{xxx} - 6v^2 v_x \qquad \text{(mKdV)},$$

$$w_t = w_{xxx} - 3\frac{w_x w_{xx}}{w} \qquad \text{(KdV eig.)},$$

$$\varphi_t = \varphi_x\{\varphi; x\}, \quad \text{where} \quad \{\varphi; x\} := \left(\frac{\varphi_{xx}}{\varphi_x}\right)_x - \frac{1}{2}\left(\frac{\varphi_{xx}}{\varphi_x}\right)^2 \qquad \text{(KdV sing.)},$$

$$s^2 s_t = s^2 s_{xxx} - 3ss_x s_{xx} + \frac{3}{2}{s_x}^3 \qquad \text{(int. sol KdV)},$$

$$\rho_t = \rho^3 \rho_{\bar{x}\bar{x}\bar{x}} \qquad \text{(Dym)}.$$

The Bäcklund transformations linking these equations among them, following the order in the Bäcklund chart itself, are:

$$(a) \quad u + v_x + v^2 = 0, \qquad (b) \quad v - \frac{w_x}{w} = 0, \tag{31}$$

$$(c) \; w^2 - \varphi_x = 0, \qquad (d) \quad s - \varphi_x = 0, \tag{32}$$

and the *reciprocal* transformation:

$$(e) \quad \bar{x} := D^{-1}s(x), \;\; \rho(\bar{x}) := s(x), \;\; \text{where } D^{-1} := \int_{-\infty}^{x} d\xi. \tag{33}$$

The latter exchanges the role played by the dependent and independent variables between them; hence, $\bar{x} = \bar{x}(s, x)$ and $\rho(\bar{x}) := \rho(\bar{x}(s, x))$. Details on the reciprocal transformation (e) are analysed in [10, 24] while a general introduction on reciprocal transformations together with various applications are illustrated in [58].

Notably, the Bäcklund chart represents a useful tool to reveal that the nonlinear evolution equations it links enjoy invariances, which might be new or already known. In particular, the invariance M under the Möbius group of transformations exhibited by the KdV singularity manifold equation (KdV sing.), allows to further extend the Bäcklund chart as indicated in the following Fig.4.

$$\begin{array}{ccccccccccc}
\boxed{\text{KdV}(u)} & \overset{(a)}{-} & \boxed{\text{mKdV}(v)} & \overset{(b)}{-} & \boxed{\text{KdV eig.}(w)} & \overset{(c)}{-} & \boxed{\text{KdV sing.}(\varphi)} & \overset{(d)}{-} & \boxed{\text{int. sol KdV}(s)} & \overset{(e)}{-} & \boxed{\text{Dym}(\rho)} \\
AB_1 \updownarrow & & AB_2 \updownarrow & & AB_3 \updownarrow & & M \updownarrow & & AB_4 \updownarrow & & AB_5 \updownarrow \\
\boxed{\text{KdV}(\tilde{u})} & \overset{(a)}{-} & \boxed{\text{mKdV}(\tilde{v})} & \overset{(b)}{-} & \boxed{\text{KdV eig.}(\tilde{w})} & \overset{(c)}{-} & \boxed{\text{KdV sing.}(\tilde{\varphi})} & \overset{(d)}{-} & \boxed{\text{int. sol KdV}(\tilde{s})} & \overset{(e)}{-} & \boxed{\text{Dym}(\tilde{\rho})}
\end{array}$$

FIGURE 4. Abelian KdV-type hierarchies Bäcklund chart: induced invariances.

Indeed, the Möbius invariance allows to obtain the auto-Bäcklund transformations $AB_k, k = 1 \ldots 5$. The obtained auto-Bäcklund transformations AB_1 and AB_2 are the well known ones admitted by the KdV and the mKdV hierarchies [48, 7, 24]. The auto-Bäcklund transformation AB_3 denotes an invariance enjoyed by the KdV eingenfunction equation [9]. Finally, AB_4, and AB_5, respectively, are auto-Bäcklund transformations, enjoyed by, respectively, the int. sol. KdV and Dym equations, according to [24]. Notably, the constructed Bäcklund chart not only can be extended in $2+1$- dimensions [50] but also can be regarded as a *constrained* version of such an extension [51]. Hence, both in $1+1$ [26, 30] as well as in $2+1$ dimensions [54], solutions to Dym equations problems can be obtained. Furthermore, according to [24], the Hamiltonian and bi-Hamiltonian structure of all the nonlinear evolution equations in the Bäcklund chart can be constructed from those ones admitted by the KdV equation [43, 23, 25, 27]. As discussed in [8], a Bäcklund chart connecting the Caudrey-Dodd-Gibbon-Sawata-Kotera and Kaup-Kupershmidt hierarchies [19, 61, 36] whose base member is a 5th order nonlinear evolution equations similar to the one connecting KdV-type equations can be constructed [56, 10].

All the links in the Bäcklund chart, in Fig.4, are valid for the whole hierarchies of nonlinear evolution equations generated via application of the admitted recursion operators. Hence, each element in Fig.4 can be interpreted as the whole corresponding hierarchy. The recursion operators, respectively, are [24, 9]

$$
\begin{aligned}
&\Phi_{\mathrm{KdV}}(u) = D^2 + 2DuD^{-1} + 2u && \text{(KdV)},\\
&\Phi_{\mathrm{mKdV}}(v) = D^2 - 4DvD^{-1}vD && \text{(mKdV)},\\
&\Phi_{\mathrm{KdV\ eig.}}(w) = \frac{1}{2w}Dw^2\left[D^2 + 2U + D^{-1}2UD\right]\frac{1}{w^2}D^{-1}2w && \text{(KdV eig.)},\\
&\Phi_{\mathrm{KdVsing}}(\varphi) = \varphi_x\left[D^2 + \{\varphi;x\} + D^{-1}\{\varphi;x\}D\right]\frac{1}{\varphi_x} && \text{(KdV sing.)},\\
&\Phi_{\mathrm{KdVsol}}(s) = Ds\left[D^2 + S + D^{-1}SD\right]\frac{1}{s}D^{-1} && \text{(int. sol KdV)},\\
&\Phi_{Dym}(\rho) = \rho^3 D_{\bar{x}}^3\rho D_{\bar{x}}^{-1}\rho^{-2} && \text{(Dym)},
\end{aligned}
$$

where, in the Dym equation, $D_{\bar{x}}^{-1}$ denotes the inverse operator of $D_{\bar{x}}$ while, respectively U and S, in the KdV eigenfunction (KdV eig.) and in the interacting soliton KdV (int. sol KdV), are defined via

$$
D_{\bar{x}} := \frac{d}{d\bar{x}}\ ,\ U := \frac{w_{xx}}{w} - 2\frac{w_x^2}{w^2}\ ,\ S := \left(\frac{s_x}{s}\right)_x - \frac{1}{2}\left(\frac{s_x}{s}\right)^2\ . \tag{34}
$$

The links among such hierarchies of nonlinear evolution equations, are indicated in (31)-(33).

4.2. **Non-Abelian Bäcklund chart.** This subsection is concerned about the Bäcklund chart connecting operator KdV-type equations. Accordingly, the results comprised in [12, 15, 17], are summarised in the following picture.

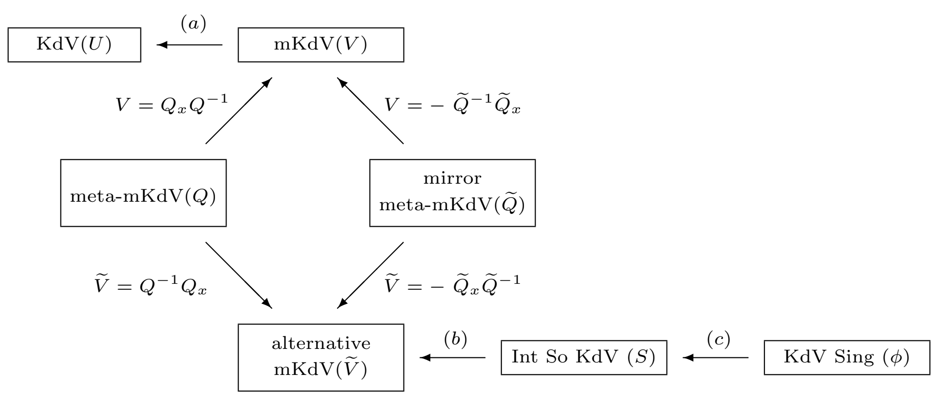

FIGURE 5. KdV-type hierarchies Bäcklund chart: the non-Abelian case.

In Fig.5, the third order nonlinear operator evolution equations [3] are, in turn,

$$
\begin{aligned}
&U_t = U_{xxx} + 3\{U, U_x\} && \text{(KdV)},\\
&V_t = V_{xxx} - 3\{V^2, V_x\} && \text{(mKdV)},\\
&Q_t = Q_{xxx} - 3Q_{xx}Q^{-1}Q_x && \text{(meta-mKdV)},\\
&\widetilde{Q}_t = \widetilde{Q}_{xxx} - 3\widetilde{Q}_x\widetilde{Q}^{-1}\widetilde{Q}_{xx} && \text{(mirror meta-mKdV)},
\end{aligned}
$$

[3]All the unknown are denoted via capital case letters with the only exception of the KdV sing. equation since Φ is used to indicate recursion operators.

$$\widetilde{V}_t = \widetilde{V}_{xxx} + 3[\widetilde{V}, \widetilde{V}_{xx}] - 6\widetilde{V}\widetilde{V}_x\widetilde{V} \qquad \text{(amKdV)},$$
$$\phi_t = \phi_x\{\phi; x\}, \quad \text{where} \ \ \{\phi; x\} = \big(\phi_x^{-1}\phi_{xx}\big)_x - \frac{1}{2}\big(\phi_x^{-1}\phi_{xx}\big)^2 \qquad \text{(KdV sing.)},$$
$$S_t = S_{xxx} - \frac{3}{2}\big(S_x S^{-1} S_x\big)_x \qquad \text{(int. sol KdV)}.$$

while the Bäcklund transformations linking the KdV with the mKdV, the amKdV with the int.sol KdV and the latter with the KdV sing. are, respectively

$$U = -\,(V^2 + V_x) \qquad (a)$$
$$\widetilde{V} = \frac{1}{2}S^{-1}S_x \qquad (b)$$
$$S = \phi_x \ . \qquad (c)$$

Again, all the nonlinear evolution equations in the Bäcklund chart admit a recursion operator; hence, the Bäcklund chart can be extended to the whole corresponding hierarchies. Indeed, the recursion operator themselves can be constructed on application of formula (8) according to [21]. In this, way, in [15, 17] the recursion operators of the newly inserted nonlinear evolution equations in the Bäcklund chart are obtained. The recursion operators, in turn, are the following ones.

$$\Phi_{\mathrm{KdV}}(U) = D^2 + 2A_U + A_{U_x}D^{-1} + C_U D^{-1} C_U D^{-1} \qquad \text{(KdV)},$$
$$\Phi_{\mathrm{mKdV}}(V) = (D - C_V D^{-1} C_V)(D - A_V D^{-1} A_V) \qquad \text{(mKdV)},$$
$$\Phi_{\mathrm{amKdV}}(\widetilde{V}) = (D + 2C_{\widetilde{V}})(D - 2R_{\widetilde{V}})(D + C_{\widetilde{V}})^{-1}(D + 2L_{\widetilde{V}})D(D + C_{\widetilde{V}})^{-1} \qquad \text{(amKdV)},$$
$$\Phi_{\mathrm{mmKdV}}(Q) = R_Q D^{-1}(D + C_{Q_xQ^{-1}})(D - A_{Q_xQ^{-1}}D^{-1}A_{Q_xQ^{-1}})(D - C_{Q_xQ^{-1}})R_{Q^{-1}} \qquad \text{(mmKdV)},$$
$$\Phi_{\mathrm{mmmKdV}}(\widetilde{Q}) = L_{\widetilde{Q}}D^{-1}(D - C_{\widetilde{Q}^{-1}\widetilde{Q}_x})(D - A_{\widetilde{Q}^{-1}\widetilde{Q}_x}D^{-1}A_{\widetilde{Q}^{-1}\widetilde{Q}_x})(D + C_{\widetilde{Q}^{-1}\widetilde{Q}_x})L_{\widetilde{Q}^{-1}} \qquad \text{(mmmKdV)},$$
$$\Phi_{\mathrm{KdVsing}}(\phi) = L_{\phi_x}\mathbb{D}^{-1}(\mathbb{D} - A_{N(\phi_x)})(\mathbb{D} - C_{N(\phi_x)}\mathbb{D}^{-1}C_{N(\phi_x)})(\mathbb{D} + A_{N(\phi_x)})L_{\phi_x^{-1}} \qquad \text{(KdV sing.)}.$$

where *mmKdV* and *mmmKdV*, respectively, denote *meta-mKdV*, *mirror meta-mKdV* and

$$\mathbb{D} := D + C_{N(\phi_x)},\ N(\phi_x) = \frac{1}{2}(\phi_x)^{-1}\phi_{xx},\ C_W := \big[W, \cdot\big],\ A_W := \big\{W, \cdot\big\},\ \text{arbitrary } W\ . \tag{35}$$

All the recursion operators are hereditary [12, 15, 17, 65], therefore, all the links in Fig.5 hold true for the whole corresponding hierarchies. Furthermore, in analogy with what happens in the commutative case, we expect that known invariances allow to prove new ones. Specifically, in [15] the non-Abelian KdV sing. equation is proved to be invariant under a non-Abelian counterpart of the Möbius group of transformations. This results, also extended to the corresponding hierarchy. Further results are obtained in [17].

4.3. **Non-Abelian versus Abelian Bäcklund charts.** This subsection collects a few remarks on the comparison between the Abelian and non-Abelian Bäcklund charts connecting KdV- type equations. They, respectively, are depicted in Fig.3 and in Fig.5. Notably, in the non-commutative case two different equations, namely meta-mKdV and mirror meta-mKdV,

$$Q_t = Q_{xxx} - 3Q_{xx}Q^{-1}Q_x \qquad \text{(meta-mKdV)},$$
$$\widetilde{Q}_t = \widetilde{Q}_{xxx} - 3\widetilde{Q}_x\widetilde{Q}^{-1}\widetilde{Q}_{xx} \qquad \text{(mirror meta-mKdV)},$$

represent non-commutative counterparts of the nonlinear equation for the KdV eigenfunction, termed, for short, *KdV eigenfunction* equation (KdV eig.),

$$w_t = w_{xxx} - 3\frac{w_x w_{xx}}{w} \ . \qquad (KdV eig.)$$

The latter, inserted in the commutative Bäcklund chart in [9], finds its meaning in connection with the Lax pair formulation of the KdV equation [38, 45, 49, 66]. Conversely, the non-commutative equations, denoted as meta-mKdV and mirror meta-mKdV, are introduced in the non-commutative Bäcklund chart in [17]. They both represent novel nonlinear evolution equations. Both these equations admit a hereditary recursion operator, constructed in [17] together with solutions. Notably, the hereditariness of the admitted recursion operator follows from the hereditariness of the KdV recursion operator [65]. Furthermore, when commutativity is assumed, the mKdV and amKdV equations coincide, hence the Bäcklund chart as Fig.5 reduces to the Abelian Bäcklund chart in Fig.4 without the Dym equation. On the other hand, also in the non-Abelian case, we expect the invariance under the Möbius group of transformations, suitably defined, allows to *double* the Bäcklund chart. Hence, in both commutative as well as non-commutative cases new and/or well known invariances [17] exhibited by the nonlinear evolution equations which appear the Bäcklund chart can be obtained, or, if already known, recovered.

Furthermore, it should be noted that the Dym hierarchy is inserted in the Abelian Bäcklund chart. Conversely, at the moment, a non-commutative counterpart the reciprocal transformation remains to be investigated. Solutions to non-commutative problems can be obtained [13, 62, 63, 64] and are currently under investigation.

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