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Title	Cosmological models with the energy density of random fluctuations and the Hubble-constant problem
Author(s)	Tomita, Kenji
Citation	Progress of Theoretical and Experimental Physics (2017), 2017(8)
Issue Date	2017-08
URL	http://hdl.handle.net/2433/237299
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Type	Journal Article
Textversion	publisher

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Cosmological models with the energy density of random fluctuations and the Hubble-constant problem

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The fluctuation energy is derived from adiabatic random fluctuations due to second-order perturbation theory, and the evolutionary relation for it is expressed in the form of $\rho_f = \rho_f(\rho)$, where ρ and ρ_f are the densities of ordinary dust and the fluctuation energy, respectively. The pressureless matter as a constituent of the universe at the later stage is assumed to consist of ordinary dust and the fluctuation energy. Next, cosmological models including the fluctuation energy as a kind of dark matter are derived using the above relation, and it is found that the Hubble parameter and the other model parameters in the derived models can be consistent with the recent observational values. Moreover, the perturbations of ρ and ρ_f are studied.
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Subject Index   E50, E60, E64

1.   Introduction

At the later stage of the universe, the main constituent is considered to be a pressureless matter consisting of ordinary dust. It is well known that the universe has random fluctuations in its density, which were caused by quantum fluctuations at the early stage [1–5], and their amplitude and spectrum have been studied through precise mesurements of fluctuations in the cosmic microwave background radiation (CMB) by the WMAP [6] and Planck [7,8] collaborations. However, the mean energy density corresponding to the fluctuations has not yet been derived, and so their dynamical influence on the universe has also not yet been clarified.

   In a previous paper [9], we tried to derive the energy density of random fluctuations using the general-relativistic second-order nonlinear perturbation theory [10,11], in which the random density fluctuations are given as first-order density perturbations with the specified spectrum, and the homogeneous energy density ρ_f was derived as the (spatially) averaged value of the second-order density perturbations. Moreover, the corresponding second-order metric perturbations and its spatial average were also derived. By adding the contribution of second-order homogeneous perturbations to the background model parameters, we renormalized the model parameters from background ones to modified ones. As a result of this procedure, we found the possibility of solving the Hubble-constant problem, in which the contradiction between the measured Hubble constant and the background Hubble constant was shown [7,8,12–17]. In the previous paper, it was found that the renormalized Hubble constant can become nearly equal to the measured Hubble constants.

   In this paper, we treat the fluctuation energy as a kind of dark matter and construct cosmological models involving it as part of the constituent. In Sect. 2 we express the fluctuation energy density ρ_f as a function of the ordinary dust density ρ, using the result of calculations in the second-order

perturbation theory in the basic background models. In Sect. 3, we derive cosmological flat models including pressureless matter, whose density is the sum of the densities of ordinary dust (ρ) and the fluctuation energy (ρ_f). The revised model parameters in these models are compared with those in the basic models without the fluctuation energy. In Sect. 4, we discuss the perturbations in the models with the fluctuation energy. In Sect. 5, we give some concluding remarks. In Appendix A, the formula of the fluctuation energy is shown.

2.   Evolutionary relation for the fluctuation energy

First, to derive the fluctuation energy, we assume two basic background models (Model 1 and Model 2) with

$$(\Omega_M^b, \Omega_\Lambda^b) = (0.22, 0.78) \text{ and } H_0^b = 67.3 \text{ km s}^{-1} \text{ Mpc}^{-1} \quad \text{for Model 1,} \tag{1}$$

and

$$(\Omega_M^b, \Omega_\Lambda^b) = (0.24, 0.76) \text{ and } H_0^b = 67.3 \text{ km s}^{-1} \text{ Mpc}^{-1} \quad \text{for Model 2,} \tag{2}$$

where

$$\Omega_M^b = \frac{8\pi G \rho_0^b}{3(H_0^b)^2} = \frac{1}{3}\frac{\rho_0^b}{(H_0^b)^2} \quad \text{and} \quad \Omega_\Lambda^b = \frac{\Lambda c^2}{3(H_0^b)^2} = \frac{1}{3}\frac{\Lambda}{(H_0^b)^2}, \tag{3}$$

ρ^b is the density of ordinary dust in the basic background models, H_0^b is the Hubble parameter H^b at the present epoch t_0^b, and $8\pi G = c = 1$. In the previous paper [9], only Model 1 was taken as the background model. Here we also consider Model 2 for reference. The Hubble parameter H^b satisfies

$$(H^b)^2 = \frac{1}{3}(\rho^b + \Lambda). \tag{4}$$

Using the transfer function (BBKS) for cold dark matter adiabatic fluctuations [5], we derived the second-order density perturbations $\delta_2\rho$ and the spatial average $\langle\delta_2\rho\rangle$ as a function of the cosmic time t^b in the previous paper. The formula for $\langle\delta_2\rho\rangle$ is shown in Appendix A. The latter is represented here as the fluctuation energy ρ_f ($\equiv \langle\delta_2\rho\rangle$). In this paper, we eliminate t^b from ρ_f and ρ^b, and represent ρ_f as the evolutionary function ($\rho_f(\rho^b)$) of ρ^b. Moreover, the ratio of their values is expressed as

$$\beta(\rho^b) \equiv \rho_f(\rho^b)/\rho^b. \tag{5}$$

The value β at $\rho^b \to \infty$ vanishes and the present values are

$$\beta(\rho_0^b) = 0.552 \text{ and } 0.685 \tag{6}$$

for Models 1 and 2, respectively. Their numerical values in Models 1 and 2 are shown as functions of ρ^b in Figs. 1 and 2, respectively.

The functional relation $\beta(x)$ is expressed approximately using two analytic functions in Models 1 and 2 in terms of $u \equiv x/[3(H_0^b)^2]$ as follows.
Model 1:

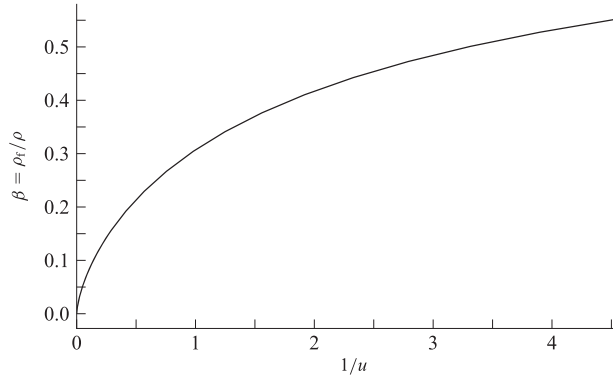$$\beta(x) = 0.292(1/u)^{0.400}[1 + 0.065/u - 0.0137/u^2] \tag{7}$$

Fig. 1. β is expressed as a function of $1/u$ in Model 1. The ordinate is β $(\equiv \rho_f/\rho^b)$ and $u \equiv \rho^b/[3(H_0^b)^2]$.

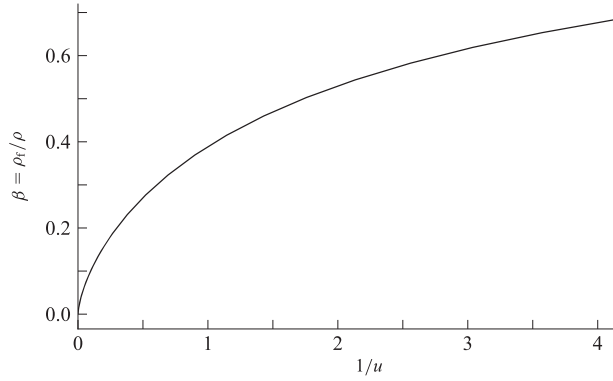**Fig. 2.** β is expressed as a function of $1/u$ in Model 2. The ordinate is β $(\equiv \rho_f/\rho^b)$ and $u \equiv \rho^b/[3(H_0^b)^2]$.

for $(0.22)^{-1} \geq 1/u \geq 0.982$ $(1 \geq a^b \geq 0.6)$, and

$$\beta(x) = 0.383(1/u)^{0.665}[1 - 0.261/u + 0.0627/u^2] \tag{8}$$

for $0.982 \geq 1/u \geq 0$ $(0.6 \geq a^b \geq 0)$, where a^b is the scale factor with the present value $a_0^b = 1$.
Model 2:

$$\beta(x) = 0.362(1/u)^{0.400}[1 + 0.0884/u - 0.0166/u^2] \tag{9}$$

for $(0.24)^{-1} \geq 1/u \geq 0.9$ $(1 \geq a^b \geq 0.6)$, and

$$\beta(x) = 0.459(1/u)^{0.665}[1 + 0.074/u - 0.251/u^2] \tag{10}$$

for $0.900 \geq 1/u \geq 0$ $(0.6 \geq a^b \geq 0)$.

3. Cosmological models with the fluctuation energy and the model parameters

To derive a spatially flat model with the fluctuation energy, we consider the line element

$$ds^2 = g_{\mu\nu}dx^\mu dy^\nu = a^2(\eta)[-d\eta^2 + \delta_{ij}dx^i dx^j], \tag{11}$$

where the Greek and Roman letters denote $0, 1, 2, 3$ and $1, 2, 3$, respectively. The conformal time $\eta(= x^0)$ is related to the cosmic time t by $dt = a(\eta)d\eta$.

In this paper, the fluctuation energy is regarded as a kind of dark matter, and is assumed to move together with ordinary dust. Then the velocity vector and energy–momentum tensor of pressureless matter are expressed in comoving coordinates as

$$u^0 = 1/a, \quad u^i = 0 \tag{12}$$

and

$$T_0^0 = -\rho_{\mathrm{T}}, \quad T_i^0 = 0, \quad T_j^i = 0 \tag{13}$$

with $\rho_{\mathrm{T}} \equiv \rho + \rho_{\mathrm{f}}$, where ρ_{T}, ρ, and ρ_{f} are the total density of pressureless matter, the ordinary dust density, and the fluctuation energy density, respectively, and we assume

$$\rho_{\mathrm{f}} = \rho \, \beta(\rho) \tag{14}$$

as the approximate equation of state for the fluctuation energy, where the function $\beta(\rho)$ is specified by Eq. (5) with Figs. 1 and 2, and Eqs. (7)–(10).

From the Einstein equations, we obtain

$$\rho_{\mathrm{T}} a^2 = 3(a'/a)^2 - \Lambda a^2, \tag{15}$$

and the energy–momentum conservation ($T^{\mu\nu}_{;\nu} = 0$) gives the relation

$$\rho_{\mathrm{T}} a^3 = \rho_{\mathrm{T}}(t_0), \tag{16}$$

where $a = 1$ at the present epoch ($t = t_0$) and a prime denotes $\partial/\partial\eta$. In the previous paper [9], the renormalization of the Hubble constant was done using the spatial average of the second-order metric perturbation. In this paper, the Hubble parameter is derived only through considering the fluctuation energy ρ_{f} as part of the total energy. Then the Hubble parameter $H(\equiv \dot{a}/a = a'/a^2)$ satisfies

$$H^2 = \frac{1}{3}(\rho_{\mathrm{T}} + \Lambda) = \frac{1}{3}(\rho + \rho_{\mathrm{f}} + \Lambda) \tag{17}$$

and we have the relations for the model parameters

$$\Omega_M \equiv \Omega_{Md} + \Omega_{Mf}, \tag{18}$$

and

$$\Omega_{Md} \equiv \frac{1}{3}\frac{\rho(t_0)}{(H_0)^2}, \quad \Omega_{Mf} \equiv \frac{1}{3}\frac{\rho_{\mathrm{f}}(t_0)}{(H_0)^2}, \tag{19}$$

where H_0 is H at the present epoch (t_0). This model reduces to the basic background models in Sect. 2 in the limit $a \to 0$, because $\rho_{\mathrm{f}}/\rho \to 0$.

From Eqs. (15) and (18), the equation for a is

$$a' = H_0[a(\Omega_M + \Omega_\Lambda a^3)]^{1/2}, \tag{20}$$

and $a(t)$ is determined by specifying Ω_M, Ω_Λ, and H_0, and solving this equation.

Now let us derive the model parameters $(\Omega_M, \Omega_\Lambda, H_0)$ in the present model as functions of $(\Omega_M^b, \Omega_\Lambda^b, H_0^b)$ in the basic models. Here the Hubble parameters are represented by H and H^b at epochs with scale factors a and a^b, respectively, and their ratio α is expressed as

$$\alpha^2 \equiv (H/H^b)^2 = \frac{(1+\beta)\rho + \Lambda}{\rho^b + \Lambda} \tag{21}$$

using Eqs. (4) and (17). This equation is rewritten as

$$\rho/\rho^b = [(\alpha^2 - 1)\Lambda/\rho^b + \alpha^2]/(1+\beta). \tag{22}$$

At the present epoch with $a = a^b = 1$, we have

$$(\alpha_0)^2 = \Omega_\Lambda^b + (1+\beta_0)\Omega_M^b(\rho/\rho^b)_0 \tag{23}$$

or

$$(\rho/\rho^b)_0 = X/(1+\beta_0), \tag{24}$$

where (α_0, β_0) is the present counterpart of (α, β) and

$$X \equiv [(\alpha_0)^2 - 1]\Omega_\Lambda^b / \Omega_M^b + (\alpha_0)^2. \tag{25}$$

Here we express Ω_{Md}, Ω_{Mf}, and Ω_Λ in terms of Ω_M^b and Ω_Λ^b. Using Eq. (23), we obtain

$$(\Omega_{Md}, \Omega_{Mf}, \Omega_\Lambda) = \frac{1}{3(H_0)^2}(\rho_0, \rho_{f0}, \Lambda) = \frac{(\rho_0, \rho_{f0}, \Lambda)}{3(\alpha_0)^2(H_0^b)^2}. \tag{26}$$

Using Eq. (24), moreover, we obtain

$$(\Omega_{Md}, \Omega_{Mf}, \Omega_\Lambda) = \frac{\Omega_M^b}{(\alpha_0)^2}\left(\frac{X}{1+\beta_0}, \frac{\beta_0 X}{1+\beta_0}, \frac{\Omega_\Lambda^b}{\Omega_M^b}\right). \tag{27}$$

For the density parameter of the pressureless matter $\Omega_M (\equiv \Omega_{Md} + \Omega_{Mf})$, we have

$$(\Omega_M, \Omega_\Lambda) = \frac{\Omega_M^b}{(\alpha_0)^2}(X, \Omega_\Lambda^b / \Omega_M^b), \tag{28}$$

and for ordinary dust, we have

$$(\Omega_{Md}, \Omega_{Mf} + \Omega_\Lambda) = \frac{\Omega_M^b}{(\alpha_0)^2}\left(\frac{X}{1+\beta_0}, \frac{\beta_0}{1+\beta_0}X + \Omega_\Lambda^b / \Omega_M^b\right). \tag{29}$$

Here we consider the correspondence between the ordinary dust density in the model with $\rho_f \neq 0$ and that in the basic model ($\rho_f = 0$), so that we may clarify the additional effect of the fluctuation energy. First we take the correspondence in which the present densities of ordinary dust are equal, i.e.,

$$(\rho/\rho^b)_0 = 1. \tag{30}$$

Then we obtain $X = 1 + \beta_0$ and

$$(\alpha_0)^2 = \Omega_M^b(1+\beta_0) + \Omega_\Lambda^b \tag{31}$$

from Eq. (23). For inserting $\beta_0 \equiv \beta(\rho_0)$ and the model parameters of the two basic models, therefore, we obtain

$$\alpha_0 = 1.059,$$

$$(\Omega_M, \Omega_\Lambda) = (0.305, 0.695) \text{ and } H_0 = 71.3 \text{ km s}^{-1} \text{ Mpc}^{-1} \tag{32}$$

for Model 1, and

$$\alpha_0 = 1.079,$$

$$(\Omega_M, \Omega_\Lambda) = (0.347, 0.653) \text{ and } H_0 = 72.6 \text{ km s}^{-1} \text{ Mpc}^{-1} \tag{33}$$

for Model 2.

For the present ordinary dust density ratio $(\rho/\rho^b)_0$, which is not equal to 1, we have $X = (\rho/\rho^b)_0(1 + \beta_0)$ from Eq. (24), and, using Eq. (23) for α_0, we obtain the following parameters for the model parameters of the two basic models and several values of $(\rho/\rho^b)_0$:

$$(\rho/\rho^b)_0 = 1.181, \quad \alpha_0 = 1.088,$$

$$(\Omega_M, \Omega_\Lambda) = (0.341, 0.659), \text{ and } H_0 = 73.2 \text{ km s}^{-1} \text{ Mpc}^{-1} \tag{34}$$

for Model 1, and

$$(\rho/\rho^b)_0 = 1.110, \quad \alpha_0 = 1.099,$$

$$(\Omega_M, \Omega_\Lambda) = (0.371, 0.629), \text{ and } H_0 = 74.0 \text{ km s}^{-1} \text{ Mpc}^{-1}, \tag{35}$$

and

$$(\rho/\rho^b)_0 = 1.276, \quad \alpha_0 = 1.130,$$

$$(\Omega_M, \Omega_\Lambda) = (0.404, 0.596), \text{ and } H_0 = 76.0 \text{ km s}^{-1} \text{ Mpc}^{-1} \tag{36}$$

for Model 2.

Thus, we obtained model parameters in the models with fluctuation energy by specifying the basic model parameters and $(\rho/\rho^b)_0$ for their correspondence. The above model parameters with the fluctuation energy are comparable with the observed ones [7,8,12,13,15–17]. Those with $(\rho/\rho^b)_0 = 1$ in Model 2 and $(\rho/\rho^b)_0 = 1.181$ in Model 1 are near to the observed ones with $H_0 = 73$–$74 \text{ km s}^{-1} \text{ Mpc}^{-1}$.

Next let us study the behaviors of models in the past in comparison with the basic models. Here $\alpha \ (\equiv H/H^b)$ is obtained from Eq. (21) as

$$\alpha^2 = \frac{(\rho_T)_0/a^3 + \Lambda}{(\rho^b/\rho_T)(\rho_T)_0/a^3 + \Lambda} = \frac{(1 + \beta_0)(\rho/\rho^b)_0 \Omega_M^b + \Omega_\Lambda^b a^3}{\frac{1+\beta_0}{1+\beta} \frac{(\rho/\rho^b)_0}{\rho/\rho^b} \Omega_M^b + \Omega_\Lambda^b a^3}, \tag{37}$$

where $\beta(\rho)$ is given by Eqs. (7)–(10), and

$$a^3 = \rho_T(t_0)/\rho_T = (\rho_0/\rho) \frac{1 + \beta_0}{1 + \beta(\rho)}. \tag{38}$$

To evaluate α in the past for $(\rho/\rho^b)_0 = 1$, we take the correspondence between a and a^b, in such a way that $\rho/\rho^b = 1$ also in the past. Then we have

$$\alpha^2 = \frac{(1 + \beta_0)\Omega_M^b + \Omega_\Lambda^b a^3}{\frac{1+\beta_0}{1+\beta} \Omega_M^b + \Omega_\Lambda^b a^3}, \tag{39}$$

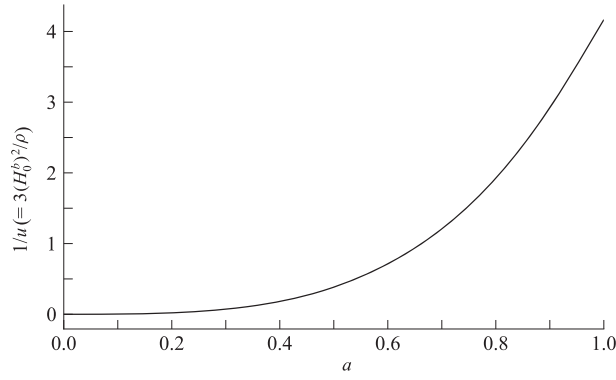so that $\beta \to 0$ and $\alpha \to 1$ for $a \to 0$.

Fig. 3. The $(1/u - a)$ relation in the case of $(\rho/\rho^b)_0 = 1$. The ordinate is $1/u$, where $u \equiv \rho/[3(H_0^b)^2]$. a is the scale factor and $a = 1$ at the present epoch.
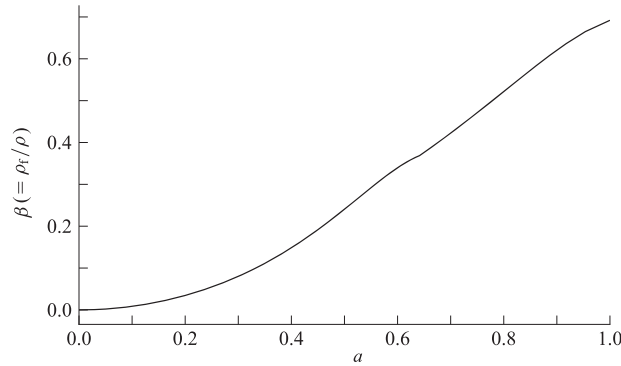

Fig. 4. The $(\beta - a)$ relation in the case of $(\rho/\rho^b)_0 = 1$. The ordinate is β ($\equiv \rho_f/\rho$).

To evaluate α in the past for $(\rho/\rho^b)_0 \neq 1$, we take the correspondence in such a way that

$$\rho/\rho^b = [(\rho/\rho^b)_0 - 1](\rho_0^b/\rho^b) + 1. \tag{40}$$

Then we find that $\rho/\rho^b \to 1$ and $\beta \to 0$ for $a \to 0$, and, from Eq. (37), that $\alpha \to 1$ for $a \to 0$.

The a dependences of $1/u, \beta$, and α in the case of $(\rho/\rho^b)_0 = 1$ for Model 2 (with the model parameter (33)) are shown in Figs. 3, 4, and 5, respectively, where $u \equiv \rho/[3(H_0^b)^2]$. At the early stage with $a < 0.6$, the role of ρ_f is effective and α increases with a, but at the later stage with $1.0 > a > 0.6$, Λ is dominant and α decreases slowly after a peak.

The a dependences of $1/u, \beta$, and α in the case of $(\rho/\rho^b)_0 \neq 1$ are also found to be similar to those in the case of $(\rho/\rho^b)_0 = 1$, owing to the above correspondence. Here the a dependence of α in the case of $(\rho/\rho^b)_0 \neq 1$ for Model 1 (with the model parameter (34)) is shown in Fig. 6.

Moreover, let us define the time-dependent model parameters $\Omega_M(t)$ and $\Omega_M^b(t)$ (representing those in the past) by

$$\Omega_M(t) \equiv \frac{\rho_T}{3H^2} \quad \text{and} \quad \Omega_M^b(t) \equiv \frac{\rho^b}{3(H^b)^2}. \tag{41}$$

Then $\Omega_M = \Omega_M(t_0)$ and $\Omega_M^b = \Omega_M^b(t_0)$, and we have the ratio

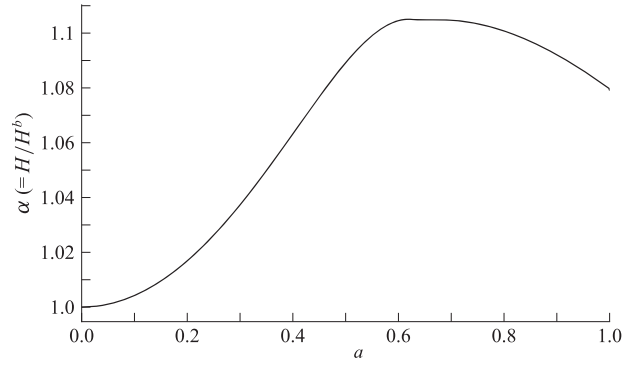$$\Omega_M(t)/\Omega_M^b(t) = (\rho/\rho^b)(1 + \beta)/\alpha^2. \tag{42}$$

Fig. 5. The $(\alpha - a)$ relation in the case of $(\rho/\rho^b)_0 = 1$. The ordinate is $\alpha \ (\equiv H/H^b)$. a is the scale factor and $a = 1$ at the present epoch.
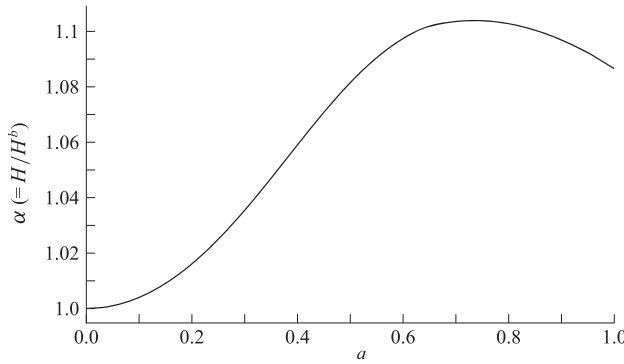

Fig. 6. The $(\alpha - a)$ relation in the case of $(\rho/\rho^b)_0 = 1.181$. The ordinate is $\alpha \ (\equiv H/H^b)$.
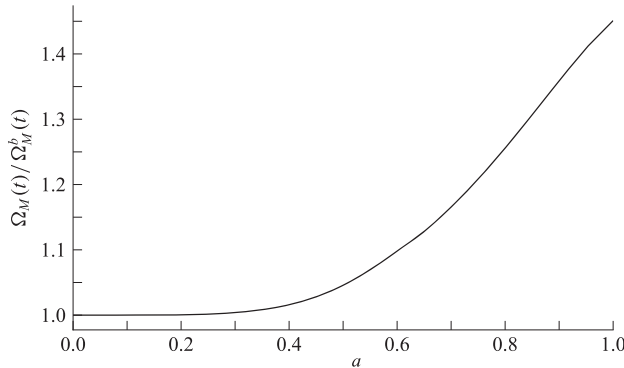

Fig. 7. The $(\Omega_M(t)/\Omega_M^b(t) - a)$ relation in the case of $(\rho/\rho^b)_0 = 1$. The ordinate is $\Omega_M(t)/\Omega_M^b(t)$.

This ratio tends to 1 for $a \to 0$. The a dependence of $\Omega_M(t)/\Omega_M^b(t)$ is shown in Fig. 7 for the model parameter (33).

It is concluded, therefore, that at the later stage the models with fluctuation energy can have a Hubble constant ($H_0 = 73$–74 km s^{-1} Mpc^{-1}) larger than that ($H_0^b = 67.3$ km s^{-1} Mpc^{-1}) in the basic models, while, at the early stage with large densities, both models have the same Hubble constants (in such a way that $H/H^b \to 1$ for $a \to 0$). This shows that the Hubble-constant problem [7,8,12,13,15–17] can be solved by taking the fluctuation energy into account.

4. Perturbations in cosmological models with the fluctuation energy

The behaviors of linear perturbations in cosmological models with pressureless matter are well known and expressed using the gauge-invariant treatment [18,19].

Here we assume that an accurate background model has been obtained and consider the perturbations to it. The gauge-invariant density perturbation ϵ_{T} for the total density ρ_{T} ($\equiv \rho + \rho_{\mathrm{f}}$) satisfies the equation

$$\epsilon_{\mathrm{T}}'' + \frac{a'}{a}\epsilon_{\mathrm{T}}' - \frac{1}{2}(\rho_{\mathrm{T}}a^2)\epsilon_{\mathrm{T}} = 0. \tag{43}$$

The evolutionary relation for the fluctuation energy is assumed to hold in the weak inhomogeneities. Then the gauge-invariant density perturbation ϵ for ordinary dust satisfies

$$\epsilon_{\mathrm{T}} = (1 + d\rho_{\mathrm{f}}/d\rho)\epsilon = (1 + \beta + \rho d\beta/d\rho)\epsilon, \tag{44}$$

and the perturbation ϵ_{f} of the fluctuation energy is expressed as

$$\epsilon_{\mathrm{f}} \equiv \epsilon_{\mathrm{T}} - \epsilon = \epsilon_{\mathrm{T}}(\beta + \rho d\beta/d\rho)/(1 + \beta + \rho d\beta/d\rho), \tag{45}$$

and

$$\epsilon = \epsilon_{\mathrm{T}}/(1 + \beta + \rho d\beta/d\rho). \tag{46}$$

At the early stage of $\beta \ll 1$, $\epsilon \simeq \epsilon_{\mathrm{T}}$ and $\epsilon_{\mathrm{f}} \ll \epsilon$, but at the later stage of $\beta \sim 1$, ϵ and ϵ_{f} are comparable.

5. Concluding remarks

The existence of random fluctuations is beyond doubt and their amplitudes are also well known [1–5]. We must take their energy (the fluctuation energy) into account, to clarify the dynamical evolution of the universe. This paper is the first step to considering it as a kind of dark matter.

At the stage of $a \ll 1$, the fluctuation energy ρ_{f} is negligibly small compared with the density ρ of ordinary dust, but at the present epoch it occupies about 36–41% of the total density of pressureless matter, depending on the basic models. The fluctuation energy was considered in this paper as part of the dark matter, which cannot be touched but contributes to the formation and evolution of astronomical objects at the later stage. The essential difference between the model with the fluctuation energy and the basic models is the quantitative large change in the dark matter.

In this paper we tentatively adopted Model 1 and Model 2 as the basic models, to derive the fluctuation energy using second-order perturbation theory. The derived model parameters depend sensitively on their basic model parameters, the present ordinary dust density ratio $(\rho/\rho^b)_0$, and the upper limit $x_{\max}$ for the integrations A and B (in Appendix A). Therefore, they should be selected so that the derived model parameters may be fitted as well as possible with the observational ones.

In the previous paper [9], we took the effect of fluctuation energy into account, by renormalizing the model parameters of a basic background model by adding the second-order density and metric perturbations to the background quantities. That method is different from the present one, in which the cosmological models are constructed by taking the fluctuation energy into account as part of the pressureless matter. However, we could obtain similar model parameters that are consistent with their observational values.

The accuracy for the second-order perturbations ρ_f $(\equiv \delta_2 \rho)$ is good at the early stage of the universe, because $\beta \equiv \rho_f/\rho \ll 1$, but it becomes worse with the expansion of the universe. At the present epoch, β is still smaller than 1, but not so small, i.e., 0.552 and 0.685 for the two basic models, as Eq. (6) shows. So, to derive a more accurate model at the stage of $a \simeq 1$, we should correct $\beta(x)$ in Eqs. (7)–(10) by constructing higher-order general-relativistic perturbation theories.

The contributions of the super-sample modes (i.e., large-scale modes longer than the survey scales) to the mean density fluctuations and the power spectrum in the finite-volume survey have recently been studied by several authors [20,21]. They are not equal to the backreaction of long-wavelength random fluctuations, but they may be closely connected with it, and so with the present analyses. If so, the general-relativistic second-order perturbations or the nonlinear perturbations in the post-Newtonian approximation may also play important roles in their treatments (in a way similar to our treatment in the previous paper [9]). This is because the large-scale modes cross the Hubble-scale length during their evolution from the very early stage to the present epoch [22].

Acknowledgements

Appendix. Second-order density perturbations corresponding to first-order random fluctuations

In Sect. 3 of the previous paper [9], we obtained the formula for the spatial average of the second-order density perturbations in the basic models. It is expressed as

$$\langle \underset{2}{\delta} \rho/\tilde{\rho} \rangle = \frac{4\pi}{3} \, (K_{\mathrm{eq}})^4 \, \mathcal{P}_{\mathcal{R}0} \, \frac{[1 - Y(a)]}{(\Omega_M/a + \Omega_\Lambda a^2)} \left[\frac{11}{2} \, (K_{\mathrm{eq}})^{-2} A + Z(a) B \right], \tag{A1}$$

where $\mathcal{P}_{\mathcal{R}0} = 2.2 \times 10^{-9}$, $\tilde{\rho} = \rho + \Lambda$, $H_0^b = 100h$, and $K_{\mathrm{eq}} \equiv k_{\mathrm{eq}}/H_0^b = 219(\Omega_M^b h)$.

For $h = 0.673$, we have

$$K_{\mathrm{eq}} = 32.4(\Omega_M^b/0.22). \tag{A2}$$

For the transfer function $T_s(x)$, A and B are expressed as

$$A \equiv \int_{x_{\mathrm{min}}}^{x_{\mathrm{max}}} dx \, x \, T_s^2(x), \quad B \equiv \int_{x_{\mathrm{min}}}^{x_{\mathrm{max}}} dx \, x^3 \, T_s^2(x), \tag{A3}$$

where $x \equiv k/k_{\mathrm{eq}}$ for the wave-number k, and the upper and lower limits of the integrations are specified by $x_{\mathrm{max}} = 5.7$ and $x_{\mathrm{min}} = 0.01$.

The definitions of $Y(a)$ and $Z(a)$ can be found in the previous paper [9].

References

[1] S. Weinberg, *Cosmology* (Oxford University Press, New York, 2008).
[2] A. R. Liddle and D. H. Lyth, *Cosmological Inflation and Large-Scale Structure* (Cambridge University Press, New York, 2000).
[3] J. A. Peacock, *Cosmological Physics* (Cambridge University Press, Cambridge, UK, 1999).
[4] L. Amendola and S. Tsujikawa, *Dark Energy* (Cambridge University Press, Cambridge, UK, 2010).
[5] J. M. Bardeen, J. R. Bond, N. Kaiser, and A. S. Szalay, Astrophys. J. **304**, 15 (1986).
[6] G. Hinshaw et al., Astrophys. J. Suppl. **208**, 19 (2013).
[7] P. A. R. Ade et al., Astron. Astrophys. **571**, A16 (2014).
[8] P. A. R. Ade et al., Astron. Astrophys. **594**, A3 (2016).

[9] K. Tomita, Prog. Theor. Exp. Phys. **2017**, 053E01 (2017) [arXiv:1702.07821 [astro-ph.CO]] [Search INSPIRE].

[10] K. Tomita, Phys. Rev. D **71**, 083504 (2005).

[11] K. Tomita, Prog. Theor. Phys. **37**, 831 (1967).

[12] A. G. Riess, L. Macri, S. Casertano, H. Lampeitl, H. C. Ferguson, A. V. Filippenko, S. W. Jha, W. Li, and R. Chornock et al., Astrophys. J. **730**, 119 (2011).

[13] W. L. Freedman, B. F. Madore, V. Scowcroft, C. Burns, A. Monson, S. E. Persson, M. Seibert, and J. Rigby, Astrophys. J. **758**, 24 (2012).

[14] W. L. Freedman, Nature Astron. **1**, 0169 (2017).

[15] S. H. Suyu et al., Astrophys. J. **766**, 70 (2013).

[16] J. L. Bernal et al., J. Cosmol. Astropart. Phys. **10**, 019 (2016).

[17] Z. Berezhiani et al., Phys. Rev. D **92**, 061303 (2015).

[18] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).

[19] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984).

[20] M. Takeda and W. Hu, Phys. Rev. D **87**, 123504 (2013).

[21] Y. Li, W. Hu, and M. Takeda, Phys. Rev. D **90**, 103530 (2014).

[22] T. Baldauf, U. Seljak, L. Senatore, and M. Zaldarriaga, J. Cosmol. Astropart. Phys. **10**, 031 (2011).