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# A PENALTY METHOD FOR THE TIME-DEPENDENT STOKES PROBLEM WITH THE SLIP BOUNDARY CONDITION AND ITS FINITE ELEMENT APPROXIMATION

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ABSTRACT. We consider the finite element method for the time-dependent Stokes problem with the slip boundary condition in a smooth domain. To avoid a variational crime of numerical computation, a penalty method is applied, which also facilitates the numerical implementation. For the continuous problems, the convergence of the penalty method is investigated. Then, we consider the P1/P1-stabilization or P1b/P1 finite element approximations with penalty and time-discretization. For the penalty term, we propose the reduced and non-reduced integration schemes, and obtain the error estimate for velocity and pressure. The theoretical results are verified by numerical experiments.

## 1. INTRODUCTION

We consider the time-dependent Stokes problem in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$) with boundary $\partial\Omega = \gamma \cup \Gamma$, $\overline{\gamma} \cap \overline{\Gamma} = \emptyset$, which reads as:

$$(1.1) \qquad \textbf{(P)} \quad \begin{cases} u_t - \nu\Delta u + \nabla p = f, \quad \nabla \cdot u = 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \gamma \times (0,T), \\ u \cdot n = 0, \quad (I - n \otimes n)\sigma(u,p)n = 0 & \text{on } \Gamma \times (0,T), \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases}$$

where $0 < T < \infty$, $u$ and $p$ represent the velocity and pressure of the fluid, respectively, $\nu$ denotes the viscosity constant, $n$ is the unit outer normal vector to $\Gamma$, and $\sigma(u,p) = -pI + \nu(\nabla u + \nabla u^T)$ represents the stress tensor.

The slip boundary conditions $(1.1)_3$ have many applications for the real flow problems [19, 16, 12, 21]. In applying the finite element method (FEM) to **(P)**, however, there exist some numerical difficulties to deal with the slip boundary condition when $\Gamma$ is smooth. In FEM, $\Omega$ is approximated by a polygon or polyhedron $\Omega_h$ with the boundaries $\gamma_h \approx \gamma$ and $\Gamma_h \approx \Gamma$. Let $n_h$ denote the unit outer normal vector to $\Gamma_h$. If the slip boundary condition is implemented by $u_h \cdot n_h = 0$ on $\Gamma_h$, then it reduces to the homogeneous Dirichlet boundary

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condition $u_h|_{\Gamma_h} = 0$, because $n_h$ is discontinuous at the vertices of $\Gamma_h$, which is called the variational crime.

To overcome the variational crime, in [24, 23], the slip boundary condition is implemented by $(u_h \cdot n)(P) = 0$ for every boundary node $P$, where $\Omega$ is a spherical shell and the exact value of $n$ is easy to obtain. Using the quadratic approximation, the implementation $u_h(P) \cdot n(G_h(P)) = 0$ for each node or barycentre $P$ of the boundary element on $\Gamma_h$ is considered in [1], where $G_h : \Gamma_h \to \Gamma$ is an abstract transformation. Noting that $n(G_h(P))$ and $n(P)$ are nontrivial to calculate for a general $\Omega$, we can use some average of $n_h$ near $P$ for approximation (see [2, 5]), which shows good convergence properties. However, the rigorous error analysis is difficult and seems still unknown in the literature. In addition, to implement $(u_h \cdot n)(P) = 0$ in finite element code seems not easy, which requires more techniques than the Dirichlet boundary condition (see [1, 7]).

Instead of implementing the slip boundary condition directly, we consider a penalty method, the implementation of which can be easily achieved by the popular FEM software Freefem++ [9] or FEniCS [17]. Moreover, the penalty method avoids the variational crime and has good convergence properties.

The idea of the penalty method is to approximate $(u \cdot n)|_\Gamma = 0$ by the bilinear form $\frac{1}{\epsilon}\int_\Gamma (u_\epsilon \cdot n)(v \cdot n)\, d\Gamma$ for any test function $v \in H^1(\Omega)$, where $\epsilon$ is a penalty parameter with $0 < \epsilon \ll 1$. Substituting $v = u_\epsilon$ and passing to the limit $\epsilon \to 0$, we can prove the convergence $(u_\epsilon \cdot n)|_\Gamma \to 0$.

There exists some literature work on the penalty method. First, let us pay attention to the error estimate of $\|u - u_\epsilon\|$. For the stationary Stokes/Navier-Stokes problems, the sub-optimal error estimate $\|u - u_\epsilon\|_{H^1} \le C\sqrt{\epsilon}$ is proved under a priori estimate $\|\sigma(u,p)\|_{L^2(\Gamma)} \le C$; whereas the optimal error estimate $\|u - u_\epsilon\|_{H^1} \le C\epsilon$ requires a priori estimate $\|\sigma(u,p)\|_{H^{\frac{1}{2}}(\Gamma)} + |k_\epsilon| \le C$ and the estimate of error of pressure using the inf-sup condition (2.2) (cf. [4, 6, 30]), where $k_\epsilon$ is a constant from the pressure $p_\epsilon$ of the penalty problem. Owing to the compatibility of the initial value and the boundary condition for **(P)** and $(\mathbf{P}_\epsilon)$, we only have the regularity with weight $\sqrt{t}$ for $u_{tt}$ and $u_{\epsilon tt}$ near $t = 0$. As a result, the analysis method of the stationary problem cannot be directly applied to the case of time-dependent problem. In this paper, we show a priori estimates of **(P)** and $(\mathbf{P}_\epsilon)$ under various assumptions on the regularity of given data, with the help of which we derive the sub-optimal $O(\sqrt{\epsilon})$ and quasi-optimal $O(\epsilon|\log\epsilon|)$ error estimates for penalty.

Let us turn our attention to the finite element approximation and error analysis. For the stationary problems without penalty, Verfürth [27, 28, 29] obtains the error $O(h^{\frac{1}{2}})$ in energy norm using the Lagrange approach. It claims the error estimate $O(h)$ in [28, Theorem 5.1] by virtue of the estimate: $\left|\int_{\Gamma_h} u \cdot (n_h - n \cdot \pi^{-1})\sigma_h\right| \le Ch\|u\|_{H^{\frac{1}{2}}(\Gamma_h)}\|\sigma_h\|_{H^{-\frac{1}{2}}(\Gamma_h)}$, which seems non-trivial since $n_h$ is discontinuous on $\Gamma_h$. If the right-hand side is estimated by $\|\cdot\|_{L^2(\Gamma_h)}$, it yields the error estimate $O(h^{\frac{1}{2}})$. Then, under the assumption that there exists an approximation of $n$ better than $n_h$, Knobloch [15] derives the error estimate $O(h)$ for linear approximation and $O(h^{\frac{3}{2}})$ for quadratic approximation. And under the assumption that $n(G_h)$ is prescribed, Bäncsh and Deckelnick [1] prove $O(h^{\frac{3}{2}})$ for P2/P1-element approximation. For the stationary problems with penalty method, Dione and Urquiza [6] also consider the P2/P1-element approximation, and deduce $O(h^{\frac{2}{3}})$ using the technique from [1]. We mention that the error estimate of [6] reduces to $O(h^{\frac{1}{2}})$ if the contribution $\int_{v_h} \frac{1}{\sqrt{\epsilon}}\|(u_\epsilon - \bar{v}_h)\|_{L^2(\Gamma)}$ (see (4.13) of [6]) is taken into account in the error analysis (see [16, Proposition 4.2]). In [13, 30], the P1/P1-stabilization (or P1b/P1) approximation is considered, and the penalty term is implemented by reduced and non-reduced integration schemes.The authors show the error estimate $O(h^{\frac{1}{2}})$, which is improved to $O(h)$ for the two-dimensional case with reduced integration scheme.

All the above results are concerned with the stationary problem. In the present paper, we investigate the P1/P1-stabilization (or P1b/P1) full-discrete finite element approximation for the time-dependent problem with penalty. Introducing the projection operators of velocity and pressure (by the result of [13, 30]), under some assumptions on a priori estimates of (**P**) (cf. [25]), we derive the error estimate $O(\tau + h + \sqrt{\epsilon} + h/\sqrt{\epsilon})$. Taking $\epsilon = h$, we have the convergence order $O(\tau + \sqrt{h})$. For the two-dimensional case with reduced integration scheme, the error estimate is improved to $O(\tau + h + \sqrt{\epsilon} + h^2/\sqrt{\epsilon})$, which becomes $O(\tau + h)$ if $\epsilon = h^2$.

The paper is organized as follows. In Section 2, we introduce the penalty problem ($\mathbf{P}_\epsilon$), and derive a priori estimates for (**P**) and ($\mathbf{P}_\epsilon$)under various assumptions on the regularity of the initial value and force. In Section 3, we investigate the convergence behaviour of the penalty method for continuous problem. We deduce the sub-optimal $O(\sqrt{\epsilon})$ and quasi-optimal $O(\epsilon|\log \epsilon|)$ error estimates for penalty. Section 4 is devoted to the numerical analysis for the P1/P1-stabilization (or P1b/P1) finite element approximation with penalty and time-discretization. Two integration schemes (reduced and non-reduced) are considered for the penalty term, and we derive the error estimate. The numerical experiments are presented in Section 5.

**Notation.** Throughout this paper, the norms of the Sobolev spaces $H^k(\omega)$ and $W^{k,p}(\omega)$ are denoted by $\|\cdot\|_{H^k(\omega)}$ and $\|\cdot\|_{W^{k,p}(\omega)}$, respectively. The inner product of $L^2(\omega)$ or $L^2(\omega)^N$ is denoted by $(\cdot,\cdot)_\omega$. We will use the abbreviation $L^m(H^k(\omega))$ to imply $L^m(0,T;H^k(\omega))$, $L^m(0,t;H^k(\omega))$, $L^m(0,t;H^k(\omega)^N)$ or $L^m(0,T;H^k(\omega)^N)$. Sometimes, we omit $\omega$ in the above notations when $\omega = \Omega$. We introduce the notation $v_n = v \cdot n$ and $v_T = (I - n \otimes n)v$ to represent the normal and tangential component of $v$ on $\Gamma$, respectively. We use $C$ to denote generic constants independent of $\epsilon$, $h$ and $\tau$. We also use $C(a,b)$ to emphasize the constant is dependent on $a$ and $b$.

## 2. The penalty problem and a priori estimates

**2.1. Function spaces and bilinear forms.** We introduce the function spaces:

$$\begin{aligned}
&V = \{v \in H^1(\Omega)^N \mid v = 0 \text{ on } \gamma\}, \quad V_n = \{v \in V \mid v_n = 0 \text{ on } \Gamma\},\\
&H^\sigma = \{v \in L^2(\Omega)^N \mid \nabla \cdot v = 0\}, \quad H^\sigma_n = \{v \in H^\sigma \mid v_n|_\Gamma = 0 \text{ holds weakly}\},\\
&V^\sigma = \{v \in V \mid \nabla \cdot v = 0\}, \quad V^\sigma_n = V_n \cap V^\sigma, \quad Q = L^2(\Omega),\\
&\mathring{Q} = L^2_0(\Omega) = \{q \in L^2(\Omega) \mid (q,1) = 0\}, \quad \Lambda = H^{\frac{1}{2}}(\Gamma), \quad \Lambda^* = H^{-\frac{1}{2}}(\Gamma),
\end{aligned}$$

where $X^*$ denotes the dual space of a Banach space $X$.

For any $\omega \subset \mathbb{R}^N$, we set the bilinear forms:

$$a_\omega(u,v) := \frac{\nu}{2}(\mathcal{E}(u),\mathcal{E}(v))_\omega \qquad \text{for } u,v \in H^1(\omega)^N,$$
$$b_\omega(v,p) := (-\nabla\cdot v, q)_\omega \qquad \text{for } v \in H^1(\omega)^N,\ q \in L^2(\omega),$$
$$c(\lambda,\mu) := (\lambda,\mu)_\Gamma \qquad \text{for } \lambda \in \Lambda,\ \mu \in \Lambda^*,$$

where $\mathcal{E}(u) = \nabla u + \nabla u^T$ and $(\cdot,\cdot)_\omega$ denotes the inner product of $L^2(\omega)$ or the dual product between $\Lambda$ and $\Lambda^*$. We introduce some inequalities for the above bilinear forms.

(1) Korn's inequality: there exists a constant $C$ depending on $\Omega$ such that

$$a_\Omega(v,v) \geq C\|v\|_{H^1}^2, \quad \forall v \in V. \tag{2.1}$$

(2) Inf-sup condition: there exists a constant $C$ depending on $\Omega$ such that

$$C\|q\|_{L^2} \leq \sup_{v\in H_0^1(\Omega)^N} \frac{b_\Omega(v,q)}{\|v\|_{H^1}} \quad \forall q \in \mathring{Q}. \tag{2.2}$$

The variational form for $(\mathbf{P})$ reads as: for all $t \in (0,T)$,

$$\begin{cases} (u_t(t),v) + a_\Omega(u(t),v) + b_\Omega(v,p(t)) + c(\lambda(t),v_n) = (f(t),v) & \forall v \in V, \\ b_\Omega(u(t),q) = 0 & \forall q \in Q, \\ c(u_n(t),\mu) = 0 & \forall \mu \in \Lambda^*, \end{cases} \tag{2.3}$$

where $\lambda(t) := -\sigma(u(t),p(t))n\cdot n$ is the normal component of traction tensor on $\Gamma$, and $u(x,0) = u_0$. The unique existence of weak solution for $(\mathbf{P})$ follows from the standard theory (see [26, §1, Chapter 3]). In fact, given $u_0 \in H_n^\sigma$ and $f \in L^2(V_n^{\sigma*})$, then there exists a unique weak solution $u \in C([0,T];H_n^\sigma) \cap L^2(0,T;V_n^\sigma)$ for $(\mathbf{P})$ , i.e. $u$ satisfies: $u(x,0) = u_0$, and for all $t \in (0,T)$,

$$(u_t(t),v) + a_\Omega(u(t),v) = (f(t),v), \quad \forall v \in V_n^\sigma. \tag{2.4}$$

2.2. **The penalty method.** Let $\epsilon$ be the penalty parameter with $0 < \epsilon \ll 1$, and $u_{\epsilon 0}$ be an initial value approximating to $u_0$. The penalty problem in variational form is given by: for all $t \in (0,T)$, find $(u_\epsilon(t),p_\epsilon(t)) \in V \times Q$ with $u_{\epsilon t}(t) \in V^*$ and $u_\epsilon(x,0) = u_{\epsilon 0}$ such that $\forall (v,q) \in V \times Q$,

$$\begin{cases} (u_{\epsilon t}(t),v) + a_\Omega(u_\epsilon(t),v) + b_\Omega(v,p_\epsilon(t)) + \epsilon^{-1}c(u_{\epsilon n}(t),v_n) = (f(t),v), \\ b(u_\epsilon(t),q) = 0. \end{cases}$$

The strong form of the penalty problem reads as:

$$(\mathbf{P}_\epsilon)\quad \begin{cases} u_{\epsilon t} - \nu\Delta u_\epsilon + \nabla p_\epsilon = f, \quad \nabla\cdot u_\epsilon = 0 & \text{in } \Omega\times(0,T), \\ u_\epsilon = 0 & \text{on } \gamma\times(0,T), \\ \sigma(u_\epsilon,p_\epsilon)n + \epsilon^{-1}u_{\epsilon n}n = 0 & \text{on } \Gamma\times(0,T), \\ u_\epsilon(x,0) = u_{\epsilon 0} & \text{in } \Omega. \end{cases} \tag{2.5}$$

**Proposition 2.1.** *Given $u_{\epsilon 0} \in H^\sigma$ and $f \in L^2(V^{\sigma*})$, there exists a unique weak solution $u_\epsilon \in C([0,T];H^\sigma) \cap L^2(V^\sigma)$ for $(\mathbf{P}_\epsilon)$. i,e, $u_\epsilon$ satisfies: $u_\epsilon(x,0) = u_{\epsilon 0}$ and for all $t \in (0,T)$,*

$$(u_{\epsilon t}(t),v) + a_\Omega(u_\epsilon(t),v) + \epsilon^{-1}(u_{\epsilon n}(t),v_n)_\Gamma = (f(t),v), \quad \forall v \in V^\sigma.$$

2.3. **A priori estimates for (P) and ($\mathbf{P}_\epsilon$).** To obtain the error estimates of the penalty method i.e. $\|u - u_\epsilon\|$, we need a priori estimates for (**P**) and ($\mathbf{P}_\epsilon$).

2.3.1. *A priori estimate for (**P**).*

**Proposition 2.2.** (1) *Under $u_0 \in H^\sigma_n$ and $f \in L^2(V^{\sigma *}_n)$, we have:*

$$\|u\|^2_{L^\infty(L^2)} + \|u\|^2_{L^2(H^1)} \le C(\|f\|^2_{L^2(V^{\sigma *}_n)} + \|u_0\|^2_{L^2}) =: C_1(f, u_0).$$

(2) *Under $u_0 \in V^\sigma_n$ and $f \in L^2(L^2)$, we have:*

$$\|u_t\|^2_{L^2(L^2)} + \|u\|^2_{L^\infty(H^1)} \le C(\|f\|^2_{L^2(L^2)} + \|u_0\|^2_{H^1}) =: C_2(f, u_0).$$

(3) *Under $u_0 \in V^\sigma_n \cap H^2(\Omega)^N$, $f \in C([0,T]; L^2)$ and $f_t \in L^2(0,T; L^2)$, we have:*

$$\|u_t\|^2_{L^\infty(L^2)} + \|u_t\|^2_{L^2(H^1)} \le C_{31}(f, u_0), \tag{2.6a}$$

$$\|\sqrt{t} u_{tt}\|^2_{L^2(L^2)} + \|\sqrt{t} u_t\|^2_{L^2(H^1)} \le C\|\sqrt{t} f\|^2_{L^2(L^2)} + C_{31}(f, u_0), \tag{2.6b}$$

*where $C_{31}(f, u_0) := C(\|f_t\|^2_{L^2(V^{\sigma *}_n)} + \|u_0\|^2_{H^2} + \|f\|^2_{C([0,t];L^2)})$. In addition, if $u_0 \in H^3(\Omega)^N$ and $f(0) \in H^1(\Omega)^N$, then we have:*

$$\|u_{tt}\|^2_{L^2(L^2)} + \|u_t\|^2_{L^2(H^1)} \le (\|f_t\|^2_{L^2(L^2)} + \|u_0\|^2_{H^3} + \|f(0)\|^2_{H^1}) =: C_{32}(f, u_0). \tag{2.7}$$

*Remark* 2.1 (Regularity for $u$). By a similar argument of [10, Theorems 2.4 and 2.5], we can show the regularity $\sup_{0<t<T} t^{2n+m-2}\|D^n_t u\|^2_{H^m} < \infty$ when $\Omega$ and $f$ are sufficiently smooth, which implies that we can obtain any regularity of $u$ in time interval $(t_a, T)$ for some $t_a > 0$.

*Remark* 2.2 (Regularity for $p$). Consider the stationary Stokes problem with the slip boundary condition:

$$\begin{cases} -\Delta u^* + \nabla p^* = f^*, \quad \nabla \cdot u^* = 0 \quad \text{in } \Omega, \\ u^* = 0 \quad \text{on } \gamma, \quad u^*_n = 0, \quad (I - n \otimes n)\sigma(u^*, p^*)n = 0 \quad \text{on } \Gamma. \end{cases}$$

For sufficiently smooth $\gamma$ and $\Gamma$, given $f^* \in H^m(\Omega)^N$ ($m \in \mathbb{N}$), the following regularity holds (cf. [20]): $\|u^*\|_{H^{m+2}} + \|p^*\|_{H^{m+1}} \le C\|f^*\|_{H^m}$. As a result, we conclude from Proposition 2.2 (2) that:

$$\|u\|_{L^2(H^2)} + \|p\|_{L^2(H^1)} \le C_2(f, u_0).$$

And it follows from (2.6) and (2.7) that:

$$\|u\|_{C([0,T];H^2)} + \|p\|_{C([0,T];H^1)} \le C_{31}(f, u_0), \tag{2.8a}$$

$$\|u_t\|_{L^2(H^2)} + \|p_t\|_{L^2(H^1)} \le C_{32}(f, u_0). \tag{2.8b}$$

2.3.2. *A priori estimate for ($\mathbf{P}_\epsilon$).*

**Proposition 2.3.** (1) *Under $u_{\epsilon 0} \in H^\sigma$ and $f \in L^2(V^{\sigma *})$, we have:*

$$\|u_\epsilon\|^2_{L^\infty(L^2)} + \|u_\epsilon\|^2_{L^2(H^1)} + \epsilon^{-1}\|u_{\epsilon n}\|^2_{L^2(L^2(\Gamma))} \le C_1(f, u_{\epsilon 0}).$$

(2) *Under $u_{\epsilon 0} \in V^\sigma$ with $\|u_{\epsilon 0} \cdot n\|_{L^2(\Gamma)} \le C\sqrt{\epsilon}$ and $f \in L^2(L^2)$, we have:*

$$\|u_{\epsilon s}\|^2_{L^2(L^2)} + \|u_\epsilon\|^2_{L^\infty(H^1)} + \epsilon^{-1}\|u_{\epsilon n}\|^2_{L^\infty(L^2(\Gamma))} \le C_2(f, u_{\epsilon 0}) + C\epsilon^{-1}\|u_{\epsilon 0}\|^2_{L^2(\Gamma)}.$$

(3) *Under* $u_{\epsilon 0} \in V^\sigma \cap H^2(\Omega)^N$, $\|u_{\epsilon 0}\cdot n\|_{H^{\frac{1}{2}}(\Gamma)} \le C\epsilon$, $f \in C([0,T];L^2)$ *and* $f_t \in L^2(L^2)$, *we have:*

(2.9a) $$\|u_{\epsilon t}\|^2_{L^\infty(L^2)} + \|u_{\epsilon t}\|^2_{L^2(H^1)} \le C_{31}(f,u_{\epsilon 0}) + C\|\epsilon^{-1}u_{\epsilon 0}\cdot n\|^2_{H^{\frac{1}{2}}(\Gamma)},$$

(2.9b) $$\|\sqrt{t}u_{\epsilon tt}\|^2_{L^2(L^2)} + \|\sqrt{t}u_{\epsilon t}\|^2_{L^\infty(H^1)} \le C_{32}(f,u_{\epsilon 0}) + C\|\epsilon^{-1}u_{\epsilon 0}\cdot n\|^2_{H^{\frac{1}{2}}(\Gamma)}.$$

*Remark* 2.3 (Regularity for $u_\epsilon$). By a similar argument to [10, Theorems 2.4 and 2.5], we can obtain any regularity for $u_\epsilon$ away from $t=0$. However, we have a breakdown of regularity for $u$ on $\partial\Omega$ at $t=0$. In order to derive $\|u_{\epsilon tt}\|_{L^2(L^2)} \le C$, we need $u_{\epsilon t}(0) \in H^1(\Omega)^N$ and $\epsilon^{-1}\|u_{\epsilon t}(0)\cdot n\|_{L^2(\Gamma)} \le C$, which cannot be realistically assumed. Hence, we only have $\sqrt{t}u_{\epsilon tt} \in L^2(L^2)$.

*Remark* 2.4 (Regularity for $p_\epsilon$). Consider the stationary Stokes problem with penalty:

$$\begin{cases} -\Delta u^*_\epsilon + \nabla p^*_\epsilon = f^*, \quad \nabla\cdot u^*_\epsilon = 0 \quad \text{in } \Omega, \\ u^*_\epsilon = 0 \quad \text{on } \gamma, \quad \sigma(u^*_\epsilon, p^*_\epsilon)n + \epsilon^{-1}u^*_{\epsilon n}n = 0 \quad \text{on } \Gamma. \end{cases}$$

For sufficiently smooth $\gamma$ and $\Gamma$, given $f^* \in H^m(\Omega)^N$ ($m \in \mathbb{N}$), we have the following regularity (cf. [30, Appendix]): $\|u^*_\epsilon\|_{H^{m+2}} + \|p^*_\epsilon\|_{H^{m+1}} \le C\|f^*\|_{H^m}$. Then, we obtain from (2.9) that

(2.10a) $$\|u_\epsilon\|_{C([0,T];H^2)} + \|p_\epsilon\|_{C([0,T];H^1)} \le C_{31}(f,u_0) + C\|\epsilon^{-1}u_{\epsilon 0}\cdot n\|_{H^{\frac{1}{2}}(\Gamma)},$$

(2.10b) $$\|\sqrt{t}u_{\epsilon t}\|_{L^2(H^2)} + \|\sqrt{t}p_{\epsilon t}\|_{L^2(H^1)} \le C_{32}(f,u_{\epsilon 0}) + C\|\epsilon^{-1}u_{\epsilon 0}\cdot n\|_{H^{\frac{1}{2}}(\Gamma)}.$$

## 3. The error estimate of penalty method

With the help of a priori estimates in the previous section, we investigate the convergence of the penalty method.

**3.1. The sub-optimal error estimate.** Setting $\lambda_\epsilon := \frac{1}{\epsilon}u_{\epsilon n}$, we rewrite (2.5) as

(3.1) $$\begin{cases} (u_{\epsilon t}, v) + a_\Omega(u_\epsilon, v) + b_\Omega(v, p_\epsilon) + c(\lambda_\epsilon, v_n) = (f,v) & \forall v \in V, \\ b(u_\epsilon, q) = 0 & \forall q \in Q, \\ c(u_{\epsilon n}, \mu) = \epsilon c(\lambda_\epsilon, \mu) & \forall \mu \in \Lambda^*. \end{cases}$$

Since $u_{\epsilon n}|_\Gamma \neq 0$, we see that $p_\epsilon(t) \notin \mathring{Q}$. We set

$$k_\epsilon(t) = \frac{1}{|\Omega|}\int_\Omega p_\epsilon(t)\,dx, \quad \mathring{p}_\epsilon(t) = p_\epsilon(t) - k_\epsilon(t) \in \mathring{Q}.$$

Recalling that $\lambda(t) = -\sigma(u(t),p(t))n\cdot n$, we set:

$$e_u(t) := u(t) - u_\epsilon(t), \quad e_p(t) := p(t) - \mathring{p}_\epsilon(t), \quad e_\lambda(t) := \lambda(t) - (\lambda_\epsilon(t) - k_\epsilon(t)).$$

We state the sub-optimal error estimate result.

**Theorem 3.1.** (1) *Under the initial error* $\|u_0 - u_{\epsilon 0}\|_{L^2} \le C_{i1}\sqrt{\epsilon}$, $u_0 \in V^\sigma_n$ *and* $f \in L^2(L^2)$, *we have*

(3.2) $$\|e_u\|_{L^\infty(L^2)} + \|e_u\|_{L^2(H^1)} + \sqrt{\epsilon}\|\lambda - \lambda_\epsilon\|_{L^2(L^2(\Gamma))} \le C\sqrt{\epsilon}.$$

(2) *In addition, if we assume* $\|u_0-u_{\epsilon 0}\|_{H^1} \le C_{i1}\sqrt{\epsilon}$, $\|u_{\epsilon 0}\cdot n\|_{L^2(\Gamma)} \le C\epsilon$, $u_0 \in V_n^\sigma \cap H^3(\Omega)^N$, $f(0) \in H^1(\Omega)^N$ *and* $f_t \in L^2(L^2)$, *then we have*

$$\|e_{ut}\|_{L^2(L^2)} + \|e_u\|_{L^\infty(H^1)} + \sqrt{\epsilon}\|\lambda - \lambda_\epsilon\|_{L^\infty(L^2(\Gamma))} \le C\sqrt{\epsilon}. \tag{3.3}$$

3.2. **The quasi-optimal error estimate.** Under stronger assumptions than in Theorem 3.1, we prove the quasi-optimal error estimate.

**Theorem 3.2.** *We take the assumption of Theorem 3.1 (2). Moreover we assume that* $\|u_0 - u_{\epsilon 0}\|_{L^2} \le C_{i2}\epsilon$, $\|u_{\epsilon 0}\cdot n\|_{H^{\frac{1}{2}}(\Gamma)} \le C\epsilon$ *and* $f \in C([0,T]; L^2)$. *Then we have:*

$$\|e_u\|_{L^\infty(L^2)} + \|e_u\|_{L^2(H^1)} + \|\sqrt{t}e_u\|_{L^\infty(H^1)} + \|\sqrt{t}e_{ut}\|_{L^2(L^2)} \le C\epsilon|\log\epsilon|. \tag{3.4}$$

We explain the main difficulties and introduce a sketch of the proof of Theorem 3.2. In the case of the stationary Stokes problem, the estimates of $\|e_p\|_{L^2}$ and $\|e_\lambda\|_{H^{-\frac{1}{2}}(\Gamma)}$ follow from the $H^1$-norm estimate of $e_u$ by the inf-sup conditions of $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$. However, for the non-stationary problem, we have to deal with the estimates of $e_{ut}(t)$, $e_p$ and $e_\lambda$ at the same time.

To prove (3.4), we consider the energy estimate of $e_u$ in $L^2(H^1)$ and $L^2(L^2)$ norms, and the energy estimate of $\sqrt{t}e_{ut}$ in $L^2(L^2)$ norm and $\sqrt{t}e_u$ in $L^\infty(H^1)$ norm. Since we have the a-priori estimates (2.9b) and (2.10b) with weight $\sqrt{t}$ near $t=0$, we divide the estimate of $e_u$ into three cases: (1) $0 < t \le \epsilon$, (2) $\epsilon < t \le 1$ and (3) $t > 1$. For case (1), the desired result follows from the energy estimate of $e_u$, the sub-optimal error estimate (3.3) and $t \le \epsilon$. In case (2), we apply the a-priori estimates (2.9b) and (2.10b). Owing to the weight $\sqrt{t}$ and $\epsilon < t \le 1$, we get the error bound $O(\epsilon|\log\epsilon|)$. The case (3) is easy, since we have the regularity without weight $\sqrt{t}$ according to Remarks (2.2) and (2.3). Combining these three cases, we can conclude (3.4).

*Remark* 3.1. Because of the nonlocal compatibility condition $\|u_{\epsilon t}(0)\|_{H^1(\Omega)} \le C$ is unrealistic to assume, we only have a priori estimate for $u_{\epsilon tt}$ with weight $\sqrt{t}$ (see Proposition 2.3 (3)). Moreover, the initial error $\|\lambda(0) - \epsilon^{-1}u_{\epsilon 0}\cdot n + k_\epsilon(0)\|_{L^2(\Gamma)} \le C\sqrt{\epsilon}$ seems non-trivial to ensure. For the above two reasons, we obtain the error estimate for $e_{ut}$ with weight $\sqrt{t}$, and derive the error estimate $O(\epsilon|\log\epsilon|)$ instead of $O(\epsilon)$.

## 4. The finite element approximation

We introduce a regular triangulation $\mathcal{T}_h$ to $\Omega_h$, where $h := \max_{K\in\mathcal{T}_h}\operatorname{diam}(K)$ denotes the mesh size. In this paper, the P1/P1-stabilization (or P1b/P1) finite element approximation is considered. We set the finite element spaces for P1/P1 (or P1b/P1) element:

$$V_h = \{v_h \in C(\overline{\Omega_h})^N \mid v_h \in P_1(K)^N \ \forall K \in \mathcal{T}_h, v_h = 0 \text{ on } \gamma_h\}, \quad \text{for P1/P1},$$

$$V_h = \{v_h \in C(\overline{\Omega_h})^N \mid v_h \in P_1(K)^N \oplus B(K)^N \ \forall K \in \mathcal{T}_h, v_h = 0 \text{ on } \gamma_h\}, \quad \text{for P1b/P1},$$

$$Q_h = \{q_h \in C(\overline{\Omega_h})^N \mid q_h \in P_1(K) \ \forall K \in \mathcal{T}_h\}, \quad \mathring{Q}_h = Q_h \cap L^2_0(\Omega_h),$$

where $P_1(K)$ is the set of linear polynomials in a triangle $K$ and $B(K)$ stands for the bubble function space on $K$. We denote by $\mathcal{S}_h$ the triangulation of $\Gamma_h$ inherited from $\mathcal{T}_h$. For the Dirichlet boundary condition $u|_\gamma = 0$, the error owing to the approximation $u_h|_{\gamma_h} = 0$ has been well studied in the literature. In this paper, we focus on dealing with the slip boundary condition. In the following argument, we ignore the difference between $\gamma$ and $\gamma_h$ (i.e. we assume $\gamma = \gamma_h$) for simplicity.

We consider the backward approximation for time differentiation. For an integer $M \in \mathbb{N}_+$ ($M \gg 1$), we denote by $\tau := \frac{T}{M}$ the time-step size. For $t_m = m\tau$ with $m = 0, 1, \dots, M$, we set $(u^m, p^m) := (u(t_m), p(t_m))$. We use $\partial_\tau u^m := \frac{u^m - u^{m-1}}{\tau}$ to denote the backward approximation. Given the initial value $u_{0h} \in V_h$, the finite element approximation problem reads as:

$$(4.1) \qquad (\mathbf{P}_{\epsilon,h}) \quad \begin{cases} \text{find } (u_h^m, p_h^m) \in V_h \times Q_h,\ m = 1, \dots, M, \text{ such that} \\ (\partial_\tau u_h^m, v_h)_{\Omega_h} + a_{\Omega_h}(u_h^m, v_h) + b_{\Omega_h}(v_h, p_h^m) \\ \quad + \epsilon^{-1} c_h(u_h^m \cdot n_h, v_h \cdot n_h) = (\tilde{f}^m, v_h)_{\Omega_h}, \quad \forall v_h \in V_h, \\ b_{\Omega_h}(u_h^m, q_h) = \eta h^2 (\nabla p_h^m, \nabla q_h)_{\Omega_h}, \quad q_h \in Q_h, \end{cases}$$

where $\tilde{f}$ is a continuous extension of $f$ to $\Omega_h$ (note that $\Omega \neq \Omega_h$), $\eta = 0$ for P1b/P1 element and $\eta = 1$ for P1/P1 element. We assume $f \in C([0,T]; L^2)$ so that $\tau \sum_{m=1}^{M} \|\tilde{f}^m\|_{L^2(\Omega_h)}^2 \le C$. The bilinear form $c_h(\cdot,\cdot)$ is defined below.

We consider two types of $c_h(\cdot,\cdot)$ to approximate $c(\cdot,\cdot)$: for any $\lambda_h, \mu_h \in \Lambda_h = \{v_h \cdot n_h \text{ on } \Gamma_h \mid v_h \in V_h\}$,

$$c_h(\lambda_h, \mu_h) = \begin{cases} c_h^N(\lambda_h, \mu_h) := (\lambda_h, \mu_h)_{\Gamma_h}, \\ c_h^R(\lambda_h, \mu_h) := \sum_{S \in \mathcal{S}_h} |S| \lambda_h(m_S) \mu_h(m_S), \end{cases}$$

where $m_S$ is the barycentre of $S$. We set $\|\cdot\|_{c_h}^2 := c_h(\cdot,\cdot)$. $c_h^R(\cdot,\cdot)$ is the barycentre formula approximating to $c_h^N(\cdot,\cdot)$.

We introduce some inequalities for the bilinear forms $a_{\Omega_h}(\cdot,\cdot)$ and $b_{\Omega_h}(\cdot,\cdot)$.

(1) Korn's inequality (cf. [3, 14]): there exists a constant $C$ such that

$$(4.2) \qquad a_{\Omega_h}(v_h, v_h) \ge C \|v_h\|_{H^1(\Omega_h)}^2 \quad \forall v_h \in V_h.$$

(2) Inf-sup condition (cf. [8, 22]): there exists a constant $C$ such that

$$(4.3) \qquad \sup_{v_h \in \mathring{V}_h} \frac{b_{\Omega_h}(v_h, q_h)}{\|v_h\|_{V_h}} + C\eta h \|\nabla q_h\|_{L^2(\Omega_h)} \ge C \|q_h\|_{L^2(\Omega_h)} \quad \forall q_h \in \mathring{Q}_h,$$

where $\mathring{V}_h := \{v_h \in V_h \mid v_h = 0 \text{ on } \Gamma_h\}$.

**Proposition 4.1.** *There exists a unique solution $\{(u_h^m, p_h^m)\}_{m=1}^M \subset V_h \times Q_h$ for $(\mathbf{P}_{\epsilon,h})$ satisfying:*

$$(4.4) \qquad \begin{aligned} &\|u_h^m\|_{L^2(\Omega_h)}^2 + 2\tau \sum_{j=1}^{m} \left[ \|u_h^j - u_h^{j-1}\|_{L^2(\Omega_h)}^2 + \|u_h^j\|_{H^1(\Omega_h)}^2 + \eta h^2 \|\nabla p_h^j\|_{L^2(\Omega_h)}^2 \right] \\ &+ \epsilon^{-1} 2\tau \sum_{j=1}^{m} \|u_h^j \cdot n_h\|_{c_h}^2 \le C \|u_h^0\|_{L^2(\Omega_h)}^2 + C\tau \sum_{j=1}^{m} \|\tilde{f}^j\|_{L^2(\Omega_h)}^2. \end{aligned}$$

*Assume that $u_h^0$ satisfies $\epsilon^{-1}\|u_h^0 \cdot n_h\|_{c_h}^2 \le C$. And for P1/P1 element, we assume there exists a $p_h^0 \in Q_h$ such that $b_{\Omega_h}(u_h^0, q_h) = \eta h^2 (\nabla p_h^0, \nabla q_h)_{\Omega_h}$ for all $q_h \in Q_h$. For P1b/P1 element, we* *assume* $b_{\Omega_h}(u_h^0, q_h) = 0$ *for all* $q_h \in Q_h$. *Then we have:*

$$
\begin{aligned}
&\tau \sum_{j=1}^{m} \|\partial_\tau u_h^j\|_{L^2(\Omega_h)}^2 + \|u_h^m\|_{H^1(\Omega_h)} + \epsilon^{-1}\|u_h^m \cdot n_h\|_{c_h}^2 + \eta h^2 \|\nabla p_h^m\|_{L^2(\Omega_h)}^2 \\
&+ \sum_{j=1}^{m} \Big[\eta h^2 \|\nabla (p_h^j - p_h^{j-1})\|_{L^2(\Omega_h)}^2 + \epsilon^{-1}\|(u_h^j - u_h^{j-1}) \cdot n_h\|_{c_h}^2 + \|u_h^j - u_h^{j-1}\|_{H^1(\Omega_h)}\Big] \\
&\leq C \left( \tau \sum_{j=1}^{m} \|\tilde{f}^j\|_{L^2(\Omega_h)}^2 + \|u_h^0\|_{H^1(\Omega_h)}^2 + \epsilon^{-1}\|u_h^0 \cdot n_h\|_{c_h}^2 + \eta h^2 \|\nabla p_h^0\|_{L^2(\Omega_h)}^2 \right).
\end{aligned}
\tag{4.5}
$$

To obtain the error estimate for finite element discretization, we introduce a projection lemma, which is a direct result from [13, 30] for the stationary case.

**Lemma 4.2** ([13, Theorems 4.1 and 5.1 and their proofs]). *Let* $(\tilde{u}^m, \tilde{p}^m)$ *be a continuous extension of* $(u^m, p^m)$ *to* $\tilde{\Omega} := \Omega \cup \Omega_h$ *with* $\tilde{f}^m = \tilde{u}_t^m - \nu\Delta\tilde{u}^m + \nabla\tilde{p}^m$ *for* $m = 1, \dots, M$. *There exists a unique* $(P^u\tilde{u}^m, P^p\tilde{p}^m) \in V_h \times Q_h$ *such that*

$$
\begin{aligned}
&a_{\Omega_h}(P^u\tilde{u}^m, v_h) + b_{\Omega_h}(v_h, P^p\tilde{p}^m) + \epsilon^{-1}c_h(P^u\tilde{u}^m \cdot n_h, v_h \cdot n_h) = (\tilde{f}^m - \tilde{u}_t^m, v_h) \quad \forall v_h \in V_h, \\
&b_{\Omega_h}(P^u\tilde{u}^m, q_h) = \eta h^2 (\nabla P^p\tilde{p}^m, \nabla q_h)_{\Omega_h} \quad \forall q_h \in Q_h.
\end{aligned}
$$

*Moreover, the following error estimate holds.*

(i) *For the non-reduced integration* $c_h(\cdot,\cdot) = c_h^N(\cdot,\cdot)$,

$$\|P^u\tilde{u}^m - \tilde{u}^m\|_{V_h} + \|P^p\tilde{p}^m - \tilde{p}^m\|_{Q_h/\mathbb{R}} + \eta h \|\nabla P^p\tilde{p}^m\|_{L^2(\Omega_h)} \leq C\left(h + \sqrt{\epsilon} + h/\sqrt{\epsilon}\right).$$

(ii) *For the reduced integration* $c_h(\cdot,\cdot) = c_h^R(\cdot,\cdot)$,

$$\|P^u\tilde{u}^m - \tilde{u}^m\|_{V_h} + \|P^p\tilde{p}^m - \tilde{p}^m\|_{Q_h/\mathbb{R}} + \eta h \|\nabla P^p\tilde{p}^m\|_{L^2(\Omega_h)} \leq C\left(h + \sqrt{\epsilon} + h^\beta/\sqrt{\epsilon}\right),$$

*where* $\beta = 2$ *if* $N = 2$ *and* $\beta = 1$ *if* $N = 3$.

*Remark* 4.1. In [13] the above error estimates (i)(ii) are obtained under the implicit assumption that $k_h^m := \frac{1}{|\Omega_h|}\int_{\Omega_h} P^p\tilde{p}^m\, dx$ is bounded independently of $\epsilon$ and $h$. In fact, with a careful examination, we have $k_h^m \leq C(1 + \frac{h^2}{\epsilon})$. However, with this a-priori estimate, we can still obtain the error estimates (i)(ii).

We state the error estimate result for the finite element approximation with penalty. For $(\tilde{u}^m, \tilde{p}^m)$ given by Lemma 4.2, we set the error functions:

$$e_{h,u}^m := u_h^m - \tilde{u}^m, \quad e_{h,p}^m := p_h^m - \tilde{p}^m.$$

We make the assumptions on $(u, p)$ and initial error $\|\tilde{u}_0 - u_h^0\|_{L^2(\Omega_h)}$:

**($A_e$1)** $u \in C^2([0,T]; L^2) \cap C^1([0,T]; W^{2,r})$, where $r = \infty$ if $c_h(\cdot,\cdot) = c_h^R(\cdot,\cdot)$ with $N = 2$, otherwise $r = 2$.

**($A_e$2)** $\|\tilde{u}_0 - u_h^0\|_{L^2(\Omega_h)} \leq Ch$. For P1b/P1-element, $b_{\Omega_h}(u_h^0, q_h) = 0$ for all $q_h \in Q_h$.

*Remark* 4.2 (Regularity assumption for FEM). As stated in Remark 2.1, the assumptions **($A_e$1)** implies nonlocal compatibilty conditions for $f(0)$ and $u_0$. But **($A_e$1)** can be achieved in time interval $(t_a, T)$ for some $t_a > 0$ with smooth $f$ and $u_0$. As an analogue to [25], we assume **($A_e$1)** and deduce the error estimate for finite element approximation. We refer the readers to [11, 18] for the smoothing property and error estimate for finite element method.

**Theorem 4.3.** *Under the assumptions* $(\mathbf{A_e1})(\mathbf{A_e2})$ *we have:*

$$\|e^m_{h,u}\|^2_{L^2(\Omega_h)} + \tau \sum_{j=1}^{m} \|e^j_{h,u}\|^2_{V_h} \le C(\tau + h + \sqrt{\epsilon} + h^\beta/\sqrt{\epsilon})^2, \tag{4.6a}$$

$$\begin{aligned} &\tau \sum_{j=1}^{m} t_{j-1}\|\partial_\tau e^j_{h,u}\|^2_{L^2(\Omega_h)} + t_{m-1}\|e^m_{h,u}\|^2_{V_h} + \tau \sum_{j=1}^{m} t_{j-1}\|\partial_\tau e^j_{h,p}\|^2_{Q_h/\mathbb{R}} \\ \le& C(\tau + h + \sqrt{\epsilon} + h^\beta/\sqrt{\epsilon})^2, \end{aligned} \tag{4.6b}$$

*where* $\beta = 1$ *for* $c_h(\cdot,\cdot) = c^N_h(\cdot,\cdot)$ *with* $N = 2, 3$, *and* $c_h(\cdot,\cdot) = c^R_h(\cdot,\cdot)$ *with* $N = 3$. *It can be improved to* $\beta = 2$ *when* $c_h(\cdot,\cdot) = c^R_h(\cdot,\cdot)$ *and* $N = 2$.

*Remark* 4.3. In view of the error estimate results (4.6a) and (4.6b), we have the optimal choice of $\epsilon$ and $h$.

(1) For the non-reduced integration $c_h(\cdot,\cdot) = c^N_h(\cdot,\cdot)$, we choose $\epsilon = Ch$ and obtain the error $O(\sqrt{h} + \tau)$.
(2) For the reduced integration $c_h(\cdot,\cdot) = c^R_h(\cdot,\cdot)$, when $N = 3$ we choose $\epsilon = Ch$ and obtain the error $O(\sqrt{h} + \tau)$; when $N = 2$, we set $\epsilon = Ch^2$ and obtain the error $O(h + \tau)$.

## 5. The numerical experiment

We consider **(P)** in an annular domain $\Omega = \{(x,y) \mid 1 \le x^2 + y^2 < 4\}$ with boundaries $\Gamma = \{(x,y) \mid x^2 + y^2 = 4\}$ and $\gamma = \{(x,y) \mid x^2 + y^2 = 1\}$. $f$ and $u_0$ are given such that we have the exact solution:

$$u(x,y,t) = ((t^2+1)y(x^2+y^2-1), -(t^2+1)x(x^2+y^2-1)), \quad p(x,y,t) = (t^2+1)xy.$$

It is easy to see that $n = \frac{1}{2}(x,y)^T$ and $u_n = 0$ on $\Gamma$. In view of $g := (I - n \otimes n)\sigma(u,p)n \ne 0$ on $\Gamma$, we have to add $\int_\Gamma g \cdot v_T \, d\Gamma$ to the right-hand side of $(2.3)_1$ and $(3.1)_1$. Correspondingly, we add $\int_{\Gamma_h} \tilde{g}^m \cdot v_{hT} \, d\Gamma_h$ to the right-hand side of $(4.1)_1$, $\tilde{g}^m := (I - n_h \otimes n_h)\sigma(u(t_m), p(t_m))n_h$ is an approximation of $g(t_m)$.

We solve **(P)** by the penalty method with finite element approximation, and test both the non-reduced ($c^N(\cdot,\cdot)$) and reduced ($c^R(\cdot,\cdot)$) integration schemes for the penalty term. In the following, we show the error of numerical solutions for the case of P1/P1 element. For the case of P1b/P1 element, the numerical results are almost the same.

First, fixing $h$ and $\tau$, we plot the errors of the non-reduced and reduced schemes in Figure 1, where N and R stand for the non-reduced and reduced scheme, respectively. From this, we can observe that the orders of convergence of both the schemes are almost $O(\epsilon)$, which verifies our theoretical results (see Theorem 3.2). Note that the error is saturated as $\epsilon$ decreases because $h$ and $\tau$ are fixed. Moreover, we observe that for the non-reduced integration scheme, the convergence fails for $\epsilon \ll h$, which does not occur for the reduced integration scheme. It suggests that the reduced scheme is more stable for small $\epsilon$ than the non-reduced one.

Next, we plot the errors depending on $h$ in Figure 2. According to Theorem 4.3 and Remark 4.3, the optimal choice is to let $\epsilon = Ch$ for non-reduced scheme and $\epsilon = Ch^2$ for non-reduced scheme ($N = 2$). We observe that the convergence orders of the non-reduced scheme are $O(h)$, which is better than our theoretical result $O(\sqrt{h})$ (see Remark 4.3). For the

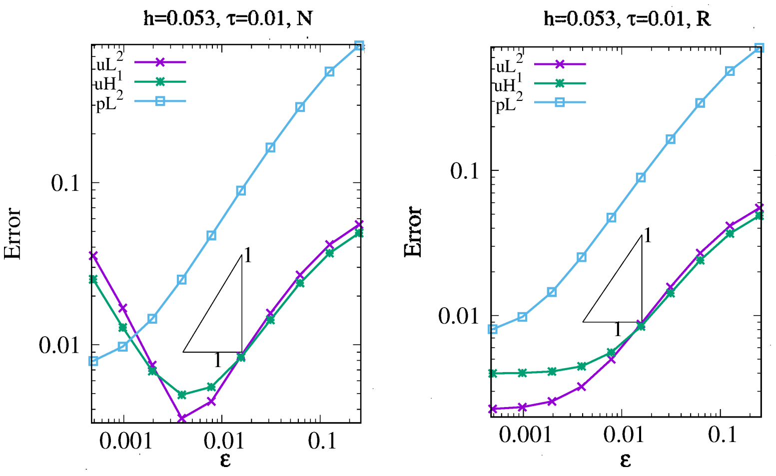


FIGURE 1. The errors of velocity in the $L^2$ and $H^1$ norms and pressure in the $L^2$ norm (denoted by $uL^2$, $uH^1$ and $pL^2$ respectively) are plotted for different $\epsilon$ with $h$ and $\tau$ fixed. The slopes represent the order $O(\epsilon)$.

reduced scheme, we see that the convergence order of velocity in the $H^1$ norm is $O(h)$, which corresponds to our theoretical result (see Remark 4.3). Moreover, the numerical experiment shows the convergence order of velocity in the $L^2$ norm is $O(h^2)$. Because we have fixed $\tau = 0.01$, the $L^2$ error is saturated as $h$ decreases (see Figure 2 (right)).

At last, we verify the errors depending on $\tau$. Theorem 4.3 shows that for fixed $\epsilon$ and $h$, the convergence orders are estimated to be $O(\tau)$, which is confirmed by our numerical examples, see Figure 3.

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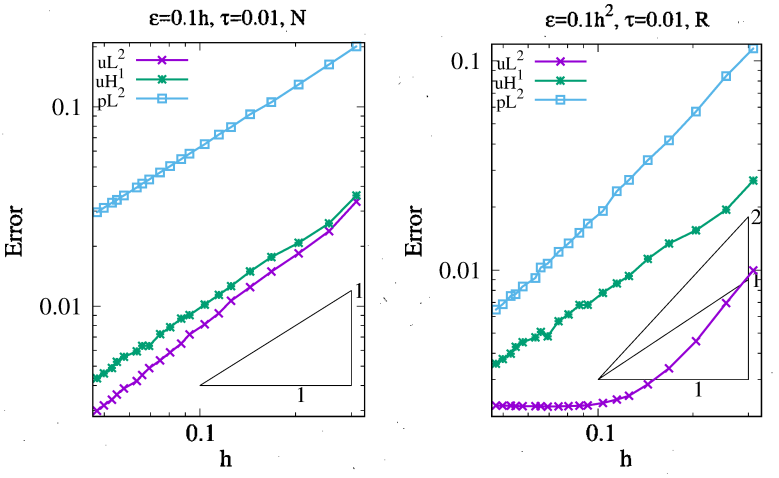


FIGURE 2. The relative errors are plotted for different $h$. We set $\epsilon = 0.1h$ for the non-reduced scheme and $\epsilon = 0.1h^2$ for the reduced scheme and fix $\tau = 0.01$. The slope in the left figure represents the order $O(h)$. The lower slope in the right figure represents the order $O(h)$, the higher one represents $O(h^2)$.

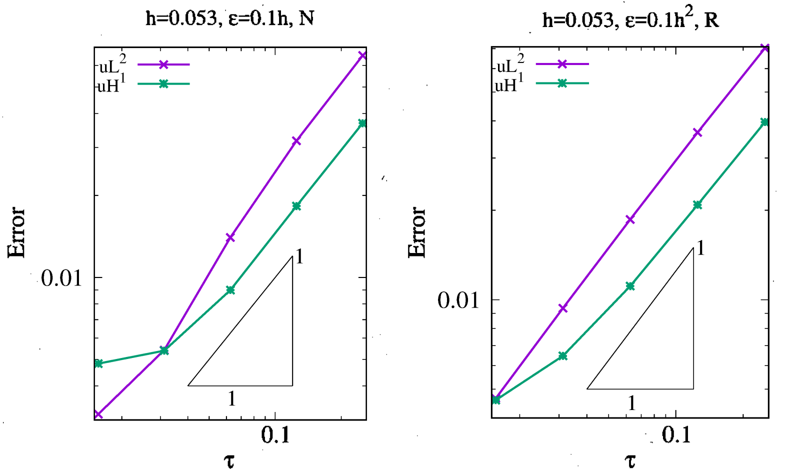


FIGURE 3. The errors are plotted for different $\tau$ with $h$ and $\epsilon$ fixed. The slopes represent the order $O(\tau)$.

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