Title	Instanton moduli spaces on 4-manifolds with periodic end and an obstruction of the existence of embeddings (Intelligence of Low-dimensional Topology)
Author(s)	Taniguchi, Masaki
Citation	数理解析研究所講究録 = RIMS Kokyuroku (2017), 2052: 42-47
Issue Date	2017-10
URL	http://hdl.handle.net/2433/237110
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
第2052巻 2017年 42-47

Instanton moduli spaces on 4-manifolds with periodic end and an obstruction of the existence of embeddings

Masaki Taniguchi
Tokyo University

1 Introduction

The gauge theory has been an important tool for the study of 4-dimensional manifolds since the early 1980s, when Donaldson solved long-standing problems in topology. The purpose of our study are developing gauge theory on non-compact 4-manifolds and its application to low dimensional topology. The statement of our main theorem is the compactness of ASD-moduli spaces for non-compact 4-manifolds with periodic end and cylindrical end. Furthermore, we construct an obstruction of embeddings of 3-manifolds into 4-manifolds with some homological condition as an application of the main theorem. There are two backgrounds for our study.

First background is the transition of development of gauge theory on non-compact 4-manifolds. Gauge theory on 4-manifolds with cylindrical end did very well. We give the typical study of gauge theory on 4-manifolds with periodic end. In [T87], Taubes studied the ASD-moduli space for 4-manifolds with periodic end under some assumption. More explicitly, Taubes showed that the ASD-moduli spaces for 4-manifold $M = K \cup_Y W_0 \cup_Y W_1 \cdots$ (where K and W_0 are compact 4-manifolds, W_i is a copy of W_0) has a natural compactification under the assumption that there is no non trivial representation $\pi_1(W_i) \to SU(2)$. Moreover there is another study for gauge theory on 4-manifolds with periodic end. In [L16], Lin also give a natural compactification of Seiberg-Witten moduli spaces for 4-manifolds $Y \times \mathbb{R}_{\leq 0} \cup W_0 \cup W_1 \cdots$ under the assumption that W_0 has a positive scalar curvature. They assume the strong condition on the segment W_i of periodic end. One of the purpose is treating more general segment W_i.

Second, there is the background which relates the invariants of a homology $S^3 \times S^1$. For 4-manifolds with $b^+ = 0$, the Donaldson invariant and Seiberg-Witten invariant can not be defined. Because the moduli spaces of solutions have the quotient singularity for such 4-manifold. But for a homology $S^3 \times S^1$, the gauge theoric invariants are defined by counting the solutions except for these singularity. In [FO93], the Furuta-Ohta invariant

λ_{FO} is defined by using ASD moduli spaces for a homology $S^3 \times S^1$. In [MRS11], Mrowka-Ruberman-Saveliev invariant λ_{MRS} is also defined by using the SeibergWitten moduli spaces. When a homology $S^3 \times S^1$ is equal to $Y \times S^3$ for some homology S^3 Y, these invariants are essentially equal to the Casson invariant of Y. On the other hand, Casson invariant has a categorification by using the moduli spaces on the $\mathbb{Z}$ covering space of $Y \times S^1$. It is called Floer theory which is developing gauge theory on 4-manifolds with cylindrical end. ([Fl88], [KM08]) There is a natural question: Is there a refinement of $\lambda_{FO}(\lambda_{MRS}\ )$ by developing gauge theory on $\mathbb{Z}$ covering of X. This question suggests an extension of Floer theory.

2 Notations and Assumptions

We define several notations for any manifold Z. We denote by P_Z the product $SU(2)$ bundle.

$$\mathcal{A}(Z) := \{SU(2)\text{-connections on } P_Z\},$$
$$\mathcal{A}^{\text{flat}}(Z) := \{SU(2)\text{-flat connections on } P_Z\} \subset \mathcal{A}(Y),$$
$$\widetilde{\mathcal{B}}(Z) := \mathcal{A}(Z)/\text{Map}_0(Z, SU(2)),$$
$$\widetilde{R}(Z) := \mathcal{A}^{\text{flat}}/\text{Map}_0(Z, SU(2)) \subset \widetilde{\mathcal{B}}(Z),$$

and

$$R(Z) := \mathcal{A}^{\text{flat}}(Z)/\text{Map}_0(Z, SU(2)),$$

where $\text{Map}_0(Z, SU(2))$ is a set of smooth functions with mapping degree 0. When Z is equal to an oriented homology 3-sphere Y, the Chern-Simons functional $cs_Y : \mathcal{A}(Y) \to \mathbb{R}$ is defined by

$$cs_Y(a) := \frac{1}{8\pi^2}\int_Y Tr(a \wedge da + \frac{2}{3} a \wedge a \wedge a).$$

It is known that cs decends to a map $\widetilde{\mathcal{B}}(Y) \to \mathbb{R}$, which we denote by the same notation cs_Y.

Notation 2.1. *We denote the number of elements in $R(Y)$ by l_Y. If $R(Y)$ is not a finite set, we set $l_Y = \infty$.*

We will use the following assumption on Y in our main theorem(Theorem 5.1) and the compactness theorem(Theorem 4.1).

Assumption 2.2. *All $SU(2)$ flat connections on Y are non-degenerate, i.e. the first cohomology group of the next twisted de Rham complexes vanish.*

$$0 \to \Omega^0(Y) \otimes \mathfrak{su}(2) \xrightarrow{d_a} \Omega^1(Y) \otimes \mathfrak{su}(2) \xrightarrow{d_a} \Omega^2(Y) \otimes \mathfrak{su}(2) \xrightarrow{d_a} \Omega^3 \otimes \mathfrak{su}(2) \to 0$$

for $[a] \in R(Y)$.

Example 2.3. *All flat connections on the Brisckorn homology three sphere* $\Sigma(p,q,r)$ *are non-degenerate.([FS90])*

Under Assumption 2.2, l_Y is finite ([T90]).

3 Chern-Simons functional for homology $S^3 \times S^1$.

For a homology $S^1 \times S^3$ which we denote by X, we generalize the Chern-Simons functional to a functional cs_X on the flat connections on X. In our construction, cs_X cannot be extended to a functional for arbitrary $SU(2)$ connections on X.

Let X be a homology $S^3 \times S^1$, i.e. , X is a closed 4-manifold equipped with an isomorphism $\phi : H_*(X) \to H_*(S^3 \times S^1)$ in this paper. Then X has an orientation induced by the standard orientation of $S^3 \times S^1$ and ϕ.

Proposition 3.1. *Let* X *be a homology* $S^1 \times S^3$. *There is a well-defined map* $cs_X : \widetilde{R}(X) \times \widetilde{R}(X) \to \mathbb{R}$ *satisfying the following condition.*

When X *is equal to* $Y \times S^1$ *with an oriented homology 3-sphere* Y, *the map* $cs_X : \widetilde{R}(X) \times \widetilde{R}(X) \to \mathbb{R}$ *essentially coincides with the restriction of Chern-Simons functional* cs_Y *on* Y *by the following sense. For* $[a] \in \widetilde{R}(Y \times S^1)$, *the restriction* $[i^*a] \in \widetilde{R}(Y)$ *satisfies*

$$cs_Y([i^*a]) = cs_{Y \times S^1}([a], [\theta])$$

where i *is a inclusion* $Y = Y \times 1 \to Y \times S^1$ *and* θ *is the product* $SU(2)$ *connection.*

4 Compactness of ASD-moduli space for 4-manifolds with periodic ends.

The compactness of ASD-moduli spaces for non-compact 4-manifolds is treated in [Fl88], [Fu90],[Do02] for cylindrical end case and in [T87] for periodic end case. In [Fu90] and [T87], they consider the ASD-moduli spaces with the connections asymptotically convergent to the trivial connection on the end. We also follow their strategy by using $Q_X^{2l_Y+3}$ defined by using value of cs_X. More explicitly, in this section we explain a compactness result for the instanton moduli spaces for a non-compact manifold W^+ with periodic end.

Let Y be a oriented homology S^3. Let W_0 be an oriented homology cobordism from Y to $-Y$. We get an oriented compact 4-manifold X by pasting Y and $-Y$ of W_0. We denote by W^+ and W the following non-compact 4-manifolds.

$$\begin{cases} W^+ := W_0 \cup_Y W_1 \cup_Y \dots, \\ W := Y \times (-\infty, 0] \cup W^+, \end{cases}$$

where W_i is a copy of W_0. For a fixed Riemannian metric g_Y on Y, we choose a Riemannian metric g_W on W with the conditions which are $g_W|_{Y\times(-\infty,-1]} = g_Y \times g_{\mathbb{R}}^{\mathrm{stan}}$ and $g_W|_{W^+}$ is a periodic metric. There is a natural orientation on W^+ and W induced by the orientations of W_0. The infinite cyclic covering space of X can be written by

$$\widetilde{X} \cong \dots W_{-1} \cup_Y W_0 \cup_Y W_1 \cup_Y \dots,$$

where W_i is also copy of W_0. Let T be the deck transformation of $\widetilde{X}$ which maps each W_i to W_{i+1}. By restriction, T has an action on W^+. We use the following smooth function on W^+

$$\tau : W^+ \to \mathbb{R} ,$$

satisfying $\tau(T|_{W^+}(x)) = \tau(x) + 1$ for $x \in W^+$.

By pasting W_0 with itself along ifts boundary Y and $-Y$, we get a homology $S^3 \times S^1$ which we denote by X . We consider the product $SU(2)$-bundle P_{W^+} on W^+. For $q \geq 3$ and $\delta > 0$, we define the ASD-moduli space $M^{W^+}_\delta$ by

$$M^{W^+}_\delta := \left\{ \theta + c \in \Omega^1(W^+) \otimes \mathfrak{su}(2)_{L^2_{q,\delta}} \middle| F^+(\theta + c) = 0 \right\} / \mathcal{G},$$

where $\mathcal{G}$ is the gauge group

$$\mathcal{G} := \left\{ g \in \mathrm{Aut}(P_{W^+}) \subset \mathrm{End}(\mathbb{C}^2)_{L^2_{q+1,\mathrm{loc}}} \middle| dg \in L^2_{q,\delta} \right\},$$

and the action of $\mathcal{G}$ is given by the pull-back of connections. For $f \in \Omega^i(W^+) \otimes \mathfrak{su}(2)$ with compact support, we define $L^2_{q,\delta}$ norm by the following formula

$$||f||^2_{L^2_{q,\delta}} := \sum_{0 \leq j \leq q} \int_{W^+} e^{\delta\tau} |\nabla^j_\theta f|^2 d\mathrm{vol},$$

where ∇_θ is the covariant derivertive with respect to the product connection. We use the periodic metric $| - |$ which is induced from the Riemannian metric g_W. Its completion is denoted by $\Omega^i(W^+) \otimes \mathfrak{su}(2)_{L^2_{q,\delta}}$.

We define the following invariants.

- $Q^i_X \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ for $i \in \mathbb{N}$ and X by using the value of cs_X in Section 3. When X is a homotopy $S^3 \times S^1$, $Q^i_X = \infty$ for all $i \in \mathbb{N}$.

Theorem 4.1. *Under Assumption* 2.2, *the following statement holds. There exist* $\delta' > 0$ *satisfying the following property. Suppose that* δ *is a non-negative number less than* δ', *and* $\{A_n\}$ *is a sequence in* $M_\delta^{W^+}$ *satisfying* $\sup_{n\in\mathbb{N}} ||F(A_n)||^2_{L^2(W^+)} < \min\{8\pi^2, Q_X^{2l_Y+3}\}$

Then for some subsequence $\{A_{n_j}\}$, *a positive integer* N_0 *and some gauge transformations* $\{g_j\}$ *on* $W_{N_0} \cup_Y W_{N_0+1} \cdots$, *the sequences* $\{g_j^* A_{n_j}\}$ *converges to some* A_∞ *in* $L^2_{q,\delta}(W_{N_0} \cup_Y W_{N_0+1} \cdots))$.

5 Application

We construct an obstruction of embeddings f of Y into X satisfying $f_*[Y] = 1 \in H_3(X)$ as an element in the filtered instanton Floer cohomology. We use information of the compactness of ASD-moduli spaces for periodic end-4-maniofold in a crucial step of our construction.

We introduce the following invariants. Here we do not use Assumption 2.2.

- The filtered instanton Floer cohomology $HF^i_r(Y)$ for Y and $r \in \mathbb{R} \backslash cs_Y(\widetilde{R}(Y)) \cup \{\infty\}$ satisfying $HF^i_\infty(Y) = HF^i(Y)$.
- The class $[\theta^r] \in H^1_r(Y)$ for Y and $r \in \mathbb{R} \setminus cs_Y(\widetilde{R}(Y)) \cup \{\infty\}$ satisfying $[\theta^\infty] = [\theta] \in HF^1(Y)$.

Our main theorem is:

Theorem 5.1. *Under Assumption* 2.2, *if there exists an embedding* f *of* Y *into* X *with* $f_*[Y] = 1 \in H_3(X)$ *then* $[\theta^r]$ *vanishes for any* $r \in [0, Q_X^{2l_Y+3}] \cap (\mathbb{R} \setminus cs_Y(\widetilde{R}(Y)) \cup \{\infty\})$

In particular, if there exists an element $r \in [0, Q_X^{2l_Y+3}] \cap (\mathbb{R} \backslash cs_Y(\widetilde{R}(Y)) \cup \{\infty\})$ satisfying $0 \neq [\theta^r]$, Theorem 5.1 implies that there is no embedding from Y to X with $f_*[Y] = 1 \in H_3(X)$.

Example 5.2. *Let* X *be a homotopy* $S^3 \times S^1$. *There is no embedding* f *of* $\Sigma(p, q, kpq-1)$ *into* X *satisfying* $f_*[\Sigma(p, q, kpq-1))] = 1 \in H_3(X)$ *for coprime pair of positive integers* (p, q) *and any positive integer* k.

Because X is a homotopy $S^3 \times S^1$, $Q^i_X = \infty$ for $i \in \mathbb{N}$. When $r = \infty$, $[\theta^r] = [\theta]$ holds, Froyshov showed $0 \neq \theta \in HF^1(-\Sigma(p, q, pqk-1)$ for (p, q, k) in Example 5.2 by using the property of h-invariant in [Fr02]. So we can apply Theorem 5.1 for such pair (X, Y).

References

[Do02] S. K. Donaldson Floer homology groups in Yang-Mills theory Cambridge U.P. (2002)

[Fl88] Andreas Floer, An instanton-invariant for 3-manifolds, Comm. Math. Phys. 118 (1988), no. 2, 215-240.

[FO93] M.Furuta,H.Ohta, Differentiable structures on punctured 4-manifolds. Topology Appl. 51 (1993), 291-301

[Fr02] K. A. Froyshov. Equivariant aspects of Yang-Mills Floer theory. Topology, 41:525-552, (2002).

[FS90] R. Fintushel and R. J. Stern, Instanton homology of Seifert fibred homology three spheres, Proc. London Math. Soc. (3) (1990)

[FS92] R. Fintushel, R. Stern, Integer graded instanton homology groups for homology three-spheres, Topology 31 (1992), no. 3, 589-604.

[Fu90] Mikio Furuta, Homology cobordism group of homology 3-spheres, Invent. Math. 100 (1990) no. 2, 339-355.

[KM08] P. B. Kronheimer and T. S. Mrowka, Monopoles and Three-Manifolds, Cambridge University Press, Cambridge, UK, 2008.

[L16] J.Lin, The Seiberg-Witten equations on end-periodic manifolds and positive scalar curvature metrics , arXiv:1603.03698 [math.GT] (2016)

[MRS11] T. Mrowka, D. Ruberman, N. Saveliev, Seiberg-Witten equations, end-periodic Dirac operators, and a lift of Rohlin's invariant, J. Differential Geom. 88 (2011), 333-377

[T87] C.H. Taubes, Gauge theory on asymptotically periodic 4-manifolds, J. Differential Geom. 25 (1987) 363-430.

[T90] C. H. Taubes. Casson ’ s invariant and gauge theory. J. Differential Geom. 31(2) (1990)

東京大学数理科学研究科　谷口　正樹